

Set-decomposition of normal rectifiable G -chains *via* an abstract decomposition principle

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Abstract

We introduce the notion of *set-decomposition* of a normal G -flat chain A in \mathbb{R}^n as a sequence $A_j = A \llcorner S_j$ associated to a Borel partition S_j of \mathbb{R}^n such that $\mathbb{N}(A) = \sum \mathbb{N}(A_j)$. We show that any *normal rectifiable* G -flat chain admits a decomposition in set-indecomposable sub-chains. This generalizes the decomposition of sets of finite perimeter in their “measure theoretic” connected components due to Ambrosio, Caselles, Masnou and Morel. It can also be seen as a variant of the decomposition of integral currents in indecomposable components by Federer.

As opposed to previous results, we do not assume that G is boundedly compact. Therefore we cannot rely on the compactness of sequences of chains with uniformly bounded \mathbb{N} -norms. We deduce instead the result from a new abstract decomposition principle.

As in earlier proofs a central ingredient is the validity of an isoperimetric inequality. We obtain it here using the finiteness of some h -mass to replace integrality.

1 Introduction

The aim of this note is to extend the notion of decomposition of normal currents from the integral setting [Fed69, ACMM01, BPR20, BDNP22] to the general setting of normal rectifiable G -flat chains. This work is motivated by [GM22] where we use the decomposition result to study the rectifiability properties of tensor flat chains.

In order to state our main result, let us start with some notation and definitions. Let G be a complete Abelian normed group and let $0 \leq k \leq n$. We denote by $\mathcal{F}_k^G(\mathbb{R}^n)$ the group of k -chains in \mathbb{R}^n with coefficients in G as introduced by Fleming in [Fle66]. However, as in [Whi99a, Whi99b] we do not assume that chains are compactly supported. The mass of a chain A is denoted $\mathbb{M}(A)$ and $\mathcal{M}_k^G(\mathbb{R}^n)$ is the subgroup of finite mass k -chains. The restriction of $A \in \mathcal{M}_k^G(\mathbb{R}^n)$ to a Borel set $S \subset \mathbb{R}^n$ is denoted $A \llcorner S$. By definition, $A \in \mathcal{M}_k^G(\mathbb{R}^n)$ is rectifiable if $A = A \llcorner \Sigma$ for some countably k -rectifiable set $\Sigma \subset \mathbb{R}^n$. By [Whi99a, Section 6], we can identify every rectifiable k -chain with a measure $w\xi\mathcal{H}^k \llcorner \Sigma$ where $w : \mathbb{R}^n \rightarrow G$ is Borel measurable and ξ is a Borel measurable field of unit k -vectors orienting Σ . Eventually, we set $\mathbb{N}(A) := \mathbb{M}(A) + \mathbb{M}(\partial A)$ and denote $\mathcal{N}_k^G(\mathbb{R}^n) = \{A \in \mathcal{F}_k^G(\mathbb{R}^n) : \mathbb{N}(A) < \infty\}$ the subgroup of normal k -chains.

Flat chains with real or integer coefficients were introduced as a particular class of currents in [FF60]. Later, in [Fle66] Fleming proposed a theory for flat chains with coefficients in a

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(commutative normed) group, still in an Euclidean ambient space, which have been further developed by White in [Whi99a, Whi99b]. Shortly after, [AK00] introduced a theory of currents in an ambient metric space. Both this theory and the theory of Fleming and White have been generalized to form a theory of G -flat chains in metric spaces in [DPH12].

Even in the case of Euclidean ambient spaces, the topological and geometrical structure of rectifiable and even finite mass chains is still under investigation (see *e.g.* [AM17, You18, AM22]). In this note we introduce the notion of set-decomposition of normal flat chains and prove that every normal and rectifiable chain can be decomposed in indecomposable components.

Definition 1.1. Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$.

(1) A *set-decomposition* of A is a sequence (finite or countable) of normal chains A_j such that there exists a Borel partition S_j of \mathbb{R}^n with $A_j = A \llcorner S_j$ for every j and $\mathbb{N}(A) = \sum \mathbb{N}(A_j)$.

We say that each A_j is a *set-subchain* of A .

(2) We say that A is *set-indecomposable* if the only set-decompositions of A are trivial, that is, for any set-decomposition A_j of A , there holds $A_j = A$ for some index j and $A_j = 0$ for the others.

Remark 1.2. Notice that by definition if A is rectifiable then for every Borel set S , $A \llcorner S$ is also rectifiable. In particular every set-decomposition of a rectifiable chain is made of rectifiable subchains.

Theorem 1.3. Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$, if A is rectifiable then it admits a set-decomposition in set-indecomposable subchains.

Such decomposition is also called a maximal set-decomposition of A .

We obtain Theorem 1.3 as a corollary of the abstract decomposition Lemma 2.1, stated and established in Section 2. Let us give some comments.

(a) When A is an integral current, a set-decomposition is in general coarser than a decomposition into indecomposable integral currents introduced by Federer [Fed69, 4.2.25]. For instance, if A' is the integral 1-current with multiplicity 1 associated with a smooth oriented Jordan curve, then $A := 2A'$ is set-indecomposable in $\mathcal{N}_1^{\mathbb{Z}}(\mathbb{R}^n)$ but admits the decomposition $(A', A', 0, \dots)$ in the sense of Federer.

(b) In the case $k = n$ and $G = \mathbb{Z}$, if $A = \llbracket E \rrbracket$, where E is a set of finite perimeter, our definition corresponds to the decomposition of E into its measure theoretic connected components introduced in [ACMM01] and Theorem 1.3 generalizes [ACMM01, Theorem 1].

(c) For $k = 0$, the set-decomposition in set-indecomposable subchains of a normal rectifiable 0-chain is essentially unique. Indeed, any normal rectifiable 0-chain is of the form $A = \sum g_j \llbracket x^j \rrbracket$ where $g_j \in G$ is such that $\sum |g_j|_G < \infty$ and $x^j \in \mathbb{R}^n$ is a sequence of *pairwise distinct* points. The set-indecomposable 0-chains are the chains $g \llbracket x \rrbracket$ for $g \in G$, $x \in \mathbb{R}^n$. It follows that the sequence $(g_1 \llbracket x^1 \rrbracket, g_2 \llbracket x^2 \rrbracket, \dots)$ is a set-decomposition in set-indecomposable subchains of the above 0-chain A . Moreover, all the maximal set-decompositions are obtained by rearranging this latter and possibly inserting and removing zeros.

At the other end, any n -chain A is rectifiable and the group of normal rectifiable n -chains is the group of normal n -chains. We believe that the set-decomposition of normal n -chains in set-indecomposable subchains is also essentially unique. We establish this fact in the particular case $G = \mathbb{R}$ (and thus also $G = \mathbb{Z}$), see the statement of Proposition 4.1.

On the contrary, for $1 \leq k \leq n - 1$, the decomposition in set-indecomposable subchains is in general not unique even up to rearrangements. For instance, set $n = 2$ and $G = \mathbb{Z}$ and consider the polyhedral 1-chains with multiplicity 1, A^h and A^v , where

* A^h is supported by the horizontal segment $[-1, 1] \times \{0\}$ and is oriented by e_1 ,

* A^v is supported by the vertical segment $\{0\} \times [-1, 1]$ and is oriented by e_2 .

Setting,

$$A := A^h + A^v, \quad A^+ := A \llcorner \{x_2 > x_1\}, \quad A^- := A \llcorner \{x_2 < x_1\},$$

we see that $(A^h, A^v, 0, \dots)$ and $(A^+, A^-, 0, \dots)$ are two distinct set-decompositions of A in set-indecomposable subchains.

(d) For a decomposition process to result in an *at most countable number* of indecomposable parts, we need some principle which prevents big pieces from crumbling into dust. In [Fed69, ACMM01], this role is played by some isoperimetric inequalities which gives a *superlinear* estimate of the mass of an object by the mass of its boundary. Here we use Lemma 3.2 which is of the same nature. Indeed, with Step 1 of the proof of Theorem 1.3, we have that if A is normal and rectifiable, there exists an increasing and strictly subadditive cost function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $h'(0^+) = \infty$ such that $\mathbb{M}_h(A) < \infty$ (see Definition 3.1 for the definition of \mathbb{M}_h). The isoperimetric inequality of Lemma 3.2 (which extends Almgren's isoperimetric inequality [Alm86], see Remark 3.3) then provides a nondecreasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\eta(m) \rightarrow 0$ as $m \downarrow 0$ and for every k -chain A'

$$\mathbb{F}(A') \leq \eta(\mathbb{M}(A')) (\mathbb{M}_h(A') + \mathbb{N}(A')). \quad (1.1)$$

(e) Unlike the references mentioned above our proof of Theorem 1.3 does not use any compactness theorem of the form:

$$\text{“for } \Lambda \geq 0, \text{ the set } \{A \in \mathcal{N}_k^G(\mathbb{R}^n) : \text{supp } A \subset \overline{B}_\Lambda, \mathbb{N}(A) \leq \Lambda\} \text{ is compact in } \mathbb{F}\text{-norm.”} \quad (1.2)$$

This statement is true if and only if G is boundedly compact in which case it is a classical consequence of the deformation theorem. We use here instead the convergence in strong norm of monotone sequences.

