KAKEYA MAXIMAL INEQUALITY IN THE HEISENBERG GROUP

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ABSTRACT. We define the Heisenberg Kakeya maximal functions $M_\delta f$, $0 < \delta < 1$, by averaging over δ-neighborhoods of horizontal unit line segments in the Heisenberg group $\mathbb{H}^1$ equipped with the Korányi distance $d_{\mathbb{H}}$. We show that for $p \in [1,\infty]$ and $\varepsilon > 0$

$$\|M_\delta f\|_{L^p(S^1)} \leq C(p,\varepsilon)\delta^{-\alpha(p)-\varepsilon}\|f\|_{L^p(\mathbb{H}^1)}, \quad f \in L^p(\mathbb{H}^1),$$

where $\alpha(p) = \max\{4/p - 1, 1/p\}$ and, moreover, that $\alpha(p)$ is the smallest exponent for which the bound holds. The proof is based on a recent variant, due to Pramanik, Yang, and Zahl, of Wolff's circular maximal function theorem for a class of planar curves related to Sogge's cinematic curvature condition. As an application of our Kakeya maximal inequality for $p = 3$, we recover the sharp lower bound for the Hausdorff dimension of Heisenberg Kakeya sets of horizontal unit line segments in $(\mathbb{H}^1, d_{\mathbb{H}})$, first proven by Liu.

1. INTRODUCTION

This paper concerns the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ with the product

$$(x_1,x_2,x_3)\cdot(x_1',x_2',x_3') = \Big(x_1+x_1', x_2+x_2', x_3+x_3'+\tfrac{1}{2}[x_1x_2' - x_2x_1']\Big)$$

and the *Korányi metric* $d_{\mathbb{H}}(x,y) = \|y^{-1}\cdot x\|_{\mathbb{H}}$, where $\|x\|_{\mathbb{H}} = \big((x_1^2+x_2^2)^2 + 16x_3^2\big)^{1/4}$. We introduce the *Heisenberg Kakeya maximal function* $M_\delta f : S^1 \to [0,\infty]$,

$$M_\delta f(e) = \sup_{y\in\mathbb{H}^1} \frac{1}{|T_\delta(y,e)|}\int_{T_\delta(y,e)} |f|, \quad e \in S^1. \tag{1.1}$$

Here $T_\delta(y,e)$ is the *Heisenberg δ-tube* of length 1 at y in direction e, as defined in Definition 2.1, and integration is with respect to Lebesgue measure on $\mathbb{R}^3$. The coaxial lines of such tubes are *horizontal lines* in the sense of Heisenberg geometry, or, equivalently, lines in $\mathcal{L}_{SL(2)}$ in the terminology of [23, 5, 11]. We use a δ-incidence result for arcs of parabolas in $\mathbb{R}^2$ to deduce information about Heisenberg Kakeya maximal functions. More precisely, we apply a special case of Pramanik, Yang, and Zahl's recent generalization [20] of Wolff's circular maximal function bound [24] to prove the following result.

Theorem 1.2. *For all $p \in [1,\infty]$, $\varepsilon > 0$, and $\delta \in (0,1)$*

$$\|M_\delta f\|_{L^p(S^1)} \lesssim_{p,\varepsilon} \delta^{-\alpha(p)-\varepsilon}\,\|f\|_{L^p(\mathbb{H}^1)}, \quad f \in L^p(\mathbb{H}^1),$$

where $\alpha(p) = \max\{4/p-1, 1/p\}$. Moreover, for $p \in [1,\infty]$, $\alpha(p)$ is the smallest exponent for which the inequality holds.

Date: September 25, 2023.

2020 *Mathematics Subject Classification.* (Primary) 28A80 (Secondary) 28A78, 42B25, 43A80.

Key words and phrases. Kakeya inequality, Heisenberg groups, maximal functions.

K.F. was supported by the Academy of Finland grants 321696, 328846. A.P. was supported by the INdAM-GNAMPA Project 2022 Problemi al bordo e applicazioni geometriche, codice CUP_E55F22000270001. P.W. was supported by the Warwick Mathematics Institute Centre for Doctoral Training, and gratefully acknowledges funding from the University of Warwick and the UK Engineering and Physical Sciences Research Council (Grant number: EP/W524645/1).

In particular, we prove the *Heisenberg Kakeya maximal function inequality*

$$\|M_\delta f\|_{L^3(S^1)} \lesssim_\varepsilon \delta^{-1/3-\varepsilon} \|f\|_{L^3(\mathbb{H}^1)}, \quad f \in L^3(\mathbb{H}^1). \tag{1.3}$$

1.1. **Implications of the main result and related work.** Our definition of Heisenberg Kakeya maximal functions is inspired by the *Euclidean* Kakeya maximal functions $\mathcal{K}_\delta f$ introduced by Bourgain [2]. The prominent *Kakeya maximal conjecture* states that

$$\|\mathcal{K}_\delta f\|_{L^n(S^{n-1})} \lesssim_{n,\varepsilon} \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)} \quad \text{for all } \varepsilon > 0,\ 0 < \delta < 1.$$

This is currently known only for $n = 2$, where it was first proven by Córdoba [4]. For a survey of the important developments related to Euclidean Kakeya maximal inequalities, we refer the reader to, for instance, [25, 26, 18]. As far as we know, Theorem 1.2 does not entail progress related to the Kakeya maximal conjecture in $\mathbb{R}^3$. Its setting differs from that of standard Euclidean Kakeya inequalities in two crucial aspects: The "δ-tubes" used in the definition of the Heisenberg Kakeya maximal function

(i) are defined using the *Korányi distance* and hence have volume comparable to δ^3,
(ii) arise as *Heisenberg left translates* of a one-dimensional family of Heisenberg tubes $\{T_\delta(0,e) : e \in S^1\}$ pointing in the *horizontal directions* $\{e\} \times \{0\}$, $e \in S^1$.

Our definition is tailored to the geometry of $\mathbb{H}^1$. As a corollary of (1.3), we recover a result, originally due to Liu [17], which states that *Heisenberg Kakeya sets* of horizontal unit line segments in $\mathbb{H}^1$ (in the sense of Definition 4.1) have Hausdorff dimension at least 3 with respect to $d_{\mathbb{H}}$. This bound is sharp, as evidenced by the horizontal plane $\{x_3 = 0\}$. In contrast, a union of horizontal line segments whose directions range in a positive measure subset of S^2 in the classical sense must necessarily have Hausdorff dimension 4 with respect to $d_{\mathbb{H}}$. This follows from the full dimensionality of such unions with respect to Euclidean metric, proven recently by Orponen and the first author in [5], and with a different technique by Katz, Wu, and Zahl in [11]. The proof in [5] used a Marstrand-type theorem for a restricted family of orthogonal projections onto *planes* in $\mathbb{R}^3$, while the results in the present paper and in [17] are conceptually related to a projection theorem for *lines*, which seems well suited when the Heisenberg metric is used, see Section 1.2.

We briefly discuss the relation with other results in the literature. The version of Wolff's circular maximal function by Pramanik, Yang, and Zahl [20] was recently also applied in [23] to prove a special case of the Kakeya conjecture in $\mathbb{R}^3$, the *sticky Kakeya set conjecture*, and in [11] for the $SL(2)$ *Kakeya conjecture*. Since in our case, the coaxial lines of the relevant tubes are horizontal lines, we do not need the full strength of [20], but only a special case for quadratic functions. On the other hand, the proof of Theorem 1.2 necessitates considerations related to the geometry of $(\mathbb{H}^1, d_{\mathbb{H}})$, as mentioned in (i) -(ii).

Certain lower bounds for the Hausdorff dimension with respect to $d_{\mathbb{H}}$ of *standard* (not Heisenberg) Kakeya sets in $\mathbb{R}^{2n+1}$ were deduced by Venieri [21] from known bounds for the *standard* Kakeya maximal functions. On the other hand, Euclidean Kakeya problems with a restricted set of directions were studied in [19, 22, 12, 7], but Heisenberg Kakeya sets lie outside this scope. The reason is that the left translate $y \cdot I_e$ of a segment I_e with respect to the Heisenberg product need not be parallel to I_e in the Euclidean space $\mathbb{R}^3$. The Heisenberg Kakeya maximal functions introduced in the present paper are closer in spirit to the Nikodym maximal functions defined by Kim in [13, 14], but he works with different types of tubes and his results do not seem to have direct implications regarding the Heisenberg Kakeya inequality studied here. Finally, we mention that Venieri formulated in [22] an axiomatic framework for deriving Kakeya-type inequalities and Hausdorff dimension lower bounds for Kakeya-type sets using ideas by Bourgain and

Wolff, which however does not seem to yield Theorem 1.2. Nonetheless, the arguments we use to derive the sharp Hausdorff dimension bound for Heisenberg Kakeya sets fit well in this axiomatic framework, cf., e.g., the proof of [22, Theorem 4.1].

