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Équations aux dérivées partielles/***Partial differential equations***

Small energy solutions to the Ginzburg-Landau equation

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Abstract. *We generalize to arbitrary dimensions a result obtained by F.H. Lin and T. Rivière [4],[5] in dimension three for solutions to the Ginzburg-Landau equation,as well as,in arbitrary dimensions, in the case of minimizing solutions.*

Solutions d'énergie petite de l'équation de Ginzburg-Landau

Résumé. Nous généralisons en dimension quelconque un résultat de F.H. Lin et T. Rivière [4],[5] démontré en dimension trois, pour des solutions de l'équation de Ginzburg-Landau, ainsi que pour les solutions minimisantes en dimension quelconque.

Version française abrégée

Soient N un entier supérieur à deux, et $B(0,1)$ la boule unité de $\mathbf{R}^N$. On considère, pour un petit paramètre $0 < \varepsilon < 1$, les solutions u de l'équation de Ginzburg-Landau :

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \qquad \text{dans} \quad B(0,1) \ . \tag{1}$$

Notre résultat principal est le suivant:

Théorème 1. – *Il existe une constante $C > 0$ dépendant seulement de N telle que, si*

$$\int_{B(0,1)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \ \le \eta \log \frac{1}{\varepsilon} \ , \tag{2}$$

alors

$$|u(0)| \ge 1 - C\eta^{\frac{1}{4N(N+2)}} \ .$$

Ici η désigne un nombre positif arbitraire.

Le cas $N = 2$ découle directement des estimations en [1]. Le cas $N = 3$ et le cas minimisant pour N quelconque ont été prouvés par F.H. Lin et T. Rivière [5],[6]. Notre approche reprend en partie la stratégie développée dans [5], [6]. Elle s'appuie sur une identité de Pohozaev, une formule de monotonie pour l'équation (1) (voir [1],[5],[6],[7]), et sur une décomposition de Hodge adéquate de la quantité $u \times \nabla u$ (comme dans [1]), qui représente, en gros, le gradient de la phase de u.

1. Introduction

Let N be an integer larger than two, and let $B(0,1)$ be the unit ball of $\mathbf{R}^N$. For $0<\varepsilon<1$ a small parameter, we consider solutions u of the Ginzburg-Landau equation:

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2) \qquad \text{in } \; B(0,1)\;. \tag{1}$$

Our main result is the following:

Theorem 1. – *There exists a constant $C>0$ depending only on N such that if*

$$\int_{B(0,1)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \;\le \eta \log\frac{1}{\varepsilon}\;, \tag{2}$$

where η is any arbitrary positive constant, then

$$|u(0)| \ge 1 - C\eta^{\frac{1}{4N(N+2)}}.$$

The case $N=2$ can be easily derived from estimates in [1]. The case $N=3$ and the case of minimizing solutions (N arbitrary) were derived by F.H. Lin and T. Rivière [5],[6].

By scaling, it follows from Theorem 1 that if if u is a solution of (1) on the ball $B(0,R)$, and if

$$R^{2-N}\int_{B(0,R)} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \;\le \eta \log\frac{R}{\varepsilon}\;,$$

then, as before, $|u(0)| \ge 1 - C\eta^{\frac{1}{4N(N+2)}}$. .It suffices to make the changes of variable $x \to Rx$ and of parameter $\varepsilon \to R\varepsilon$.

Our proof borrows many ideas and the general outline from [5], [6]. We will sketch the main ideas involved. If not otherwise stated, C will denote any arbitrary absolute constant, depending only on N. We will denote $B_r(x_0)$ the ball of radius r centered at the point x_0. In case $x_0=0$, we simply write B_r. We will also assume $\eta<1/4$. In Section 2 we recall a few preliminary estimates, already derived in [1],[2], [3], [5], [6],[7].

