Thèse pour l'obtention du titre de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Spécialité : **Mathématiques Appliquées**

par

Michael Goldman

Quelques applications des fonctions à variation bornée en dimension finie et infinie.

Soutenue le vendredi 9 décembre 2011 devant le jury composé de :

Antonin Chambolle	Directeur de thèse
Luigi Ambrosio	Rapporteur
Guy Bouchitté	Rapporteur
Vicent Caselles	Examinateur
Christian Léonard	Examinateur
Régis Monneau	Examinateur
Matteo Novaga	Examinateur

Remerciements

Mes remerciements vont tout particulièrement à mon directeur de thèse Antonin Chambolle qui m'a guidé dans mes premiers pas dans le monde de la recherche depuis mon stage de L3 jusqu'à l'aboutissement de cette thèse. Il a toujours su me proposer des sujets de recherche passionnants, souvent à l'interface entre mathématiques pures et appliquées. Il m'a permis de voir à quel point la frontière entre ces deux mondes était perméable. Travailler avec lui a toujours été extrêmement enrichissant pour moi. Si son soutien et ses conseils ont toujours été précieux, j'ai également beaucoup apprécié la confiance et la très grande liberté qu'il m'a accordées. En particulier, je lui suis profondément reconnaissant de m'avoir mis en contact avec Matteo Novaga. Cette thèse doit beaucoup au travail que j'ai effectué avec eux deux. Matteo a été pour moi de fait, un co-directeur et je le remercie chaleureusement pour sa générosité, son investissement et pour les très nombreuses idées et problèmes qu'il a partagés avec moi dans un bureau, à la terrasse d'un café à Tours, Paris, Pise et Padoue ou encore chez lui où j'ai eu le plaisir d'être hébergé.

Je voudrais remercier Luigi Ambrosio d'avoir accepté de rapporter cette thèse. C'est à la fois un grand honneur et un plaisir particulier pour moi que de l'avoir dans mon jury. Outre le fait que certains de ses livres ou articles ont été sur ma table de chevet pendant une bonne partie de ces dernières années, je lui suis reconnaissant pour les deux merveilleux cours que j'ai suivis avec lui (dont un sur le sujet de cette thèse) ainsi que pour son invitation à venir à Pise pendant six mois au cours de ma thèse et pour l'intérêt qu'il a porté à mes travaux. Je remercie également Guy Bouchitté de m'avoir fait l'honneur de rapporter cette thèse.
Vicent Caselles dont le nom est cité quasiment à chacune des pages de cette thèse m'a fait un grand plaisir d'accepter de faire partie du jury. Je remercie de même Régis Monneau de faire partie de ce jury ainsi que pour l'intérêt qu'il a porté à mes travaux. Je remercie Christian Léonard pour sa participation à mon jury.

La liste des autres personnes à remercier serait infinie mais je pense que je me dois de commencer par Khaled Jalalzai et Gilles Thouroude avec qui j'ai passé un nombre d'heures non dénombrable à parler de nos problèmes mathématiques ainsi que d'autre sujets plus ou moins professionnels.

Je voudrais remercier au sein du CMAP, Benoît Merlet pour nos discussions sur les varifolds, les champs à divergence nulle où encore la nature des gens, François Alouges pour le temps qu'il a pris pour me parler de la méthode des volumes finis ainsi que pour sa sollicitude, Grégoire Allaire pour avoir toujours eu un mot gentil à mon égard, en particulier en ce qui concerne le Groupe de Travail de calcul des variations, Robert Brizzi, Vincent Giovangigli et Kamel Hamdache pour leur bonne humeur de karatekas. Je remercie également tous les doctorants et Post-doc avec qui il a été très agréable de partager des repas, des sorties et de grandes discussions sur le café, le RER B ou le Magnan pendant ces années. J'ai une pensée particulière pour mes co-bureaux, la balle rouge et les clubs de golf. Je voudrais enfin souligner l'efficacité et la gentillesse du personnel administratif du CMAP, en particulier de Nasséra, Alexandra, Christian et Anna sans qui bien des choses auraient été plus compliquées. Je voudrais profiter de l'occasion pour remercier Michel Multan de la bibliothèque centrale de l'X pour son incroyable efficacité.

Une partie importante de cette thèse a été réalisée lors de mon séjour à Pise. J'y ai trouvé un environnement extrêmement stimulant tant d'un point de vue intellectuel qu'humain. Je voudrais remercier à nouveau Luigi Ambrosio pour son invitation, Giuseppe Buttazzo pour les quelques discussions fort intéressantes que nous avons eues ensemble ainsi que tous les doctorants et Post-doc à la passion débordante pour les mathématiques que j'ai pu y rencontrer.

Je voudrais remercier Antoine Lemenant, Filippo Santambroggio et Jean-François Babadjian pour m'avoir soutenu dans l'organisation du Groupe de Travail de calcul des variations. Je remercie Hugues Talbot et Pierre Bousquet pour leur bienveillance.

Je remercie chaleureusement tous les instituteurs, enseignants et professeurs que j'ai eus depuis le CP jusqu'à la thèse et qui ont contribué à me donner le goût pour les mathématiques.

Je remercie Alain Truong et tout le club de Lam Son Vo Dao de l'X pour m'avoir apporté un équilibre important.

Je voudrais finalement remercier tous ceux, famille et amis qui à Paris, Lyon, Pise ou ailleurs ont été à mes côtés pendant ces années avec amour et amitié.

Table des matières

Chapitre 1

Introduction

1.1 Introduction générale

Le but de cette thèse est d'étudier quelques problèmes dans lesquels interviennent les fonctions à variation bornée et les ensembles de périmètre fini. Depuis leur introduction par Camille Jordan en 1881, les fonctions à variation bornée se sont imposées comme un outil très puissant pour modéliser de nombreux phénomènes et affronter quantité de problèmes mathématiques. C'est surtout à partir des années 50, avec les travaux de De Giorgi, qu'elles ont pris une place centrale dans le calcul des variations.

Outre le problème des surfaces minimales, les fonctions à variation bornée ont permis d'étudier de nombreux phénomènes tels que les transitions de phases, les fractures, la segmentation ou d'autres applications en traitement d'images, des problèmes de discontinuités libres tels que certaines questions de plasticité ou de la théorie des cristaux liquides. C'est également un bon cadre pour étudier des questions de nature géométrique telles que les diverses variantes du problème isopérimétrique. Pour plus de détails ainsi que pour d'autres exemples d'applications, on pourra consulter [10, 32, 104].

Nous verrons ici quelques-unes des utilisations des fonctions à variation bornée en traitement d'images, en géométrie et en probabilités. Le plan de la thèse est le suivant :

- Dans le chapitre 2, nous étudions une méthode dite Primale-Duale proposée par Appleton et Talbot pour résoudre de nombreux problèmes

de traitement d'images tels que le débruitage ou la segmentation. Nous donnons un sens rigoureux à leur approche en l'interprétant comme une méthode d'Arrow-Hurwicz pour la recherche de points selle. Nous démontrons ensuite que le problème de Cauchy étudié est bien posé. Pour le problème de débruitage nous prouvons la convergence vers l'équilibre . Nous donnons également des estimations *a posteriori* et terminons le chapitre par une étude numérique. Ces résultats proviennent de l'article [86].
- Dans le chapitre 3, nous démontrons l'existence, génériquement dans L^∞, de surfaces fermées compactes de courbure moyenne prescrite en milieu périodique. Nous étudions également le comportement asymptotique des solutions ainsi construites lorsque leur volume tend vers l'infini. Ce chapitre est basé sur un article écrit en collaboration avec Matteo Novaga [88].
- Dans le chapitre 4, nous démontrons un théorème de type Modica-Mortola dans les espaces de Wiener. Nous calculons la Γ-limite de la fonctionnelle d'Allen-Cahn dans ce contexte. Contrairement à ce qui se passe dans le cadre euclidien, cette Γ-limite n'est pas un multiple du périmètre. En effet, nous montrons qu'elle coïncide avec une fonctionnelle bien connue de certains probabilistes intéréssés par l'étude de phénomènes de diffusion. L'un des outils principaux de ce travail est la méthode de symétrisation d'Ehrhard. Nous effectuons donc une étude poussée de celle-ci. Ce chapitre est issu d'un travail écrit avec Matteo Novaga [89].
- Dans le dernier chapitre de la thèse, nous étudions la convexité des solutions de certains problèmes variationnels en dimension infinie. Ce dernier chapitre contient d'une part une preuve utilisant les techniques développées par Alvarez Lasry et Lions [7] et d'autre part une preuve moins générale mais de nature plus géométrique inspirée par les techniques de Korevaar [99]. La première partie de ce chapitre provient de travaux effectués avec Antonin Chambolle et Matteo Novaga [47]. La deuxième partie est issue de la note [87]

Les concepts de Γ-convergence, de semi-continuité, de convexité ainsi que les méthodes de symétrisations sont, avec les fonctions à variation bornée, au coeur de cette thèse. Nous rappellons donc dans les deux prochaines parties de cette introduction certaines de ces notions. Nous faisons ensuite une introduction plus détaillée de chacuns des chapitres de cette thèse.

1.2 Définitions et propriétés principales des fonctions BV

Nous rappellons brièvement dans cette partie les définitions et propriétés principales des fonctions à variation bornée et des ensembles de périmètre fini. Pour en savoir d'avantage sur ce sujet, le lecteur pourra se référer aux livres [10] ou [83] ainsi qu'à l'ouvrage récent [104].

Nous désignerons ici par Ω un ensemble ouvert de $\mathbb{R}^m$.

Définition 1.2.1. *Soit $BV(\Omega)$ l'espace des fonctions u appartenant à L^1 pour lesquelles,*

$$\int_\Omega |Du| := \sup_{\substack{\xi \in \mathcal{C}_c^1(\Omega) \\ |\xi|_\infty \leq 1}} \int_\Omega u \operatorname{div} \xi < +\infty.$$

Muni de la norme $|u|_{BV} = \int_\Omega |Du| + |u|_{L^1}$, l'espace $BV(\Omega)$ est un espace de Banach.

Proposition 1.2.2. *Soit $u \in L^1(\Omega)$ alors $u \in BV(\Omega)$ si et seulement si, sa dérivée au sens des distributions Du est une mesure de Radon finie. De plus, la variation totale de Du est alors égale à $\int_\Omega |Du|$. Nous noterons parfois $|Du|(\Omega) = \int_\Omega |Du|$.*

L'un des aspects importants de l'espace BV est l'existence de bonnes propriétes de compacité, d'approximation et de semicontinuité.

Proposition 1.2.3. *Soit Ω un ouvert borné à bord Lipschitz et $u_n \in BV(\Omega)$ une suite bornée dans $BV(\Omega)$ alors la suite u_n est compacte pour la topologie $L^1(\Omega)$.*

Proposition 1.2.4. *Soit $u \in BV(\Omega)$ alors il existe une suite $u_n \in \mathcal{C}^\infty(\Omega)$ telle que u_n converge vers u dans L^1 et*

$$\int_\Omega |Du_n| \to \int_\Omega |Du|.$$

Proposition 1.2.5. *Soit $u_n \in BV(\Omega)$ telle que u_n converge vers $u \in L^1(\Omega)$ alors*

$$\varliminf_{n\to\infty} \int_\Omega |Du_n| \geq \int_\Omega |Du|.$$

Définition 1.2.6. *Si E est un ensemble de $\mathbb{R}^m$ tel que $\chi_E \in BV(\Omega)$, nous dirons que E est un ensemble de périmètre fini dans Ω et nous noterons $P(E,\Omega) := \int_\Omega |D\chi_E|$. Lorsque $\Omega = \mathbb{R}^m$, nous dirons simplement que E est un ensemble de périmètre fini et désignerons son périmètre $P(E,\mathbb{R}^m)$ par $P(E)$.*

Un lien important entre les ensembles de périmètre fini et les fonctions à variation bornée est donné par la formule de la coaire.

Théorème 1.2.7. *Soit $u \in BV(\Omega)$ et B un borélien de Ω alors*

$$|Du|(B) = \int_{\mathbb{R}} |D\chi_{\{u>t\}}|(B)\, dt.$$

Lorsque $B = \Omega$, cette égalité s'écrit

$$|Du|(\Omega) = \int_{\mathbb{R}} P(\{u > t\}, \Omega)\, dt.$$

Pour les ensembles de périmètre fini, il est possible de définir une notion de frontière, de normale et de plan tangent au sens de la mesure. C'est pourquoi, on désigne fréquemment cette branche des mathématiques sous le nom de Théorie Géométrique de la mesure.

Définition 1.2.8. *Soit E un ensemble de périmètre fini et soit $t \in [0;1]$. On définit alors l'ensemble*

$$E^{(t)} := \left\{ x \in \mathbb{R}^m \ / \ \lim_{r\downarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

Nous appellerons frontière au sens de la mesure l'ensemble $\partial E := \left(E^{(0)} \cup E^{(1)}\right)^c$. On peut également définir la frontière réduite de E par

$$\partial^* E := \left\{ x \in supp|D\chi_E| \ / \ \nu(x) := -\lim_{r\downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \ \textit{existe et} \ |\nu(x)| = 1 \right\}.$$

Pour nous, ν sera donc la normale sortante à l'ensemble E. Un théorème profond de De Giorgi démontre que les ensembles de périmètre fini possèdent une certaine régularité. Nous noterons dans la suite $\mathcal{H}^{m-1}$ la mesure de Hausdorff $m-1$ dimensionnelle.

Théorème 1.2.9. *Soit E un ensemble de périmètre fini. La frontière réduite $\partial^* E$ vérifie alors :*
- *$\partial^* E$ est dénombrablement $m-1$ rectifiable,*
- *$\partial E = \partial^* E$ à un ensemble de mesure $\mathcal{H}^{m-1}$ nulle près,*

- $|D\chi_E| = \mathcal{H}^{m-1} \llcorner \partial^* E$,
- *pour tout* $x \in \partial^* E$, *les ensembles* $(E - x)/r$ *convergent localement au sens des mesures vers le demi-espace orthogonal à* $\nu(x)$ *lorsque* r *tend vers 0.*

L'inégalité isopérimétrique est une pierre angulaire de cette théorie. Elle résume le fait que les boules sont les ensembles de périmètre minimal parmi tous les ensembles ayant un volume fixé.

Théorème 1.2.10. *Il existe une constante* $C(m)$ *dépendant uniquement de la dimension de l'espace ambient tel que pour tout ensemble de périmètre fini* E *on ait :*

$$P(E) \geq C(m)|E|^{\frac{m-1}{m}}.$$

De plus, on a l'inégalité isopérimétrique relative : pour tout ouvert à bord Lipschitz Ω, *il existe une constante* $C(\Omega, m)$ *telle que*

$$\frac{|\Omega \cap E|}{|\Omega|} \cdot \frac{|\Omega \backslash E|}{|\Omega|} \leq C(\Omega, m) P(E, \Omega).$$

1.3 Γ-convergence et enveloppes semi-continues

Nous rappellons ici les définitions et les quelques propriétés essentielles de la Γ-convergence et de la relaxation. Ces deux concepts apparaîtrons à de très nombreuses reprises dans cette thèse. Pour de plus amples détails, nous renvoyons aux livres [32] et [58].

La Γ-convergence est une notion introduite par De Giorgi pour traiter des problèmes de convergence en calcul des variations.

Définition 1.3.1. *Soit* X *un espace topologique et soit* $F_n : X \to \overline{\mathbb{R}}$ *une suite de fonctions. La* Γ*-limite inférieure et la* Γ*-limite supérieure de la suite* F_n *sont définies par :*

$$(\Gamma - \varliminf_{n\to\infty} F_n)(x) := \sup_{U \in \mathcal{N}(x)} \varliminf_{n\to\infty} \inf_{y \in U} F_n(y)$$
$$(\Gamma - \varlimsup_{n\to\infty} F_n)(x) := \sup_{U \in \mathcal{N}(x)} \varlimsup_{n\to\infty} \inf_{y \in U} F_n(y)$$

où $\mathcal{N}(x)$ *est l'ensemble des voisinages de* x *dans* X. *Lorsque ces deux quantités coïncident, on dit que la suite* F_n Γ*-converge.*

Lorsque X est un espace métrique, il existe une caractérisation séquentielle de la Γ-convergence.

Théorème 1.3.2. *Soit X un espace métrique. Une suite de fonctions F_n Γ-converge vers $F : X \to \overline{\mathbb{R}}$ si et seulement si, les deux conditions suivantes sont vérifiées :*

- *pour toute suite x_n convergeant vers x, on a $\underline{\lim}_{n\to\infty} F_n(x_n) \geq F(x)$,*
- *pour tout $x \in X$ il existe une suite x_n convergeant vers x avec $\overline{\lim}_{n\to\infty} F_n(x_n) \leq F(x)$.*

Dans la pratique, il suffit de vérifier la deuxième propriété pour un espace dense en énergie dans X.

L'intérêt principal de la Γ-convergence provient du théorème suivant.

Théorème 1.3.3. *Si une suite de fonctionnelles F_n Γ-converge vers F et si x_n est une suite de minimiseurs de F_n telle que x_n converge vers x alors x est un minimiseur de F.*

L'un des exemples les plus classiques de Γ-convergence est celui de Modica-Mortola [110]. Cet exemple réapparaîtra d'ailleurs fréquemment dans la thèse. Ce résultat stipule que l'énergie d'Allen-Cahn,

$$\int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\, dx$$

Γ-converge lorsque ε tend vers 0, vers un multiple du périmètre. Ici W désigne un potentiel double puit, typiquement $W(t) = (1-t^2)^2$. Ce résultat montre que les liens entre les solutions de l'équation d'Allen-Cahn,

$$-\Delta u + W'(u) = 0$$

et les surfaces minimales sont très forts. L'étude de ces liens a d'ailleurs été l'objet de nombreux travaux. Voir par exemple [120, 124].

Nous définissons maintenant l'enveloppe semi-continue d'une fonction. Cette notion apparaît naturellement en calcul des variations. En effet, lorsque l'on essaye d'appliquer la méthode directe pour minimiser une fonctionnelle, on se rend compte que celle qui est effectivement minimisée n'est pas l'énergie initiale mais son enveloppe semi-continue.

Définition 1.3.4. *Soit X un espace topologique. L'enveloppe semi-continue (ou fonction relaxée) d'une fonction $F : X \to \overline{\mathbb{R}}$, est la plus grande fonction semi-continue inférieurement qui soit en desous de F.*

Lorsque X est un espace métrique, comme pour la Γ-convergence il existe une caractérisation séquentielle de la relaxation.

Proposition 1.3.5. *Soit X un espace métrique. Pour toute fonction F : $X \to \overline{\mathbb{R}}$, et tout $x \in X$, la fonction relaxée $\overline{F}$ est donnée par*

$$\overline{F}(x) = \inf \left\{ \varliminf_{n \to \infty} F(x_n) \ : \ x_n \to x \right\}.$$

Le lien entre relaxation et Γ-convergence est le suivant :

Proposition 1.3.6. *Soit F une fonction de X dans $\mathbb{R}$. La Γ-limite de la suite constante égale à F n'est autre que son enveloppe semi-continue.*

Parmi les nombreux problèmes de relaxation, on peut citer celui du calcul de l'enveloppe semi-continue de fonctionnelles intégrales de la forme

$$J(u) := \int_\Omega F(x, u, \nabla u) \ dx.$$

Ce problème, qui a connu de très nombreux développements depuis les années 80, reviendra fréquemment tout au long de cette thèse. On pourra consulter à ce sujet le livre [36], ou bien [10, Chapitre 5] ou encore les articles [28, 29, 63].

1.4 Les fonctions à variation bornée en traitement d'images

Depuis l'avènement des ordinateurs et l'explosion de leurs capacités de calcul dans les années 60, le traitement d'images est devenue une discipline de recherche très importante, à cheval entre les mathématiques appliquées et l'informatique. On y utilise des outils mathématiques très divers tels que les statistiques, l'analyse harmonique, les équations aux dérivées partielles ou le calcul des variations. Ce domaine est extrêmement vaste et en constante évolution de sorte qu'il n'existe pas, à ma connaissance, de livre complet et vraiment à jour sur le sujet. Nous renvoyons toutefois le lecteur intéressé aux livres [19], [64], [105] ou [111] pour un panorama plus précis.

L'une des grandes difficultés en traitement d'images provient de l'immense variété existante dans les images naturelles comme le montre par exemple la figure 1.1. Il est donc très difficile de modéliser correctement les images. L'un des modèles mathématiques les plus utilisés aujourd'hui consiste à considérer les images comme étant des fonctions à variation bornée. On sait toutefois que celui-ci n'est pas parfait (voir [91] à ce sujet). Il possède l'avantage d'admettre les images ayant des objets aux contours bien nets tels les images cartoon mais a du mal à tenir compte des textures fines.

FIG. 1.1: Quelques exemples d'images naturelles

Ce modèle d'images a été utilisé dans de nombreux problèmes tels que le débruitage, le zoom, la segmentation, l'inpainting ou l'estimation de disparité pour la reconstruction d'images en 3D. On pourra trouver bien d'autres applications des fonctions à variation bornée en imagerie dans [45]. Dans toutes ces applications, on se ramène à l'étude d'un problème variationnel du type :

$$\min_{u \in BV(\Omega)} J(u) := \int_\Omega g(x)|Du| + G(u), \tag{1.1}$$

où Ω est le domaine de définition de l'image (typiquement $\Omega = [0,1]^2$) et où G est un terme d'attache à la donnée. L'un des exemples les plus classiques où l'on rencontre ce genre de régularisations est celui du débruitage. En effet, Rudin, Osher et Fatemi ont proposé en 1992 dans [125] de minimiser la fonctionnelle (dite de ROF) :

$$\min_{u \in BV(\Omega)} \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega |u - f|^2 \, dx,$$

afin de corriger l'image bruitée f. Ce problème à été étudié de façon approfondie par Chambolle et Lions dans [50]. On renvoie au travail de Cécile

Louchet [103] pour une interprétation de ce modèle en termes statistiques. On peut voir sur la figure 1.2 le résultat obtenu à l'aide de cette méthode.

FIG. 1.2: Débruitage par la fonctionnelle de ROF.

Un autre problème très classique entrant dans cette catégorie est celui de la segmentation par contours actifs géodésiques. Le problème est le suivant. Partant d'une image f, on cherche à trouver les bords de l'un des objets contenus dans l'image. On suppose de plus qu'une partie S de l'objet est connue ainsi qu'une partie T en dehors de celui-ci. L'idée des contours actifs géodésiques introduits par Caselles et al. dans [41] est que les bords de l'objet doivent passer par les zones où l'image f a un fort gradient. On associe à l'image f, une fonction g positive qui est petite là où le gradient de l'image est grand et inversement. On cherche alors une courbe fermée qui passe par les zones où la valeur de g est faible. On veut donc trouver E un ensemble qui minimise :

$$\min_{\substack{S\in E\\ T\in E^c}} \int_{\partial^* E} g(s) d\mathcal{H}^{m-1}. \tag{1.2}$$

On utilise généralement pour g des fonctions de la forme :

$$g = \frac{1}{1+|\nabla(G_\sigma * f)|} + \varepsilon.$$

Le terme $G_\sigma * f$, qui est la convolution de l'image par une gaussienne de variance σ permet de traiter des images dégradées par du bruit. Le terme $\varepsilon > 0$ pénalise la longueur de la courbe ce qui permet d'obtenir des courbes régulières. La figure 1.3 montre ce que l'on trouve pour la segmentation d'une levure. Le carré blanc représente l'ensemble S et les bords de l'image correspondent à l'ensemble T. Si l'on pose

$$BV_D(\Omega) := \{u \in BV(\Omega) \,/\, u = 1 \text{ sur } S,\, u = 0 \text{ sur } T \text{ et } 0 \le u \le 1\},$$

on peut voir à l'aide de la formule de la coaire, que les minimiseurs de (1.2) sont les lignes de niveaux des minimiseurs de

$$\min_{u \in BV_D(\Omega)} \int_\Omega g(x)|Du|. \tag{1.3}$$

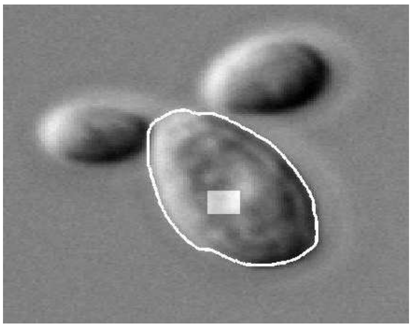

FIG. 1.3: Segmentation d'une levure.

La non-différentiabilité de la fonctionnelle apparaissant dans (1.1), rend difficile la mise en oeuvre d'algorithmes efficaces pour résoudre numériquement la minimisation. Le nombre important de publications récentes à ce sujet montre que ce problème est loin d'être considéré comme résolu.

Depuis le premier schéma numérique réellement efficace proposé par Chambolle dans [44], quasiment tous les algorithmes ont fait intervenir d'une façon ou d'une autre la dualité. On pourra trouver dans [45] un guide (déjà en partie dépassé !) pour se repérer dans cette jungle. Le Chapitre 2 de cette thèse contient une étude détaillée de l'une de ces méthodes dite Primale-Duale continue introduite par Appleton et Talbot dans [17] pour minimiser (1.1). Ce type de méthodes est aujourd'hui considéré comme l'un des plus performant.

Les deux auteurs de [17], étaient à l'origine motivés par la résolution de (1.2) en se basant sur une analogie avec les méthodes dites de "max flow/ min cut". Ce type de méthodes a été utilisé en segmentation d'images depuis que Boykov et Kolmogorov ont remarqué dans [31] que la discrétisation de (1.2) se réduit à la question classique de recherche de coupures minimales dans les graphes.

1.4.1 Coupures minimales et flots maximaux sur les graphes

Soit G un graphe orienté de sommets V et d'arêtes E. A chaque arête $e \in E$ on associe une capacité $C(e) \geq 0$. Pour une partition (V_1, V_2) donnée de V, on définit la coupure Γ comme l'ensemble des arêtes de E ayant un sommet dans V_1 et l'autre dans V_2. Le coût de Γ est alors égal à la somme des capacités des arêtes de Γ. Soient s et t deux sommets fixés de V. On cherche alors parmi les coupures qui séparent s et t i.e. telles que $s \in V_1$ et $t \in V_2$, celle qui minimise le coût. Pour le résumer en une formule, on cherche

$$\min_{\Gamma} \sum_{e \in \Gamma} C(e).$$

La recherche de coupures minimales se fait à l'aide du théorème "de max flow/ min cut".
Un flot ξ depuis une source s vers un puit t est une fonction de E dans $\mathbb{R}^+$ ayant les propriétés suivantes :

- Loi de Kirchhoff ou conservation du flot : pour tout sommet autre que la source ou le puit, le flot entrant est égal au flot sortant. Si on note e_v l'ensemble des arêtes ayant v comme sommet, on a donc pour v différent de s ou t, $\sum_{e \in e_v} \pm\xi(e) = 0$ où le signe est plus si l'arête est entrante en v et moins si elle est sortante.
- Le flot dans une arête ne peut dépasser sa capacité. Autrement dit, pour tout $e \in E$, $\xi(e) \leq C(e)$.

La valeur du flot est définie comme la somme des flots sortants de s. Le théorème du "max flow/ min cut" établit une correspondance entre les flots ayant une valeur maximale et les coupures minimales. Les arêtes saturées par un flot maximal i.e. celles où $\xi(e) = C(e)$, forment une coupure minimale.

Une méthode rapide pour calculer un flot maximal est la méthode du "prefow-push". Le principe est l'abandon de la contrainte de conservation du flot (on parle alors de préflot) et l'introduction d'une variable auxiliaire.

Pour la démonstration du théorème, la description du "preflow-push" ainsi que pour plus de précisions sur l'optimisation sur les graphes, on pourra par exemple consulter le livre de Ahuja, Magnanti et Orlin [1].

Les méthodes d'optimisation sur les graphes aboutissent en un nombre fini d'opérations à une minimisation exacte et globale. C'est l'avantage sur les méthodes d'optimisation continue. Le prix à payer est l'apparition de biais dûs la discrétisation.

1.4.2 L'approche d'Appleton et Talbot

Pour éviter ce biais, Appleton et Talbot ont étudié dans leur article [17] le problème continu directement. Leur idée est de pousser l'analogie avec le discret. Ils utilisent pour cela une version continue du théorème de "max flow / min cut" démontrée par Strang dans [127]. Celui-ci démontre en effet que si l'on pose,

$$X_N := \{\xi \,/\, |\xi| \leq g,\, \xi \cdot \nu = 0 \text{ sur } \partial\Omega \backslash (S \cup T) \text{ et } \operatorname{div} \xi = 0\},$$

alors

$$\min_{\substack{S \subset E \\ T \subset E^c}} \int_{\partial^* E} g(x) d\mathcal{H}^{m-1} = \sup_{\xi \in X_N} \int_S \xi \cdot \nu d\mathcal{H}^{m-1}.$$

On rappelle qu'ici ν désigne la normale sortante à l'ensemble E. L'idée d'Appleton et Talbot est alors de trouver un flot maximal en utilisant un analogue continu de la méthode de "preflow-push". Ils introduisent alors une variable auxiliaire u et proposent de résoudre le système :

$$\begin{cases} \partial_t u = \operatorname{div}(\xi) \\ \\ \partial_t \xi = Du \qquad\qquad |\xi| \leq g. \end{cases}$$

La solution de (1.2) est alors donnée par l'une des lignes de niveau de $\bar{u} = \lim_{t \to \infty} u(t)$.

On peut toutefois donner une interprétation alternative de cette approche qui explique le terme de méthode Primale-Duale.

Afin de simplifier la présentation, nous décrivons uniquement le cas homogène $g \equiv 1$. Pour le cas général, il suffit d'utiliser les formules de dualité contenues par exemple dans [28] ou [29]. En écrivant la définition de la variation totale, on voit que le problème (1.3) peut se réécrire comme un problème de recherche de point selle :

$$\begin{aligned} \min_{u \in BV_D(\Omega)} \int_\Omega |Du| &= \min_{u \in BV_D(\Omega)} \sup_{\substack{\xi \in \mathcal{C}_c^\infty(\Omega) \\ |\xi|_\infty \leq 1}} - \int_\Omega u \operatorname{div} \xi \, dx \\ &= \min_{u \in BV_D(\Omega)} \sup_{\xi \in \mathcal{C}_c^\infty(\Omega)} - \int_\Omega u \operatorname{div} \xi \, dx - I_{B(0,1)}(\xi), \end{aligned}$$

où $I_{B(0,1)}$ est la fonction indicatrice de la boule unité de L^∞ (qui vaut 0 si $|\xi|_\infty \leq 1$ et plus l'infini sinon). On applique alors une méthode d'Arrow-Hurwicz [18] pour trouver ce point selle, à savoir, une descente de gradient pour la variable primale u et une montée de gradient dans la variable duale ξ (d'où le terme Primale-Duale). Le système à résoudre est alors,

$$\begin{cases} \partial_t u = \operatorname{div}(\xi) \\ \partial_t \xi = Du - \partial I_{B(0,1)}(\xi). \end{cases}$$

Ceci montre que, du moins formellement, la solution $\bar{u}$ trouvée par Appleton et Talbot, n'est rien d'autre que la solution du problème (1.3). Cette approche Primale-Duale s'étend de façon immédiate au problème plus général (1.1). Notons également que cette méthode est l'analogue continu de l'algorithme proposé par Chan et Zhu [135] pour minimiser la variation totale.

1.4.3 Contributions de la thèse

Dans le Chapitre 2, nous donnons un sens rigoureux à ce système d'EDP hyperboliques à l'aide de la théorie des opérateurs maximaux monotones [33]. Le principal résultat que nous obtenons est l'existence et l'uncité d'une solution au problème de Cauchy :

Théorème 1.4.1. *Pour tout* (u_0, ξ_0)*, il existe un unique couple* $(u(t), \xi(t))$ *tel que*

$$\begin{cases} \partial_t u \in \operatorname{div}(\xi) - \partial G(u) \\ \partial_t \xi \in Du - \partial I_{B(0,1)}(\xi) \\ (u(0), \xi(0)) = (u_0, \xi_0). \end{cases} \tag{1.4}$$

De plus, l'énergie $|\frac{d^+u}{dt}|_2^2 + |\frac{d^+\xi}{dt}|_2^2$ *est décroissante.*

La question de la convergence vers un point selle est en général assez délicate. Néanmoins, dans le cas du débruitage par ROF nous avons réussi à l'obtenir. De plus, nous avons prouvé des estimations *a posteriori* inconnues jusque-là.

Proposition 1.4.2. *Soit* $\bar{u}$ *l'unique minimiseur de ROF alors toute solution de* (1.4) *converge en norme* L^2 *vers* $\bar{u}$*. On a également l'estimation,*

$$|u-\bar{u}|_2 \leq \frac{1}{2}\left(\frac{1}{\lambda}|\partial_t u|_2 + \sqrt{\frac{|\partial_t u|_2^2}{\lambda^2} + \frac{8|\Omega|^{\frac{1}{2}}}{\lambda}|\partial_t \xi|_2}\right).$$

Il est intéressant de noter que cette estimation *a posteriori* s'étend à l'algorithme de Chan et Zhu. Cette estimation est nouvelle même dans ce contexte.

En ce qui concerne le cas général, bien que la convergence vers un point selle de la solution semble difficile à démontrer, nous avons été en mesure d'obtenir des estimations *a posteriori*.

Proposition 1.4.3. *Pour tout point selle* $(\bar{u}, \bar{\xi})$ *et tout* (u_0, ξ_0)*, la solution* $(u(t), \xi(t))$ *de* (1.4) *vérifie,*

$$|J(u) - J(\bar{u})| \leq \left(\sqrt{|u_0-\bar{u}|_2^2 + |\xi_0-\bar{\xi}|_2^2}\right)|\partial_t u|_2 + 2|\Omega|^{\frac{1}{2}}|\partial_t \xi|_2.$$

Ces différentes estimations *a posteriori* permettent de donner des critères d'arrêt fiables pour les schémas numériques.

1.5 Surfaces de courbure moyenne prescrite

Dans le Chapitre 3 de cette thèse, nous nous intéressons à un problème de nature plus géométrique. La question est la suivante. Soit g une fonction de $\mathbb{R}^m$ dans $\mathbb{R}$ donnée, on veut savoir s'il existe des hypersurfaces de $\mathbb{R}^m$ (des sous-variétés de dimension $m-1$ de $\mathbb{R}^m$) ayant en chaque point une courbure moyenne égale à g. Cette question a été posée plus ou moins en ces termes par S.T. Yau dans [134]. Ce problème généralise l'étude des surfaces minimales et des surfaces de courbure de moyenne constante.

1.5.1 Surfaces minimales et problème de Plateau

Le problème de Plateau consiste à trouver parmi les surfaces ayant un bord prescrit, celle qui a une aire minimale. Ce problème a été tout d'abord étudié expérimentalement par le physicien belge Joseph Plateau au XIXème siècle. On peut voir sur la figure 1.4 un exemple d'une telle surface. Le très beau livre d'Hildebrandt et Tromba [93] contient de très nombreux autres exemples.

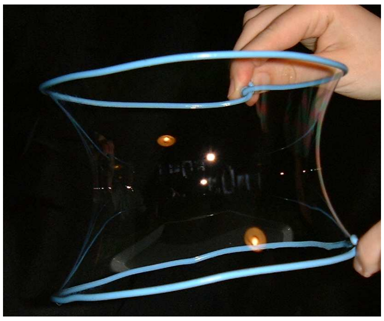

FIG. 1.4: Une surface minimale

On peut démontrer qu'une solution du problème de Plateau a une courbure moyenne nulle. On appelle alors surface minimale toute surface ayant une courbure moyenne nulle. Les hyperplans constituent l'exemple le plus simple de telles surfaces. Les premiers résultats d'existence pour les surfaces minimales ont été obtenus par Douglas et Radò en 1931 dans la classe des surfaces paramétriques. L'étude de ces surfaces minimales est depuis devenu un champ d'investigations mathématiques très vaste (voir [119] par exemple).

Cette première approche souffre toutefois de nombreuses limitations. Elle possède par exemple, un caractère fortement non intrinsèque. De plus, elle demande d'imposer *a priori* le genre de la solution recherchée et une grande régularité sur les objets considérés. Enfin, et c'est peut être l'une des limitations les plus sérieuses, les surfaces minimales ne sont que des points critiques de l'aire et non des minima. Pour pallier à toutes ces restrictions, De Giorgi, Federer, Fleming et Almgren ont été amenés à introduire dans les années 60, de nouvelles notions de surfaces telles que les ensembles de périmètre fini, les courants rectifiables et les varifolds. Dans cette thèse nous nous limiterons à l'étude des ensembles de périmètre fini et nous renvoyons aux livres introductifs [112] et [3] ainsi qu'au livre monumental de Federer [73] au sujet des deux autres notions faibles de surfaces.

La question importante qui se pose une fois que l'on a introduit ces nouveaux concepts de surfaces est de savoir s'il est possible de démontrer que les solutions faibles du problème de Plateau sont en fait des surfaces régulières et donc des solutions classiques. Cette question a été résolue dans les travaux de De Giorgi, Bombieri, Giusti, Simons et Federer. Le théorème central de cette théorie est le suivant :

Théorème 1.5.1. *Soit E un ensemble de périmètre fini minimisant localement le périmètre alors :*
- *l'ensemble $\partial^* E$ est localement analytique,*
- *l'ensemble singulier $\Sigma = \partial E \backslash \partial^* E$ est un fermé de ∂E,*
- *si la dimension de l'espace est inférieure à 7 alors $\Sigma = \emptyset$,*
- *si $m \geq 8$ alors la dimension de Hausdorff de Σ est inférieure ou égale à $m - 8$.*

De plus, l'exemple du cône de Simons montre que le théorème est optimal. Pour une démonstration de ce théorème nous conseillons de consulter l'excellent livre de Giusti [83] ou celui plus récent de Maggi [104]. Cette théorie de la régularité a ensuite été étendue à des surfaces minimisants des fonctionnelles plus générales ainsi que pour des surfaces quasi-minimisantes. On pourra consulter à ce sujet [4], [66].

1.5.2 Surfaces de courbure moyenne constante et problème isopérimétrique

Le problème isopérimétrique ressemble beaucoup au problème de Plateau. Il s'agit de trouver parmi les surfaces enfermant un volume fixe, celle qui possède l'aire la plus petite.
Il est "connu" depuis des siècles que les solutions de ce problème doivent être les sphères (selon la légende, la reine Didon, fondatrice de Carthage le savait déjà !). Toutefois, pendant très longtemps il n'existait pas de preuve de ce fait en dehors de la classe des ensembles lisses. L'une des premières grandes réussites de De Giorgi a été de démontrer ce théorème dans la classe très vaste des ensembles de périmètre fini dans [61] (voir également [62] pour une traduction en anglais). La démonstration se base sur l'utilisation de la symétrisation de Steiner. On verra au Chapitre 4 comment l'utilisation d'une autre symétrisation très semblable permet d'obtenir le même type de résultats dans un contexte gaussien.

Si l'on considère des ensembles qui sont minimiseurs locaux du périmètre sous contrainte de volume, on peut démontrer que ceux-ci possèdent la même régularité que les ensembles minimisants le périmètre sans contrainte (à savoir que ce sont des ensembles analytiques en dehors d'un ensemble singulier de dimension $m-8$). On pourra consulter à ce sujet les articles récents [121] et [133] ou les travaux originaux de Gonzalez, Massari et Tamanini [90]. On voit alors que ces ensembles ont une courbure moyenne constante. Cependant, un théorème d'Alexandrov [2] assure que les seules hypersurfaces compactes plongées de $\mathbb{R}^m$ ayant une courbure moyenne constantes sont les sphères.

1.5.3 Le problème général

Il n'existe pas de réponse générale à la question posée par S.T. Yau concernant l'existence de surfaces ayant une courbure moyenne prescrite. Toutefois, des résultats existent sous différentes hypothèses sur la fonction g donnée.

Une première famille de résultats a été obtenue par Bakelman, Kantor, Treibergs, Wei et Huang dans [20], [130] et [94]. Ceux-ci utilisent une approche paramétrique. Partant d'une surface de référence et sous l'hypothèse que g ressemble à la courbure moyenne de cette surface, ils construisent une solution au problème paramétré par la surface de référence. L'un des résultats représentatifs de cette approche est le suivant

Théorème 1.5.2. [130] *Soit* $U := \{r_1 < |x| < r_2\}$ *avec* $0 < r_1 \leq 1 \leq r_2$ *et* $0 < g \in \mathcal{C}^1(\bar{U})$ *tel que*
- $\dfrac{\partial}{\partial \rho}\rho g(\rho x) \leq 0$ *pour tout* $\rho x \in U$,
- $g(x) > \frac{1}{|x|}$ *pour* $|x| = r_1$ *et* $g(x) < \frac{1}{|x|}$ *pour* $|x| = r_2$,

alors il existe $\alpha \in (0,1)$ *tel qu'il existe une surface plongée, paramétrée par la sphère unité et ayant comme courbure moyenne* g.

Les deux autres approches pour ce problème sont de nature variationnelle. Elles partent de l'observation que, comme pour les surfaces minimales et les surfaces de courbure moyenne constante, les surfaces de courbure moyenne prescrite peuvent être vues comme des ensembles stationnaires pour l'énergie

$$F(E) := P(E) - \int_E g\,dx. \tag{1.5}$$

Sans hypothèses très fortes sur g, il n'est cependant pas possible de garantir l'existence de minimiseurs pour F.

La première de ces deux approches est une approche paramétrique restreinte aux dimensions 2 et 3. Pour simplifier sa présentation, nous nous restreindrons ici au cas de la dimension 2. On identifie alors $\mathbb{R}^2$ et le plan complexe $\mathbb{C}$. L'idée est de considérer pour chaque courbe fermée, une paramétrisation $u : [0,1] \to \mathbb{C}$ de celle-ci et d'observer que la courbe a pour courbure g si et seulement si g vérifie l'équation différentielle :

$$\ddot{u} = iL(u)g(u)\dot{u}, \tag{1.6}$$

où

$$L(u) := \left(\int_0^1 |\dot{u}|^2\right)^{\frac{1}{2}}.$$

Soit $Q : \mathbb{R}^2 \to \mathbb{R}^2$ un champ de vecteurs tel que $\operatorname{div} Q = g$. On pose alors

$$S(u) := \int_0^1 iQ(u) \cdot \dot{u}.$$

L'équation (1.6) est l'équation d'Euler-Lagrange de la fonctionnelle

$$F_g(u) := L(u) - S(u).$$

Remarquons que la fonctionnelle F_g ressemble beaucoup à celle de (1.5). L'existence d'une solution à (1.6) s'obtient alors en utilisant le Lemme du Col (voir par exemple [71] à ce sujet).

En dimension 3, la paramétrisation par longueur d'arc est remplacée par l'utilisation de paramétrisations conformes. Une difficulté née alors de l'invariance du problème par transformation conforme. Cette méthode ne peut cependant pas s'étendre en dimension supérieure car il n'existe pas d'équivalent pour la notion de paramétrisation conforme en grande dimension. Remarquons également que cette approche fournit l'existence de courbes (ou de surfaces de $\mathbb{R}^3$) paramétrées, ce qui n'exclut pas les auto-intersections. Nous renvoyons le lecteur intéréssé aux articles de Bethuel, Caldiroli, Guida, Rolando et Musina ainsi qu'à la thèse de Kirsch, [24, 92, 115, 98].

La deuxième approche, qui est celle que nous suivrons dans la thèse, consiste à minimiser la fonctionnelle F de (1.5) sous certaines contraintes. Il s'agit alors de montrer que ces contraintes n'influencent pas trop l'équation d'Euler-Lagrange vérifiée par les solutions. L'article fondateur dans cette direction est sans conteste celui de Caffarelli et De La Llave [37]. Dans ce travail, les auteurs considèrent une fonction g périodique de moyenne nulle sur le carré unité $Q := [0,1)^m$, vérifiant pour un certain $0 < \Lambda < 1$, la condition

$$\int_E g\,dx \leq (1-\Lambda)P(E,Q) \qquad \forall E \subset Q. \tag{1.7}$$

Cette condition (1.7) est par exemple vérifiée si la norme L^m de g est assez petite. Dans l'article [37] c'est en fait cette condition plus forte qui est requise. La condition (1.7) apparaît elle seulement dans [52]. Sous ces conditions, les auteurs démontrent l'existence de minimiseurs plane-like de F. Un tel minimiseur est un ensemble qui est localement de périmètre fini, dont le périmètre augmente lorsque l'on effectue des perturbations compactes et dont la frontière est comprise entre deux plans parallèles. Plus précisément on a le théorème,

Théorème 1.5.3. *Soit $g \in L^m(Q)$ périodique de moyenne nulle et vérifiant la condition* (1.7) *alors il existe $M > 0$ tel que pour toute direction $\omega \in \mathbb{S}^{m-1}$, il existe E minimiseur local de F tel que*

$$\{x \cdot \omega \geq -M\} \subset E \subset \{x \cdot \omega \leq M\}.$$

Par la théorie de la régularité pour les surfaces minimales, si $m \leq 7$ et si $g \in \mathcal{C}^{0,\alpha}(Q)$ alors ces plane-like minimiseurs sont réguliers et ont donc une courbure moyenne égale à g.