1.2. **Outline of the proofs.** The case $p = 3$ of Theorem 1.2 (inequality (1.3)) can be obtained by duality from the following discretized Heisenberg Kakeya inequality.

Theorem 1.4 (Kakeya inequality for Heisenberg tubes). *Let $\delta \in (0,1)$ and assume that $\mathcal{T}$ is a family of Heisenberg δ-tubes pointing in δ^2-separated directions of S^1. Then, for all $\varepsilon > 0$,*

$$\int \left(\sum_{T \in \mathcal{T}} \chi_T \right)^{3/2} \lesssim_\varepsilon \delta^{3-\varepsilon} \mathrm{card}(\mathcal{T}). \tag{1.5}$$

The fact that Theorem 1.4 can be applied to (maximal) families of tubes pointing in δ^2-separated directions – and not only to sparser collections of directionally δ-separated tubes – is crucial in deriving (1.3) from Theorem 1.4 with the help of Lemma 3.1. This is a particular feature of the Heisenberg group, see Remark 3.18. The inequalities stated in Theorem 1.2 for $1 < p < 3$ and $3 < p < \infty$ then follow by interpolation between (1.3) and (trivial) estimates for $p = 1$ and $p = \infty$, respectively.

To understand why Theorem 1.4 holds for Heisenberg δ-tubes pointing in δ^2-separated directions, we reduce its proof to a Kakeya-type inequality for neighborhoods of arcs of parabolas in $\mathbb{R}^2$, see Corollary 2.19. A crucial tool for this reduction is the *vertical Heisenberg projection*

$$\pi_{\mathbb{W}} : \mathbb{H}^1 \to \mathbb{W}, \quad \pi_{\mathbb{W}}((0, x_2, x_3) \cdot (x_1, 0, 0)) = (0, x_2, x_3), \tag{1.6}$$

where "$\cdot$" denotes the group product. This bears similarities with the use of the *twisted projections* in [23]; see [23, Definition 6.1] for $f(z) = z/2$ and its relation with [23, (7.4)]. Lemma 2.8 shows that Heisenberg δ-tubes in δ^2-separated directions are mapped by $\pi_{\mathbb{W}}$ into $\sim \delta^2$-neighborhoods of graphs of quadratic polynomials with $\sim \delta^2$-separated leading coefficients, at least under suitable assumptions on the position and direction of the tubes. This allows to deduce Theorem 1.4 from Corollary 2.19 applied at scale "δ^2". Finally, Corollary 2.19 is obtained as a simple special case of [20, Theorem 1.7].

Liu's earlier work on Heisenberg Kakeya sets [17] was based on a Marstrand-type theorem for the almost sure Hausdorff dimension of orthogonal projections from $\mathbb{R}^3$ onto one-dimensional subspaces foliating the surface of a cone in $\mathbb{R}^3$. This projection theorem was first proven by Käenmäki, Orponen, and Venieri in [10], and recently sharpened and extended to a larger class of families of one-dimensional subspaces by Pramanik, Yang, and Zahl using the Kakeya-type inequality in [20, Theorem 1.7] mentioned above. The projection theorem was independently and simultaneously extended in [6] with different methods, but here we discuss especially the approaches in [10, 20] since they were both inspired by Wolff's work on the circular Kakeya problem. In particular, Liu's proof of the dimension bound for Kakeya sets is ultimately based on Wolff's results for δ-annuli in $\mathbb{R}^2$, albeit indirectly via the projection theorem. By using the recent generalization [20] of Wolff's work, which directly applies to arcs of parabolas in $\mathbb{R}^2$, we are able to make use of planar incidence geometry in a more direct way and obtain a stronger conclusion in the form of a Heisenberg Kakeya maximal inequality.

1.3. **Structure of the paper.** Section 2 contains preliminaries on Heisenberg δ-tubes and cinematic functions. In Section 3, we prove the main results of the paper, the Heisenberg

Kakeya inequalities in Theorems 1.2 and 1.4. Finally, in Section 4, we apply (1.3) to give a new proof for Liu's theorem on the dimension of Heisenberg Kakeya sets.

1.4. **Notation.** If $f, g \geq 0$, the notation $f \lesssim g$ denotes the existence of a positive constant C such that $f \leq Cg$. The notation $f \lesssim_\kappa g$ means that C may depend on a parameter "κ". Finally, $f \sim g$ is an abbreviation of $f \lesssim g \lesssim f$. We denote by $|E|$ the d-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$. The s-dimensional Hausdorff measure on $\mathbb{H}^1$ with respect to $d_{\mathbb{H}}$ is denoted by $\mathcal{H}^s_{\mathbb{H}}$, or $\mathcal{H}^s$ if the metric is clear from the context. Korányi balls are denoted by $B_{\mathbb{H}}(x, r)$ or $B(x, r)$, and Euclidean balls by $B_E(x, r)$.

Acknowledgements. Part of this work was done while K.F. and P.W. visited the Hausdorff Research Institute for Mathematics, Bonn, during the trimester *Interactions between Geometric measure theory, Singular integrals, and PDE*. The hospitality of the institute is gratefully acknowledged. Part of this work was done while P.W. was a master student at the University of Trento. We also wish to thank Tuomas Orponen and Joshua Zahl for inspiring discussions in Bonn that gave an important impetus for the present paper. Finally, we wish to thank the referee for the useful comments that have helped improve the manuscript.

2. Preliminaries on Heisenberg tubes and cinematic functions

2.1. **Heisenberg tubes.** We consider the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \cdot)$ with the left-invariant Korányi metric as defined in Section 1. For a thorough introduction to this space, we refer the reader to [3]. To define Heisenberg Kakeya maximal functions, we need *Heisenberg tubes*, whose definition and elementary properties we now discuss.

Definition 2.1 (Heisenberg tubes)**.** Let $0 < \delta < 1$. Given $y \in \mathbb{H}^1$ and $e \in S^1$, the *Heisenberg δ-tube* $T_\delta(y, e)$ is the δ-neighborhood (in the metric $d_{\mathbb{H}}$) of the horizontal line segment $y \cdot I_e$, where $I_e = \{(se, 0) : s \in [-1/2, 1/2]\}$. We also say that $T_\delta(y, e)$ *points in direction* e.

It is often convenient to write Heisenberg δ-tubes in the following more explicit form.

Lemma 2.2. *For every $y \in \mathbb{H}^1$, $e \in S^1$, and $\delta > 0$,*

$$T_\delta(y, e) = \{y \cdot (se, 0) \cdot B(0, \delta) : \ s \in [-1/2, 1/2]\},$$

where $B(0, \delta) = \{x \in \mathbb{H}^1 : \ \|x\|_{\mathbb{H}} \leq \delta\}$.

Proof. To prove the inclusion "$\subseteq$", we consider an arbitrary point $x \in T_\delta(y, e)$. Then there exists $s \in [-1/2, 1/2]$ such that

$$\|[y \cdot (se, 0)]^{-1} \cdot x\|_{\mathbb{H}} = d_{\mathbb{H}}(x, y \cdot (se, 0)) \leq \delta.$$

Writing $x = y \cdot (se, 0) \cdot [y \cdot (se, 0)]^{-1} \cdot x$ then proves the desired inclusion. For the reverse inclusion, it suffices to observe for all $z \in B(0, \delta)$ that

$$d_{\mathbb{H}}(y \cdot (se, 0) \cdot z, y \cdot (se, 0)) = \|z\|_{\mathbb{H}} \leq \delta. \qquad \square$$

The Heisenberg Kakeya maximal function $M_\delta f$ is defined by taking averages over Heisenberg δ-tubes with respect to the 3-dimensional Lebesgue measure on the underlying $\mathbb{R}^3$. We recall that this measure is invariant under translations with respect to the Heisenberg group product, and it agrees up to a positive and finite multiplicative factor with the 4-dimensional Hausdorff measure $\mathcal{H}^4$ with respect to $d_{\mathbb{H}}$. The space $(\mathbb{H}^1, d_{\mathbb{H}})$ is topologically 3-dimensional, but Ahlfors 4-regular. While a Euclidean δ-tube of length 1 in $\mathbb{R}^3$ has volume $\sim \delta^2$, the volume of Heisenberg δ-tubes is $\sim \delta^3$.