2. Some preliminary estimates

Lemma 1. – ([1],[2]) *Assume u verifies (1) . Then*

$$u \in \mathbf{C}^\infty(B(0,1)), \quad |u| \le C\,, \quad |\nabla u| \le \frac{C}{\varepsilon} \quad \text{on } B(0,\frac{1}{2}). \tag{3}$$

Lemma 2. – (Pohozaev's identity) *Let $x_0 \in B(0,1)$, $r>0$ such that $B(x_0,r) \subset B(0,1)$. Define* $(N-2)F_\varepsilon(x_0,r) \equiv \dfrac{N-2}{2}\displaystyle\int_{B_r(x_0)} |\nabla u|^2 + \frac{N}{4\varepsilon^2}\int_{B_r(x_0)} (1-|u|^2)^2$. *One then has*

$$(N-2)F_\varepsilon(x_0,r) = r\int_{\partial B_r(x_0)} \frac{|\nabla_\tau u|^2}{2} - \frac{1}{2}\left|\frac{\partial u}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2. \tag{4}$$

A straightforward consequence of Lemma 2 is the following (see [7])

LEMMA 3. – (Monotonicity formula) *With the notations of Lemma 2, it holds*

$$\frac{d}{dr}\left(r^{2-N}F_\varepsilon(x_0,r)\right) = r^{2-N}\int_{\partial B_r(x_0)} |\partial_n u|^2 + (2(N-2)\varepsilon^2)^{-1}(1-|u|^2)^2. \tag{5}$$

Integrating (5) we obtain the crucial estimate

$$\int_{B_r(x_0)} \rho^{2-N}\left(|\partial_\rho u|^2 + (2(N-2)\varepsilon^2)^{-1}(1-|u|^2)^2\right) \le r^{2-N}F_\varepsilon(x_0,r), \tag{6}$$

where $\rho = |x-x_0|$ (note that ρ^{2-N} is, up to a multiplicative constant, the Green function for the Laplacian: this remark turns out to be fundamental in our analysis).

COROLLARY 1. – *Let $0<\delta<1/16$ and assume $\varepsilon \le \frac{\delta^2}{4}$. If u verifies (1) and (2), then there exists some $r_0 \in [\varepsilon^{1/2}, \delta/2]$, depending on u, such that, for some absolute constant C and $U_0 = B_{\delta^{-1}r_0}\setminus B_{\delta^2 r_0}$,*

$$\int_{U_0} \rho^{2-N}\left(|\partial_\rho u|^2 + (2(N-2)\varepsilon^2)^{-1}(1-|u|^2)^2\right) \le C\eta|\log\delta|. \tag{7}$$

Proof: (see [2], [7]) let $3k+1$ be the largest integer smaller than $|\log(2\varepsilon^{1/2})||\log\delta|^{-1}$, so that $k \ge C|\log\varepsilon||\log\delta|^{-1}$. Let $V_j = B_{\delta^{-3j-3}\varepsilon^{1/2}}\setminus B_{\delta^{-3j}\varepsilon^{1/2}}$. Since the V_j are disjoint, by (6) and (12)

$$\sum_{j=0}^{k}\int_{V_j}\rho^{2-N}\left(|\partial_\rho u|^2 + (2(N-2)\varepsilon^2)^{-1}(1-|u|^2)^2\right) \le F_\varepsilon(0,1) \le 3\eta|\log\varepsilon|.$$

Hence for some j, the integral $\int_{V_j}\ldots$ is smaller than $\eta|log\epsilon|/(k+1)$, and the result (7) follows with $r_0 = \delta^{-3j+2}\varepsilon^{1/2}$.

COROLLARY 2. – *There exists $r_1 \in [\delta^2 r_0, 2\delta^2 r_0]$ and $r_2 \in [(4\delta)^{-1}r_0, (2\delta)^{-1}r_0]$ such that,*

$$\int_{\partial B_{r_i}} |\partial_\rho u|^2 + (2(N-2)\varepsilon^2)^{-1}(1-|u|^2)^2 \le C\eta|\log\delta| r_i^{N-3}, \quad i=1,2\ , \tag{8}$$

$$r_i^{3-N}\int_{\partial B_{r_i}} |\nabla_\tau u|^2 \le C\left(\eta|\log\delta| + (\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0)\right), \quad i=1,2\ . \tag{9}$$

