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Introduction to the Study of the Signature Function.

Stage de M1

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Preface.

Last year, for the first time, I worked six weeks in a mathematical laboratory. This year I had the great oportunity to do it again. I have learnt a lot from this new experience and especially thanks to my advisor Selim Esedoglu.

First of all, from the mathematical point of view, I discovered what investing new ideas and new fields looked like, with its moments of excitement when everything turns well, and its moments of frustration when things seem harder. Beside that I have also learnt a lot, concerning matlab, xfig or the bibliographical work for example. Thanks to this internship, I have been able to improve my english (oral by talking with people and written by writing this report), discover a different society and different people. For me it was the first oportunity to discover the United States and its inhabitants. I also had the wonderful chance of experimenting the co-ops. I lived these six weeks in a house with thirty people (mainly americans but not only) who made me feel really at home by their openness and kindness. I have more than appreciated this co-opers way of living.

I would like to thank Selim Esedoglu for having accepted to be my advisor during this internship. I am very grateful to him for having introduced me to this very interesting problem and showing me a new field of mathematics. I have more than apreciated the regular occasions that we had to speak and exchange ideas. He was always very enthusiastic and encouraging even when I was doubting. Every time I saw him, he had a lot of new ideas which made the progress of this internship easier. The topic treated in this report is mainly original and might lead me to publish an article with my advisor.

1 Introduction.

In image analysis, one of the main problems is called the image segmentation problem. It consists of finding an effective way of extracting automatically, the contours of the different objects in the image. The first thing one has to do before one solves this problem is to model it.

We will represent an image as a function g of $[0,1]^2$ to $[0,1]$. We will then suppose that we are given a functional F, defined on a set of admissible sets E, which evaluates the difference between the set of discontinuities of g and the boundary of E. Hence minimizing F will give us the set of discontinuities of g, which represents the edges of the objects of our image. In 1989 David Mumford and Jayant Shah introduced in [8] one of these functionals. The complete study of the Mumford-Shah functional is still not finished but see [1] for an introduction to its study. The Mumford-Shah functional does not however utilise any prior shape information. In order to avoid this problem, we can add to the functional a term which should take into account some a priori knowledge about the image.

Let us suppose that we know that an object of shape represented by the set Ω is contained in the image. However, we will also suppose that we do not know its position and orientation. Hence, the term that we want to add should be invariant under solid motion (rotation and translation) and be zero for Ω.

For $\mu \geq 0$ let $F_\Omega(\mu)$ be the "amount" of lines that have an intersection with Ω whose length is greater than μ. We will explain in the sequel what we mean by "amount". F_Ω will be called the Signature Function of the shape Ω. We will show that the signature function is invariant under solid motions and thus, adding the term $||F_E(\mu)-F_\Omega(\mu)||_{L^2}$ to the Mumford-Shah functional will have the desired effect.

The natural question is whether Ω is uniquely determined by F_Ω. We can observe that such a results would imply that $||F_E(\mu) - F_\Omega(\mu)||_{L^2}$ is a distance on shapes. Unfortunately we do not have a proof of such a result. We will however see that it is likely to be true, at least for polygons. We will also study the function F_Ω and show that it yields a lot of information about Ω. We will finish this study by showing a continuity result for F_Ω, when Ω varies.

1.1 Point configurations and distribution of pairwise distances.

Before looking at our problem, let us consider a finite dimensional equivalent problem studied by Mireille Boutin and Gregor Kemper few years ago (see [3] and [4]).

Let $p_1, ..., p_n$ be n points of $\mathbb{R}^m$. Let D be the set of all pairs (d,p) where d is equal to $||p_i - p_j||$ for some (i,j) and p is the number of occurrences of d.

Definition 1 *D is called the distribution of distances of $p_1, ..., p_n$.*

Example 1 *for the unit square, $D = \{(1,4); (\sqrt{2}, 2)\}$.*

Definition 2 *We say that the n-point configuration $p_1, ..., p_n$ is reconstructible if for every $q_1, ..., q_n$ having the same distribution of distances there exist a rigid motion M and a permutation π of the labels $\{1..n\}$ such that $Mp_i = q_{\pi(i)}$ for every i.*

We then have this surprising theorem (see [3])

Theorem 1 *Let n be a positive integer with $n \leq 3$ or $n \geq m + 2$. Then there exists f, a non-zero polynomial of $m \times n$ variables such that every n-point configuration $p_1, ..., p_n$ with $f(p_1, ..., p_n) \neq 0$ is reconstructible.*

Corollary 1 *The set of reconstructible points of the plan is a dense open set whose complementary is of measure zero.*

Proof : The sequent lemma yields the proof :

Lemma 1 *For every non-zero real polynomial P of n variables, $N = \{x \in \mathbb{R}^n / P(x) \neq 0\}$ is a dense and open subset of $\mathbb{R}^n$ whose complementary is of measure zero.*

Proof : P is continuous so $N = P^{-1}(\mathbb{R}\backslash\{0\})$ is open.

Let $M = \mathbb{R}^n \backslash N$. Let us prove by induction on n that $\mathcal{H}^n(M) = 0$ where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure (i.e. the Lebesgue measure).

If $n = 1$, then P is a non-zero polynomial of one variable. Hence it has a finite number of roots. M is thus a finite set and hence of measure 0.

Suppose that the property is true for $n - 1$.

$$\mathcal{H}^n(M) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} 1_M(x_1, .., x_n) dx_1 ... dx_{n-1} \right) dx_n$$

We first show that $V = \{x_n / Q_{x_n}(x_1, ..x_{n-1}) = P(x_1, ..x_n)$ is the zero polynomial$\}$ is a finite set. Suppose that the contrary holds. Then with the multi-index notation we have :

$$\begin{aligned} P &= \sum_{|\alpha| \leq N} a_\alpha x^\alpha \\ &= \sum_{|\alpha_1| + \alpha_2 \leq N} a_\alpha (x_1..x_{n-1})^{\alpha_1} x_n^{\alpha_2} \\ &= \sum_{|\alpha_1| \leq N} \left(\sum_{\alpha_2 \leq N - |\alpha_1|} a_\alpha x_n^{\alpha_2} \right) (x_1..x_{n-1})^{\alpha_1} \\ &= \sum_{|\alpha_1| \leq N} R_{\alpha_1}(x_n)(x_1..x_{n-1})^{\alpha_1} \end{aligned}$$

and $R_{\alpha_1}(l) = 0$ for all $l \in V$ and $|\alpha_1| \leq N$. But R_{α_1} is a polynomial of one variable so if V is infinite, $R_{\alpha_1} = 0$ for all $|\alpha_1| \leq N$ so $P = 0$ which is absurd.

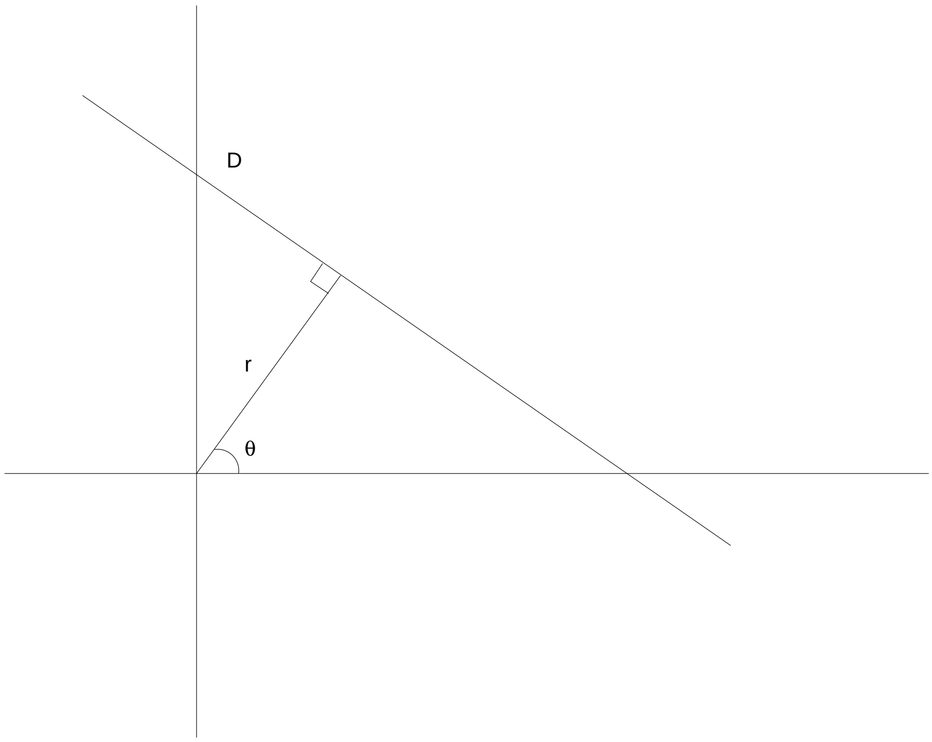

Figure 1:

Let $M_{x_n} = \{x \in \mathbb{R}^{n-1} / \, Q_{x_n}(x) = 0\}$. Then,

$$\mathcal{H}^n(M) = \int_{\mathbb{R}\setminus V} \left(\int_{\mathbb{R}^{n-1}} 1_{M_{x_n}}(x_1, ..x_{n-1}) dx_1 ... dx_{n-1} \right) dx_n$$

which is positive by the induction hypothesis.

If N were not dense, then its complement M would have non empty interior and by the way, positive measure.

We see that this corollary answers the uniqueness question for the discrete case.

1.2 The Radon Transform.

Before starting the study of our problem, we need to have a better definition of F_Ω.

Let D be a line of the plane which does not intersect the origin O. Let (r, θ) be the polar coordinates of the orthogonal projection of O on this line (see Figure 1). This gives us a parametrization of the lines which are not passing through the origin. We then have :

$$D = D_{r,\theta} = \{ \begin{pmatrix} t\cos\theta + r\sin\theta \\ -t\sin\theta + r\cos\theta \end{pmatrix} \; , \; t \in \mathbb{R} \}$$

This formula defines also $D_{r,\theta}$ when r is negative so that we have $D_{-r,\theta} = D_{r,\pi+\theta}$.

Definition 3 *For $f \in L^1(\mathbb{R}^2)$ let the Radon transform of f be*

$$\begin{aligned}(Rf)(r,\theta) &= \int_{\mathbb{R}} f(t\cos\theta + r\sin\theta, -t\sin\theta + r\cos\theta)dt \\ &= \int_{D_{r,\theta}} f(x)\,dx\end{aligned}$$

In the case $f = 1_\Omega$, we can observe that

$$R1_\Omega(r,\theta) = \int_{D_{r,\theta}} 1_\Omega(r,\theta) = |D_{r,\theta} \cap \Omega| = \mathcal{H}^1(D_{r,\theta} \cap \Omega)$$

If we denote by $A_\Omega(\mu) = \{(r,\theta)\,/\,|D_{r,\theta} \cap \Omega| \geq \mu\} = \{(r,\theta)\,/\,R1_\Omega(r,\theta) \geq \mu\}$ or A_Ω if there is no ambiguity, then :

$$\begin{aligned}F_\Omega(\mu) &= \int_0^\infty \int_0^{2\pi} 1_{A_\Omega(\mu)}(r,\theta) \\ &= \int_{\mathbb{R}} \int_0^{\pi} 1_{A_\Omega(\mu)}(r,\theta) \\ &= \int_{\mathbb{R}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1_{A_\Omega(\mu)}(r,\theta)\end{aligned}$$

The Radon transform was first introduced by Radon in 1919. Since then, it has proven to be very useful in many fields. In mathematics it is used in image analysis (tomography) and in harmonic analysis. It finds applications in astronomy, optics, physics, geophysics and in a lot of other fields. However, one of the most important applications of the Radon Transform can be found in medicine.

It can be shown that when a body B of density b(x) is crossed by an X-ray of initial intensity I_0 and final intensity I then $Rb = log(\frac{I_0}{I})$. Hence, by using X-rays, physicians are able to find the Radon transform of the density. The obvious question for them is whether they can found from this, the density itself.

Radon showed that it is indeed possible, in the sense that the Radon transform is invertible. There is also this stronger result :

Theorem 2 (Fourier Slice Theorem) *for every $f \in L^1(\mathbb{R}^2)$*

$$\int_{\mathbb{R}} (Rf)(\theta,\tau)exp(i\tau t)d\tau = \mathcal{F}(f)(t(\cos\theta, \sin\theta))$$

where $\mathcal{F}$ is the Fourier transform.

The definition of the Radon transform can be extended to more general classes of functions, like distributions (see [10]), defined on more general sets, like manifolds (see [6]).

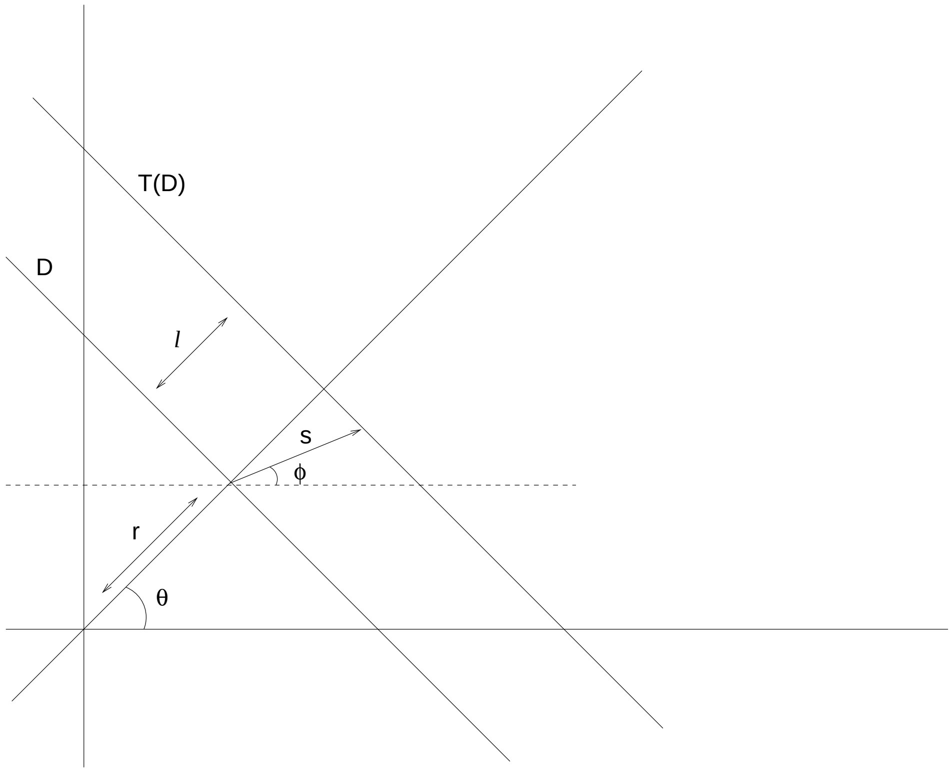

Figure 2:

2 Behavior of the signature under transformations of the shape.

Before starting to investigate the other properties of F_Ω let us see how a rotation, a translation, a reflection or an homothetic transformation of Ω affects it.