Lemma 2.3 (Volume of a Heisenberg tube). *For each $\delta \in (0,1)$, Heisenberg δ-tubes have volume*

$$|T_\delta(y,e)| \sim \delta^3, \quad y \in \mathbb{H}^1,\ e \in S^1.$$

Proof. Given a tube $T = T_\delta(y,e)$ as in the statement of the lemma, let $\{x_1,\ldots,x_N\}$ be a maximal δ-separated subset of its core segment $y \cdot I_e$. Then $N \sim \delta^{-1}$, because $(y \cdot I_e, d_{\mathbb{H}})$ is isometric to $([0,1],|\cdot|)$. Now the open balls $B_{\mathbb{H}}(x_i,\delta/2)$, $i = 1,\ldots,N$, are pairwise disjoint, have volume comparable to δ^4 and are all contained in T, so that $|T| \gtrsim \delta^3$. On the other hand, the union of the balls $B_{\mathbb{H}}(x_i, 2\delta)$, $i = 1,\ldots,N$, covers T, hence $|T| \lesssim \delta^3$. □

2.2. **From Heisenberg tubes to neighborhoods of parabolas.** We now discuss connections between Heisenberg δ-tubes in $\mathbb{H}^1$ and Euclidean δ^2-neighborhoods of arcs of parabolas in $\mathbb{R}^2$. For $\delta = 0$, i.e., for horizontal line segments and parabolic arcs, such connections were used by Liu [17] to prove the dimension bound for Heisenberg Kakeya sets. The relevant parabolas take a particularly simple form if Heisenberg tubes are described by parameters a, b, c and y_2 as in the following lemma.

Lemma 2.4. *If $e = (\cos\varphi, \sin\varphi) \in S^1$ for $\varphi \in (0,\pi)$ and $y = (y_1,y_2,y_3)$, then*

$$T_\delta(y,e) = \{(b,0,c)\cdot(as,s,0)\cdot B(0,\delta) :\ s \in [s_-,s_+]\}, \tag{2.5}$$

where $s_\pm = y_2 \pm \frac{1}{2\sqrt{1+a^2}}$ and

$$a = \frac{\cos\varphi}{\sin\varphi}, \quad b = y_1 - \left(\tfrac{\cos\varphi}{\sin\varphi}\right) y_2, \quad c = y_3 - \tfrac{1}{2}y_1y_2 + \tfrac{1}{2}\left(\tfrac{\cos\varphi}{\sin\varphi}\right) y_2^2. \tag{2.6}$$

Proof. By Lemma 2.2, it suffices to observe that the horizontal core segment of $T_\delta(y,e)$ can be parameterized as follows

$$y \cdot I_e = \{(y_1,y_2,y_3)\cdot(as,s,0) :\ s \in (1+a^2)^{-1/2}[-1/2,1/2]\}, \tag{2.7}$$

which was already shown in [17, Lemma 2.1] (up to an obvious change in the roles of the first and second coordinate axis). Then (2.5) follows since

$$\begin{aligned}(y_1,y_2,y_3)\cdot(as,s,0) &= (y_1 + as, y_2 + s, y_3 + \tfrac{1}{2}[y_1 - ay_2]s)\\ &= ([y_1 - ay_2] + a(y_2+s), y_2 + s, y_3 - \tfrac{1}{2}[y_1 - ay_2]y_2 + \tfrac{1}{2}[y_1 - ay_2](y_2+s))\\ &= (b + a(y_2+s), y_2+s, c + \tfrac{1}{2}b(y_2+s)).\end{aligned}$$

□

The next result relates Heisenberg tubes to Euclidean neighborhoods of arcs of parabolas in $\mathbb{R}^2$ via the vertical Heisenberg projection in (1.6) onto the plane $\mathbb{W} = \{x_1 = 0\}$. Explicitly, in coordinates,

$$\pi_{\mathbb{W}} : \mathbb{H}^1 \to \mathbb{W}, \quad \pi_{\mathbb{W}}(x_1,x_2,x_3) = \left(0, x_2, x_3 + \tfrac{1}{2}x_1x_2\right).$$

A related statement appeared in [1, Lemma 4.5], but our setting is a little different. For $(a,b,c) \in \mathbb{R}^3$, we let $\gamma_{(a,b,c)}$ be the parabola in $\mathbb{R}^2$ parameterized by

$$\gamma_{(a,b,c)}(s) := (s, \tfrac{a}{2}s^2 + bs + c), \quad s \in \mathbb{R}.$$

By $[\Gamma]^r$ we denote the Euclidean r-neighborhood of a set $\Gamma \subset \mathbb{R}^2$.

Lemma 2.8 (Projections of segments and tubes). *Let $0 < \delta < 1$. For all $e = (\cos\varphi, \sin\varphi)$ with $\varphi \in (0,\pi)$ and $y = (y_1,y_2,y_3) \in \mathbb{R}^3$, we have that*

$$\pi_{\mathbb{W}}(y \cdot I_e) = \gamma_{(a,b,c)}([s_-,s_+]) \quad \text{and} \quad \pi_{\mathbb{W}}(T_\delta(y,e)) \subset \left[\gamma_{(a,b,c)}([s_- - \delta, s_+ + \delta])\right]^r,$$

where $s_\pm$ and (a,b,c) are as in Lemma 2.4, and $r \sim (1+|a|)\delta^2$.

Proof. First, to prove the claim concerning the projected segment, we recall from the proof of Lemma 2.4 that

$$y \cdot I_e = \{(as + b, s, c + \tfrac{b}{2}s) : s \in [s_-, s_+]\}.$$

It follows immediately that

$$\pi_{\mathbb{W}}\left(y \cdot I_{e_a}\right) = \{(s, \tfrac{a}{2}s^2 + bs + c) : s \in \left[s_-, s_+\right]\} = \gamma_{(a,b,c)}\left(\left[s_-, s_+\right]\right),$$

where we have identified $\{0\} \times \mathbb{R}^2$ with $\mathbb{R}^2$ in the obvious way. Next, to prove the claim about the projected tube, we fix $\delta \in (0,1)$ and consider an arbitrary point

$$(b, 0, c) \cdot (as, s, 0) \cdot z \in y \cdot I_e \cdot B(0, \delta) \overset{\text{Lem. 2.2}}{=} T_\delta(y, e_a),$$

where $s \in [s_-, s_+]$ and $z \in B(0, \delta)$. By similar computations as before

$$\begin{aligned}
&\pi_{\mathbb{W}}\left((b, 0, c) \cdot (as, s, 0) \cdot z\right) \\
&\quad = \left(s + z_2, c + \tfrac{b}{2}s + z_3 + \tfrac{1}{2}[b + as]z_2 - \tfrac{1}{2}sz_1 + \tfrac{1}{2}[b + as + z_1][s + z_2]\right) \\
&\quad = \left(s + z_2, \tfrac{a}{2}s^2 + bs + c + [b + as]z_2 + z_3 + \tfrac{1}{2}z_1z_2\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma_{(a,b,c)}(s + z_2) &= \left(s + z_2, \tfrac{a}{2}[s + z_2]^2 + b[s + z_2] + c\right) \\
&= (s + z_2, \tfrac{a}{2}s^2 + bs + c + [b + as]z_2 + \tfrac{a}{2}z_2^2).
\end{aligned}$$

Since $\|z\|_{\mathbb{H}} \leq \delta$, it follows that

$$\left|\pi_{\mathbb{W}}\left((b, 0, c) \cdot (as, s, 0) \cdot z\right) - \gamma_{(a,b,c)}(s + z_2)\right| = \left|z_3 + \tfrac{z_1z_2}{2} - \tfrac{a}{2}z_2^2\right| \lesssim (1 + |a|)\delta^2.$$

This shows that $\pi_{\mathbb{W}}\left(T_\delta(y, e)\right)$ is contained in the Euclidean r-neighborhood of the parabola $\gamma_{(a,b,c)}$ for some $r \sim (1 + |a|)\delta^2$. To conclude the proof, it suffices now to observe that $s + z_2 \in [s_- - \delta, s_+ + \delta]$. □

Let $\mathbb{W}$ be the vertical plane $\{x_1 = 0\}$ and $\mathbb{L}$ the x_1-axis. Together with Lemma 2.8, the next result will allow us to reduce the Heisenberg Kakeya inequality in Theorem 1.4 to a corresponding inequality for neighborhoods of parabolic arcs in $\mathbb{R}^2$ by means of the Fubini-type formula

$$\int_{\mathbb{H}^1} h(x)dx = \int_{\mathbb{W}} \int_{\mathbb{L}} h((0, y, t) \cdot (x, 0, 0))\, d(y, t)\, dx. \tag{2.9}$$

which holds for nonnegative measurable functions h since $\Phi(x, y, t) = (0, y, t) \cdot (x, 0, 0) = (x, y, t - \frac{1}{2}xy)$ has Jacobi determinant 1. The use of (2.9) in our context requires us to understand how the fibres $\pi_{\mathbb{W}}^{-1}(w) = w \cdot \mathbb{L}$, for $w \in \mathbb{W}$, intersect a Heisenberg δ-tube.

Lemma 2.10. *Let $\mathbb{W}$ be the vertical plane $\{x_1 = 0\}$ and $\mathbb{L}$ the x_1-axis. Then*

$$\mathcal{H}^1\left(T_\delta(y, e) \cap \pi_{\mathbb{W}}^{-1}(w)\right) \lesssim \delta$$

for all $y \in \mathbb{H}^1$, $w \in \mathbb{W}$, $\delta \in (0,1)$ and $e \in S^1$ making angle at most $\pi/4$ with the x_2-axis.