(f) Some generalizations of the results of [Fed69, ACMM01] exist in the context of real currents in metric spaces of [AK00]. Namely, [BPR20, Theorem 2.14] generalizes the decomposition of sets with finite perimeter in a (doubling) metric measure space and [BDNP22, Theorem 3.2] generalizes the decomposition of integral currents in metric spaces. This suggests that a version of Theorem 1.3 should hold true within the theory of G -currents in metric spaces developed in [DPH12].

(g) In connection with (d), let us point out that if we fix a cost function h as above (satisfying in particular $h'(0^+) = \infty$) we can define the h -mass of any flat chain as the lower-semicontinuous envelope of \mathbb{M}_h restricted to polyhedral chains. By [Whi99b, Theorem 8.1] every normal chain with finite h -mass is rectifiable (with h -mass coinciding with \mathbb{M}_h by [Whi99a, CDRMS17]). Therefore Theorem 1.3 provides a decomposition in indecomposable components for normal chains of finite h -mass. Partly due to their connection with branched transport models, this type of functionals has received a lot of attention in the past few years, see *e.g.* [BW18, CFM19, CDRM21]. It is however worth noticing that our notion of set-decomposition (and indecomposability) is independent of the choice of h .

The main contribution of this note is the fact that we obtain the decomposition result Theorem 1.3 without the closure/compactness property (1.2). For comparison, let us sketch an alternative proof in a case where this latter holds true, namely when $G = (\mathbb{R}, +)$ equipped with the usual norm and (for simplicity) when A is compactly supported. This proof is close in spirit to the one of [ACMM01, Theorem 1].

Alternative proof of Theorem 1.3 when $G = \mathbb{R}$ and A is compactly supported.

Let $A \in \mathcal{N}_k^{\mathbb{R}}(\mathbb{R}^n)$ be rectifiable and compactly supported. We introduce the set

$$\mathcal{D} := \{A_j \text{ set-decomposition of } A \text{ such that } \mathbb{N}(A_j) \text{ is nonincreasing}\}.$$

This set is not empty as it contains $(A, 0, \dots)$. Let us endow the space of sequences $v_j \in \mathbb{R}$ indexed by $j \geq 1$ with the lexicographic ordering, *i.e.* $(v'_j) <_{\text{lex}} (v_j)$ if there exists $j_0 \geq 1$ such that $v'_j = v_j$ for $1 \leq j < j_0$ and $v'_{j_0} < v_{j_0}$. We consider the optimization problem

$$(v_j) := \inf_{\text{lex.}} \{(\mathbb{N}(A_j)) : A_j \in \mathcal{D}\}. \quad (1.3)$$

Since the $\mathbb{N}(A_j)$'s are nonnegative, the infimum is well defined and $v_j \geq 0$ for every j .

We claim that if A_j is a minimizer of (1.3) then each A_j is set-indecomposable. To see this, we assume by contradiction that for some $j_0 \geq 1$, A_{j_0} admits a nontrivial set-decomposition (A'_{j_0}, A''_{j_0}) , that is $\max(\mathbb{N}(A'_{j_0}), \mathbb{N}(A''_{j_0})) < \mathbb{N}(A_{j_0})$. Substituting (A'_{j_0}, A''_{j_0}) for A_{j_0} in the sequence A_j and then rearranging the terms in decreasing order of \mathbb{N} -norms we obtain a new set-decomposition of A which contradicts the minimality of A_j .

To complete the proof we establish that (1.3) does admit a minimizer.

Step 1. (\mathbb{F} -compactness of minimizing sequences).

Let $(A_j^m)_{m \geq 1}$ be a minimizing sequence for (1.3). Let us first fix $j \geq 1$. The sequence $(A_j^m)_m$ satisfies $\mathbb{N}(A_j^m) \leq \mathbb{N}(A)$ and $\text{supp } A_j^m \subset \text{supp } A$ so by (1.2) there exists a normal chain A_j such that up to extraction, $A_j^m \rightarrow A_j$ in \mathbb{F} -norm. Using a diagonal argument, we may assume that $A_j^m \rightarrow A_j$ as $m \uparrow \infty$ in \mathbb{F} -norm for every $j \geq 1$. Moreover, by lower semicontinuity of the masses under \mathbb{F} -convergence,

$$\mathbb{N}(A_j) \leq \liminf_{m \uparrow \infty} \mathbb{N}(A_j^m). \quad (1.4)$$

By Lemma A.1 there exists a cost function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, increasing, concave such that $h(0) = 0$, $h'(0^+) = \infty$ and $\mathbb{M}_h(A) < \infty$ (in particular, h is strictly subadditive). We deduce from [CDRMS17, Propositions 2.6&2.7] that the A_j 's are rectifiable and

$$\mathbb{M}_h(A_j) \leq \liminf_{m \uparrow \infty} \mathbb{M}_h(A_j^m). \quad (1.5)$$

Moreover by Lemma 3.2 there exists a nondecreasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\eta(m) \downarrow 0$ as $m \downarrow 0$ such that (1.1) holds with $A' = A_j^m$ for any $j, m \geq 1$.

Step 2. (Uniform \mathbb{F} -summability and mass identities). Let $j_0, m \geq 1$. Since $\mathbb{N}(A) = \sum_j \mathbb{N}(A_j^m)$ and $(\mathbb{N}(A_j^m))_j$ is nonincreasing, we have for $j \geq j_0$,

$$\mathbb{M}(A_j^m) \leq \mathbb{N}(A_j^m) \leq \mathbb{N}(A_{j_0}^m) \leq \frac{1}{j_0} \sum_{i=1}^{j_0} \mathbb{N}(A_i^m) \leq \frac{\mathbb{N}(A)}{j_0}.$$

Using (1.1), we compute,

$$\begin{aligned} \sum_{j \geq j_0} \mathbb{F}(A_j^m) &\leq \sum_{j \geq j_0} \eta(\mathbb{M}(A_j^m)) (\mathbb{N}(A_j^m) + \mathbb{M}_h(A_j^m)) \\ &\leq \eta\left(\frac{\mathbb{N}(A)}{j_0}\right) \sum_{j \geq j_0} (\mathbb{N}(A_j^m) + \mathbb{M}_h(A_j^m)) \leq \eta\left(\frac{\mathbb{N}(A)}{j_0}\right) (\mathbb{N}(A) + \mathbb{M}_h(A)) \xrightarrow{j_0 \uparrow \infty} 0. \end{aligned}$$

Hence, the series $\sum_j A_j^m$ converges in \mathbb{F} -norm uniformly with respect to m . As a consequence, we can pass to the limit in $A = \sum_j A_j^m$ and deduce the identity,

$$A = \sum_{j \geq 1} A_j. \quad (1.6)$$

Then, by the triangle inequality for \mathbb{N} and \mathbb{M}_h (see (3.1)), there holds,

$$\mathbb{N}(A) \leq \sum_{j \geq 1} \mathbb{N}(A_j) \quad \text{and} \quad \mathbb{M}_h(A) \leq \sum_{j \geq 1} \mathbb{M}_h(A_j). \quad (1.7)$$

By definition, we have for $m \geq 1$, $\mathbb{N}(A) = \sum_j \mathbb{N}(A_j^m)$ and $\mathbb{M}_h(A) = \sum_j \mathbb{M}_h(A_j^m)$. Together with (1.4), (1.5), Fatou and (1.7), this leads to

$$\mathbb{N}(A) = \sum_{j \geq 1} \mathbb{N}(A_j) \quad \text{and} \quad \mathbb{M}_h(A) = \sum_{j \geq 1} \mathbb{M}_h(A_j). \quad (1.8)$$

Thus (1.4) improves to $\mathbb{N}(A_j) = \lim_{m \uparrow \infty} \mathbb{N}(A_j^m)$ and we get eventually,

$$\mathbb{N}(A_j) = v_j \quad \text{for every } j \geq 1, \text{ where the } v_j\text{'s are given by (1.3)}. \quad (1.9)$$

Step 3. (Conclusion by strict subadditivity of h).

At this point, we know that the A_j 's are normal rectifiable chains satisfying (1.6), (1.8) and (1.9). To conclude that the sequence A_j is a minimizer of (1.3) we still have to show that it is a set-decomposition of A .