Proposition 1 (Invariance under solid motion.) *For every measurable set Ω and every solid motion T, $F_\Omega = F_{T(\Omega)}$.*

Proof : Every solid motion is a composition of a rotation by a translation. Let us show first that F_Ω is invariant under translations.

Let T be the translation of $v_{s,\phi} = (s\cos\phi, s\sin\phi)$ and $\Sigma = T(\Omega)$ (see Figure 2). For every line $D_{r,\theta}$, $\mathcal{H}^1(D_{r,\theta} \cap \Omega) = \mathcal{H}^1((D_{r,\theta} + v_{s,\phi}) \cap (\Omega + v_{s,\phi}))$. But

$$D_{r,\theta} + v_{s,\phi} = D_{r+s\cos(\theta-\phi),\theta}.$$

So if

$$\Psi(r,\theta) = (r+l,\theta) = (r+s\cos(\theta-\phi),\theta)$$

then Ψ is clearly bijective and from one side we have:

$$\begin{aligned} A_\Sigma =& \{(r,\theta)/\ |D_{r,\theta}\cap\Sigma| \geq \mu\} \\ =& \{\Psi\circ\Psi^{-1}(r,\theta)/\ |D_{\Psi^{-1}(r,\theta)}\cap\Omega| \geq \mu\} \\ =& \Psi(A_\Omega) \end{aligned}$$

From the other side :

$$D\Psi = \begin{pmatrix} 1 & -s\sin(\theta-\phi) \\ 0 & 1 \end{pmatrix}$$

So $\det D\Psi = 1$ and we have the invariance under translation.

If r is a rotation around the origin then clearly $F_\Omega = F_{r(\Omega)}$. If r is a rotation of center Γ and angle α then if τ is the translation which sends Γ on the origin we have $r = \tau^{-1}\tilde{r}\tau$, where $\tilde{r}$ is the rotation of angle α around the origin. Hence, by the preceding argument, F_Ω is invariant under rotations.

By the same type of arguments, we can show that :

Proposition 2 *F_Ω is invariant under reflections.*

We have also a result for the scaling transforms.

Proposition 3 *For all $a > 0$ and Ω measurable, $F_{a\Omega}(\mu) = aF_\Omega(\frac{\mu}{a})$*

Proof : for every line, $aD_{r,\theta} = D_{ar,\theta}$ so

$$|D_{r,\theta}\cap a\Omega| = a|D_{\frac{r}{a},\theta}\cap\Omega|$$

And hence,

$$|D_{r,\theta}\cap a\Omega| \geq \mu \Leftrightarrow |D_{\frac{r}{a},\theta}\cap\Omega| \geq \frac{\mu}{a}$$

If we set $\Psi(r,\theta) = (ar,\theta)$, whose jacobian is a, we have

$$A_{a\Omega}(\mu) = \Psi(A_\Omega(\frac{\mu}{a}))$$

So $F_{a\Omega}(\mu) = aF_\Omega(\frac{\mu}{a})$

3 Examples and uniqueness for convex polygons.

Before discussing the uniqueness, let us look at some examples.

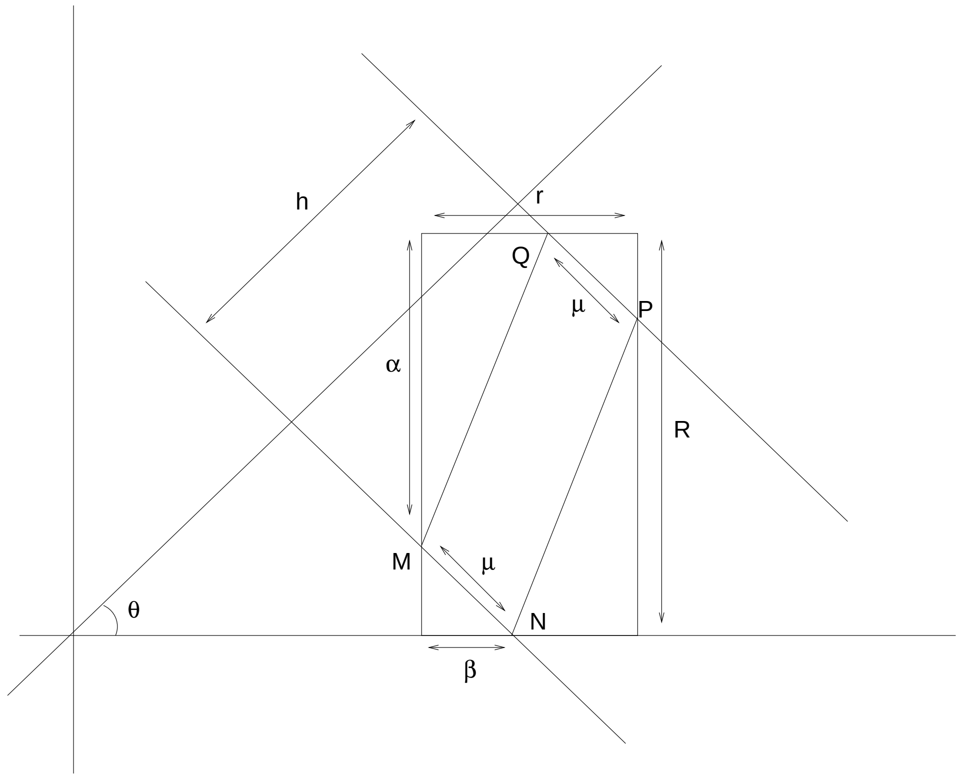

Figure 3:

3.1 The rectangle.

By the Proposition 1, we can restrict ourselves to the situation of Figure 3, where $R \geq r$. By the symmetry of this problem we can restrict ourselves to $\pi \in [0, \frac{\pi}{2}]$.

Let $h(\theta,\mu) = h_\mu(\theta) = \int_{\mathbb{R}} |D_{r,\theta} \cap \Omega| dr$. When no confusion can be made, we will also write $h(\theta,\mu) = \; h(\theta)$. Then the area $\mathcal{A}$ of the parallelogram MNPQ is equal to $\mu h(\theta)$ and is also equal to the area of the rectangle minus the area $\mathcal{B}$ of the four triangles.

We have $\cos\theta = \frac{R-\alpha}{\mu}$ so $\alpha = R - \mu\cos\theta$, and $\beta = \sqrt{\mu^2 - (R-\alpha)^2} = \mu|\sin\theta|$. This calculation holds only if $0 \leq \alpha \leq R$ and $0 \leq \beta \leq r$ Hence only if

$$\frac{R}{\mu} \geq \cos\theta \qquad \text{and} \qquad \frac{r}{\mu} \geq |\sin\theta|$$

For these θ, $\mathcal{B} = (R-\alpha)\beta + (r-\beta)\alpha$. Hence $h(\theta) = |\sin\theta|(R - 2\mu\cos\theta) + r\cos\theta$. For the other θ, $h(\theta) = 0$. So,

$$F_\Omega(\mu) = 2\int_0^{\frac{\pi}{2}} h(\theta,\mu) = \begin{cases} 2\,(R+r-\mu) & \text{if } \frac{R}{\mu} \geq \frac{r}{\mu} \geq 1 \\ 2R\left(1 - \sqrt{1-\frac{r^2}{\mu^2}}\right) & \text{if } R \geq \mu \geq r \\ 2\left(\mu - R\sqrt{1-\frac{r^2}{\mu^2}} - r\sqrt{1-\frac{R^2}{\mu^2}}\right) & \text{if } \sqrt{R^2+r^2} \geq \mu \geq R \\ 0 & \text{either} \end{cases}$$

The above calculations showed that :

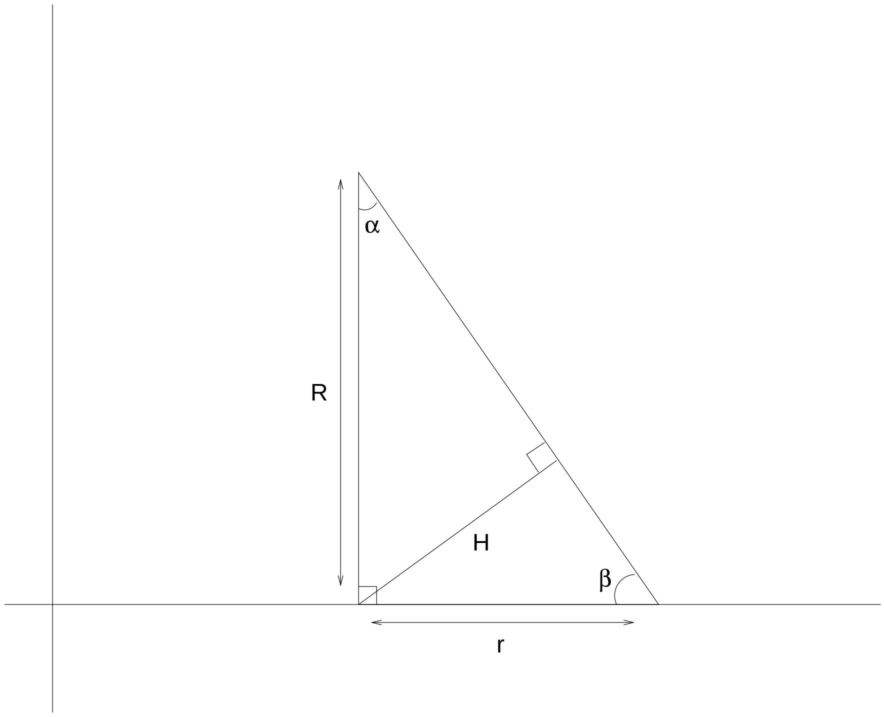

Figure 4:

Theorem 3 *Rectangles are uniquely determined by their signature.*

We can observe that F_Ω is linear between 0 and r, continuous on $]0,+\infty]$, smooth away from R, r and $\sqrt{R^2+r^2}$. We can also see that F_Ω tends to the perimeter of Ω when μ tends to zero.

3.2 The right triangle.

A similar calculation (see Appendix A) for the right triangle T (see Figure 4) of sides R and r gives :

a) for $0 \leq \mu \leq \frac{Rr}{\sqrt{R^2+r^2}}$,
$F_T(\mu) = \sqrt{R^2+r^2} + R + r - \frac{\mu}{2}(3 + \frac{\pi}{2}\frac{r^2+R^2}{rR} + \frac{r}{R}\alpha + \frac{R}{r}(\frac{\pi}{2} - \alpha))$

b) for $\frac{Rr}{\sqrt{R^2+r^2}} \leq \mu \leq r$,
$F_T(\mu) = \sqrt{R^2+r^2} + R + r - \frac{\mu}{2}(3 + \frac{\pi}{2}\frac{r^2+R^2}{rR} + \frac{r}{R}\alpha + \frac{R}{r}(\frac{\pi}{2} - \alpha)) + \mu\frac{R^2+r^2}{2rR}(\pi - 2\arcsin\frac{rR}{\mu\sqrt{r^2+R^2}}) - \sqrt{r^2+R^2}\sqrt{1 - \frac{r^2R^2}{\mu^2(r^2+R^2)}}$

c) for $r \leq \mu \leq R$,
$F_T(\mu) = \sqrt{R^2+r^2} + R - \frac{\mu}{2}(2 + \frac{R^2+r^2}{rR}(\arcsin\frac{rR}{\sqrt{R^2+r^2}} - \alpha) - \frac{R^2-r^2}{rR}\alpha + R\sqrt{1-\frac{r^2}{\mu^2}} + \frac{\mu R}{r}\arcsin\frac{r}{\mu})$

d) for $R \leq \mu \leq \sqrt{R^2 + r^2}$,
$F_T(\mu) = \sqrt{R^2 + r^2} - \frac{\mu}{2}(1 + \frac{r^2 - R^2}{rR}\alpha + 2 \arcsin \frac{r}{\mu} - 2 \arccos \frac{R}{\mu}) - \frac{1}{2}(R\sqrt{1 - \frac{r^2}{\mu^2}} + r\sqrt{1 - \frac{R^2}{r^2}})$

d) for $\sqrt{R^2 + r^2} \leq \mu$,
$F_T(\mu) = 0$

This computation showed again that :

Theorem 4 *Right triangles are uniquely determined by their signature function.*

We can again verify that F_T is linear between 0 and r, continuous on $]0, +\infty]$, smooth away from R, r, $\sqrt{R^2 + r^2}$ and $\frac{Rr}{\sqrt{R^2+r^2}}$. We can also see that when μ tends to zero, F_T tends to the perimeter of the triangle. The value $H = \frac{rR}{\sqrt{R^2+r^2}}$ corresponds to the height descended from the right angle (see Figure 4).

3.3 Discussion about uniqueness.

From these two examples and some others (see Figure 5 to Figure 18) , we can make a conjecture :

Conjecture : For every convex polygon Ω, F_Ω is continuous, has a linear part and is smooth away from the inter-vertex distances and away from the distances between the vertices and the sides of Ω.

We will prove the first part of the conjecture (everything, including the continuity, but the smoothness). We will also see that for all convex shapes, F_Ω is continuous and tends to the perimeter of Ω when μ tends to zero. We can hope that if we prove the last part of the conjecture, regarding the location of the singularities, it will gives us a result on uniqueness. In fact, it might be possible to link it to Theorem 1. Unfortunately, we don't actually have such a proof. However, we can give an incomplete idea of how the proof shall work.