L'idée de la démonstration de ce théorème est de regarder les minimiseurs de F avec une contrainte d'inclusion entre deux plans parallèles. On augmente alors la distance entre ces deux plans jusqu'à ce qu'il n'y ait plus de contact entre les minimiseurs et les plans. On obtient ainsi des minimiseurs libres du problème.

On peut se demander s'il existe dans ce contexte des solutions compactes. En général ce n'est pas le cas comme le démontre un résultat de Barles, Cesaroni et Novaga [22].

Proposition 1.5.4. *Soit g une fonction lipschitz vérifiant les hypothèses précédentes et telle que $|g|_{lip} \leq \delta$ pour un certain δ assez petit. Si g ne dépend pas de la dernière variable alors il n'existe pas de surface compacte de courbure moyenne g plongée dans $\mathbb{R}^m$.*

Novaga et Valdinoci ont cependant démontré dans [117] qu'il était possible de trouver de telles solutions compactes de façon générique au sens L^1.

Théorème 1.5.5. *Soit g une fonction périodique $\mathcal{C}^2$ de moyenne nulle et de norme L^∞ assez petite alors pour tout $\varepsilon > 0$ il existe $g_\varepsilon \in \mathcal{C}^\infty(Q)$ périodique de moyenne nulle telle que*

- $|g_\varepsilon|_{L^\infty} \leq |g|_{L^\infty}$,
- $|g_\varepsilon - g|_{L^1} \leq \varepsilon$,
- *il existe un ensemble compact ayant pour courbure moyenne g_ε.*

L'idée de la démonstration est à nouveau de minimiser F sous contraintes d'inclusions entre une sous-solution et une sur-solution du problème.

Ce type de résultats d'existence de surfaces de courbure moyenne prescrite est fortement lié à des questions d'homogénisation de fronts. En effet, ces ensembles servent de barrières pour des évolutions de type mouvement par courbure moyenne avec forçage. On pourra par exemple consulter [22, 118] à ce sujet.

Notons enfin qu'il existe un analogue de ces questions pour l'équation d'Allen-Cahn. En effet, comme le montre le résultat de Γ-convergence de Modica-Mortola [110] déjà cité, il existe un lien très fort entre cette équation et les surfaces minimisantes. L'existence de solutions de type plane-like ou bump a été étudiée par exemple dans [116, 117].

1.5.4 Contributions de la thèse

L'un des défauts du résultat de Novaga et Valdinoci réside dans le caractère L^1 de l'approximation. En effet, les hypersurfaces sont des ensembles de mesure de Lebesgue nulle et il est donc possible de changer de manière drastique une fonction sur un petit voisinage d'une telle hypersurface sans s'éloigner beaucoup en norme L^1. La question qui se pose, et qui est déjà posée dans [117], est de savoir s'il est possible d'obtenir un résultat analogue au théorème 1.5.5 mais en renforçant l'approximation. Dans le chapitre 3 nous démontrons le théorème suivant

Théorème 1.5.6. *Soit $g \in \mathcal{C}^{0,\alpha}(Q)$ une fonction périodique de moyenne nulle vérifiant* (1.7)*. Si $m \leq 7$, alors pour tout $\varepsilon > 0$ il existe $\varepsilon' \in [0, \varepsilon]$ tel qu'il existe une hypersurface compacte vérifiant*

$$\kappa = g + \varepsilon'.$$

Dans l'énoncé du théorème et dans la suite de la thèse, la lettre κ désigne la courbure moyenne. L'idée est de considérer le problème sous contrainte de volume :

$$f(v) := \min_{|E|=v} P(E) - \int_E g. \tag{1.8}$$

La première étape consiste à montrer que le minimium est bien atteint pour chaque v par un ensemble compact. Pour cela, on restreint le problème spatialement aux ensembles inclus dans une grande boule dont on fait tendre le rayon vers l'infini (ce qui est très similaire au raisonnement de [37]).

Notons qu'au cours de la démonstration de l'existence de minimiseurs compacts, on démontre une inégalité généralisant l'inégalité d'Alexandrov-Fenchel (voir [126]) aux ensembles non convexes.

Lemme 1.5.7. *Soit $E \subset \mathbb{R}^m$ un ensemble compact de frontière $\mathcal{C}^2$ alors*

$$\frac{m-1}{m} P(E)^2 \geq |E| \int_{\partial E} \kappa \, d\mathcal{H}^{m-1}.$$

Soit donc E un minimiseur compact de F sous contrainte de volume. Il vérifie alors l'équation d'Euler-Lagrange

$$\kappa = g + \lambda$$

où λ est une constante. On démontre ensuite que la fonction isovolumétrique f est lipschitz et qu'en tout point de différentiabilité,

$$f'(v) = \lambda.$$

Pour prouver le théorème 1.5.6, il suffit donc de montrer qu'on peut toujours trouver un volume v tel que $f'(v)$ est aussi petit que l'on veut. Cette dernière propriété découle du fait que grâce à l'hypothèse (1.7),

$$f(v) \simeq v^{\frac{m-1}{m}}.$$

On voit donc que plus ε sera petit et plus les surfaces construites ainsi auront tendance à contenir un grand volume.

On peut alors se demander quel est le comportement asymptotique des ensembles que nous avons construit lorsque le volume tend vers l'infini. Soit $\phi_g : \mathbb{R}^m \to [0, +\infty)$, la fonction définie par

$$\phi_g(p) := \min_{u \in BV(Q)} \int_Q |Du + p| + \int_Q p \cdot x - g(x) \, dx.$$

On pose

$$W_g := \left\{ x \in \mathbb{R}^m : \max_{\phi_g(y) \leq 1} x \cdot y \leq 1 \right\}.$$

W_g est la forme de Wulff associée à ϕ_g. D'après, [132, 129], W_g est l'unique minimiseur à translation et homothétie près du périmètre anisotrope

$$\int_{\partial^* E} \phi_g(\nu) d\mathcal{H}^{m-1}$$

sous contrainte de volume. On peut démontrer que :

Théorème 1.5.8. *Soit $m \leq 7$. Pour $v > 0$, soit E_v un minimiseur compact avec contrainte de volume de* (1.5). *Il existe alors des points $z_v \in \mathbb{R}^m$ tels qu'en définissant*

$$\widetilde{E}_v := \left(\frac{|W_g|}{v}\right)^{\frac{1}{m}} E_v + z_v,$$

on ait

$$\lim_{v \to +\infty} \left|\widetilde{E}_v \Delta W_g\right| = 0.$$

La preuve de ce résutat est basée sur un théorème de Γ-convergence contenu dans [52]. La difficulté à surmonter est le manque de compacité de la suite $\widetilde{E}_v$. Ceci est réalisé par un raisonnement de type concentration-compacité (voir [102] à ce sujet).

1.6 Fonctions à variation bornée dans les espaces de Wiener

Depuis les premiers articles de Fukushima et Hino [79, 80], de nombreux travaux ont été menés pour étendre la théorie des fonctions à variation bornée aux espaces de Wiener. Ces espaces, très utilisés en probabilités, sont des espaces de Banach équipés d'une mesure gaussienne. Depuis leur introduction par Gross, ils ont été l'objet d'innombrables travaux. Ils sont en effet le cadre adapté pour le calcul des variations stochastique ou calcul de Malliavin. On pourra voir à ce sujet [106] ou [59] ainsi que [27].

L'extension à ces espaces de Wiener des fonctions à variation bornée et des ensembles de périmètre fini se justifie principalement par l'étude des phénomènes de concentration dans les équations de diffusion stochastiques. Celle-ci est intimement liée avec les propriétés isopérimétriques qui, comme nous l'avons déjà dit, s'expriment naturellement dans le langage des ensembles de périmètre fini. Il existe toutefois bien d'autres applications potentielles de cette théorie telles que l'analyse des semi-groupes en dimension infinie ou celle des équations différentielles définies par un champ de vecteurs BV.

La théorie des fonctions à variation bornée a été étendue à des espaces métriques très généraux (voir [14]). Cependant, ce qui permet d'établir une "bonne" théorie dans ces espaces métriques est l'hypothèse que la mesure équipant l'espace est doublante. Ceci veut dire que l'on peut contrôler la mesure des boules de taille $2r$ par celle des boules de taille r. L'une des spécificités des espaces de Wiener est justement qu'ils sont munis d'une mesure qui n'est pas doublante. L'une des difficultés dans ce contexte non localement compact, est la non validité des théorèmes de dérivation de Besicovitch et de dualité de Riesz qui sont à la base de nombreux théorèmes sur les fonctions à variation bornée dans le cadre euclidien.

Dans celui-ci, on sait depuis longtemps qu'il est équivalent de définir les fonctions à variation bornée de diverses manières. On peut les voir soit comme les fonctions ayant une dérivée au sens des distributions qui est une mesure de Radon bornée, soit comme des fonctions dont la variation est finie. Celle-ci peut être également définie de multiples façons, soit par dualité, en tant que suprémum, soit en tant que relaxée de la variation définie pour les fonctions lisses, soit encore à l'aide du semi-groupe de la chaleur. L'une des premières pierres dans l'édification de la théorie des fonctions BV dans le cadre gaussien a été de montrer que ces différentes définitions y coïncidaient également [12]. Le semi-groupe jouant le rôle du semi-groupe de la chaleur étant ici le semi-groupe d'Ornstein-Uhlenbeck défini sur l'espace de Wiener X par la formule de Mahler,

$$T_t u(x) := \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)\, d\gamma(y).$$

De nombreux travaux ont également été menés pour trouver une bonne notion de frontière réduite pour les ensembles de périmètre fini ainsi qu'une preuve de la rectifiabilité de celle-ci. Feyel et De La Pradelle [74] ont les premiers réussi à définir une notion de frontière réduite dans ce contexte ainsi qu'une définition de mesure de Hausdorff de codimension 1. Ils ont démontré que pour un ensemble de périmètre fini E, la mesure $D_\gamma \chi_E$ était concentrée sur cette frontière réduite et coïncidait alors avec la mesure de Hausdorff restreinte à celle-ci. Ambrosio, Miranda et Pallara [13] ont ensuite prouvé la rectifiabilité au sens de Sobolev de cette frontière réduite. Afin d'éliminer en partie le caractère non intrinsèque de la définition de frontière réduite de [74], Ambrosio et Figalli [9] ont donné une définition de $E^{\frac{1}{2}}$ utilisant le semi-groupe de Ornstein-Uhlenbeck. De nombreuses questions restent toutefois ouvertes comme la rectifiabilité lipschitz de la frontière réduite ou encore un analogue du théorème de blow-up de De Giorgi.

Avant d'aller plus avant, rappellons quelques définitions concernant les espaces de Wiener.

Définition 1.6.1. *Un espace de Wiener est un espace de Banach X muni* *d'une mesure gaussienne non dégénérée γ. Ceci veut dire que pour tout $x^* \in X^*$, la mesure image de γ par x^* est une mesure gaussienne sur $\mathbb{R}$.*

On peut montrer que les fonctions $x \to \langle x^*, x\rangle$ sont dans $L^2_\gamma(X)$. On considère alors l'espace $\mathcal{H} \subset L^2_\gamma(X)$ défini comme étant la fermeture dans $L^2_\gamma(X)$ de celles-ci. L'espace de Cameron-Martin H défini comme l'ensemble

$$H := \left\{ \int_X x\hat{h}(x)\; d\gamma(x) \;/\; \hat{h} \in \mathcal{H} \right\}$$

joue un rôle fondamental dans l'étude des espaces de Wiener. C'est en effet un espace de Hilbert inclus dans X tel que la mesure γ est invariante par les rotations de H. On peut voir que c'est également l'espace des directions dans lesquelles il faut calculer les dérivées et on notera alors ∇_H le gradient par rapport aux directions de H.

1.7 Approximation et relaxation du périmètre dans les espaces de Wiener

1.7.1 Isopérimétrie et symétrisation de Ehrhard

Ainsi que nous l'avons déjà indiqué plus haut, l'une des justifications principales de l'introduction des fonctions à variation bornée dans le contexte gaussien vient de l'étude des inégalités isopérimétriques. Il a été démontré de façon indépendante par Borell, Sudakov et Tsirel'son que parmi les ensembles contenant un volume fixé, les demi espaces sont ceux qui ont un périmètre minimal. Autrement dit, si l'on note

$$\Phi(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}}\; dx,$$

et

$$\mathcal{U}(v) := \Phi'(\Phi^{-1}(v))$$

alors $\mathcal{U}(v)$ est le périmètre du demi-espace de volume v et donc pour tout ensemble E, si l'on note P_γ le périmètre dans ce cadre gaussien, on a

$$P_\gamma(E) \geq \mathcal{U}(\gamma(E)).$$

La première démonstration de cette inégalité était basée sur un passage à la limite dans une inégalité isopérimétrique sur les sphères. Plus tard, Ehrhard [67] a utilisé une technique de symétrisation pour redémontrer ce résultat. Son idée est de partir d'un ensemble quelconque, de choisir une direction et de remplacer toutes les sections orthogonales de cet ensemble par des demi-espaces de même volume (voir figure 1.5). Il montre alors que cette symétrisation fait décroître le périmètre tout en conservant le volume. Pour une démonstration usant le langage de la théorie de la mesure et des ensembles de périmètre fini, on pourra consulter [56].

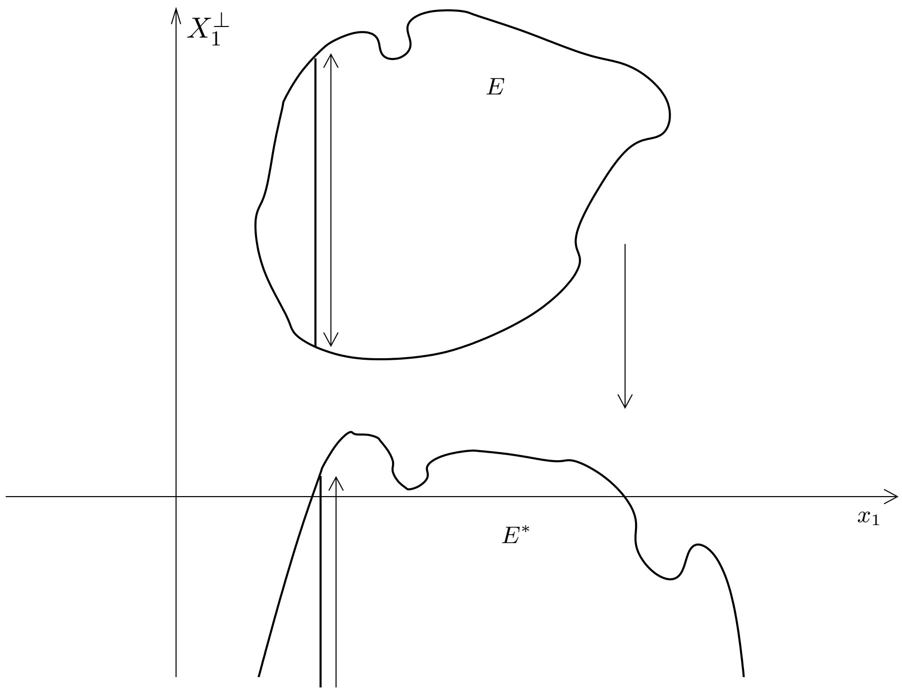

FIG. 1.5: La symétrisation de Ehrhard.

Une troisième démonstration de cette inégalité a été donnée par Bobkov [26]. Cette dernière démonstration se base sur une inégalité ponctuelle ainsi que sur l'utilisation d'un théorème central limite. Cette preuve a ensuite inspiré le travail pionnier de Bakry et Ledoux [21] où ce type d'inégalités isopérimétriques sont démontrées pour des opérateurs de diffusion généraux. L'idée est de démontrer en fait l'inégalité fonctionnelle suivante :

$$\mathcal{U}\left(\int_X u d\gamma\right) \leq \int_X \sqrt{\mathcal{U}(u)^2 + |\nabla_H u|_H^2} d\gamma \tag{1.9}$$

et de remarquer que si on met dans cette inégalité $u = T_t \chi_E$ alors le membre de gauche tend vers $\mathcal{U}(\gamma(E))$ quand t tend vers 0 tandis que le membre de droite converge vers $P_\gamma(E)$ ce qui donne l'inégalité isopérimétrique. Notons que l'article de Bakry et Ledoux a été le point de départ de nombreux travaux sur les inégalités fonctionnelles et leurs liens avec la courbure des espaces sous-jacents. On pourra consulter à ce sujet le livre de Villani [131].

1.7.2 Contributions de la thèse

Dans le Chapitre 4 de cette thèse, nous démontrons un résultat de type Modica-Mortola dans les espaces de Wiener. La démonstration de ce résultat permet de faire un lien entre la symétrisation d'Ehrhard et la démonstration de l'inégalité isopérimétrique par Bobkov.

Inspiré par le cas euclidien, on peut se demander s'il est vrai que la fonctionnelle

$$\int_X \frac{\varepsilon}{2}|\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon}\, d\gamma$$

Γ-converge vers un multiple du périmètre lorsque ε tend vers 0. Malheureusement, en dimension infinie, il n'y a pas de compacité forte des minimiseurs de cette fonctionnelle. Ceci est dû au fait que dans les espaces de Wiener, il n'est plus vrai que d'une suite bornée dans BV, on puisse extraire une sous-suite convergeant fortement dans L^1. On voit donc que la bonne topologie pour calculer la Γ-limite est la topologie faible de $L^2_\gamma(X)$. Le problème étant que dans cette topologie, le périmètre n'est pas semi-continu car la classe des ensembles de périmètre fini n'est pas un fermé de $L^2_\gamma(X)$ pour cette topologie. La Γ-limite ne peut donc pas être un multiple du périmètre ! Nous démontrons que cette Γ-limite est en fait un multiple de

$$\overline{F}(u) := \begin{cases} \displaystyle\int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2} d\gamma & \text{si } 0 \leq u \leq 1 \quad \gamma - a.e. \\ +\infty & \text{sinon.} \end{cases}$$

Cette fonction $\overline{F}$ n'est rien d'autre que le second membre de l'inégalité (1.9) prouvée par Bobkov. C'est également la fonctionnelle relaxée du périmètre. La preuve de ce théorème est basée principalement sur la symétrisation de Ehrhard car $\overline{F}(u)$ correspond au périmètre d'un ensemble symétrique ayant des sections de volume $u(x)$.

Nous montrons également que si l'on considère le même problème de Γ-convergence avec contrainte de volume, alors à moins d'une rotation, les minimiseurs de l'énergie d'Allen-Cahn convergent fortement vers un demi-espace.

Proposition 1.7.1. *Soit $s \in [0,1]$ et soit u_ε un minimiseur de*

$$\min_{\int_X u\,d\gamma = s} \int_X \left(\frac{\varepsilon}{2} |\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon} \right) d\gamma$$

alors $u_\varepsilon = v_\varepsilon(\hat{h}_\varepsilon(x))$ pour un certain $\hat{h}_\varepsilon \in \mathcal{H}$ avec $|h_\varepsilon|_H = 1$ et un certain v_ε minimiseur du problème unidimensionnel

$$\min_{\int_{\mathbb{R}} v d\gamma_1 = s} \int_{\mathbb{R}} \frac{\varepsilon}{2} v'^2 d\gamma + \int_{\mathbb{R}} \frac{W(v)}{\varepsilon} d\gamma_1.$$

en particulier, v_ε converge vers la fonction caractéristique d'une demi-droite.

Ceci montre que le manque de compacité des minimiseurs u_ε provient essentiellement de l'invariance du problème par rotation de l'espace. La démonstration de cette proposition utilise un analogue de l'inégalité de Pólya-Szegö dans le cas gaussien. On définit la symétrisée de Ehrhard d'une fonction u en transformant chaque sous-niveau de u en un demi-espace de même volume. Cette transformation est l'équivalent de la symétrisée de Schwarz dans le contexte gaussien. On a alors,

Proposition 1.7.2. *Soit $u \in H^1_\gamma(X)$, et soit u^* sa symétrisée de Ehrhard alors $u^* \in H^1_\gamma(X)$ et*

$$\int_X |\nabla_H u^*|_H^2 \, d\gamma_1 \leq \int_X |\nabla_H u|_H^2 \, d\gamma.$$

L'égalité a lieu si et seulement si

$$u = \tilde{u}\left(\hat{h}(x)\right) \qquad \text{pour un certain } \hat{h} \in \mathcal{H}\,,$$

où $\hat{h}$ peut être choisi unitaire.

Nous étudions par ailleurs les cas d'unicité lorsque l'on considère le problème de courbure moyenne prescrite.

Proposition 1.7.3. *Soit $g \in L^2_\gamma(X)$ alors les propositions suivantes sont équivalentes :*

– *la fonctionnelle*

$$F_g(E) := P_\gamma(E) + \int_E g d\gamma \tag{1.10}$$

possède un unique minimiseur dans la classe des ensembles de périmètre fini ;

– *la fonctionnelle*

$$\overline{F}_g(u) := \overline{F}(u) + \int_X ugd\gamma \tag{1.11}$$

a un unique minimiseur dans $BV_\gamma(X)$.

De plus, lorsque l'une des deux conditions est vérifiée, les deux minimiseurs coïncident. Enfin, si u_ε *est une suite de* $H^1_\gamma(X)$ *vérifiant*

$$\sup_\varepsilon \left(F_\varepsilon(u_\varepsilon) + \int_X u_\varepsilon g d\gamma \right) \leq C$$

pour un certain $C > 0$, *alors* u_ε *possède une sous-suite convergeant fortement vers* χ_E *dans* $L^2_\gamma(X)$, *où* E *est le minimiseur commun de* (1.10) *et* (1.11).

Nous démontrons également dans ce chapitre un analogue du théorème de Bernstein dans l'espace de Wiener :

Proposition 1.7.4. *Les demi-espaces sont les seuls minimiseurs locaux du périmètre avec contrainte de volume.*

1.8 Convexité des solutions de certains problèmes variationnels en dimension infinie

1.8.1 Le cas Euclidien

L'étude des propriétés qualitatives des solutions d'EDP ou de problèmes variationnels a occupé et occupe toujours de très nombreux mathématiciens. La convexité est l'une des caractéristiques géométriques qui ont fait l'objet de nombreuses recherches. Parmi les diverses méthodes existantes pour affronter cette question, il existe deux classes de méthodes très importantes. La première est basée sur des techniques de symétrisations telles que la symétrisation de Schwarz et de Steiner dont on a brièvement parlé dans la section précédente. La deuxième classe est constituée par les méthodes reposant sur un principe du maximum ou un principe de comparaison. Nous nous focaliserons ici sur le deuxième ensemble de méthodes. On pourra trouver dans le livre de Kawohl [97] un large panorama sur ces deux approches.

Dans le chapitre 5 de cette thèse, nous utiliserons deux méthodes différentes, mais apparentées, pour démontrer la convexité de solutions de certains problèmes variationnels. La première de ces méthodes est dûe à Korevaar

[99]. L'idée est de considérer une solution classique u d'une EDP elliptique sur un domaine Ω avec contact vertical au bord et de poser

$$C(t,x,y) := u(tx+(1-t)y) - tu(x) - (1-t)u(y).$$

La convexité de u est alors équivalente à

$$C(t,x,y) \leq 0, \qquad \forall (t,x,y) \in [0,1] \times \bar{\Omega} \times \bar{\Omega}.$$

On procède par l'absurde. Si u n'est pas convexe alors le maximum de C doit être strictement positif. La condition de contact vertical au bord, exclut que ce maximum puisse être atteint au bord du domaine. En utilisant ensuite l'équation satisfaite par u à l'intérieur du domaine, on arrive également à exclure que ce maximum puisse y être positif.
Illustrons la deuxième partie de l'argument sur un cas très simple. Soit u la solution de

$$-\Delta u + u = g \tag{1.12}$$

où g est une fonction convexe. Soit (t,x,y) un point où le maximum de C est atteint. On considére alors la fonction définie par $\phi(\tau) := C(t, x+\tau, y+\tau)$. La dérivée seconde de ϕ en 0 est négative donc

$$D^2u(tx+(1-t)y) - tD^2u(x) - (1-t)D^2u(y) \leq 0.$$

Ceci qui donne en prenant la trace et en utilisant l'équation (1.12),

$$0 < C(t,x,y) \leq g(tx+(1-t)y) - tg(x) - (1-t)g(y) \leq 0$$

d'où une contradiction.

La deuxième méthode, qui s'applique de façon beaucoup plus générale que celle de Korevaar, est dûe à Alvarez, Lasry et Lions [7]. Leur approche se place dans le cadre des solutions viscosité. Rappellons en la définition.

Définition 1.8.1. *Soit $F(x,r,p,X) : \Omega \times \mathbb{R} \times \mathbb{R}^m \times S(m) \to \mathbb{R}$ un opérateur elliptique. On a alors les définitions suivantes :*
- *une fonction u est sous-solution de viscosité de l'équation $F(x,u,\nabla u, D^2u) = 0$ si u est semi-continue supérieurement et si pour tout $\varphi \in \mathcal{C}^2(\Omega)$ et tout $x \in \Omega$, si $u - \varphi$ a un maximum local en x et $u(x) = \varphi(x)$ alors*

$$F(x, \varphi(x), \nabla\varphi(x), D^2\varphi(x) \leq 0,$$

– *une fonction u est sur-solution de viscosité l'équation $F(x, u, \nabla u, D^2 u) = 0$ si u est semi-continue inférieurement et si pour tout $\varphi \in \mathcal{C}^2(\Omega)$ et tout $x \in \Omega$, si $u - \varphi$ a un minimum local en x et $u(x) = \varphi(x)$ alors*

$$F(x, \varphi(x), \nabla\varphi(x), D^2\varphi(x) \geq 0.$$

Une fonction qui est à la fois sous-solution et sur-solution est appelée solution de viscosité de l'équation $F(x, u, \nabla u, D^2 u) = 0$.

Dans la définition précédente, $S(m)$ désigne l'ensemble des matrices symétriques d'ordre m. La théorie des solutions viscosité permet d'obtenir des résultats d'existence pour des équations elliptiques et paraboliques très générales. On pourra consulter [57] pour une bonne introduction à ce sujet.

L'idée de [7] est de considérer une équation elliptique au sens où F est une fonction continue tel que,

$$F(x, r, p, X) \leq F(x, r, p, Y) \qquad \text{si } X \geq Y$$

vérifiant de plus que la fonction

$$(x, r, X) \to F(x, r, p, X^{-1}) \tag{1.13}$$

est concave pour tout $(x, r, p, X) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m \times S(m)^{++}$, où $S(m)^{++}$ est l'ensemble des matrices symétriques définies positives d'ordre m. Ils montrent alors que si u est sur-solution de viscosité de l'équation alors son enveloppe convexe est également sur-solution. Si l'équation vérifie un principe de comparaison et si u est également sous-solution de l'équation (et donc en fait solution) u est en-dessous de son enveloppe convexe. Étant également audessus, u coïncide avec son enveloppe convexe et est donc elle-même convexe. Par cette méthode, Alvarez, Lasry et Lions démontrent le théorème suivant

Théorème 1.8.2. *Soit Ω un ensemble convexe borné et soit F un opérateur elliptique vérifiant la condition* (1.13) *et pour lequel le principe de comparaison pour les solutions de l'équation avec contact vertical au bord soit vérifié alors toute solution viscosité de l'équation est convexe.*

Nous renvoyons à nouveau à [57] en ce qui concerne les conditions assurant que le principe de comparaison soit valable. Notons également que le théorème précédent a été légèrement étendu par Imbert dans [95]. On pourra également voir l'article [25] où une approche similaire permet de traiter des équations avec conditions de Dirichlet.

Dans [6], Alter, Caselles et Chambolle ont utilisé ces résultats de convexité pour démontrer le théorème suivant :

Théorème 1.8.3. *Soit Ω un ensemble convexe borné de classe $\mathcal{C}^{1,1}$ et soit K son unique ensemble de Cheeger alors pour tout $v \in [|K|, |\Omega|]$, la solution de*

$$\min_{E\subset\Omega, |E|=v} P(E)$$

est unique et convexe.

Un ensemble de Cheeger de Ω est un ensemble minimisant

$$\min_{E\subset\Omega} \frac{P(E)}{|E|}. \tag{1.14}$$

L'unicité de l'ensemble de Cheeger est démontrée dans [5]. Cette approche a ensuite été utilisée par Caselles et Chambolle [40] pour démontrer le théorème analogue suivant :

Théorème 1.8.4. *Soit g une fonction convexe telle que $\overline{\lim}_{x\to\infty} \frac{g(x)}{|x|} = L < +\infty$ alors il existe v_0 tel que pour tout $v \in]v_0, +\infty[$, il existe une unique solution de*

$$\min_{|E|=v} P(E) + \int_E g(x)\, dx. \tag{1.15}$$

De plus cette solution est convexe.

Remarquons que si g est la fonction indicatrice de Ω alors les problèmes (1.14) et (1.15) sont identiques. On peut également noter que dans [40], ce résultat bien que nulle part énoncé clairement, est en fait démontré pour des périmètres anisotropes très généraux. Les auteurs de [40] étaient en fait à l'origine motivés par l'étude du mouvement par courbure moyenne anisotrope.

La démonstration du théorème 1.8.4, est basée sur l'idée suivante. On considère tout d'abord pour $\lambda \in \mathbb{R}$, le problème auxiliaire :

$$\min P(E) + \int_E (g - \lambda)\, dx. \qquad (P_\lambda)$$

Grâce à la formule de la coaire, on peut démontrer que les solutions du problème (P_λ) sont les lignes de niveaux de l'unique minimiseur local $\bar{u}$ à croissance L de la fonctionnelle

$$\int_{\mathbb{R}^m} |Du| + \frac{1}{2}\int_{\mathbb{R}^m} |u-g|^2. \tag{1.16}$$

Ce minimiseur local est solution de l'équation

$$-\operatorname{div}\frac{Du}{|Du|}+u=g.$$

Par approximation et à l'aide des résultats d'Alvarez, Lasry et Lions [7], on peut démontrer que $\bar{u}$ est convexe. Ses lignes de niveaux sont donc également convexes. On obtient ainsi en posant $v_0=|\{\bar{u}=\inf\bar{u}\}|$, pour $v\in]v_0,+\infty[$, que les solutions de (1.15) sont convexes.

Observons que si dans [77], Figalli et Maggi ont démontré que les solutions de (1.15) étaient convexes pour de petits volume v, la convexité de ces solutions pour tout volume reste ouverte.

Pour les espaces de Wiener, le seul résulat de convexité pour des solutions de problèmes variationnels est dûs à Caselles, Miranda, et Novaga. Ceux-ci généralisent dans [43] le théorème 1.8.3.

1.8.2 Contributions de la thèse

Dans le chapitre 5 de la thèse, nous nous intéressons à l'analogue du théorème 1.8.4 dans les espaces de Wiener. Nous présentons deux démonstrations de ce théorème, la première basée sur la méthode d'Alvarez, Lasry et Lions et l'autre sur celle de Korevaar. L'idée est, dans les deux cas, de considérer comme précédemment, le problème

$$\min_{u\in BV_\gamma(X)}\int_X F(D_\gamma u)+\frac{1}{2}\int_X(u-g)^2\,d\gamma \tag{1.17}$$

pour $F=|\cdot|_H$. Une fois la convexité du minimiseur de (1.17) prouvée, celle des minimiseurs de

$$\min P_\gamma(E)+\int_E g(x)\,d\gamma \tag{1.18}$$

s'obtient aisément en procédant comme dans le cas euclidien. Dans les deux démonstrations, l'idée est d'approcher le problème (1.17) en dimension infinie par des problèmes de dimension finie pour lesquels il est possible d'adapter les techniques de Korevaar ou d'Alvarez, Lasry et Lions afin d'obtenir la convexité des solutions des problèmes approchés. Dans la première de ces méthodes, on montre que de façon très générale :

Théorème 1.8.5. *Soit F une fonction convexe coercive finie et soit $g\in L^2_{\gamma_m}(\mathbb{R}^m)$ une fonction convexe alors le minimiseur de*

$$\min_{u\in BV_{\gamma_m}}\int_{\mathbb{R}^m}F(D_{\gamma_m}u)+\frac{1}{2}\int_{\mathbb{R}^m}(u-g)^2d\gamma_m \tag{1.19}$$

est convexe.

La preuve est une adaptation non triviale de la méthode d'Alvarez, Lasry et Lions. La difficulté vient en partie de la non utilisation de principes de comparaison ainsi que de la construction pas évidente de sur et sous-solutions.

À l'aide de la deuxième stratégie de démonstration, il n'a été possible de prouver le théorème 1.8.5 que dans les cas particuliers de la variation totale, de l'énergie de Dirichlet (c'est-à-dire $F = |\cdot|^2$) et pour la fonctionnelle de l' “aire”, $F(p) = \sqrt{\varepsilon^2 + |p|^2}$. En effet, pour cette dernière fonction, on commence par considérer le problème restreint spatialement

$$\min_{u=M \text{ sur } \partial B_R} \int_{B_R} \sqrt{\varepsilon^2 + |D_{\gamma_m} u|^2} + \frac{1}{2} \int_{B_R} (u-g)^2 \, d\gamma_m.$$

On montre, en utilisant un principe de comparaison que pour M assez grand, la solution u_M de ce problème a un contact vertical avec le bord. En utilisant un résultat de Giaquinta, Modica et Souček [81], on obtient que u_M est lisse à l'intérieur de B_R. La difficulté provient du fait que pour appliquer le théorème de Korevaar, il faudrait savoir que u_M est continue jusqu'au bord de B_R. Pour contourner cette difficulté, on montre que le sous-graphe de u_M est solution d'un certain problème de surfaces minimales avec obstacles, ce qui permet d'en obtenir la régularité jusqu'au bord. On applique alors une version géométrique de l'argument de Korevaar pour obtenir la convexité de u_M. On laisse ensuite R tendre vers l'infini pour obtenir la convexité de la solution de (1.19) pour $F(p) = \sqrt{\varepsilon^2 + |p|^2}$. En faisant ensuite tendre ε vers zéro ou vers l'infini on prouve la convexité des solutions pour les deux autres fonctionnelles. Cette méthode ne s'étend malheureusement pas à des énergies plus générales car celles-ci donnent naissance à des problèmes d'obstacles pour des surfaces minimales par rapport à des périmètres anisotropes pour lesquels la régularité nécessaire fait défaut.

On démontre ensuite par Γ-convergence le théorème suivant

Théorème 1.8.6. *Soit $F : H \to \mathbb{R}$ une fonction convexe propre semi-continue inférieurement et soit $g \in L^2_\gamma(X)$ une fonction convexe alors la solution de* (1.17) *est convexe.*

L'une des étapes clefs dans la preuve de ce théorème est la généralisation de théorèmes de représentation de fonctionnelles intégrales aux espaces de Wiener.

Théorème 1.8.7. *Soit $F : H \to \mathbb{R}$ une fonction convexe semi-continue et bornée inférieurement st soit $\mu = \mu^a\gamma + \mu^s$ une mesure de Radon bornée sur X alors*

$$\int_X F(\mu^a)\ d\gamma + \int_X F^\infty(\frac{d\mu^s}{d|\mu^s|})d|\mu^s| = \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma.$$

Ici $\mathcal{F}\mathcal{C}_b^1(X,H)$ est l'espace des fonctions $\mathcal{C}^1$ bornées de X dans H dépendant d'un nombre fini de variables et dont l'image est également de dimension finie. Ces fonctions sont dites cylindriques.

Chapitre 2

Continuous Primal-Dual methods for Image Processing

Abstract

In this chapter we study a continuous Primal-Dual method proposed by Appleton and Talbot and generalize it to other problems in image processing. We interpret it as an Arrow-Hurwicz method which leads to a better description of the system of PDEs obtained. We show existence and uniqueness of solutions and get a convergence result for the denoising problem. Our analysis also yields new *a posteriori* estimates. We also discuss the numerical scheme and its link with other existing methods.

Résumé

Dans ce chapitre, nous étudions la méthode Primale-Duale continue proposée par Appleton et Talbot. Nous la généralisons à d'autres problèmes de traitement d'images et l'interprétons comme une méthode de Arrow-Hurwicz. Ceci permet d'aboutir à une meilleure compréhension du système d'EDP obtenu. Nous prouvons l'existence et l'unicité d'une solution au problème de Cauchy à l'aide de la théorie des opérateurs maximaux monotones. Nous donnons également un preuve de convergence de la solution lorsque le problème considéré est le débruitage à l'aide de la fonctionnelle de Rudin-Osher-Fatemi. Notre analyse permet d'obtenir de nouvelles estimations *a posteriori*. Nous discutons finallement l'implémentation numérique de cette méthode ainsi que son lien avec d'autres algorithmes existants.

2.1 Introduction

In imaging, duality has been recognized as a fundamental ingredient for designing numerical schemes solving variational problems involving a total variation term. Primal-Dual methods were introduced in the field by Chan, Golub and Mulet in [53]. Afterwards, Chan and Zhu [135] proposed to rewrite the discrete minimization problem as a min-max and solve it using an Arrow-Hurwicz [18] algorithm, which is a gradient ascent in one direction and a gradient descent in the other. Just as for the simple gradient descent, one can think of extending this method to the continuous framework. This is in fact what is done by the algorithm previously proposed by Appleton and Talbot in [17], derived by analogy with discrete graph cuts techniques. The first to notice the link between their method and Primal-Dual schemes were Chambolle et al. in [46].
Besides its intrinsic theoretical interest, considering the continuous framework has also pratical motivations. Indeed, as illustrated by Appleton and Talbot in [17], this approach leads to higher quality results compared with fully discrete schemes such as those proposed by Chan and Zhu. We will numerically illustrate this in the final part of this chapter.

This chapter proposes to study the continuous Primal-Dual algorithm following the philosophy of the work done for the gradient flow by Caselles and its collaborators (see the book of Andreu et al. [15] and the references therein). We give a rigorous definition of the system of PDEs which is obtained and show existence and uniqueness of a solution to the Cauchy problem. We prove strong L^2 convergence to the minimizer for the Rudin-Osher-Fatemi model and derive some *a posteriori* estimates. As a byproduct of our analysis we also obtain *a posteriori* estimates for the numerical scheme proposed by Chan and Zhu.

This chapter is based on the paper [86].

2.1.1 Presentation of the problem

Many problems in image processing can be seen as minimizing in $BV^2 := BV \cap L^2$ an energy of the form

$$J(u) = \int_\Omega |Du| + G(u) + \int_{\partial\Omega_D} |u - \varphi|. \tag{2.1}$$

We assume that Ω is a bounded Lipschitz open set of $\mathbb{R}^m$ (in applications for image processing, usually $m = 2$ or $m = 3$) and that $\partial\Omega_D$ is a subset of $\partial\Omega$. The function φ being given in $L^1(\partial\Omega_D)$, the term $\displaystyle\int_{\partial\Omega_D} |u - \varphi|$ is a Dirichlet condition on $\partial\Omega_D$. We call $\partial\Omega_N$ the complement of $\partial\Omega_D$ in $\partial\Omega$ and assume that G is convex and continuous in L^2 with

$$G(u) \leq C(1 + |u|_2^p) \qquad \text{with } 1 \leq p \leq +\infty.$$

In this chapter we note $|u|_2$ the L^2 norm of u. According to Giaquinta et al. [81] we have,

Proposition 2.1.1. *The functional J is convex and lower-semi-continuous (lsc) in L^2.*

In the following, we also assume that J attains its minimum in $BV \cap L^2$. This is for example true if G satisfies some coercivity hypothesis or if G is non negative.

Two fundamental applications of our method are image denoising via total variation regularization and segmentation with geodesic active contours.

In the first problem, one starts with a corrupted image $f = \bar{u} + n$ and wants to find the clean image $\bar{u}$. Rudin, Osher and Fatemi proposed to look for an approximation of $\bar{u}$ by minimizing

$$\int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (u - f)^2.$$

This corresponds to $G(u) = \frac{\lambda}{2} \int_\Omega (u - f)^2$ and $\partial\Omega_D = \emptyset$ in (2.1). For a comprehensive introduction to this subject, we refer to the lecture notes of Chambolle et al. [45]. Figure 2.1 shows the result of denoising using the algorithm of Chan and Zhu.

The issue in the second problem is to extract automatically the boundaries of an object within an image. We suppose that we are given two subsets S and T of $\partial\Omega$ such that S lies inside the object that we want to segment and T lies outside. Caselles et al. proposed in [41] to associate a positive function g to the image in a way that g is high where the gradient of the image is low and vice versa. The object is then segmented by minimizing

$$\min_{E \supset S,\, E^c \supset T} \int_{\partial E} g(s)ds. \tag{2.2}$$

Figure 2.1: Denoising using the ROF model

In order to simplify the notation, we will deal only with $g = 1$ in the following. It is however straightforward to extend our discussion to general (continuous) g. The energy we want to minimize is thus $\int_\Omega |D\chi_E|$. This functional is non convex but by the coarea formula (see Ambrosio-Fusco-Pallara [10]), it can be relaxed to functions $u \in [0,1]$.
Let $\varphi = 1$ on S and $\varphi = 0$ on T. Letting $\partial\Omega_D = S \cup T$ and f be an L^2 function, our problem can be seen as a special case of the prescribed mean curvature problem (in our original segmentation problem, $f = 0$),

$$\inf_{\substack{0 \le u \le 1 \\ u=\varphi \text{ in } \partial\Omega_D}} \int_\Omega |Du| + \int_\Omega fu. \tag{2.3}$$

If u is a solution of (2.3), a minimizer E of (2.2) is then given by any superlevel of u, namely $E = \{u > s\}$ for any $s \in]0,1[$. This convexification argument is somewhat classical but more details can be found in the lecture notes [45, Section 3.2.2].
It is however well known that in general the infimum is not attained because of the lack of compactness for the boundary conditions in BV. Following the ideas of Giaquinta et al. [81] we have to relax the boundary conditions by adding a Dirichlet term $\int_{\partial\Omega_D} |u - \varphi|$ to the functional. We also have to deal with the hard constraint, $0 \le u \le 1$. This last issue will be discussed afterwards but it brings some mathematical difficulties that we were not able to solve. Fortunately, our problem is equivalent (see [46]) to the minimization of the unconstrained problem

$$J(u) = \inf_{u \in BV(\Omega)} \int_\Omega |Du| + \int_{\partial\Omega_D} |u - \varphi| + \int_\Omega f^+|u| + \int_\Omega f^-|1-u|.$$

Here $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.
We give in Figure 2.2 the result of this segmentation on yeasts. The small

square is the set S and the set T is taken to be the image boundary. The study of this problem was in fact our first motivation for this work.

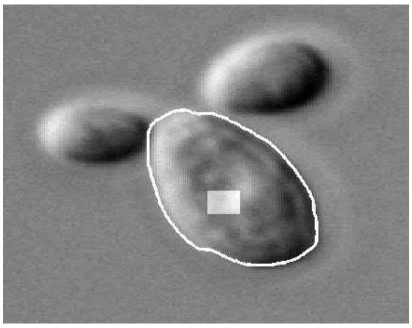

Figure 2.2: Yeast segmentation

2.1.2 Idea of the Primal-Dual method

Formally, the idea behind the Primal-Dual method is using the definition of $\int_\Omega |Du|$ (see Definition 1.2.1) in order to write J as

$$J(u) = \sup_{\substack{\xi \in \mathcal{C}_c^1(\Omega) \\ |\xi|_\infty \leq 1}} K(u, \xi),$$

where $K(u,\xi) = -\int_\Omega u \operatorname{div}(\xi) + \int_{\partial\Omega_D} |u - \varphi| + G(u)$. Then, finding a minimum of J is equivalent to finding a saddle point of K. This is done by a gradient descent in u and a gradient ascent in ξ.
Let $I_{B(0,1)}(\xi)$ be the indicator function of the unit ball in L^∞ (it takes the value 0 if $|\xi|_\infty \leq 1$ and $+\infty$ otherwise) and ∂ denote the subdifferential (see Ekeland-Temam [69] for the definition). As

$$K(u,\xi) = -\int_\Omega u \operatorname{div}(\xi) + \int_{\partial\Omega_D} |u - \varphi| + G(u) - I_{B(0,1)}(\xi).$$

we have $\nabla_u K \simeq -\operatorname{div}\xi + \partial G(u)$ and $\nabla_\xi K \simeq Du - \partial I_{B(0,1)}(\xi)$. We are thus led to solve the system of PDEs:

$$\begin{cases} \partial_t u = \operatorname{div}(\xi) - \partial G(u) \\ \partial_t \xi = Du - \partial I_{B(0,1)}(\xi) \\ + \text{ boundary conditions.} \end{cases} \tag{2.4}$$

This system is almost the one proposed by Appleton and Talbot in [17] for the segmentation problem.