Let Σ be a countably k -rectifiable set of \mathbb{R}^n such that for every $j \geq 1$, $A_j = A_j \llcorner \Sigma$ and let us write

$$A = w \xi \mathcal{H}^k \llcorner \Sigma \quad \text{and} \quad A_j = w_j \xi \mathcal{H}^k \llcorner \Sigma \quad \text{for } j \geq 1,$$

where w, w_j are Borel measurable functions on \mathbb{R}^n and ξ is a Borel measurable field of unit k -vectors orienting Σ . From (1.6)&(1.8) we have $w(x) = \sum w_j(x)$ for \mathcal{H}^k -almost every $x \in \Sigma$. Using the fact that h is increasing and subadditive, we compute,

$$\begin{aligned} \mathbb{M}_h(A) &= \int_{\Sigma} h(|w|) d\mathcal{H}^k = \int_{\Sigma} h \left(\left| \sum_{j \geq 1} w_j \right| \right) d\mathcal{H}^k \\ &\leq \int_{\Sigma} h \left(\sum_{j \geq 1} |w_j| \right) d\mathcal{H}^k \leq \sum_j \int_{\Sigma} h(|w_j|) d\mathcal{H}^k = \sum \mathbb{M}_h(A_j). \end{aligned}$$

By (1.8) the inequalities are identities and since h is strictly subadditive, we conclude that for \mathcal{H}^k -almost every $x \in \Sigma$, there exists $j_0 \geq 1$ such that $|w_{j_0}(x)| = |w(x)|$ and $w_j(x) = 0$ for $j \neq j_0$. Since $\sum w_j = w$ \mathcal{H}^k -almost everywhere on Σ , we have in fact $w_{j_0}(x) = w(x)$. Hence A_j is a set-decomposition of A which proves the result. \square

In this proof, we can substitute for the coefficient group \mathbb{R} any boundedly compact Abelian normed group. In the general case, Steps 2&3 are still valid but the proof fails at Step 1 where (1.2) is required. It turns out that we can overcome this difficulty by observing that we only need compactness of nonincreasing sequences of set-subchains of A and that this latter property holds true. Indeed, if A has finite mass (resp. finite h -mass) and $A_j = A \llcorner S_j$ with S_j a nonincreasing sequence of Borel subsets of \mathbb{R}^n we have by the monotone convergence theorem, $A_j \rightarrow A \llcorner \cap S_j$ in mass (resp. in h -mass).

In the next section we establish Lemma 2.1. It provides an abstract decomposition principle in Abelian normed groups, assuming a general version of (1.1) and a closure property for non-increasing sequences in the subset chosen for the decompositions.

In Section 3, we prove the isoperimetric inequality for normal rectifiable chains (Lemma 3.2) and then Theorem 1.3.

In Section 4 we state and prove Proposition 4.1 about the uniqueness of the maximal set-decomposition when G is a subgroup of $(\mathbb{R}, +)$ endowed with the usual norm.

Eventually, we establish in appendix a simple ‘‘higher integrability’’ lemma used in the proof of Theorem 1.3.

2 An abstract decomposition Lemma

Lemma 2.1. *Let $(\mathcal{G}, +, \nu)$ be a complete Abelian normed group and let $\mathcal{S} \subset \mathcal{G}$ such that $0 \in \mathcal{S}$.*

- (i) *A sequence $a_j \in \mathcal{S}$ is a decomposition of $b \in \mathcal{G}$ if $b = \sum a_j$ and $\nu(b) = \sum \nu(a_j)$. In such case we write $a_j \preceq b$, for every $j \geq 1$.*
- (ii) *b is an atom if $b \in \mathcal{S}$ and any decomposition of b is trivial, that is $a \preceq b$ implies $a = 0$ or $a = b$.*

We make the following assumptions.

- (H1) *The limit of nonincreasing sequences $b_j \in \mathcal{G}$ (that is $b_1 \succeq b_2 \succeq \dots$) belong to \mathcal{S} .*
- (H2) *There exists another norm ϕ on \mathcal{G} and a nondecreasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{s \rightarrow 0} \eta(s) = 0$ such that $\phi(a) \leq \eta(\nu(a)) \nu(a)$ for every $a \in \mathcal{S}$.*

Then, if $b \in \mathcal{G}$ admits at least one decomposition, it admits a decomposition in atoms.

Remark 2.2.

(1) Since $0 \in \mathcal{S}$ any element $a \in \mathcal{S}$ admits the trivial decomposition $(a, 0, 0, \dots)$ and the lemma implies that under Assumptions (H1)&(H2) any element of \mathcal{S} admits a decomposition in atoms.¹ Also notice that 0 is always an atom.

(2) Let us stress again that (H1) is the only closure property that we consider in the lemma, and that it only concerns monotone sequences. In the proof of Theorem 1.3, (H1) follows from the monotone convergence theorem of measure theory.

(3) As already mentioned, the lemma provides an alternative proof to the existence of decompositions in indecomposable components of a normal integral current supported in some compact K ([Fed69, 4.2.25], [BDNP22, Theorem 3.2]). For this, we take $\mathcal{S} = \mathcal{G}$ as the group of normal integral currents supported in K , $\nu = \mathbb{N}$ and $\phi = \mathbb{F}$. In this setting, (H1) follows from the completeness of (\mathcal{G}, ν) and (H2) from the isoperimetric inequality.

Before proving the lemma, let us discuss some consequences of the assumptions. It is convenient to consider a broader notion of decomposition. For this, we complete the definitions (i)(ii) in the lemma by:

- (iii) We say that a sequence $a_j \in \mathcal{G}$ is a pseudo-decomposition of $b \in \mathcal{G}$ if $b = \sum a_j$ and $\nu(b) = \sum \nu(a_j)$. In such case we write for every $j \geq 1$, $a_j \preceq_\psi b$.

A decomposition is then a pseudo-decomposition whose components lie in \mathcal{S} .

Observations 2.3.

(1) Let $a, b \in \mathcal{G}$ with $a \preceq b$ (or $a \preceq_\psi b$). By definition, there exists a (pseudo-)decomposition a_j of b with $a = a_{j_0}$ for some $j_0 \geq 1$. Rearranging the first j_0 terms, we may always assume $a_1 = a$.

(2) The relation \preceq_ψ defines a partial order on \mathcal{G} . More precisely, for $a, b, c \in \mathcal{G}$, there hold,

$$a \preceq_\psi b \text{ and } b \preceq_\psi a \iff a = b, \quad a \preceq_\psi b \text{ and } b \preceq_\psi c \implies a \preceq_\psi c. \quad (2.1)$$

As a consequence, $(\mathcal{G}, \preceq_\psi)$ is a partially ordered set. Similarly (\mathcal{S}, \preceq) is a partially ordered set.

¹Conversely if any element of \mathcal{S} admits a decomposition in atoms, then if $b \in \mathcal{G}$ admits a decomposition (b_1, b_2, \dots) we obtain a decomposition of b by collecting the decompositions in atoms of the b_j 's.

Let us prove (2.1). First any $a \in \mathcal{G}$ admits the pseudo-decomposition $(a, 0, 0, \dots)$ so $a \preceq_\psi a$. Conversely, let $a, b \in \mathcal{G}$ such that $a \preceq_\psi b$ and $b \preceq_\psi a$. Let a_j be a pseudo-decomposition of b such that $a_1 = a$ and let b_i be a pseudo-decomposition of a such that $b_1 = b$. We compute,

$$\nu(b) = \nu(a) + \sum_{j \geq 2} \nu(a_j) = \nu(b) + \sum_{i \geq 2} \nu(b_i) + \sum_{j \geq 2} \nu(a_j).$$

Hence $b_i = a_j = 0$ for, $i, j \geq 2$ and $a = b$. This proves the equivalence in the left of (2.1). Let us now establish the implication in the right. Let $a, b, c \in \mathcal{G}$ and assume that $a \preceq_\psi b \preceq_\psi c$. Denoting a_i and b_j some pseudo-decompositions of b and c with $a_1 = a$ and $b_1 = b$, we have

$$c = b + \sum_{j \geq 2} b_j = a + \sum_{i \geq 2} a_i + \sum_{j \geq 2} b_j,$$

with

$$\nu(c) = \nu(b) + \sum_{j \geq 2} \nu(b_j) = \nu(a) + \sum_{i \geq 2} \nu(a_i) + \sum_{j \geq 2} \nu(b_j).$$

Setting $d_1 = a$ and then $d_{2j} = a_{j+1}$, $d_{2j+1} = b_{j+1}$ for $j \geq 1$, we get that d_j is a pseudo-decomposition of c with $d_1 = a$, hence $a \preceq_\psi c$ as claimed.

(3.a) If $a \preceq_\psi b$, then $(a, b - a, 0, 0, \dots)$ is pseudo-decomposition of b . Indeed if a_j is a pseudo-decomposition of b with $a_1 = a$, then $b - a = \sum_{j \geq 2} a_j$ and by the triangle inequality,

$$\nu(a) + \nu(b - a) \geq \nu(b) = \sum_{j \geq 2} \nu(a_j) = \nu(a) + \sum_{j \geq 2} \nu(a_j) \geq \nu(a) + \nu(b - a).$$

Hence $\nu(b) = \nu(a) + \nu(b - a)$ and $(a, b - a, 0, 0, \dots)$ is a pseudo-decomposition of b .