Proof: a simple mean-value argument and (7) yield (8). For (9) use (4) and (5), so that

$$r_i^{3-N}\int_{\partial B_{r_i}}|\nabla_\tau u|^2 \le r_i^{2-N}(N-2)F_\varepsilon(0,r_i)+(1/2)r_i^{3-N}\int_{\partial B_{r_i}}|\partial_\rho u|^2 \le (N-2)\frac{F_\varepsilon(0,\delta^{-1}r_i)}{(\delta^{-1}r_i)^{N-2}}+C\eta|\log\delta|.$$

The main ingredient in the proof of Theorem 1 is

PROPOSITION 1. – *Choose* $\delta = \eta^{1/8N}$, *and let* $W_0 = B_{2r_0} \setminus B_{r_0} \subset U_0$. *Then we have*

$$\int_{W_0} \frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \leq C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right] . \tag{10}$$

3. Proof of Proposition 1

We use the same Hodge type decomposition as in [1]: remark that

$$4|u|^2|\nabla u|^2 = 4|u \times \nabla u|^2 + |\nabla |u|^2|^2 . \tag{11}$$

Estimate of $|u \times \nabla u|$. The starting point, as in [1], is the equation $\mathrm{div}(u \times \nabla u) = 0$, obtained through exterior multiplication by u of equation (1). Using the formalism of differential forms we write the previous relation as

$$d^*(u \times du) = 0 \text{ in } \Omega , \qquad \text{where } \ du = \sum \partial_i u dx_i , \qquad d^* = \pm \star d \star , \tag{12}$$

and $\star$ denotes the Hodge star operator. We introduce the auxiliary Neumann problem

$$\Delta\xi = 0 \qquad \text{on } U_1 \equiv B_{r_2} \setminus B_{r_1} , \qquad \partial_n \xi = u \times \partial_n u \quad \text{on } \partial U_1 . \tag{13}$$

The solution ξ clearly exists, since $\int_{\partial U_1} u \times \partial_n u = 0$ by (12). Moreover, combining (8) and standard elliptic estimates, We obtain

$$r_0^{2-N} \int_{U_1} |\nabla \xi|^2 \leq C\eta |log\delta|. \tag{14}$$

By construction we have now $d^*((u \times du - d\xi)\mathbf{1}_{U_1}) = 0$ on $\mathbf{R}^N$, where $\mathbf{1}_{U_1}$ denotes the characteristic function of U_1. By the classical Hodge theory (see [4]), there exists some 2-form φ on $\mathbf{R}^N$ such that (here $1 < p < \infty$)

$$d^*\varphi = (u \times du - d\xi)\mathbf{1}_{U_1} , \quad d\varphi = 0 , \quad \varphi \in H^1 , \quad ||\nabla\varphi||_p \leq C_p(||\nabla u||_{L^p(U_1)} + ||\nabla \xi||_{L^p(U_1)}) . \tag{15}$$

In view of the identity $u \times du = d^*\varphi + d\xi$, and since we already have derived a good bound (14) for $\nabla\xi$,we observe, that it in order to bound $|u \times du|$, it suffices to bound $|\nabla\varphi|$. We therefore look for an elliptic equation for φ. For that purpose, let $0 < \beta < 1/2$ to be determined later, and let $f : \mathbf{R} \to [1, 1/(1-\beta)]$ be a smooth function such that $f(t) = 1/t$ if $t \geq 1-\beta$, $f(t) = 1$ if $t \leq 1 - 2\beta$, and $|f'| \leq 4$.

Define on $\mathbf{R}^N$ the function α such that $\alpha(x) = f(|u(x)|)^2$ on U_1, $\alpha \equiv 1$ otherwise, so that $0 \leq \alpha - 1 \leq 4\beta$ on $\mathbf{R}^N$. Note that $f(|u|)^2 u \times du = f(|u|)u \times d(f(|u|)u)$, hence (15) gives

$$\begin{aligned} d(\alpha d^*\varphi) = \omega_1 + \omega_2 + \omega_3 \equiv & \sum_{i<j} \mathbf{1}_{U_1}(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} dx_i \wedge dx_j + \\ & + (\sigma_{\partial B_{r_2}} - \sigma_{\partial B_{r_1}})f(|u|)u \times du \wedge d\rho - d(\mathbf{1}_{U_1}\alpha d\xi) , \end{aligned}$$

where $\sigma_{\partial B_{r_i}}$ stands for the surface measure on ∂B_{r_i}, $i = 1, 2$. We claim that