Let us remark that, at least formally, the differential operator $A(u,\xi) = \begin{pmatrix} -\operatorname{div}\xi + \partial G(u) \\ -Du + \partial I_{B(0,1)}(\xi) \end{pmatrix}$ verifies by Green's formula and the monotonicity of the subdifferential (see Proposition 2.2.3),

$$\langle A(u,\xi),(u,\xi)\rangle = \langle \partial G(u), u\rangle + \langle \partial I_{B(0,1)}(\xi), \xi\rangle \geq 0$$

which means that A is monotone (see Definition 2.2.2).
In the next section we recall some facts about the theory of maximal monotone operators and its applications for finding saddle points. In the last section we use it to give a rigorous meaning to the hyperbolic system (2.4) together with existence and uniqueness of solutions of the Cauchy problem.

2.2 Maximal Monotone Operators

Following Brézis [33], we briefly present in the first part of this section the theory of maximal monotone operators. In the second part we show how this theory sheds light on the general Arrow-Hurwicz method. We mainly give results found in Rockafellar's paper [123].

2.2.1 Definitions and first properties of maximal monotone operators

Definition 2.2.1. *Let X be a Hilbert space. An operator is a multivaluated mapping A from X into $\mathcal{P}(X)$. We call $D(A) = \{x \in X \,/\, A(x) \neq \varnothing\}$ the domain of A and $R(A) = \bigcup_{x\in X} A(x)$ its range. We identify A and its graph in $X \times X$.*

Definition 2.2.2. *An operator A is monotone if :*

$$\forall x_1, x_2 \in D(A), \qquad \langle A(x_1) - A(x_2), x_1 - x_2\rangle \geq 0$$

or more precisely if for all $x_1^* \in A(x_1)$ and $x_2^* \in A(x_2)$,

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0.$$

It is maximal monotone if it is maximal in the set of monotone operators. The maximality is to be understood in the sense of graph inclusion.

One of the essential results for us is the maximal monotonicity of the subgradient for convex functions.

Proposition 2.2.3. [33] *Let φ be a proper lower-semi-continuous convex function on X then $\partial\varphi$ is a maximal monotone operator.*

Before stating the main theorem of this theory, namely the existence of solutions of the Cauchy problem $-u' \in A(u(t))$ we need one last definition.

Definition 2.2.4. *Let A be maximal monotone. For $x \in D(A)$ we call $A^{\circ}(x)$ the projection of* 0 *on $A(x)$ (it exists since $A(x)$ is closed and convex, see Brzis [33, p. 20]).*

We now turn to the theorem.

Theorem 2.2.5. [33] *Let A be maximal monotone then for all $u_0 \in D(A)$, there exists a unique function $u(t)$ from $[0, +\infty[$ into X such that*

- *$u(t) \in D(A)$ for all $t > 0$,*
- *$u(t)$ is Lipschitz continous on $[0, +\infty[$, i.e $u' \in L^{\infty}(0, +\infty; X)$ (in the sense of distributions) and*

$$\left|u'\right|_{L^{\infty}(0,+\infty;X)} \leq |A^{\circ}(u_0)|,$$

- *$-u'(t) \in A(u(t))$ for almost every t,*
- *$u(0) = u_0$.*

Moreover u verifies,

- *u has a right derivative for every $t \in [0, +\infty[$ and $-\dfrac{d^{+}u}{dt} \in A^{\circ}(u(t))$,*
- *the function $t \to A^{\circ}(u(t))$ is right continuous and $t \to |A^{\circ}(u(t))|$ is non increasing,*
- *if u and $\hat{u}$ are two solutions then $|u(t) - \hat{u}(t)| \leq |u(0) - \hat{u}(0)|$.*

2.2.2 Application to Arrow-Hurwicz methods

Let us now see how this theory can be applied for tracking saddle points. As mentioned before, we here follow [123]. We start with some definitions.

Definition 2.2.6. *Let $X = Y \oplus Z$ where Y and Z are two Hilbert spaces. A proper saddle function on X is a function K such that :*

- *for all $y \in Y$, the function $K(y, \cdot)$ is convex,*
- *for all $z \in Z$, the function $K(\cdot, z)$ is concave,*
- *there exists $x = (y, z)$ such that $K(y, z') < +\infty$ for all $z' \in Z$ and $K(y', z) > -\infty$ for all $y' \in Y$. The set of x for which it holds, is called the effective domain of K and is written* $\operatorname{dom} K$.

Definition 2.2.7. *A point $(y, z) \in X$ is called a saddle point of K if*

$$K(y, z') \leq K(y, z) \leq K(y', z) \qquad \forall y' \in Y, \forall z' \in Z.$$

We then have,

Proposition 2.2.8. *A point (y, z) is a saddle point of a saddle function K, if and only if*

$$K(y, z) = \sup_{z' \in Z} \inf_{y' \in Y} K(y', z') = \inf_{y' \in Y} \sup_{z' \in Z} K(y', z').$$

The proof of this proposition is easy and can be found in Rockafellar's book [122, p.380] .
The next theorem shows that the Arrow-Hurwicz method always provides a monotone operator.

Theorem 2.2.9. [123] *Let K be a proper saddle function. For $x = (y, z)$ let*

$$T(x) = \left\{ (y^*, z^*) \in Y \oplus Z / \begin{array}{c} y^* \text{ is a subgradient of } K(\cdot, z) \text{ in } y \\ z^* \text{ is a subgradient of } -K(y, \cdot) \text{ in } z \end{array} \right\}.$$

Then T is a monotone operator with $D(T) \subset$ $\operatorname{dom} K$.

We can now characterize the saddle points of K using the operator T.

Proposition 2.2.10. [123] *Let K be a proper saddle function then a point x is a saddle point of K if and only if $0 \in T(x)$.*

Remark 2.2.11. *This property is to be compared with the minimality condition $0 \in \partial f(x)$ for convex functions f.*

The next theorem shows that for regular enough saddle functions, the corresponding operator T is maximal.

Theorem 2.2.12. [123] *Let K be a proper saddle function on X. Suppose that K is lsc in y and upper-semi-continuous in z then T is maximal monotone.*

Proof. We just sketch the proof because it will inspire us in the following. The idea is to use the equivalent theorem for convex functions. For this we "invert" the operator T in the second variable. Let

$$H(y, z^*) = \sup_{z \in X} \langle z^*, z\rangle + K(y, z).$$

The proof is then based on the following lemma :

Lemma 2.2.13. [123] *H is a convex lsc function on X and*

$$(y^*, z^*) \in T(y, z) \Leftrightarrow (y^*, z) \in \partial H(y, z^*).$$

It is then not too hard to prove that T is maximal.

□

2.3 Study of the Primal-Dual Method

In this section, unless otherwise stated, everything holds for general functionals J of the type (2.1).

Before starting the study of the Primal-Dual method, let us remind some facts about pairings between measures and bounded functions.

Following Anzellotti [16], we define $\int_\Omega [\xi, Du]$ which has to be understood as $\int_\Omega \xi \cdot Du$, for functions u with bounded variation and bounded functions ξ with divergence in L^2.

Definition 2.3.1.
- *Let $X^2 = \left\{\xi \in (L^\infty(\Omega))^m \,/\, \operatorname{div} \xi \in L^2(\Omega)\right\}$.*
- *For $(u, \xi) \in BV^2 \times X^2$ we define the distribution $[\xi, Du]$ by*

$$\langle [\xi, Du], \varphi\rangle = -\int_\Omega u\varphi \operatorname{div}(\xi) - \int_\Omega u\, \xi \cdot \nabla\varphi \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Theorem 2.3.2. [16] *The distribution $[\xi, Du]$ is a bounded Radon measure on Ω and if ν is the outward unit normal to Ω, we have Green's formula,*

$$\int_\Omega [\xi, Du] = -\int_\Omega u \operatorname{div}(\xi) + \int_{\partial\Omega} (\xi \cdot \nu) u.$$

We will need in the following some approximation Lemmas which can be found in [16]

Proposition 2.3.3. [16, Prop 2.1 and Lem. 2.2] *Let* $u \in BV^2(\Omega)$ *and* $\xi \in X^2$ *with* $|\xi|_\infty \leq 1$*, then we can find* $\xi_n \in X^2 \cap \mathcal{C}(\Omega)$ *with* $|\xi_n|_\infty \leq 1$ *and* $[\xi_n, Du]$ *tending to* $[\xi, Du]$ *in the sense of weak convergence of measures.*

Proposition 2.3.4. [16, Lem. 5.2 and Lem. 1.8] *Let* Ω *be any open set in* $\mathbb{R}^m$ *and let* $u \in BV^2(\Omega)$ *be fixed then there exists a sequence of functions* $u_n \in \mathcal{C}^\infty \cap BV(\Omega)$ *such that*

- $u_n \to u$ *in* $L^2(\Omega)$,
- $\displaystyle\int_\Omega |Du_n| \to \int_\Omega |Du|$,
- $\displaystyle\int_\Omega |Du_n - h|\, dx \to \int_\Omega |Du - h\, dx|$ *for all* $h \in L^1(\Omega)$.

Moreover if $\xi \in X^2$ *then*

$$\int_\Omega [\xi, Du_n] \to \int_\Omega [\xi, Du].$$

Proposition 2.3.5. [16, Lem. 5.5] *Let* Ω *be a bounded open set with Lipschitz boundary then for any* $u \in L^1(\partial\Omega)$ *and for any* $\varepsilon > 0$*, there exists* $w \in W^{1,1}(\Omega) \cap \mathcal{C}(\Omega)$ *such that*

- $w = u$ *on* $\partial\Omega$,
- $\int_\Omega |Dw| \leq \int_{\partial\Omega} |u| + \varepsilon$,
- $w(x) = 0$ *if* $dist(x, \partial\Omega) > \varepsilon$,
- $|w|_2 \leq \varepsilon$.

We will need the following lemma.

Proposition 2.3.6. *Let* $u \in BV(\Omega)$ *then*

$$\int_\Omega |Du| = \sup_{\substack{\xi \in X^2 \\ |\xi|_\infty \leq 1}} \int_\Omega [\xi, Du].$$

Proof. By the definition of the total variation,

$$\int_\Omega |Du| \leq \sup_{\substack{\xi \in X^2 \\ |\xi|_\infty \leq 1}} \int_\Omega [\xi, Du].$$

The other inequality follows from [16, Cor. 1.6] which states that

$$\int_\Omega [\xi, Du] \leq |\xi|_\infty \int_\Omega |Du|.$$

$\square$

The next proposition gives a characterization of the minimizers of the functional J.

Proposition 2.3.7. *Let* $J(u) = \int_\Omega |Du| + G(u) + \int_{\partial\Omega_D} |u-\varphi|$ *then* u *is a minimizer of* J *in* BV^2 *if and only if there exists* $\xi \in X^2$ *with* $|\xi|_\infty \leq 1$ *such that*

$$\begin{cases} \operatorname{div}(\xi) \in \partial G(u) \\ \displaystyle\int_\Omega |Du| = \int_\Omega [\xi, Du] \\ \xi \cdot \nu = 0 \ \text{in } \partial\Omega_N \ \text{ and } \ (\xi\cdot\nu) \in \operatorname{sign}(\varphi - u) \ \text{in } \partial\Omega_D. \end{cases}$$

We do not give the proof of this proposition here since it can be either found in Andreu et al. [15] p.143 or derived more directly using the techniques we used in Proposition 2.3.8 and Proposition 2.3.9.

With these few propositions in mind we can turn back to the analysis of the Primal-Dual method. As noticed in the introduction, finding a minimizer of J is equivalent to finding a saddle point of

$$K(u,\xi) = \int_\Omega [Du,\xi] + G(u) + \int_{\partial\Omega_D} |u-\varphi| - I_{B(0,1)}(\xi).$$

The saddle function K does not fulfill the assumptions of Theorem 2.2.12 since it is not lsc in u. However staying in the spirit of Lemma 2.2.13, we set

$$\begin{aligned} H(u,\xi^*) &= \sup_{\substack{\xi\in X^2 \\ |\xi|_\infty \leq 1}} \langle \xi, \xi^* \rangle + K(u,\xi) \\ &= \sup_{\substack{\xi\in X^2 \\ |\xi|_\infty \leq 1}} \langle \xi, \xi^* \rangle + \int_\Omega [Du,\xi] + G(u) + \int_{\partial\Omega_D} |u-\varphi| \\ &= \int_\Omega |Du + \xi^*| + G(u) + \int_{\partial\Omega_D} |u-\varphi|. \end{aligned}$$

Where the last equality is obtained as in Proposition 2.3.6. The function H is then a convex lsc function on $L^2 \times (L^2)^m$ hence ∂H is maximal monotone. We are now able to define a maximal monotone operator T by

$$T(u,\xi) = \left\{ (u^*, \xi^*) \,/\, (u^*, \xi) \in \partial H(u,\xi^*) \right\}.$$

In order to compute ∂H, which gives the expression of T, we use the characterization of the subdifferential

$$(u^*, \xi) \in \partial H(u, \xi^*) \quad \Longleftrightarrow \quad \langle u^*, u\rangle + \langle \xi^*, \xi\rangle = H(u, \xi^*) + H^*(u^*, \xi).$$

A first step is thus to determine what H^* is.

Proposition 2.3.8. *We have*

$$D(H^*) = \big\{(u^*, \xi) \,/\, u^* \in L^2(\Omega) \ \textit{and}\ \xi \in X^2 \,,\ \xi \cdot \nu = 0 \ \textit{in}\ \partial\Omega_N \,,\ |\xi|_\infty \leq 1\big\}$$

and

$$H^*(u^*, \xi) = G^*(u^* + \mathrm{div}(\xi)) - \int_{\partial\Omega_D} (\xi \cdot \nu)\varphi.$$

Proof. We start by computing the domain of H^*.
If $(u^*, \xi) \in D(H^*)$ then there exists a constant C such that for every $(u, \xi^*) \in BV^2 \times (L^2)^m$,

$$\langle u^*, u\rangle + \langle \xi^*, \xi\rangle - H(u, \xi^*) \leq C.$$

Restraining to $u \in H^1(\Omega)$ with $u_{|\partial\Omega_D} = 0$ and $\xi^* \in (L^2)^m$, we find that

$$\langle u^*, u) + \langle \xi^*, \xi\rangle - \int_\Omega |\nabla u + \xi^*| - G(u) \leq C$$

from which

$$\langle \nabla u + \xi^*, \xi\rangle - \langle \nabla u, \xi\rangle + \langle u^*, u\rangle - \int_\Omega |\nabla u + \xi^*| - G(u) \leq C.$$

Setting $\xi' = \nabla u + \xi^*$ and taking the supremum over all $\xi' \in (L^2)^m$ we have that $|\xi|_\infty \leq 1$ and for all $u \in H^1(\Omega)$ with $u_{|\partial\Omega_D} = 0$,

$$-\langle \nabla u, \xi\rangle + \langle u^*, u\rangle \leq C + G(u).$$

Taking now $\tilde{u} = \lambda u$ with λ positive and recalling the form of G, it can be shown letting λ tends to infinity, that for every $u \in H^1$ with $u_{|\partial\Omega_D} = 0$,

$$-\langle \nabla u, \xi\rangle + \langle u^*, u\rangle \leq C|u|_2.$$

This implies that $u^* + \mathrm{div}\, \xi \in L^2$ hence $\mathrm{div}\, \xi \in L^2$. Then by Green's formula in $H^1(\mathrm{div})$ (see Dautray-Lions [60] p.205) we have $\xi \cdot \nu = 0$ in $\partial\Omega_N$.

Let us now compute H^*.
Let $(u^*, \xi) \in D(H^*)$,

$$H^*(u^*, \xi) = \sup_{\xi^* \in L^2} \sup_{u \in BV^2} \left\{ \langle u^*, u \rangle + \langle \xi^*, \xi \rangle - \int_\Omega |Du + \xi^*| - G(u) - \int_{\partial\Omega_D} |u - \varphi| \right\}.$$

Let $\xi^* \in L^2$ be fixed. Then by Proposition 2.3.4, for every $u \in BV^2$ there exists $u_n \in \mathcal{C}^\infty \cap BV^2$ such that

$$u_n \xrightarrow{L^2} u\,, \quad (u_n)_{|\partial\Omega_D} = u_{|\partial\Omega_D} \qquad \text{and}$$
$$\int_\Omega |Du_n + \xi^*| \to \int_\Omega |Du + \xi^*|.$$

We can thus restrict the supremum to functions u of class $\mathcal{C}^\infty(\Omega)$. We then have

$$\begin{aligned}
H^*(u^*, \xi) &= \sup_{u \in BV^2 \cap \mathcal{C}^\infty} \sup_{\xi \in L^2} \left\{ \langle u^*, u \rangle + \langle \xi^*, \xi \rangle - \int_\Omega |Du + \xi^*| - G(u) - \int_{\partial\Omega_D} |u - \varphi| \right\} \\
&= \sup_{u \in BV^2 \cap \mathcal{C}^\infty} \left\{ \langle u^*, u \rangle - \langle \nabla u, \xi \rangle - G(u) - \int_{\partial\Omega_D} |u - \varphi| \right\} \\
&= \sup_{u \in BV^2} \left\{ \langle u^*, u \rangle - \int_\Omega [Du, \xi] - G(u) - \int_{\partial\Omega_D} |u - \varphi| \right\} \\
&= \sup_{u \in BV^2} \left\{ \langle u, u^* + \operatorname{div} \xi \rangle - G(u) - \int_{\partial\Omega_D} \{ |u - \varphi| + (\xi \cdot \nu) u \} \right\}.
\end{aligned}$$

Beware that $u \in BV^2 \cap \mathcal{C}^\infty$ implies that $\nabla u \in L^1$ and not $\nabla u \in L^2$ but the density of L^2 in L^1 allows us to pass from the first equality to the second. The third equality follows from Proposition 2.3.4. We now have to show that we can take the supremum in the interior of Ω and on the boundary $\partial\Omega_D$ separately.

Let f be in $L^1(\partial\Omega)$ and v be in $L^2(\Omega)$. We want to find $u_\varepsilon \in BV^2$ converging to v in L^2 and such that $(u_\varepsilon)_{|\partial\Omega_D} = f$.

By Proposition 2.3.5 there is a $w_\varepsilon \in W^{1,1}$ with $(w_\varepsilon)_{|\partial\Omega_D} = f$ and $|w_\varepsilon|_2 \leq \varepsilon$. By density of $\mathcal{C}_c^\infty(\Omega)$ in L^2 we can find $v_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ with $|v_\varepsilon - v|_2 \leq \varepsilon$ We can then take $u_\varepsilon = v_\varepsilon + w_\varepsilon$.

This shows that

$$\begin{aligned} H^*(u^*,\xi) &= \sup_{u\in L^2(\Omega)} \{\langle u, u^* + \operatorname{div}\xi\rangle - G(u)\} - \inf_{u\in L^1} \int_{\partial\Omega_D} \{|u-\varphi| + (\xi\cdot\nu)u\} \\ &= G^*(u^* + \operatorname{div}(\xi)) - \int_{\partial\Omega_D} (\xi\cdot\nu)\varphi. \end{aligned}$$

□

We can now compute T

Proposition 2.3.9. *Let $(u,\xi) \in BV^2 \times X^2$ then, $(u^*,\xi^*) \in T(u,\xi)$ if and only if*

$$\begin{cases} u^* + \operatorname{div}(\xi) \in \partial G(u) \\ \displaystyle\int_\Omega |\xi^* + Du| = \langle \xi^*, \xi\rangle + \int_\Omega [\xi, Du] \\ \xi\cdot\nu = 0 \text{ in } \partial\Omega_N \text{ and } (\xi\cdot\nu) \in \operatorname{sign}(\varphi - u) \text{ in } \partial\Omega_D. \end{cases}$$

Proof. Let us first note that,

$$G(u) + G^*(u^* + \operatorname{div}(\xi)) \geq \langle u, u^* + \operatorname{div}(\xi)\rangle \tag{2.5}$$

$$\int_\Omega |Du + \xi^*| \geq \int_\Omega [\xi, Du] + \int_\Omega \xi^*\xi \tag{2.6}$$

$$|u-\varphi| \geq (\xi\cdot\nu)(\varphi - u) \tag{2.7}$$

where the second inequality is obtained by arguing as in Proposition 2.3.6. By definition, $(u^*,\xi^*) \in T(u,\xi)$ if and only if

$$\begin{aligned} \langle u, u^*\rangle + \langle \xi, \xi^*\rangle &= H(u,\xi^*) + H^*(u^*,\xi) \\ &= \int_\Omega |Du + \xi^*| + G(u) + \int_{\partial\Omega_D} |u-\varphi| \\ &\quad + G^*(u^* + \operatorname{div}(\xi)) - \int_{\partial\Omega_D} (\xi\cdot\nu)\varphi. \end{aligned}$$

This shows that (2.5), (2.6) and (2.7) must be equalities which is exactly

$$\begin{cases} u^* + \operatorname{div}(\xi) \in \partial G(u) \\ \displaystyle\int_\Omega |\xi^* + Du| = \langle \xi^*, \xi\rangle + \int_\Omega [\xi, Du] \\ (\xi\cdot\nu) \in \operatorname{sign}(\varphi - u) \text{ in } \partial\Omega_D. \end{cases}$$

Moreover, $\xi\cdot\nu = 0$ in $\partial\Omega_N$ because $(u,\xi) \in D(T)$. □

Remark 2.3.10.

- *The condition* $(\xi \cdot \nu) \in \mathrm{sign}(\varphi - u)$ *in* $\partial\Omega_D$ *is equivalent to*

$$\int_{\partial\Omega_D} |u-\varphi| + (\xi\cdot\nu)u = \inf_v \int_{\partial\Omega_D} |v-\varphi| + (\xi\cdot\nu)v$$

  because inequality (2.7) *holds true for every* v *and is an equality for* u*.*

- *Whenever it has a meaning, it can be shown that the condition*

$$\int_\Omega |\xi^* + Du| = \langle \xi^*, \xi\rangle + \int_\Omega [\xi, Du]$$

  is equivalent to

$$\xi^* + Du \in \partial I_{B(0,1)}(\xi)$$

  so that we will not distinguish between these two notations.

- *This analysis shows why the constraint* $u \in [0,1]$ *is hard to deal with. In fact, it imposes that* $\mathrm{div}(\xi)$ *is a measure but not necessarily a* L^2 *function. It is not easy to give a meaning to* $\int_\Omega Du\cdot\xi$ *or to* $(\xi\cdot\nu)$ *on the boundary for such functions. However, when dealing with numerical implementations, it is better to keep the constraint on* u*.*

We can summarize those results in the following theorem which says that the Primal-Dual Method is well-posed.

Theorem 2.3.11. *For all* $(u_0, \xi_0) \in \mathrm{dom}(T)$*, there exists a unique* $(u(t), \xi(t))$ *such that*

$$\begin{cases} \partial_t u \in \mathrm{div}(\xi) - \partial G(u) \\ \partial_t \xi \in Du - \partial I_{B(0,1)}(\xi) \\ (\xi\cdot\nu) \in \mathrm{sign}(\varphi - u) \text{ in } \partial\Omega_D \qquad \xi\cdot\nu = 0 \text{ in } \partial\Omega_N \\ (u(0), \xi(0)) = (u_0, \xi_0). \end{cases} \tag{2.8}$$

Moreover, the energy $|\frac{d^+u}{dt}|_2^2 + |\frac{d^+\xi}{dt}|_2^2$ *is non increasing and if* $(\bar{u}, \bar{\xi})$ *is a saddle point of* K, $|u-\bar{u}|_2^2 + |\xi - \bar{\xi}|_2^2$ *is also non increasing.*

Proof. The operator T is maximal monotone hence Theorem 2.2.5 applies and gives the result. □

Remark 2.3.12. *This theorem also shows that whenever J has a minimizer, K has saddle points. This is because stationary points of the system* (2.8) *are minimizers of J (verifying the Euler-Lagrange equation for J, see Proposition 2.3.7).*

For the Rudin-Osher-Fatemi model, one can show that there is convergence of u to the minimizer of the functional J and obtain *a posteriori* estimates.

Proposition 2.3.13. *Let $G = \frac{\lambda}{2}\int_\Omega (u-f)^2$ and $\partial\Omega_D = \emptyset$. Then if $\bar{u}$ is the minimizer of J, every solution of* (2.8) *converges in L^2 to $\bar{u}$. Furthermore,*

$$|u-\bar{u}|_2 \leq \frac{1}{2}\left(\frac{1}{\lambda}|\partial_t u|_2 + \sqrt{\frac{|\partial_t u|_2^2}{\lambda^2} + \frac{8|\Omega|^{\frac{1}{2}}}{\lambda}|\partial_t \xi|_2}\right).$$

Proof. Let $(\bar{u},\bar{\xi})$ be such that $0 \in T(\bar{u},\bar{\xi})$. Let $e(t) = |u(t)-\bar{u}|_2^2$ and $g(t) = |\xi(t)-\bar{\xi}|_2^2$. We show that

$$\frac{1}{2}(e+g)' \leq -\lambda e. \tag{2.9}$$

Indeed, by definition of the flow,

$$\int_\Omega [\xi, Du] - \langle \xi, \partial_t \xi\rangle \geq \int_\Omega [\bar{\xi}, Du] - \langle \bar{\xi}, \partial_t \xi\rangle \qquad \text{and}$$
$$\int_\Omega [\bar{\xi}, D\bar{u}] - \langle \bar{\xi}, \partial_t \bar{\xi}\rangle \geq \int_\Omega [\xi, D\bar{u}] - \langle \xi, \partial_t \bar{\xi}\rangle.$$

Summing these two we find,

$$\int_\Omega [\xi-\bar{\xi}, D(u-\bar{u})] \geq \langle \xi-\bar{\xi}, \partial_t\xi - \partial_t\bar{\xi}\rangle.$$

We thus have

$$\begin{aligned}\frac{1}{2}(e+g)' &= \langle u-\bar{u}, \partial_t u - \partial_t \bar{u}\rangle + \langle \xi-\bar{\xi}, \partial_t\xi - \partial_t\bar{\xi}\rangle \\ &\leq \langle u-\bar{u}, \operatorname{div}(\xi-\bar{\xi}) - \lambda(u-\bar{u})\rangle + \int_\Omega [\xi-\bar{\xi}, D(u-\bar{u})] \\ &= -\lambda e.\end{aligned}$$

The functions e and g are Lipschitz continuous. Let L be the Lipschitz constant of e and let $h = e+g$.

Let us show by contradiction that e tends to zero when t tends to infinity.

Suppose that there exist $\alpha > 0$ and $T > 0$ such that $e \geq \alpha$ for all $t > T$, then we would have $h' \leq -\lambda\alpha$ and h would tend to minus infinity which is impossible by positivity of h. Hence

$$\forall \alpha > 0\ \forall T > 0\ \exists t \geq T \text{ such that } e(t) \leq \alpha.$$

Suppose now the existence of $\varepsilon > 0$ such that for all $T \geq 0$ there exists $t \geq T$ with $e(t) \geq \varepsilon$.
By continuity of e, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to+\infty} t_n = +\infty$ such that

$$e(t_{2n}) = \frac{\varepsilon}{2} \qquad e(t_{2n+1}) = \varepsilon.$$

Moreover, on $[t_{2n-1}, t_{2n}]$, we have $e(t) \geq \frac{\varepsilon}{2}$. We then find that

$$\begin{aligned} |e(t_{2n}) - e(t_{2n-1})| \leq L(t_{2n} - t_{2n-1}) \quad &\text{so} \\ \frac{\varepsilon}{2L} \leq t_{2n} - t_{2n-1}. \end{aligned}$$

From this we see that,

$$\begin{aligned} h(t_{2n+2}) &= h(t_{2n+1}) + \int_{t_{2n+1}}^{t_{2n+2}} h'(t)\,dt \\ &\leq h(t_{2n+1}) - \varepsilon\lambda(t_{2n+2} - t_{2n+1}) \\ &\leq h(t_{2n}) - \frac{\lambda\varepsilon^2}{2L}. \end{aligned}$$

This shows that $\lim\limits_{t\to+\infty} e(t) = 0$.

We now prove the *a posteriori* error estimate. We have that

$$\begin{aligned} u &= f + \frac{1}{\lambda}(\operatorname{div}\xi - \partial_t u) \\ \bar{u} &= f + \frac{1}{\lambda}\operatorname{div}\bar{\xi}, \end{aligned}$$

which leads to

$$\begin{aligned} |u - \bar{u}|_2^2 &= \frac{1}{\lambda}\langle \operatorname{div}(\xi - \bar{\xi}) - \partial_t u, u - \bar{u}\rangle \\ &= \frac{1}{\lambda}\left[\langle \operatorname{div}(\xi - \bar{\xi}), u - \bar{u}\rangle - \langle \partial_t u, u - \bar{u}\rangle\right] \\ &= \frac{1}{\lambda}\left[-\langle \xi - \bar{\xi}, Du - D\bar{u}\rangle - \langle \partial_t u, u - \bar{u}\rangle\right] \\ &\leq \frac{1}{\lambda}\left[\int_\Omega |Du| - \int_\Omega [\xi, Du] + |\partial_t u|_2 |u - \bar{u}|_2\right],] \end{aligned}$$

where the last inequality follows from $\int_\Omega [\bar{\xi}, Du] \leq \int_\Omega |Du|$ and $\int_\Omega \bar{\xi} \cdot D\bar{u} = \int_\Omega |D\bar{u}| \geq 0$.
Studying the inequality $X^2 \leq A + BX$, we can deduce that

$$|u - \bar{u}|_2 \leq \frac{1}{2}\left(\frac{1}{\lambda}|\partial_t u|_2 + \sqrt{\frac{|\partial_t u|_2^2}{\lambda^2} + \frac{4}{\lambda}\Big(\int_\Omega |Du| - \int_\Omega [\xi, Du]\Big)}\right).$$

The estimate follows from the fact that

$$\begin{aligned}
\int_\Omega |-\partial_t \xi + Du| &= \int_\Omega [\xi, Du] - \int_\Omega \partial_t \xi \cdot \xi \quad \text{thus} \\
\int_\Omega |Du| - \int_\Omega |\partial_t \xi| &\leq \int_\Omega [\xi, Du] - \int_\Omega \partial_t \xi \cdot \xi \quad \text{hence} \\
\int_\Omega |Du| - \int_\Omega [\xi, Du] &\leq 2\int_\Omega |\partial_t \xi| \leq 2|\Omega|^{\frac{1}{2}}|\partial_t \xi|_2.
\end{aligned}$$

□

Remark 2.3.14. *Since there is a priori no uniqueness of the calibrating field $\bar{\xi}$, it is not so clear whether $\xi(t)$ converges to $\bar{\xi}$.*

Following the same lines, we can show *a posteriori* error estimates for general finite difference scheme. Indeed if ∇^h is any discretization of the gradient and if div^h is defined as $-(\nabla^h)^*$, the associated algorithm is

$$\begin{cases} \xi^n = P_{B(0,1)}(\xi^{n-1} + \delta\tau^n \nabla^h u^{n-1}) \\ u^n = u^{n-1} + \delta t^n (\operatorname{div}^h \xi^n - \lambda(u^{n-1} - f)), \end{cases} \tag{2.10}$$

where $P_{B(0,1)}(\xi)_{i,j} = \dfrac{\xi_{i,j}}{\max(|\xi_{i,j}|, 1)}$ is the componentwise projection of ξ on the unit ball. This algorithm is exactly the one proposed by Chan and Zhu in [135]. We can associate at this system a discrete energy,

$$J_h(u) = \sum_{i,j} |\nabla^h u|_{i,j} + \frac{\lambda}{2}\sum_{i,j} |u_{i,j} - f_{i,j}|^2.$$

The algorithm (2.10) could have been directly derived from this discrete energy using the method of Chan and Zhu [135] (which is just the discrete counterpart of our continuous method). Hence, the next proposition gives a stopping criterion for their algorithm.

Proposition 2.3.15. *Let $N \times M$ be the size of the discretization grid and $\bar{u}$ be the minimizer of J_h then*

$$|u^n - \bar{u}|_2 \leq \frac{1}{2}\left(\frac{1}{\lambda}|\partial_t u^n|_2 + \sqrt{\frac{|\partial_t u^n|_2^2}{\lambda^2} + \frac{8\sqrt{N \times M}}{\lambda}|\partial_t \xi^n|_2}\right),$$

where $\partial_t u^n = \dfrac{u^{n+1} - u^n}{\delta t^{n+1}}$ *and* $\partial_t \xi^n = \dfrac{\xi^{n+1} - \xi^n}{\delta \tau^{n+1}}$.

Proof. For notational convenience, we present the proof for $\lambda = 1$. Let $\bar{u}$ be the minimizer of J_h then there exists $\bar{\xi}$ such that $|\bar{\xi}|_\infty \leq 1$ and

$$\begin{cases} \sum_{i,j} |\nabla^h \bar{u}|_{i,j} = \langle \nabla^h \bar{u}, \bar{\xi} \rangle \\ \bar{u} = \operatorname{div}^h \bar{\xi} + f. \end{cases}$$

Recalling that $u^n = f + \operatorname{div}^h \xi^{n+1} - \partial_t u^n$ we get

$$\begin{aligned} |u^n - \bar{u}|^2 &= \langle \operatorname{div}^h(\xi^{n+1} - \bar{\xi}) - \partial_t u^n, u^n - \bar{u} \rangle \\ &= -\langle \xi^{n+1} - \bar{\xi}, \nabla^h u^n - \nabla^h \bar{u} \rangle - \langle \partial_t u^n, u^n - \bar{u} \rangle \\ &\leq \langle \bar{\xi} - \xi^{n+1}, \nabla^h u^n \rangle + |\partial_t u^n||u^n - \bar{u}|. \end{aligned}$$

We have that $\xi^{n+1} = P_{B(0,1)}(\xi^n + \delta\tau^{n+1}\nabla^h u^n)$ hence by definition of the projection,

$$\forall \bar{\xi} \in B(0,1) \qquad \langle \xi^{n+1} - (\xi^n + \delta\tau^{n+1}\nabla^h u^n), \bar{\xi} - \xi^{n+1} \rangle \geq 0.$$

This gives us

$$\langle \nabla^h u^n, \bar{\xi} - \xi^{n+1} \rangle \leq \langle \partial_t \xi^n, \bar{\xi} - \xi^n \rangle.$$

Combining this with $\langle \partial_t \xi^n, \bar{\xi} \rangle - \langle \partial_t \xi^n, \xi^n \rangle \leq 2\sqrt{N \times M}|\partial_t \xi^n|$ (which holds by Cauchy-Schwarz's inequality, $|\bar{\xi}|_\infty \leq 1$ and $|\xi^n|_\infty \leq 1$), we find that

$$|u^n - \bar{u}|^2 \leq 2\sqrt{N \times M}|\partial_t \xi^n| + |\partial_t u^n||u^n - \bar{u}|.$$

The announced inequality easily follows. $\square$

Remark 2.3.16. *When the paper [86] (from which this chapter is taken), was finished and in contrast with the continuous framework, no fully satisfactory statement was known in the discrete framework. Some partial results were however available (see for example [70] or [51]). Very recently, Jalalzai proved in his PhD Thesis [96] an analogous of inequality* (2.9) *for the algorithm of Chan and Zhu which as in the continuous case implies convergence.*

For the general problem, there is no uniqueness for the minimizer (for example in the segmentation problem) and hence convergence may not occur or be hard to prove. Indeed, even when uniqueness holds, we can have non vanishing oscillations. For example in the simpler one dimensional problem

$$\min_{u \in BV([0,1])} \int_0^1 |u'|$$

the unique minimizer is $u = 0$ but $u(t,x) = \frac{1}{2}\cos(\pi x)\sin(\pi t)$ and $\xi(t,x) = \frac{1}{2}\sin(\pi x)\cos(\pi t)$ gives a solution to the associated PDE system which does not converge to a saddle point. In this example, the energy is constant hence does not converges to zero. We can however show general *a posteriori* estimates for the energy.

Proposition 2.3.17. *For every saddle point* $(\bar{u}, \bar{\xi})$ *and every* (u_0, ξ_0)*, the solution* $(u(t), \xi(t))$ *of* (2.8) *satisfies*

$$|J(u) - J(\bar{u})| \leq \left(\sqrt{|u_0 - \bar{u}|_2^2 + |\xi_0 - \bar{\xi}|_2^2}\right) |\partial_t u|_2 + 2|\Omega|^{\frac{1}{2}} |\partial_t \xi|_2.$$

Proof. Let $(\bar{u}, \bar{\xi})$ be a saddle point and $(u(t), \xi(t))$ be a solution of (2.8). Then

$$J(u) - J(\bar{u}) = \int_\Omega |Du| + \int_{\partial\Omega_D} |u - \varphi| - \int_\Omega |D\bar{u}| - \int_{\partial\Omega_D} |\bar{u} - \varphi| + G(u) - G(\bar{u}).$$

By definition of the operator T we have

$$\begin{aligned} \int_\Omega [\xi, Du] - \int_\Omega \partial_t \xi \cdot \xi &= \int_\Omega |Du - \partial_t \xi| \\ &\geq \int_\Omega |Du| - \int_\Omega |\partial_t \xi|. \end{aligned}$$

This shows that

$$\int_\Omega |Du| \leq \int_\Omega [\xi, Du] + 2 \int_\Omega |\partial_t \xi|. \tag{2.11}$$

On the other hand,

$$\int_\Omega [\xi, Du] + \int_{\partial\Omega_D} |u - \varphi| = -\int_\Omega u \operatorname{div} \xi + \int_{\partial\Omega_D} \{(\xi \cdot \nu) u + |u - \varphi|\}.$$

Applying $\int_{\partial\Omega_D} \{(\xi\cdot\nu)u + |u-\varphi|\} = \inf_v \int_{\partial\Omega_D} \{(\xi\cdot\nu)v + |v-\varphi|\}$ (remember the Remarks after Proposition 2.3.9) to $v=\bar{u}$ we have

$$\begin{aligned}\int_\Omega [\xi, Du] + \int_{\partial\Omega_D} |u-\varphi| - \int_{\partial\Omega_D} |\bar{u}-\varphi| &\leq -\int_\Omega u \operatorname{div}\xi + \int_{\partial\Omega_D} (\xi\cdot\nu)\bar{u} \\ &= -\int_\Omega u \operatorname{div}\xi + \int_\Omega \bar{u}\operatorname{div}\xi + \int_\Omega [\xi, D\bar{u}] \\ &= \int_\Omega (\bar{u}-u)\operatorname{div}\xi + \int_\Omega [\xi, D\bar{u}].\end{aligned}$$

This and (2.11) show that

$$J(u)-J(\bar{u}) \leq \int_\Omega (\bar{u}-u)\operatorname{div}\xi + \int_\Omega [\xi, D\bar{u}] + 2\int_\Omega |\partial_t\xi| - \int_\Omega |D\bar{u}| + G(u) - G(\bar{u}).$$

If we now use the definition of the subgradient to get

$$G(u) - G(\bar{u}) \leq \langle \operatorname{div}(\xi) - \partial_t u, u-\bar{u}\rangle$$

we find with Cauchy-Schwarz's inequality,

$$\begin{aligned}J(u)-J(\bar{u}) &\leq 2|\Omega|^{\frac{1}{2}}|\partial_t\xi|_2 + \int_\Omega (\bar{u}-u)\partial_t u + \int_\Omega [\xi, D\bar{u}] - \int_\Omega |D\bar{u}| \\ &\leq 2|\Omega|^{\frac{1}{2}}|\partial_t\xi|_2 + |\bar{u}-u|_2|\partial_t u|_2\end{aligned}$$

which gives the estimate, recalling that $\sqrt{|u-\bar{u}|_2^2 + |\xi-\bar{\xi}|_2^2}$ is non increasing. □

Remark 2.3.18. *Supported by numerical evidence, we can conjecture that whenever the constraint on ξ is saturated somewhere, convergence of u occurs. It might however also be necessary to add the constraint $u\in[0,1]$ in order to have this convergence.*

Considering a finite difference scheme, just as for the Rudin-Osher-Fatemi model, we can define a discrete energy J_h and show the corresponding *a posteriori* estimate.

Proposition 2.3.19. *If $\bar{u}$ is a minimizer of J_h and (u^n, ξ^n) is defined by*

$$\begin{cases}\xi^n = P_{B(0,1)}(\xi^{n-1} + \delta\tau^n\nabla^h u^{n-1}) \\ u^n = u^{n-1} + \delta t^n(\operatorname{div}^h \xi^n - p^n).\end{cases}$$

with $p^n \in \partial G^h(u^{n-1})$ then

$$|J_h(u^n) - J_h(\bar{u})| \leq 2\sqrt{N\times M}|\partial_t\xi^n| + |\partial_t u^n||u^{n-1}-\bar{u}|.$$

The proof being very similar to the proof of Proposition 2.3.15 we omit it.

Remark 2.3.20.

- *The boundary conditions are hidden here in the operator* ∇^h.
- *In the discrete framework, the estimate involves* $|u^n - \bar{u}|$ *which cannot easily be bounded by the initial error.*
- *In this still more general framework than in* (2.10) *no good convergence result is known. For some partial results for variants of this algorithm, we again refer to Chambolle and Pock [51].*

2.4 Numerical Experiments

2.4.1 The numerical scheme

We explicit here the scheme proposed by Appleton and Talbot in [17] to solve the system (2.4). We give it only for the denoising problem since there is no difficulty to extend it to segmentation. The only subtlety lies in the choice of the weight function. For a discussion about this, we refer to [17] or to [85]. We recall that for the ROF model, the system we want to solve is formally:

$$\begin{cases} \partial_t u = \mathrm{div}(\xi) - \lambda(u - f) \\ \partial_t \xi = Du \qquad |\xi|_\infty \leq 1. \end{cases}$$

We then discretize u and ξ on a regular grid; u^n, living on the vertices and (ξ^n_x, ξ^n_y) on the edges of this grid. The scheme then writes for u^n,

$$u^{n+1}(i,j) = (1-\lambda\delta t)\, u^n(i,j) + \delta t \left[\xi^n_x(i+\frac{1}{2},j) - \xi^n_x(i-\frac{1}{2},j) + \xi^n_y(i,j+\frac{1}{2}) \right. \\ \left. -\xi^n_y(i,j-\frac{1}{2}) + \lambda f(i,j) \right].$$

We let

$$\begin{aligned} \xi'^{n+1}_x(i+\frac{1}{2},j) &= \xi^n_x(i+\frac{1}{2},j) + \delta t \left[u^{n+1}(i+1,j) - u^{n+1}(i,j) \right] \\ \xi'^{n+1}_y(i,j+\frac{1}{2}) &= \xi^n_y(i,j+\frac{1}{2}) + \delta t \left[u^{n+1}(i,j+1) - u^{n+1}(i,j) \right]. \end{aligned}$$

We apply then the constraint on the norm of ξ^n. Let first

$$|\xi_x^{n+1}(i,j)|' = \max(-\xi_x'^{n+1}(i-\frac{1}{2},j), 0, \xi_x'^{n+1}(i+\frac{1}{2},j))$$
$$|\xi_y^{n+1}(i,j)|' = \max(-\xi_y'^{n+1}(i,j-\frac{1}{2}), 0, \xi_y'^{n+1}(i,j+\frac{1}{2})).$$

We also let

$$v^{n+1}(i,j) = \max(\sqrt{(|\xi_x^{n+1}(i,j)|')^2 + (|\xi_y^{n+1}(i,j)|')^2}, 1)$$

and then

$$|\xi_x^{n+1}(i,j)| = \frac{|\xi_x^{n+1}(i,j)|'}{v^{n+1}(i,j)} \qquad |\xi_y^{n+1}(i,j)| = \frac{|\xi_y^{n+1}(i,j)|'}{v^{n+1}(i,j)}.$$

Finally we compute

$$\xi_x^{n+1}(i-\frac{1}{2},j) = \max(\xi_x'^{n+1}(i-\frac{1}{2},j), -|\xi_x^{n+1}(i,j)|)$$
$$\xi_x^{n+1}(i+\frac{1}{2},j) = \min(\xi_x'^{n+1}(i+\frac{1}{2},j), |\xi_x^{n+1}(i,j)|)$$
$$\xi_y^{n+1}(i,j-\frac{1}{2}) = \max(\xi_y'^{n+1}(i,j-\frac{1}{2}), -|\xi_y^{n+1}(i,j)|)$$
$$\xi_y^{n+1}(i,j+\frac{1}{2}) = \min(\xi_y'^{n+1}(i,j+\frac{1}{2}), |\xi_y^{n+1}(i,j)|).$$

2.4.2 The experiments

To illustrate the relevance of our *a posteriori* estimates, we first consider the simple example of denoising a rectangle (see Figure 2.3). We then compare the *a posteriori* error bound with the "true" error. We use the relative L^2 error defined as $\frac{|u^n - \bar{u}|}{|\bar{u}|}$ and ran the algorithm of Chan and Zhu with $\lambda = 0.005$ and fixed time steps verifying $\lambda\delta t = 1$ and $\delta\tau = \frac{\lambda}{5}$. With this choice of parameters convergence is guaranteed by the work of Esser et al. [70]. The minimizer $\bar{u}$ is computed by the algorithm after 50000 iterations. Figure 2.4 shows that the *a posteriori* bound is quite sharp.