If moreover a admits a pseudo-decomposition c_i , then the sequence $(b - a, c_1, c_2, \dots)$ is a pseudo-decomposition of b . Indeed, we have similarly $b = (b - a) + \sum c_i$ and

$$\nu(b - a) + \sum \nu(c_i) \geq \nu(b) = \nu(b - a) + \nu(a) = \nu(b - a) + \sum \nu(c_i),$$

and $\nu(b) = \nu(b - a) + \sum \nu(c_i)$.

(3.b) Applying this principle recursively, if $b_1 \succeq_\psi b_2 \succeq_\psi \dots$ then for $j \geq 2$, $(b_j, (b_{j-1} - b_j), (b_{j-1} - b_{j-2}), \dots, (b_1 - b_2))$ is a decomposition of b_1 , in particular,

$$b_1 = b_j + \sum_{1 \leq i < j} (b_{i+1} - b_i), \quad (2.2)$$

$$\nu(b_1) = \nu(b_j) + \sum_{1 \leq i < j} \nu(b_{i+1} - b_i). \quad (2.3)$$

(3.c) By (2.3) we see that the series $\sum (b_j - b_{j+1})$ is absolutely converging and since \mathcal{G} is complete the sum admits a limit c_∞ . In light of (2.2) we see that the sequence b_j also converges and its limit is $b_\infty := b_1 - c_\infty$. Notice that this justifies the existence of this limit which is implicitly assumed in hypothesis (H1).

(3.d) Passing to the limit in (2.2)&(2.3) we obtain that

$$(b_\infty, b_1 - b_2, b_2 - b_3, \dots)$$

is a pseudo-decomposition of b_1 .

Also remark that we can start the nonincreasing sequence from any $j > 1$ rather than from 1. We deduce that for $j \geq 1$,

$$(b_\infty, b_j - b_{j+1}, b_{j+1} - b_{j+2}, \dots)$$

is a pseudo-decomposition of b_j .

(4) With the same triangle inequality based arguments as above, if b_j is a pseudo-decomposition of b and for each j , $(b_{j,i})_i$ is a pseudo-decomposition of b_j then for any bijection:

$$r \in \{1, 2, 3, \dots\} \mapsto (\bar{j}(r), \bar{i}(r)) \in \{1, 2, 3, \dots\} \times \{1, 2, 3, \dots\},$$

the sequence defined by

$$a_r := b_{\bar{j}(r), \bar{i}(r)} \quad \text{for } r \geq 1,$$

is a pseudo-decomposition of b_j . Besides, if for every j , $(b_{j,i})_i$ is a decomposition of b_j (that is $b_{j,i} \in \mathcal{S}$ for every i, j) then a_r is a decomposition of b .

Proof of Lemma 2.1. In the proof, we use the preceding observations without explicit mention.

Step 1 (Definition and properties of $q(\cdot)$). We define for $b \in \mathcal{G}$, the quantity,

$$q(b) := \inf \{ \sup \nu(b_j) : (b_j) \text{ decomposition of } b \},$$

with the convention $q(b) = \infty$ if b does not admit a decomposition. In the other cases, since $\sum \nu(b_j) = \nu(b) < \infty$, the supremum $\sup \nu(b_j)$ is a maximum.

Let us establish some properties of q . First any $b \in \mathcal{S}$ admits the decomposition $(b, 0, 0, \dots)$ and we deduce,

$$q(b) \leq \nu(b) \quad \text{for every } b \in \mathcal{S}. \quad (2.4)$$

In the rest of the proof we assume that $b \in \mathcal{G}$ admits at least one decomposition. Equivalently,

$$q(b) < \infty.$$

We claim that if b_j is a pseudo-decomposition of b then,

$$q(b) \leq \sup q(b_j). \quad (2.5)$$

To establish (2.5) we assume without loss of generality that $\sup q(b_j)$ is finite. Let $\varepsilon > 0$ and for $j \geq 1$, let $(b_{j,1}, b_{j,2}, \dots)$ be a decomposition of b_j such that $\max_i \nu(b_{j,i}) \leq q(b_j) + \varepsilon$. Rearranging the countable family $b_{j,i}$, $i, j \geq 1$ to form a sequence, we obtain a decomposition a_r of b with $\sup_r \nu(a_r) \leq \max_j q(b_j) + \varepsilon$. This yields $q(b) \leq \sup_j q(b_j) + \varepsilon$ and then $q(b) \leq \sup q(b_j)$ since $\varepsilon > 0$ is arbitrary.

Observe that if b_j is decomposition of b , we have $q(b_j) \leq \nu(b_j)$ by (2.4) and since $\nu(b_j) \rightarrow 0$, the supremum in (2.5) is a maximum.

We now prove the following.

$$\forall \varepsilon > 0 \exists b_1 \preceq b \text{ such that } \begin{cases} q(b) \leq q(b_1) \leq \nu(b_1) \leq q(b) + \varepsilon, \\ q(b - b_1) \leq q(b_1). \end{cases} \quad (2.6)$$

Let b_j be a decomposition of b with $\max \nu(b_j) \leq q(b) + \varepsilon$. Since $q(b_j) \leq \nu(b_j)$ and $\sum \nu(b_j) < \infty$, the sequence $q(b_j)$ reaches its maximum. Reordering if necessary, we assume $q(b_1) = \max q(b_j)$. There holds,

$$q(b) \stackrel{(2.5)}{\leq} \max q(b_j) = q(b_1) \stackrel{(2.4)}{\leq} \nu(b_1) \leq q(b) + \varepsilon.$$

Moreover, by (2.5) again applied to the decomposition $(b_j)_{j \geq 2}$ of $b - b_1$, we have $q(b - b_1) \leq \max_{j \geq 2} q(b_j) \leq q(b_1)$. This proves (2.6).

Step 2 (Extraction of a "big atom"). Let $b \in \mathcal{G}$ with $q(b) < \infty$. Let us establish that:

$$\text{there exists an atom } a \preceq b \text{ such that } q(b - a) \leq \nu(a). \quad (2.7)$$

Let $\varepsilon_j > 0$ with $\varepsilon_j \rightarrow 0$. We build recursively $b_0 \succeq b_1 \succeq b_2 \succeq \dots$ by setting $b_0 = b$ and then by applying (2.6) to b_j with $\varepsilon = \varepsilon_j$. We have for $j \geq 0$,

$$q(b_j) \leq q(b_{j+1}) \leq \nu(b_{j+1}) \leq q(b_j) + \varepsilon_j, \quad (2.8)$$

$$q(b_j - b_{j+1}) \leq q(b_{j+1}). \quad (2.9)$$

Applying (H1) to the nonincreasing sequence b_j , there exists $a \in \mathcal{S}$ such that $b_j \rightarrow a$ and by Observations 2.3(3.d)&(4) we have $a \preceq b_j$ for every $j \geq 0$ and in particular $a \preceq b = b_0$. Moreover, from (2.8), $q(b_j)$ is nondecreasing and

$$\sup_{j \geq 0} q(b_j) = \lim_{j \uparrow \infty} q(b_j) = \lim_{j \uparrow \infty} \nu(b_j) = \inf_{j \geq 0} \nu(b_j) = \nu(a). \quad (2.10)$$

Next, for $j \geq 1$, the sequence $(b_i - b_{i+1})_{i \geq j}$ is a pseudo-decomposition of $b_j - a$ and we deduce from (2.5) that

$$q(b_j - a) \leq \max_{i \geq j} q(b_i - b_{i+1}). \quad (2.11)$$

Applying this inequality with $j = 0$, we get (recall $b_0 = b$),

$$q(b - a) \leq \max_{i \geq 0} q(b_i - b_{i+1}) \stackrel{(2.9)}{\leq} \sup_{i \geq 0} q(b_{i+1}) \stackrel{(2.10)}{=} \nu(a).$$

This proves the inequality in (2.7).

We still have to check that a is an atom. For this, we first prove:

$$\lim_{j \uparrow \infty} q(b_j - a) = 0. \quad (2.12)$$

Combining (2.11) and (2.5) we have, $q(b_j - a) \leq \max_{i \geq j} \nu(b_i - b_{i+1})$ and since $\sum_{i \geq 0} \nu(b_i - b_{i+1}) < \infty$, the right-hand side goes to 0 as $j \uparrow \infty$. This proves (2.12).