For sake of simplicity, we will only deal here with the case of the triangle. The reasoning can however be extended to general polygons. Let B be the set of all inter-vertex distances and distances of all the vertices to all the sides of Ω. We want to prove that F_Ω is smooth away from B.

Let us call $f(r, \theta) = |D_{r,\theta} \cap \Omega|$. Suppose that $\mu \in]\mu_*, \mu^*[$, where μ_* and μ^* are two consecutive values of B then, by the continuity of F_Ω (that will be proven later on (see Theorem 9)),

$$F_\Omega(\mu) = \mathcal{H}^2(\{(r, \theta)/f(r, \theta) \in [\mu_*, +\infty[\}) + \mathcal{H}^2(\{(r, \theta)/f(r, \theta) \in]\mu, \mu_*[\})$$

In order to continue we will need the co-area formula (see [1]). To state it we need the notion of 2-rectifiable sets.

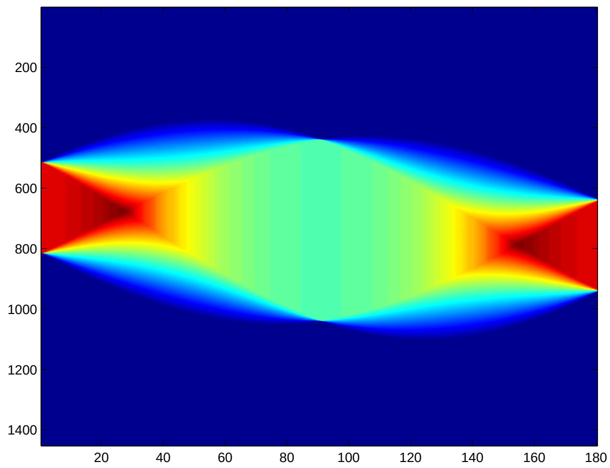

Figure 5: Radon transform of a rectangle.

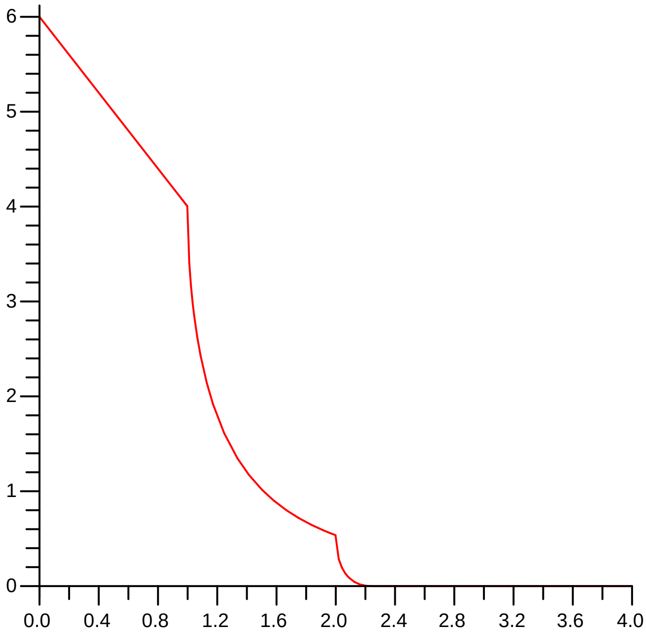

Figure 6: Signature Function for the rectangle with $r = 1$ and $R = 2$.

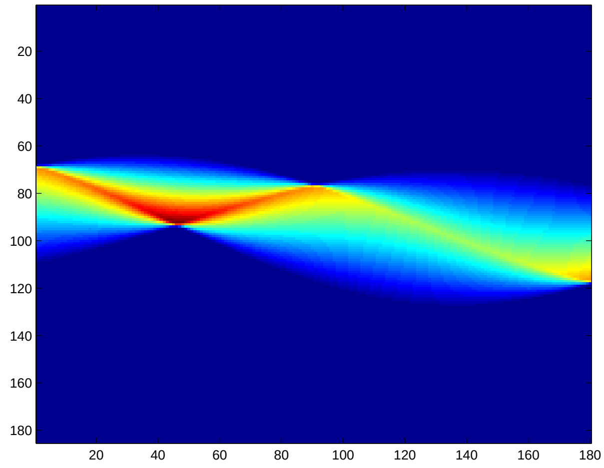

Figure 7: Radon transform of a right triangle.

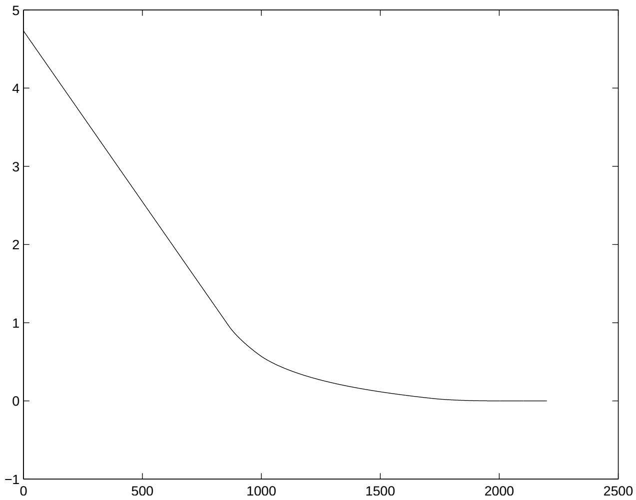

Figure 8: Signature Function for the right triangle with $r = 1$ and $R = \sqrt{3}$.

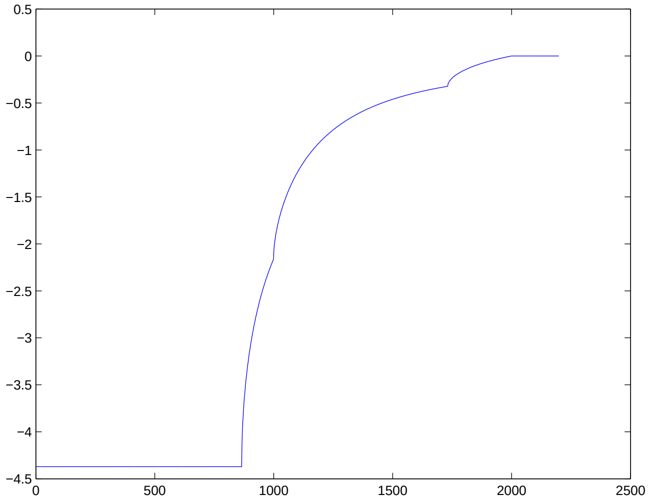

Figure 9: The derivative of the Signature function of the right triangle with $r = 1$ and $R = \sqrt{3}$.

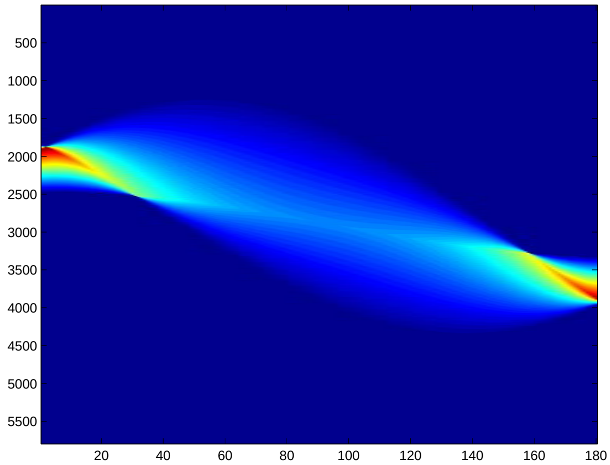

Figure 10: Radon transform of a random triangle.

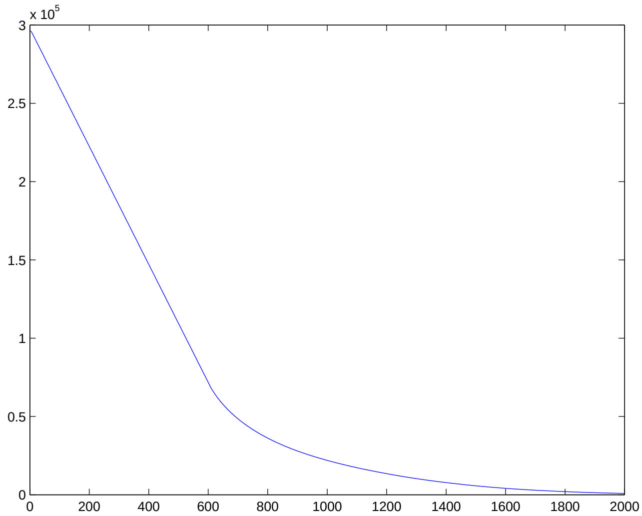

Figure 11: Signature Function for a random triangle.

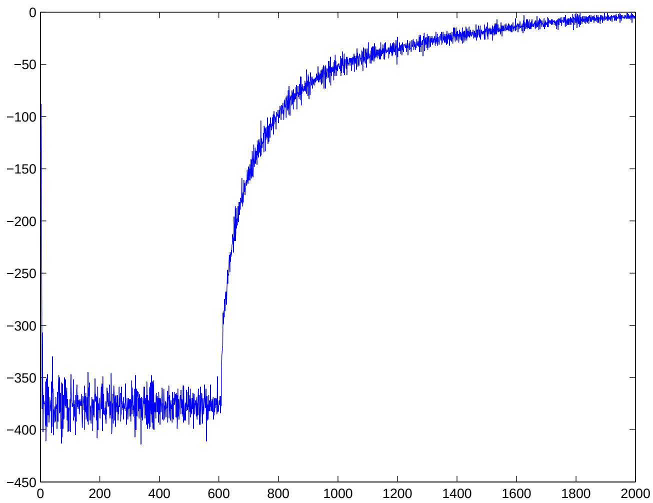

Figure 12: The derivative of the Signature function of a random triangle.

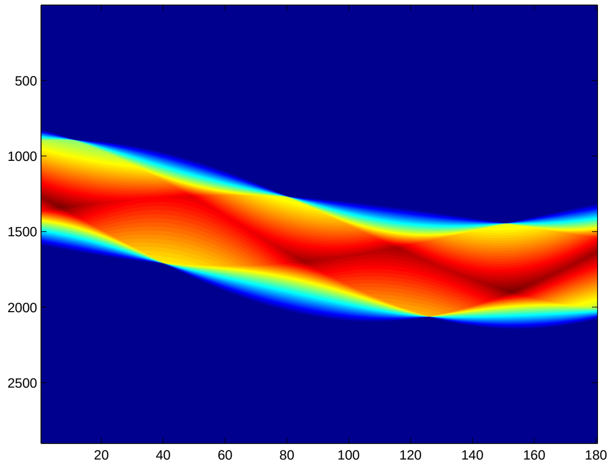

Figure 13: Radon transform of a pentagon.

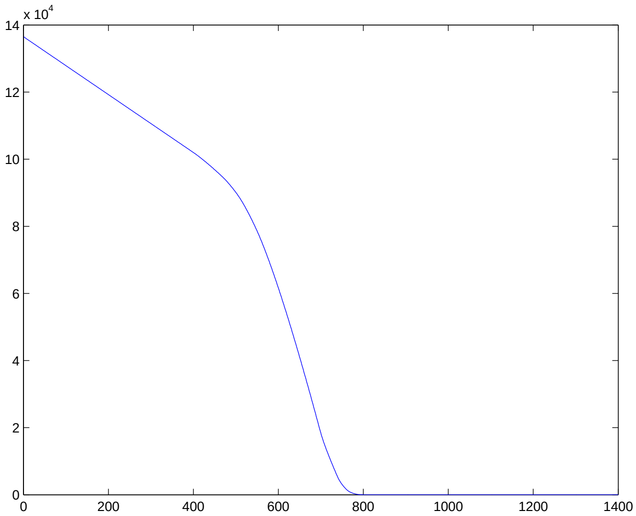

Figure 14: Signature Function of a pentagon.

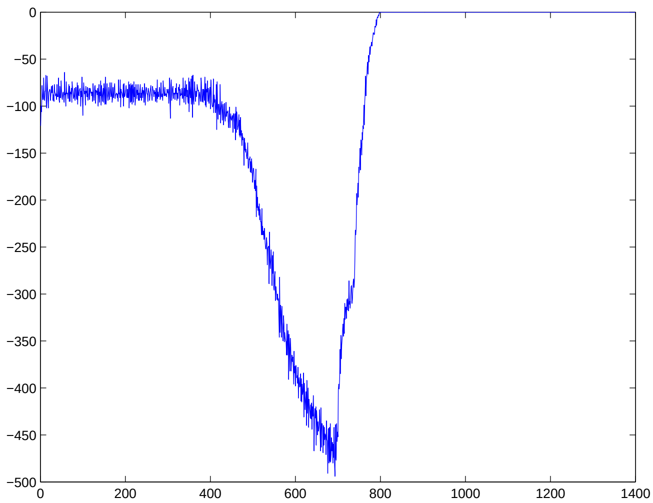

Figure 15: The derivative of the Signature function of a pentagon.

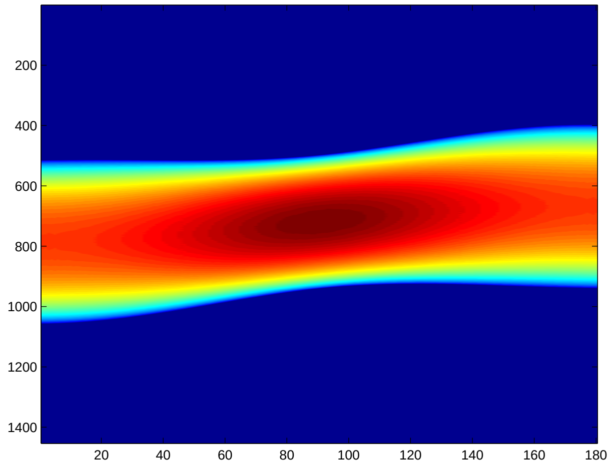

Figure 16: Radon transform of an ellipse.