Remark 2.11. Heisenberg projections can be defined for arbitrary vertical planes in $\mathbb{H}^1$, and Lemma 2.10 holds in this generality with obvious modifications. Indeed, for each $O \in SO(2)$, the map $R_O : (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \mapsto (Ox', x_3)$ is a $(d_{\mathbb{H}}, d_{\mathbb{H}})$-isometry, and $R_O \circ \pi_{\mathbb{W}} = \pi_{R_O(\mathbb{W})} \circ R_O$. Thus, Lemma 2.10 holds more generally for $\mathbb{L} \subset \mathbb{R}^2 \times \{0\}$ a 1-dimensional subspace and $\mathbb{W} = \mathbb{L}^\perp$ the orthogonal complement in $\mathbb{R}^3$ of $\mathbb{L}$.

Proof of Lemma 2.10. We can assume w.l.o.g. $y \in \mathbb{W}$; indeed $\mathcal{H}^1\left(T_\delta(y,e) \cap \pi_{\mathbb{W}}^{-1}(w)\right) = \mathcal{H}^1\left(T_\delta(w^{-1} \cdot y, e) \cap \mathbb{L}\right)$. Hence we will show for an arbitrary δ-tube $T = T_\delta((y_1,y_2,y_3),e)$ pointing in direction $e \in S^1$ that

$$\mathcal{H}^1\left(\{s \in \mathbb{R} : (s,0,0) \in T\}\right) \lesssim \delta. \tag{2.12}$$

We employ the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\} \equiv \mathbb{R}^2$ to reduce the verification of (2.12) to planar Euclidean geometry. Clearly, if $(s,0,0) \in T$, then $(s,0) \in P(T)$. Moreover, since $P : (\mathbb{H}^1, d_{\mathbb{H}}) \to (\mathbb{R}^2, |\cdot|)$ is 1-Lipschitz, the projection $P(T)$ is contained in the infinite strip

$$S := S_\delta^{\mathbb{R}^2}((y_1,y_2),e) := [(y_1,y_2) + \mathrm{span}(e)]^\delta,$$

that is, in the Euclidean δ-neighborhood of the line $P(y) + \mathrm{span}(e)$. Hence,

$$\mathcal{H}^1\left(\{s \in \mathbb{R} : (s,0,0) \in T\}\right) \leq \mathcal{H}^1\left(\{s \in \mathbb{R} : (s,0) \in S\}\right).$$

Since the strip S has width 2δ and points in direction e, trigonometry shows that the the x_1-axis intersects S in an interval of length $2\delta/|\langle e, e_2\rangle|$. By the assumption on the angle between e and the x_2-axis $\mathrm{span}(e_2)$, we know that $|\langle e, e_2\rangle| \gtrsim 1$, and (2.12) follows. □

2.3. **Kakeya inequality for parabolas.** In this section, we recall a special case of a recent Kakeya-type inequality by Pramanik, Yang, and Zahl [20] that, in the proof of Theorem 1.4, will be applied to arcs of parabolas arising in Lemma 2.8.

Remark 2.13. The scope of the Kakeya-type inequality in [20, Theorem 1.7] is broader than what is required for our application; it is formulated for a class of C^2 *cinematic functions*. This condition is related to Sogge's *cinematic curvature condition*, which was used by Kolasa and Wolff [15]. We could likely also have employed earlier work by Zahl [27, 28], which holds under the cinematic curvature assumption instead of [20, Theorem 1.7]. However, the main results in [27, 28] are not directly applicable in our setting because we are dealing with polynomials with a "dimensionality" or "non-concentration" condition for the coefficients of the quadratic term which would force us to make adaptations similar to the ones made in [16, Lemma B.2].

The arcs of the parabolas $\gamma_{(a,b,c)}$ in Lemma 2.8 are instances of the kind of curves studied in [20]. To see this, for $r > 0$ and $z = (a,b,c) \in \mathbb{R}^3$, we define

$$f_z(s) = \tfrac{a}{2}s^2 + bs + c, \quad s \in [-r,r]$$

and we let $\gamma \in C^2([-r,r];\mathbb{R}^3)$, $\gamma(s) = (s^2/2, s, 1)$. It is useful to note that $f_z(s) = \langle \gamma(s), z\rangle$ for $s \in [-r,r]$ and $z \in \mathbb{R}^3$, and

$$\mathrm{span}\{\gamma(s), \dot\gamma(s), \ddot\gamma(s)\} = \mathbb{R}^3, \quad s \in [-r,r]. \tag{2.14}$$

Hence the following result is applicable to γ as above.

Theorem 2.15 (Pramanik-Yang-Zahl). *Let $\varepsilon > 0$ and $r > 0$. Let I be a compact interval and let $\gamma : I \to \mathbb{R}^3$ be a C^2 curve satisfying $\mathrm{span}\{\gamma(s), \dot\gamma(s), \ddot\gamma(s)\} = \mathbb{R}^3$, $s \in I$. Then there exists $\delta_0 > 0$, depending only on ε, r, and $\gamma : I \to \mathbb{R}^3$, such that the following holds for all $0 < \delta \leq \delta_0$. Let $Z_\delta \subset [-r,r]^3 \subset \mathbb{R}^3$ be a set with the non-concentration condition*

$$\mathrm{card}(Z_\delta \cap B) \leq \delta^{-\varepsilon}(\rho/\delta) \tag{2.16}$$

for all Euclidean balls $B \subset \mathbb{R}^3$ of radius $\rho \geq \delta$. Then

$$\int_{[0,1]^2} \left(\sum_{z \in Z_\delta} \chi_{\Gamma_z^\delta}\right)^{3/2} \leq \delta \cdot \delta^{-C\varepsilon} \mathrm{card}(Z_\delta),$$

where $C > 0$ is a constant depending on $\gamma : I \to \mathbb{R}^3$, and Γ_z^δ is the Euclidean δ-neighborhood of the graph $\Gamma_z = \{(s, \langle \gamma(s), z\rangle) : s \in I\}$ in $\mathbb{R}^2$.

Theorem 2.15 is essentially a special case of [20, Proposition 2.1] with parameters $\alpha = \zeta = 1$ and $E = [0,1]^2$ since $[0,1]^2$ is a $(\delta, 1; 1)_1 \times (\delta, 1; 1)_1$ quasi-product in the terminology of [20]; see Remark 1 below Theorem 1.7 therein. We briefly comment on the three points in which our formulation differs slightly from the statement of [20, Proposition 2.1]:

(1) [20, Proposition 2.1] was formulated for curves γ in S^2. However, it is clear from the proof that the argument works also for curves in $\mathbb{R}^3$ satisfying all the other assumptions. Indeed, the assumption $\mathrm{span}\{\gamma(s), \dot\gamma(s), \ddot\gamma(s)\} = \mathbb{R}^3$ ensures that

$$|\langle \gamma(s), z\rangle| + |\langle \dot\gamma(s), z\rangle| + |\langle \ddot\gamma(s), z\rangle| \neq 0, \quad z \in \mathbb{R}^3 \setminus \{0\},\ s \in I. \tag{2.17}$$

Then, for all $z, z' \in \mathbb{R}^3$, we have by compactness of S^2 and I that

$$|z - z'| \overset{(2.17)}{\lesssim}_{\gamma, I} \min_{s\in I} \sum_{k=0}^{2} |\langle \gamma^{(k)}(s), z - z'\rangle| \leq \sum_{k=0}^{2} \max_{s \in I} |\langle \gamma^{(k)}(s), z - z'\rangle| \lesssim_{\|\gamma\|_{C^2(I)}} |z - z'|.$$

The remaining proof proceeds exactly as in [20].

(2) [20, Proposition 2.1] was formulated for $Z_\delta \subset B_E(0,1)$ instead of $Z_\delta \subset [-r,r]^3$, but this change only influences the cinematic constant "K" of the family $\mathcal{F}$ in the proof of [20, Proposition 2.1], and hence the constant δ_0.

(3) [20, Proposition 2.1] was formulated with an additional δ-separateness assumption for Z_δ. However, the version we stated above can easily be reduced to this case since (2.16) ensures that every δ-ball in $\mathbb{R}^3$ contains at most $\delta^{-\varepsilon}$ elements of Z_δ. Then a standard coloring argument allows us to write Z_δ as a union of $M \sim \delta^{-\varepsilon}$ sets $Z_{\delta,1}, \ldots, Z_{\delta,M}$ such that each $Z_{\delta,i}$ is δ-separated, see for instance [9, p.101]. Thus [20, Proposition 2.1] can be applied to each set $Z_{\delta,i}$ individually, and the above version follows (with a larger constant C) by triangle inequality.