Let now (c_i) be a decomposition of a and assume without loss of generality that $\max \nu(c_i) = \nu(c_1)$. For $j \geq 0$, the sequence $(b_j - a, c_1, c_2, \dots)$ is pseudo-decomposition of b_j . We compute,

$$q(b_j) \stackrel{(2.5)(2.4)}{\leq} \max \left(q(b_j - a), \max_i \nu(c_i) \right) = \max(q(b_j - a), \nu(c_1)).$$

Sending j to ∞ , by (2.10), the left-hand side converges towards $\nu(a) = \sum \nu(c_i)$ and, by (2.12), the right-hand side towards $\nu(c_1)$. We obtain,

$$\sum \nu(c_i) \leq \nu(c_1),$$

hence $c_i = 0$ for $i \geq 2$ and a is an atom. The claim (2.7) is established.

Step 3 (Conclusion). Let $b \in \mathcal{G}$ such that $q(b) < \infty$. We build recursively two sequences $a_j \preceq b$, $b_j \preceq_\psi b$ indexed by $j \geq 0$ such that the a_j 's are atoms and for $j \geq 1$, $(a_1, a_2, \dots, a_j, b_j, 0, \dots)$ is a pseudo-decomposition of b . For this, we start with $a_0 := 0$, $b_0 := b$ and then for $j \geq 0$ we apply (2.7) to b_j to get an atom $a_{j+1} \preceq b_j$. We then set $b_{j+1} := b_j - a_{j+1}$ so that (a_{j+1}, b_{j+1}) is a pseudo-decomposition of b_j and proceed to the next step. By construction, $(a_1, a_2, \dots, a_{j+1}, b_{j+1}, 0, \dots)$ is a pseudo-decomposition of b . Moreover, by (2.7), for $j \geq 0$

$$q(b_{j+1}) \leq \nu(a_{j+1}). \quad (2.13)$$

In particular $q(b_{j+1}) < \infty$ and we can apply (2.7) to b_{j+1} and continue the construction. The pseudo-nonincreasing sequence b_j converges to some $b_\infty \in \mathcal{S}$ and (a_1, a_2, \dots) is a decomposition of $b - b_\infty = b_0 - b_\infty$ in atoms.

To conclude, we establish that $b_\infty = 0$. Let us fix $j \geq 1$ and let $(b_{j,1}, b_{j,2}, \dots)$ be a decomposition of b_j such that $\nu(b_{j,i}) \leq 2q(b_j)$ for $i \geq 1$. We compute, using (H2),

$$\phi(b_j) \leq \sum_{i \geq 1} \phi(b_{j,i}) \leq \sum_{i \geq 1} \eta(\nu(b_{j,i})) \nu(b_{j,i}) \leq \eta(2q(b_j)) \nu(b_j).$$

Using (2.13) and $\nu(b_j) \leq \nu(b)$, we get

$$\phi(b_j) \leq \eta(2\nu(a_j)) \nu(b).$$

Since $\sum \nu(a_j) \leq \nu(b) < \infty$ we have $\nu(a_j) \rightarrow 0$, hence $\eta(2\nu(a_j)) \rightarrow 0$ and we obtain $\phi(b_j) \rightarrow 0$. Using (H2) again we write

$$\phi(b_\infty) \leq \phi(b_j - b_\infty) + \phi(b_j) \leq \eta(\nu(b)) \nu(b_j - b_\infty) + \phi(b_j) \xrightarrow{j \uparrow \infty} 0,$$

and since ϕ is a norm we conclude that $b_\infty = 0$. This proves the lemma. \square

Remark 2.4. It transpires from the proof that we do need the whole “normed group” structure of \mathcal{G} . Indeed, the inequality $\nu(c-a) \leq \nu(c) + \nu(a)$ is never used. By inspecting the demonstration, we see that the lemma is still correct if $(\mathcal{G}, +, \nu)$ is a complete normed commutative *monoid*. In this setting, the identities of the form $b = c - a$ in the proof should be read as $a + b = c$. For instance, the term $b - a$ of Observations 2.3 (3.a) should be defined as $\sum_{j \geq 2} a_j$ where (a, a_2, a_3, \dots) is a pseudo-decomposition of b .

3 Proof of Theorem 1.3

As in the introduction, $(G, +, |\cdot|_G)$ is a complete Abelian normed group and $0 \leq k \leq n$. Let us first establish an isoperimetric inequality for normal rectifiable G -flat chains. In light of Almgren’s isoperimetric inequality [Alm86, Whi99a], the result is not that surprising and might not be new. Its proof is based on the deformation theorem of White [Whi99a] and follows the steps of the proof of Federer’s isoperimetric inequality for integral currents.

Let us recall the definition of the h -mass of a rectifiable chain, see [Whi99a, Section 6].

Definition 3.1.

Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a lower semicontinuous and subadditive function satisfying $h(0) = 0$. The h -mass of a rectifiable k -chain $A = w\xi\mathcal{H}^k \llcorner \Sigma$ is defined as

$$\mathbb{M}_h(A) := \int_{\Sigma} h(\rho) d\mathcal{H}^k,$$

where $\rho := |w|_G$.

The condition $h(0) = 0$ ensures that the definition does not depend on the choice of (Σ, w) . The lower-semicontinuity and subadditivity properties are necessary to get good properties of \mathbb{M}_h with respect to convergence and projections/deformations. Let us recall in particular that under these assumptions \mathbb{M}_h is countably subadditive (see [Whi99a, Section 6]), that is,

$$\mathbb{M}_h\left(\sum B_j\right) \leq \sum \mathbb{M}_h(B_j) \quad \text{for } B_j \in \mathcal{M}_k^G(\mathbb{R}^n), \text{ rectifiable.} \quad (3.1)$$

Of course, the case $h(s) = s$ corresponds to the usual mass.

Lemma 3.2 (Isoperimetric inequality for normal rectifiable chains).

Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be lower semicontinuous, subadditive and such that $h(0) = 0$, $h > 0$ on $(0, +\infty)$ and $h'(0^+) := \lim_{s \downarrow 0} h(s)/s = \infty$. There exists a nondecreasing function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ only depending on n and h such that $\lim_{m \downarrow 0} \eta(m) = 0$ and:

$$\mathbb{F}(A) \leq \eta(\mathbb{M}(A)) (\mathbb{M}_h(A) + \mathbb{N}(A))$$

for every normal rectifiable chain $A \in \mathcal{N}_k^G(\mathbb{R}^n)$.

Proof. Let h and A as in the statement of the lemma and let $\varepsilon > 0$ to be fixed later. Applying the deformation theorem [Whi99a, Theorem 1.1], there exist a constant $c \geq 1$ only depending on n and chains $P \in \mathcal{P}_k^G(\mathbb{R}^n)$, $R \in \mathcal{F}_k^G(\mathbb{R}^n)$ and $S \in \mathcal{F}_{k+1}^G(\mathbb{R}^n)$ with the following properties:

- (a) $A = P + R + \partial S$.
- (b) $P = \sum g_F F$, with $g_F \in G$ and where we sum over a countable set of essentially disjoint oriented k -cubes F with side length ε .
- (c) $\mathbb{M}(P) \leq c\mathbb{M}(A)$.
- (d) $\mathbb{M}_h(P) \leq c\mathbb{M}_h(A)$.
- (e) $\mathbb{M}(R) + \mathbb{M}(S) \leq c\varepsilon\mathbb{N}(A)$.

Let us denote $\nu(A) := \mathbb{M}_h(A) + \mathbb{N}(A)$. By (a) and (e), there holds,

$$\mathbb{F}(A) \leq \mathbb{M}(P) + c\varepsilon\nu(A). \quad (3.2)$$

To estimate the first term, we write, using the formula (b) and the inequality (c),

$$\mathbb{M}(P) = \varepsilon^k \sum_F |g_F|_G \leq c\mathbb{M}(A). \quad (3.3)$$

We deduce

$$\max_F |g_F|_G \leq c\mathbb{M}(A)\varepsilon^{-k}. \quad (3.4)$$

We now set $\tilde{\eta}(0) := 0$ and for $m > 0$,

$$\tilde{\eta}(m) := \sup \left\{ \frac{s}{h(s)} : 0 < s \leq m \right\}.$$

From the assumption on h , $\tilde{\eta}$ is nondecreasing and $\lim_{m \downarrow 0} \tilde{\eta}(m) = 0$. We compute, using (b),

$$\begin{aligned} \mathbb{M}(P) &= \varepsilon^k \sum_F |g_F|_G \leq \varepsilon^k \sum_F \tilde{\eta}(|g_F|_G) h(|g_F|_G) \stackrel{(3.4)}{\leq} \tilde{\eta}(c\mathbb{M}(A)\varepsilon^{-k}) \varepsilon^k \sum_F h(|g_F|_G) \\ &= \tilde{\eta}(c\mathbb{M}(A)\varepsilon^{-k}) \mathbb{M}_h(P) \stackrel{(d)}{\leq} c\tilde{\eta}(c\mathbb{M}(A)\varepsilon^{-k}) \nu(A). \end{aligned}$$