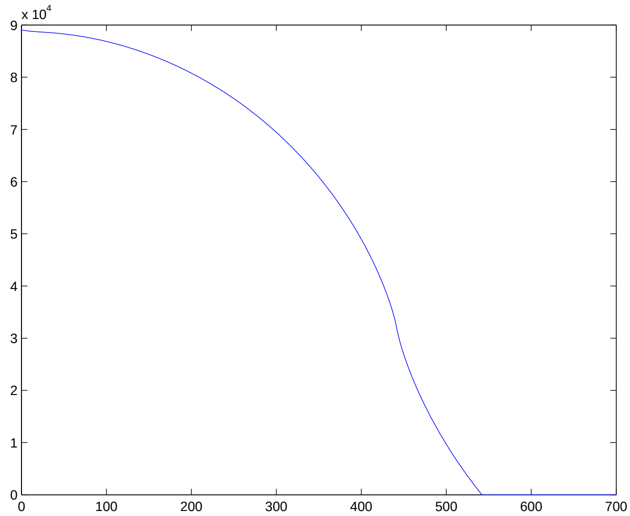

Figure 17: Signature Function of an ellipse.

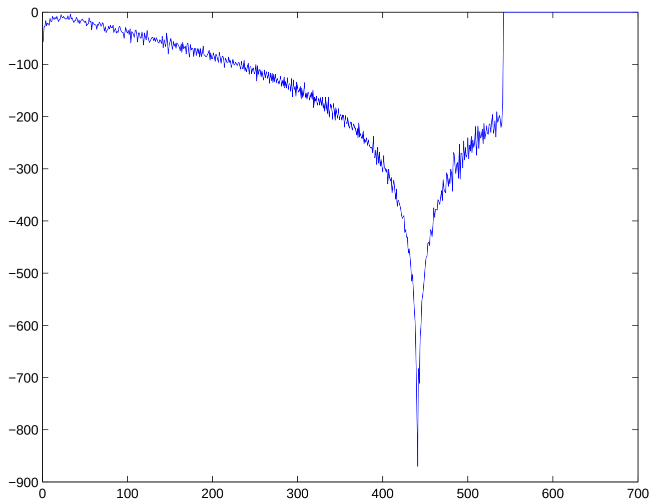

Figure 18: The derivative of the Signature function of an ellipse.

Definition 4 (Rectifiability) *We will say that a set $E \subset \mathbb{R}^2$ is 2-rectifiable if :*

$$\lim_{\rho \to 0} \int_{E_{x,\rho}} \phi(y) d\mathcal{H}^2 = \int_{\mathbb{R}^2} \phi(y) d\mathcal{H}^2 \qquad \forall \phi \in \mathcal{C}_c(\mathbb{R}^2) \text{ for } \mathcal{H}^2 - a.e\ x \in E$$

Where $E_{x,\rho} = \frac{E-x}{\rho}$.

Theorem 5 (Co-area Formula) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a lipschitz function. and E a 2-rectifiable set then :*

$$\int_E |\nabla f| d\mathcal{H}^2 = \int_{\mathbb{R}} \mathcal{H}^1(E \cap f^{-1}(t)) dt$$

Observation : ∇f exists almost everywhere according to Rademacher's theorem.

If we can then show that f is smooth away from B, which might come from the characterization of the singularity set of the Radon transform (see [10]), then $E = f^{-1}(]\mu, \mu^*[)$ will be open, hence 2-rectifiable. If we apply now the co-area formula to f and E we get :

$$\int_E |\nabla f| = \int_{\mu}^{\mu^*} \mathcal{H}^1(r, \theta / |D_{r,\theta} \cap \Omega = \nu) d\nu$$

So if we can link $\int_E |\nabla f|$ to $\mathcal{H}^2(E) = \mathcal{H}^2(\{(r,\theta)/\, f(r,\theta) \in]\mu, \mu_*[\})$, we will then have an expression of the type,

$$F_{\Omega}(\mu) = C + \int_{\mu}^{\mu^*} \mathcal{H}^1(r, \theta / \,|D_{r,\theta} \cap \Omega = \nu) d\nu \quad \forall \mu \in]\mu_*, \mu^*[$$

Now if we take a line $D_{r,\theta}$ which intersects Ω in two points x and y then $f(r,\theta) = ||x - y||$, which is smooth in (x, y) because $x < y$.

If x is one of the vertices, then y is not his projection on the opposite side, neither an other vertex of the triangle (if it were, then $||x - y||$ would have been one of the sides length which is in B). Hence locally, $||x - y||$, as a function of y, is strictly increasing or strictly decreasing. Its derivative according to y is then different from zero.

If neither x or y are vertices, then, suppose $x \in [AB]$ and $y \in [BC]$ (see Figure 19). As (AB) and (BC) are not parallel, either x is not the orthogonal projection of y on (AB) either y is not the orthogonal projection of x on (BC). Suppose that y is not the orthogonal projection of x on (BC). Then as before, the partial derivative of $||x - y||$ according to y is not 0.

Suppose that we can show that (x, y) are depending by a diffeomorphism of (r, θ) so that at least one of the two partial derivatives $\frac{\partial f}{\partial r}$ or $\frac{\partial f}{\partial \theta}$ is non zero. For sake

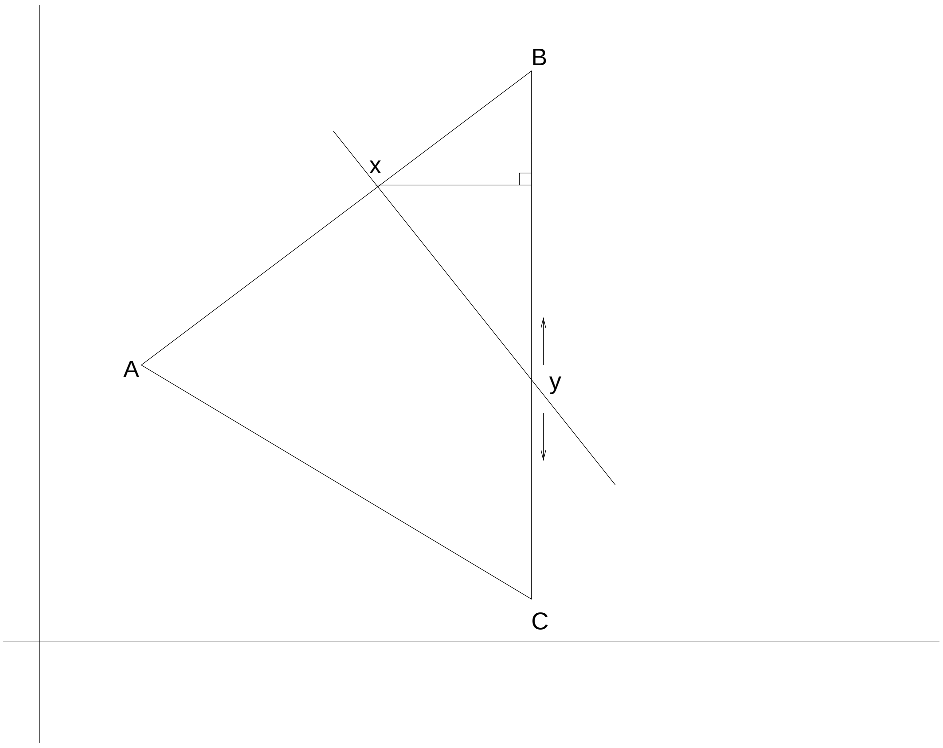

Figure 19:

of simplicity, suppose that $\frac{\partial f}{\partial \theta}$ is always non zero. Hence we can apply the implicit function theorem. :

For every (r_0, θ_0) there exists a neighborhood $V \times W$ of (r_0, θ_0), a neighborhood U of $\mu_0 = f(r_0, \theta_0)$ and a function $\phi \in \mathcal{C}^\infty(V, U)$ such that :

$$f(r,\theta) = \mu \Leftrightarrow \theta = \phi(r,\mu) \qquad \forall (r,\theta,\mu) \in V \times W \times U$$

Suppose that in addition, we can do it globally, then

$$F_\Omega(\mu) = C + \int_\mu^{\mu^*} \mathcal{H}^1(r, \phi(r,\nu))d\nu \quad \forall \mu \in]\mu_*, \mu^*[$$

which is a $\mathcal{C}^\infty$ expression of μ.

We can see that for the points of B, the smoothness of f will not be guarantied. Even if this problem shall be removed, we see that for the x, y of the previous reasoning such that $||x - y|| = \mu \in B$, we may have $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ zero (as it is the case when x is a vertex and y its projection on the opposite side).

Of course we made a large number of simplifications and assumptions that have to be adapted or proved, in order to have a real proof. We can also note that even if we prove this conjecture, it will only gives us the possible candidates for the singularities. One has still to show that these points are indeed singularities.

Proposition 4 *For every convex polygon Ω, let $\mu^* = \min\{\mu \,/\, \mu \in B\}$. Then F_Ω is linear on $]0, \mu^*[$.*

Proof : We will prove it for triangles, but the reasoning extends itself easily for all convex polygons.
For Ω the triangle ABC (see Figure 20), let $\varepsilon > 0$ be such that $0 < \mu + \varepsilon < \mu^*$. Then, by the continuity of F_Ω (see Theorem 9),

$$-\delta = F_\Omega(\mu + \varepsilon) - F_\Omega(\mu) = -\mathcal{H}^2(r, \theta \,/\, |D_{r,\theta} \cap \Omega| \in]\mu, \mu + \varepsilon[)$$

As $\mu + \varepsilon < \mu^*$, $D_{r,\theta} \cap \Omega = [a, b]$ where a and b are not vertices of Ω. Let

$$\begin{aligned} \Sigma_1 =& \{r, \theta / \, a \in [AB] \text{ and } b \in [BC]\} \\ \Sigma_2 =& \{r, \theta / \, a \in [AB] \text{ and } b \in [AC]\} \\ \Sigma_3 =& \{r, \theta / \, a \in [AC] \text{ and } b \in [BC]\}. \end{aligned}$$

We can hence write δ as $\delta_1 + \delta_2 + \delta_3$ where $\delta_i = \mathcal{H}^2(C_i)$ with

$$C_i = \{r, \theta / |D_{r,\theta} \cap \Omega| \in]\mu, \mu + \varepsilon[\} \cap \Sigma_i$$

For every $i = 1..3$, as for F_Ω, δ_i is invariant under solid motion, so that we can bring us back to the situation of Figure 20 and study only δ_1.

Suppose that $\alpha < \frac{\pi}{2}$. We can introduce a new parametrization, given by the angle ϕ of $D_{r,\theta}$ and $[BC]$, and x, the orthogonal projection of a on $[BC]$. The two parametrizations are linked by :

$$(r, \theta) = (x(\sin\phi + \tan\alpha\cos\phi), \frac{\pi}{2} - \phi) = \Psi(x, \phi)$$

Ψ is a bijection from $E = \left(]-\frac{\pi}{2}, -\alpha[\cup]0, \frac{\pi}{2}[\right) \times]0, +\infty[$ to Σ_1 , whose jacobian is equal to $|\sin\phi + \tan\alpha\cos\phi|$. So separately on $]0, \frac{\pi}{2}[\times]0, +\infty[$ and $]-\frac{\pi}{2}, -\alpha[\times]0, +\infty[$ it is a diffeomorphism.

Let $D^{x,\phi}$ be the line whose parameters are x, ϕ and $B(\mu) = \{x, \phi / \, |D^{x,\phi} \cap \Omega| \in]\mu, \mu + \varepsilon[\}$. Then $C_1(\mu) = \Psi(B(\mu))$ and :

$$\begin{aligned} \delta_1 &= \int_{C_1} 1_{C_1}(r, \theta) \\ &= \int_{C_1} 1_{\Psi(B(\mu))}(\Psi \circ \Psi^{-1})(r, \theta) \\ &= \int_{B(\mu)} |\sin\phi + \tan\alpha\cos\phi| 1_{B(\mu)}(x, \phi) \end{aligned}$$

Let $\Xi_\eta(x, \phi) = (y, \phi) = (x + \eta\frac{\sin\phi}{\tan\alpha}, \phi)$ then (see Figure 22) if $0 < \mu + \eta < \mu^*$,

$$|D^{x,\phi} \cap \Omega| = \mu \Leftrightarrow |D^{\Xi_\eta(x,\phi)} \cap \Omega| = \mu + \eta$$

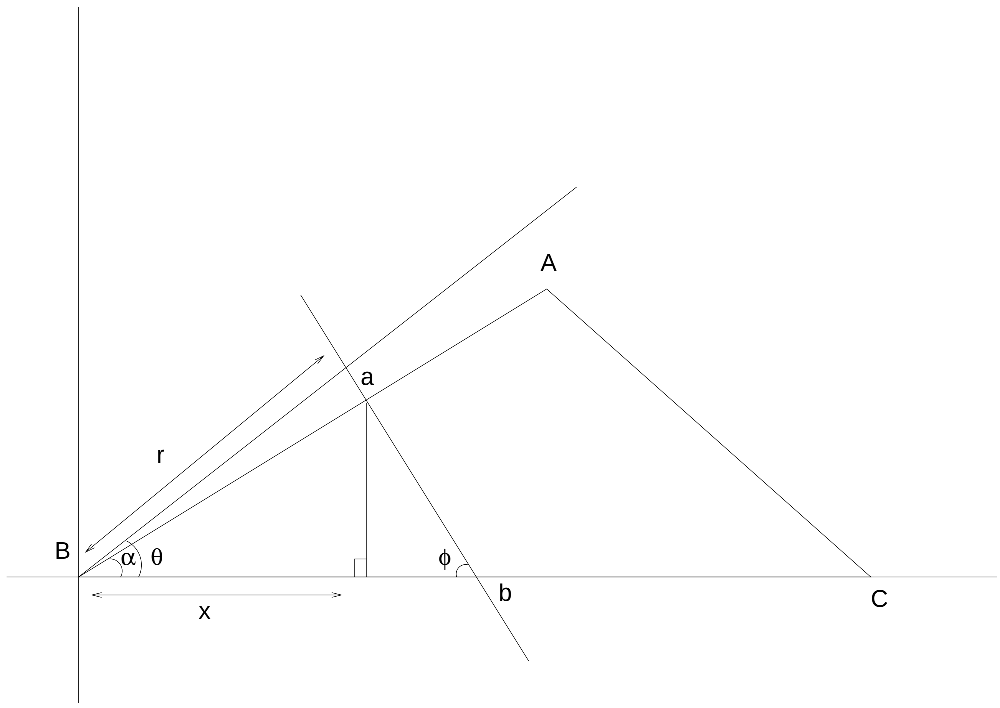

Figure 20:

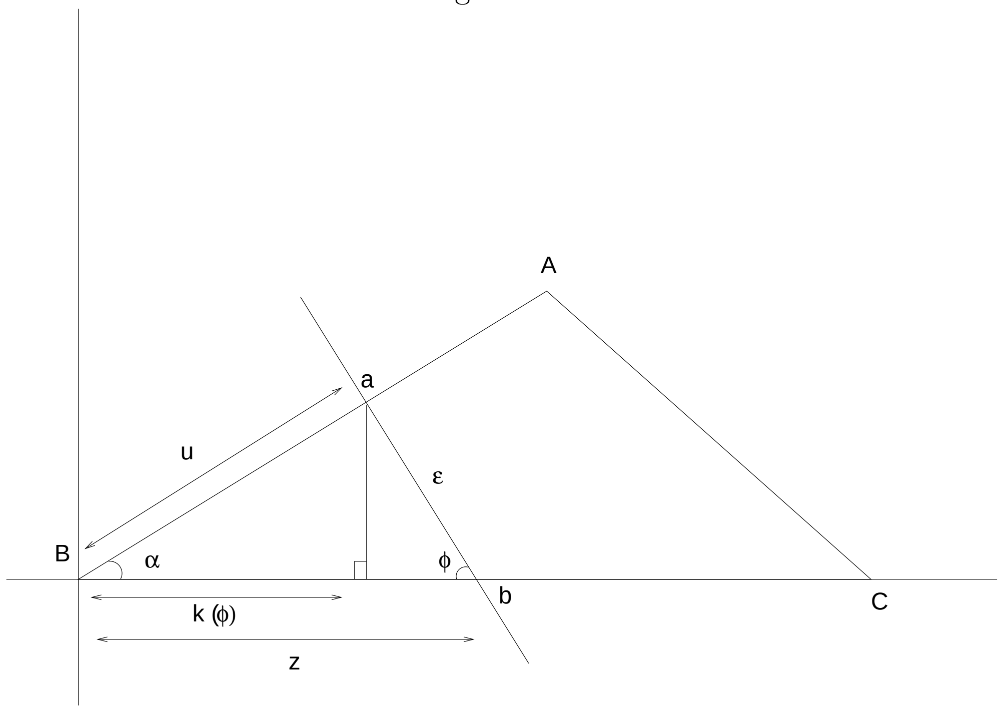

Figure 21:

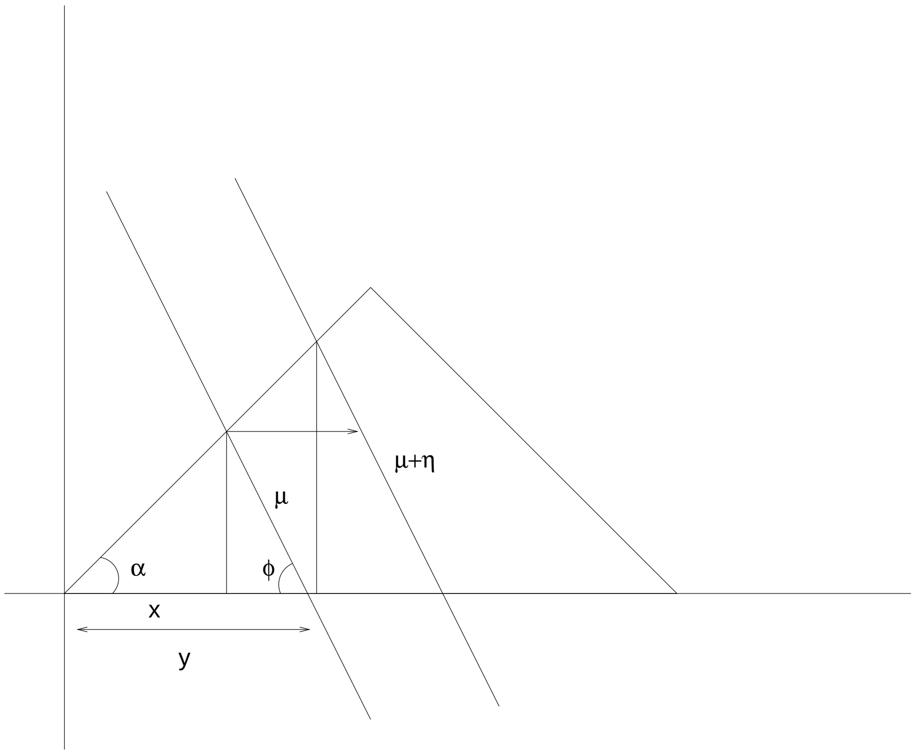

Figure 22:

For $\eta = -\mu$ we have thus,

$$B(\mu) = \{x, \phi / \, |D^{\Xi_\eta(x,\phi)} \cap \Omega| \in]0, \varepsilon[\} = \Xi_\eta^{-1}(B(0))$$

But $|Jac(\Xi_\eta)| = 1$ so

$$\delta_1 = \int_{B(0)} |\sin\phi + \tan\alpha\cos\phi| 1_{B(0)}(x, \phi)$$

For every ϕ, let $k(\phi) = \int_{\mathbb{R}} 1_{B(0)}(x, \phi)dx$ (see Figure 21). If $z < |BC|$ and $u < |AB|$ i.e. if $\frac{\sin(\phi+\alpha)}{\sin\alpha}\varepsilon < |BC|$ and $\frac{|\sin\phi|}{\sin\alpha}\varepsilon < |AB|$, which is the case for every ϕ if ε is small enough, then $k(\phi) = \varepsilon\frac{|\sin\phi|}{\tan\alpha}$. Therefore :

$$\delta_1 = \frac{\varepsilon}{\tan\alpha}\left(\int_{-\alpha}^{\frac{\pi}{2}} \sin^2\phi + \tan\alpha\sin\phi\cos\phi d\phi + \int_0^{\frac{\pi}{2}} \sin^2\phi + \tan\alpha\sin\phi\cos\phi d\phi\right)$$

So that we get after some computations,

$$\delta_1 = \varepsilon\left(\frac{1}{2\tan\alpha}(\pi - \alpha + \frac{\sin 2\alpha}{2}) + \frac{1}{4}(1 - \cos 2\alpha)\right)$$

For $\alpha \neq \frac{\pi}{2}$, similar calculations give the same formula (for $\alpha = \frac{\pi}{2}$, we intend it to be $\frac{\varepsilon}{2}$).

For $\varepsilon < 0$, by noting $\varepsilon' = -\varepsilon$ and doing the same reasoning, we get that $\frac{\delta}{\varepsilon}$ is constant and hence that F_Ω is linear on $]0, \mu^*[$.

Corollary 2 *For every convex polygon Ω, with angles $\alpha_1..\alpha_n$, F_Ω is linear on $]0,\mu^*[$ with a slope equal to :*

$$-\left(\sum_{i=1}^{n}\frac{1}{2\tan\alpha_i}(\pi-\alpha_i+\frac{\sin 2\alpha_i}{2})+\frac{1}{4}(1-\cos 2\alpha_i)\right)$$

We can observe that this result is coherent with the computations we made for the rectangle and the right triangle.

Before looking at the properties of the function F_Ω for general convex shapes, let us give some properties of the convex sets.

4 Distances and convex bodies.

We will admit in this section most of the results. However, we will always give references to find the proofs. If nothing is specified, they can be found in [2]. [5] can also be consulted.

4.1 Convex bodies.

Definition 5 *A convex body is a compact convex set with non empty interior and piecewise $\mathcal{C}^1$ boundary.*

This first property is a classical exercise on convex sets.

Proposition 5 *For every convex body Ω, $\overline{\overset{\circ}{\Omega}} = \overline{\Omega} = \Omega$. Furthermore we have $\partial\Omega = \partial\overset{\circ}{\Omega}$.*

Definition 6 *A support plan Π of a convex body Ω at a point $x \in \Omega$, is a plan such that $x \in \Pi$ and Ω lies entirely (in the large sense) in one of the half spaces delimited by Π.*

Proposition 6 *For every point $x \in \partial\Omega$, there exists at least one support plan.*

We can characterize the convex sets by a property of their boundary :

Proposition 7 *Ω is convex if and only if $\partial\Omega$ is a close Jordan curve, such that its intersection with every line is a segment.*

The support function is very important in the study of convex sets.

Definition 7 *For every convex body Ω, the support function of Ω is $H_\Omega(v) = \sup_{x\in\Omega} < x \,|\, v >$.*

Theorem 6 *For two convex bodies Ω and Σ, if $H_\Omega = H_\Sigma$ then $\Omega = \Sigma$.*

Using the support function, we can prove this surprising theorem (see [9]) :

Theorem 7 *For every convex body Ω,*

$$\mathcal{H}^2(r,\theta/\, D_{r,\theta}\cap\Omega \neq \emptyset) = Perimeter(\Omega)$$

Observation : this theorem also holds for non-smooth, compact convex sets.

4.2 Distances.

We want to have a notion of distance between two convex sets (and even between two general sets) in order to show a property of continuity of F_Ω when Ω varies. We will discuss here two different distances. We will give some of their properties and show relations between them.

The first distance that we will study is the Hausdorff distance :

Definition 8 *Let $B(\Omega,\varepsilon) = \{y \in \mathbb{R}^2 / \exists x \in \Omega, ||x-y|| < \varepsilon\}$. Then the Hausdorff distance between two sets Ω and Σ is equal to :*

$$d(\Omega,\Sigma) = inf\,(r/\,\Omega \subset B(\Sigma,r) \quad and \quad \Sigma \subset B(\Omega,r))$$

Proposition 8 *The Hausdorff distance is a distance on the non empty compact sets.*

For a proof see [7].

We also have this nice link between Hausdorff distance and support functions :

Proposition 9 *For every convex bodies Ω and Σ, we have*

$$d(\Omega,\Sigma) = \max_{S(0,1)} |H_\Omega(v) - H_\Sigma(v)|$$

where $S(0,1)$ is the unit sphere.

We also have a link between the Hausdorff distance of two sets and the Hausdorff distance of their boundaries.

Proposition 10 *For every convex bodies Ω and Σ, $d(\partial\Omega,\partial\Sigma) \leq d(\Omega,\Sigma)$.*

Proof : let $r = d(\Omega,\Sigma)$ and let us show that $\partial\Sigma \subset B(\partial\Omega, r)$.

Let $x \in \partial\Sigma$. Suppose $B(x,r) \cap \partial\Omega = \emptyset$.

Let Π_x be the support plan of Σ in x, and v be the orthogonal vector to Π_x such that $x + tv \notin \Sigma$ for all $t > 0$.

As $x \in B(\Omega,r)$ and $B(x,r) \cap \partial\Omega = \emptyset$, $B(x,r) \subset \Omega$. Ω is closed hence $x + rv \in \Omega$ but $d(x + rv, \Sigma) = r$ which is absurd.

One of the most powerful theorems concerning the Hausdorff distance is called the Blaschke selection theorem. It gives a property of compactness for convex bodies.

Theorem 8 (The Blaschke Selection Theorem.) *Every infinite set of uniformly bounded (for the Hausdorff distance) convex bodies is relatively compact for the Hausdorff distance.*

We then get easily this corollary :

Corollary 3 *For every convex body Ω there exists a sequence of convex polygons Q_n that converge to Ω for the Hausdorff distance and such that Ω is included in Q_n for every n.*

Let us now consider a second distance.

Definition 9 *For Ω and Σ two sets, let $\delta(\Omega,\Sigma)=\mathcal{H}^2(\Omega\Delta\Sigma)$.*

Even if δ is not a distance on compact sets, it can be shown that it is a distance on convex bodies. We then have this relation between d and δ :

Proposition 11 *d and δ are defining the same topology on the convex bodies.*

Proof : a) Let us first show that if Ω_n tends to Ω for the Hausdorff distance then it also converges to Ω for the δ one :

We will first show that almost everywhere, 1_{Ω_n} tends to 1_Ω.

If $x\notin\Omega$ then let $\varepsilon=d(x,\Omega)$.

There exists $N\in\mathbb{N}$ such that for every $n\geq N$ $d(\Omega_n,\Omega)<\varepsilon$. Then for $n\geq N$, $x\notin\Omega_n$ so $1_{\Omega_n}(x)=0=1_\Omega(x)$.

If $x\in\overset{\circ}{\Omega}$, then there exists $r>0$ such that $\overline{B}(x,r)\subset\overset{\circ}{\Omega}$. Let $\varepsilon=\min\limits_{y\in\overline{B}(x,r)}d(y,\partial\Omega)$. Then $\varepsilon>0$ and if N is such that for every $n\geq N$, $d(\Omega_n,\Omega)<\frac{\varepsilon}{2}$, we have that for every $n\geq N$, $d(\partial\Omega_n,\partial\Omega)<\frac{\varepsilon}{2}$ by Proposition 10. Therefore, for all $y\in B(x,r)$,

$$d(y,\partial\Omega_n)=d(y,z)\geq d(y,v)-d(v,z)\geq\varepsilon-d(v,z)\geq\frac{\varepsilon}{2}.$$

Where $z\in\partial\Omega_n$ and $v\in\partial\Omega$. Hence $B(x,r)\cap\partial\Omega_n=\emptyset$. But here exists $y\in\overset{\circ}{\Omega}_n$ with $||x-y||<\frac{\varepsilon}{2}$. x and y are thus in the same connected compound of $\mathbb{R}^2\backslash\partial\Omega$. So $x\in\overset{\circ}{\Omega}$ and $1_{\Omega_n}(x)=1=1_\Omega(x)$.

Hence, for almost every x, $|1_{\Omega_n}(x)-1_\Omega(x)|$ tends to zero. We can now conclude by the Dominated Convergence Theorem that $\mathcal{H}^2(\Omega_n\Delta\Omega)$ tends to 0 when n tends to infinity.

b) Let us now show that if $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of convex bodies converging for δ to a convex body Ω then it also converges to it for the Hausdorff distance.

Suppose that $d(\Omega_n,\Omega)$ does not tends to zero, let us show that $\delta(\Omega_n,\Omega)$ does not tends to zero either.