$$|\omega_1| \leq C\beta^{-2}\varepsilon^{-2}(1-|u|^2)^2 . \tag{16}$$

Indeed, if $|u| \geq 1-\beta$ then $(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} = \frac{u}{|u|}_{x_i} \times \frac{u}{|u|}_{x_j} = 0$, otherwise we have $\beta^2 \leq (1-|u|^2)^2$, and by (3) $|\omega_1| \leq C/\varepsilon^2$. Hence $|\omega_1| \leq C\varepsilon^{-2}\beta^{-2}\beta^2 \leq C\beta^{-2}\varepsilon^{-2}(1-|u|^2)^2$.

Since $d\varphi = 0$, $\Delta\varphi = dd^*\varphi = d(\alpha d^*\varphi) + d((1-\alpha)d^*\varphi)$, we are led to write $\varphi = \sum_{i=1}^4 \varphi_i$, where $\varphi_i = G * \omega_i$ for $i = 1,2,3$ and $G = c_N|x|^{2-N}$ is the fundamental solution of the Laplacian in $\mathbf{R}^N$, so that $\Delta\varphi_i = \omega_i$ for $i = 1,2,3,4$. The form φ_4 represents the remainder, i.e $\varphi_4 = \varphi - \sum_{i=1}^3 \varphi_i$. Next we will estimate each $||\nabla\varphi_i||_{L^2(W_0)}$ separately for $i = 1,2,3$. While the estimate for φ_2 ,φ_3 and φ_4 are more or less standard, the estimate for φ_1 is the central observation of the paper.

Estimate for φ_2: we have (see Appendix)

$$(17) \qquad ||\nabla\varphi_2||^2_{L^2(R^N)} \leq C\sum_{i=1}^2 r_i \int_{\partial B_{r_i}} |\nabla_\tau u|^2 \leq C\sum_{i=1}^2 (r_i^{N-2}\eta|\log\delta| + F_\varepsilon(x_0, \delta^{-1}r_0)) \ ,$$

$$(18) \qquad \int_{W_0} |\nabla\varphi_2|^2 \leq C\left(\delta^2 r_0^{N-2}\eta|\log\delta| + \delta^N F_\varepsilon(x_0,\delta^{-1}r_0)\right) \ .$$

Estimate for φ_3: standard estimates lead to $||\nabla\varphi_3||_{L^2(R^N)} \leq ||\alpha||_{L^\infty(U_1)}||\nabla\xi||_{L^2(U_1)}$, and by (14)

$$(19) \qquad r_0^{2-N}||\nabla\varphi_3||^2_{L^2(R^N)} \leq C\eta|log\delta|.$$

Estimate for φ_1: it uses the monotonicity formula in a crucial way. We claim that

$$(20) \qquad ||\nabla\varphi_1||^2_{L^2(R^N)} \leq C\beta^{-4}\eta|\log\delta|\left[F_\varepsilon(0,\delta^{-1}r_0) + C\eta r_0^{N-2}\delta^{2-N}|\log\delta|\right] \ .$$

Indeed, by the maximum principle, $|\varphi_1|$ achieves its supremum on U_1. For $x \in U_1$

$$|\varphi_1(x)| \leq C\beta^{-2}\int_{U_1}|x-y|^{2-N}\varepsilon^{-2}(1-|u|^2)^2 dy \qquad \text{(by definition and by (16))} \ .$$

Let $A_x = B(x,(4\delta)^{-1}r_0)\cap U_1$, $D_x = U_1 \setminus A_x$. The monotonicity formula (6) applied with $x_0 = x$ yields

$$\int_{A_x}|x-y|^{2-N}\varepsilon^{-2}(1-|u|^2)^2 dy \leq ((4\delta)^{-1}r_0)^{2-N}F_\varepsilon(x,(4\delta)^{-1}r_0) \leq C(\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0) \ ,$$

while by Corollary 1 we have

$$\int_{D_x}|x-y|^{2-N}\varepsilon^{-2}(1-|u|^2)^2 dy \leq C\int_{U_0}|y|^{2-N}\varepsilon^{-2}(1-|u|^2)^2 dy \leq C\eta|\log\delta| \ .$$

It follows

$$||\varphi_1||_\infty \leq C\beta^{-2}\left((\delta^{-1}r_0)^{2-N}(F_\varepsilon(0,\delta^{-1}r_0) + C\eta|\log\delta|\right).$$

On the other hand, combining Corollary 1 with (16), we deduce

$$\int_{R^N}|\omega_1| \leq C\beta^{-2}\eta|log\delta|.$$

Finally, we derive estimate (20) by multiplying the equation $\Delta\varphi_1 = \omega_1$ by φ_1, integrating by parts, and combining the two previous inequalities .