Putting this estimate in (3.2), we get

$$\mathbb{F}(A) \leq c \left[\tilde{\eta}(c\mathbb{M}(A)\varepsilon^{-k}) + \varepsilon \right] \nu(A).$$

Then, taking the infimum over $\varepsilon > 0$ and setting for $m > 0$,

$$\eta(m) := c \inf_{\varepsilon > 0} \left[\tilde{\eta}(c\varepsilon m) + \varepsilon \right], \quad (3.5)$$

we obtain,

$$\mathbb{F}(A) \leq \eta(\mathbb{M}(A))\nu(A) = \eta(\mathbb{M}(A))(\mathbb{M}_h(A) + \mathbb{N}(A)).$$

Eventually, since $\tilde{\eta}$ is nondecreasing, η is nondecreasing. Moreover, given $\delta > 0$, using $\tilde{\eta}(s) \rightarrow 0$ as $s \downarrow 0$ we can choose ε and then m small enough so that $\tilde{\eta}(cm\varepsilon^{-k}) + \varepsilon < \delta$. We deduce that $\eta(m) \rightarrow 0$ as $m \downarrow 0$. This proves the lemma. \square

Remark 3.3. Assuming that h is increasing and using \mathbb{M}_h instead of \mathbb{M} in (3.3) we can replace (3.4) by

$$\max_F |g_F|_G \leq h^{-1}(c\mathbb{M}_h(A)\varepsilon^{-k}).$$

Therefore, setting $\tilde{\eta}_*(m) := h^{-1}(m)/m$ and then defining η_* by (3.5) with $\tilde{\eta}_*$ instead of $\tilde{\eta}$ we have also the inequality

$$\mathbb{F}(A) \leq \eta_*(\mathbb{M}_h(A))(\mathbb{M}_h(A) + \mathbb{N}(A)).$$

In the particular case $h(m) = m^\alpha$ for $\alpha \in (0, 1)$ this yields

$$\mathbb{F}(A) \leq c\mathbb{M}_h(A)^{\frac{1-\alpha}{\alpha+k(1-\alpha)}}(\mathbb{M}_h(A) + \mathbb{N}(A)).$$

Sending α to 0, so that in the limit $\mathbb{M}_h(A) = \text{Size}(A)$ we recover Almgren's isoperimetric inequality [Alm86] (see also [Whi99a, Theorem 6.2]).

We can now prove the main result.

Proof of Theorem 1.3. Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$ be a normal rectifiable k -chain.

Step 1. We first claim that:

$$\text{There exists } h \in C(\mathbb{R}_+, \mathbb{R}_+) \text{ increasing, concave and such that } \begin{cases} h(0) = 0, \\ h'(0^+) = \infty, \\ \mathbb{M}_h(A) < \infty. \end{cases}$$

With the notation of Definition 3.1, $\mathbb{M}(A) = \int \rho d\mathcal{H}^k \llcorner \Sigma < \infty$ and for $h \in C(\mathbb{R}_+, \mathbb{R}_+)$,

$$\mathbb{M}_h(A) = \int h(\rho) d\mathcal{H}^k \llcorner \Sigma.$$

The claim is then a direct application of Lemma A.1 established in appendix, with the nonnegative function $f = \rho$ and the measure $\mu = \mathcal{H}^k \llcorner \Sigma$.

Remark that h is strictly subadditive so that it satisfies the assumptions of Lemma 3.2.

Step 2. For $B \in \mathcal{F}_k(\mathbb{R}^n, G)$, we set

$$\nu(B) := \mathbb{M}_h(B) + \mathbb{N}(B), \tag{3.6}$$

and define

$$\mathcal{G} := \{B \in \mathcal{F}_k(\mathbb{R}^n, G), \text{ rectifiable and such that } \nu(B) < \infty\}. \tag{3.7}$$

The mapping ν is obviously a norm on \mathcal{G} (recall (3.1)). Let us show that (\mathcal{G}, ν) is complete. Let B_j be a Cauchy sequence in (\mathcal{G}, ν) . In particular, B_j is a Cauchy sequence in $(\mathcal{N}_k^G(\mathbb{R}^n), \mathbb{N})$ which is complete and there exists $B_\infty \in \mathcal{N}_k^G(\mathbb{R}^n)$ such that $\mathbb{N}(B_j - B_\infty) \rightarrow 0$.

Now, there exists a k -rectifiable set $\Sigma \subset \mathbb{R}^n$ such that $B_j = B_j \llcorner \Sigma$ for every $j \geq 1$. Using transparent notation, we write $B_j = w_j \mathcal{H}^k \llcorner \Sigma$ and we observe that the convergence $\mathbb{M}(B_j - B_\infty) \rightarrow 0$ rewrites as

$$w_j \rightarrow w_\infty \quad \text{in } L^1\left((\Sigma, \mathcal{H}^k), (G, |\cdot|_G)\right). \tag{3.8}$$

Similarly, denoting $|g|_G^* := h(|g_G|)$ for $g \in G$, the fact that B_j is a Cauchy sequence with respect to \mathbb{M}_h rewrites as

$$w_j \text{ is a Cauchy sequence in } L^1 \left((\Sigma, \mathcal{H}^k), (G, |\cdot|_G^*) \right).$$

The latter group being complete, there exists w_∞^* such that

$$w_j \rightarrow w_\infty^* \quad \text{in } L^1 \left((\Sigma, \mathcal{H}^k), (G, |\cdot|_G^*) \right).$$

Besides, with (3.8) we have $w_\infty^* = w_\infty$ \mathcal{H}^k -almost everywhere on Σ and we see that $\mathbb{M}_h(B_j - B_\infty) \rightarrow 0$. This proves that (\mathcal{G}, ν) is complete.

Now we set

$$\mathcal{S} := \{A \llcorner S : S \subset \mathbb{R}^n \text{ Borel set such that } \mathbb{N}(A \llcorner S) < \infty\}. \quad (3.9)$$

Notice that by Remark 1.2 every element of \mathcal{S} is rectifiable.

We claim that for $B \in \mathcal{G}$ and $A_1, A_2, \dots \in \mathcal{S}$ we have (using the notation of Lemma 2.1 with (3.6)(3.7)&(3.9)),

$$A_j \text{ decomposition of } B \text{ as in Lemma 2.1} \iff A_j \text{ set-decomposition of } B. \quad (3.10)$$

Assuming that (A_1, A_2, \dots) is a set-decomposition of B , we write $A_j = B \llcorner S_j$ where S_j is a Borel partition of \mathbb{R}^n . We have obviously $\mathbb{M}_h(B) = \sum \mathbb{M}_h(A_j)$ and by definition $\mathbb{N}(B) = \sum \mathbb{N}(A_j)$, hence (A_1, A_2, \dots) is a decomposition of B in the sense of Lemma 2.1.

Conversely, if (A_1, A_2, \dots) is a decomposition of B in the sense of Lemma 2.1, then $B = \sum A_j$ and $\nu(B) = \sum \nu(A_j)$. Since, by the triangle inequality,

$$\lambda(B) \leq \sum \lambda(A_j) \quad \text{for } \lambda = \mathbb{M}, \mathbb{M}_h \text{ and } \mathbb{M}(\partial \cdot),$$

the identity $\nu(B) = \sum \nu(A_j)$ yields

$$\lambda(B) = \sum \lambda(A_j) \quad \text{for } \lambda = \mathbb{M}, \mathbb{M}_h \text{ and } \mathbb{M}(\partial \cdot).$$

Hence $\mathbb{N}(B) = \sum \mathbb{N}(A_j)$.

It remains to check that the A_j 's belong to \mathcal{S} . With the notation of Definition 3.1, we set

$$\mu_A := \rho \mathcal{H}^k \llcorner \Sigma,$$

so that $\mathbb{M}(A \llcorner S) = \mu_A(S)$ for every Borel set $S \subset \mathbb{R}^n$.

Writing $B = A \llcorner S$ and similarly $A_j = A \llcorner S_j$ for $j \geq 1$ for some Borel subsets $S, S_j \subset \mathbb{R}^n$, the convergence of $\sum A_j$ towards B in mass is equivalent to $\sum \mathbf{1}_{S_j} \rightarrow \mathbf{1}_S$ in $L^1(\mathbb{R}^n, \mu_A)$. We deduce that, up to μ_A -negligible sets, S_j is a partition of S . This proves (3.10).

The proof that B is an atom in the sense of Lemma 2.1 if and only if B is set-indecomposable is similar.

Step 3. By the previous step the theorem follows from Lemma 2.1 (and Remark 2.2 (1)) applied to $(\mathcal{G}, +, \nu)$ and \mathcal{S} provided that (H1)&(H2) hold true.

First, choosing the norm $\phi = \mathbb{F}$, Assumption (H2) is a direct consequence of Lemma 3.2.