The second experiment was performed on the yeast segmentation of Figure 2.2. The solution was computed with the algorithm of Chan and Zhu using as weight function g the one proposed by Appleton and Talbot [17]. We used the error $|J_h(u^n) - J_h(\bar{u})|$, this time and ran the algorithm with $\delta t = 0.2$ and $\delta\tau = 0.2$. For this problem there is no proof of convergence of the algorithm. The minimizer $\bar{u}$ was computed by the algorithm after 50000

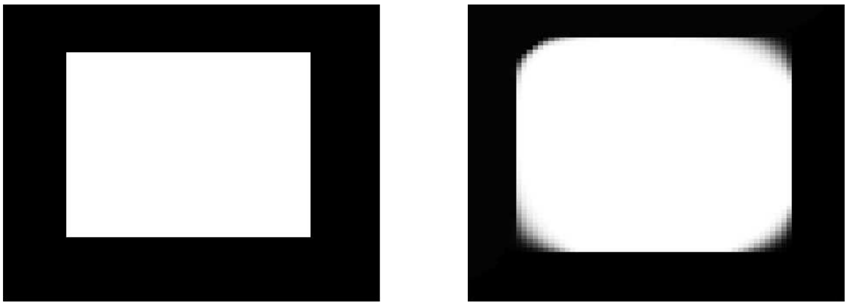

Figure 2.3: Denoising of a rectangle using the ROF model

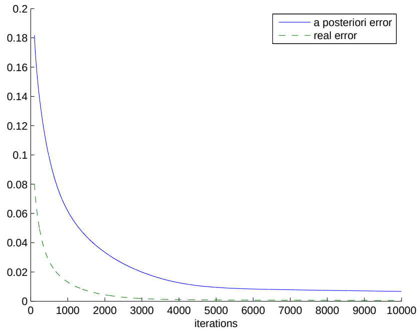

Figure 2.4: Comparison of the relative L^2 error with the predicted *a posteriori* bound.

iterations. We can see in Figure 2.5 that for this problem, the *a posteriori* estimate is not so sharp. We must also notice that in general we do not know $\bar{u}$.

In the third example, we compare the results obtained by the algorithm of Appleton and Talbot (see [17]) with those obtained by a classical discretization of the total variation. In Figure 2.6, we can see the denoising of a disk with these two methods for $\lambda = 0.003$. We used the algorithm of Chan and Zhu [135] to compute the minimization of the discrete total variation.
Looking at the top right corner (see Figure 2.7), we can see that the result is more accurate and less anisotropical for the algorithm of Appleton and Talbot than for the scheme of Chan and Zhu. These results are to be compared with those obtained by Chambolle et al. for the so-called "upwind" discrete BV norm in [49].

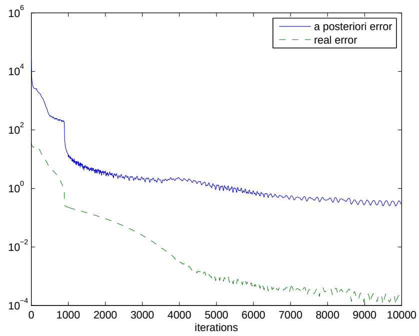

Figure 2.5: Comparison for the segmentation problem.

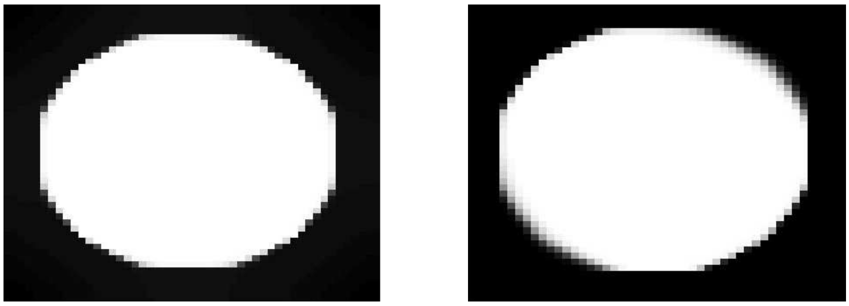

Figure 2.6: Denoising of a disk using the algorithm of Appleton-Talbot (left) and Chan-Zhu (right)

Finally in Figure 2.8, we show how this scheme can be applied for computing three dimensional minimal surfaces.

2.5 Conclusion and perspectives

In this chapter we have shown the well posedness of the continuous Primal-Dual method proposed by Appleton and Talbot for solving problems arising in imaging. We have also proved for the ROF model, that in the continuous setting there is convergence towards the minimizer. We then derived some *a posteriori* estimates. Numerical experiments have illustrated that if these estimates are quiet sharp for the ROF model, they should be improved for applications to other problems.
This continuous framework leaves the way open to a wide variety of numerical schemes, ranging from finite differences to finite volumes. Indeed, by designing algorithms solving the system of PDEs (2.8) one can expect to

Figure 2.7: Top right corner of the denoised disk, Appleton-Talbot (left) and Chan-Zhu (right)

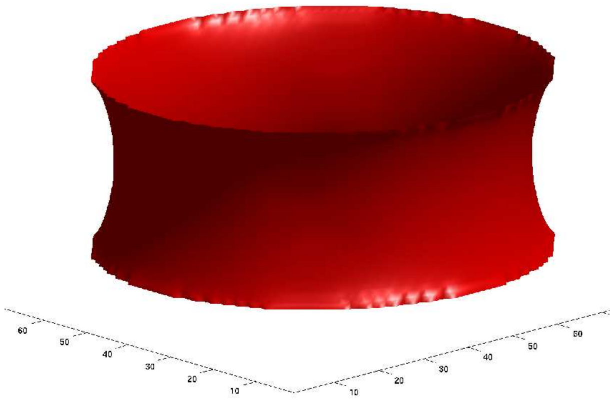

Figure 2.8: A minimal surface computed with the algorithm of Appleton and Talbot

find accurate algorithms for computing solutions of variational problems involving a total variation term. It would be interesting to investigate further in this direction.

Chapitre 3

Volume-constrained minimizers for the prescribed curvature problem in periodic media

Abstract

We establish existence of compact minimizers of the prescribed mean curvature problem with volume constraint in periodic media. As a consequence, we construct compact approximate solutions to the prescribed mean curvature equation. We also show convergence after rescaling of the volume-constrained minimizers towards a suitable Wulff Shape, when the volume tends to infinity.

Résumé

Dans ce chapitre, nous prouvons l'existence de solutions compactes au problème de courbure moyenne prescrite avec contrainte de volume dans un milieu périodique. Gràce à ce résultat, nous sommes en mesure de construire des solutions approchées de l'équation de courbure moyenne prescrite. Nous étudions par ailleurs le comportement asymptotique de ces minimiseurs lorsque leur volume tend vers l'infini.

3.1 Introduction

In recent years, a lot of attention has been drawn towards the problem of constructing surfaces with prescribed mean curvature. More precisely, given an assigned function $g : \mathbb{R}^m \to \mathbb{R}$, the problem is finding a hypersurface having mean curvature κ satisfying

$$\kappa = g. \tag{3.1}$$

To the best of my knowledge, this problem was first posed by S.T. Yau in [134], under the additional constraint of the hypersurface being diffeomorphic to a sphere, and a solution was provided in [130, 94] when the function g satisfies suitable decay conditions at infinity, namely that it decays faster than the mean curvature of concentric spheres. Another approach was presented in [24, 92], by means of conformal parametrizations and a clever use of the mountain pass lemma. A serious limitation of this method is the impossibility to extend it to dimension higher than three, due to the lack of a good equivalent of a conformal parametrization.

Motivated by some homogenization problems in front propagation [117], in this chapter we look for solutions to (3.1) without any topological constraint but with a periodic function g, so that in particular, it does not decay to zero at infinity. A natural idea is to look for critical points of the prescribed curvature functional

$$F(E) = P(E) - \int_E g\, dx,$$

as it is well-known that such critical points solve (3.1), whenever they are smooth [83]. Observe that, in general, it is not possible to construct solutions of (3.1) by a direct minimization of the functional F, because such minimizers may not exist or be empty.

The first result in this setting was obtained by Caffarelli and de la Llave in [37] (see also [52]) where the authors construct planelike solutions of (3.1) under the assumption that g is small and has zero average, by minimizing F among sets with boundary contained in a given strip, and then show that the constraint does not affect the curvature of the solution.

Here we are interested instead in compact solutions of (3.1). This problem seems difficult in this generality and only some preliminary results, in the two-dimensional case, are presently available [98]. However, the following perturbative result has been proved in [117]: given a periodic function g with zero average and small L^∞-norm and ε arbitrarily small, there exists a compact solution of

$$\kappa = g_\varepsilon$$

where $\|g_\varepsilon - g\|_{L^1} \leq \varepsilon$. Since the L^1-norm does not seem very well suited for this problem, a natural question raised in [117] was whether the same result holds when the L^1-norm is replaced by the L^∞-norm.

In this chapter we answer this question. More precisely, we prove the following result (see Theorem 3.4.4): let g be a periodic Hölder continuous function with zero average on the unit cell $Q = [0,1]^m$ and such that

$$\int_E g\, dx \leq (1-\Lambda) P(E,Q) \qquad \forall E \subset Q \tag{3.2}$$

for some $\Lambda > 0$, where $P(E,Q)$ is the relative perimeter of E in Q. Then for every $\varepsilon > 0$ there exist $0 < \varepsilon' < \varepsilon$ and a compact solution of

$$\kappa = g + \varepsilon'. \tag{3.3}$$

We observe that (3.2) is the same assumption made in [52] in order to prove existence of planelike minimizers. This condition is for instance verified if $||g||_{L^m(Q)}$ is smaller than the isoperimetric constant of Q, and allows g to take large negative values.

We construct approximate solutions of (3.3) as volume constrained minimizers of F for big volumes. This motivates the study of the isovolumetric function $f : [0, +\infty) \to \mathbb{R}$ defined as

$$f(v) = \min_{|E|=v} F(E). \tag{3.4}$$

As a by-product of our analysis, we are able to characterize the asymptotic shape of minimizers as the volume tends to infinity, showing that they converge after appropriate rescaling to the Wulff Shape (i.e. the solution of the isoperimetric problem) relative to an anisotropy ϕ_g depending on g. We mention that, in the small volume regime, the contribution of g becomes irrelevant and the minimizers converge to standard spheres (see [77] and references therein).

The plan of the chapter is the following: in Section 3.2 we show existence of compact minimizers of (3.4). In Section 3.3 we prove that the function f is locally Lipschitz continuous and link its derivative to the curvature of the minimizers. We also provide an example of a function f which is not differentiable everywhere. Let us notice that in these first two parts no assumption is made on the average of g or on its size. In Section 3.4 we use the isovolumetric function to find solutions of (3.3). Eventually, in Section 3.4.1 we investigate the behavior of the constrained minimizers of (3.4) as the volume goes to infinity.

This chapter is based on a joint work with Matteo Novaga [88].

Notation and general assumptions. We shall assume that g is a $\mathcal{C}^{0,\alpha}$ periodic function, with periodicity cell $Q = [0,1[^m$. We shall also suppose that the dimension of the ambient space is smaller or equal to 7, so that quasi-minimizers of the perimeter have boundary of class $\mathcal{C}^{2,\alpha}$ [83]. We believe that this restriction is not relevant for the results of this work, but we were not able to remove it. For a set of finite perimeter we denote by $P(E)$ its perimeter and by $\partial^* E$ its reduced boundary (see [83] for precise definitions). Given an open set Ω, we denote by $P(E,\Omega)$ the relative perimeter of E in Ω. We take as a convention that the mean curvature (which we define as the sum of all principal curvatures) of a convex set is positive. If ν is the outward normal to a set with smooth boundary, this amounts to say that the mean curvature κ is equal to $\mathrm{div}(\nu)$.

3.2 Existence of minimizers

In this section we prove existence of compact volume-constrained minimizers of F, by showing that for every volume v, the problem is equivalent to the unconstrained problem

$$\min_{E\subset\mathbb{R}^m} F_\mu(E) = \min_{E\subset\mathbb{R}^m} P(E) - \int_E g\,dx + \mu\big||E| - v\big|, \tag{3.5}$$

for $\mu > 0$ large enough. We start by studying (3.5), showing existence of smooth compact minimizers. We then show that there exists μ_0 such that, for $\mu \geq \mu_0$, every compact minimizer of F_μ has volume v. In particular, this will provide existence of minimizers of (3.4), since $f(v) \leq \min_E F_\mu(E)$ for every $\mu \geq 0$.

Denoting by Q_R the cube $[-R/2, R/2]^m$ of sidelength R, we consider the spatially constrained problem

$$\min_{E\subset Q_R} F_\mu(E). \tag{3.6}$$

Having restrained our problem to a bounded domain, we gain compactness of minimizing sequences and thus existence of minimizers for (3.6) by the direct method [83]. We want to show that these minimizers do not depend on R for R big enough. In order to do so, we need density estimates as [37].

Proposition 3.2.1. *There exist two constants $C(m)$ and γ depending only on the dimension m such that, if we set $r_0(\mu) = \dfrac{C(m)}{\mu + \|g\|_\infty}$, then for every minimizer E of (3.6) and every $x \in \mathbb{R}^m$,*

- $|E \cap B_r(x)| \geq \gamma r^m$ *for every $r \leq r_0$ if $|B_r(x) \cap E| > 0$ for any $r > 0$,*
- $|B_r(x) \backslash E| \geq \gamma r^m$ *for every $r \leq r_0$ if $|B_r(x) \backslash E| > 0$ for any $r > 0$.*

Proof. Let $x \in \partial^* E$ then by minimality of E we have

$$P(E) - \int_E g\,dx + \mu\big||E| - v\big| \leq P(E \backslash B_r(x)) - \int_{E \backslash B_r(x)} g\,dx + \mu\big||E \backslash B_r(x)| - v\big|,$$

hence

$$\begin{aligned} P(E) &\leq \int_{E \cap B_r} g\,dx + P(E \backslash B_r) + \mu\big||E| - |E \backslash B_r|\big| \\ &= \int_{E \cap B_r} g\,dx + P(E \backslash B_r) + \mu|E \cap B_r| \\ &\leq |E \cap B_r|(\|g\|_\infty + \mu) + P(E \backslash B_r). \end{aligned}$$

On the other hand we have

$$P(E) = \mathcal{H}^{m-1}(\partial^* E \cap B_r) + \mathcal{H}^{m-1}(\partial^* E \cap B_r^c)$$

and

$$P(E \backslash B_r) = \mathcal{H}^{m-1}(E \cap \partial B_r) + \mathcal{H}^{m-1}(\partial^* E \cap B_r^c).$$

From these inequalities we get

$$\mathcal{H}^{m-1}(\partial^* E \cap B_r) \leq \mathcal{H}^{m-1}(E \cap \partial B_r) + (\|g\|_\infty + \mu)|E \cap B_r|.$$

Letting $U(r) = |E \cap B_r|$ and using the isoperimetric inequality [83], we have

$$\begin{aligned} c(m)U(r)^{\frac{m-1}{m}} &\leq P(E \cap B_r) \\ &= \mathcal{H}^{m-1}(\partial^* E \cap B_r) + \mathcal{H}^{m-1}(\partial B_r \cap E) \\ &\leq 2\mathcal{H}^{m-1}(\partial B_r \cap E) + (\|g\|_\infty + \mu)U(r). \end{aligned}$$

Recalling that $\mathcal{H}^{m-1}(\partial B_r \cap E) = U'(r)$ for a.e. $r > 0$, we find

$$c(m)U(r)^{\frac{m-1}{m}} \leq 2U'(r) + (\|g\|_\infty + \mu)U(r). \tag{3.7}$$

The idea is that, when U is small, the term $U^{\frac{m-1}{m}}$ dominates the term which is linear in U so that we can get rid of it. Letting ω_m be the volume of the unit ball and $r \leq \omega_m^{-\frac{1}{m}} \left(\frac{c(m)}{2(\mu + \|g\|_\infty)}\right)$, we then have

$$U(r) \leq |B_r| = \omega_m r^m \leq \left(\frac{c(m)}{2(\mu + \|g\|_\infty)}\right)^m.$$

Raising each side of the inequality to the power $-\frac{1}{m}$ and multiplying by U we get

$$U(r)^{\frac{m-1}{m}} \geq \frac{2(\mu + \|g\|_\infty)}{c(m)} U$$

and from this

$$\frac{c(m)}{2} U(r)^{\frac{m-1}{m}} - (\mu + \|g\|_\infty) U \geq 0$$

thus finally

$$c(m) U(r)^{\frac{m-1}{m}} - (\mu + \|g\|_\infty) U \geq \frac{c(m)}{2} U(r)^{\frac{m-1}{m}}.$$

Putting this back in (3.7) and letting $C(m) = c(m)\omega_m^{-\frac{1}{m}}/2$ we have

$$\frac{c(m)}{4} U(r)^{\frac{m-1}{m}} \leq U'(r) \qquad \forall r \leq \frac{C(m)}{(\mu + \|g\|_\infty)}.$$

If we set $V(r) = U^{\frac{1}{m}}(r)$ we have

$$V'(r) = \frac{1}{m} U'(r) U^{\frac{1-m}{m}}(r) \geq \frac{c(m)}{4m}.$$

Integrating we get

$$V(r) \geq \frac{c(m)}{4m} r \qquad \text{hence} \qquad U(r) \geq \left(\frac{c(m)}{4m}\right)^m r^m.$$

The second inequality is obtained by repeating the argument with $E \cup B_r(x)$ instead of $E \backslash B_r(x)$. $\square$

We now estimate the error made by relaxing the constraint on the volume.

Lemma 3.2.2. *For every set of finite perimeter E and every $\mu > \|g\|_\infty$ we have*

$$\big| |E| - v \big| \leq \frac{F_\mu(E) + v\|g\|_\infty}{\mu - \|g\|_\infty}.$$

Proof. If $|E| > v$ we have

$$F_\mu(E) = P(E) - \int_E g + \mu(|E| - v)$$

thus

$$\mu(|E| - v) \le F_\mu(E) + \|g\|_\infty |E|$$

and from this we find

$$(\mu - \|g\|_\infty)(|E| - v) \le F_\mu(E) + v\|g\|_\infty.$$

Dividing by $\mu - \|g\|_\infty$ we get

$$\big||E| - v\big| \le \frac{F_\mu(E) + v\|g\|_\infty}{\mu - \|g\|_\infty}.$$

If $|E| \le v$ we similarly get

$$(\mu + \|g\|_\infty)(|E| - v) \le F_\mu(E) + v\|g\|_\infty$$

hence

$$\big||E| - v\big| \le \frac{F_\mu(E) + v\|g\|_\infty}{\mu + \|g\|_\infty} \le \frac{F_\mu(E) + v\|g\|_\infty}{\mu - \|g\|_\infty}.$$

□

We now prove that the minimizers do not depend on R, for R big enough. Here the periodicity of g is crucial.

Proposition 3.2.3. *For every $\mu > \|g\|_\infty$, there exists $R_0(\mu)$ such that for every $R \ge R_0$, there exists a minimizer E_R of* (3.6) *verifying diam$(E_R) \le R_0$. Equivalently we have*

$$\min_{E \subset Q_R} F_\mu(E) = \min_{E \subset Q_{R_0}} F_\mu(E)$$

for all $R \ge R_0$.

Proof. Let E_R be a minimizer of (3.6). Let Q be the unit square and

$$N = \sharp\{z \in \mathbb{Z}^m \ / \ |\{z + Q\} \cap E_R| \ne 0\}.$$

We want to bound N from above by a constant independent of R.

Let $r_0 = \frac{C(m)}{\mu + \|g\|_\infty}$ as in Proposition 3.2.1. For all $x \in E_R$ we have

$$|E_R \cap B_r(x)| \ge \gamma r^m \qquad \forall r \le r_0.$$

Letting $r_1 = \min(r_0, \frac{1}{2})$, for all $x \in \mathbb{R}^m$ we have

$$\sharp\{z \in \mathbb{Z}^m \ / \ \{z+Q\} \cap B_{r_1}(x) \neq \emptyset\} \leq 2^m.$$

Therefore, we can find at least $N/2^m$ points x_i in E_R such that $B_{r_1}(x_i) \cap B_{r_1}(x_j) = \emptyset$ for every $i \neq j$ and such that $x_i \in Q+z_i$ with $|\{z_i+Q\} \cap E_R| \neq 0$ for some $z_i \in \mathbb{Z}$.
We thus have

$$|E_R| \geq \sum_i |B_{r_1}(x_i) \cap E_R| \geq \frac{N}{2^m}\gamma r_1^m.$$

This gives us

$$N \leq \frac{2^m |E_R|}{\gamma r_1^m}.$$

Letting B^v be a ball of volume v, by Lemma 3.2.2 and $F_\mu(E_R) \leq F_\mu(B^v)$, we have

$$\begin{aligned} \big||E_R| - v\big| &\leq \frac{F_\mu(B^v) + v\|g\|_\infty}{\mu - \|g\|_\infty} \\ &\leq \frac{c(m)v^{\frac{m-1}{m}} + 2v\|g\|_\infty}{\mu - \|g\|_\infty}. \end{aligned}$$

This shows that

$$|E_R| \leq v + \frac{c(m)v^{\frac{m-1}{m}} + 2v\|g\|_\infty}{\mu - \|g\|_\infty}$$

so that N is bounded by a constant independent of R.

We now prove that $\mathrm{diam}(E_R) \leq C(m)N$. Indeed let $x \in E_R$ and let $P_0 = [0,1] \times [-R/2, R/2]^{m-1}$ be a slice of Q_R orthogonal to the direction e_1. For $i \in \mathbb{Z}$ we also set $P_i = P_0 + ie_1$. Our aim is showing that E_R is contained in a box of size N in the direction e_1. Up to translation we can suppose that $E_R \cap P_i = \emptyset$ for all $i < 0$. We want to show that we can choose $E_R \subset \bigcup_{0 \leq i \leq N} P_i$.

Let $I \leq R$ be the least integer such that $E_R \subset \bigcup_{0 \leq i \leq I} P_i$ and suppose $I \geq N$. Because of the definition of N, there is at most N slices P_i such that $P_i \cap E_R \neq \emptyset$. Hence there exists i between 0 and N such that $P_i \cap E_R = \emptyset$. Let $E_i^+ = \bigcup_{j>i} E_R \cap P_j$ and $E_i^- = \bigcup_{j<i} E_R \cap P_j$ then if we set $\widetilde{E}_R = E_i^- \cup \{E_i^+ - e_1\}$ we have $F_\mu(\widetilde{E}_R) = F_\mu(E_R)$ and $\widetilde{E}_R \subset \bigcup_{0 \leq i \leq I-1} P_i$ giving the claim by iterating the procedure (see Figure 3.1).

The same argument applies to any orthonormal direction e_k, hence $E_R \subset Q_{2N}$. □

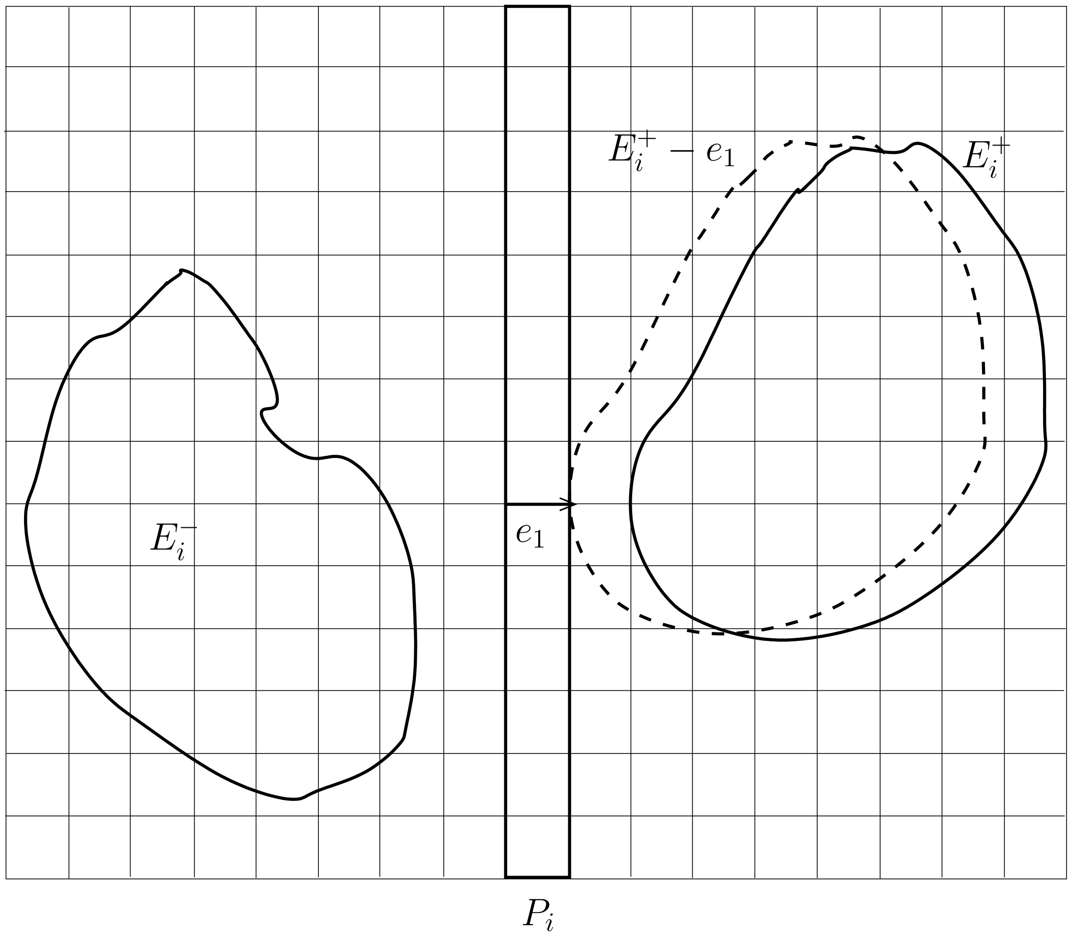

Figure 3.1: The construction in the proof of Proposition 3.2.3.

We now prove existence of minimizers for F_μ.

Proposition 3.2.4. *For $\mu > \|g\|_\infty$, there exists a bounded minimizer of F_μ. Moreover such minimizer has essential boundary $\partial^* E$ of class $\mathcal{C}^{2,\alpha}$, where α is the Hölder exponent of the function g. If we further assume that $m \leq 7$, then the singular part of ∂E is empty and thus E is smooth.*

Proof. By Proposition 3.2.3 there exists R_0 such that $E_R \subset B_{R_0}$ for every $R > 0$. Suppose now that there exists E with $F_\mu(E) < F_\mu(E_{R_0})$. Then there exists $\varepsilon > 0$ such that

$$F_\mu(E) + \varepsilon \leq F_\mu(E_{R_0}).$$

Let us show that there exists $R > R_0$ such that

$$F_\mu(E \cap B_R) + \frac{\varepsilon}{2} \le F_\mu(E_{R_0}).$$

We start by noticing that $|E \cap B_R|$ tends to $|E|$ and that $\int_{E\cap B_R} g\,dx$ tends to $\int_E g\,dx$ when $R \to +\infty$. On the other hand,

$$P(E \cap B_R) = \mathcal{H}^{m-1}(E \cap \partial B_R) + \mathcal{H}^{m-1}(\partial^* E \cap B_R)$$

and we have

$$\lim_{R\to+\infty} \mathcal{H}^{m-1}(\partial^* E \cap B_R) = P(E)$$

and

$$\lim_{R\to+\infty} \int_0^R \mathcal{H}^{m-1}(E \cap \partial B_s)ds = \lim_{R\to+\infty} |E \cap B_R| = |E|.$$

The last equality shows that $\mathcal{H}^{m-1}(E \cap \partial B_R)$ is integrable so that, for every $R > 0$, there exists $R' > R$ such that $\mathcal{H}^{m-1}(E \cap \partial B_{R'})$ is arbitrarily small. This implies that we can find a R large enough so that

$$F_\mu(E \cap B_R) + \frac{\varepsilon}{2} \le F_\mu(E_{R_0}).$$

The minimality of E_{R_0} yields to a contradiction.

We now focus on the regularity. Let E be a minimizer of F_μ then for every G,

$$P(E) - \int_E g\,dx + \mu\big||E| - v\big| \le P(G) - \int_G g\,dx + \mu\big||G| - v\big|.$$

Hence

$$\begin{aligned} P(E) &\le P(G) + \|g\|_\infty |E\Delta G| + \mu\big||E| - |G|\big| \\ &\le P(G) + (\|g\|_\infty + \mu)|E\Delta G|. \end{aligned}$$

E is thus a quasi-minimizer of the perimeter so that, by classical regularity theory [83] (see also [113]), we get that $\partial^* E$ is of class $\mathcal{C}^{2,\alpha}$. $\square$

Before stating the equivalence between the constrained and unconstrained problems, we prove a generalization of the Alexandrov-Fenchel inequality (see Schneider [126]) for smooth non convex sets which will be useful for us and, we believe, is of independent interest.

Lemma 3.2.5. *Let $E \subset \mathbb{R}^m$ be a compact set with $\mathcal{C}^2$ boundary, then*

$$\frac{m-1}{m} P(E)^2 \geq |E| \int_{\partial E} \kappa \, d\mathcal{H}^{m-1}. \tag{3.8}$$

Proof. Let $\varphi(t) = \left|(1-t)E + tB\right|^{\frac{1}{m}}$ where B is the unit ball. The function φ is concave by the Brunn-Minkowski inequality [35]. If we set

$$\psi(t) = |E + tB|.$$

We have

$$\varphi(t) = (1-t)\psi\left(\frac{t}{1-t}\right)^{\frac{1}{m}}.$$

We can now compute $\varphi''(0)$. The first derivative of φ is given by

$$\varphi'(t) = -\psi\left(\frac{t}{1-t}\right)^{\frac{1}{m}} + \frac{1}{m(1-t)}\psi'\left(\frac{t}{1-t}\right)\psi\left(\frac{t}{1-t}\right)^{\frac{1-m}{m}}.$$

Differentiating again we find

$$\varphi''(t) = \frac{1}{(1-t)^3}\psi''\left(\frac{t}{1-t}\right)\psi\left(\frac{t}{1-t}\right)^{\frac{1-m}{m}} + \frac{1-m}{m(1-t)^3}\psi'^2\left(\frac{t}{1-t}\right)\psi\left(\frac{t}{1-t}\right)^{\frac{1-2m}{m}},$$

which gives

$$\varphi''(0) = \frac{\psi(0)^{\frac{1-2m}{m}}}{m}\left(\psi''(0)\psi(0) - \frac{m-1}{m}\psi'^2(0)\right).$$

The concavity of φ thus implies

$$\psi''(0)\psi(0) \leq \frac{m-1}{m}\psi'^2(0).$$

As E is smooth, for t small we have

$$E + tB = E \cup \{\, x + s\nu(x) \,\text{with}\; x \in \partial E, s \in [0,t] \,\}$$

thus

$$|E + tB| = |E| + tP(E) + \frac{t^2}{2}\int_{\partial E} \kappa \, d\mathcal{H}^{m-1} + o(t^2).$$

This shows that $\psi'(0) = P(E)$ and $\psi''(0) = \displaystyle\int_{\partial E} \kappa \, d\mathcal{H}^{m-1}$ giving the desired result. $\square$

We are finally in position to prove existence of minimizers of problem (3.4).

Theorem 3.2.6. *Let $m \leq 7$, then for all $v > 0$ there exists a compact minimizer E_v of (3.4) with ∂E_v of class $\mathcal{C}^{2,\alpha}$. Moreover, E_v is also a minimizer of F_μ for all*

$$\mu \geq C_1(m)\|g\|_\infty + C_2(m)v^{-\frac{1}{m}} \tag{3.9}$$

where $C_1(m)$ and $C_2(m)$ are two positive constants depending only on m.

Proof. Letting E_μ be a bounded and smooth minimizer of F_μ, given by Proposition 3.2.4, We will show that $|E_\mu| = v$, for μ large enough. Let μ be larger than $\|g\|_\infty$ and suppose by contradiction $|E_\mu| \neq v$. Then, if $|E_\mu| > v$, the Euler-Lagrange equation for F_μ writes

$$\kappa_{E_\mu} = g - \mu$$

where κ_{E_μ} is the mean curvature of E_μ. But this is impossible since $\mu > \|g\|_\infty$, which would lead to $\kappa_{E_\mu} < 0$, contradicting the compactness of E_μ.

Thus for $\mu > \|g\|_\infty$, we have $|E_\mu| < v$ and

$$\kappa_{E_\mu} = g + \mu.$$

Using inequality (3.8) with $E = E_\mu$, and the fact that $|E_\mu| \geq v/2$ by Lemma 3.2.2, we get

$$\begin{aligned} F_\mu(E_\mu) &\geq \frac{m}{m-1}(\mu - \|g\|_\infty)|E_\mu| - \|g\|_\infty |E_\mu| \\ &\geq \frac{m}{m-1}(\mu - \|g\|_\infty)\frac{v}{2} - \|g\|_\infty v. \end{aligned}$$

On the other hand, $F_\mu(E_\mu) \leq F_\mu(B^v)$, where B^v is a ball of volume v, so that

$$C(m)v^{\frac{m-1}{m}} + \|g\|_\infty v \geq F_\mu(B^v) \geq \frac{m}{m-1}(\mu - \|g\|_\infty)\frac{v}{2} - \|g\|_\infty v$$

and we finally obtain

$$\mu \leq C_1(m)\|g\|_\infty + C_2(m)v^{-\frac{1}{m}}.$$

□

Remark 3.2.7. *The minimizer E_v satisfies the Euler-Lagrange equation*

$$\kappa_E = g + \lambda_v \qquad \text{with } |\lambda_v| \leq \mu,$$

where μ verifies (3.9)*. In particular, λ_v and thus also $\|\kappa_E\|_\infty$ are uniformly bounded in v, for $v \in [\varepsilon, +\infty)$.*

The regularity of ∂E_v also follows from the works of Rigot [121] and Xia [133] on quasi-minimizers of the perimeter with a volume constraint.

3.3 Properties of the isovolumetric function

We show here some of the properties of the isovolumetric f defined by (3.4).

Proposition 3.3.1. *The function f is sub-additive and locally Lipschitz continuous. Let v be a point of differentiability of f and E_v be a minimizer of (3.4) then $f'(v) = \lambda_v$ where λ_v is the Lagrange multiplier associated to E_v, that is, $\kappa_{E_v} = g + \lambda_v$. As a consequence, λ_v is unique for almost every $v > 0$, in the sense that it does not depend on the specific minimizer E_v.*

Proof. Let E_v and $E_{v'}$ be compact minimizers associated to v and v'. Up to a translation we can suppose that $F(E_v \cup E_{v'}) = F(E_v) + F(E_{v'})$, so that

$$f(v + v') \leq F(E_v \cup E_{v'}) = F(E_v) + F(E_{v'}) = f(v) + f(v')$$

and f is sub-additive.

By Theorem 3.2.6, for every $\alpha > 0$ there exists μ_α such that, for every $v \geq \alpha$, the constrained problem (3.4) and the relaxed one (3.5) are equivalent for $\mu \geq \mu_\alpha$. Let $v, v' \in [\alpha, +\infty)$, then

$$f(v) = F(E_v) \leq P(E_{v'}) - \int_{E_{v'}} g\,dx + \mu_\alpha|v - v'| = f(v') + \mu_\alpha|v - v'|$$

thus $|f(v) - f(v')| \leq \mu_\alpha|v - v'|$ and f is Lipschitz continuous on $[\alpha, +\infty)$.

We now compute the derivative of f. For $v, \varepsilon > 0$ we have

$$f(v + \varepsilon) - f(v) \leq F((1 + \varepsilon/v)^{\frac{1}{m}} E_v) - F(E_v).$$

Let $\delta_\varepsilon = (1 + \varepsilon/v)^{\frac{1}{m}} - 1$; then $(1 + \varepsilon/v)^{\frac{1}{m}} E_v = E_v + \delta_\varepsilon E_v$. Recalling that $\kappa_{E_v} = g + \lambda_v$ we get

$$\begin{aligned}
P((1+\delta_\varepsilon)E_v) &= P(E_v) + \delta_\varepsilon \int_{\partial E_v} \kappa_{E_v} x \cdot \nu \, d\mathcal{H}^{m-1} + o(\delta_\varepsilon) \\
&= P(E_v) + \delta_\varepsilon \int_{\partial E_v} g(x) x \cdot \nu \, d\mathcal{H}^{m-1} + \delta_\varepsilon \int_{\partial E_v} \lambda_v x \cdot \nu \, d\mathcal{H}^{m-1} + o(\delta_\varepsilon) \\
&= P(E_v) + \delta_\varepsilon \int_{\partial E_v} g(x) x \cdot \nu \, d\mathcal{H}^{m-1} + \delta_\varepsilon \lambda_v d|E_v| + o(\delta_\varepsilon)
\end{aligned}$$

and

$$\int_{(1+\delta_\varepsilon)E_v} g = \int_{E_v} g\,dx + \delta_\varepsilon \int_{\partial E_v} g(x) x \cdot \nu \, d\mathcal{H}^{m-1} + o(\delta_\varepsilon).$$

From this we obtain

$$F((1 + \varepsilon/v)^{\frac{1}{m}} E_v) - F(E_v) = \delta_\varepsilon v d \lambda_v + o(\delta_\varepsilon).$$

As $\delta_\varepsilon = \varepsilon/(vm) + o(\varepsilon)$, we find

$$\limsup_{\varepsilon\to 0^+} \frac{f(v+\varepsilon)-f(v)}{\varepsilon} \leq \lambda_v$$
$$\liminf_{\varepsilon\to 0^-} \frac{f(v+\varepsilon)-f(v)}{\varepsilon} \geq \lambda_v.$$

In particular, if f is differentiable in v we have

$$f'(v) = \lambda_v.$$

□

In fact, the isovolumetric function f is slightly more regular.

Proposition 3.3.2. *Let $\lambda_v^{\max}$ and $\lambda_v^{\min}$ be respectively the bigger and the smaller Lagrange multipliers associated with v then f has left and right derivatives in v and*

$$\lim_{h\to 0^+} \frac{f(v+h)-f(v)}{h} = \lambda_v^{\min} \leq \lambda_v^{\max} = \lim_{h\to 0^-} \frac{f(v+h)-f(v)}{h}. \tag{3.10}$$

The proof is based on the following lemma:

Lemma 3.3.3. *Let v_n be a sequence converging to v. Then there exist sets E_n with $|E_n| = v_n$ and*

$$f(v_n) = F(E_n),$$

and a set E with $|E| = v$ and

$$f(v) = F(E),$$

such that, up to extraction, E_n tends to E in the L^1-topology, ∂E_n tends to ∂E in the Hausdorff sense, and λ_n tends to λ, where λ_n (resp. λ) is the Lagrange multiplier corresponding to E_n (resp. to E).

Proof. By Theorem 3.2.6, we can find minimizers E_n of (3.4), with $|E_n| = v_n$. Moreover, by Proposition 3.2.3 we can assume that $E_n \subset B_R$ with R independent of n. Since $P(E_n)$ is uniformly bounded from above, it then follows that there exists a (not relabelled) subsequence of E_n converging in the L^1-topology to a set $E \subset B_R$ with volume $v = \lim_n v_n$. Moreover, by the lower-semi-continuity of the perimeter and the continuity of f, the set E verifies

$$f(v) = F(E).$$

Let us now prove that the convergence also occurs in the sense of Hausdorff.

Let $\varepsilon > 0$ be fixed and let $x \in E \cap \{y \ / \ d(y, \partial E) > \varepsilon\}$. If x is not in E_n then by Proposition 3.2.1 we have

$$|E_n \Delta E| \geq |B_\varepsilon(x) \backslash E_n| \geq \gamma \varepsilon^m.$$

This is impossible if n is big enough because $|E_n \Delta E|$ tends to zero. Similarly, we can show that for n big enough, all the points of $E^c \cap \{y \ / \ d(y, \partial E) > \varepsilon\}$ are outside E_n. This shows that $\partial E_n \subset \{y \ / \ d(y, \partial E) \leq \varepsilon\}$. Inverting the rôles of E_n and E, the same argument proves that $\partial E \subset \{y \ / \ d(y, \partial E_n) \leq \varepsilon\}$ giving the Hausdorff convergence of ∂E_n to ∂E. Now if λ_n is the Lagrange multiplier associated with E_n, it is uniformly bounded and we can extract a converging subsequence which converges to some $\lambda \in \mathbb{R}$.

To conclude the proof we must show that $\kappa_E = g + \lambda$. As proved for instance in [128], for every $x \in \partial E$ there exists $r > 0$ such that for n large enough the set $B_r(x) \cap \partial E_n$ is the graph of a function φ_n, and the set $B_r(x) \cap \partial E$ is the graph of a function φ, in a suitable coordinate system. We then have that φ_n tends uniformly to φ, as $n \to +\infty$, and

$$-\operatorname{div}\left(\frac{\nabla \varphi_n}{\sqrt{1 + |\nabla \varphi_n|^2}}\right) = g(x, \varphi_n(x)) + \lambda_n \tag{3.11}$$

for all n big enough. By elliptic regularity [38], we can pass to the limit in (3.11) and obtain that ϕ solves

$$-\operatorname{div}\left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}}\right) = \kappa_E = g(x, \varphi(x)) + \lambda.$$

□

Proof of Proposition 3.3.2. Let $v > 0$ and let

$$\lambda = \liminf_{\varepsilon \to 0+} f'(v + \varepsilon) \tag{3.12}$$

Notice that, for every $\varepsilon > 0$, there exists a $v_\varepsilon \in]v, v + \varepsilon[$ such that

$$f'(v_\varepsilon) \leq \frac{f(v + \varepsilon) - f(v)}{\varepsilon}. \tag{3.13}$$

From (3.13) we get

$$\lambda \leq \liminf_{\varepsilon \to 0+} \frac{f(v + \varepsilon) - f(v)}{\varepsilon}.$$

Let ε_n be a sequence realizing the infimum in (3.12) and let $E_n \subset B_R$ be a set of volume $v_n = v + \varepsilon_n$ such that

$$f(v_n) = F(E_n).$$

By Lemma 3.3.3 the sets E_n converge, up to a subsequence in the L^1-topology, to a limit set E, with $|E| = v$ and $\kappa_E = g + \lambda$, where $\lambda = \lim_n \lambda_n$. Reasoning as in Proposition 3.3.1, we see that

$$\liminf_{\varepsilon \to 0+} \frac{f(v+\varepsilon) - f(v)}{\varepsilon} \geq \lambda \geq \limsup_{\varepsilon \to 0^+} \frac{f(v+\varepsilon) - f(v)}{\varepsilon}$$

hence f admits a right derivative which is equal to λ_v^{min}. Analogously one can show that f has a left derivative equal to λ_v^{max}. □

Remark 3.3.4. *f is differentiable at any local minimum so that, if equation* (3.1) *has no solution, either f is increasing on $[0, +\infty)$, or there exists $\overline{v} > 0$ such that f is increasing on $[0, \overline{v}]$, decreasing on $[\overline{v}, +\infty)$, and is not differentiable at $\overline{v}$.*

We now give an example of a isovolumetric function f which has a point of nondifferentiability. It is not clear to which extent this is a generic phenomenon.

Example. Consider a periodic function g which is equal to 0 everywhere in the unit cell Q, except in the neighborhood of two points a and b. Around these points, g is taken to be equal to radial parabolas centered at the point, one parabola high and thin, and the other small and large (see Figure 3.2).

It is shown in [77] that, when the volume v is sufficiently small, the minimizer E_v is connected. Since the bound on v depends only on $\|g\|_\infty$, which can be fixed as small as we want, we can suppose that the minimizers E_v are connected and are located near a or b. By the isoperimetric inequality [83] we then get that E_v is a disk with volume v centered at a or b, and will be denoted by $D_v(a)$, $D_v(b)$, respectively.