Remark 2.18. Condition (2.16) holds whenever the set Z_δ satisfies a non-concentration condition in one of the coordinates. Indeed, assume that $Z_\delta \subset [-r,r]^3$ is such that $|a - a'| \geq c_0\delta$ for all distinct $z = (a,b,c)$ and $z' = (a',b',c') \in Z_\delta$ and a universal constant $c_0 > 0$. This implies that $A := \{a \in \mathbb{R} : (a,b,c) \in Z_\delta \text{ for some } b, c \in \mathbb{R}\}$ is $c_0\delta$-separated and, moreover, for each $a \in A$ there is exactly one pair $(b,c) \in \mathbb{R}^2$ for which $(a,b,c) \in Z_\delta$. Now if $B = B_E(z_0, \rho) \subset \mathbb{R}^3$ is an Euclidean ball with $\rho \geq \delta$ and $a_0 \in \mathbb{R}$ denotes the first coordinate of z_0, then clearly

$$\mathrm{card}(Z_\delta \cap B) \leq \mathrm{card}(A \cap [a_0 - \rho, a_0 + \rho]) \lesssim_{c_0, r} \frac{\rho}{\delta}.$$

Corollary 2.19 (Kakeya inequality for parabolas). *Let $r > 0$, $c_0 > 0$ and $\delta_0 > 0$, and let $I \subset \mathbb{R}$ be a compact interval. Denote $\gamma(s) := (s^2/2, s, 1)$. Let $Z_\delta \subseteq [-r,r]^3$ satisfy $|a - a'| \geq c_0\delta$ for distinct $(a,b,c), (a',b',c') \in Z_\delta$ and some $\delta \in (0, \delta_0)$. Set $\Gamma_z = \{(s, \langle\gamma(s), z\rangle) : s \in I\}$, $z \in Z_\delta$. Then for all $\varepsilon > 0$*

$$\int_{[0,1]^2} \left(\sum_{z \in Z_\delta} \chi_{\Gamma_z^\delta} \right)^{3/2} \leq C_\varepsilon(c_0, \delta_0, r, I)\delta \cdot \delta^{-\varepsilon} \mathrm{card}(Z_\delta),$$

where Γ_z^δ is the Euclidean δ-neighborhood of Γ_z, and $C_\varepsilon(c_0, \delta_0, r, I) > 0$ is a constant.

Proof. It follows from the discussion before and after Theorem 2.15, that this theorem is applicable to the specific curve γ in the corollary. Fix $\varepsilon > 0$ arbitrarily. Let $\delta_1(\varepsilon) = \delta_1(\varepsilon, r, I) > 0$ be such that the thesis of Theorem 2.15 holds for all $\delta \in (0, \delta_1(\varepsilon))$.

By Remark 2.18 and the assumptions on Z_δ, we know that there is a constant $C_0 = C_0(r, c_0) > 0$ such that

$$\operatorname{card}(Z_\delta \cap B) \leq C_0 \rho/\delta,$$

for all balls $B \subseteq \mathbb{R}^3$ of radius $\rho \geq \delta$. Let $\delta_\varepsilon \in (0, \delta_1(\varepsilon)]$ be such that $C_0 \leq \delta^{-\varepsilon}$ for all $\delta \in (0, \delta_\varepsilon)$. If $\delta \in (0, \delta_\varepsilon)$, then Z_δ satisfies all the assumptions of Theorem 2.15, yielding the desired inequality. If $\delta \in [\delta_\varepsilon, \delta_0)$, then $\operatorname{card}(Z_\delta) \lesssim \max\{r/(c_0\delta), 1\} \lesssim_{\varepsilon,c_0,r,I} 1$. Hence

$$\int_{[0,1]^2} \left(\sum_{z \in Z_\delta} \chi_{\Gamma_z^\delta} \right)^{3/2} \leq \operatorname{card}(Z_\delta)^{1/2+1} \lesssim_{\varepsilon,c_0,r,I} \operatorname{card}(Z_\delta) \lesssim_{\varepsilon,c_0,r,I} \delta^{1-\varepsilon} \operatorname{card}(Z_\delta). \qquad \square$$

3. Linear Kakeya inequality

It is well known that $L^p \to L^p$-bounds for the Euclidean Kakeya maximal operators are equivalent to estimates on the $L^{p'}$-interaction of Euclidean δ-separated δ-tubes, where $p' = p/(p-1)$. We establish a similar result (Proposition 3.7) for the Heisenberg Kakeya maximal operator M_δ, defined in (1.1), and then prove Theorems 1.4 and 1.2. The main novelty related to the Heisenberg group lies in the proof of Theorem 1.4. The proofs of Propositions 3.5 and 3.7 follow almost verbatim their Euclidean analogs as presented in [18]. The most obvious difference is the fact that we consider δ^2-separated (Heisenberg) δ-tubes, rather than δ-separated δ-tubes. This mismatch of exponents originates in the following lemma, see also Remark 3.18.

Lemma 3.1. *There exists a constant $c_1 \in (0,1)$ such that*

$$M_\delta f(e) \lesssim M_{2\delta} f(e'), \tag{3.2}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $\delta \in (0,1)$ and $e, e' \in S^1$ with $|e - e'| \leq c_1 \delta^2$.

Proof. Throughout the proof, we denote by "e" both an element $e \in S^1$ and its embedding $(e, 0) \in \mathbb{R}^2 \times \{0\}$ into $\mathbb{H}^1$. It is well known that there exists a constant $C > 1$ such that

$$d_{\mathbb{H}}(e, e') \leq C\sqrt{|e - e'|}, \quad e, e' \in S^1. \tag{3.3}$$

This can be seen explicitly by considering arbitrary points $e = (a, b), e' = (a', b') \in S^1$ and noting that $|ab' - ba'| = |(a - a')b' - (b - b')a'| = |\langle e - e', (b', -a')\rangle| \leq |e - e'|$ and so

$$d_{\mathbb{H}}(e, e')^4 = |e - e'|^4 + 16|\tfrac{1}{2}[ab' - ba']|^2 \lesssim |e - e'|^2.$$

We set $c_1 := 1/C^2$ for a fixed $C > 1$ as in (3.3), and now show that we can cover a Heisenberg δ-tube with another Heisenberg 2δ-tube whenever their directions are $c_1\delta^2$-close. That is, we claim that for each $e, e' \in S^1$, $\delta > 0$ and $y \in \mathbb{H}^1$ it holds

$$|e - e'| \leq c_1 \delta^2 \quad \Rightarrow \quad T^\delta(y, e) \subseteq T^{2\delta}(y, e'). \tag{3.4}$$

Let $e, e' \in S^1$ be such that $|e - e'| \leq c_1\delta^2$ and note that $d_{\mathbb{H}}(e, e') \leq \delta$. Let $y \in \mathbb{H}^1, T = T_\delta(y, e)$ and set $T' := T_{2\delta}(y, e')$. For $x \in T$ there is $s_x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$ such that $d_{\mathbb{H}}(x, y \cdot s_x e) \leq \delta$. Since $d_{\mathbb{H}}(y \cdot s_x e, y \cdot s_x e') = |s_x| d_{\mathbb{H}}(e, e') \leq \delta/2$, we have

$$\operatorname{dist}_{\mathbb{H}}(x, y \cdot I_{e'}) \leq d_{\mathbb{H}}(x, y \cdot s_x e) + d_{\mathbb{H}}(y \cdot s_x e, y \cdot s_x e') \leq \delta + \delta/2$$

and so $T \subseteq T'$. We can finally prove (3.2). If T and T' are as before and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, then

$$\frac{1}{|T|} \int_T |f| \leq \frac{1}{|T|} \int_{T'} |f| \lesssim M_{2\delta} f(e'),$$

and taking the supremum over $y \in \mathbb{H}^1$ yields the claim. $\square$

We next consider the operator norm of the Heisenberg Kakeya maximal operator, where $\|\cdot\|_{L^p(S^1)}$ is computed with respect to the standard length measure σ on S^1, and $\|\cdot\|_{L^p(\mathbb{H}^1)} = \|\cdot\|_{L^p(\mathbb{R}^3)}$ with the Lebesgue measure.

Proposition 3.5. *Let $M > 0$, $1 < p < \infty$, $p' = p/(p-1)$ and $\delta \in (0,1)$. Suppose that for all $t_1, \dots, t_m > 0$ with $\sum_{j=1}^m t_j^{p'} \delta^2 \le 1$ and Heisenberg δ-tubes $T_1, \dots, T_m$ in δ^2-separated directions it holds*

$$\|\sum_{j=1}^m t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \le M.$$

Then $\|M_{\delta/2}\|_{p \to p} \lesssim_p \delta^{-1} M$.