There exists $\varepsilon>0$ such that for every $N>0$ there exists $n>N$ with $d(\Omega_n,\Omega)>\varepsilon$. Hence $\Omega\not\subset B(\Omega_n,\varepsilon)$.

Therefore there exists $x\in\Omega$ with $B(x,\varepsilon)\cap\Omega_n=\emptyset$. So

$$\delta(\Omega_n,\Omega)\geq\mathcal{H}^2(B(x,\varepsilon)\cap\Omega).$$

Let us show that $A=\inf_{x\in\Omega}\mathcal{H}^2(B(x,\varepsilon)\cap\Omega)$ is positive. We will first prove that $x\to\mathcal{H}^2(B(x,\varepsilon)\cap\Omega)$ is continuous :

$$\begin{aligned}|\mathcal{H}^2(B(x,\varepsilon)\cap\Omega)-\mathcal{H}^2(B(z,\varepsilon)\cap\Omega)| &= |\int_\Omega 1_{B(x,\varepsilon)}-1_{B(z,\varepsilon)}|\\ &\leq\int_{\mathbb{R}^2}1_{B(x,\varepsilon)\Delta B(z,\varepsilon)}\\ &=\mathcal{H}^2(B(x,\varepsilon)\Delta B(z,\varepsilon))\end{aligned}$$

Which tends to zero when z tends to x.

Hence $O_\mu = \{x \,/\, \mathcal{H}^2(B(x,\varepsilon)\cap\Omega) > \mu\}$ is open for every $\mu > 0$. By the convexity of Ω and Proposition 5, for every $x \in \Omega$:

$$\mathcal{H}^2(B(x,\varepsilon \cap \Omega) > 0$$

Hence,

$$\Omega \subset \bigcup_{\mu>0} O_\mu$$

By the compactness of Ω, we can extract from $(O_\mu)_{\mu\in\mathbb{R}^+}$ a finite covering of Ω, $O_{\mu_1}, .., O_{\mu_n}$. If $\mu = \min \mu_i$, as O_μ is a decreasing family, we have

$$\Omega \subset O_\mu$$

and therefore $A \geq \mu$ which shows that For every $N > 0$ there exists $n > N$ such that $\delta(\Omega_n, \Omega) > A$. So Ω_n does not tends to Ω for the δ distance.

5 Properties of the Signature Function.

Now that we have all these definitions and properties about convex bodies, we are able to prove some properties of the function F_Ω. The main result of this part will be the continuity of F_Ω for all convex bodies. In addition, we will also show two other results. First we will prove that the limit of F_Ω when μ tends to zero is equal to the perimeter of Ω. Then we will show that F_Ω is positive for μ minor than the diameter of Ω and zero for μ major than this diameter.

Proposition 12 *F_Ω is a non-increasing function.*

Proof : clear.

Proposition 13 *For every convex body Ω, the limit of $F_\Omega(\mu)$ when μ tends to zero is equal to the perimeter of Ω.*

Proof : First, let us show that $\lim_{\mu\to 0} F_\Omega(\mu) = \mathcal{H}^2(r,\theta \,/\, R1_\Omega(r,\theta) > 0) = \mathcal{H}^2(B)$.

For every sequence $(a_n)_{n\in\mathbb{N}}$ which tends to zero, $B = \bigcap_{n\geq 1} A(a_n)$ with $F_\Omega(a_n) = \mathcal{H}^2(A(a_n))$ and $A(a_n)$ increasing sets. Hence by a classical result of integration theory,

$$\lim_{n\to+\infty} F_\Omega(a_n) = \mathcal{H}^2(B)$$

By Theorem 7, we now only have to show that

$$L = \mathcal{H}^2(r,\theta/\,|D_{r,\theta} \cap \Omega| = 0 \text{ and } D_{r,\theta} \cap \Omega \neq \emptyset) = 0$$

If $D_{r,\theta} \cap \Omega$ contains more than two points then it contains a whole segment by the convexity of Ω and hence has positive length.

Let $D_{r,\theta}$ be a line which intersects Ω in exactly one point x, which has to be on the boundary of Ω.

If the sets of non-smooth points of $\partial\Omega$ is $\{x_1..x_n\}$, then we can write L as $L_{smooth} + \sum_i L_i$, where L_{smooth} stands for the lines which intersect $\partial\Omega$ on a point of smoothness and L_i stands for the lines passing through x_i.

If x is a point where the boundary is smooth, then $D_{r,\theta}$ has to be the tangent to Ω in x (if it were not, then $D_{r,\theta}$ would cross Ω in more than one point). As $\partial\Omega$ is $\mathcal{C}^1$ by parts, $\mathcal{H}^2(\partial\Omega) = 0$ and hence $L_{smooth} = 0$.

For every i and every θ, there exists only one r, such that $D_{r,\theta}$ passes through x_i so

$$L_i = \int_\theta \int_r 1_{D_{r,\theta}\cap\Omega = x_i}(r,\theta) = 0$$

Therefore, $L = 0$.

Proposition 14 *Let Ω be a convex body, then if diam$(\Omega) = d = \sup_{x,y\in\Omega} ||x-y||$ then for $\mu < d$, $F_\Omega(\mu)$ is positive and for $\mu > d$, $F_\Omega(\mu)$ is equal to zero.*

Proof : the second part of the proposition is clear by the fact that $|D_{r,\theta} \cap \Omega| \leq d$.

To prove the second part, as F_Ω is non-increasing, it is sufficient to show that :

$$\forall \varepsilon > 0 \quad F_\Omega(d-\varepsilon) > 0$$

Ω is compact so there exist $x, y \in \partial\Omega$ such that $d = ||x-y||$. By Proposition 5, $x, y \in \partial \overset{\circ}{\Omega}$. Hence, there exist $z_1 \in \overset{\circ}{\Omega} \cap B(x, \frac{\varepsilon}{2})$ and $z_2 \in \overset{\circ}{\Omega} \cap B(y, \frac{\varepsilon}{2})$. We then have :

$$||z_1 - z_2|| \geq ||x-y|| + ||z_1 - x|| + ||z_2 - y|| \geq d - \varepsilon$$

As $[z_1, z_2] \subset \overset{\circ}{\Omega}$ we can assume that $||z_1 - z_2|| = d - \varepsilon$.

There exists $\alpha > 0$ such that $B(z_1, \alpha) \subset \overset{\circ}{\Omega}$ and $B(z_2, \alpha) \subset \overset{\circ}{\Omega}$.

If we call v the unit vector orthogonal to $z_1 - z_2$, then both $y_1 = z_1 + \frac{\alpha}{2}v$ and $y_2 = z_2 + \frac{\alpha}{2}v$ are in Ω. By convexity, the rectangle $R = z_1z_2y_2y_1$ is also included in Ω. By the computations of 3.1, $F_R(d-\varepsilon) > 0$. Furthermore, we clearly have that $F_\Omega \geq F_R$; so that $F_\Omega(d-\varepsilon) > 0$.

We can now show the continuity of F_Ω.

Theorem 9 *For every convex body Ω, the function F_Ω is continuous.*

Proof : F_Ω is discontinuous at μ if and only if $\mathcal{H}^2(r, \theta / |D_{r,\theta} \cap \Omega| = \mu) > 0$. which means that

$$\int_0^\pi \int_{\mathbb{R}} 1_{|D_{r,\theta}\cap\Omega| = \mu}(r,\theta) > 0$$

Therefore there exists $U \subset [0, \pi[$, with $\mathcal{H}^1(U) > 0$ such that

$$\forall \theta \in U \quad \int_{\mathbb{R}} 1_{|D_{r,\theta}\cap\Omega| = \mu}(r,\theta) = h_\mu(\theta) > 0$$

Lemma 2 *For all $\theta \in [0, \pi[$, there exist real numbers $\alpha \leq A \leq B \leq \beta$ such that $|D_{r,\theta} \cap \Omega|$ is :*

a) *zero on $]-\infty, \alpha[\cap]\beta, +\infty[$.*

b) *increasing on $[\alpha, A]$.*

c) *constant on $]A, B[$.*

d) *decreasing on $[B, \beta]$.*

Proof of the lemma : By a rotation we can restrict ourselves to the case of $\theta = 0$ (see Figure 23). Let $M_r = |D_{r,0} \cap \Omega|$ and $D_{r,0} \cap \Omega = [\begin{pmatrix} r \\ a_r \end{pmatrix}, \begin{pmatrix} r \\ b_r \end{pmatrix}]$ with $b_r \geq a_r$.
Let r_1, r_2 such that $M_{r_1} = M_{r_2} = M_R = \max M_r$. Then the parallelogram $P = b_{r_1} b_{r_2} a_{r_2} a_{r_1}$ is included in Ω. Thus for every $r \in [r_1, r_2]$, $M_r \geq |P \cap D_{0,r}| = M_R$. Hence,

$$M_r = M_R \quad \forall r \in [r_1, r_2]$$

So if $A = \inf\{r \,/\, M_r = M_R\}$ and $B = \sup\{r \,/\, M_r = M_R\}$, for $r \in]A, B[$, $M_r = M_R$, which shows the point c).

The same reasoning shows that if $r_1 < r_2$ with $M_{r_1} > 0$ and $M_{r_2} > 0$ then for all $r \in [r_1, r_2], M_r > 0$ which shows a).

Let us now prove b) : we will prove that if $\alpha < r_1 < r_2 < A \leq R_1$, with $M_{R_1} = M_R$ then $M_{r_1} < M_{r_2} < M_R$.

As before, the parallelogram $P = b_{r_1} b_{R_1} a_{R_1} a_{r_1} \subset \Omega$. Hence $M_{r_2} \geq |P \cap D_{r_2,0}| = ||u - v||$. Where

$$u = \frac{b_{R_1} - b_{r_1}}{R_1 - r_1}(r_2 - r_1) + b_{r_1}$$

and

$$v = \frac{a_{R_1} - a_{r_1}}{R_1 - r_1}(r_2 - r_1) + a_{r_1}$$

So that $||u - v|| = \frac{r_2 - r_1}{R_1 - r_1}(M_R - M_{r_1}) + M_{r_1} > M_{r_1}$.
The same way we can show d).

Lemma 3 *If $h_\mu(\theta) > 0$ then there exist two parallel segments in $\partial\Omega$.*

Proof of the lemma : As in the previous lemma we can assume that $\theta = 0$. We will use here, the same notations as above.

$h_\mu(0) > 0$ means that there exists a set $V \subset \mathbb{R}$ of positive measure, with $M_r = \mu$ for all $r \in V$.

By the preceding lemma, that implies that $M_R = \mu$ and $A \neq B$.

For $A < R_1 < R_2 < B$ we therefore have that the parallelogram $P = b_{R_1} b_{R_2} a_{R_2} a_{R_1} \subset \Omega$.

But for all $r \in [R_1, R_2]$, $|P \cap D_{r,0}| = M_R = |\Omega \cap D_{r,0}|$, so by the convexity of $P \cap D_{r,0}$ and $\Omega \cap D_{r,0}$, we have that $P \cap D_{r,0} = \Omega \cap D_{r,0}$ and hence $P \cap (\bigcup_{r=R_1}^{R_2} D_{r,\theta}) = \Omega \cap (\bigcup_{r=R_1}^{R_2} D_{r,\theta})$. Therefore,

$$[\begin{pmatrix} R_1 \\ b_{R_1} \end{pmatrix}, \begin{pmatrix} R_2 \\ b_{R_2} \end{pmatrix}] \cup [\begin{pmatrix} R_1 \\ a_{R_1} \end{pmatrix}, \begin{pmatrix} R_2 \\ a_{R_2} \end{pmatrix}] \subset \partial\Omega$$

Let $O = \{(a,b) / [a,b] \subset \partial\Omega\}$ and $\sim$ be the equivalence relation, being parallel. Let then $Q = O/_\sim$.

Lemma 4 *If $h_\mu(\theta)$ and $h_\mu(\varphi)$ are positive ($\theta \neq \varphi$) then the points of Q associated to θ and φ by lemma 3 are distinct.*

Proof of the lemma : Let us prove it by absurd.

There exists ψ such that the two lines associated to θ are $D_{r_1,\psi}$ and $D_{r_2,\psi}$ with $r_1 > r_2$. If the two points are equal, then the two lines associated to φ are $D_{r_3,\psi}$ and $D_{r_4,\psi}$ with $r_3 > r_4$.

As these four lines have to be support planes of Ω we have :

$$r_1 \geq r_3 > r_4 \geq r_2$$
$$r_3 \geq r_1 > r_2 \geq r_4$$

Hence $r_3 = r_1$ and $r_4 = r_2$ which means that the lines are pairwise equals.

By invariance under rotation and translation, we can assume that $\psi = \frac{\pi}{2}$ and $r_2 = 0$ (see Figure 24). Then if r is such that

$$|D_{r,\theta} \cap \Omega| = \mu \quad \text{with} \quad D_{r,\theta} \cap \Omega = [\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ r_1 \end{pmatrix}]$$

Then we must have $\mu = ||b - a|| = \sqrt{r_1^2 + (x-y)^2}$. As $(x-y)^2 = \mu^2 \cos^2\theta$, this implies that $\mu^2 \sin^2\theta = r_1^2$.

We must also have $\mu^2 \sin^2\varphi = r_1^2$ which is impossible if $\theta \neq \varphi$ (both of them are supposed to be in $[0, \pi[$).

We can now finish the proof of the theorem : if F_Ω is not continuous in μ, then by Lemma 3, there must exist a non countable number of lines $D_{r,\theta}$ verifying :

$$D_{r,\theta} \cap \partial\Omega = [A, B] \qquad \text{with} \quad A \neq B.$$

By Lemma 4, for different couples (r,θ), the segments are different (and can only intersect in their end points). Furthermore, as $]A, B[$ contains points with rational coordinates, we see that it is absurd.