Let us establish (H1). Let $B_1 \succeq B_2 \succeq \dots$ be a nonincreasing sequence. Referring to Observations 2.3 (3.c), there exists $B_\infty \in \mathcal{G}$ such that $\nu(B_j - B_\infty) \rightarrow 0$. Next, reasoning as above, there exists a nonincreasing sequence S_j of Borel subsets of \mathbb{R}^n such that $B_j = A \llcorner S_j$. Defining $S_\infty := \cap S_j$ and $B_\infty^* := A \llcorner S_\infty$, we have by the monotone convergence theorem,

$$\lim \mathbb{M}(B_j - B_\infty^*) = 0.$$

Consequently, $B_\infty = B_\infty^* = A \llcorner S_\infty$ and $B_\infty \in \mathcal{S}$. This proves (H1).

As a conclusion the theorem follows from Lemma 2.1. \square

4 Uniqueness of the set-decomposition of normal n -chains with real coefficients

As announced in the introduction, we are able to establish the uniqueness of the maximal set-decomposition of normal codimension 0 chains when $(G, +, |\cdot|_G)$ is a closed subgroup of $(\mathbb{R}, +, |\cdot|)$.

Proposition 4.1. *If G is the additive group \mathbb{R} endowed with the standard norm, then the decomposition of a normal G -flat n -chain in \mathbb{R}^n in set-indecomposable subchains is unique up to rearranging the sequence and adding or deleting zeros.*

The result also applies to $G = \mathbb{Z}$ by the embedding $\mathcal{N}_n^{\mathbb{Z}}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_n^{\mathbb{R}}(\mathbb{R}^n)$. The proof of the proposition is based on the coarea formula for functions of bounded variations and on the uniqueness of the decomposition of a set of finite perimeter in its measure theoretic connected components provided by [ACMM01, Theorem 1]. We first reformulate the proposition as a result about functions of bounded variation (Theorem 4.2 below).

It is well-known that the space of \mathbb{R} -flat n -chains in \mathbb{R}^n in the sense of [Fle66] identifies with a subspace of k -currents in \mathbb{R}^n , namely, the closure of the space of normal n -currents with respect to the norm

$$W(T) := \sup \langle T, \omega \rangle,$$

where the supremum is taken over the smooth and compactly supported differential n -forms ω over \mathbb{R}^n such that $\|\omega\|_{\infty} \leq 1$. This space obviously identifies isometrically with $L^1(\mathbb{R}^n)$ and denoting f_A the function corresponding to a \mathbb{R} -flat n -chain A , we have $\mathbb{F}(A) = \mathbb{M}(A) = \|f_A\|_{L^1}$ and ∂A is the $(n-1)$ -current $\sum_{i=1}^n \partial_{x_i} f_A \mathcal{H}^n e_{\bar{i}}$ where e_1, \dots, e_n is the standard basis of \mathbb{R}^n and

$$e_{\bar{i}} := e_1 \wedge \dots \wedge e_{i-1} \wedge e_{i+1} \wedge \dots \wedge e_n.$$

Using the Hodge star operator $e_{\bar{i}} \mapsto e_i$, ∂A identifies with the distribution ∇f_A . Moreover, the n -current A is normal if and only if f_A is a function with bounded variation and we have the identity $\mathbb{M}(\partial A) = |\nabla f_A|_{TV}$ where here and below the total variation of a vector valued Borel measure $\mu \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^d)$ is computed with respect to the Euclidian norm in \mathbb{R}^d , that is,

$$|\mu|_{TV} := \sup \left\{ \sum_{S_j} \|\mu(S_j)\|_{\ell^2(\mathbb{R}^d)} : S_j \text{ Borel partition of } \mathbb{R}^n \right\}.$$

Next, given a Borel set $S \subset \mathbb{R}^n$, we have $f_{A \llcorner S} = \mathbf{1}_S f_A$ and a set-decomposition of A corresponds to a finite or countable Borel partition \mathcal{B} of \mathbb{R}^n such that

$$|\nabla f_A|_{TV} = \sum_{S \in \mathcal{B}} |\nabla[\mathbf{1}_S f_A]|_{TV}. \quad (4.1)$$

Denoting $\Omega := \{x \in \mathbb{R}^n : f_A(x) \neq 0\}$, we have $f_A = \mathbf{1}_{\Omega} f_A$ so we may only consider Borel partitions of Ω . A set-decomposition corresponds to a Borel partition of Ω satisfying (4.1) and A is indecomposable if for every Borel set $S \subset \Omega$ there holds

$$|\nabla f_A|_{TV} = |\nabla[\mathbf{1}_S f_A]|_{TV} + |\nabla[\mathbf{1}_{\Omega \setminus S} f_A]|_{TV} \implies \mathcal{H}^n(S) = 0 \text{ or } \mathcal{H}^n(\Omega \setminus S) = 0.$$

Let us state in terms of BV -functions both the existence result of Theorem 1.3 in the case $k = n$, $G = \mathbb{R}$ and the uniqueness result (still to be proved) of Proposition 4.1.

Theorem 4.2. *Let $f \in BV(\mathbb{R}^n)$, there exists a Borel partition \mathcal{B} of $\Omega := \{x \in \mathbb{R}^n : f(x) \neq 0\}$ such that*

$$|\nabla f|_{TV} = \sum_{S \in \mathcal{B}} |\nabla[\mathbf{1}_S f]|_{TV} \quad (4.2)$$

and such that for any Borel partition \mathcal{B}' with the same properties and any $S \in \mathcal{B}$ we have $\mathcal{H}^n(S \setminus S') = 0$ for some S' in \mathcal{B}' . In other words, \mathcal{B} is the finest Borel partition of Ω satisfying (4.2).

Proof of Theorem 4.2.

Step 0 (conventions). In this proof we identify Borel subsets of Ω which only differ by a Lebesgue null set and we make an abuse of notation by writing $S' \subset S$ if $S' \setminus S$ is a null set. With this convention, given $\mathcal{B}, \mathcal{B}'$ two families of Borel subsets of Ω , we write $\mathcal{B} \in \mathcal{B}'$ whenever

for every element $S \in \mathcal{B}$ there exists $S' \in \mathcal{B}'$ such that $S \subset S'$.

This defines a partial order on the families of Borel subsets of \mathbb{R}^n . The theorem states that the collection of Borel partitions of Ω satisfying (4.2) admits a least element for the relation \in . Similarly, with this vocabulary [ACMM01, Theorem 1] states that a set of finite perimeter $E \subset \mathbb{R}^n$ admits a Borel partition \mathcal{B}_E whose elements are called the M-connected components of E such that

$$P(E) = \sum_{F \in \mathcal{B}_E} P(F)$$

and such that if \mathcal{B}'_E is any other Borel partition of E ,

$$P(E) = \sum_{F \in \mathcal{B}'_E} P(F) \implies \mathcal{B}_E \in \mathcal{B}'_E. \quad (4.3)$$

Step 1. Let $f \in BV(\mathbb{R}^n)$. For $t \in \mathbb{R} \setminus \{0\}$, we set

$$E_t := \begin{cases} \{x \in \mathbb{R}^n : f(x) > t\} & \text{if } t > 0, \\ \{x \in \mathbb{R}^n : f(x) < t\} & \text{if } t < 0. \end{cases}$$

For almost every t , E_t is a set of finite perimeter and denoting by $P(E)$ the perimeter of $E \subset \mathbb{R}^n$ (that is the \mathcal{H}^{n-1} -measure of the reduced boundary of E), the mapping $t \mapsto P(E_t)$ is measurable and by the coarea formula [AFP00, Theorem 3.40],

$$|\nabla f|_{TV} = \int_{\mathbb{R}} P(E_t) dt. \quad (4.4)$$

Let us denote by \mathcal{B}_t the collection of the M-connected components of E_t given by [ACMM01, Theorem 1]. For almost every $t \in \mathbb{R}$, \mathcal{B}_t is a finite or countable Borel partition of E_t and

$$P(E_t) = \sum_{F \in \mathcal{B}_t} P(F).$$

Let \mathcal{B} be a Borel partition of Ω such that (4.2) holds true. Since for $t \in \mathbb{R} \setminus \{0\}$ and $S \in \mathcal{B}$,

$$E_t \cap S = \begin{cases} \{x \in \mathbb{R}^n : \mathbf{1}_S(x) f(x) > t\} & \text{if } t > 0, \\ \{x \in \mathbb{R}^n : \mathbf{1}_S(x) f(x) < t\} & \text{if } t < 0, \end{cases}$$

we have again by the coarea formula,

$$|\nabla[\mathbf{1}_S f]|_{TV} = \int P(E_t \cap S) dt.$$

With (4.2) and (4.4) this leads to

$$\int_{\mathbb{R}} P(E_t) dt = \int_{\mathbb{R}} \sum_{S \in \mathcal{B}} P(E_t \cap S) dt.$$