Proof. Let $c_1 \in (0,1)$ be as in Lemma 3.1. Let $\{e_1, \dots, e_m\} \subset S^1$ be a maximal $c_1(\delta/2)^2$-separated set. Let $f \in L^p(\mathbb{R}^3)$ and note that $M_{\delta/2} f(e) \lesssim M_\delta f(e_j)$ for each $e \in B_E(e_j, c_1(\delta/2)^2)$. We thus have

$$\|M_{\delta/2} f\|_{L^p(S^1)}^p = \int_{S^1} M_{\delta/2} f(e)^p \, d\sigma(e) \le \sum_{j=1}^m \int_{B_E(e_j, c_1(\delta/2)^2)} M_{\delta/2} f(e)^p \, d\sigma(e) \lesssim_p \sum_{j=1}^m \delta^2 M_\delta f(e_j)^p.$$

By a dual characterisation of the ℓ^p-norm there are $b_1, \dots, b_m \ge 0$ such that $\sum_{j=1}^m b_j^{p'} = 1$ and $\sum_{j=1}^m M_\delta f(e_j)^p = \left(\sum_{j=1}^m b_j M_\delta f(e_j)\right)^p$; thus

$$\|M_{\delta/2} f\|_{L^p(S^1)} \lesssim_p \delta^{2/p} \sum_{j=1}^m b_j M_\delta f(e_j). \tag{3.6}$$

Let $T_1, \dots, T_m$ be Heisenberg δ-tubes having directions respectively $e_1, \dots, e_m$, satisfying $⨍_{T_j} |f| \ge (1/2) M_\delta f(e_j)$. Then (3.6) gives

$$\begin{aligned}\|M_{\delta/2} f\|_{L^p(S^1)} &\lesssim_p \delta^{2/p} \delta^{-3} \sum_{j=1}^m b_j \int_{T_j} |f| = \delta^{-1} \delta^{-2/p'} \int \sum_{j=1}^m b_j \chi_{T_j} |f| \\ &\le \delta^{-1} \|\sum_{j=1}^m \delta^{-2/p'} b_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)}.\end{aligned}$$

Note that $T_1, \dots, T_m$ are Heisenberg δ-tubes, but have directions which are only $c_1(\delta/2)^2$-separated, while we need them to be δ^2-separated. (Recall that $c_1 \in (0,1)$.) However, one can show that there are $J_1, \dots, J_k \subseteq \{1, \dots, m\}$ with $k \lesssim \delta^2 / \big(c_1(\delta/2)^2\big) \lesssim 1$, such that $\{e_j : j \in J_i\}$ is δ^2-separated for each i, and $\cup_{i=1}^k J_i = \{1, \dots, m\}$. Since $\sum_{j \in J_i} (\delta^{-2/p'} b_j)^{p'} \delta^2 \le \sum_{j=1}^m (\delta^{-2/p'} b_j)^{p'} \delta^2 = 1$, we have by assumption that

$$\begin{aligned}\|M_{\delta/2} f\|_{L^p(S^1)} &\lesssim_p \delta^{-1} \sum_{i=1}^k \|\sum_{j \in J_i} \delta^{-2/p'} b_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \|f\|_{L^p(\mathbb{R}^3)} \\ &\le \delta^{-1} M k \|f\|_{L^p(\mathbb{R}^3)} \lesssim \delta^{-1} M \|f\|_{L^p(\mathbb{R}^3)}.\end{aligned}$$

□

Proposition 3.7. *Let $M \ge 1$, $\beta \in \mathbb{R}$, $1 < p < \infty$, and $p' = p/(p-1)$. Then the following are equivalent:*

- *for all $\varepsilon > 0$, $\delta \in (0,1)$ and every family $\mathcal{T}$ of Heisenberg δ-tubes having δ^2-separated direction it holds*

$$\|\sum_{T \in \mathcal{T}} \chi_T\|_{L^{p'}(\mathbb{R}^3)} \lesssim_{p,\beta,\varepsilon} M \delta^{\beta - \varepsilon} (\delta^2 \mathrm{card}(\mathcal{T}))^{1/p'}; \tag{3.8}$$

- *for all* $\varepsilon > 0$ *and* $\delta \in (0,1)$ *it holds*

$$\|M_\delta\|_{p\to p} \lesssim_{p,\beta,\varepsilon} M\delta^{\beta-1-\varepsilon}. \tag{3.9}$$

Remark 3.10. Note that for β too large or too small inequality (3.8) becomes respectively false or trivial. For example, if $\beta > 1/p'$, then (3.8) is false, as can be seen taking $\sim \delta^{-2}$ disjoint Heisenberg δ-tubes; while it follows from the triangle inequality if $\beta \le 3/p' - 2$. The same can be said about (3.9); see the proof of Theorem 1.2 for more details.

Proof. We start by showing that (3.8) implies (3.9). Note that $\|M_\delta\|_{p\to p} \lesssim_p 1 \lesssim_\beta M\delta^{\beta-1-\varepsilon}$ for all $\delta \in [1/2, 1)$ and $\varepsilon > 0$, so it is enough to show that (3.9) holds for $\delta \in (0, 1/2)$ and $\varepsilon > 0$. From Proposition 3.5 it then suffices to prove that for all $\delta \in (0,1)$, $t_1, \dots, t_m > 0$ with $\sum_{j=1}^m t_j^{p'} \delta^2 \le 1$ and δ^2-separated Heisenberg δ-tubes $T_1, \dots, T_m$, it holds

$$\|\sum_{j=1}^m t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \lesssim_{p,\beta,\varepsilon} M\delta^{\beta-\varepsilon}. \tag{3.11}$$

Let $t_1, \dots, t_m$ and $T_1, \dots, T_m$ be as above. Suppose we have proven (3.11) under the additional assumption $t_j \ge \delta^{3/p+\beta-1}$ for all j. Then

$$\begin{aligned}\|\sum_{j=1}^m t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} &\le \|\sum_{j: t_j < \delta^{3/p+\beta-1}} t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} + \|\sum_{j: t_j \ge \delta^{3/p+\beta-1}} t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \\ &\lesssim_{p,\beta,\varepsilon} \delta^{3/p+\beta-1} m \delta^{3/p'} + M\delta^{\beta-\varepsilon} \lesssim \delta^{3/p-3+3/p'}\delta^\beta + M\delta^{\beta-\varepsilon} \lesssim M\delta^{\beta-\varepsilon},\end{aligned}$$

where we have used $m \lesssim \delta^{-2}$ (δ^2-separated directions) and $M\delta^{-\varepsilon} \ge 1$ for all $\delta \in (0,1)$. We can thus assume w.l.o.g. $\delta^{3/p+\beta-1} \le t_j \le \delta^{-2/p'}$. (If $3/p + \beta - 1 < -2/p'$, then this condition is vacuous and (3.11) follows from the triangle inequality, as we have just seen.) For each $k \in \mathbb{Z}$, let $J_k := \{j : 2^{k-1} < t_j \le 2^k\}$ and note that $\operatorname{card}(\{k : J_k \ne \varnothing\}) \lesssim_{p,\beta} \log(1/\delta) + 1$. Then from (3.8) we get

$$\|\sum_{j=1}^m t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \le \sum_{k\in\mathbb{Z}} 2^k \|\sum_{j\in J_k} \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \lesssim_{p,\beta,\varepsilon} M\delta^{\beta-\varepsilon/2}\delta^{2/p'} \sum_{k\in\mathbb{Z}} 2^k (\operatorname{card}(J_k))^{1/p'}.$$

Since $2^{kp'}\operatorname{card}(J_k) \lesssim_p \sum_{j=1}^m t_j^{p'} \le \delta^{-2}$, it follows that

$$\sum_{k\in\mathbb{Z}} 2^k (\operatorname{card}(J_k))^{1/p'} \lesssim_p \delta^{-2/p'} \operatorname{card}(\{k : J_k \ne \varnothing\}) \lesssim_{p,\beta} \delta^{-2/p'}(\log(1/\delta) + 1).$$

Finally, we have

$$\|\sum_{j=1}^m t_j \chi_{T_j}\|_{L^{p'}(\mathbb{R}^3)} \lesssim_{p,\beta,\varepsilon} M\delta^{\beta-\varepsilon/2}\delta^{2/p'}\delta^{-2/p'}(\log(1/\delta)+1) \lesssim_\varepsilon M\delta^{\beta-\varepsilon},$$

which concludes the proof of the first part of the statement. We now show that (3.9) implies (3.8). We are assuming that (3.9) holds for all $\delta \in (0,1)$, but adjusting the implicit constant we see that it actually holds also for $\delta \in (0,2)$. (We made a similar remark at the beginning of this proof.) Let $\delta \in (0,1)$ and $\mathcal{T}$ be a family of Heisenberg δ-tubes having δ^2-separated directions and let $e_T \in S^1$ denote the direction of T, for each $T \in \mathcal{T}$. Let $g \in L^p(\mathbb{R}^3)$, $g \ge 0$, with $\|g\|_{L^p(\mathbb{R}^3)} \le 1$. Set $c_2 := \min\{c_1, 1/2\}$, where $c_1 \in (0,1)$ is as in

Lemma 3.1. Then

$$\int g \sum_{T\in\mathcal{T}} \chi_T \sim \sum_{T\in\mathcal{T}} \delta^3 \fint_T g \leq \sum_{T\in\mathcal{T}} \delta^3 M_\delta g(e_T) \lesssim \delta^3 \sum_{T\in\mathcal{T}} \fint_{B_E(e_T,c_2\delta^2)} M_{2\delta} g \, d\sigma$$
$$\lesssim \delta \left[\sigma\left(\bigcup_{T\in\mathcal{T}} B_E(e_T,c_2\delta^2)\right)\right]^{1/p'} \|M_{2\delta} g\|_{L^p(S^1)} \lesssim_{p,\beta,\varepsilon} M\delta^{\beta-\varepsilon}(\delta^2 \mathrm{card}(\mathcal{T}))^{1/p'}.$$

Taking the supremum over the g as above, we obtain (3.8). □

We are now ready to prove Theorem 1.4, which we restate.