Therefore, for small volumes the global minimizer is $D_v(a)$ and, once the equality

$$\int_{D_v(a)} g = \int_{D_v(b)} g$$

is attained, it switches to the disk $D_v(b)$. When this transition occurs, there is a jump singularity of the derivative f'.

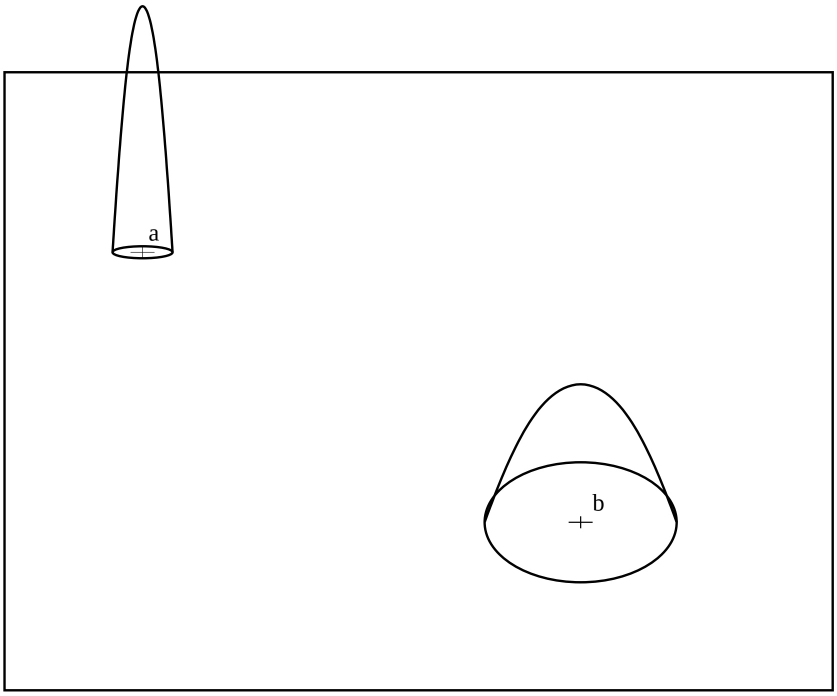

Figure 3.2: Example of a function f with a point of nondifferentiability.

3.4 Existence of surfaces with prescribed mean curvature

In this section we shall assume that g has zero average and satisfies

$$\int_E g \leq (1-\Lambda)P(E,Q) \qquad \forall E \subset Q \tag{3.14}$$

for some $\Lambda > 0$. Notice that (3.14) is always satisfied if $\|g\|_{L^m(Q)}$ is small enough, and is precisely the assumption needed in [52] (see also [37]) to prove existence of planelike minimizers of F. Notice also that, if g satisfies (3.14), then the inequality in (3.14) holds for all sets $E \subset \mathbb{R}^m$ of finite perimeter. In particular, this implies the following estimate on the function f:

$$c\, v^{\frac{m-1}{m}} \leq f(v) \leq C\, v^{\frac{m-1}{m}} \qquad \text{for some } 0 < c < C. \tag{3.15}$$

In the sequel we will need a representation result for the functional F, due to Bourgain and Brezis [30].

Theorem 3.4.1. *Let g be a function verifying* (3.14) *then there exists a periodic and continuous function σ with* $\max \sigma(x) < 1$ *satisfying* $\operatorname{div} \sigma = g$.

The energy F can thus be written as an anisotropic perimeter:

$$F(E) = \int_{\partial^* E} (1 + \sigma(x) \cdot \nu) \,.$$

Theorem 3.4.1 implies that

$$\Lambda P(E) \leq F(E) \leq 2P(E) \tag{3.16}$$

for all sets E of finite perimeter.

The next Lemma gives an upper bound on the number of “large” connected components of a volume-constrained minimizer.

Lemma 3.4.2. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function with zero average and satisfying* (3.14)*. Let E_v be a compact minimizer of* (3.4)*, and let E_i be the connected components of E_v. We can order the sets E_i in such a way that $|E_i|$ is decreasing in i. Given $\delta > 0$ let*

$$N_\delta = \left[1 + \left(\frac{C}{c} \right)^m \frac{1}{\delta^m} \right].$$

Then

$$\sum_{i=N_\delta}^{\infty} |E_i| \leq \delta v. \tag{3.17}$$

Proof. Let $x_i = \dfrac{|E_i|}{v} \in [0,1]$. Recalling (3.15), we have

$$c v^{\frac{m-1}{m}} \sum_{i=1}^{\infty} x_i^{\frac{m-1}{m}} \leq \sum_{i=1}^{\infty} f(|E_i|) = f(v) \leq C v^{\frac{m-1}{m}},$$

hence

$$\sum_{i=1}^{\infty} x_i^{\frac{m-1}{m}} \leq \frac{C}{c} \qquad \text{and} \qquad \sum_{i=1}^{\infty} x_i = 1.$$

Let now M be the smallest integer such that

$$\sum_{i=M+1}^{\infty} x_i < \delta,$$

we want to prove that $M < N_\delta$. Indeed, we have

$$\delta \leq \sum_{n=M}^{\infty} x_i = \sum_{n=M}^{\infty} x_i^{\frac{1}{m}} x_i^{\frac{m-1}{m}} \leq x_M^{\frac{1}{m}} \sum_{n=M}^{\infty} x_i^{\frac{m-1}{m}} \leq \frac{C}{c} x_M^{\frac{1}{m}}.$$

We then obtain

$$x_M \geq \left(\frac{c}{C}\right)^m \delta^m.$$

Hence, as

$$1 \geq \sum_{i=1}^{M} x_i \geq \sum_{i=1}^{M} x_M = M x_M,$$

by the decreasing property of x_i, we get

$$1 \geq M x_M \geq M \left(\frac{c}{C}\right)^m \delta^m,$$

which gives

$$M \leq \left(\frac{C}{c}\right)^m \frac{1}{\delta^m} < N_\delta.$$

□

3.4.1 Compact solutions with big volume.

From (3.15), Proposition 3.3.2 and Remark 3.3.4, we immediately obtain the following result.

Proposition 3.4.3. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function of zero average satisfying* (3.14). *Assume that $f'(v) \leq 0$ for some $v > 0$. Then there exists $w > 0$ such that $f'(w) = 0$, therefore problem* (3.1) *admits a compact solution.*

Theorem 3.4.4. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function with zero average and satisfying* (3.14). *There exist $v_n \to +\infty$ and compact minimizers E_n of* (3.4) *such that $|E_n| = v_n$ and E_n solves*

$$\kappa = g + \lambda_n$$

with $\lambda_n \geq 0$ and $\lambda_n \to 0$ as $n \to +\infty$.

Proof. Two situations can occur:
*C*ase 1. There exists a sequence $\tilde{v}_n \to +\infty$ such that $f'(\tilde{v}_n) \leq 0$. Recalling (3.15) we have $f(v) \geq c v^{\frac{m-1}{m}}$, which implies that we can find $v_n \geq \tilde{v}_n$ such that f has a local minimum in v_n, hence $\lambda_v = f'(v_n) = 0$.
*C*ase 2. There exists $v_0 > 0$ such that $f'(v) > 0$ for every $v \geq v_0$. By (3.15) we have $f(v) \leq C v^{\frac{m-1}{m}}$, and

$$f(v) = f(v_0) + \int_{v_0}^{v} f'(s)\, ds.$$

It follows that there exists a sequence $v_n \to +\infty$ such that

$$\lim_{n \to +\infty} f'(v_n) = 0.$$

□

Corollary 3.4.5. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function with zero average and satisfying* (3.14)*. Then for every $\varepsilon > 0$ there exists $\varepsilon' \in [0, \varepsilon]$ such that there exists a compact solution of*

$$\kappa = g + \varepsilon'.$$

Notice that for a general function g we cannot let $\varepsilon' = 0$ in Corollary 3.4.5. Indeed, as shown in [22], there are no compact solutions to (3.1) for periodic functions g, of zero average, which are translation invariant in some direction and of sufficiently small lipschitz norm.

We expect that condition (3.14) is not necessary for the thesis of Corollary 3.4.5 to hold, as suggested by the following result:

Theorem 3.4.6. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function with zero average and such that $g|_{\partial Q} = 0$. Then for every $\varepsilon > 0$ there exists a compact solution of*

$$\kappa = g + \varepsilon.$$

Proof. Fix $\varepsilon > 0$. For $N \in \mathbb{N}$ we let E_N be a minimizer of the problem

$$\min_{E \subset Q_N} P(E) - \int_E (g(x) + \varepsilon)\, dx.$$

Since $g|_{\partial Q} = 0$, by strong maximum principle, E_N is contained in the interior of Q_N and either $E_N = \emptyset$ or ∂E_N is a $\mathcal{C}^{2,\alpha}$ solution of $\kappa = g + \varepsilon$.

However, from the inequality

$$P(E_N) - \int_{E_N} (g(x) + \varepsilon)\, dx \leq P(Q_N) - \varepsilon N^m + = N^{m-1}\left(2^m - \varepsilon N\right) < 0$$

which holds for all $N > 2^m/\varepsilon$, it follows $E_N \neq \emptyset$. □

3.4.2 Asymptotic behavior of minimizers.

For $\varepsilon > 0$ and $E \subset \mathbb{R}^m$ of finite perimeter, we let

$$F_\varepsilon(E) = \varepsilon^{(m-1)} F\left(\varepsilon^{-1} E\right) = P(E) - \frac{1}{\varepsilon} \int_E g\left(\frac{x}{\varepsilon}\right)\, dx.$$

Notice that, given a minimizer E_v of (3.4), the set εE_v is a volume-constrained minimizer of F_ε. We recall from [52, Theorem 2] the following result.

Theorem 3.4.7. *Let g be a periodic $\mathcal{C}^{0,\alpha}$ function with zero average and satisfying* (3.14)*. Then there exists a convex positively one-homogeneous function $\phi_g : \mathbb{R}^m \to [0, +\infty)$, with $\phi_g(x) > 0$ for all $x \neq 0$, such that the* *functionals F_ε Γ-converge, with respect to the L^1-convergence of the characteristic functions, to the anisotropic functional*

$$F_0(E) = \int_{\partial^* E} \phi_g(\nu)\, d\mathcal{H}^{m-1} \qquad E \subset \mathbb{R}^m \text{ of finite perimeter.}$$

We remark that, with a minor modification of the proof, the result of Theorem 3.4.7 also holds if we restrict the functionals F_ε and F_0 to set of prescribed volume. In particular, by a general property of Γ-converging sequences [58], we have the following consequence of Theorem 3.4.7.

Corollary 3.4.8. *Let $\widetilde{E}_\varepsilon$ be minimizers of F_ε with volume constraint $|\widetilde{E}_\varepsilon| = v$, then*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\widetilde{E}_\varepsilon) \le \min_{|\widetilde{E}|=v} F_0(\widetilde{E}). \tag{3.18}$$

Moreover, if $|\widetilde{E}_\varepsilon \Delta \widetilde{E}| \to 0$ for some $\widetilde{E} \subset \mathbb{R}^m$, as $\varepsilon \to 0$, then $|\widetilde{E}| = v$ and $\widetilde{E}$ is a volume-constrained minimizer of F_0. More generally, if $\widetilde{E}_\varepsilon \to \widetilde{E}$ in the L^1_{loc} topology, then $\widetilde{E}$ is a minimizer of F_0 with volume constraint $|\widetilde{E}| \le v$.

Given the function ϕ_g as above, we let

$$W_g = \left\{ x \in \mathbb{R}^m : \max_{\phi_g(y)\le 1} x\cdot y \le 1 \right\}$$

be the Wulff Shape corresponding to ϕ_g. It is well-known that W_g is the unique minimizer of F_0 with volume constraint, up to homothety and translation [132, 129]. It is not very difficult to see [48] that

$$\phi_g(p) = \sup\left\{ \left(\int_Q \xi\right)\cdot p \;/\; \xi \in L^2(Q),\ \operatorname{div}\xi = g \ \text{ and } |\xi|_\infty \le 1 \right\}$$

from which it follows by standard calculus on polar functions that

$$W_g = \left\{ \int_Q \xi \;/\; \xi \in L^2(Q),\ \operatorname{div}\xi = g \ \text{ and } |\xi|_\infty \le 1 \right\}.$$

By Theorem 3.4.7 we can characterize the asymptotic shape of the constrained minimizers as the volume tend to infinity.

Theorem 3.4.9. *Let $m \le 7$. For $v > 0$ we let E_v be volume-constrained minimizers of* (3.4)*, whose existence is guaranteed by Theorem 3.2.6. Then, there exist points $z_v \in \mathbb{R}^m$ such that letting*

$$\widetilde{E}_v = \left(\frac{|W_g|}{v}\right)^{\frac{1}{m}} E_v + z_v$$

it holds

$$\lim_{v\to+\infty} \left|\widetilde{E}_v \Delta W_g\right| = 0. \tag{3.19}$$

Proof. Notice first that $\widetilde{E}_v$ is a minimizer of $F_{(\frac{|W_g|}{v})^{\frac{1}{m}}}$, with volume constraint $|\widetilde{E}_v| = |W_g|$. Moreover, by (3.15) the perimeter of $\widetilde{E}_v$ is uniformly bounded in v.

Case 1. Let us consider the case $m = 2$. Assume first that $\widetilde{E}_v$ is connected. Then we have

$$\mathrm{d}iam(\widetilde{E}_v) \leq P(\widetilde{E}_v)/\pi,$$

hence the sets $\widetilde{E}_v$ are all contained, up to a translation, in a fixed ball centered in the origin. By the compactness theorem for sets of finite perimeter [83], there exist a bounded set $\widetilde{E}_\infty$ of finite perimeter and a sequence $v_k \to \infty$ such that $|\widetilde{E}_\infty| = |W_g|$ and

$$\lim_{k\to+\infty} \left|\widetilde{E}_{v_k} \Delta \widetilde{E}_\infty\right| = 0.$$

Since by Theorem 3.4.7 the set $\widetilde{E}_\infty$ is also a volume-constrained minimizer of F_0, by uniqueness of the minimizer it follows that $\widetilde{E}_\infty$ is equal to W_g up to a translation.

We now consider the general case when the sets $\widetilde{E}_v$ are not necessarily connected. In particular we can write $\widetilde{E}_v = \cup_{i\geq 1} \widetilde{E}_v^i$, with $|\widetilde{E}_v^i|$ a decreasing sequence and $\sum_{i\geq 1} |\widetilde{E}_v^i| = 1$. Reasoning as before, there exists a sequence $v_k \to +\infty$ such that for all $i \in \mathbb{N}$ the sets $\widetilde{E}_{v_k}^i$ converge to $\rho_i W_g$, up to a translation, where $\rho_i \in [0,1]$ is a decreasing sequence. Moreover, by Lemma 3.4.2, for all $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that $\sum_{i=N_\delta}^\infty |\widetilde{E}_v^i| \leq \delta |W_g|$ for all $\delta > 0$, which implies in the limit

$$\sum_{i=1}^\infty \rho_i^2 = 1. \tag{3.20}$$

We claim that $\rho_1 = 1$ and $\rho_i = 0$ for all $i > 1$. Indeed, from (3.18) we have

$$F_0(W_g) \geq \limsup_{k\to+\infty} F_{\left(\frac{|W_g|}{v_k}\right)^{\frac{1}{2}}} (\widetilde{E}_{v_k}) \geq \sum_{i=1}^{+\infty} F_0(\rho_i W_g) = F_0(W_g) \sum_{i=1}^{+\infty} \rho_i\,.$$

Recalling (3.20), this implies

$$\sum_{i=1}^{+\infty} \rho_i = \sum_{i=1}^{+\infty} \rho_i^2 = 1$$

which proves the claim.

Case 2. We now turn to the general case. Let $v_k \to +\infty$ and let $\varepsilon_k = (|W_g|/v_k)^{\frac{1}{d}}$. For all k, let $\{Q_{i,k}\}_{i\in\mathbb{N}}$ be a partition of $\mathbb{R}^m$ into disjoint cubes of equal volume larger than $2|W_g|$, such that the sets $\widetilde{E}_{v_k} \cap Q_{i,k}$ are of decreasing measure, and let $x_{i,k} = |\widetilde{E}_{v_k} \cap Q_{i,k}|/|W_g|$. By the isoperimetric inequality [83], there exist $0 < c < C$ such that

$$\begin{aligned} c\sum_i x_{i,k}^{\frac{m-1}{m}} &= c\sum_i \min\left(\frac{|\widetilde{E}_{v_k}\cap Q_{i,k}|}{|W_g|}, \frac{|Q_{i,k}\backslash \widetilde{E}_{v_k}|}{|W_g|}\right)^{\frac{m-1}{m}} \\ &\leq \sum_i P(\widetilde{E}_{v_k}, Q_{i,k}) \\ &\leq \sum_i \frac{1}{\Lambda}\int_{\partial\widetilde{E}_{v_k}\cap Q_{i,k}} \left(1+\sigma\left(\frac{x}{\varepsilon_k}\right)\cdot\nu\right) d\mathcal{H}^{m-1} \\ &\leq \frac{1}{\Lambda}F_{\varepsilon_k}(\widetilde{E}_{v_k}) \leq C \end{aligned}$$

hence

$$\sum_{i=1}^{+\infty} x_{i,k} = 1 \qquad \textit{and} \qquad \sum_{i=1}^{+\infty} x_{i,k}^{\frac{m-1}{m}} \leq \frac{C}{c}.$$

Reasoning as in Lemma 3.4.2 we obtain that for all $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that

$$\sum_{i=N_\delta}^{\infty} x_{i,k} \leq \delta. \tag{3.21}$$

Up to extracting a subsequence, we can suppose that $x_{i,k} \to \alpha_i^m \in [0,1]$ as $k \to +\infty$ for every $i \in \mathbb{N}$, so that by (3.21) we have

$$\sum_i \alpha_i^m = 1. \tag{3.22}$$

Let $z_{i,k} \in Q_{i,k}$. Up to extracting a further subsequence, we can suppose that $d(z_{i,k}, z_{j,k}) \to c_{ij} \in [0, +\infty]$, and

$$\left(\widetilde{E}_{v_k} - z_{i,k}\right) \to E_i \quad \text{in the } L^1_{\text{loc}}\text{-convergence}$$

for every $i \in \mathbb{N}$ (see Figure 3.3). By Corollary 3.4.8 we thus have

$$E_i = \rho_i W_g \qquad \rho_i \in [0,1].$$

We say that $i \sim j$ if $c_{ij} < +\infty$ and we denote by $[i]$ the equivalence class of i. Notice that E_i equals E_j up to a traslation, if $i \sim j$. We want to prove that

$$\sum_{[i]} \rho_i^m \geq 1, \tag{3.23}$$

where the sum is taken over all equivalence classes. For all $R > 0$ let $Q_R = [-R/2, R/2]^m$ be the cube of sidelength R. Then for every $i \in \mathbb{N}$,

$$|E_i| \geq |E_i \cap Q_R| = \lim_{k \to +\infty} \left| \left(\widetilde{E}_{v_k} - z_{i,k} \right) \cap Q_R \right|.$$

If j is such that $j \sim i$ and $c_{ij} \leq \frac{R}{2}$, possibly increasing R we have $Q_{j,k} - z_{i,k} \subset Q_R$ for all $k \in \mathbb{N}$, so that

$$\lim_{k \to +\infty} \left| \left(\widetilde{E}_{v_k} - z_{i,k} \right) \cap Q_R \right| \geq \lim_{k \to +\infty} \sum_{c_{ij} \leq \frac{R}{2}} |\widetilde{E}_{v_k} \cap Q_{j,k}| = \sum_{c_{ij} \leq \frac{R}{2}} \alpha_j^m |W_g|.$$

Letting $R \to +\infty$ we then have

$$|E_i| \geq \sum_{i \sim j} \alpha_j^m |W_g|$$

hence, recalling (3.22),

$$\sum_{[i]} |E_i| \geq |W_g|,$$

thus proving (3.23).

Let us now show that

$$\sum_{[i]} \rho_i^{m-1} = 1. \tag{3.24}$$

Up to passing to a subsequence, from now on we shall assume that $c_{ij} = +\infty$ for all $i \neq j$. Let $I \in \mathbb{N}$ be fixed. Then for every $R > 0$ there exists $K \in \mathbb{N}$ such that for every $k \geq K$ and i, j less than I, we have

$$d(z_{i,k}, z_{j,k}) > R.$$

For $k \geq K$ we thus have

$$\begin{aligned}
F_{\varepsilon_k}(\widetilde{E}_{v_k}) \geq & \sum_{i=1}^{I} \int_{\partial \widetilde{E}_{v_k} \cap (B_R + z_{i,k})} \left(1 + \sigma \left(\frac{x}{\varepsilon_k} \right) \cdot \nu \right) d\mathcal{H}^{m-1} \\
= & \sum_{i=1}^{I} \int_{\partial (\widetilde{E}_{v_k} - z_{i,k}) \cap B_R} \left(1 + \sigma \left(\frac{x}{\varepsilon_k} \right) \cdot \nu \right) d\mathcal{H}^{m-1} \\
= & \sum_{i=1}^{I} F_{\varepsilon_k}(\widetilde{E}_{v_k} - z_{i,k}, B_R)
\end{aligned}$$

where

$$F_\varepsilon(E, B_R) = \int_{\partial E \cap B_R} \left(1 + \sigma \left(\frac{x}{\varepsilon_k} \right) \cdot \nu \right) d\mathcal{H}^{m-1}.$$

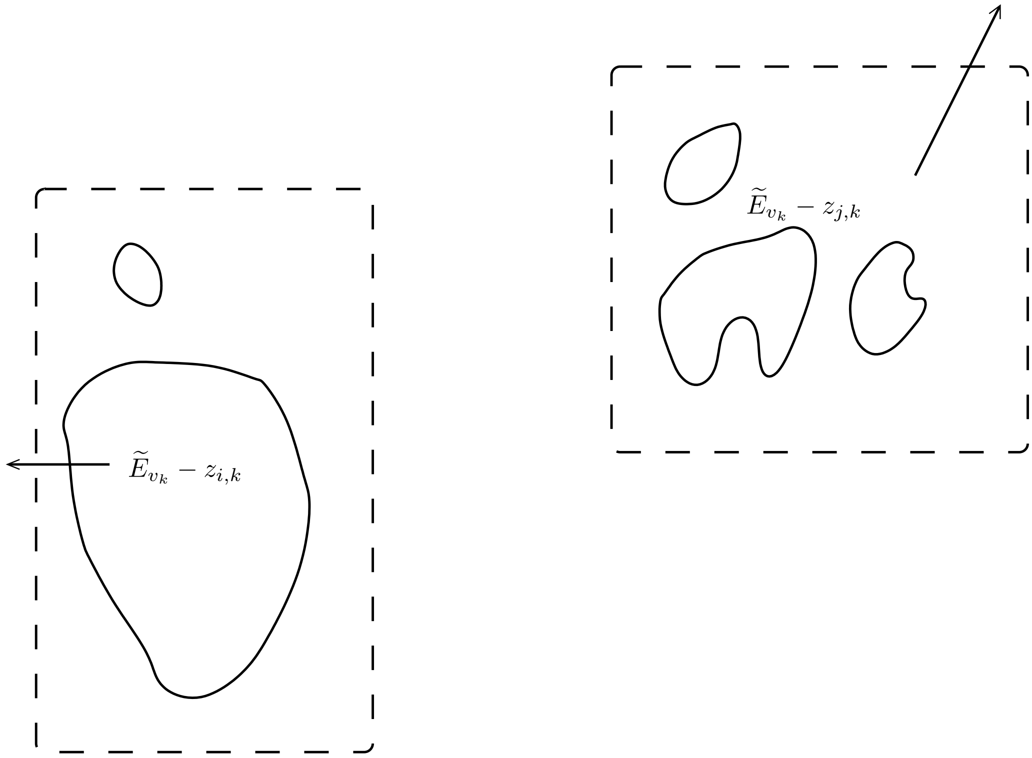

Figure 3.3: The construction in the proof of Theorem 3.4.9.

From this, (3.18) and the Γ-convergence of $F_\varepsilon(\cdot, B_R)$ to $F_0(\cdot, B_R)$, we get

$$F_0(W_g) \geq \limsup_{\varepsilon_k \to 0} F_{\varepsilon_k}(\widetilde{E}_{v_k}) \geq \sum_{i=1}^{I} \liminf_{\varepsilon_k \to 0} F_{\varepsilon_k}(\widetilde{E}_{v_k} - z_{i,k}, B_R) \geq \sum_{i=1}^{I} F_0(E_i, B_R).$$

For $R > \mathrm{d}iam(W_g)$ we have $F_0(E_i, B_R) = F_0(E_i)$ because $E_i = \rho_i W_g$ and therefore

$$F_0(W_g) \geq \sum_{i=1}^{I} F_0(E_i) = \sum_{i=1}^{I} \rho_i^{m-1} F_0(W_g).$$

Letting $I \to +\infty$ we get (3.24).

Recalling (3.23), from (3.24) we then obtain

$$\sum_i \rho_i^{m-1} = \sum_i \rho_i^m = 1.$$

As before, this implies $\rho_1 = 1$ and $\rho_i = 0$ for all $i > 1$, thus giving

$$\lim_{k \to +\infty} \left| \left(\widetilde{E}_{v_k} - z_{1,k} \right) \Delta W_g \right| = 0.$$

By the uniqueness of the limit this shows that the whole sequence $\widetilde{E}_v$ tends to W_g as $v \to +\infty$, up to suitable translations. □

Remark 3.4.10. *Let us point out that, if uniform density estimates for $\widetilde{E}_v$ were available, we would get Hausdorff convergence instead of L^1 convergence in* (3.19)*, showing in particular that the sets $\widetilde{E}_v$ are connected for v large enough (see [114]). We believe that such estimates are true even if we were not able to prove them.*

Remark 3.4.11. *The asymptotic behavior of minimizers of* (3.4)*, in the small volume regime, have been considered in [77], where the authors prove a result similar to Theorem 3.4.9, with the Wulff Shape W_g replaced by the Euclidean ball, showing in particular that the volume term becomes irrelevant for small volumes.*

Remark 3.4.12. *Notice that the results of this chapter can be extended with minor modifications of the proofs to anisotropic perimeters of the form*

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu) d\mathcal{H}^{m-1}$$

where $\phi : \mathbb{R}^m \to [0, +\infty)$ is a smooth and uniformly convex norm on $\mathbb{R}^m$, with $m \leq 3$ [4].

3.5 Conclusion and perspectives

In this chapter we used a variational approach to prove existence of closed hypersurfaces of prescribed mean curvature in periodic media up to an arbitrary small L^∞ term. We also characterized the limit of these sets when their volume goes to infinity as Wulff shapes for anisotropic perimeters. However, this work leaves some questions open.

First, we can wonder if it is possible to prove existence of the volume constrained minimizers E_v independently of the dimension of the ambiant space. It is very unlikely that this existence should depend on the regularity of the sets E_v. Our approach seems also to fail covering the case of g with positive mean which intuitively looks easier. An other interesting question, raised in Remark 3.4.10, is the study of the Hausdorff convergence of the rescaled sets $\widetilde{E}_v$ towards the Wulff shape. A question raised to us by Buttazzo is whether the limiting anisotropy ϕ_g can ever be in fact isotropic. This question has strong links with problems of homogenization of Riemannian metrics and weak KAM theory. Finally, we can wonder if an analogous approach can lead to the construction of approximated bump solutions of the forced Allen-Cahn equation

$$-\Delta u + W'(u) = g.$$

Chapitre 4

Approximation and relaxation of perimeter in the Wiener space

Abstract

In this chapter, we characterize the relaxation of the perimeter in an infinite dimensional Wiener space, with respect to the weak L^2-topology. We also show that the rescaled Allen-Cahn functionals approximate this relaxed functional in the sense of Γ-convergence.

Résumé

Dans ce chapitre nous calculons la relaxée du périmètre dans un espace de Wiener de dimension infinie. Celle-ci se trouve être une fonctionnelle bien connue de certains probabilistes. Nous démontrons également un résultat de Γ-convergence pour la fonctionnelle d'Allen-Cahn correspondante.

4.1 Introduction

Extending the variational methods and the geometric measure theory from the Euclidean to the Wiener space has recently attracted a lot of attention. In particular, the theory of functions of bounded variation in infinite dimensional spaces started with the works by Fukushima and Hino [79, 80]. Since then, the fine properties of BV functions and sets of finite perimeter have been investigated in [12, 13, 9, 8]. We point out that this theory is closely related to older works by M. Ledoux and P. Malliavin [100, 106].
In the Euclidean setting it is well-known that the perimeter can be approximated by means of more regular functionals of the form

$$\int \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dx$$

when ε tends to zero, in the sense of Γ-convergence with respect to the strong L^1-topology [110, 109]. An important ingredient in this proof is the compact embedding of BV in L^1.
A natural question is whether a similar approximation property holds in the infinite dimensional case. The main goal of this chapter is answering to this question by computing the Γ-limit, as $\varepsilon \to 0$, of the Allen-Cahn-type functionals (see Section 4.2 for precise definitions)

$$F_\varepsilon(u) = \int_X \left(\frac{\varepsilon}{2} |\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon} \right) d\gamma.$$

In the Wiener space there are two possible definitions of gradient, and consequently two different notions of Sobolev spaces, functions of bounded variation and perimeters [12, 8]. In one definition the compact embedding of $BV_\gamma(X)$ in $L^1_\gamma(X)$ still holds [12, Th. 5.3] and the Γ-limit of F_ε is, as expected, the perimeter up to a multiplicative constant. We do not reproduce here the proof of this fact, since it is very similar to the Euclidean one.
A more interesting situation arises when we consider the other definition of gradient, which gives rise to a more invariant notion of perimeter and is therefore commonly used in the literature [79, 80, 12]. In this case, the compact embedding of $BV_\gamma(X)$ in $L^1_\gamma(X)$ does not hold anymore. In particular sequences with uniformly bounded F_ε-energy are not generally compact in the (strong) L^1_γ-topology, even though they are bounded in $L^2_\gamma(X)$, and hence compact with respect to the weak $L^2_\gamma(X)$-topology. This suggests that the right topology for considering the Γ-convergence should rather be the weak $L^2_\gamma(X)$-topology.

A major difference with the finite dimensional case is the fact that the perimeter function defined by

$$F(u) = \begin{cases} P_\gamma(E) & \text{if } u = \chi_E \\ +\infty & \text{otherwise} \end{cases}$$

is no longer lower semicontinuous in this topology, and therefore cannot be the Γ-limit of the functionals F_ε. The problem is that the sets of finite perimeter are not closed under weak convergence of the characteristic functions. However, it is possible to compute the relaxation $\overline{F}$ of F (Theorem 4.4.4), which reads:

$$\overline{F}(u) = \begin{cases} \displaystyle\int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2}\, d\gamma & \text{if } 0 \le u \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Such functional is quite familiar to people studying log–Sobolev and isoperimetric inequalities in Wiener spaces [21, 26, 39].
Our main result is to show that the Γ-limit of F_ε, with respect to the weak $L^2_\gamma(X)$-topology, is a multiple of $\overline{F}$ (Theorem 4.5.3). The proof relies on the interplay between symmetrization, semicontinuity and isoperimetry.

The plan of the chapter is the following. In Section 4.2 we recall some basic facts about Wiener spaces and functions of bounded variation. In Section 4.3 we give the main properties of the Ehrhard symmetrizations. We also prove a Pólya-Szegö inequality and a Bernstein-type result in the Wiener space (Propositions 4.3.12 and 4.3.5), which we believe to be interesting in themselves. In Section 4.4, we use the Ehrhard symmetrization to compute the relaxation of the perimeter (Theorem 4.4.4). Finally, in Section 4.5 we compute the Γ-limit of the functionals F_ε (Theorem 4.5.3) and discuss some consequences of this result.

The results of this chapter are contained in a joint work with M. Novaga [89].

4.2 Wiener space and functions of bounded variation

A clear and comprehensive reference on the Wiener space is the book by Bogachev [27] (see also [106]). We follow here closely the notation of [12]. Let X be a separable Banach space and let X^* be its dual. We say that X

is a Wiener space if it is endowed with a non-degenerate centered Gaussian probability measure γ. That amounts to say that γ is a probability measure for which $x^*\sharp\gamma$ is a centered Gaussian measure on $\mathbb{R}$ for every $x^* \in X^*$. The non-degeneracy hypothesis means that γ is not concentrated on any proper subspace of X.
As a consequence of Fernique's Theorem [27, Th. 2.8.5], for every $x^* \in X^*$, the function $R^*x^*(x) = \langle x^*, x\rangle$ is in $L^2_\gamma(X) = L^2(X,\gamma)$. Let $\mathcal{H}$ be the closure of R^*X^* in $L^2_\gamma(X)$; the space $\mathcal{H}$ is usually called the reproducing kernel Hilbert space of γ. Let R, the operator from $\mathcal{H}$ to X, be the adjoint of R^* that is, for $\hat{h} \in \mathcal{H}$,

$$R\hat{h} = \int_X x\hat{h}(x)\,d\gamma$$

where the integral is to be intended in the Bochner sense. It can be shown that R is a compact and injective operator. We will let $Q = RR^*$so that for every x^*, $y^* \in X^*$,

$$\langle Qx^*, y^*\rangle = \int_X \langle x^*, x\rangle\langle y^*, x\rangle\ d\gamma.$$

We denote by H the space $R\mathcal{H} \subset X$. This space is called the Cameron-Martin space. It is a separable Hilbert space with the scalar product given by

$$[h_1, h_2]_H = \langle \hat{h}_1, \hat{h}_2\rangle_{L^2_\gamma(X)}$$

if $h_i = R\hat{h}_i$. We will denote by $|\cdot|_H$ the norm in H. The space H is a dense subspace of X, with compact embedding, and $\gamma(H) = 0$ if X is of infinite dimension.
For $x_1^*, .., x_m^* \in X^*$ we denote by $\Pi_{x_1^*,..,x_m^*}$ the projection from X to $\mathbb{R}^m$ given by

$$\Pi_{x_1^*,..,x_m^*}(x) = (\langle x_1^*, x\rangle, .., \langle x_m^*, x\rangle).$$

We will also denote it by Π_m when specifying the points x_i^* is unnecessary. Two elements x_1^* and x_2^* of X^* will be called orthonormal if the corresponding $h_i = Qx_i^*$ are orthonormal in H. We will fix in the following an orthonormal basis of H given by $h_i = Qx_i^*$.
We also denote by $H_m = \mathrm{span}(h_1, .., h_m) \simeq \mathbb{R}^m$ and $X_m^\perp = \mathrm{Ker}(\Pi_m) = \overline{H_m^\perp}^X$, so that $X \cong \mathbb{R}^m \oplus X_m^\perp$. The map Π_m induces the decomposition $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m, $\gamma_m^\perp$ Gaussian measures on $\mathbb{R}^m$, $X_m^\perp$ respectively.

Proposition 4.2.1 ([27]). *Let $\hat{h}_1, .., \hat{h}_m$ be in $\mathcal{H}$ then the image measure of γ under the map*

$$\Pi_{\hat{h}_1,..,\hat{h}_m}(x) = (\hat{h}_1(x), .., \hat{h}_m(x))$$

is a Gaussian in $\mathbb{R}^m$*. If the* $\hat{h}_i$ *are orthonormal, then such measure is the standard Gaussian measure on* $\mathbb{R}^m$*.*

Given $u \in L^2_\gamma(X)$, we will consider the canonical cylindrical approximation $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X_m^\perp} u(\Pi_m(x), y)\, d\gamma_m^\perp(y).$$

Notice that $\mathbb{E}_m u$ is a cylindrical functions depending only on the first m variables, and $\mathbb{E}_m u$ converges to u in $L^2_\gamma(X)$.
We will denote by $\mathcal{FC}_b^1(X)$ the space of cylindrical $\mathcal{C}^1$ bounded functions that is the functions of the form $v(\Pi_m(x))$ with v a $\mathcal{C}^1$ bounded function from $\mathbb{R}^m$ to $\mathbb{R}$. We denote by $\mathcal{FC}_b^1(X, H)$ the space generated by all functions of the form Φh, with $\Phi \in \mathcal{FC}_b^1(X)$ and $h \in H$.
We now give the definitions of gradients, Sobolev spaces functions of bounded variation. Given $u : X \to \mathbb{R}$ and $h = R\hat{h} \in H$, we define

$$\frac{\partial u}{\partial h}(x) = \lim_{t \to 0} \frac{u(x + th) - u(x)}{t}$$

whenever the limit exists, and

$$\partial_h^* u = \frac{\partial u}{\partial h} - \hat{h} u.$$

We define $\nabla_H u : X \to H$, the gradient of u by

$$\nabla_H u = \sum_{i=1}^{+\infty} \frac{\partial u}{\partial h_i} h_i$$

and the divergence of $\Phi : X \to H$ by

$$\operatorname{div}_\gamma \Phi = \sum_{i=1}^{+\infty} \partial_{h_i}^* [\Phi, h_i]_H.$$

The operator $\operatorname{div}_\gamma$ is the adjoint of the gradient so that for every $u \in \mathcal{FC}_b^1(X)$ and every $\Phi \in \mathcal{FC}_b^1(X, H)$, the following integration by parts holds:

$$\int_X u \operatorname{div}_\gamma \Phi \, d\gamma = -\int_X [\nabla_H u, \Phi]_H d\gamma. \tag{4.1}$$

The ∇_H operator is thus closable in $L^2_\gamma(X)$ and we will denote by $H^1_\gamma(X)$ its closure in $L^2_\gamma(X)$. From this, formula (4.1) still holds for $u \in H^1_\gamma(X)$ and $\Phi \in \mathcal{FC}_b^1(X, H)$. Analogously, we define the Sobolev spaves $W^{1,p}_\gamma(X)$ for $p \geq 1$ (these spaces are denoted by $\bar{D}^{1,p}(X, \gamma)$ in [12]).

Following [79, 12], given $u \in L^1_\gamma(X)$ we say that $u \in BV_\gamma(X)$ if

$$\int_X |D_\gamma u|_H = \sup\left\{\int_X u \operatorname{div}_\gamma \Phi \, d\gamma;\ \Phi \in \mathcal{FC}^1_b(X,H),\ |\Phi|_H \le 1\ \forall x \in X\right\} < +\infty.$$

We will also denote by $|D_\gamma u|(X)$ the total variation of u. If $u = \chi_E$ is the characteristic function of a set E we will denote $P_\gamma(E)$ its total variation and say that E is of finite perimeter if $P_\gamma(E)$ is finite. As shown in [12] we have the following properties of $BV_\gamma(X)$ functions.

Theorem 4.2.2. *Let $u \in BV_\gamma(X)$ then the following properties hold:*

- *$D_\gamma u$ is a countably additive measure on X with finite total variation and values in H (we will note the space of these measures by $\mathcal{M}(X,H)$), such that for every $\Phi \in \mathcal{FC}^1_b(X)$ we have:*

$$\int_X u\, \partial^*_{h_j} \Phi \, d\gamma = -\int_X \Phi d\mu_j \qquad \forall j \in \mathbb{N}$$

  where $\mu_j = [h_j, D_\gamma u]_H$.

- $|D_\gamma u|(X) = \inf \underline{\lim}\{\int_X |\nabla_H u_i|_H d\gamma \ :\ u_j \in H^1_\gamma(X),\ u_j \to u \textit{ in } L^1_\gamma(X)\}$.

We next introduce the the Ornstein-Uhlenbeck semigroup. Let $u \in L^1_\gamma(X)$ then

$$T_t u(x) := \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y).$$

Proposition 4.2.3. *The Ornstein-Uhlenbeck semigroup satisfies:*

- *if $u \in L^1_\gamma(X)$ then $T_t u \in W^{1,1}_\gamma(X)$,*
- *if $u \in L^p_\gamma(X)$ then $T_t u$ converges in $L^p_\gamma(X)$ to u when t goes to zero,*
- *for every $\Phi \in \mathcal{FC}^1_b(X,H)$, and $u \in L^2_\gamma(X)$,*

$$\int_X T_t u \operatorname{div}_\gamma \Phi \, d\gamma = e^{-t} \int_X u \operatorname{div}_\gamma T_t \Phi \, d\gamma, \tag{4.2}$$

- *if $\Phi \in \mathcal{FC}^1_b(X,H)$ then $T_t\Phi \in \mathcal{FC}^1_b(X,H)$*
- *for every convex function $F : H \to \mathbb{R}$, and every Φ,*

$$\int_X F(T_t\Phi) d\gamma \le \int_X F(\Phi)\, d\gamma.$$

Remark 4.2.4. *Notice that (4.2) holds more generally for u in the Orlicz space $L \log^{\frac{1}{2}} L$ but not for a general u in $L^1_\gamma(X)$ (see [12]).*

Proposition 4.2.5. *Let $u = v(\Pi_m)$ be a cylindrical function then $u \in BV_\gamma(X)$ if and only if $v \in BV_{\gamma_m}(\mathbb{R}^m)$. We then have*

$$\int_X |D_\gamma u|_H = \int_{\mathbb{R}^m} |D_{\gamma_m} v|.$$

Proposition 4.2.6 (Coarea formula [10])**.** *If $u \in BV_\gamma(X)$ then for every borel set $B \subset X$,*

$$|D_\gamma u|(B) = \int_{\mathbb{R}} P_\gamma(\{u > t\}, B)\, dt. \tag{4.3}$$

In Proposition 4.3.12, we will need the following extension of Proposition 4.2.6.

Lemma 4.2.7. *For every function $u \in BV_\gamma(X)$ and every non-negative Borel function g,*

$$\int_X g(x)\, d|D_\gamma u|(x) = \int_{\mathbb{R}} \left(\int_X g(x)\, d|D_\gamma \chi_{E_t}|(x) \right) dt \tag{4.4}$$

where $E_t := \{u > t\}$.

Proof. The proof of this lemma mimic the standard proof in the Euclidean case [55, Th.2.2]. By [72, Ch.1,Th.7] we can write g as

$$g = \sum_{i=1}^{+\infty} \frac{1}{i} \chi_{A_i}$$

where the $A_i \subset X$ are Borel sets. Using the coarea formula (4.3), we then get

$$\begin{aligned}
\int_X g(x) d|D_\gamma u|(x) &= \sum_{i=1}^{+\infty} \frac{1}{i} |D_\gamma u|(A_i) \\
&= \sum_{i=1}^{+\infty} \frac{1}{i} \int_{\mathbb{R}} |D_\gamma \chi_{E_t}|(A_i)\, dt \\
&= \int_{\mathbb{R}} \left(\int_X \sum_{i=1}^{+\infty} \frac{1}{i} \chi_{A_i} d|D_\gamma \chi_{E_t}|(x) \right) dt \\
&= \int_{\mathbb{R}} \int_X g(x)\, d|D_\gamma \chi_{E_t}|(x)\, dt.
\end{aligned}$$

□

In [12] it is also shown that sets with finite Gaussian perimeter can be approximated by smooth cylindrical sets.

Proposition 4.2.8. *Let $E \subset X$ be a set of finite Gaussian perimeter then there exists smooth sets $E_m \subset \mathbb{R}^m$ such that $\Pi_m^{-1}(E_m)$ converges in $L^1_\gamma(X)$ to E and $P_\gamma(\Pi_m^{-1}(E_m)) = P_{\gamma_m}(E_m)$ converges to $P_\gamma(E)$ when m tends to infinity.*

Note that, for half-spaces, the perimeter can be exactly computed [12, Cor. 3.11].

Proposition 4.2.9. *Let $h = R\hat{h} \in H$ and $c \in \mathbb{R}$ then the half-space*

$$E = \{x \in X \ : \ \hat{h}(x) \leq c\}$$

has perimeter

$$P_\gamma(E) = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2|h|_H^2}}.$$

The following result can be found in [27, Corollary 4.4.2].

Proposition 4.2.10. *Let u be a convex function from X to $\mathbb{R} \cup \{+\infty\}$, let $F(t) = \gamma(\{u \leq t\})$ and $t_0 = \inf\{t \ : \ F(t) > 0\}$, then F is continuous on $\mathbb{R}\backslash\{t_0\}$. As a consequence $\gamma(\{u = t\}) = 0$ for every $t \neq t_0$.*

In the finite dimensional setting, we will keep the same notations as in the infinite dimensional one. Notice that in $\mathbb{R}^m$, the following equality holds:

$$\operatorname{div}_\gamma \Phi = \operatorname{div} \Phi - \langle x, \Phi \rangle.$$

We see that functions in $BV_{\gamma_m}(\mathbb{R}^m)$ are in $BV_{\text{loc}}(\mathbb{R}^m)$ and that $D_{\gamma_m} u = \gamma Du$ so that most of the properties of classical BV functions extend to $BV_{\gamma_m}(\mathbb{R}^m)$ (see [10]).