Since for every t , $E_t = \cup_S (E_t \cap S)$ we have $P(E_t) \leq \sum_S P(E_t \cap S)$ and the preceding inequality enforces

$$P(E_t) = \sum_{S \in \mathcal{B}} P(E_t \cap S) \quad \text{for almost every } t \in \mathbb{R} \setminus \{0\}.$$

By (4.3) this implies $\mathcal{B}_t \in \{E_t \cap S : S \in \mathcal{B}\} \in \mathcal{B}$ and we conclude that

$$\text{for almost every } t \in \mathbb{R} \setminus \{0\} \quad \mathcal{B}_t \in \mathcal{B}. \quad (4.5)$$

Step 2. Now we claim that the collection \mathcal{B}_t for $t \neq 0$ admits a \subseteq -maximal element \mathcal{B}_0 . Indeed, for $x, y \in \Omega$ let us write $x \sim y$ whenever there exists $t = t_{x,y} \neq 0$ such that E_t is a set of finite perimeter and x and y are both points of density of the same $F \in \mathcal{B}_t$. Remark that if $0 < t_1 < t_2$ or $t_2 < t_1 < 0$ then $\mathcal{B}_{t_2} \in \mathcal{B}_{t_1}$ so the relation \sim is an equivalence relation on

$$\Omega_0 := \bigcup_{t \neq 0} \bigcup_{F \in \mathcal{B}_t} \{x \text{ point of density of } F\}$$

which is of full measure in Ω . Moreover we can impose that the $t_{x,y}$'s above lie in a countable set T such that $\sup T \cap (-\infty, 0) = \inf T \cap (0, +\infty) = 0$. It follows that the each equivalence class of Ω_0/\sim writes as countable union of sets with finite perimeter and in particular $\mathcal{B}_0 := \Omega_0/\sim$ is a Borel partition of Ω_0 .

By construction and (4.5) we have $\mathcal{B}_0 \in \mathcal{B}$ for any Borel partition \mathcal{B} of Ω satisfying (4.2). Let us check that $\mathcal{B} = \mathcal{B}_0$ satisfies (4.2). For $t \neq 0$ and $S \in \mathcal{B}_0$, we define

$$S_t := S \cap E_t = \begin{cases} \{x \in \mathbb{R}^n : \mathbf{1}_S(x)f(x) > t\} & \text{if } t > 0, \\ \{x \in \mathbb{R}^n : \mathbf{1}_S(x)f(x) < t\} & \text{if } t < 0, \end{cases}$$

On the one hand, we have for almost every t and any $S \in \mathcal{B}_0$,

$$|\nabla[\mathbf{1}_S f]|_{TV} = \int_{\mathbb{R}} P(S_t) dt. \quad (4.6)$$

On the other hand, defining $\mathcal{B}'_t := \{S_t : S \in \mathcal{B}_0\}$ we have by definition of \mathcal{B}_0 that $\mathcal{B}_t \in \mathcal{B}'_t$ for almost every $t \in \mathbb{R} \setminus \{0\}$ so

$$P(E_t) = \sum_{S \in \mathcal{B}_0} P(S_t).$$

Integrating over $t \in \mathbb{R}$, we get

$$|\nabla f|_{TV} = \sum_{S \in \mathcal{B}_0} \int P(S_t) dt \stackrel{(4.6)}{=} \sum_{S \in \mathcal{B}_0} |\nabla[\mathbf{1}_S f]|_{TV}.$$

This proves the theorem. \square

We believe that Proposition 4.1 generalizes to any Abelian normed groups G but addressing the general case would take us too far afield, however an idea would be to identify the normal n -chain A with a G -valued function $f_A : \mathbb{R}^n \rightarrow G$ such that f_A is of bounded variation in the sense of [Amb90]. Such identification is established in [DPH14, Theorem 4.1] and to complete the program, we need the identity

$$\mathbb{M}(\partial A) = |\nabla f_A|_{TV} := \sup \sum_j |\nabla[\phi_j \circ f_A]|(S_j), \quad (4.7)$$

where the supremum is taken over all possible the countable Borel partitions S_j of \mathbb{R}^n and over the sequences of 1-Lipschitz continuous functions $\phi_j : G \rightarrow \mathbb{R}$. Unfortunately, the result of [DPH14] only provides a bilipschitz group isomorphism $A \in \mathcal{N}_n^G(\mathbb{R}^n) \rightarrow BV(\mathbb{R}^n, G)$ and we only have at hand a two-sided estimate in place of the identity (4.7).

A Higher integrability lemma

Lemma A.1. *Let (Ω, μ) be a measure space. Given a μ -integrable function $f : \Omega \rightarrow \mathbb{R}_+$, there exists $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous, increasing, concave such that $h(0) = 0$, $h'(0^+) := \lim_{s \downarrow 0} h(s)/s = \infty$ and*

$$\int h(f) d\mu < \infty.$$

Proof. This is the consequence of La Vallée Poussin's higher summability property for absolutely converging series, that is, if $a_j \geq 0$ is such that $\sum a_j < \infty$ then there exists a sequence $1 = b_0 < b_1 < \dots < b_j < b_{j+1} < \dots \rightarrow \infty$ such that $\sum a_j b_j < \infty$. Applying this to the series

$$\sum_{j \geq 0} \int_{\{2^{-j-1} < f \leq 2^{-j}\}} f d\mu = \int f d\mu < \infty,$$

we get $1 = b_0 < b_1 < \dots < b_j < b_{j+1} < \dots \rightarrow \infty$ such that,

$$\sum_{j \geq 0} b_j \int_{\{2^{-j-1} < f \leq 2^{-j}\}} f d\mu < \infty.$$

Defining $c_0 := 1$ and then recursively $c_j := \min(\sqrt{2}, b_j/b_{j-1}) c_{j-1}$ for $j \geq 1$, we have $1 = c_0 < \dots < c_{j-1} < c_j < \dots$ and by induction $c_j \leq b_j$ for $j \geq 0$. Consequently,

$$\sum_{j \geq 0} c_j \int_{\{2^{-j-1} < f \leq 2^{-j}\}} f d\mu < \infty. \quad (\text{A.1})$$

Notice also that by induction there holds,

$$c_{j+i} \leq 2^{i/2} c_j \quad \text{for } i, j \geq 0. \quad (\text{A.2})$$

Moreover,

$$c_j = \prod_{i=1}^j \min\left(\sqrt{2}, \frac{b_i}{b_{i-1}}\right) \xrightarrow{j \uparrow \infty} \infty. \quad (\text{A.3})$$

Indeed, denoting $\Lambda := \{i \geq 1 : b_i \geq \sqrt{2} b_{i-1}\}$. If on the one hand Λ is finite, then for $j \geq j_0 := \max \Lambda$, $c_j = (c_{j_0}/b_{j_0}) b_j$ hence $c_j \rightarrow \infty$ as $j \rightarrow \infty$. If on the other hand, Λ is infinite, there holds

$$c_j \geq \left(\sqrt{2}\right)^{|\Lambda \cap [1, j]|} \xrightarrow{j \uparrow \infty} \infty.$$

Summing up, we have $1 = c_0 < \dots < c_j < c_{j+1} < \dots \rightarrow \infty$ and (A.1), (A.2) & (A.3) hold. Let us define $g : (0, \infty) \rightarrow [1, \infty)$ by

$$g(s) := \begin{cases} c_j & \text{for } s \in (2^{-j-1}, 2^{-j}], j \geq 1, \\ c_0 = 1 & \text{for } s > 1/2. \end{cases}$$

We notice that g is nonincreasing and that by (A.3) $g(s) \rightarrow \infty$ as $s \downarrow 0$. Let us set, for $s \geq 0$,

$$h(s) := \int_0^s g(t) dt,$$

(observe that by (A.2), $g(s) \leq s^{-1/2}$ for $0 < s \leq 1/2$ so that h is well-defined). We have that $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, concave, increasing and such that $h(0) = 0$ and $h'(0^+) = \infty$. Eventually, we compute for $j \geq 1$ and $s \in (2^{-j-1}, 2^{-j}]$,

$$\begin{aligned} h(s) &\leq h(2^{-j}) = \sum_{i \geq j} 2^{-i-1} c_i = 2^{-j-1} \sum_{i \geq 0} 2^{-i} c_{j+i} \\ &\stackrel{(A.2)}{\leq} 2^{-j-1} c_j \sum_{i \geq 0} 2^{-i/2} = (2 + \sqrt{2}) 2^{-j-1} c_j \leq (2 + \sqrt{2}) s c_j. \end{aligned}$$

Consequently, $h \circ f \leq (2 + \sqrt{2}) c_j f$ in the domain $\{2^{-j-1} < f \leq 2^{-j}\}$. We conclude by (A.1) that $h \circ f$ is μ -integrable which proves the lemma. \square

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