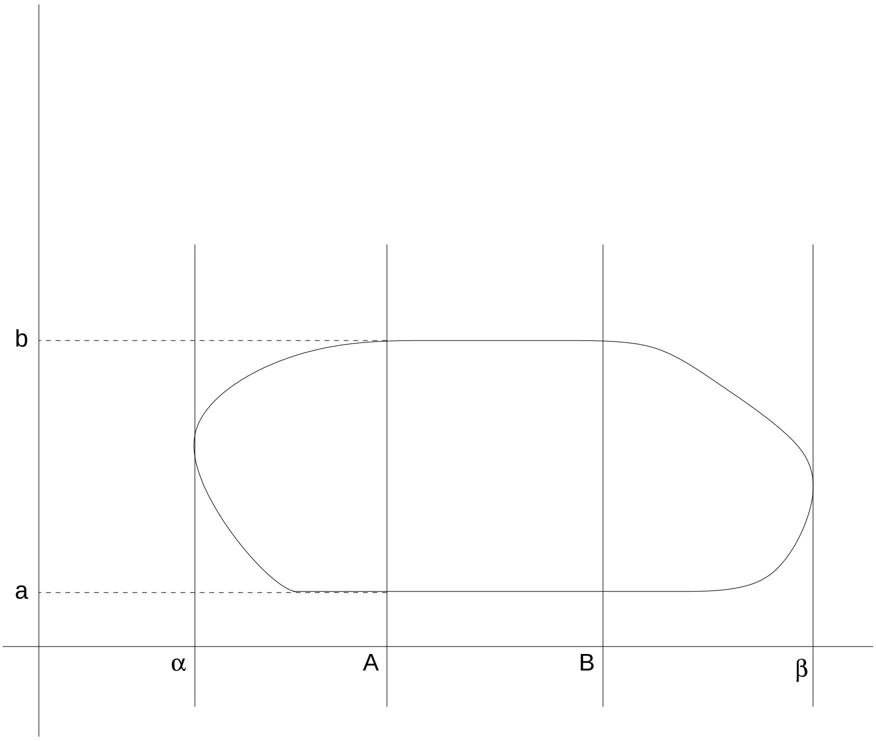

Figure 23:

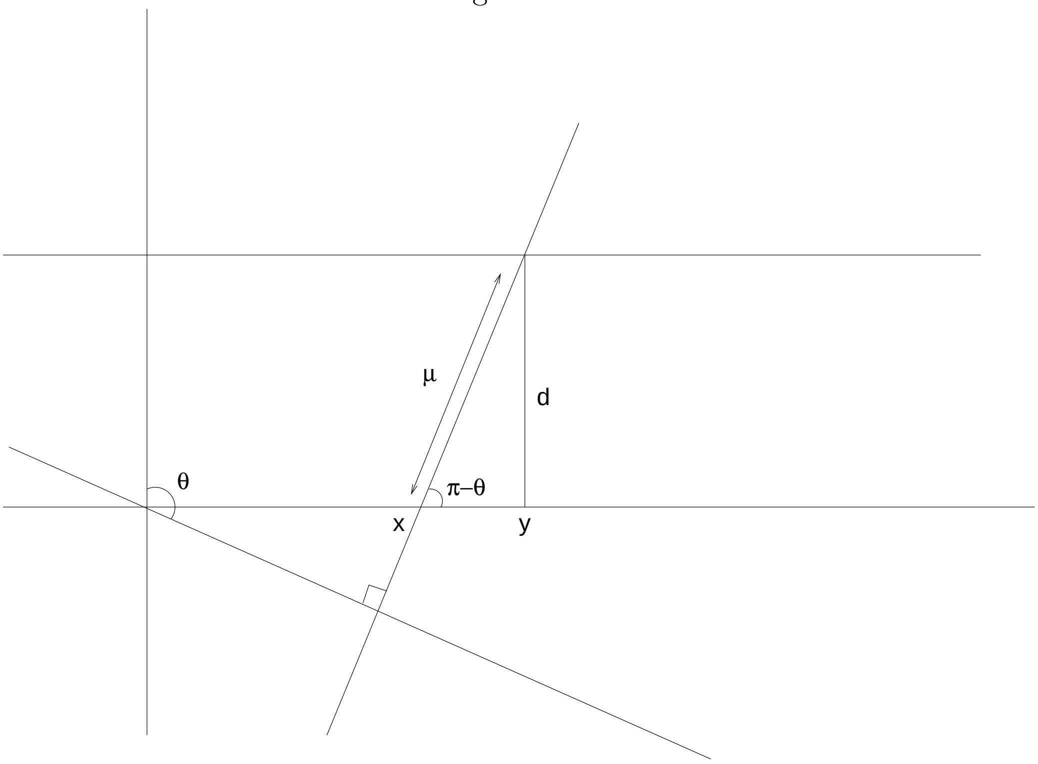

Figure 24:

6 Signature functions and distances.

We will now see how F_Ω changes when Ω changes. The main result of this part will be that for any convex body, F_Ω is continuous in Ω for the δ distance. However, before discussing this result we will give a formula which links the measure of a set to the norm of its signature function.

Theorem 10 *For every measurable set* Ω*,*

$$\mathcal{H}^2(\Omega) = \frac{1}{\pi}||F_\Omega||_{L^1}$$

To prove this theorem, we will need the following theorem (see [11]) :

Theorem 11 *For all* (X, μ) *measurable set and* f *positive measurable function,*

$$\int_X f d\mu = \int_{\mathbb{R}^+} \mu(f > t) dt \tag{1}$$

Proof of the Theorem 10 : By Fubini's theorem, for every θ :

$$\mathcal{H}^2(\Omega) = \int_{\mathbb{R}} \left(\int_{D_{r,\theta}} 1_\Omega(x)\, dx \right) dt$$

Hence by (1) applied to $f(r) = \int_{D_{r,\theta}} 1_\Omega(x)\, dx$, we have :

$$\mathcal{H}^2(\Omega) = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} 1_{R1_\Omega > t}(r, \theta)\, dr \right) dt$$

By integrating between 0 and π we get :

$$\pi\mathcal{H}^2(\Omega) = \int_{\mathbb{R}^+} \left(\int_0^\pi \int_{\mathbb{R}} 1_{R1_\Omega > t}(r, \theta)\, dr\, d\theta \right) dt$$

As F_Ω is non-increasing, it has at most a numerable points of discontinuity and hence there exists at most a numerable set of t such that $\mathcal{H}^2(r, \theta /\, R1_\Omega(r, \theta) = t)$ is not zero. Furthermore, for almost every $t \in \mathbb{R}^+$:

$$\mathcal{H}^2(r, \theta /\, R1_\Omega(r, \theta) > t) = \mathcal{H}^2(r, \theta /\, R1_\Omega(r, \theta) \geq t)$$

and

$$\pi\mathcal{H}^2(\Omega) = ||F_\Omega||_{L^1}$$

Observation : if we apply this theorem to $\Omega\Delta\Sigma$, we get that $\delta(\Omega, \Sigma) = \frac{1}{\pi}|||F_{\Omega\Delta\Sigma}||_{L^1}$.

Theorem 12 *For every measurable sets* Ω *and* Σ *we have :*

$$|F_\Omega(\mu) - F_\Sigma(\mu)| \leq \frac{2\pi}{\varepsilon}\delta(\Omega, \Sigma) + |F_\Sigma(\mu - \varepsilon) - F_\Sigma(\mu + \varepsilon)| \quad \forall \mu > 0, \varepsilon \in]0, \mu[\tag{2}$$

Proof : for every θ, let $S_\theta = \{r \,/\, R1_{\Omega(r,\theta)} \geq \mu \text{ and } R1_\Sigma(r,\theta) \leq \mu - \varepsilon\}$ and $G^\theta_\Omega(\mu) = \mathcal{H}^2(r \,/\, R1_\Omega(r,\theta) \geq \mu)$. So that $\int_0^\pi G^\theta_\Omega(\mu) d\theta = F_\Omega(\mu)$. We then have :

$$\mathcal{H}^2(S_\theta) \geq G^\theta_\Omega(\mu) - G^\theta_\Sigma(\mu - \varepsilon)$$

If $(r, \theta) \in S_\theta$ then

$$\begin{aligned} |(\Omega \Delta \Sigma) \cap D_{r,\theta}| &= |\Omega \cap D_{r,\theta}| + |\Sigma \cap D_{r,\theta}| - 2|(\Omega \cap \Sigma) \cap D_{r,\theta}| \\ &\geq |\Omega \cap D_{r,\theta}| - |\Sigma \cap D_{r,\theta}| \\ &\geq \varepsilon \end{aligned}$$

By Fubini's Theorem, we then have :

$$\begin{aligned} \mathcal{H}^2(\Omega \Delta \Sigma) &= \int_\mathbb{R} \int_{D_{r,\theta}} |1_\Omega(x) - 1_\Sigma(x)| \, dx \, dr \\ &\geq \int_{S_\theta} \int_{D_{r,\theta}} |1_\Omega(x) - 1_\Sigma(x)| \, dx \, dr \\ &\geq \varepsilon \, \mathcal{H}^2(S_\theta) \\ &\geq \varepsilon \, [G^\theta_\Omega(\mu) - G^\theta_\Sigma(\mu - \varepsilon)] \end{aligned}$$

So by integrating the above inequality and repeating the argument inverting Ω and Σ, we get :

$$\begin{aligned} \pi \, \delta(\Omega, \Sigma) &\geq \varepsilon [F_\Omega(\mu) - F_\Sigma(\mu - \varepsilon)] \\ \pi \, \delta(\Omega, \Sigma) &\geq \varepsilon [F_\Sigma(\mu) - F_\Omega(\mu - \varepsilon)] \end{aligned}$$

By applying the second line to $\nu = \mu + \varepsilon$ we have that :

$$F_\Sigma(\mu + \varepsilon) - \frac{\pi}{\varepsilon} \delta(\Omega, \Sigma) \leq F_\Omega(\mu) \leq F_\Sigma(\mu - \varepsilon) + \frac{\pi}{\varepsilon} \delta(\Omega, \Sigma)$$

As F_Σ is non-increasing, $F_\Sigma(\mu)$ is also in $[F_\Sigma(\mu + \varepsilon) - \frac{\pi}{\varepsilon} \delta(\Omega, \Sigma), F_\Sigma(\mu - \varepsilon) + \frac{\pi}{\varepsilon} \delta(\Omega, \Sigma)]$. Hence :

$$|F_\Omega(\mu) - F_\Sigma(\mu)| \leq \frac{2\pi}{\varepsilon} \delta(\Omega, \Sigma) + |F_\Sigma(\mu - \varepsilon) - F_\Sigma(\mu + \varepsilon)|$$

Corollary 4 *Let Ω be a measurable set with finite diameter, then for every sequence Ω_n tending to Ω for the δ norm, if $d = \sup_n diam(\Omega_n) < +\infty$ and $A = \max(||F_\Omega||_{L^\infty}, \sup_n ||F_{\Omega_n}||_{L^\infty}) < +\infty$, then F_{Ω_n} tends to F_Ω in L^1.*

Proof : let $b = max(d, diam(\Omega))$ then for $\mu \geq b$, $F_{\Omega_n}(\mu) = F_\Omega(\mu) = 0$. So by (2) we have :

$$\begin{aligned} ||F_\Omega - F_{\Omega_n}||_{L^1} \leq \int_0^\varepsilon |F_\Omega(\mu) - F_{\Omega_n}(\mu)| \, d\mu + &\frac{2\pi b}{\varepsilon} \delta(\Omega_n, \Omega) \\ &+ \int_\varepsilon^b |F_\Omega(\mu - \varepsilon) - F_\Omega(\mu + \varepsilon)| \, d\mu \quad \forall \varepsilon > 0 \end{aligned}$$

Let $U = \int_\varepsilon^b |F_\Omega(\mu-\varepsilon) - F_\Omega(\mu+\varepsilon)|\, d\mu \leq \int_0^b |F_\Omega(\mu) - F_\Omega(\mu+2\varepsilon)|\, d\mu$ which by the Dominated Convergence Theorem tends to zero when ε tends to zero (we need here to use the fact that F_Ω is continuous almost everywhere).

Let $V = \int_0^\varepsilon |F_\Omega(\mu) - F_{\Omega_n}(\mu)| d\mu \leq 2\varepsilon A$.

For $\eta > 0$ let $\varepsilon > 0$ be such that $U + V \leq \frac{\eta}{2}$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\frac{2\pi b}{\varepsilon}\,\delta(\Omega_n, \Omega) \leq \frac{\eta}{2}$.

Then for every $n \geq N$,

$$||F_\Omega - F_{\Omega_n}||_{L^1} \leq U + V + \frac{2\pi b}{\varepsilon}\delta(\Omega_n, \Omega) \leq \eta$$

For convex bodies we have a stronger result :

Corollary 5 *Let Ω be a measurable set of finite diameter, for which F_Ω is continuous (Ω a convex body for example). Let ω be the modulus of continuity of F_Ω. Then for every measurable set Σ :*

$$||F_\Omega - F_\Sigma||_{L^\infty} \leq \frac{4\pi}{\varepsilon}\delta(\Omega, \Sigma) + \omega(\varepsilon) \quad \forall \varepsilon > 0 \tag{3}$$

In particular, for every sequence Ω_n tending to Ω for δ, F_{Ω_n} tends to F_Ω in L^∞.

Proof : If F_Ω is continuous then it is uniformly continuous on $[0, diam(\Omega)]$ and 0 on $[diam(\Omega), +\infty[$. Hence it is uniformly continuous on $\mathbb{R}^+$.
$|F_\Sigma(\mu-\varepsilon) - F_\Sigma(\mu+\varepsilon)| \leq \omega(2\varepsilon)$ so, by (2) we have the result.

Observation : When we say that F_Ω is continuous, we mean that it is continuous on $]0, +\infty[$ with a finite limit in zero. We then let $F_\Omega(0) = \lim\limits_{\mu\to 0} F_\Omega(\mu)$.

A Appendix A.

Let us now compute F_Ω for a right triangle of sides R and r, with $R \geq r$.

As for the rectangle, we will try to calculate $h(\theta)$ for $\theta \in [0, \pi]$. We will split the calculations in two parts, $\theta \in [0, \frac{\pi}{2}]$ and $\theta \in [\frac{\pi}{2}, \pi]$, so that F_Ω is equal to $F_1 + F_2$.

A.1 $\theta \in [0, \frac{\pi}{2}]$.

We will calculate here F_1. Two cases are to be distinguished.

A.1.1 $\theta \in [\alpha, \frac{\pi}{2}]$.