4.3 The Ehrhard symmetrization

The Ehrhard symmetrization has been introduced by Ehrhard in [67] for studying the isoperimetric inequality in a Gaussian setting. We recall the definition and the main properties of such symmetrization.

Definition 4.3.1. *We define the functions Φ and α by*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \qquad \text{and} \qquad \alpha(x) = \Phi^{-1}(x).$$

we then let $\mathcal{U}(x) = \Phi' \circ \alpha(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2(x)}{2}}$.

Notice that $\Phi(t)$ is the volume of the half-space $\{\hat{h}(x) < t\}$ and that $\mathcal{U}(x)$ is the perimeter of a half-space of volume x.

Lemma 4.3.2. *Let $\hat{h}_1$, $\hat{h}_2 \in \mathcal{H}$, with $|h_1|_H = |h_2|_H = 1$, and suppose that there exist C_1, $C_2 \in \mathbb{R}$ such that*

$$\{\hat{h}_1 < C_1\} \subset \{\hat{h}_2 < C_2\}.$$

Then $\hat{h}_1 = \hat{h}_2$.

Proof. Assume by contradiction $\hat{h}_1 \neq \hat{h}_2$, and let $\eta > 0$ be such that $|\hat{h}_1 - \hat{h}_2|_{L^2_\gamma(X)} \geq \eta$. We shall bound from below by a positive constant the quantity

$$\gamma\left(\left\{\hat{h}_1(x) < C_1\right\} \cap \left\{\hat{h}_2(x) \geq C_2\right\}\right)$$

thus contradicting the inclusion

$$\left\{\hat{h}_1 < C_1\right\} \subset \left\{\hat{h}_2 < C_2\right\}.$$

Letting h be a unitary vector in H orthogonal to h_1, we can write

$$h_2 = \lambda h_1 + \beta h$$

with $\lambda^2 + \beta^2 = 1$. Up to exchanging h with $-h$, we can also assume that $\beta \geq 0$. We then have $|h_1 - h_2|_H = 2(1-\lambda)$ and thus $-1 \leq \lambda \leq 1 - \frac{\eta}{2}$.
Let us first suppose that $-1 \leq \lambda \leq -\frac{1}{2}$, then

$$\left\{\hat{h}_1(x) < \min\left(C_1, -\frac{C_2}{\lambda}\right)\right\} \cap \left\{\hat{h}(x) \geq 0\right\} \subset \left\{\hat{h}_1(x) < C_1\right\} \cap \left\{\hat{h}_2(x) \geq C_2\right\}.$$

As $\hat{h}_1$ and $\hat{h}$ are orthogonal we have $\Pi_{\hat{h}_1,\hat{h}}\sharp\gamma = \gamma_2$ and thus

$$\begin{aligned}\gamma\left(\left\{\hat{h}_1(x) < \min(C_1, -\frac{C_2}{\lambda})\right\} \cap \left\{\hat{h}(x) \geq 0\right\}\right) &= \frac{1}{2}\Phi(\min(C_1, -C_2/\lambda)) \\ &\geq \frac{1}{2}\Phi(\min(C_1, 2C_2)).\end{aligned}$$

Hence, for $-1 \leq \lambda \leq -\frac{1}{2}$,

$$\gamma\left(\left\{\hat{h}_1(x) < C_1\right\} \cap \left\{\hat{h}_2(x) \geq C_2\right\}\right) \geq \frac{1}{2}\,\Phi(\min(C_1, 2C_2)).$$

If now $-\frac{1}{2} \leq \lambda \leq 1 - \frac{\eta}{2}$, we can assume that η is such that $1 - \frac{\eta}{2} \geq \frac{1}{2}$. Let us start by computing the Fourier transform of $\Pi_{\hat{h}_1,\hat{h}_2}\sharp\gamma$. Denoting by $\tilde{\mu}$ the

Fourier transform of a measure μ (see [27, Sec. 1.2]) and letting $\Pi := \Pi_{\hat{h}_1,\hat{h}_2}$, for every $(z_1, z_2) \in \mathbb{R}^2$ we have

$$\begin{aligned}
\widetilde{\Pi\sharp\gamma}(z_1, z_2) &= \int_{\mathbb{R}^2} e^{iz\cdot x} d\Pi\sharp\gamma(x) \\
&= \int_X e^{iz\cdot\Pi(x)} d\gamma(x) \\
&= \int_X e^{i[z_1\hat{h}_1(x)+z_2\hat{h}_2(x)]} d\gamma(x) \\
&= \int_X e^{i[(z_1+z_2\lambda)\hat{h}_1(x)+z_2\beta\hat{h}(x)]} d\gamma(x) \\
&= \int_{\mathbb{R}^2} e^{i[(z_1+z_2\lambda)x_1+z_2\beta x_2]} d\gamma_2(x_1, x_2) \\
&= \widetilde{\gamma_2}(z_1 + \lambda z_2, \beta z_2) \\
&= e^{-\frac{1}{2}[(z_1+\lambda z_2)^2+\beta^2 z_2^2]} \\
&= e^{-\frac{1}{2}[z_1^2+z_2^2+2\lambda z_1 z_2]}.
\end{aligned}$$

Thus, if we set $K := \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$, we have $\widetilde{\Pi\sharp\gamma}(z) = e^{-\frac{1}{2}z^t K z}$. It follows that $\Pi\sharp\gamma$ is a centered Gaussian measure with density $\frac{1}{2\pi\sqrt{\det K}} e^{-\frac{1}{2}z^t K^{-1} z}$ and thus

$$\Pi\sharp\gamma(z_1, z_2) = \frac{\sqrt{1-\lambda^2}}{2\pi} e^{-\frac{1}{2}[z_1^2+z_2^2-2\lambda z_1 z_2]} dz.$$

We now compute

$$\begin{aligned}
\gamma\left(\left\{\hat{h}_1(x) < C_1\right\} \cap \left\{\hat{h}_2(x) \geq C_2\right\}\right) &= \int_X \chi_{\{\hat{h}_1(x)<C_1\}}(x)\chi_{\{\hat{h}_2(x)\geq C_2\}}(x)\, d\gamma(x) \\
&= \int_{\mathbb{R}^2} \chi_{\{z_1<C_1\}}(z)\chi_{\{z_2\geq C_2\}}(z)\, d\Pi\sharp\gamma(z) \\
&= \int_{-\infty}^{C_1}\int_{C_2}^{+\infty} \frac{\sqrt{1-\lambda^2}}{2\pi} e^{-\frac{1}{2}[z_1^2+z_2^2-2\lambda z_1 z_2]} dz_1 dz_2 \\
&\geq \frac{1}{2\pi}\sqrt{\frac{3}{4}} \int_{-\infty}^{C_1}\int_{C_2}^{+\infty} e^{-\frac{1}{2}z_1^2} e^{-\frac{1}{2}z_2^2} e^{\lambda z_1 z_2} dz_1 dz_2.
\end{aligned}$$

Finally, when $\lambda z_1 z_2 \geq 0$, we can bound $e^{\lambda z_1 z_2}$ from below by 1, and when $\lambda z_1 z_2 \leq 0$ we can bound it form below by $e^{-\frac{1}{2}|z_1 z_2|}$ so that we can always bound from below

$$\gamma\left(\{\hat{h}_1(x) < C_1\} \cap \{\hat{h}_2(x) \geq C_2\}\right)$$

by a positive constant. $\square$

We now define the Ehrhard symmetrization.

Definition 4.3.3. *Let $E \subset X$ and let $m \in \mathbb{N}$. The Ehrhard symmetral of E along the first m variables is defined as (see Figure 4.1):*

$$E^* := \begin{cases} \left\{(x, x_m, x_m^\perp) \in \mathbb{R}^{m-1} \times \mathbb{R} \times X_m^\perp \ : \ x_m < \alpha(\mathbb{E}_{m-1}\chi_E(x))\right\} & \text{if } m > 1 \\ \\ \{x \in X \ : \ \langle x_1^*, x\rangle < \alpha(\gamma(E))\} & \text{if } m = 1. \end{cases}$$

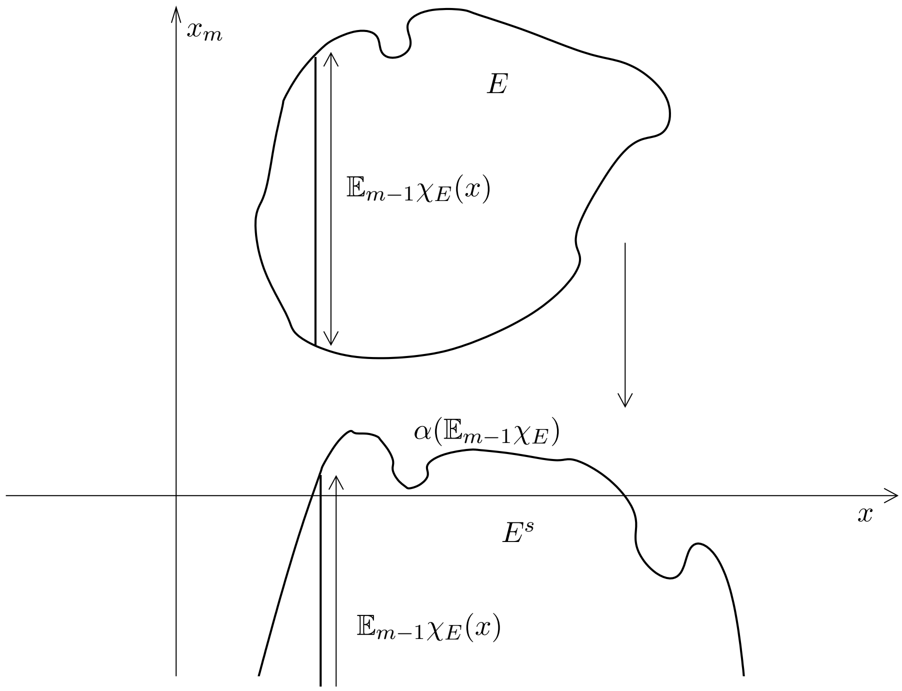

Figure 4.1: The Ehrhard symmetrization.

The interest of this symmetrization is that it decreases the Gaussian perimeter, while keeping the volume fixed.

Proposition 4.3.4. *Let E be a set of finite perimeter and E^* be an Ehrhard symmetral of E, then*

$$\gamma(E^*) = \gamma(E), \tag{4.5}$$

$\mathbb{E}_{m-1}\chi_{E^*} = \mathbb{E}_{m-1}\chi_E$ *and*

$$P_\gamma(E^*) \le P_\gamma(E). \tag{4.6}$$

In particular, we have the isoperimetric inequality

$$P_\gamma(E) \ge \mathcal{U}(\gamma(E)),$$

with equality if and only if E is a half-space.

For the proof we refer to [26, 39], and to [12] for the extension to infinite dimensions.
We can also prove a stronger result which is a kind of Bernstein Theorem in this setting.

Proposition 4.3.5. *The half-spaces are the only local minimizers of the Gaussian perimeter with volume constraint.*

Proof. Let $E \subset X$ be a local minimizer of the (Gaussian) perimeter and let $v = \gamma(E)$. This means that, for every $R > 0$ and every set F of finite perimeter, with $\gamma(F) = v$ and $E \Delta F \subset B_R$ (where B_R denotes the ball of radius R centered at 0), we have

$$P_\gamma(E) \leq P_\gamma(F).$$

If E is not an half space then, by Proposition 4.3.4, there exists $\eta > 0$ such that

$$P_\gamma(E) \geq \mathcal{U}(v) + \eta.$$

Let α_R be such that

$$\gamma(E \cap B_R) = \gamma(\{\langle x_1^*, x\rangle < \alpha_R\} \cap B_R).$$

We have that α_R tends to $\alpha(v)$ when R goes to infinity and $P_\gamma(\{\langle x_1^*, x\rangle < \alpha_R\})$ tends to $P_\gamma(\{\langle x_1^*, x\rangle < \alpha(v)\})$. Letting

$$F_R = (\{\langle x_1^*, x\rangle < \alpha_R\} \cap B_R) \cup (E \cap B_R^c)$$

we get

$$\begin{aligned}
\mathcal{U}(v) + \eta \leq P_\gamma(E) \leq P_\gamma(F_R) &\leq P_\gamma(\{\langle x_1^*, x\rangle < \alpha_R\} \cap B_R) + P_\gamma(E \cap B_R^c) \\
&\leq P_\gamma(\{\langle x_1^*, x\rangle < \alpha_R\}) + P_\gamma(B_R) + P_\gamma(E \cap B_R^c) \\
&\leq P_\gamma(\{\langle x_1^*, x\rangle < \alpha(v)\}) + \varepsilon(R) \\
&= \mathcal{U}(v) + \varepsilon(R),
\end{aligned}$$

where we used various time the inequality (see [83])

$$P_\gamma(E \cup F) + P_\gamma(E \cap F) \leq P_\gamma(E) + P_\gamma(F)$$

and where $\varepsilon(R)$ is a function which goes to zero when R goes to infinity. We thus found a contradiction. $\square$

Remark 4.3.6. *In the Euclidean setting, half-spaces are the only local minimizers of the perimeter only in dimension lower than 8 (see [83]). Notice also that if we drop the volume constraint, half spaces are no longer local minimizers for the Gaussian perimeter, since there are no nonempty local minimizers.*

In the sequel we will also need another transformation which from a finite dimensional function gives an Ehrhard symmetric set whose sections have volume prescribed by the original function. More precisely:

Definition 4.3.7. *Given a measurable function $v : \mathbb{R}^m \to [0,1]$, we define its Ehrhard set $ES_m(v) \subset X$ by*

$$ES_m(v) := \left\{(x, x_{m+1}, x_{m+1}^{\perp}) \in \mathbb{R}^m \times \mathbb{R} \times X_{m+1}^{\perp} \ : \ x_{m+1} < \alpha(v(x))\right\}.$$

Given a measurable cylindrical function $u : X \to [0,1]$ depending only on the first m variables, that is, $u = v \circ \Pi_m$ for some $v : \mathbb{R}^m \to [0,1]$, we set

$$ES_m(u) := ES_m(v).$$

The link between Ehrhard sets and Ehrhard symmetrization is the following:

Proposition 4.3.8. *Let E be a set of finite perimeter and E^* be its Ehrhard symmetrization with respect to the first $(m+1)$ variables, then*

$$E^* = ES_m(\mathbb{E}_m(\chi_E)).$$

In the next proposition we compute the perimeter of Ehrhard sets. It slightly extends a result in [56].

Proposition 4.3.9. *Let $u \in BV_{\gamma_m}(\mathbb{R}^m)$ with $0 \le u \le 1$, then*

$$P_\gamma(ES_m(u)) = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m$$

where

$$\int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2} d\gamma_m = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2}\, d\gamma_m + |D^s_\gamma u|(X)$$

and $D_\gamma u = \nabla u\, \gamma + D^s_\gamma u$ is the Radon-Nikodym decomposition of $D_\gamma u$.

Proof. By [56, Th. 4.3] the result holds for $u \in H^1_{\gamma_m}(\mathbb{R}^m)$. We will show by approximation that the same holds for $u \in BV_{\gamma_m}(\mathbb{R}^m)$.
Let $E = ES_m(u)$, then we can find sets E_n such that $\gamma(E_n \Delta E) \to 0$ and $P_\gamma(E_n) \to P_\gamma(E)$ as $n \to +\infty$, and all the E_n have smooth boundary and are

Ehrhard symmetric. Thus, for every $n \in \mathbb{N}$, there exists a smooth function u_n such that $0 \le u_n \le 1$, $E_n = ES_m(u_n)$, $u_n \to u$ in $L^1_{\gamma_m}(\mathbb{R}^m)$, and

$$P_\gamma(E_n) = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u_n)^2 + |D_{\gamma_m} u_n|^2}\, d\gamma_m.$$

Since, by Proposition 4.4.2, the functional $\int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2} d\gamma_m$ is lower semicontinuous in $L^1_{\gamma_m}(\mathbb{R}^m)$, we get

$$\begin{aligned} P_\gamma(E) &= \lim_{n\to\infty} P_\gamma(E_n) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u_n)^2 + |D_{\gamma_m} u_n|^2}\, d\gamma_m \\ &\ge \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m. \end{aligned}$$

The other inequality follows as in [56]. Let $\widetilde{E} = \Pi_{m+1}(E) \subset \mathbb{R}^{m+1}$ and observe that $\gamma_{m+1}(\widetilde{E}) = \gamma(E)$ and $P_{\gamma_{m+1}}(\widetilde{E}) = P_\gamma(E)$. By Vol'pert Theorem [10, Th. 3.108] there exists a set $B \subset \mathbb{R}^m$ such that for every $x \in B$, $\nu^{\widetilde{E}}_{m+1}(x, \alpha(u_E(x)))$ exists and is not equal to zero, where $\nu^{\widetilde{E}}_{m+1}$ denotes the last coordinate of the unit external normal to $\partial^* \widetilde{E}$. By [56, Lemma 4.4], γ_m-almost every $x \in B$ is a point of approximate differentiability for u. By Lemma 4.5 and 4.6 of [56] we then have

$$\begin{aligned} P_{\gamma_{m+1}}(\widetilde{E}) &= P_{\gamma_{m+1}}(\widetilde{E}, B \times \mathbb{R}) + P_{\gamma_{m+1}}(\widetilde{E}, B^c \times \mathbb{R}) \\ &\le \int_B \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2} d\gamma_m + \int_{B^c} |D_{\gamma_m} u| + \int_{B^c} \mathcal{U}(u)\, d\gamma_m\,. \end{aligned}$$

As $\gamma_m(B^c) = 0$, we find that

$$\int_B \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2} d\gamma_m + \int_{B^c} |D_{\gamma_m} u| = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}^2(u) + |\nabla u|^2} d\gamma_m + |D^s_{\gamma_m} u|(\mathbb{R}^m)$$

and thus $P_\gamma(E) = P_{\gamma_{m+1}}(\widetilde{E}) \le \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2} d\gamma_m$. □

The last transformation that we consider is the analog of the Schwarz symmetrization in the Gaussian setting, and was first introduced by Ehrhard in [68].

Definition 4.3.10. *Let $u \in X \to \mathbb{R}$ be a measurable function and let $m \in \mathbb{N}$ be fixed. We define the m-dimensional Ehrhard symmetrization u^* of u as follows:*

- *for all $t \in \mathbb{R}$ we let E^*_t be the Ehrhard symmetrization of $E_t := \{u > t\}$ with respect to the first m variables;*

- *we let* $u^*(x) := \inf\{t\ :\ x \in E_t^*\}$.

As (4.5) implies $\gamma(\{u^* > t\}) = \gamma(\{u > t\})$ for all $t \in \mathbb{R}$, from the Layer Cake formula it follows that, if $u \in L^2_\gamma(X)$, then $u^* \in L^2_\gamma(X)$ and

$$\int_X |u^*|^2 d\gamma = \int_X |u|^2\, d\gamma\,. \tag{4.7}$$

Indeed, we have

$$\begin{aligned}\int_X |u|^2 d\gamma &= 2\int_0^{+\infty} t\,\gamma(\{u > t\})\,dt - 2\int_{-\infty}^0 t\,\gamma(\{u < t\})\,dt\\ &= 2\int_0^{+\infty} t\,\gamma(\{u^* > t\})\,dt - 2\int_{-\infty}^0 t\,\gamma(\{u^* < t\})\,dt\\ &= \int_X |u^*|^2 d\gamma.\end{aligned}$$

Lemma 4.3.11. *Let* $u, v : X \to [0, +\infty)$ *belonging to* $L^2_\gamma(X)$*, then*

$$\|u^* - v^*\|_{L^2_\gamma(X)} \le \|u - v\|_{L^2_\gamma(X)}. \tag{4.8}$$

Proof. The proof is a straightforward adaptation of the analogous proof for the Schwarz symmetrization [101, Th. 3.4].
Recalling (4.7) with $p = 2$, we have only to show that

$$\int_X uv d\gamma \le \int_X u^* v^* d\gamma. \tag{4.9}$$

Again by the Layer Cake formula we have

$$\int_X uv d\gamma = \int_0^{+\infty}\int_0^{+\infty}\int_X \chi_{\{u>t\}}(x)\chi_{\{v>s\}}(x)d\gamma(x)\,dt\,ds.$$

Thus (4.9) would follow from the same inequality for sets, that is,

$$\gamma\,(A \cap B) \le \gamma\,(A^* \cap B^*)\,. \tag{4.10}$$

Let $x_m \in \mathbb{R}^m$ and assume that

$$\int_{X_m^\perp} \chi_A(x_m, y) d\gamma_m^\perp(y) \ge \int_{X_m^\perp} \chi_B(x_m, y) d\gamma_m^\perp(y)$$

then by definition of the Ehrhard symmetrization we have

$$B^* \cap (x_m + X_m^\perp) \subset A^* \cap (x_m + X_m^\perp)$$

and therefore

$$\begin{aligned}
\int_{X_m^\perp} \chi_{A^*}(x_m, y)\chi_{B^*}(x_m, y) d\gamma_m^\perp(y) &= \int_{X_m^\perp} \chi_{A^*}(x_m, y) d\gamma_m^\perp(y) \\
&= \int_{X_m^\perp} \chi_A(x_m, y) d\gamma_m^\perp(y) \\
&\geq \int_{X_m^\perp} \chi_A(x_m, y)\chi_B(x_m, y) d\gamma_m^\perp(y)
\end{aligned}$$

This inequality also holds if $\int_{X_m^\perp} \chi_B(x_m, y) d\gamma_m^\perp(y) \geq \int_{X_m^\perp} \chi_A(x_m, y) d\gamma_m^\perp(y)$ so that finally

$$\begin{aligned}
\gamma\left(A^* \cap B^*\right) &= \int_{\mathbb{R}^m} \int_{X_m^\perp} \chi_{A^*}(x, y)\chi_{B^*}(x, y) d\gamma_m^\perp(y) d\gamma_m(x) \\
&\geq \int_{\mathbb{R}^m} \int_{X_m^\perp} \chi_A(x, y)\chi_B(x, y) d\gamma_m^\perp(y) d\gamma_m(x) \\
&= \gamma\left(A \cap B\right)
\end{aligned}$$

which gives (4.10). $\square$

As for the Schwarz symmetrization, a Pólya-Szegö principle holds for the Ehrhard symmetrization.

Proposition 4.3.12. *Let $u \in H_\gamma^1(X)$, let $m \in \mathbb{N}$ and let u^* be the m-dimensional Ehrhard symmetrization of u. Then $u^* \in H_\gamma^1$ and*

$$\int_X |\nabla_H u^*|_H^2 \, d\gamma \leq \int_X |\nabla_H u|_H^2 \, d\gamma. \tag{4.11}$$

Moreover, if $m = 1$ and equality holds in (4.11), then

$$u = \tilde{u}\left(\hat{h}(x)\right) \qquad \textit{for some } \hat{h} \in \mathcal{H}\,,$$

and $\hat{h}$ can be chosen to be a unitary vector.

Proof. In [68, Th. 3.1], inequality (4.11) is proven for Lipschitz functions, in finite dimensions. We extend it by approximation to Sobolev functions. We can assume $u \geq 0$, since we have $(u^\pm)^* = (u^*)^\pm$, where $u^\pm, (u^*)^\pm$ denote the positive and negative part of u and u^*, respectively.
Let $u_n \in \mathcal{FC}_c^1(X)$ be positive functions converging to u in $H_\gamma^1(X)$, then by (4.8), u_n^* converges to u^* in $L_\gamma^2(X)$ and thus by the lower semicontinuity of the $H_\gamma^1(X)$ norm we have

$$\int_X |\nabla_H u^*|_H^2 \leq \varliminf_{n\to\infty} \int_X |\nabla_H u_n^*|_H^2 \leq \varliminf_{n\to\infty} \int_X |\nabla_H u_n|_H^2 = \int_X |\nabla_H u|_H^2.$$

We now turn to the equality case for one-dimensional symmetrizations. For this we closely follow [39] and give an alternative proof of (4.11), based on ideas of Brothers and Ziemer [34] for the Schwarz symmetrization.
Let $u \in H^1_\gamma(X)$ and $\mu(t) = \gamma(\{u > t\}) = \gamma(\{u^* > t\})$. By the coarea formula (4.4), for all $t \in \mathbb{R}$ we have

$$\mu(t) = \gamma(\{u > t\} \cap \{\nabla_H u = 0\}) + \int_t^{+\infty} \left(\int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_\tau}| \right) d\tau.$$

Hence

$$-\mu'(t) \geq \int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_t}| \qquad \textit{for a.e. } t \in \mathbb{R}. \tag{4.12}$$

Since u^* is a function depending only on one variable, arguing as in [55] we get

$$\frac{d}{dt} \gamma(\{u^* > t\} \cap \{\nabla_H u^* = 0\}) = 0 \qquad \textit{for a.e. } t \in \mathbb{R}.$$

As u^* is monotone we have that $|\nabla_H u^*|_H$ is constant on $\{u^* = t\} \cap \{\nabla_H u^* \neq 0\}$. Observe also that, being u^* one-dimensional, $\{u^* = t\}$ has a well defined meaning. We thus find:

$$-\mu'(t) = \frac{P_\gamma(\{u^* > t\})}{|\nabla_H u^*|_{\{u^* = t\}}} \qquad \textit{for a.e. } t \in \mathbb{R},$$

which implies, recalling (4.12),

$$\frac{P_\gamma(\{u^* > t\})}{|\nabla_H u^*|_{\{u^* = t\}}} \geq \int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_t}| \qquad \textit{for a.e. } t \in \mathbb{R}. \tag{4.13}$$

Let us note that as in [39, Lem. 4.2], using (4.4) with $g = \chi_{\{\nabla_H u = 0\}}$ we find

$$\int_X \chi_{\{\nabla_H u = 0\}} |\nabla_H u|_H d\gamma = 0 = \int_{\mathbb{R}} \int_X \chi_{\{\nabla_H u = 0\}} d|D_\gamma \chi_{E_t}|(x) \, dt$$

and thus for almost every $t \in \mathbb{R}$,

$$\int_X \chi_{\{\nabla_H u = 0\}} d|D_\gamma \chi_{E_t}|(x) = 0.$$

This shows that for almost every $t \in \mathbb{R}$, $\nabla_H u(x) \neq 0$ for $|D_\gamma \chi_{E_t}|$-almost every $x \in X$ and thus

$$\int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} d|D_\gamma \chi_{E_t}|(x) = \int_X \frac{1}{|\nabla_H u|_H} d|D_\gamma \chi_{E_t}|(x) \qquad \textit{for a.e. } t \in \mathbb{R}. \tag{4.14}$$

By (4.4), (4.6),(4.13) and (4.14), we eventually get

$$
\begin{aligned}
\int_X |\nabla_H u^*|^2 d\gamma &= \int_{\mathbb{R}} |\nabla_H u^*|_{\{u^*=t\}} P_\gamma(\{u^* > t\}) dt \\
&= \int_{\mathbb{R}} \frac{P_\gamma(\{u^* > t\})^2}{\left(\frac{P_\gamma(\{u^*>t\})}{|\nabla_H u^*|_{\{u^*=t\}}}\right)} dt \\
&\leq \int_{\mathbb{R}} \frac{P_\gamma(\{u > t\})^2}{\int_X \frac{1}{|\nabla_H u|_H} d|D_\gamma \chi_{E_t}|(x)} \, dt \\
&\leq \int_{\mathbb{R}} \int_X |\nabla_H u|_H \; d|D_\gamma \chi_{E_t}|(x) \; dt \\
&= \int_X |\nabla_H u|_H^2 d\gamma \, .
\end{aligned}
$$

As a consequence, if equality holds in (4.11), then equality holds in the Gaussian isoperimetric inequality, that is,

$$P_\gamma(u > t) = P_\gamma(u^* > t) \qquad \textit{for a.e. } t \in \mathbb{R}.$$

This implies that almost every level-set of u is a half-space, i.e. for almost every $t \in \mathbb{R}$ there exists $\hat{h}_t \in \mathcal{H}$ such that $\{u > t\} = \{\hat{h}_t < \alpha(\mu(t))\}$, and without loss of generality we can assume that $|h_t|_H = 1$. Such half-spaces being nested, by Lemma 4.3.2 we have that $\hat{h}_t$ does not depend on t and thus $u(x) = v(\hat{h}(x))$. □

Remark 4.3.13. *We notice that the fact that equality in* (4.11) *implies that u is one-dimensional is a specific feature of the Gaussian setting, and the analogous statement does not hold for the Schwarz symmetrization in the Euclidean case [34]. Indeed, this property is a consequence of the fact that Gaussian measures, differently from the Lebesgue measure, are not invariant under translations.*

4.4 Relaxation of perimeter

In this section we compute the relaxation of the perimeter functional

$$F(u) := \begin{cases} P_\gamma(E) & \text{if } u = \chi_E \\ +\infty & \text{otherwise} \end{cases}$$

with respect to the weak $L^2_\gamma(X)$-topology. The fact that F is not lower semicontinuous can be easily checked by taking the sequence $E_n = \{\langle x_n^*, x\rangle < 0\}$. Indeed, the characteristic functions of these sets weakly converge to the constant function $1/2$, which is not a characteristic function, while the perimeter of E_n is constantly equal to $1/\sqrt{2\pi}$.

We will show that the relaxation of F is equal to

$$\overline{F}(u) := \begin{cases} \displaystyle\int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2 d\gamma} & \text{if } 0 \le u \le 1 \quad \gamma - a.e. \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2 d\gamma} = \int_X \sqrt{\mathcal{U}^2(u) + |\nabla_H u|_H^2} d\gamma + |D_\gamma^s u|(X)$$

with $D_\gamma u = \nabla_H u d\gamma + D_\gamma^s u$. Observe that the functional $\overline{F}$ already appears in the seminal work of Bakry and Ledoux [21] and in the earlier work of Bobkov [26] in the context of log-Sobolev inequalities. This functional has been also studied in [39]. See also [12, Remark 4.3] where it appears in a setting closer to ours.

We now show a representation formula for $\overline{F}$ which is reminiscent of the definition of the total variation and of the nonparametric area functional (see [83]). We start with a preliminary result.

Lemma 4.4.1. *Let $g \in L^\infty(X)$ with $g \ge 0$, let $\mu \in \mathcal{M}(X, H)$, and define*

$$\tilde{f}(g, \mu) := \sqrt{g^2 + |h|_H^2}\, d\gamma + |\mu^s|\,,$$

where $\mu = h\,\gamma + \mu^s$. There holds

$$\tilde{f}(g,\mu)(X) = \sup_{\substack{\Phi \in L^1_\mu(X,H) \\ \xi \in L^1_\mu(X)}} \left\{ \int_X [\Phi, d\mu]_H + \int_X g\,\xi\, d\gamma :\ |\Phi|_H^2 + |\xi|^2 \le 1 \ in\ X \right\}. \tag{4.15}$$

Proof. The proof is adapted from [63].

Notice first that, for $(\lambda, p) \in \mathbb{R} \times H$, the function $f(\lambda, p) := \sqrt{\lambda^2 + |p|_H^2}$ defines a norm on the product space $\mathbb{R} \times H$. Moreover, if we let $f_\lambda(p) := \sqrt{\lambda^2 + |p|_H^2}$, then the convex conjugate of f_λ is $f_\lambda^*(\Phi) = -\lambda\sqrt{1 - |\Phi|_H^2}$. We divide the proof into three steps.

*S*tep 1. Let

$$M(g,\mu) = \sup_{\Phi\in L^1_\mu(X,H)} \left\{ \int_X [\Phi, d\mu]_H + \int_X g\sqrt{1-|\Phi|_H^2}\, d\gamma : \ |\Phi|_H \le 1 \text{ in } X \right\}.$$

We will show that

$$M(g,h\gamma) = \int_X f(g,h)d\gamma. \tag{4.16}$$

By definition of convex conjugate, it is readily checked that $M(g,h\gamma) \le \int_X f(g,h)d\gamma$. We thus turn to the other inequality. By definition of the Bochner integral, for every $\delta > 0$, there exists $h_i \in H$ and $A_i \subset X$ with A_i disjoints Borel sets and $i \in [1,n]$ such that if we set

$$\theta = \sum_{i=1}^n \chi_{A_i} h_i$$

then $|\theta - h|_{L^1_\gamma} \le \delta$. Analogously there exists $\eta_i \in X$ such that setting

$$\tilde{g} = \sum_{i=1}^n \chi_{A_i} \eta_i$$

we have $|\tilde{g} - g|_{L^1_\gamma} \le \delta$. By the observation at the beginning of the proof and the triangle inequality we get

$$|f(\tilde{g},\theta) - f(g,h)| \le f(\tilde{g}-g, \theta-h)| \le |\tilde{g}-g| + |\theta-h|_H.$$

For every i, by definition of convex conjugate, there exists $\xi_i \in H$ with $|\xi_i|_H \le 1$ such that

$$f(\eta_i,h_i) \le [\xi_i,h_i]_H + \eta_i\sqrt{1-|\xi_i|_H^2} + \delta.$$

From this, setting $\Phi = \sum_{i=1}^n \chi_{A_i}\xi_i$ we have

$$\begin{aligned}
\int_X f(g,h)d\gamma &\le \int_X f(\tilde{g},\theta)d\gamma + 2\delta \\
&= \sum_{i=1}^n \int_{A_i} f(\eta_i,h_i)d\gamma + 2\delta \\
&\le \sum_{i=1}^n \int_{A_i} [\xi_i,h_i]_H + \eta_i\sqrt{1-|\xi_i|_H^2} d\gamma + 3\delta \\
&= \int_X [\Phi,\theta]_H + \tilde{g}\sqrt{1-|\Phi|_H^2} d\gamma + 3\delta.
\end{aligned}$$

Since $\left| \tilde{g}\sqrt{1-|\Phi|_H^2} + g\sqrt{1-|\Phi|_H^2} \right| \le |\tilde{g}-g|$ we find

$$\begin{aligned}\int_X f(g,h)d\gamma &\le \int_X \Phi \cdot h - g\sqrt{1-|\Phi|_H^2}d\gamma + 5\delta \\ &\le M(g,h\gamma) + 5\delta.\end{aligned}$$

Since δ is arbitrary we have $M(g,h\gamma) = \int_X f(g,h)d\gamma$.

*S*tep 2. The proof proceeds exactly as in [63] and we only sketch it. Recalling (4.16), it remains to show that

$$M(g,h\gamma+\mu^s) = M(g,h\gamma) + |\mu^s|(X).$$

One inequality is easily obtained, since

$$\begin{aligned}M(g,h\gamma+\mu^s) &= \sup_\Phi \int_X [\Phi,h]_H d\gamma + \int_X \Phi\cdot d\mu^s + \int_X g(x)\sqrt{1-|\Phi|_H^2}d\gamma \\ &\le \left(\sup_\Phi \int_X [\Phi,h]_H d\gamma + \int_X g(x)\sqrt{1-|\Phi|_H^2}d\gamma\right) + \int_X |d\mu^s| \\ &= M(g,h\gamma) + |\mu^s|(X).\end{aligned}$$

For the opposite inequality, let $\delta > 0$ be fixed then there exists Φ_1 and Φ_2 such that

$$\begin{aligned}M(g,h\gamma) &\le \int_X [\Phi_1,h]_H d\gamma + \int_X g(x)\sqrt{1-|\Phi_1|_H^2}d\gamma + \delta \\ |\mu^s|(X) &\le \int_X [\Phi_2, d\mu^s]_H + \delta.\end{aligned}$$

Taking Φ equal to Φ_2 on a sufficiently small neighborhood of the support of μ^s and equal to Φ_1 outside this neighborhood, we get

$$\begin{aligned}M(g,h\gamma)+|\mu^s|(X) &\le \int_X [\Phi,h]_H d\gamma + \int_X g(x)\sqrt{1-|\Phi|_H^2}d\gamma + \int_X [\Phi, d\mu^s]_H + C\delta \\ &\le M(g,h\gamma+\mu^s) + C\delta\end{aligned}$$

which gives the opposite inequality.

*S*tep 3. In order to conclude the proof, it is enough to notice that for every $\Phi \in L^1_\mu(X,H)$, with $|\Phi|_H \le 1$, we have

$$\begin{aligned}&\sup_{\xi\in L^1_\mu(X)} \left\{ \int_X [\Phi, d\mu]_H + \int_X g\,\xi\, d\gamma :\ |\Phi|_H^2 + |\xi|^2 \le 1 \ in\ X \right\} \\ &\quad = \int_X [\Phi, d\mu]_H + \int_X g\sqrt{1-|\Phi|_H^2}\, d\gamma.\end{aligned}$$

□

Proposition 4.4.2. *Let $u \in BV_\gamma(X)$ then*

$$\overline{F}(u) = \sup_{\substack{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H) \\ \xi \in \mathcal{F}\mathcal{C}_b^1(X)}} \left\{ \int_X (u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi)\, d\gamma : \quad |\Phi(x)|_H^2 + |\xi(x)|^2 \le 1 \; \forall x \in X \right\}. \tag{4.17}$$

Proof. We apply Lemma 4.4.1 with $\mu = Du$ and $g = \mathcal{U}(u)$. Since μ is tight [12], the space $\mathcal{F}\mathcal{C}_b^1(X,H)$ is dense in $L^1_\mu(X,H)$ so that we can restrict the supremum in (4.17) to smooth cylindrical functions Φ, ξ. □

Remark 4.4.3. *Since $\mathcal{U}$ is concave, the duality formula* (4.17) *is not sufficient to prove that $\overline{F}$ is lower semicontinuous for the weak $L^2_\gamma(X)$-topology. It shows however the lower-semicontinuity of $\overline{F}$ in the strong $L^2_\gamma(X)$-topology.*

We now prove that $\overline{F}$ is the lower semicontinuous envelope of F.

Theorem 4.4.4. *$\overline{F}$ is the relaxation of F in the weak $L^2_\gamma(X)$-topology.*

Proof. Let us first notice that F takes finite values only on functions of the closed unit ball of $L^2_\gamma(X)$ which is metrizable for the weak convergence. Therefore the relaxation and the sequential relaxation in the weak topology of $L^2_\gamma(X)$ coincide.

Let χ_{E_n} be a sequence of sets weakly converging in $L^2_\gamma(X)$ to $u \in BV_\gamma(X)$, with uniformly bounded perimeter. We shall show that

$$\varliminf_{n\to\infty} P_\gamma(E_n) \ge \overline{F}(u).$$

Notice that, by weak convergence, we necessarily have $0 \le u \le 1$ a.e. on X. For all $n \ge 1$ and $k \ge 2$, we let E_n^k be the Ehrhard symmetral of E_n with respect to the first k variables. Recalling the notation of Section 4.3, we have

$$P_\gamma(E_n^{k+1}) \le P_\gamma(E_n) \qquad \text{and} \qquad E_n^{k+1} = ES_k\left(\mathbb{E}_k \chi_{E_n}\right).$$

As $\displaystyle\int_X |D_\gamma \mathbb{E}_k(\chi_{E_n})|_H \le P_\gamma(E_n)$ and $\mathbb{E}_k(\chi_{E_n})$ depends only on the first k variables, by the compact embedding of $BV_{\gamma_k}(\mathbb{R}^k)$ into $L^1_{\gamma_k}(\mathbb{R}^k)$ we can extract a subsequence from $\mathbb{E}_k(\chi_{E_n})$ which converges strongly to $u^k := \mathbb{E}_k(u)$. From this we get that $E_n^{k+1} = ES_k(\mathbb{E}_k\chi_{E_n})$ tends strongly to $E^{k+1} := ES_k(u^k)$. By the lower semicontinuity of the perimeter we then have

$$\varliminf_{n\to\infty} P_\gamma(E_n) \ge \varliminf_{n\to\infty} P_\gamma(E_n^{k+1}) \ge P_\gamma(E^{k+1}).$$

For every $\varphi \in \mathcal{FC}_b^1(X)$, with φ depending only of the $j \le k$ first variables, there holds

$$\int_X \chi_{E_{k+1}}(x)\varphi(x)d\gamma(x) = \int_X u_k(x)\varphi(x)d\gamma(x) = \int_X u(x)\varphi(x)d\gamma(x),$$

which implies that the sequence $\chi_{E_{k+1}}$ tends weakly to u. In order to conclude the proof it remains to show that

$$\lim_{k\to\infty} P_\gamma(E^{k+1}) = \overline{F}(u).$$

Notice that, by Proposition 4.3.9, there holds

$$P_\gamma(E^{k+1}) = \overline{F}(u^k).$$

For every $\Phi \in \mathcal{FC}_b^1(X, H)$ and $\xi \in \mathcal{FC}_b^1(X)$, depending on the first k variables and such that the range of Φ is included in H_k, by Proposition 4.4.2, we have

$$\int_X \left(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}(u^k)\xi\right) d\gamma = \int_X (u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi)\, d\gamma \le \overline{F}(u).$$

Taking the supremum in Φ, ξ and recalling (4.17), we then get

$$\overline{F}(u^k) \le \overline{F}(u) \qquad \textit{for all } k.$$

Repeating the same argument with u^{k+1} instead of u, we obtain that $\overline{F}(u^k)$ is nondecreasing in k. Therefore there exists $\ell \ge 0$ such that

$$\lim_{k\to\infty} \overline{F}(u^k) = \lim_{k\to\infty} P_\gamma(E^{k+1}) = \ell \le \overline{F}(u).$$

Assume by contradiction that $\ell < \overline{F}(u)$. Then there exists $\delta > 0$ such that $\overline{F}(u^k) \le \overline{F}(u) - \delta$ for all k, hence there exist $N \in \mathbb{N}$, $\Phi \in \mathcal{FC}_b^1(X, H)$ and $\xi \in \mathcal{FC}_b^1(X)$, depending only on the first N variables, such that

$$\int_X \left(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}(u^k)\xi\right) d\gamma \le \overline{F}(u^k) \le \overline{F}(u) - \delta \le \int_X (u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi)\, d\gamma - \frac{\delta}{2},$$

but for $k > N$ we have

$$\int_X \left(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}(u^k)\xi\right) d\gamma = \int_X (u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi)\, d\gamma$$

which leads to a contradiction. □

Remark 4.4.5. *Theorem 4.4.4 provides an example of a nonconvex functional, namely $\overline{F}$, which is lower semicontinuous for the weak $L^2_\gamma(X)$-topology. We also know that semicontinuity does not holds for general functional of the form*

$$J(u) = \int_X f(u, D_\gamma u) d\gamma$$

since if we take for instance $f(u,p) := \sqrt{g^2(u) + |p|^2}$ with g such that $g(1/2) > \mathcal{U}(1/2)$ and $g(0) = g(1) = 0$, then, letting $u_n := \{\langle x_n^*, x\rangle < 0\}$, we have $u_n \rightharpoonup u = 1/2$ weakly in $L^2_\gamma(X)$, so that

$$J(u) = g\left(\frac{1}{2}\right) > \mathcal{U}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}} = \lim_{n\to\infty} J(u_n).$$

One could wonder what are the right hypotheses for a functional of this form to be lower semicontinuous with respect to the weak topology.

4.5 Γ-limit for the Modica-Mortola functional

Let us briefly recall the definition of Γ-convergence. We refer to [58] for a comprehensive treatment of the subject.

Definition 4.5.1. *Let X be a topological space, and let $F_n : X \to \overline{\mathbb{R}}$ be a sequence of functions. The Γ-lower limit and the Γ-upper limit of the sequence F_n is defined as*

$$(\Gamma - \underline{\lim}_{n\to\infty} F_n)(x) = \sup_{U\in\mathcal{N}(x)} \underline{\lim}_{n\to\infty} \inf_{y\in U} F_n(y)$$
$$(\Gamma - \overline{\lim}_{n\to\infty} F_n)(x) = \sup_{U\in\mathcal{N}(x)} \overline{\lim}_{n\to\infty} \inf_{y\in U} F_n(y)$$

where $\mathcal{N}(x)$ denotes the set of all open neighbourhoods of x in X. When the Γ-lower limit and the Γ-upper limit coincide, we say that the sequence F_n Γ-converges.

As for the relaxation, if X is a metric space we have a sequential caracterization of the Γ-convergence.