Estimate for φ_4: by definition $\varphi_4 = \varphi - \sum_{i=1}^3 \varphi_i$, so that $\Delta\varphi_4 = \Delta\varphi - \sum_{i=1}^3 \omega_i = d((1-\alpha)d^*\varphi)$, and one checks, using $0 \le \alpha - 1 \le 4\beta$, that

$$||\nabla\varphi_4||^2_{L^2(R^N)} \le C\beta \sum_{i=1}^{3} ||\nabla\varphi_i||^2_{L^2(R^N)} \ . \tag{21}$$

Applying (21), together with (17), (19), (20) we find

(22)

$$||\nabla\varphi_4||^2_{L^2(R^N)} \le C\beta \left[(1 + C\beta^{-4}\eta|\log\delta|)F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}(\beta^{-4}\eta^2\delta^{2-N}|log\delta|^2 + \eta|\log\delta|)\right] \ .$$

Finally combining (22), (18), (19), (20) we obtain , choosing $\beta = \eta^{1/8}$, $\delta = \eta^{1/8N}$,

$$\int_{W_0} |\nabla\varphi|^2 \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right] \ .$$

Since $|u \times du| \le |d^*\varphi| + |d\xi|$, this yields likewise

$$\int_{W_0} |u \times du|^2 \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right] \ . \tag{23}$$

Estimate of $|\nabla|u|^2|$. The equation for $|u|^2$ reads $-\Delta(1-|u|^2) + \varepsilon^{-2}(1-|u|^2)|u|^2 = |\nabla u|^2$. Multiplying by $(1-|u|^2)$ and integrating on U_1 yields

$$\int_{U_1} |\nabla|u|^2|^2 + \varepsilon^{-2}(1-|u|^2)^2|u|^2 = \int_{U_1} (1-|u|^2)|\nabla u|^2 + \int_{\partial U_1} (1-|u|^2)\partial_n|u|^2$$

By (3) we have $\left|\int_{\partial U_1} (1-|u|^2)\partial_n|u|^2\right| \le C\int_{\partial U_1} |\partial_n u|$, and (here $U_\beta = \{||u|-1| \le \beta\}$)

$$\int_{U_1} |1-|u|^2||\nabla u|^2 \le \int_{U_1\cap U_\beta} \beta|\nabla u|^2 + \beta^{-1}\int_{U_1\setminus U_\beta} C\varepsilon^{-2}(1-|u|^2)^2 \ , \tag{24}$$

so that, using (8), $\int_{U_1} |\nabla|u|^2|^2 \le \beta F_\varepsilon(0,\delta^{-1}r_0) + C\beta^{-1}\eta|\log\eta| \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right]$.

Combining this estimate with (11) and (23) we obtain $\int_{W_0} |u|^2|\nabla u|^2 \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right]$. Finally we complete the proof of Proposition 1 using (24) again.

4. Proof of Theorem 1

By the mean-value inequality applied to (10), there is some $r_3 \in [r_0, 2r_0]$ such that

$$r_3\int_{\partial B_{r_3}} |\nabla u|^2 + (4\varepsilon^2)^{-1}(1-|u|^2)^2 \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right] \ .$$

By the Pohozaev's identity (4), this implies $F_\varepsilon(0,r_0) \le F_\varepsilon(0,r_3) \le C\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right]$, so that

$$\begin{aligned} r_0^{2-N}F_\varepsilon(0,r_0) &\le Cr_0^{2-N}\eta^{1/8}\left[F_\varepsilon(0,\delta^{-1}r_0) + r_0^{N-2}\right] \le C\eta^{1/8}\delta^{2-N}\left[(\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0) + 1\right] \\ &\le C\eta^{1/4N}\left[(\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0) + 1\right] \end{aligned} \tag{25}$$