Theorem 1.4 (Kakeya inequality for Heisenberg tubes). *Let $\delta \in (0,1)$ and assume that $\mathcal{T}$ is a family of Heisenberg δ-tubes pointing in δ^2-separated directions of S^1. Then, for all $\varepsilon > 0$,*

$$\int \left(\sum_{T\in\mathcal{T}} \chi_T\right)^{3/2} \lesssim_\varepsilon \delta^{3-\varepsilon}\mathrm{card}(\mathcal{T}). \tag{1.5}$$

Proof of Theorem 1.4. Let $S_j := \{(\cos\varphi, \sin\varphi) : |\varphi - j\pi/2| \leq \pi/4\}$ for $j \in \{0,1,2,3\}$, and let $\mathcal{T}_j \subseteq \mathcal{T}$ be the set of Heisenberg tubes in $\mathcal{T}$ with direction in S_j. It is enough to show that (1.5) holds in the case $\mathcal{T} = \mathcal{T}_1$ or, equivalently, for $\mathcal{T} = \mathcal{T}_j$ for some j. (Recall that $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \mapsto (Ox', x_3)$ is a $(d_{\mathbb{H}}, d_{\mathbb{H}})$-isometry for each $O \in SO(2)$.) Indeed, we then have

$$\|\sum_{T\in\mathcal{T}} \chi_T\|_{L^{3/2}(\mathbb{R}^3)} \leq \sum_{j=0}^{3} \|\sum_{T\in\mathcal{T}_j} \chi_T\|_{L^{3/2}(\mathbb{R}^3)} \lesssim_\varepsilon \sum_{j=0}^{3} (\delta^{3-\varepsilon}\mathrm{card}(\mathcal{T}_j))^{2/3} \lesssim (\delta^{3-\varepsilon}\mathrm{card}(\mathcal{T}))^{2/3}.$$

We can now assume $\mathcal{T} = \mathcal{T}_1$. Let $\mathcal{B}$ be a collection of Heisenberg balls of radius $1/2$ covering $\mathbb{R}^3$ and such that the balls with same centers and four times larger radius have absolutely bounded overlaps:

$$\sup_{x\in\mathbb{H}^1} \mathrm{card}\left(\{B \in \mathcal{B} : \, x \in 4B\}\right) \lesssim 1.$$

If (1.5) holds with the left-hand side having integration domain an arbitrary $B \in \mathcal{B}$, we then have

$$\int \left(\sum_{T\in\mathcal{T}} \chi_T\right)^{3/2} \sim \sum_{B\in\mathcal{B}} \int_B \left(\sum_{T\in\mathcal{T}: T\cap B\neq\varnothing} \chi_T\right)^{3/2} \lesssim_\varepsilon \sum_{B\in\mathcal{B}} \delta^{3-\varepsilon}\mathrm{card}\left(\{T \in \mathcal{T} : T \cap B \neq \varnothing\}\right)$$
$$= \delta^{3-\varepsilon} \sum_{T\in\mathcal{T}} \mathrm{card}\left(\{B \in \mathcal{B} : T \cap B \neq \varnothing\}\right) \lesssim \delta^{3-\varepsilon}\mathrm{card}(\mathcal{T}),$$

as $\mathrm{card}\left(\{B \in \mathcal{B} : T \cap B \neq \varnothing\}\right)$ is bounded by an absolute constant since $T_\delta(y,e) \cap B \neq \varnothing$ implies that $y \in 4B$.

We can then focus on a single ball $B \in \mathcal{B}$ and assume that $T \cap B \neq \varnothing$ for all $T \in \mathcal{T}$. Moreover, by left translation, we can assume B to be centred at $\left(0, \frac{1}{2}, \frac{1}{2}\right)$; this ensures that $\pi_{\mathbb{W}}(B) \subseteq [0,1]^2$, where we have identified $\mathbb{W} = \{0\} \times \mathbb{R}^2$ with $\mathbb{R}^2$.

By Lemma 2.8 we know that for each $T \in \mathcal{T}$ there is a Euclidean $\sim \delta^2$-neighborhood P_T of a parabola arc such that $\pi_{\mathbb{W}}(T) \subseteq P_T$ and so $\chi_T(w \cdot l) \leq \chi_{P_T}(w)$ for all $w \in \mathbb{W}$ and

$l \in \mathbb{L} = \mathbb{R} \times \{(0,0)\}$. Also, the inclusion $\pi_{\mathbb{W}}(B) \subseteq [0,1]^2$ gives $\chi_B(w \cdot l) \le \chi_{[0,1]^2}(w)$. Thus

$$\begin{aligned} \int_B \Big(\sum_{T\in\mathcal{T}} \chi_T\Big)^{3/2} &\overset{(2.9)}{=} \int_{\mathbb{W}} \int_{\mathbb{L}} \Big(\sum_{T\in\mathcal{T}} \chi_T(w \cdot l)\Big)^{1/2+1} \chi_B(w\cdot l)\, dl\, dw \\ &\le \int_{[0,1]^2} \Big(\sum_{T\in\mathcal{T}} \chi_{P_T}(w)\Big)^{1/2} \sum_{T\in\mathcal{T}} \mathcal{H}^1\big(T \cap \pi_{\mathbb{W}}^{-1}(\{w\})\big)\, dw \\ &\overset{\text{Lem.2.10}}{\lesssim} \delta \int_{[0,1]^2} \Big(\sum_{T\in\mathcal{T}} \chi_{P_T}(w)\Big)^{3/2} dw, \end{aligned} \tag{3.12}$$

where in the last line we have also used the fact that $T \cap \pi_{\mathbb{W}}^{-1}(\{w\}) = \varnothing$ if $w \notin P_T$ and so $\mathcal{H}^1\big(T \cap \pi_{\mathbb{W}}^{-1}(\{w\})\big) \lesssim \delta \chi_{P_T}(w)$.

To conclude the proof, we show that we can apply Corollary 2.19 to the parabola neighborhoods $\{P_T : T \in \mathcal{T}\}$. To do so, we employ Lemma 2.8 and the formulae of (2.6).

For each $T \in \mathcal{T}$ there are a compact interval I and $(a,b,c) \in \mathbb{R}^3$ (both depending on T) such that P_T is a neighborhood of the parabola arc $\gamma_{(a,b,c)}(I)$. Since $\mathcal{T} = \mathcal{T}_1$ we know that $a \in [-1,1]$ and so there is an absolute constant $C_0 > 0$ such that P_T is contained in the $C_0\delta^2$-neighborhood of its "core curve" $\gamma_{(a,b,c)}(I)$. (Recall that P_T is the $\sim (1+|a|)\delta^2 \sim \delta^2$-neighborhood of $\gamma_{(a,b,c)}(I)$.) Also, $T \cap B \ne \varnothing$ and $\delta \in (0,1)$ imply that there is an absolute constant $r \ge 1$ such that $b, c \in [-r, r]$ and $I \subseteq [-r, r]$.

Let $Z \subseteq [-r,r]^3$ be the set $(a,b,c) \in \mathbb{R}^3$ corresponding to $\mathcal{T}$. Note that $a \in [-1,1] \mapsto \frac{(a,1)}{|(a,1)|}$ is Lipschitz continuous; since $\mathcal{T}$ is a family of tubes in δ^2-separated directions, it follows that there is an absolute constant $c_0 > 0$ such that $|a - a'| \ge c_0(C_0\delta^2)$ for all distinct $(a,b,c), (a',b',c') \in Z$. Thus, Corollary 2.19 applies to Z with $C_0\delta^2$ in place of δ. Taking $\delta_0 = C_0$ in Corollary 2.19, its conclusion holds for $C_0\delta^2 \in (0, C_0)$. Hence, for $\delta \in (0,1)$ and each $\varepsilon > 0$,

$$\int_{[0,1]^2} \Big(\sum_{T\in\mathcal{T}} \chi_{P_T}\Big)^{3/2} \le \int_{[0,1]^2} \Big(\sum_{z\in Z} \chi_{\Gamma_z^{C_0\delta^2}}\Big)^{3/2} \le C_\varepsilon \delta^{2-\varepsilon}\, \mathrm{card}\,(Z), \tag{3.13}$$

where $\Gamma_z^{C_0\delta^2}$ is the $C_0\delta^2$-neighborhood the parabola arc $\Gamma_z = \gamma_z([-r,r])$. Since $\mathrm{card}(Z) = \mathrm{card}(\mathcal{T})$, (3.13) and (3.12) conclude the proof. □

We can finally prove Theorem 1.2, which yields the Heisenberg Kakeya maximal function inequality.