Theorem 4.5.2. *Let X be a metric space. A sequence of functions F_n Γ-converges to $F : X \to \overline{\mathbb{R}}$ if and only if the following two conditions hold:*

- *for every sequence x_n converging to x, it holds $\underline{\lim}_{n\to\infty} F_n(x_n) \geq F(x)$*
- *for every $x \in X$ there exists a sequence x_n converging to x with $\overline{\lim}_{n\to\infty} F_n(x_n) \leq F(x)$.*

Let now $W \in C^1(\mathbb{R})$ be a double-well potential with minima in $\{0,1\}$, that is, $W(t) \geq 0$ for all $t \in \mathbb{R}$, and $W(t) = 0$ iff $t \in \{0,1\}$. We also assume $W(t) \geq C(t^2-1)$ for some $C > 0$ and $t \in \mathbb{R}$. A typical example of such potential is $W(t) = t^2(t-1)^2$.
For any $\varepsilon > 0$ we define the functionals $F_\varepsilon : L^2_\gamma(X) \to [0,+\infty]$ as

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_X \left(\frac{\varepsilon}{2}|\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon}\right) d\gamma & if\ u \in H^1_\gamma(X) \\ \\ +\infty & if\ u \in L^2_\gamma(X) \setminus H^1_\gamma(X)\,. \end{cases}$$

We are ready to prove our main Γ-convergence result.

Theorem 4.5.3. *When ε tends to zero the functionals F_ε Γ-converge, in the weak topology of $L^2_\gamma(X)$, to the functional $c_W\overline{F}$, where $c_W = \int_0^1 \sqrt{2W(t)}\,dt$.*

Proof. Notice first that the Γ-limit does not change if we restrict the domain of F_ε to the functions $u \in H^1_\gamma(X)$ such that $0 \leq u \leq 1$. This follows from the following two facts:

- for all $u \in H^1_\gamma(X)$, letting $\tilde{u} = \min(\max(u,0),1)$, we have $F_\varepsilon(\tilde{u}) \leq F_\varepsilon(u)$;
- $F_\varepsilon(u) \geq \int_X \frac{W(u)}{\varepsilon} d\gamma$ for all $u \in H^1_\gamma(X)$, which implies that the Γ-limit is concentrated on the functions $u \in L^2_\gamma(X)$ such that $u(x) \in \{0,1\}$ for a.e. $x \in X$.

Since the restricted domain is contained in the unit ball of $L^2_\gamma(X)$, which is metrizable for the weak $L^2_\gamma(X)$-topology, by Theorem 4.5.2 the Γ-limit and the sequential Γ-limit of F_ε coincide.
We now compute the Γ-liminf of F_ε.
Let $u_\varepsilon \in H^1_\gamma(X)$ be such that $0 \leq u_\varepsilon \leq 1$ and $F_\varepsilon(u_\varepsilon) \leq C$ for some $C > 0$. Then $\int_X W(u_\varepsilon)d\gamma \leq C\varepsilon$, which gives a uniform bound on $\|u_\varepsilon\|_{L^2_\gamma(X)}$ recalling that $W(u) \geq C(u^2-1)$. As a consequence, there exists a weakly converging subsequence, still denoted by u_ε. Letting u be its weak limit, from $0 \leq u_\varepsilon \leq 1$ we get $0 \leq u \leq 1$. Using the coarea formula (4.3), we obtain the estimate

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_X \left(\frac{\varepsilon}{2}|\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon}\right) d\gamma \\ &\geq \int_X \sqrt{2W(u_\varepsilon)}\,|\nabla_H u|_H\,d\gamma \\ &= \int_0^1 \sqrt{2W(t)}\,P_\gamma(\{u_\varepsilon > t\})\,dt\,. \end{aligned}$$

Fix now $\delta > 0$. From the fact that $\gamma(\{\delta \le u_\varepsilon \le 1-\delta\}) \to 0$ as $\varepsilon \to 0$, it follows that, for every sequence $t_\varepsilon \in [\delta, 1-\delta]$, then functions $\chi_{\{u_\varepsilon > t_\varepsilon\}}$ tend weakly to u in $L^2_\gamma(X)$. For every $\varepsilon > 0$ let us choose $t_\varepsilon \in [\delta, 1-\delta]$ such that

$$\int_\delta^{1-\delta} \sqrt{2W(t)} P_\gamma(\{u_\varepsilon > t\})dt \ge \left(\int_\delta^{1-\delta} \sqrt{2W(t)}dt\right) P_\gamma(\{u_\varepsilon > t_\varepsilon\}).$$

Then, by Theorem 4.4.4 we have

$$\begin{aligned} \varliminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &\ge \varliminf_{\varepsilon\to 0} \left(\int_\delta^{1-\delta} \sqrt{2W(t)}dt\right) P_\gamma(\{u_\varepsilon > t_\varepsilon\}) \\ &\ge \left(\int_\delta^{1-\delta} \sqrt{2W(t)}dt\right) \overline{F}(u)\,. \end{aligned}$$

Since δ is arbitrary we get the Γ-liminf inequality.

The Γ-limsup is done similarly to the (Euclidean) finite dimensional case [109, 110, 76]. Since $\overline{F}$ is the relaxation of F in the weak $L^2_\gamma(X)$-topology and since we can approximate sets of finite perimeter by smooth cylindrical sets by Proposition 4.2.8, for every $u \in BV_\gamma(X)$ with $0 \le u \le 1$ there exists a sequence E_n of smooth cylindrical sets with χ_{E_n} converging weakly to u and such that $P_\gamma(E_n)$ tends to $\overline{F}(u)$. This shows that we can restrict ourselves to smooth cylindrical sets for computing the Γ-limsup of F_ε.
Let $m \in \mathbb{N}$ and $E = \Pi_m^{-1}(E_m)$, where $E_m \subset \mathbb{R}^m$ is a smooth set with finite Gaussian perimeter, and let

$$d^H(x, E) := d(\Pi_m(x), E_m)$$

where $d(x, E_m)$ is the usual distance function from E_m in $\mathbb{R}^m$. Notice that

$$d^H(x, E) = \min\{|x-y|_H; y \in E, x-y \in H\},$$

moreover d^H is differentiable almost everywhere with $|\nabla_H d^H(x, E)|_H = 1$. Let $\delta > 0$, $\alpha_\delta := \max\{W(t) \ : \ t \in [0, \delta] \cup [1-\delta, 1]\}$ and define W_δ, H_δ : $[0, 1] \to \mathbb{R}$ as

$$\begin{aligned} W_\delta(t) &:= \begin{cases} \alpha_\delta & \text{if } 0 \le t \le \delta \\ W(t) & \text{if } \delta \le t \le 1-\delta \\ \alpha_\delta & \text{if } 1-\delta \le t \le 1. \end{cases} \\ H_\delta(t) &:= \int_0^t \frac{1}{\sqrt{2W_\delta(s)}} ds. \end{aligned}$$

Finally let η_δ be the usual truncated one-dimensional transition profile defined as

$$\eta_\delta(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ H_\delta^{-1}(t) & \text{if } 0 \leq t \leq H_\delta(1) \\ 1 & \text{if } t > H_\delta(1). \end{cases}$$

Observe that η_δ is a Lipschitz function which verifies $\frac{\eta_\delta'^2}{2} = W_\delta(\eta_\delta)$. We then set

$$u_\varepsilon(x) := \eta_\delta\left(\frac{d^H(x,E)}{\varepsilon}\right).$$

We finally have

$$\begin{aligned}
F_\varepsilon(u_\varepsilon) &= \int_X \left(\frac{\varepsilon}{2}|\nabla_H u_\varepsilon|_H^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right) d\gamma \\
&\leq \int_X \left(\frac{\varepsilon}{2}|\nabla_H u_\varepsilon|_H^2 + \frac{W_\delta(u_\varepsilon)}{\varepsilon}\right) d\gamma \\
&= \int_X \frac{\varepsilon}{2}\eta_\delta'^2\left(\frac{d(\Pi_m(x))}{\varepsilon}\right)\left(\frac{|\nabla_H d(\Pi_m(x))|}{\varepsilon}\right)^2 \\
&\quad + \frac{1}{\varepsilon}W_\delta\left(\frac{\eta_\delta(d(\Pi_m(x)))}{\varepsilon}\right) d\gamma \\
&= \int_{\mathbb{R}^m}\left[\frac{1}{2}\eta_\delta'^2\left(\frac{d}{\varepsilon}\right) + W_\delta\left(\eta_\delta\left(\frac{d}{\varepsilon}\right)\right)\right]\frac{|\nabla d|}{\varepsilon}\, d\gamma_m \\
&= \int_0^{H_\delta(1)}\left(\frac{\eta_\delta'^2(t)}{2} + W_\delta(\eta_\delta(t))\right) P_{\gamma_m}(\{d > \varepsilon t\})\, dt.
\end{aligned}$$

The proof is completed since for every $t \in [0, H_\delta(1)]$, $P_{\gamma_m}(\{d > \varepsilon t\})$ tends to $P_{\gamma_m}(E_m)$ as $\varepsilon \to 0$, and

$$\int_0^{H_\delta(1)}\left(\frac{\eta_\delta'^2(t)}{2} + W_\delta(\eta_\delta(t))\right) dt = \int_0^1 \sqrt{2W_\delta(t)}\, dt\,.$$

Thus we have

$$\overline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(u_\varepsilon) \leq \left(\int_0^1 \sqrt{2W_\delta(t)}\, dt\right) P_{\gamma_m}(E_m),$$

which gives the desired inequality letting $\delta \to 0$ and $m \to +\infty$. □

Remark 4.5.4. *As in the Euclidean case, a similar result can be proven for the volume constrained problems. In this case, the proof of the* Γ*-liminf is exactly the same as in Theorem 4.5.3, and the* Γ*-limsup is also very similar. The only difference comes from the fact that we have to adapt the recovery sequence to have the right volume, and this can be done as in [109] by slightly translating* η_δ*.*

We now state some simple implications of the Γ-convergence result.

Proposition 4.5.5. *Let $m \in [0,1]$ and u_ε be a minimizer of*

$$\min_{\int_X u\, d\gamma = m} \int_X \left(\frac{\varepsilon}{2} |\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon} \right) d\gamma \tag{4.18}$$

then $u_\varepsilon = v_\varepsilon(\hat{h}_\varepsilon(x))$ for some $\hat{h}_\varepsilon \in \mathcal{H}$ with $|h_\varepsilon|_H = 1$ and some v_ε minimizer of the one-dimensional problem

$$\min_{\int_{\mathbb{R}} v d\gamma_1 = m} \int_{\mathbb{R}} \frac{\varepsilon}{2} v'^2 d\gamma + \int_{\mathbb{R}} \frac{W(v)}{\varepsilon} d\gamma_1. \tag{4.19}$$

in particular, v_ε (strongly) converges to the characteristic function of a half-line.

Proof. For every $u \in H^1_\gamma(X)$, by Proposition 4.3.12, we have $\int_X u^* d\gamma = \int_X u d\gamma$ and $F_\varepsilon(u^*) \leq F_\varepsilon(u)$, with equality only if u is of the form $u(x) = v(\hat{h}(x))$ for some $\hat{h} \in \mathcal{H}$ with $|h|_H = 1$. Using that $\hat{h}$ is the limit in $L^2_\gamma(X)$ of linear functions of the form $R^* x_i^*$, it is readily seen that $\nabla_H \hat{h} = h$, and thus we get

$$\begin{aligned} F_\varepsilon(u) &= \int_X \left(\frac{\varepsilon}{2} v'(\hat{h}(x))^2 + \frac{W(v(\hat{h}(x)))}{\varepsilon} \right) d\gamma \\ &= \int_{\mathbb{R}} \left(\frac{\varepsilon}{2} v'^2 d\gamma + \int_{\mathbb{R}} \frac{W(v)}{\varepsilon} \right) d\gamma_1. \end{aligned}$$

Therefore problem (4.18) reduces to the one-dimensional problem (4.19). Using the compact embedding of $H^1_{\gamma_1}(\mathbb{R})$ in $L^2_{\gamma_1}(\mathbb{R})$ (see [12, Th. 4.10]) and the direct method of the calculus of variations, we get that (4.19) has a minimizer. Moreover, by the Γ-convergence of the one-dimensional functionals in the strong $L^2_{\gamma_1}(\mathbb{R})$-topology towards the a multiple of the perimeter (which can be obtained exactly as in the classical Modica-Mortola Theorem since compact embedding of $BV_{\gamma_1}(\mathbb{R})$ in $L^1_{\gamma_1}(\mathbb{R})$ holds), we find that every sequence of minimizers v_ε of (4.19) has a subsequence strongly converging towards the characteristic of the half-line of measure m. □

We finally give another convergence result for the prescribed curvature problem in case of uniqueness of minimizers.

Proposition 4.5.6. *Let $g \in L^2_\gamma(X)$, then the following assertions are equivalent:*

- *the functional*
$$F_g(E) = P_\gamma(E) + \int_E g d\gamma \tag{4.20}$$
has a unique minimizer in the class of sets of finite perimeter;
- *the functional*
$$\overline{F}_g(u) = \overline{F}(u) + \int_X u g d\gamma \tag{4.21}$$
has a unique minimizer in $BV_\gamma(X)$.

Moreover, when this holds the two minimizers coincides. Finally, if u_ε is a sequence in $H^1_\gamma(X)$ satisfying

$$\sup_\varepsilon \left(F_\varepsilon(u_\varepsilon) + \int_X u_\varepsilon g d\gamma \right) \leq C$$

for some $C > 0$, then u_ε has a subsequence strongly converging to χ_E in $L^2_\gamma(X)$, where E is the common minimizer of (4.20) *and* (4.21)*.*

Proof. We first notice that the problem (4.20) always has a solution. Indeed, arguing as in [43], if E_n is a minimizing sequence for (4.20), it has a subsequence weakly converging to some $u \in BV_\gamma(X)$. By the lower semicontinuity of the total variation and the coarea formula we then have

$$\begin{aligned}\inf_E \left(P_\gamma(E) + \int_E g d\gamma \right) &\geq \int_X |D_\gamma u|_H + \int_X u g d\gamma \\ &= \int_0^1 \left(P_\gamma(\{u > t\}) + \int_{\{u>t\}} g(x) d\gamma(x) \right) dt\end{aligned}$$

and thus the sets $\{u > t\}$ minimize F_g for almost every t. As $\overline{F}$ is the relaxation of the perimeter we have that the minimum values in (4.20) and (4.21) are the same and thus any minimizer of F_g is also a minimizer of $\overline{F}_g$. This shows that if uniqueness does not hold in (4.20) then it does not hold in (4.21), too. Now, if u is a minimizer of $\overline{F}_g$, applying the coarea formula once again we get

$$\begin{aligned}\inf_E F_g(E) = \overline{F}_g(u) &\geq \int_X |D_\gamma u|_H + \int_X u g d\gamma \\ &= \int_0^1 \left(P_\gamma(\{u > t\}) + \int_{\{u>t\}} g(x) d\gamma(x) \right) dt.\end{aligned}$$

As above, this implies that $\{u > t\}$ solves (4.20) for almost every t. Therefore, if the minimizer of $\overline{F}_g$ is not a characteristic function, then uniqueness

does not hold neither in (4.20) nor in (4.21). This proves the first part of the Proposition.
The second statement easily follows from Theorem 4.5.3. Indeed, as the functionals $F_\varepsilon(u) + \int_X ugd\gamma$ Γ-converge to $\overline{F}_g$ in the weak $L^2_\gamma(X)$-topology, for every sequence u_ε bounded in energy, there exists a subsequence weakly converging to χ_E (where E is the unique minimizer of (4.20) and (4.21)). However, by the lower semicontinuity of the norm,

$$m^{\frac{1}{2}} \geq \varliminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2_\gamma(X)} \geq \|\chi_E\|_{L^2_\gamma(X)} = m^{\frac{1}{2}}.$$

Thus $\|u_\varepsilon\|_{L^2_\gamma(X)}$ converges to $\|\chi_E\|_{L^2_\gamma(X)}$, which implies the strong convergence of u_ε. $\square$

Remark 4.5.7. *In chapter 5, we provide an example of functionals for which uniqueness of minimizers holds, namely*

$$P_\gamma(E) + \int_X (g - \lambda)\, d\gamma$$

where $g : X \to \mathbb{R}$ *is convex and* $\lambda \in (0, +\infty)$ *is large enough.*

4.6 Conclusion and perspectives

In this chapter we computed the Γ-limit of the Allen-Cahn functional in Wiener spaces. In order to do so we studied the connections between symetrization and isoperimetry in this setting. It has shown that in infinite dimensions, the perimeter might not be the right functional to consider when dealing with variational problems since it is not lower-semicontinuous for the weak $L^2_\gamma(X)$ convergence.

This chapter, and the next one, started the investigation of the lower-semicontinuity and representation formulas for integral functionals in the Wiener space. However there is still a lot to do in this direction. As pointed out in [11] a crucial missing point is a precise knowledge of the structure of $D_\gamma u$ for $u \in BV_\gamma(X)$.

Chapitre 5

Convex minimizers for infinite dimensional variational problems

Abstract

In this chapter, we show convexity of solutions to a class of convex variational problems in the Gauss and in the Wiener space. We give two proofs of this. The first approach relies on the method of Alvarez, Lasry and Lions and the second on the concavity maximum principle of Korevaar. An important tool in the proof is a representation formula for integral functionals in this infinite dimensional setting that we prove. It extends previous analogous results in the classical Euclidean framework.

Résumé

Dans ce chapitre nous étudions la convexité des minimiseurs de certains problèmes variationnels dans les espaces de Gauss et de Wiener. Nous donnons deux démonstrations de ce résultat. La première suit l'approche d'Alvarez, Lasry et Lions tandis que la deuxième consiste en une version géométrique du principe du maximum de Korevaar. L'un des ingrédients principaux dans ces preuves est une formule de représentation pour les fonctionnelles intégrales dans ce contexte gaussien. Celle-ci généralise une formule analogue très utilisée dans le cadre euclidien.

5.1 Introduction

The aim of this chapter is to study the convexity of the minimizers of some variational problems in Wiener spaces. In the Euclidean setting convexity is a widely discussed issue [97]. Recently, following previous work by Korevaar [99] and Alvarez, Lasry and Lions [7], Alter, Caselles and Chambolle [6, 40] showed the convexity of solutions to variational problems involving functionals with linear growth and in particular to the prescribed curvature problem.

The main goal of this chapter, is to extend these results to the (finite dimensional) Gauss space and to the (infinite dimensional) Wiener space. In this setting, very few results are currently available. To the best of our knowledge, the only result in this direction is contained in [43], where the authors proved the convexity of the solutions of the isoperimetric problem in convex domains. More explicitly they prove the following:

Theorem 5.1.1. [43] *Let C be a convex set of positive (Gaussian) measure and of finite (Gaussian) perimeter then there exists $\alpha > 0$ such that for every $v \in [\alpha, \gamma(C)]$, the solution of the constrained isoperimetric problem*

$$\min \{P_\gamma(E) \ : \ E \subseteq C \text{ and } \gamma(E) = v\}$$

has a unique solution which is convex.

We are interested in the convexity of solutions of the problem

$$\min_{\gamma(E)=v} P_\gamma(E) - \int_E g(x)\, d\gamma(x), \tag{5.1}$$

where g is a convex function.

The idea is to follow the approach of Caselles and Chambolle [40] in the Euclidean case. We will thus be naturally led to consider the variational problem

$$\min_{BV_\gamma \cap L^2_\gamma(X)} \int_X |D_\gamma u|_H + \frac{1}{2}\int_X |u-g|^2 d\gamma \tag{5.2}$$

for which we will show convexity of the minimizers. More generally, we will prove that minimizers of

$$\min_{L^2_\gamma(X)} \int_X F(D_\gamma u)\, d\gamma + \frac{1}{2}\int_X |u-g|^2 d\gamma \tag{5.3}$$

are convex if F and g are convex (see Theorems 5.3.1 and 5.4.1).

The definitions and main properties of functions of bounded variation in Wiener spaces are given in Section 4.2 and will thus not be reminded here.

One of the main ingredients in our proof is the following Theorem due to Feyel and Üstünel [75, Thm 3.1 and 4.4]:

Theorem 5.1.2. *Let $u_n \in L^2_\gamma(X)$ be a sequence of convex functions converging to u in the weak $L^2_\gamma(X)$ convergence then u is convex.*

Remark 5.1.3. *In [75], the authors introduce the notion of almost surely convex function in the sense that it coincides almost everywhere with a convex function. Since here we work with $L^2_\gamma(X)$ functions this distinction is not relevant.*

For $F : H \to \mathbb{R}$ a convex function we denote by F^∞ its recession function defined for $h \in H$ as:

$$F^\infty(h) := \lim_{t\to+\infty} \frac{F(th)}{t}.$$

The main assumptions we will use are:

(H1) $F : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lsc bounded from below and attains its minimum.

(H2) F has $p \geq 1$ growth i.e. there exists α_1, β_1, α_2 and β_2 real positive such that

$$\alpha_1 |h|_H^p + \beta_1 \geq F(h) \geq \alpha_2 |h|_H^p - \beta_2 \qquad \forall h \in H.$$

Notice that of course (H2) implies (H1). Notice also that hypothesis (H2) includes the limiting case $p = 1$ which is of particular interest for us. Under hypothesis (H1), it is not restrictive to assume that $F(0) = 0$ and $F \geq 0$. By Hahn-Banach Theorem, for every proper convex lsc function $F : H \to \mathbb{R} \cup \{+\infty\}$, there exists $q \in H$ such that $F'(h) := F(h) - [q, h]_H$ satisfies (H1).

The plan of the chapter is the following. In Section 5.2 we prove a useful representation formula for integral functionals on Wiener spaces. In Section 5.3 we show the convexity of the minima of (5.2) in finite dimension, and in Section 5.4 we investigate the convexity of the minimizers in the infinite dimensional Wiener space. Finally in Section 5.5 we give an alternative approach of the convexity of the minimizers of the total variation in the Gauss space.

The first three sections of this chapter are based on a joint work with Antonin Chambolle and Matteo Novaga [47]. The last part of the chapter comes from the note [87].

5.2 Representation formula and relaxation of integral functionals

We extend in this section a representation formula for integral functionals. We start by proving it for functionals with linear growth.

Proposition 5.2.1. *Let $F : H \to \mathbb{R}$ be a convex function satisfying*

$$\alpha|h| + \beta \geq F(h) \geq 0 \qquad \forall h \in H$$

For $\mu \in \mathcal{M}(X, H)$, with $\mu = \mu^a\gamma + \mu^s$ its Radon-Nikodym decomposition, let

$$\int_X F(\mu) := \int_X F(\mu^a)d\gamma + \int_X F^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|,$$

then there holds

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{F}\mathcal{C}^1_b(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma. \tag{5.4}$$

Proof. For $\mu \in \mathcal{M}(X, H)$, with $\mu = \mu^a\gamma + \mu^s$ its Radon-Nikodym decomposition let $\mathcal{D}_F := \{\Phi = \sum_{i=1}^n \chi_{A_i} h_i \ / n \in \mathbb{N},\ A_i$ disjoint Borel sets, $h_i \in H,\ F^*(h_i) < +\infty\}$. Then we start by proving

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)\, d\gamma. \tag{5.5}$$

The proof is adapted from [63] and is divided into three steps.

*S*tep 1. Let

$$M(\mu) := \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma.$$

We will show that for every $h \in L^1_\gamma(X)$,

$$M(h\gamma) = \int_X F(h)d\gamma. \tag{5.6}$$

By definition of convex conjugate, it is readily checked that $M(h\gamma) \leq \int_X F(h)d\gamma$. We thus turn to the other inequality. By definition of the integral, for every $\delta > 0$, there exists $h_i \in H$ and $A_i \subset X$ with A_i disjoints Borel sets and $i \in [1, n]$ such that if we set

$$\theta = \sum_{i=1}^{n} \chi_{A_i} h_i$$

then $|\theta - h|_{L^1_\gamma} \leq \delta$. As F is of linear growth it is Lipschitz continuous and thus we can assume that also

$$|F(h) - F(\theta)|_{L^1_\gamma(X)} \leq \delta.$$

For every i, by definition of convex conjugate, there exists $\xi_i \in H$ such that

$$F(h_i) \leq [\xi_i, h_i]_H - F^*(\xi_i) + \delta.$$

Notice that since F is of linear growth, the ξ_i are uniformly bounded. From this, setting $\Phi = \sum_{i=1}^{n} \chi_{A_i} \xi_i$ we have

$$\begin{aligned}
\int_X F(h) d\gamma &\leq \int_X F(\theta) d\gamma + \delta \\
&= \sum_{i=1}^{n} \int_{A_i} F(h_i) d\gamma + \delta \\
&\leq \sum_{i=1}^{n} \int_{A_i} [\xi_i, h_i]_H - F^*(\xi_i) d\gamma + 2\delta \\
&= \int_X [\Phi, \theta]_H - F^*(\Phi) d\gamma + 2\delta \\
&\leq \int_X [\Phi, h]_H - F^*(\Phi) d\gamma + C\delta \\
&\leq M(h) + C\delta.
\end{aligned}$$

Since δ is arbitrary we have $M(h\gamma) = \int_X F(h) d\gamma$.

*S*tep 2. By reproducing the proof with F^∞ instead of F, $\frac{d\mu^s}{d|\mu^s|}$ instead of h and $|\mu^s|$ instead of γ we find, using that $\mathcal{D}_{F^\infty} = \mathcal{D}_F$ (since dom $F^* =$ dom $(F^\infty)^*$ by [122, Thm. 13.3]) and $(F^\infty)^* = 0$ in its domain,

$$M_\infty(\mu^s) := \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu^s] = \int_X F^\infty \left(\frac{d\mu^s}{d|\mu^s|} \right) d|\mu^s|.$$

*S*tep 3. It remains to show that

$$M(\mu^a \gamma + \mu^s) = M(\mu^a \gamma) + M_\infty(\mu^s).$$

One inequality is easily obtained, since

$$\begin{aligned}
M(\mu^a\gamma+\mu^s) &= \sup_{\Phi\in\mathcal{D}_F}\int_X[\Phi,\mu^a]_H d\gamma+\int_X[\Phi,d\mu^s]-\int_X F^*(\Phi)d\gamma\\
&\le\left(\sup_{\Phi\in\mathcal{D}_F}\int_X[\Phi,\mu^a]_H-F^*(\Phi)d\gamma\right)+\left(\sup_{\Phi\in\mathcal{D}_F}\int_X[\Phi,d\mu^s]\right)\\
&= M(\mu^a\gamma)+M_\infty(\mu^s).
\end{aligned}$$

For the opposite inequality, let $\delta>0$ be fixed then there exists Φ_1 and Φ_2 such that

$$\begin{aligned}
M(\mu^a\gamma) &\le \int_X[\Phi_1,\mu^a]_H-F^*(\Phi_1)d\gamma+\delta\\
M_\infty(\mu^s) &\le \int_X[\Phi_2,d\mu^s]_H+\delta.
\end{aligned}$$

Taking Φ equal to Φ_2 on a sufficiently small neighborhood of the support of μ^s and equal to Φ_1 outside this neighborhood, we get

$$\begin{aligned}
M(\mu^a\gamma)+M_\infty(\mu^s) &\le \int_X[\Phi,\mu^a]_H-F^*(\Phi)d\gamma+\int_X[\Phi,d\mu^s]_H+C\delta\\
&\le M(\mu^a\gamma+\mu^s)+C\delta
\end{aligned}$$

which gives the opposite inequality and shows (5.5).

For $\Phi\in\mathcal{D}_F$, the image of Φ, being a finite number of vectors of H, is included in a finite dimensional vector space V of H. If we now consider K the convex hull of these vectors then K is a convex polytope of V. We can then write $\Phi=\sum_{i=1}^N\theta_i\tilde{h}_i$ with $\tilde{h}_i$ the extremal points of K and $\theta_i\in L^1_\gamma(X)\cap L^1_\mu(X)$ with $\theta_i\ge 0$ and $\sum_{i=1}^N\theta_i\le 1$. Arguing as in [12, Section 2.1], $\gamma+|\mu|$ being tight we can approximate θ_i in $L^1_\gamma(X)\cap L^1_\mu(X)$ with $\theta_i^k\in\mathcal{FC}_b^1(X)$ in such a way that $\theta_i^k\ge 0$ and $\sum_{i=1}^N\theta_i^k\le 1$. As F^* is bounded and continuous on K, letting $\Phi^k:=\sum_{i=1}^N\theta_i^k\tilde{h}_i$ we have $\Phi^k\in\mathcal{D}_F$ and

$$\lim_{k\to+\infty}\int_X[\Phi^k,d\mu]-\int_X F^*(\Phi^k)d\gamma=\int_X[\Phi,d\mu]-\int_X F^*(\Phi)d\gamma.$$

□

We then deduce the following corollary:

Theorem 5.2.2. *For $F:H\to\mathbb{R}\cup\{+\infty\}$ a proper lsc convex function and $\mu\in\mathcal{M}(X,H)$, with $\mu=\mu^a\gamma+\mu^s$, then again*

$$\int_X F(\mu)=\sup_{\Phi\in\mathcal{FC}_b^1(X,H)}\int_X[\Phi,d\mu]_H-\int_X F^*(\Phi)d\gamma.$$

Proof. Case 1. First assume that (H1) holds. For $n \in \mathbb{N}$ let

$$F_n(p) := \sup_{|\Phi|_H \leq n} [\Phi, p]_H - F^*(\Phi).$$

Then F_n is of linear growth and F_n is a nondecreasing sequence converging pointwise to F and thus by the monotone convergence theorem,

$$\int_X F(\mu^a)\, d\gamma = \lim_{n\to\infty} \int_X F_n(\mu^a)\, d\gamma.$$

Analogously, $(F_n)^\infty$ converges monotonically to F^∞. Indeed, since F_n is nondecreasing, $(F_n)^\infty$ is clearly nondecreasing and

$$(F_n)^\infty(p) = \lim_{t\to+\infty} \frac{F_n(tp)}{t} \leq \lim_{t\to+\infty} \frac{F(tp)}{t} = F^\infty(p).$$

On the other hand, for every $\Phi \in \text{dom } F^* = \text{dom } (F^\infty)^*$, if $n \geq |\Phi|_H$, for every $p \in H$ and $t > 0$,

$$\frac{F_n(tp)}{t} \geq [\Phi, p]_H - \frac{F^*(\Phi)}{t}$$

and thus letting t goes to infinity and then n goes to infinity as well, we find

$$\lim_{n\to\infty} (F_n)^\infty(p) \geq \sup_{\Phi\in\text{dom } F^*} [\Phi, p]_H = F^\infty(p).$$

We thus have

$$\int_X F\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| = \lim_{n\to\infty} \int_X F_n\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|.$$

By Proposition 5.2.1, for every $n \in \mathbb{N}$,

$$\int_X F_n(\mu^a) d\gamma + \int_X F_n\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| = \sup_{\substack{\Phi\in\mathcal{F}\mathcal{C}_b^1(X,H)\\ |\Phi|_\infty \leq n}} \int_X [\Phi, d\mu] - \int_X F^*(\Phi)\, d\gamma. \tag{5.7}$$

Passing to the limit when n tends to infinity we get

$$\int_X F(\mu) = \sup_{\Phi\in\mathcal{F}\mathcal{C}_b^1(X,H)} \int_X [\Phi, d\mu] - \int_X F^*(\Phi)\, d\gamma.$$

Case 2. Let now F be a generic proper lsc convex function and $q \in H$ be such that $F'(h) := F(h) - [q, h]$ satisfies (H1). It is readily seen that

$(F')^\infty(h) = F^\infty(h) - [q,h]_H$ and $(F')^*(\Phi) = F^*(\Phi+q)$. Since (5.5) holds for F',

$$\begin{aligned}\int_X F(\mu) - \int_X [q,d\mu]_H &= \int_X F'(\mu)\\ &= \sup_{\Phi\in\mathcal{FC}_b^1(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi+q)d\gamma\\ &= \sup_{\Phi\in\mathcal{FC}_b^1(X,H)} \int_X [\Phi-q, d\mu]_H - \int_X F^*(\Phi)d\gamma\\ &= \sup_{\Phi\in\mathcal{FC}_b^1(X,H)} \left\{\int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma\right\} - \int_X [q,d\mu]_H.\end{aligned}$$

□

Remark 5.2.3. *An important example of functionals covered by the Theorem is given by the functionals with $p \geq 1$ growth.*

For F a proper lsc convex function, we can define the functional on $L^2_\gamma(X)$

$$\int_X F(D_\gamma u) := \sup_{\Phi\in\mathcal{FC}_b^1(X,H)} \int_X -u\,\mathrm{div}_\gamma\,\Phi - F^*(\Phi)\ d\gamma. \tag{5.8}$$

The functional defined in this way is thus lsc in $L^2_\gamma(X)$. By (5.4), we have

$$\int_X F(D_\gamma u) = \int_X F(\nabla u)d\gamma + \int_X F^\infty\left(\frac{dD^s_\gamma u}{d|D^s_\gamma u|}\right) d|D^s_\gamma u| \tag{5.9}$$

for $u \in BV_\gamma(X)$ with $D_\gamma u = \nabla u \gamma + D^s_\gamma u$ its Radon-Nikodym decomposition.

We then have the following relaxation result:

Proposition 5.2.4. *Let F be a proper lsc convex function then the functional $\int_X F(D_\gamma u)$ is the relaxation of the functional defined as $\int_X F(\nabla_H u)d\gamma$ for $u \in W^{1,1}_\gamma(X)$. If F satisfies also (H2) then is is also the relaxation of the functional $\int_X F(\nabla_H u)d\gamma$ defined on the smaller class $\mathcal{FC}_b^1(X)$.*

Proof. Case 1. Assume first that F satisfies (H1). We start by proving that

$$\int_X F(D_\gamma u) = \inf \underline{\lim} \left\{\int_X F(\nabla_H u_n)\,d\gamma,\ u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X)\right\}. \tag{5.10}$$

Thanks to Proposition 5.2.1, the inequality '$\leq$' is obvious. To prove the opposite inequality, we proceed as in [12, Th. 4.1] by using the Ornstein-Uhlenbeck semigroup. For $u \in L^2_\gamma(X)$ and $t > 0$, thanks to Proposition 4.2.3,

$$\begin{aligned}
\int_X F(D_\gamma T_t u) &= \sup_{\Phi\in\mathcal{FC}^1_b(X,H)} \int_X -T_t u \operatorname{div}_\gamma \Phi - F^*(\Phi)\, d\gamma \\
&= \sup_{\Phi\in\mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(\Phi)\, d\gamma \\
&\leq \sup_{\Phi\in\mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(T_t\Phi)\, d\gamma \\
&\leq e^{-t} \sup_{\Phi\in\mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(T_t\Phi)\, d\gamma \\
&\leq e^{-t} \int_X F(D_\gamma u)
\end{aligned}$$

where, as $F(0) = 0$ we have $F^* \geq 0$ and thus $e^{-t}F^* \leq F^*$. This inequality shows that

$$\int_X F(D_\gamma u) \geq \inf \underline{\lim} \left\{ \int_X F(\nabla_H u_n)\, d\gamma,\ u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X) \right\}.$$

Case 2. Let F be a proper lsc convex function and $q \in H$ be such that $F'(h) = F(h) - [q,h]$ satisfies (H1) then for $u \in L^2_\gamma(X)$,

$$\int_X F(D_\gamma u) = \int_X F'(D_\gamma u) - \int_X u \operatorname{div}_\gamma p\ d\gamma.$$

Therefore, by Case 1 applied to F' we get that

$$\int_X F(D_\gamma u) = \inf \underline{\lim} \left\{ \int_X F(\nabla_H u_n)\, d\gamma,\ u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X) \right\}.$$

Case 3. If now F satisfies (H2), by the density of $\mathcal{FC}^1_b(X)$ in $W^{1,p}_\gamma(X)$ for $p \geq 1$, for every $u \in W^{1,p}_\gamma(X)$ there exists $u_n \in \mathcal{FC}^1_b(X)$ tending to u in $W^{1,p}_\gamma(X)$ and almost everywhere. Then as $F(\nabla_H u_n) \leq \alpha_2 |\nabla_H u_n|^p_H + \beta_2$, by the dominated convergence theorem,

$$\int_X F(\nabla_H u_n)\ d\gamma \to \int_X F(\nabla_H u)\ d\gamma.$$

Thus starting from $W^{1,p}_\gamma(X)$ or $\mathcal{FC}^1_b(X)$ gives the same relaxation for $\int_X F(\nabla_H u) d\gamma$.

□

5.3 The finite dimensional case

In this section we focus on the finite dimensional problem. Let $F:\mathbb{R}^m\to\mathbb{R}$ be a convex function satisfying for $p\geq 1$,

$$(H_2')\qquad \alpha_2|h|^p+\beta_2\geq F(h)\geq \alpha|h|^p-\beta\qquad \forall h\in\mathbb{R}^m.$$

As before we set

$$\int_{\mathbb{R}^m} F(D_{\gamma_m}u)\,d\gamma_m := \sup_{\Phi\in\mathcal{C}_b^1(\mathbb{R}^m)}\int_{\mathbb{R}^m}\left(-u\,\mathrm{div}_\gamma\,\Phi - F^*(\Phi)\right)\,d\gamma_m.$$

By Theorem 5.2.2 and Proposition 5.2.4,

$$\int_X F(D_{\gamma_m}u)=\int_{\mathbb{R}^m}F(\nabla u)d\gamma_m+\int_{\mathbb{R}^m}F^\infty\left(\frac{dD^s_{\gamma_m}u}{d|D^s_{\gamma_m}u|}\right)d|D^s_{\gamma_m}u|$$

and this functional also coincides with the relaxation for the $L^2_{\gamma_m}(\mathbb{R}^m)$ topology of the functional classically defined on Lipschitz functions u by $\int_{\mathbb{R}^m}F(\nabla u)d\gamma_m$. In this finite dimensional setting this representation formula is not new (see [28] and [36]).
We show in this section the convexity of the solutions of

$$\min_{u\in L^2_{\gamma_m}(\mathbb{R}^m)}\int_{\mathbb{R}^m}F(D_{\gamma_m}u)+\frac{(u-g)^2}{2}\,d\gamma_m. \tag{5.11}$$

Formally the Euler-Lagrange equation of this problem reads

$$-\mathrm{div}\,\nabla F(\nabla u)\;+\;x\cdot\nabla F(\nabla u)\;+\;u\;=\;g. \tag{5.12}$$

Theorem 5.3.1. *Let $F:\mathbb{R}^m\to\mathbb{R}$ be a convex function satisfying (H2') and $g\in L^2_{\gamma_m}(\mathbb{R}^m)$ be a convex function. The minimizer of* (5.11) *is then convex.*

Proof. We consider $F_n\to F$ a sequence of smooth, uniformly convex functions, with quadratic growth which converge locally uniformly to F. The functional $\int_{\mathbb{R}^m}F_n(\nabla u)d\gamma_m$ is then finite if and only if $u\in H^1_{\gamma_m}(\mathbb{R}^m)$.
We consider for $\varepsilon>0$ the approximation

$$g_\varepsilon(x)\;=\;\max\{g(x),-\frac{1}{\varepsilon}\}+\varepsilon x^2+\frac{1}{\varepsilon}F_n^*(\varepsilon x)$$

so that $g_\varepsilon\to g$ locally uniformly as $\varepsilon\to 0$. Indeed, it follows from the uniform convexity of F_n that F_n^* is differentiable, hence

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F_n^*(\varepsilon x)\;=\;\nabla F_n^*(0)\cdot x\;=\;0.$$

Since $F_n(p) \geq C(|p|^2+1)$, $F_n^*(q) \leq C(|q|^2-1)$ and $g_\varepsilon \in L^2_{\gamma_m}(\mathbb{R}^m)$.
In particular, letting

$$u_\varepsilon(x) \;=\; \frac{F_n^*(\varepsilon x)}{\varepsilon} + m\varepsilon - \frac{1}{\varepsilon} \;\in\; L^2_{\gamma_m}(\mathbb{R}^m)\,,$$

we have

$$-\mathrm{div}\, \nabla F_n(\nabla u_\varepsilon) + x\cdot\nabla F_n(\nabla u_\varepsilon) + u_\varepsilon \;=\; -m\varepsilon + \varepsilon x^2 + \frac{F_n^*(\varepsilon x)}{\varepsilon} + m\varepsilon - \frac{1}{\varepsilon} \;\leq\; g_\varepsilon(x)$$

hence u_ε is a classical subsolution of the approximate problem. We observe that both g_ε and u_ε have superlinear growth at infinity.
We now consider the solution $\bar{u}$ of

$$\min_{u \geq u_\varepsilon} \int_{\mathbb{R}^m} F_n(\nabla u) + \frac{(u-g_\varepsilon)^2}{2}\, d\gamma_m \tag{5.13}$$

which by definition is above u_ε.
We must show that it is a supersolution of

$$-\mathrm{div}\, \nabla F_n(\nabla u) \;+\; x\cdot\nabla F_n(\nabla u) \;+\; u \;=\; g_\varepsilon. \tag{5.14}$$

Let us first notice that by [107], the function $\bar{u}$ is Hölder continuous. Assume that $\bar{u}$ is not a supersolution of (5.13) then there exists $x_0 \in \mathbb{R}^m$ and a smooth function ϕ such that $\phi < \bar{u}$ in $\mathbb{R}^m \backslash \{x_0\}$, $\phi(x_0) = \bar{u}(x_0)$ and

$$-\mathrm{div}\, \nabla F_n(\nabla \phi) \;+\; x\cdot\nabla F_n(\nabla \phi) \;+\; \phi \;-\; g_\varepsilon < 0 \quad \text{at } x_0. \tag{5.15}$$

By the smoothness of ϕ we can assume that inequality (5.15) holds for $\phi+\delta$ in a neighborhood of x_0 for δ small. Replacing ϕ by $\phi - \eta|x-x_0|^2$ we can further assume that (5.15) holds on the open set $\{\phi+\delta > \bar{u}\}$. As $v = \max(\phi+\delta, \bar{u}) \geq u_\varepsilon$, we have

$$\int_{\mathbb{R}^m} F_n(\nabla v) + \frac{(v-g_\varepsilon)^2}{2} d\gamma_m \geq \int_{\mathbb{R}^m} F_n(\nabla \bar{u}) + \frac{(\bar{u}-g_\varepsilon)^2}{2} d\gamma_m$$

and thus

$$\int_{\{\phi+\delta>\bar{u}\}} F_n(\nabla\phi) + \frac{(\phi+\delta-g_\varepsilon)^2}{2} d\gamma_m \geq \int_{\{\phi+\delta>\bar{u}\}} F_n(\nabla \bar{u}) + \frac{(\bar{u}-g_\varepsilon)^2}{2} d\gamma_m.$$

Using that $F_n(\nabla\bar{u}) - F_n(\nabla\phi) \geq \nabla F_n(\nabla\phi)\cdot(\nabla\bar{u}-\nabla\phi)$ by convexity of F_n and

$$\frac{(\bar{u}-g_\varepsilon)^2}{2} - \frac{(\phi+\delta-g_\varepsilon)^2}{2} \geq \frac{(\phi+\delta-\bar{u})^2}{2} + (\phi+\delta-g_\varepsilon)(\bar{u}-\phi-\delta)$$

we get

$$
\begin{aligned}
0 &\geq \int_{\{\phi+\delta>\bar{u}\}} \nabla F_n(\nabla\phi)\cdot(\nabla\bar{u}-\nabla\phi) + \frac{(\phi+\delta-\bar{u})^2}{2} + (\phi+\delta-g_\varepsilon)(\bar{u}-\phi-\delta)d\gamma_m \\
&= \int_{\{\phi+\delta>\bar{u}\}} \left[-\mathrm{div}\,\nabla F_n(\nabla\phi) + x\cdot\nabla F_n(\nabla\phi) + \phi+\delta-g_\varepsilon\right](\bar{u}-\phi-\delta) + \frac{(\phi+\delta-\bar{u})^2}{2} d\gamma_m \\
&> 0
\end{aligned}
$$

and thus a contradiction. The integration by part used above is justified by the fact that $\{\phi+\delta>\bar{u}\}$ is an open set on the boundary of which $\phi+\delta$ and $\bar{u}$ agree.

Notice that using the same arguments it can be shown that there is no contact between u_ε and $\bar{u}$ so that $\bar{u}$ is in fact an unconstrained minimizer of the energy.

Now, thanks to [7, Proposition 3], given any supersolution u of (5.14), with superlinear growth, the convex envelope u^{**} is still a supersolution. Moreover, if $u \geq u_\varepsilon$, then clearly $u^{**} \geq u_\varepsilon$ (which is convex).

Hence, if we define $\tilde{u} \leq \bar{u}$ as the infimum of all supersolutions of (5.14) which are larger than u_ε, it is also the infimum of their convex envelopes (hence it is a locally uniform limit of convex supersolutions) and therefore is convex. It is also a supersolution.

Let us now show that $\tilde{u}$ is a viscosity solution. If it were not, there would exist a smooth ϕ and $x \in \mathbb{R}^m$ with $\tilde{u}(x) = \phi(x)$, and $\tilde{u} < \phi$ in $\mathbb{R}^m \setminus \{x\}$, with

$$
-\mathrm{div}\,\nabla F_n(\nabla\phi(x)) \ + \ x\cdot\nabla F_n(\nabla\phi(x)) \ + \ \phi(x) \ > \ g_\varepsilon(x).
$$

In particular, $\tilde{u}(x) > u_\varepsilon(x)$, otherwise x would also be a local maximum of $u_\varepsilon - \phi$ and the reverse inequality should hold. Now, by standard arguments, we check that $\min\{\tilde{u}, \phi-\delta\}$ is still a supersolution, larger than u_ε, if $\delta > 0$ is small enough, a contradiction.