(recall $\delta = \eta^{1/8N}$). On the other hand by monotonicity (5) and (7), $(\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0) \le r_0^{2-N}F_\varepsilon(0,r_0) + C\eta|\log\eta|$. Inserting this in (25) we obtain

$$(1 - C\eta^{1/4N})r_0^{2-N}F_\varepsilon(0,r_0) \le C\eta^{1/4N}(1+\eta|\log\eta|)\ . \tag{26}$$

Hence, if η_1 is such that $C\eta_1^{1/4N} = 1/2$, then for $\eta \le \eta_1$ we conclude

$$r_0^{2-N}F_\varepsilon(0,r_0) \le C\eta^{1/4N}\ . \tag{27}$$

Finally we invoke the monotonicity (6) once more to assert, for $\eta \le \eta_1$, by (27),

$$\varepsilon^{-N}\int_{B_\varepsilon}(1-|u|^2)^2 \le \int_{B_\varepsilon}\rho^{2-N}\varepsilon^{-2}(1-|u|^2)^2 \le \int_{B_{r_0}}\rho^{2-N}\varepsilon^{-2}(1-|u|^2)^2 \le C\eta^{1/4N}\ . \tag{28}$$

We now complete the proof as in [1]. Set $k = |u(0)|$, by (3), we have $|u(x)-u(0)| \le \frac{C}{\varepsilon}|x| \le \frac{1-k}{2}$, provided $|x| \le \frac{\varepsilon(1-k)}{2C} \equiv \gamma$. We distinguish two cases.
Case 1 : $\gamma < \varepsilon$. Then $\int_{B_\gamma}(1-|u|^2)^2 \le \int_{B_\varepsilon}(1-|u|^2)^2 \le C\varepsilon^N\eta^{\frac{1}{4N}}$.On the other hand, on B_γ we have $(1-|u|^2)^2 \ge (1-|u])^2 \ge \frac{(1-k)^2}{4}$, and therefore $\gamma^N(1-k)^2 \le C\varepsilon^N\eta^{\frac{1}{4N}}$. Hence $(1-k)^{N+2} \le C\eta^{\frac{1}{4N}}$, and the result follows.
Case 2: $\gamma \ge \varepsilon$. Then $(1-|u|^2)^2 \ge \frac{(1-k)^2}{4}$ on B_ε, and one concludes as above.

Appendix: proof of estimates (17), (18) for φ_2

Recall that $G(x) = c_N|x|^{2-N}$ denotes the fundamental solution of the Laplacian in $\mathbf{R}^N$. For $r > 0$ let $B_r \equiv B_r(0) \subset \mathbf{R}^N$ and let $\sigma_{\partial B_r}$ denote the surface measure on ∂B_r.

LEMMA A1. – *Let $r > 0$ and $g \in L^2(\partial B_r)$. Set $\varphi = G * (\sigma_{\partial B_r} g)$, i.e., for $x \in \mathbf{R}^N$, $\varphi(x) = c_N\int_{\partial B_r}|x-y|^{2-N}g(y)dy$. Then $\nabla\varphi \in L^2(\mathbf{R}^N)$ and $||\nabla\varphi||^2_{L^2(R^N)} \le Cr||g||^2_{L^2(\partial B_r)}$.*

Proof: By scaling it suffices to consider the case $r=1$. Clearly, for $x \in \mathbf{R}^N \setminus \partial B_1$,
$|\nabla\varphi(x)| \le C\int_{\partial B_1}|x-y|^{1-N}|g(y)|dy$ so that for $x \in \mathbf{R}^N \setminus \partial B_2$, we have $|\nabla\varphi(x)| \le C|x|^{1-N}|||g||_{L^1(\partial B_1)}$. Therefore

$$||\nabla\varphi||^2_{L^2(R^N\setminus B_2)} \le C||g||^2_{L^2(\partial B_1)}\ .$$