If $\theta \geq \alpha$ then we are in the situation of Figure 25 because $\delta = \theta - \alpha \geq 0$.

By the law of sines, the area $\mathcal{A}$ of the parallelogram of sides μ and b is equal to $\mu b \sin\theta$. On the other side, it is also equal to $\mu h(\theta)$. Hence,

$$h(\theta) = b \sin\theta$$

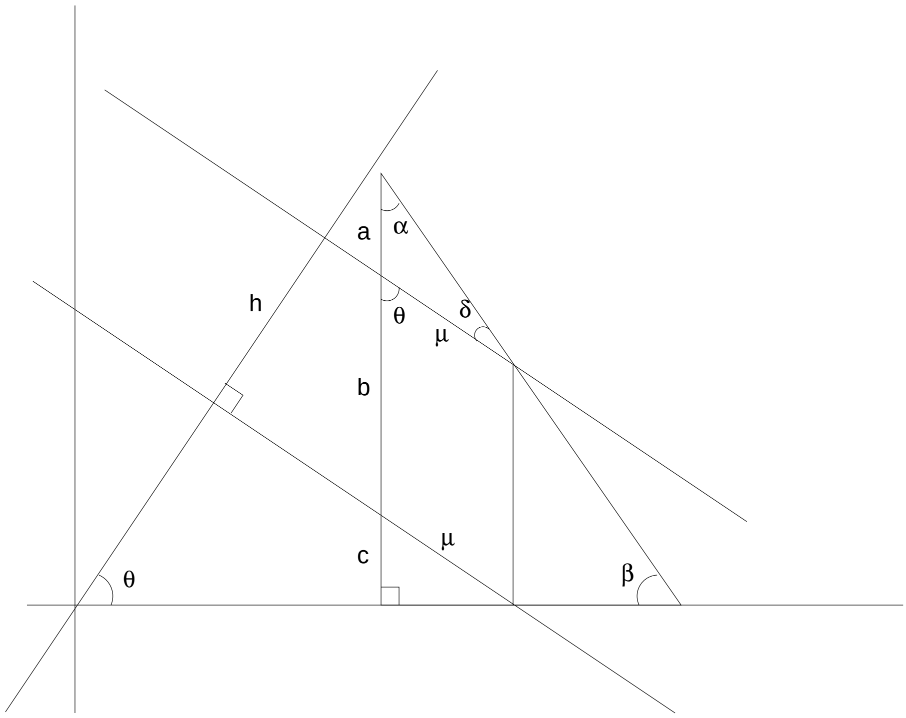

Figure 25:

$$b = R - (a + c)$$

whenever $a + c \leq R$ and is zero elsewhere. By the law of sines,

$$\frac{a}{\sin\delta} = \frac{\mu}{\sin\alpha} \quad \text{and} \quad \frac{c}{\sin(\frac{\pi}{2} - \theta)} = \mu$$

So that we have

$$a = \frac{\mu\sin(\theta - \alpha)}{\sin\alpha} \quad \text{and} \quad c = \mu\cos\theta$$

The condition on $a + c$ is thus expressed by :

$$\begin{aligned} a + c &\leq R \\ \sin(\theta - \alpha) + \sin\alpha\cos\theta &\leq \frac{R}{\mu}\sin\alpha \\ \sin\theta &\leq \frac{R}{\mu}\tan\alpha \\ &= \frac{r}{\mu} \end{aligned}$$

Which is always true if $\mu \leq r$. If $\mu \geq r$ it is true only if $\theta \in [\alpha, \arcsin(\frac{r}{\mu})]$. In this case we have :

$$h(\theta) = R(1 - \frac{\mu}{r}\sin\theta)\sin\theta$$

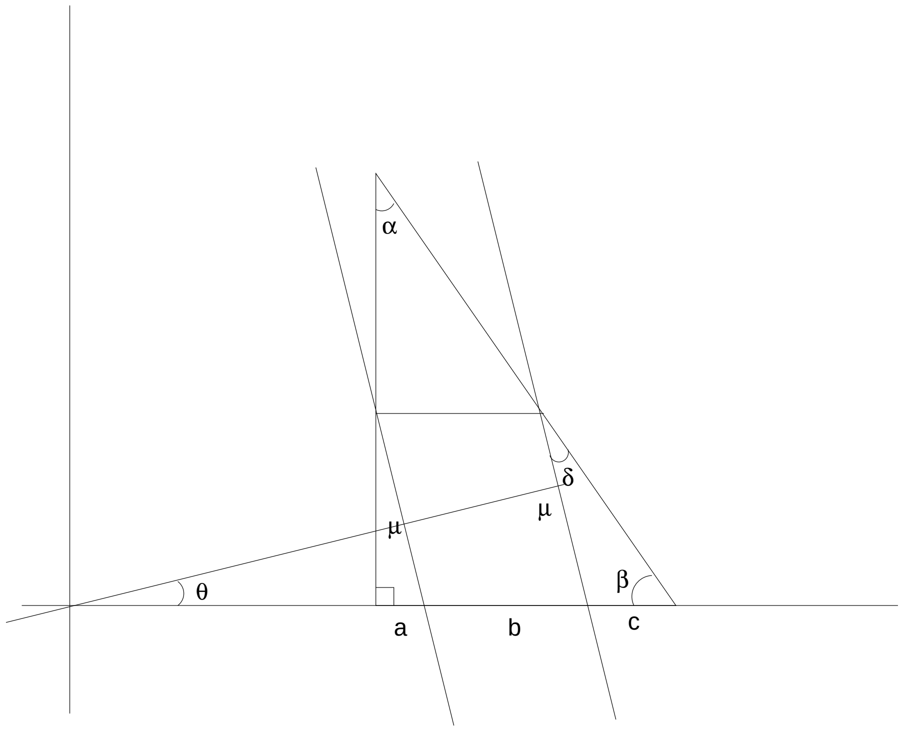

Figure 26:

A.1.2 $\theta \in [0, \alpha]$.

For $\theta \leq \alpha$ we are in the situation of Figure 26 because $\delta = \alpha - \theta \geq 0$.

Applying the same argument as before, we have :

$$\mathcal{A} = \mu h(\theta) = \mu b \cos\theta$$

so that

$$h = b\cos\theta$$
$$b = r - (a + c)$$

whenever $r \geq a + c$, and $b = 0$ in the other case. We then have :

$$\frac{a}{\sin\theta} = \mu \quad \text{and} \quad \frac{c}{\sin(\delta)} = \frac{\mu}{\sin\beta}$$

Which gives us :

$$a = \mu\sin\theta \quad \text{and} \quad c = \frac{\sin(\alpha - \theta)}{\cos\alpha}\mu$$

The condition on $a + c$ is thus expressed by :

$$\begin{aligned} a + c &\leq r \\ \cos\theta &\leq \frac{r}{\mu\tan\alpha} \\ &= \frac{R}{\mu} \end{aligned}$$

Which is always true if $\mu \leq r$. If $\mu \geq R$ it is true only if $\theta \in [\arccos(\frac{R}{\mu}), \alpha]$. In these case we have :

$$h(\theta) = r(1 - \frac{\mu}{R}\cos\theta)\cos\theta$$

A.1.3 F_1.

Adding these two computations and integrating on $[0, \frac{\pi}{2}]$ we get :

a) for $\mu \in [0, r]$,
$F_1(\mu) = \sqrt{R^2 + r^2} - \frac{1}{2}[1 + \frac{r}{R}\alpha + (\frac{\pi}{2} - \alpha)\frac{R}{r}]\mu$

b) for $\mu \in [r, R]$,
$F_1(\mu) = \sqrt{R^2 + r^2} - \frac{1}{2}[1 + \frac{r}{R}\alpha + \frac{R}{r}(\arcsin\frac{r}{\mu} - \alpha)]\mu - \frac{R}{2}\sqrt{1 - \frac{r^2}{\mu^2}}$

c) for $\mu \in [R, \sqrt{R^2 + r^2}]$,
$F_1(\mu) = \sqrt{R^2 + r^2} - \frac{1}{2}[1 + \frac{r}{R}(\alpha - \arccos\frac{R}{\mu}) + \frac{R}{r}(\arcsin\frac{r}{\mu} - \alpha)]\mu - \frac{R}{2}\sqrt{1 - \frac{r^2}{\mu^2}} - \frac{r}{2}\sqrt{1 - \frac{R^2}{\mu^2}}$

d) for $\mu \geq \sqrt{R^2 + r^2}$,
$F_1(\mu) = 0$

A.2 $\theta \in [\frac{\pi}{2}, \pi]$.

We will now compute F_2. We prefer to work with $\theta \in [0, \pi]$ so we can observe that after reflection, the situation is the same as Figure 27.

Here we have that $\delta = \pi - (\alpha + \theta)$. And as before,

$$\mathcal{A} = \mu h = \mu b \sin\delta = \mu b \sin(\alpha + \theta)$$

$$b = \sqrt{R^2 + r^2}$$

whenever $\sqrt{R^2 + r^2} \geq a + c$, and $b = 0$ in the other case. We then have :

$$\frac{a}{\sin(\frac{\pi}{2} - \theta)} = \frac{\mu}{\beta} \quad \text{and} \quad \frac{c}{\sin(\theta)} = \frac{\mu}{\sin\alpha}$$

Which gives us :

$$a = \mu\frac{\cos\theta}{\cos\alpha} \quad \text{and} \quad c = \frac{\sin(\theta)}{\sin\alpha}\mu$$

The condition on $a + c$ is thus expressed by :

$$a + c \leq \sqrt{R^2 + r^2}$$
$$\sin(\theta + \alpha) \leq \frac{\sqrt{R^2 + r^2}}{2\mu}\sin(2\alpha)$$
$$= \frac{Rr}{\sqrt{R^2 + r^2}\mu}$$

which means that :

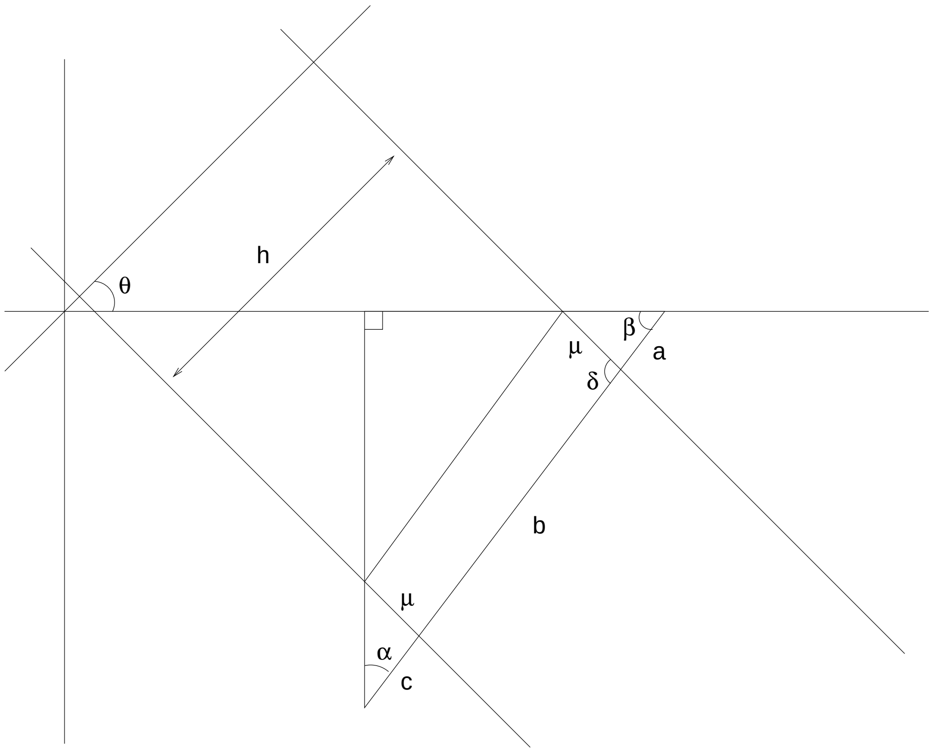

Figure 27:

a) if $\mu \leq \frac{Rr}{\sqrt{R^2+r^2}}$, it is always verified.

b) if $\mu \in [\frac{Rr}{\sqrt{R^2+r^2}}, r]$, it is verified for $\theta \in [0, \arcsin(\frac{Rr}{\mu\sqrt{R^2+r^2}}) - \alpha] \cup [\pi - \alpha - \arcsin(\frac{Rr}{\mu\sqrt{R^2+r^2}})]$.

c) if $\mu \in [r, R]$, it is verified for $\theta \in [0, \arcsin(\frac{Rr}{\mu\sqrt{R^2+r^2}}) - \alpha]$

d) it is never verified if $\mu > R$.

When this condition is fulfilled, we have :

$$h(\theta) = (\sqrt{R^2+r^2} - 2\mu\frac{\sin(\theta+\alpha)}{\sin(2\alpha)})\sin(\theta+\alpha)$$

Integrating $h(\theta)$ between 0 and $\frac{\pi}{2}$ we get :

a) if $\mu \leq \frac{Rr}{\sqrt{R^2+r^2}}$,
$F_2(\mu) = R + r - \mu(1 + \frac{\pi}{4}\frac{R^2+r^2}{Rr})$.

b) if $\mu \in [\frac{Rr}{\sqrt{R^2+r^2}}, r]$,
$F_2(\mu) = (R+r) - \mu(1 + \frac{\pi}{4}\frac{R^2+r^2}{Rr}) - \sqrt{R^2+r^2}\sqrt{1 - (\frac{Rr}{\mu\sqrt{R^2+r^2}})^2} + \frac{\mu}{2}\frac{R^2+r^2}{Rr}(\pi - 2\arcsin(\frac{Rr}{\mu\sqrt{R^2+r^2}}))$.

c) if $\mu \in [r, R]$,

$$F_2(\mu) = R - \frac{\sqrt{R^2+r^2}}{2}\sqrt{1 - (\frac{Rr}{\mu\sqrt{R^2+r^2}})^2} - \frac{\mu}{2} + \frac{\mu}{2}\frac{R^2+r^2}{Rr}(\alpha - \arcsin(\frac{Rr}{\mu\sqrt{R^2+r^2}}))$$

d) $F_2(\mu)$ is equal to zero if $\mu \geq R$.

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