Hence $\tilde{u}$ is a solution of (5.14). By [95, Theorem 4], $\tilde{u}$ is a $C^{1,1}$ function and thus by [7, Lemma 2], $\tilde{u}$ satisfies (5.14) almost everywhere (and also weakly). The function $\tilde{u}$ is therefore a critical point of the (strictly convex) energy, hence the unique solution to (5.11) (with F replaced with F_n and g with g_ε). Denote now this solution by u_ε^n.

Let us now show that we can send $\varepsilon \to 0$ and then $n \to \infty$.

Comparing the energy of u_ε^n with the energy of 0, we find that

$$
\|u_\varepsilon^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \ \leq \ 2\|g_\varepsilon\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \ \leq \ 2\|g\|_{L^2_{\gamma_m}(\mathbb{R}^m)} + 2\left\|\varepsilon x^2 + \frac{1}{\varepsilon}F_n^*(\varepsilon x)\right\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \tag{5.16}
$$

so that $\|u_\varepsilon^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)}$ is uniformly bounded. Hence, we can send $\varepsilon \to 0$ and will find that $u_\varepsilon^n \rightharpoonup u^n$. By a Theorem of Dudley [65], u_ε^n converges locally uniformly to u^n which is thus convex. By the lower-semicontinuity of the energy, u^n is the solution of problem (5.11) with F replaced with F_n.

Analogously, $u^n \to u$ locally uniformly since by (5.16), $\|u^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \leq 2\|g\|_{L^2_{\gamma_m}(\mathbb{R}^m)}$ and thus u is convex. Let us show that u is the minimizer of (5.11). We start by proving that

$$\varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) d\gamma_m \geq \int_{\mathbb{R}^m} F(\nabla u) d\gamma_m. \tag{5.17}$$

Since u^n is a sequence of convex functions converging to $u \in L^2_{\gamma_m}(\mathbb{R}^m)$ then, up to subsequence, ∇u^n converges to ∇u almost everywhere. Moreover, for all $R > 0$ there exists $C = C(R, v)$ such that $\|\nabla u^n\|_{L^\infty(B_R)} \leq C$ for all $n \in \mathbb{N}$. This is a general property of convex functions and we refer to [40, Theorem 3] for further details. By the dominated convergence Theorem, we then get

$$\varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) d\gamma_m \geq \varliminf_{n\to\infty} \int_{B_R} F_n(\nabla u^n) d\gamma_m = \int_{B_R} F(\nabla u) d\gamma_m.$$

Letting $R \to +\infty$ we obtain (5.17).
Now if v is a Lipschitz function in $L^2_{\gamma_m}(\mathbb{R}^m)$, as F_n converges locally uniformly to F,

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla v) d\gamma_m = \int_{\mathbb{R}^m} F(\nabla v) d\gamma_m$$

and thus, by the minimality of u^n and (5.17),

$$\begin{aligned} \int_{\mathbb{R}^m} F(\nabla v) + \frac{(v-g)^2}{2}\, d\gamma_m &= \lim_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla v) + \frac{(v-g)^2}{2}\, d\gamma_m \\ &\geq \varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) + \frac{(u^n-g)^2}{2}\, d\gamma_m \\ &\geq \int_{\mathbb{R}^m} F(\nabla u) + \frac{(u-g)^2}{2}\, d\gamma_m. \end{aligned}$$

Since Lipschitz functions are dense in energy in $L^2_{\gamma_m}(\mathbb{R}^m)$, we obtain that u is a minimizer of (5.11). □

Remark 5.3.2. *The proof directly extends to variational problems of the form*

$$\min_{u \in L^2(\mu)} \int_{\mathbb{R}^m} F(\nabla u) + \frac{(u-g)^2}{2}\, d\mu$$

for measures $d\mu = \mu(x)\, dx$, with $\mu(x) = e^{-(Ax,x)}$ and $A > 0$.

Remark 5.3.3. *Arguing as in the Theorem 5.4.1 of the next section, we see that this result extends to generic proper lsc convex functions F.*

5.4 The infinite dimensional case

In this final section we turn to the infinite dimensional problem.

Theorem 5.4.1. *Let $F : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function and $g \in L^2_\gamma(X)$ be a convex function then the minimizer of*

$$J(u) := \int_X F(D_\gamma u) + \frac{1}{2}\int_X (u-g)^2 d\gamma$$

is convex.

Proof. Case 1. We start by assuming that F satisfies also (H2) .

Let $g_m = \mathbb{E}_m(g)$ then g_m is a convex function. Let also $\bar{u}_m$ be the minimizer of

$$\min_{u \in L^2_\gamma(X):\, u=\mathbb{E}_m u} J_m(u) := \int_X F(D_\gamma u) + \frac{1}{2}\int_X |u-g_m|^2 d\gamma.$$

Thanks to (5.9), if u depends only on the first m variables then

$$\int_X F(D_\gamma u) = \int_X F_m(D_\gamma u) = \int_{\mathbb{R}^m} F_m(D_{\gamma_m} u)$$

where $F_m(h) = F(\Pi_m h)$. By Theorem 5.3.1, u_m is thus a convex function. As $J_m(\bar{u}_m) \leq J_m(0)$ and since $g_m \to g$ in $L^2_\gamma(X)$, $\bar{u}_m$ is bounded in $L^2_\gamma(X)$ and is thus weakly converging to $\bar{u}$ which is therefore convex by [75, Theorem 4.4].

We now show that $\bar{u}$ is the minimizer of J.

If u_m is a weakly converging sequence to $u \in L^2_\gamma(X)$, then by strong convergence of g_m to g we have

$$\varliminf_{m\to\infty} \frac{1}{2}\int_X |u_m - g_m|^2 d\gamma \geq \frac{1}{2}\int_X |u-g|^2 d\gamma.$$

By the lower semicontinuity of $\int_X F(D_\gamma u)$ (which comes from (5.8)) we then have

$$\varliminf_{m\to\infty} J_m(u_m) \geq J(u).$$

Thus if $u \in \mathcal{FC}_b^1(X)$, by minimality of $\bar{u}_m$,

$$J(u) = \lim_{m\to+\infty} J_m(u) \geq \underline{\lim}_{m\to+\infty} J_m(\bar{u}_m) \geq J(\bar{u}). \tag{5.18}$$

Since we assumed that F satisfies (H2), by Proposition 5.2.4, the space $\mathcal{FC}_b^1(X)$ is dense in energy in $L^2_\gamma(X)$ and thus inequality (5.18) proves that $\bar{u}$ is the minimizer of J in $L^2_\gamma(X)$.

Case 2. If F is a proper lsc convex function, we can approximate it by a convex function F_δ with linear growth

$$F_\delta(p) := \delta|p|_H + \inf_{q\in H}\left(\frac{1}{\delta}|p-q|_H + F(q)\right).$$

By Case 1, the minimizer u_δ of the functional with F_δ instead of F is convex. As before, we have that u_δ weakly converges to a convex function u in $L^2_\gamma(X)$. As $W^{1,1}_\gamma(X)$ is dense in energy in $L^2_\gamma(X)$, in order to conclude, it is sufficient to prove that for every $v \in W^{1,1}_\gamma(X) \cap L^2_\gamma(X)$,

$$\int_X F(\nabla_H v)d\gamma \geq \overline{\lim_{\delta\to 0}} \int_X F_\delta(\nabla_H v) \tag{5.19}$$

and

$$\underline{\lim}_{\delta\to 0} \int_X F_\delta(D_\gamma u_\delta) \geq \int_X F(D_\gamma u). \tag{5.20}$$

For inequality (5.19) we can assume that $\int_X F(\nabla_H v)d\gamma < +\infty$ then as for the Moreau regularization, $\lim_{\delta\to 0} F_\delta(p) = F(p)$ for every $p \in H$ so that for every $v \in W^{1,1}_\gamma(X)$, $F_\delta(\nabla_H v)$ converges almost everywhere to $F(\nabla_H v)$ and since $F_\delta(\nabla_H v) \leq \delta|\nabla_H v|_H + F(\nabla_H v)$, by the dominated convergence Theorem, inequality (5.19) follows.

For inequality (5.20), we start by noticing that by calculus on inf-convolutions and convex conjugates, we have,

$$F^*_\delta(q) = \inf_{\substack{|p|_H \leq \frac{1}{\delta} \\ |p-q|_H \leq \delta}} F^*(p),$$

where we take as a convention that $F^*_\delta(q) = +\infty$ if $B_{\frac{1}{\delta}} \cap B_\delta(q) = \emptyset$. Therefore, for every $q \in H$, as soon as $|q|_H \leq \frac{1}{\delta}$, we have $F^*_\delta(q) \leq F^*(q)$ and thus

$$\overline{\lim_{\delta\to 0}} F^*_\delta(q) \leq F^*(q) \qquad \forall q \in H.$$

If now $\Phi \in \mathcal{FC}_b^1(X,H)$ with $F^*(\Phi)$ integrable, we have $F_\delta^*(\Phi) \leq F^*(\Phi)$ for δ small enough and thus by the reverse Fatou lemma,

$$\overline{\lim_{\delta \to 0}} \int_X F_\delta^*(\Phi)\, d\gamma \leq \int_X \overline{\lim_{\delta\to 0}} F_\delta(\Phi) d\gamma \leq \int_X F^*(\Phi)\, d\gamma. \tag{5.21}$$

We can now conclude since for every $\Phi \in \mathcal{FC}_b^1(X,H)$ with $\int_X F^*(\Phi)\, d\gamma$ we have using (5.21),

$$\begin{aligned}\underline{\lim_{\delta\to 0}} \int_X F_\delta(D_\gamma u_\delta) &\geq \underline{\lim_{\delta\to 0}} \int_X -u_\delta \operatorname{div}_\gamma \Phi - F_\delta^*(\Phi)\, d\gamma \\ &\geq \int_X -u \operatorname{div}_\gamma \Phi - F(\Phi)\, d\gamma\end{aligned}$$

Taking then the supremum on all $\Phi \in \mathcal{FC}_b^1(X,H)$ and using (5.4), we get (5.20).

$\square$

Remark 5.4.2. *Notice that, by taking $F(h) = |h|^p$ with $p \geq 1$, Theorem 5.4.1 applies in particular to the p-Dirichlet problems*

$$\min_{L^2_\gamma(X)} \int_X |\nabla_H u|_H^p\, d\gamma + \frac{1}{2}\int_X |u-g|^2 d\gamma.$$

Remark 5.4.3. *When X is an Hilbert space, there is another definition of the gradient due to Da Prato which gives an alternative definition of Sobolev and BV spaces (see [12, Section 5]). Roughly speaking it corresponds to $Du := Q^{-\frac{1}{2}}\nabla_H u$. Theorem 5.4.1 then applies to the associated total variation since it is given by the choice*

$$F(h) = \left(\sum_{i=1}^{+\infty} \frac{1}{\lambda_i}|h_i|^2\right)^{\frac{1}{2}}$$

where the λ_i's are the eigenvalues of Q.

Remark 5.4.4. *Notice that in the proofs of Theorem 5.3.1 and 5.4.1 we made standard Γ-convergence arguments (see [32]).*

We can now use these convexity results to show the convexity of solutions of (5.1).

Theorem 5.4.5. *Let g be a convex function in $L^2_\gamma(X)$ and let u be the minimizer of (5.2). Let $\overline{\lambda} = \inf\{\lambda \ :\ \gamma(u \leq \lambda) > 0\}$. If $\overline{v} = \gamma(\{u \leq \overline{\lambda}\})$ then for every $v > \overline{v}$ there exists a unique solution to (5.1) and this solution is convex.*

Proof. The proof follows quite standard arguments so that we only sketch it (see [43] and [6] for details). Let us first consider the problem

$$\min P_\gamma(E) + \int_E (g-\lambda)d\gamma. \qquad (P_\lambda)$$

Then as in Proposition 34 of [43], by the direct method of the calculus of variations and by the co-area formula it is not difficult to show that (P_λ) has a minimum E_λ. By [43, Lemma 8] we have $E_{\lambda_1} \subset E_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$.
Setting $w(x) = \inf\{\lambda : x \in E_\lambda\}$, it is not hard to see that $w \in BV_\gamma \cap L^2_\gamma(X)$ and that w solves (5.2) (see [43] again or Lemma 3.5 in [45]). By the uniqueness of minimizers of (5.2), $w = u$ and $E_\lambda = \{u < \lambda\}$ for almost every λ (and then for every λ by an approximation procedure).
By Proposition 4.2.10, the function $\lambda \to \gamma(E_\lambda)$ is continuous on $]\overline{\lambda}, +\infty[$ and nondecreasing. Together with the inclusion property of the E_λ this implies the uniqueness of the minimizers of (P_λ). Moreover, the sets E_λ solve the problem:

$$\min_{\gamma(E_\lambda)=\gamma(E)} P_\gamma(E) + \int_E g d\gamma.$$

Vice-versa, if E_v solves (5.1) and $v > \overline{v}$ then there exists $\lambda > \overline{\lambda}$ such that $\gamma(E_\lambda) = v$ and as E_v solves (P_λ) we get $E_v = E_\lambda$. $\square$

Remark 5.4.6. *If $F : H \to \mathbb{R}$ is homogeneous of degree one and such that*

$$c|h|_H \leq F(h) \leq C|h|_H \qquad \forall h \in H,$$

then F satisfies (H2) and we can define the anisotropic perimeter P_F by

$$P_F(E) := \int_X F(D_\gamma \chi_E).$$

Repeating verbatim the proof of [72, Section 5.5], (and using that smooth cylindrical functions are dense in $BV_\gamma(X)$ by Proposition 5.2.4), we still have a coarea formula,

$$\int_X F(D_\gamma u) = \int_{\mathbb{R}} P_F(\{u < t\})\, dt \qquad \forall u \in BV_\gamma(X).$$

Using Theorem 5.4.1, it is then not difficult to extend Theorem 5.1.1 and Theorem 5.4.5 to these anisotropic perimeters P_F.

Notice that in the Wiener space, the solution of the Wulff problem

$$\min_{\gamma(E)=v} P_F(E) \qquad (5.22)$$

is quite simple. If F attains its minimum on the sphere at some direction $\nu_{\min}$ then by the isoperimetric inequality, if $E_{\nu_{\min}}$ is the half-space of volume v and normal $\nu_{\min}$ and E is any other set with volume v,

$$P_F(E_{\nu_{\min}}) = F(\nu_{\min})P_\gamma(E_{\nu_{\min}}) \leq F(\nu_{\min})P_\gamma(E) \leq P_F(E)$$

and thus $E_{\nu_{\min}}$ is the minimizer of (5.22)*. If instead F does not attain its minimium on the sphere, there is no solution to* (5.22)*.*

We can finally state a simple corollary.

Corollary 5.4.7. *Let g be a convex function in $L^2_\gamma(X)$ and let*

$$F(E) = P_\gamma(E) + \int_E g\, d\gamma.$$

Then two situations can occur:

- *If $\min F < 0$ then there exists a unique non-empty minimizer of F. Moreover this minimizer is convex.*
- *If $\min F = 0$ then there exists at most one non-empty minimizer of F which is then convex.*

Proof. The two possibilities corresponds respectively to $\overline{\lambda} < 0$ and $\overline{\lambda} \geq 0$. □

5.5 A Geometric proof for the total variation in Gauss space

The aim of this section is to show an alternative proof of Theorem 5.3.1 when F is the total variation (which was our main motivation in the previous sections) based on ideas of Korevaar [99]. More precisely, we will show that for $g \in L^2_{\gamma_m}(\mathbb{R}^m)$ a convex function then the solution of

$$\min_{BV_\gamma(\mathbb{R}^m)\cap L^2_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_{\gamma_m}u| + \frac{1}{2}\int_{\mathbb{R}^m} |u-g|^2 d\gamma_m \tag{5.23}$$

is convex. As a by-product of our analysis we will also get that the minimizer of the Ornstein-Uhhlenbeck functional

$$\min_{H^1_{\gamma_m}(\mathbb{R}^m)\cap L^2_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} \frac{|\nabla u|^2}{2} + \frac{1}{2}\int_{\mathbb{R}^m} |u-g|^2 d\gamma_m$$

is convex if g is convex.

We recall some facts about pairings between measures and bounded functions (see [16] for more details).
We define the space X_2 to be the space of bounded functions z with $\mathrm{div}_\gamma z \in L^2_{\gamma_m}(\mathbb{R}^m)$. For every smooth open set Ω, the trace $[z \cdot \nu]$ can be defined in such a way that the integration by part formula

$$\int_\Omega (z, D_{\gamma_m} u) d\gamma_m + \int_\Omega u \,\mathrm{div}_\gamma z d\gamma_m = \int_{\partial\Omega} [z \cdot \nu] u \gamma_m(x) \mathcal{H}^{m-1}$$

holds for $z \in X_2$ and $u \in BV_{\gamma_m} \cap L^2_{\gamma_m}(\mathbb{R}^m)$ where as usual $(z, D_{\gamma_m} u)$ is the measure defined by

$$\int_{\mathbb{R}^m} (z \cdot D_{\gamma_m} u) d\gamma_m = -\int_{\mathbb{R}^m} u\varphi \,\mathrm{div}_\gamma z d\gamma_m - \int_{\mathbb{R}^m} uz \cdot \nabla\varphi d\gamma_m$$

for every $\varphi \in \mathcal{C}^1_c(\mathbb{R}^m, \mathbb{R}^m)$.

5.5.1 Convexity of the minimizer

In this section we are going to prove the following result:
Let $g \in L^2_{\gamma_m}(\mathbb{R}^m)$ be a convex function then the minimizer of

$$\min_{BV_{\gamma_m} \cap L^2_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_{\gamma_m} u| + \frac{1}{2} \int_{\mathbb{R}^n} |u - g|^2 d\gamma_m \tag{5.24}$$

is a convex function.
As in many other papers involving the total variation, we are going to study first the regularized problem:

$$\min_{BV_{\gamma_m} \cap L^2_{\gamma_m}(\mathbb{R}^m)} J_\varepsilon(u) = \int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |D_\gamma u|^2} d\gamma + \frac{1}{2} \int_{\mathbb{R}^m} |u - g|^2 d\gamma_m \tag{5.25}$$

where as usual, if the Radon-Nikodym decomposition of $D_{\gamma_m} u$ is given by $D_{\gamma_m} u = \nabla u d\gamma_m + D^s_{\gamma_m} u$ we let

$$\int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |D_\gamma u|^2} d\gamma = \int_{\mathbb{R}^m} \sqrt{\varepsilon^2 + |\nabla u|^2} d\gamma_m + |D^s_{\gamma_m} u|(\mathbb{R}^m).$$

As a simple consequence of the Reshetnyak's continuity Theorem we have that J_ε is lower semicontinuous for the $L^2_{\gamma_m}(\mathbb{R}^m)$ convergence (see [10]).
We start by studying the Dirichlet problem on balls, namely

$$\min_{BV_{\gamma_m}(B_R)} \int_{B_R} \sqrt{\varepsilon^2 + |D_\gamma u|^2} d\gamma + \frac{1}{2} \int_{B_R} |u - g|^2 d\gamma_m + \int_{\partial B_R} |u - M| \gamma_m(x) d\mathcal{H}^{m-1}(x). \tag{5.26}$$

Here B_R is the ball of radius R centered in the origin and M is a constant to be chosen later. The term $\int_{\partial B_R} |u - M|\gamma_m(x) d\mathcal{H}^{m-1}(x)$ can be seen as a Dirichlet term (see [81] and [15]). In the following we will note by $F(p) = \sqrt{\varepsilon^2 + |p|^2}$.
On bounded domains, by Theorem 6.7 in [15] we can give a characterization of the minimizers of (5.26)

Theorem 5.5.1 (Characterization of the minima). *A function $u \in BV_{\gamma_m}(B_R)$ minimizes* (5.26) *if and only if $\frac{\nabla u}{\sqrt{\varepsilon^2+|\nabla u|^2}} \in X_2$ and*

$$-\operatorname{div}_\gamma\left(\frac{\nabla u}{\sqrt{\varepsilon^2+|\nabla u|^2}}\right) + u = g, \qquad \frac{\nabla u}{\sqrt{\varepsilon^2+|\nabla u|^2}} \cdot D^s_{\gamma_m} u = |D^s_{\gamma_m} u| \quad |D^s_{\gamma_m} u| - a.e.$$

$$\text{and} \qquad [\frac{\nabla u}{\sqrt{\varepsilon^2+|\nabla u|^2}} \cdot \nu] \in \operatorname{sign}(M - u) \quad \mathcal{H}^{m-1} - a.e. \text{ in } \partial B_R.$$

where ν is the outward normal to B_R.

We can prove the following comparison principle:

Proposition 5.5.2 (Comparison). *Let $g_1 \geq g_2$ and $\varphi_1 \geq \varphi_2$ then the minimizers u_i with $i = 1, 2$ of*

$$\min_{BV_{\gamma_m}(B_R)} \int_{B_R} F(D_{\gamma_m} u) d\gamma_m + \frac{1}{2}\int_{B_R} |u - g_i|^2 d\gamma_m + \int_{\partial B_R} |u - \varphi_i|\gamma_m(x) d\mathcal{H}^{m-1}(x)$$

verify $u_1 \geq u_2$.

Proof. The proof follows closely the proof of Theorem 5.16 p.145 of [15]. Let

$$\Phi_\varphi(u) = \int_{B_R} F(D_{\gamma_m} u) d\gamma_m + \int_{\partial B_R} |u - \varphi|\gamma_m(x) d\mathcal{H}^{m-1}(x).$$

By Theorem 5.5.1 we know that $p \in \partial\Phi_\varphi$ if and only if:

$$p = -\operatorname{div}_\gamma(z), \qquad z = \nabla F(\nabla u), \qquad z \cdot D^s_{\gamma_m} u = |D^s_{\gamma_m} u| \quad \text{and} \quad [z\cdot\nu] \in \operatorname{sign}(\varphi - u)$$

where we used the decomposition $D_{\gamma_m} u = \nabla u d\gamma_m + D^s_{\gamma_m} u$. We thus have that $u_i + p_i = g_i$ for $i = 1, 2$. By multiplying these two equalities by $(u_2 - u_1)^+$ (which is the positive part of $u_2 - u_1$), subtracting them and integrating, we find

$$\int_{B_R} [(u_2 - u_1) + (p_2 - p_1)]\,(u_2 - u_1)^+ d\gamma_m = \int_{B_R} (g_2 - g_1)(u_2 - u_1)^+ d\gamma_m \leq 0.$$

But on the left-hand side we have that

$$\int_{B_R} (u_2 - u_1)(u_2 - u_1)^+ d\gamma_m \geq 0.$$

So that proving $\int_{B_R} (p_2 - p_1)(u_2 - u_1)^+ d\gamma_m \geq 0$ would imply the claim. This inequality is obtained as following:

$$\begin{aligned}\int_{B_R} (p_2 - p_1)(u_2 - u_1)^+ d\gamma_m &= -\int_{B_R} \operatorname{div}_\gamma(z_2 - z_1)\,(u_2 - u_1)^+ d\gamma_m \\ &= \int_{B_R} \big((z_2 - z_1), D_{\gamma_m}(u_2 - u_1)^+\big) \\ &\quad - \int_{\partial B_R} [(z_2 - z_1)\cdot\nu](u_2 - u_1)^+ \gamma_m(x) d\mathcal{H}^{m-1}(x)\end{aligned}$$

Now, on the one hand, by Corollary C.16 of [15] (see also Proposition 2.8 of [16]),

$$\int_{B_R} \big((z_2 - z_1), D_{\gamma_m}(u_2 - u_1)^+\big) = \int_{B_R} \left((z_2 - z_1), D_{\gamma_m}(u_2 - u_1)\right) d\gamma_m.$$

Writing that $D_{\gamma_m} u_i = \nabla u_i d\gamma_m + D^s_{\gamma_m} u_i$ we find that,

$$\begin{aligned}\int_{B_R} \left((z_2 - z_1), D_{\gamma_m}(u_2 - u_1)\right)\, d\gamma_m \quad &= \quad \int_{B_R} (z_2 - z_1)\cdot(\nabla u_2 - \nabla u_1)\, d\gamma_m \\ &\quad + \int_{B_R} (z_2 - z_1)\cdot(D^s_{\gamma_m} u_2 - D^s u_1).\end{aligned}$$

By convexity of F, $\int_{B_R} (F(\nabla u_2) - F(\nabla u_1))\cdot(\nabla u_2 - \nabla u_1) d\gamma_m \geq 0$ and since $z_i \cdot D^s_{\gamma_m} u_i = |D^s_{\gamma_m} u_i|$, we have $\int_{B_R} (z_2 - z_1)\cdot(D^s_{\gamma_m} u_2 - D^s_{\gamma_m} u_1) \geq 0$.
On the other hand, by a simple argument, it can be shown that

$$\int_{\partial B_R} [(z_1 - z_2)\cdot\nu](u_2 - u_1)^+ \gamma_m(x) d\mathcal{H}^{m-1} \leq 0,$$

which ends the proof. □

With this comparison property in hands, we can prove that for M large enough, the minimizer of (5.26) makes vertical contact angle with the boundary of B_R. In the following, we will say that a function v is a supersolution of (5.26) if it minimizes the functional with $\tilde{g} \geq g$ and $\varphi \geq M$.

Proposition 5.5.3 (vertical contact angle)**.** *If* $C \geq \frac{m}{\varepsilon r} + \frac{R}{\varepsilon} + r + |g|_{L^\infty(B_R)}$, *then*

$$v(x) = \begin{cases} C - \sqrt{r^2 - (x - x_0)^2} & \text{if } x \in B_r(x_0) \\ M & \text{otherwise} \end{cases}$$

is a supersolution of (5.26) *if* $B_r(x_0) \subset B_R$*. Then for* $M > C$*, the minimizer of* (5.26) *has vertical contact angle with* ∂B_R*.*

Proof. We must show that for C large enough,

$$-\operatorname{div}_\gamma(\nabla F(\nabla v)) + v - g \geq 0.$$

A direct computation shows that in $B_r(x_0)$ we have $\nabla v = \dfrac{x - x_0}{\sqrt{r^2 - (x - x_0)^2}}$ thus

$$\nabla F(\nabla v) = \frac{x - x_0}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}}.$$

From this we get that

$$\begin{aligned} -\operatorname{div}_\gamma(\nabla F(\nabla v)) + v - g \geq & -\frac{m}{\varepsilon r} + \frac{x - x_0}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}} \cdot x + C \\ & - \sqrt{r^2 - (x - x_0)^2} - |g|_{L^\infty(B_R)} \\ \geq & -\frac{m}{\varepsilon r} - \frac{|x - x_0|}{\sqrt{\varepsilon^2 r^2 + (1 - \varepsilon^2)|x - x_0|^2}}|x| + C \\ & - r - |g|_{L^\infty(B_R)} \\ \geq & -\frac{m}{\varepsilon r} - \frac{R}{\varepsilon} + C - r - |g|_{L^\infty(B_R)} \end{aligned}$$

Thus if $C \geq \frac{m}{\varepsilon r} + \frac{R}{\varepsilon} + r + |g|_{L^\infty(B_R)}$ then v is a super-solution.
If $M > C$, then considering balls of radius r such that $\partial B_r \cap \partial B_R$ is reduced to a point, by the comparison Theorem 5.5.2, if u minimizes (5.26) then $M > C \geq v \geq u$ and thus by Theorem 5.5.1 we have

$$[\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \cdot \nu] = 1 \qquad \mathcal{H}^{m-1} - a.e. \text{ on } \partial B_R$$

which is the vertical contact angle condition. □

The interior regularity of minimizers of (5.26) easily follows by a result of Giaquinta, Modica and Soucek [81].

Proposition 5.5.4. *Let g be a $\mathcal{C}^\alpha$ function then the minimizer of* (5.26) *is* $\mathcal{C}^{2,\alpha}(B_R)$.

Proof. By Theorem 3.3 of [81] we have that minimizers of

$$\min_{BV_{\gamma_m}(B_R)} \int_{B_R} F(Du)d\gamma_m + \int_{B_R} G(x,u)d\gamma_m + \int_{\partial B_R} |u-M|\gamma_m(x)d\mathcal{H}^{m-1}(x)$$

are locally Lipschitz if $G(x,u)$ verifies the following hypothesis:

- $|\frac{\partial G}{\partial u}| + |\frac{\partial^2 G}{\partial u \partial x}| \leq C.$
- $\frac{\partial^2 G}{\partial u^2} \geq 0.$

Originally we have $G(x,u) = \frac{1}{2}|u-g(x)|^2$ which does not verifies exactly the hypothesis. However if we set $\tilde{G}(x,u) = \Psi(u) - g(x)u + \frac{1}{2}g(x)^2$ where $\Psi(u) = \frac{1}{2}u^2$ if $u \leq C$ and Ψ convex, $\mathcal{C}^2$ with linear growth at infinity then $\tilde{G}$ verifies the condition mentioned above. The Euler-Lagrange equation verified by the minimizers with $\tilde{G}$ instead of G is

$$\frac{\partial \Psi}{\partial u} + \partial\Phi_\varphi(u) = g(x). \tag{5.27}$$

Now we can apply Theorem 3.3 of [81] to find that solutions of (5.27) are locally Lipschitz. Exactly as in Proposition 5.5.2 the comparison principle holds for this equation and thus M (respectively $-M$) is a supersolution (respectively a subsolution). This implies that if $C \geq M$ solutions of (5.27) are also solutions of (5.26) which are thus locally Lipschitz. By classical regularity theory for elliptic equations (see [82]) this implies that the solutions are indeed $\mathcal{C}^{2,\alpha}(B_R)$. □

Remark 5.5.5. *This proposition in particular applies for g convex since convex functions are locally Lipschitz.*

Having only interior regularity it is not possible to directly apply the results of Korevaar [99] which need continuity up to the boundary. The idea will be to use a geometric version of Korevaar's argument to get the convexity of the minimizers.

For simplicity, in this part of the proof we focus on the case $\varepsilon = 1$. By rescaling, the general case of $\varepsilon \neq 1$ can be easily recovered (the Gaussian measure γ_m is not invariant by this scaling but it does not matter). Consider now the set (see Figure 5.1)

$$E = \left\{(x,t) \in B_R \times [-M;M] \,/\, t < u(x)\right\}. \tag{5.28}$$

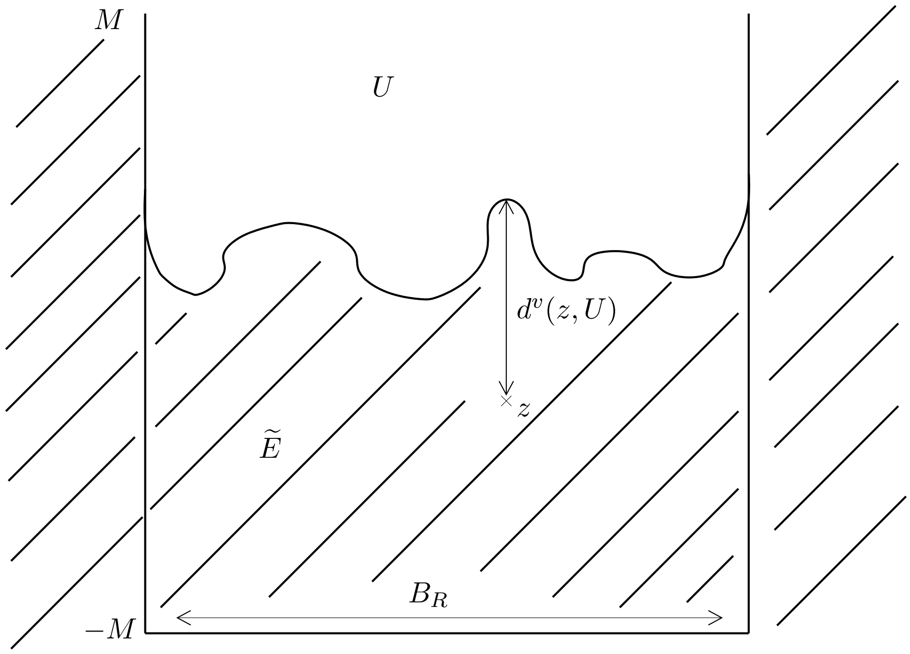

Figure 5.1: The set $\tilde{E}$

The aim is to show that E is a concave set. First we need to show that E is regular. For this we follow an idea of Giusti (see [83] and [84]) showing that E is a solution of a certain obstacle problem.
For F a set of finite perimeter in $\mathbb{R}^{m+1}$ let $\tilde{P}(F)$ be defined by

$$\tilde{P}(F) = \int_{\partial^* F} \gamma_m(x) d\mathcal{H}^m(x,t).$$

$\tilde{P}$ is thus the perimeter associated to the measure $\mu(x,t) = \gamma_m(x)dxdt$. Let now $H(x,t) = (t - g(x))\gamma_m(x)$ then we have the following:

Proposition 5.5.6. *The set $\tilde{E} = E \cup (B_R^c \times [-M;M])$, where E is defined in* (5.28)*, is a minimizer of*

$$\tilde{P}(F) + \int_F H(x,t)\, dx\, dt \tag{5.29}$$

among all sets containing $B_R^c \times [-M;M]$. As a consequence $\partial\tilde{E}$ is $\mathcal{C}^1$.

Proof. Let us define the field

$$z(x,t) = \begin{cases} \left(-\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \frac{1}{\sqrt{1+|\nabla u|^2}}\right) & (x,t) \in B_R \times]-M;M[\\ -\nu^{B_R}(x) & (x,t) \in \partial B_R \times]-M;M[\end{cases}.$$

Then z is a X_2 vector field in $B_R \times]-M;M[$ satisfying $|z|_{\mathbb{R}^{m+1}} = 1$ and $[z \cdot \nu^{\tilde{E}}] = 1$ where $\nu^{\tilde{E}}$ is the outward normal to $\tilde{E}$. Moreover if $z = (z', z_{m+1})$ with $z' \in \mathbb{R}^m$ and $z_{m+1} \in \mathbb{R}$ then setting by a slight abuse of notations

$$\operatorname{div}_\gamma z = \operatorname{div}_\gamma z' + \frac{\partial z_{m+1}}{\partial t}$$

we have $\operatorname{div}_\gamma z = g - u$. Hence if $F \Delta \tilde{E} \subset B_R \times]-M;M[$, as $t < u(x)$ in E,

$$\begin{aligned} \int_{\tilde{E}\backslash F} (\operatorname{div}_\gamma z) d\mu = \int_{\tilde{E}\backslash F} (g-u) d\mu \leq \int_{\tilde{E}\backslash F} (g(x) - t) d\mu \\ = -\int_{\tilde{E}\backslash F} H(t,x) dx dt = \int_{\tilde{E}\cap F} H dx dt - \int_{\tilde{E}} H dx dt. \end{aligned}$$

On the other hand we have:

$$\int_{\tilde{E}\backslash F} (\operatorname{div}_\gamma z) d\mu = \int_{\partial^*(\tilde{E}\backslash F)} [z \cdot \nu^{\tilde{E}\backslash F}] \gamma_m(x) d\mathcal{H}^m(x,t)$$

But $\tilde{E}\backslash F = \tilde{E} \cap F^c$ and as noticed by Figalli, Maggi and Pratelli in [78],

$$\partial^*(\tilde{E} \cap F^c) = J_{\tilde{E},F^c} \cup \left(\partial^* \tilde{E} \cap (F^c)^{(1)}\right) \cup \left(\partial^* F^c \cap \tilde{E}^{(1)}\right)$$

where $J_{\tilde{E},F^c} = \left\{x \in \partial^* \tilde{E} \cap \partial^* F^c / \nu^{\tilde{E}} = \nu^{F^c}\right\}$. Moreover we have:

$$\nu^{\tilde{E}\backslash F} = \begin{cases} \nu^{\tilde{E}} & \text{in } \partial^* \tilde{E} \cap (F^c)^{(1)} \\ \nu^{F^c} = -\nu^F & \text{in } \partial^* F^c \cap \tilde{E}^{(1)} \\ \nu^{\tilde{E}} = -\nu^F & \text{in } J_{\tilde{E},F^c} \end{cases}.$$

From this we find

$$\begin{aligned} \int_{\tilde{E}\backslash F} (\operatorname{div}_\gamma z) d\mu &= \int_{\partial^* \tilde{E} \cap F^{(0)}} \gamma_m d\mathcal{H}^m - \int_{\partial^* F \cap \tilde{E}^{(1)}} \nu^F \cdot z \gamma_m d\mathcal{H}^m + \int_{J_{\tilde{E},F^c}} [z \cdot \nu^{\tilde{E}}] \gamma_m d\mathcal{H}^m \\ &\geq \int_{\partial^* \tilde{E} \cap F^{(0)}} \gamma_m d\mathcal{H}^m - \int_{\partial^* F \cap \tilde{E}^{(1)}} \gamma_m d\mathcal{H}^m + \int_{J_{\tilde{E},F^c}} [z \cdot \nu^{\tilde{E}}] \gamma_m d\mathcal{H}^m. \end{aligned}$$

We thus find:

$$\int_{\tilde{E}\cap F} H dxdt - \int_{\tilde{E}} H dxdt \geq \int_{\partial^* \tilde{E}\cap F^{(0)}} \gamma_m d\mathcal{H}^m - \int_{\partial^* F\cap \tilde{E}^{(1)}} \gamma_m d\mathcal{H}^m + \int_{J_{\tilde{E},F^c}} [z\cdot\nu^{\tilde{E}}]\gamma_m d\mathcal{H}^m.$$

Similarly, studying what happens on $F\backslash\tilde{E}$ we get:

$$\int_{\tilde{E}\cap F} H dxdt - \int_{F} H dxdt \leq \int_{\partial^* F\cap \tilde{E}^{(0)}} \gamma_m d\mathcal{H}^m - \int_{\partial^* \tilde{E}\cap F^{(1)}} \gamma_m d\mathcal{H}^m + \int_{J_{F,\tilde{E}^c}} [z\cdot\nu^{F}]\gamma_m d\mathcal{H}^m.$$

Summing these two inequalities and using that $\int_{J_{\tilde{E},F^c}}[z\cdot\nu^{\tilde{E}}]\gamma_m d\mathcal{H}^m = \int_{J_{F,\tilde{E}^c}}[z\cdot\nu^F]\gamma_m d\mathcal{H}^m$ we have:

$$\begin{aligned}
&\int_{\partial^* F\cap(\tilde{E}^{(0)}\cup\tilde{E}^{(1)})} \gamma_m(x) d\mathcal{H}^m(x,t) + \int_F H(x,t)dxdt \geq \\
&\int_{\partial^* \tilde{E}\cap(F^{(0)}\cup F^{(1)})} \gamma_m(x) d\mathcal{H}^m(x,t) + \int_{\tilde{E}} H(x,t)dxdt.
\end{aligned}$$

Adding to this equality $\int_{\partial^*\tilde{E}\cap\partial^* F}\gamma_m(x)d\mathcal{H}^m(x,t)$ and using that $\mathcal{H}^m((A^{(1)}\cup A^{(0)}\cup\partial^* A)^c) = 0$ for every set of finite perimeter $A\subset\mathbb{R}^{m+1}$, we find as desired that

$$\int_{\partial^* F}\gamma_m(x)d\mathcal{H}^m(x,t) + \int_F H(x,t)dxdt \geq \int_{\partial^*\tilde{E}}\gamma_m(x)d\mathcal{H}^m(x,t) + \int_{\tilde{E}} H(x,t)dxdt.$$

The regularity of $\partial\tilde{E}$ follows from an old paper of Miranda [108]. We point out that in the paper cited above, the results are written for the classical perimeter without curvature terms. However, the argument is based on a blow-up procedure under which our functional reduces to the classical perimeter. □

We can now prove the concavity of $\tilde{E}$.

Proposition 5.5.7. *The set $\tilde{E}$ is concave thus u is convex.*

Proof. We will show that the set $U = \overline{E^c}$ is convex (see Figure 5.1). Let us define for every $z = (x,t) \in \overline{B}_R \times [-M; M]$ the vertical distance of z to U by

$$d^v(z,U) = \inf\left(|t-t'| \,/\, (x',t')\in U\right).$$

The function d^v is continuous since ∂U is a $\mathcal{C}^1$ surface by Proposition 5.5.6. U is a compact set thus the function

$$C(\lambda, z, z') = d^v(\lambda z + (1-\lambda)z', U) \qquad \text{for } (\lambda, z, z') \in [0;1]\times U\times U$$

attains its maximum. If this maximum is zero then U is convex and we are done. Assume on the contrary that this maximum is positive.
By the vertical contact angle condition we can assume that this maximum is attained at points z and z' in the interior of $\overline{B}_R \times [-M; M]$. Moreover, if $z = (x,t) \in U$, by decreasing t (which increases C), we can assume that $t = u(x)$. Analogously we can assume that $z' = (x', u(x'))$. Then we find

$$C(\lambda, z, z') = u(\lambda x + (1-\lambda)x') - \lambda u(x) - (1-\lambda)u(x').$$

We are thus in the situation of applying Korevaar's concavity maximum principle [99] to conclude. We briefly recall the argument for the reader's convenience.
As (λ, z, z') is a point of maximum, the gradient in x and in x' is zero and thus

$$\nabla u(\lambda x + (1-\lambda)x') = \nabla u(x) = \nabla u(x').$$

As the second derivative of $C(\lambda, (x+\tau, u(x+\tau)), (x'+\tau, u(x'+\tau)))$ is nonpositive in zero for every direction $\tau \in \mathbb{R}^m$ we get

$$D^2 u(\lambda x + (1-\lambda)x') - \lambda D^2 u(x) - (1-\lambda) D^2 u(x') \le 0.$$

Using the equation satisfied by u, this yields the desired contradiction. □

We now finally turn to the proof of our main result:

Theorem 5.5.8. *Let $g \in L^2_{\gamma_m}(\mathbb{R}^m)$ be a convex function and u be the minimizer of*

$$\min_{BV_{\gamma_m} \cap L^2_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} |D_{\gamma_m} u| + \frac{1}{2} \int_{\mathbb{R}^m} |u-g|^2 d\gamma_m$$

then u is a convex function.

Proof. By Proposition 5.5.3 we see that if u_R is the minimizer of (5.26) then it is convex. Arguing as in Theorem 5.3.1, we see that u_R converges locally uniformly to u_ε the minimizer of (5.25). Analogously, we can let ε goes to zero and get that u_ε converges to u the solution of (5.24) which is thus convex. □

Let us also notice that along the same lines we can prove the following result:

Theorem 5.5.9. *Let g be a convex $L^2_{\gamma_m}(\mathbb{R}^m)$ function then the minimizer of*

$$\min_{u \in H^1_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} \frac{|\nabla u|^2}{2} d\gamma_m + \frac{1}{2} \int_{\mathbb{R}^m} |u-g|^2 d\gamma_m$$

is convex.

Proof. Let

$$J_\lambda(u) = \lambda^2 \int_{\mathbb{R}^m} \left[\sqrt{1 + \frac{|\nabla u|^2}{\lambda^2}} - 1\right] d\gamma_m + \frac{1}{2}|u - g|^2 d\gamma_m$$

then u_λ minimizes J_λ if and only if it minimizes

$$\int_{\mathbb{R}^m} \sqrt{\lambda^2 + |\nabla u|^2} d\gamma_m + \frac{1}{2\lambda} \int_{\mathbb{R}^m} |u - g|^2 d\gamma_m.$$

Thus u_λ is convex. Using that for any p,

$$\lim_{\lambda \to \infty} \lambda^2 \left[\sqrt{1 + \frac{|p|^2}{\lambda^2}} - 1\right] = \frac{|p|^2}{2}$$

we get the conclusion. □

Remark 5.5.10. *If we want to follow this approach for more general functionals, there is a difficulty due to the lack of boundary regularity of the minimizers. More precisely, when reasoning as in Proposition 5.5.6, these functionals give rise to anisotropic perimeters, for which it is not known if the minimizers of the corresponding obstacle problem are smooth in a neighborhood of the obstacle.*

5.6 Conclusion and perspectives

In this chapter we proved convexity of the minimizers of some variational problems in Gauss and Wiener spaces. An essential tool in the proof is a representation formula for integral functional in this setting. This is a first step towards the generalization of these kind of results in infinite dimensions.

Following an approach *à la* Almgren-Taylor-Wang, as in [23] our convexity results could open the way to a study of a motion by mean curvature in Gauss and Wiener spaces which has never been investigated yet.

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Table des figures