On B_2 we write $\varphi = \varphi_1 + \varphi_2$, where φ_1 solves $\Delta\varphi_1 = g\sigma_{\partial B_1}$ in B_2, $\varphi_1 = 0$ on ∂B_2, while φ_2 solves $\Delta\varphi_2 = 0$ in B_2, $\varphi_2 = \varphi$ on ∂B_2. By the trace theorem we verify $\varphi_1 \in H_0^1(B_2)$ and

$$||\nabla\varphi_1||^2_{L^2(B_2)} \le C||g||^2_{L^2(\partial B_1)}\ .$$

Similarly, since φ_2 is harmonic in B_2, one has (observe that $||\varphi||_{L^\infty(\partial B_2)} \le C||g||_{L^2(\partial B_1)}$):

$$||\nabla\varphi_2||^2_{L^2(B_2)} \le C||\varphi||^2_{H^{1/2}(\partial B_2)} \le C||g||^2_{L^2(\partial B_1)}\ .$$

Combining the previous inequalities, the result follows easily.

Proof of (17). Write $\varphi_2 = \varphi_{2,1} + \varphi_{2,2}$, where $\varphi_{2,i} = (-1)^i\sigma_{\partial B_{r_i}}f(|u|)u \times du \wedge d\rho$ for $i = 1,2$. We apply previous lemma to obtain

$$\int_{\mathbf{R}^N}|\nabla\varphi_{2,1}|^2 \le Cr_i\int_{\partial B_{r_i}}f(|u|)^2|u|^2|du|^2 \le Cr_i\int_{\partial B_{r_i}}|\nabla u|^2\ ,$$

and (17) follows.

Proof of (18). We first bound $||\nabla\varphi_2||_{L^\infty(W_0)}$. We estimate separately $||\nabla\varphi_{2,i}||_{L^\infty(W_0)}$ for each $i=1,2$: remark that $\varphi_{2,i}$ is harmonic in $\mathbf{R}^N\setminus\partial B_{r_i}$, therefore we may use the mean-value inequality for harmonic functions.

For $i=1$ and $x\in W_0$ we have $B(x,r_0/2)\subset\mathbf{R}^N\setminus\partial B_{r_1}$. Hence, using (9),

$$\begin{aligned}|\nabla\varphi_{2,1}(x)|^2 &\le |B(x,r_0/2)|^{-1}||\nabla\varphi_{2,1}||_{L^2(R^N)} \le Cr_0^{-N}r_1\int_{\partial B_{r_1}}|\nabla u|^2\\ &\le Cr_0^{-N}(\delta^2 r_0)^{N-2}(\eta|\log\delta| + (\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0))\\ &\le Cr_0^{-N}(\delta^2 r_0^{N-2}\eta|\log\delta| + \delta^{3(N-2)}F_\varepsilon(0,\delta^{-1}r_0))\ .\end{aligned}$$

For $i=2$ and $x\in W_0$ we have $B(x,\delta^{-1}r_0/8)\subset\mathbf{R}^N\setminus\partial B_{r_2}$, so that by (9)

$$\begin{aligned}|\nabla\varphi_{2,2}(x)|^2 &\le |B(x,\delta^{-1}r_0/8)|^{-1}||\nabla\varphi_{2,2}||_{L^2(R^N)} \le Cr_0^{-N}\delta^N r_2\int_{\partial B_{r_2}}|\nabla u|^2\\ &\le Cr_0^{-N}\delta^N(\delta^{-1}r_0)^{N-2}(\eta|\log\delta| + (\delta^{-1}r_0)^{2-N}F_\varepsilon(0,\delta^{-1}r_0))\\ &\le Cr_0^{-N}(\delta^2 r_0^{N-2}\eta|\log\delta| + \delta^N F_\varepsilon(0,\delta^{-1}r_0))\ .\end{aligned}$$

Finally combining these estimates we obtain, $\forall x\in W_0$,

$$|\nabla\varphi_2(x)|^2 \le Cr_0^{-N}(\delta^2 r_0^{N-2}\eta|\log\delta| + (\delta^N+\delta^{3(N-2)})F_\varepsilon(0,\delta^{-1}r_0))\ .$$

Since $N\ge 3$, $3(N-2)\ge N$ and since $|W_0|\le Cr_0^N$, integrating the previous inequality on W_0 we obtain (18).

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