Theorem 1.2. *For all $p \in [1,\infty]$, $\varepsilon > 0$, and $\delta \in (0,1)$*

$$\|M_\delta f\|_{L^p(S^1)} \lesssim_{p,\varepsilon} \delta^{-\alpha(p)-\varepsilon}\, \|f\|_{L^p(\mathbb{H}^1)}, \qquad f \in L^p(\mathbb{H}^1),$$

where $\alpha(p) = \max\{4/p - 1, 1/p\}$. Moreover, for $p \in [1,\infty]$, $\alpha(p)$ is the smallest exponent for which the inequality holds.

Proof of Theorem 1.2. Theorem 1.4 shows that (3.8) holds with $p' = 3/2$, $M = 1$ and $\beta = 2/3$. Thus, Proposition 3.7 immediately gives

$$\|M_\delta\|_{3\to 3} \lesssim_\varepsilon \delta^{2/3-1-\varepsilon} = \delta^{-1/3-\varepsilon} \tag{3.14}$$

for all $\varepsilon > 0$ and $\delta \in (0,1)$.

For $p \in [1,\infty]$, set

$$\begin{aligned}\alpha(p) &:= \inf\{\alpha \in \mathbb{R} : \|M_\delta\|_{p\to p} \lesssim \delta^{-\alpha} \text{ holds for } \delta \in (0,1)\}\\ &= \min\{\alpha \in \mathbb{R} : \|M_\delta\|_{p\to p} \lesssim_\varepsilon \delta^{-\alpha-\varepsilon} \text{ holds for } \varepsilon > 0 \text{ and } \delta \in (0,1)\}.\end{aligned}$$

We now explain how (3.14) can be used to obtain

$$\alpha(p) = \max\{4/p - 1, 1/p\}, \quad p \in [1,\infty]. \tag{3.15}$$

On the one hand, simple examples show that $\alpha(p) \geq \max\{4/p-1, 1/p\}$. Consider first $p \in (1,\infty)$. Indeed, inequality $\alpha \geq 1/p$ can be derived considering a 3-dimensional Heisenberg Kakeya set (see Remark 4.11), or any disjoint collection of δ^2-separated Heisenberg δ-tubes (via Proposition 3.7), while $\alpha \geq 4/p - 1$ follows from $M_\delta(\chi_{B_{\mathbb{H}}(0,\delta)}) \gtrsim \delta$ and $|B_{\mathbb{H}}(0,\delta)| \sim \delta^4$, or by considering a maximal collection of δ^2-separated Heisenberg δ-tubes centred at the origin. For $p \in \{1,\infty\}$, observe that the trivial estimates $\|M_\delta\|_{1\to 1} \sim \delta^{-3}$ (consider a δ ball for the lower bound) and $\|M_\delta\|_{\infty\to\infty} = 1$ give $\alpha(1) = 3$ and $\alpha(\infty) = 0$.

On the other hand, interpolating the sharp $L^1 \to L^1$ and $L^\infty \to L^\infty$ bounds with (3.14) (real method, see e.g. [18, Theorem 2.13] or [8, Corollary 1.4.22]), yields

$$\|M_\delta\|_{p\to p} \lesssim_\varepsilon \delta^{-\max\{4/p-1,1/p\}-\varepsilon}, \quad p \in [1,\infty],$$

and thus $\alpha(p) \leq \max\{4/p-1, 1/p\}$, proving (3.15). By (3.15) the above inequality is sharp for the entire range $p \in [1,\infty]$, except possibly for the ε-loss. □

Remark 3.16. We do not know if the term $C(p,\varepsilon)\delta^{-\varepsilon}$ in Theorem 1.2 is necessary. If it can be removed in the $L^3 \to L^3$ bound (e.g. by removing it in Theorem 1.4), then Theorem 1.2 holds without it.

Remark 3.17. Analogously, for $p' \in [1,\infty]$, $\varepsilon > 0$, $\delta \in (0,1)$, and any δ^2-separated family of Heisenberg δ-tubes $\mathcal{T}$, we have

$$\|\sum_{T\in\mathcal{T}} \chi_T\|_{L^{p'}} \lesssim_{p',\varepsilon} \delta^{-\alpha'(p')-\varepsilon}(\delta^3 \mathrm{card}(\mathcal{T}))^{1/p'},$$

where $\alpha'(p') = \max\{2-3/p', 0\}$ and again it is the smallest exponent for which the above inequality holds, for every $p' \in [1,\infty]$. For $p' \in (1,\infty)$, this follows from Theorem 1.2 and Proposition 3.7, while it is trivial when $p' \in \{1,\infty\}$.

Remark 3.18. As mentioned in the introduction to this section, a peculiarity of these results is that they involve δ^2-separated Heisenberg δ-tubes, rather than δ-separated ones. There are several reasons for this:

(1) Lemma 3.1 is based on the implication (3.4), which does not hold if e, e' are only $\sim \delta$-close. To see this, consider $e = \frac{1}{\sqrt{1+\delta^2}}(1,\delta)$ and $e' = (1,0)$. Then $|e - e'| \sim \delta$ and $x = (\frac{1}{2\sqrt{1+\delta^2}}, \frac{\delta}{2\sqrt{1+\delta^2}}, 0) \in T_\delta(0,e)$, yet $x \notin T_{2\delta}(0,e')$ if δ is small enough, since

$$d_{\mathbb{H}}(x,(se,0)) \gtrsim \left|\frac{1}{2\sqrt{1+\delta^2}} - s\right| + \sqrt{\frac{\delta|s|}{2\sqrt{1+\delta^2}}} \gg \delta$$

for all sufficiently small $\delta > 0$ (uniformly in $s \in [-1/2, 1/2]$).

(2) Heisenberg δ-tubes (roughly speaking) project to $\sim \delta^2$-neighborhoods of parabolas (Lemma 2.8) while the δ-separation is preserved. In order to apply Corollary 2.19, we need the separation to be at least of the same order as the radius of the neighborhoods.

(3) Theorem 1.4 applies to the sparser δ-separated δ-tubes. However, the usual argument would not yield the sharp lower bound on the Minkowski dimension if we only considered δ-separated δ-tubes in Theorem 1.4. See Remark 4.2.

4. Conclusion about Hausdorff dimension

We recall the following definition from [17].

Definition 4.1. We say that $E \subseteq \mathbb{H}^1$ is a *Heisenberg Kakeya set* if for every $e \in S^1$ there is a $y \in \mathbb{H}^1$ such that $y \cdot I_e \subseteq E$, where $I_e = \{(se, 0) : s \in [-1/2, 1/2]\}$ is a horizontal unit line segment.

Since the projection $\pi : (x_1, x_2, x_3) \mapsto (x_1, x_2)$ maps a Heisenberg Kakeya set E onto a Kakeya set in $\mathbb{R}^2$, it follows from the validity of the Kakeya conjecture in $\mathbb{R}^2$ that E must have Euclidean Hausdorff dimension at least 2. However, since π can increase Hausdorff dimension in the metric $d_{\mathbb{H}}$, the dimension of E with respect to this metric has to be studied separately. Liu showed in [17] that Heisenberg Kakeya sets in $\mathbb{H}^1$ have Hausdorff dimension at least 3 with respect to $d_{\mathbb{H}}$. Theorem 1.2 for $p = 3$ (or for any $p \in [3, \infty)$) provides an alternative proof of this result (Proposition 4.4). For illustration, we first explain how the linear Heisenberg Kakeya inequality can be used to prove the sharp bound for the lower *Minkowski (box-counting) dimension* of Heisenberg Kakeya sets in $\mathbb{H}^1$ (Remark 4.2). This is a weaker statement since the lower Minkowski dimension of a set is always greater than or equal to its Hausdorff dimension. Nevertheless, it provides a good heuristic for the relation between the numerology of Theorem 1.4 and the dimension of Heisenberg Kakeya sets (see also Remark 3.18).

Remark 4.2. Let E^δ be the δ-enlargement w.r.t. $d_{\mathbb{H}}$ of a Heisenberg Kakeya set E. We show that $|E^\delta| \gtrsim_\varepsilon \delta^{1+\varepsilon} = \delta^{4-3+\varepsilon}$ for all $\varepsilon > 0$ and $\delta \in (0,1)$; this implies by a standard argument that E has lower Minkowski dimension (w.r.t. $d_{\mathbb{H}}$) at least 3. Fix a δ^2-separated set of directions in S^1, and for any one of them consider the unit segment contained in E having such direction. The Heisenberg δ-neighborhood of any such segment is a Heisenberg δ-tube which is contained in E^δ. Let $\mathcal{T}$ denote the collection of such tubes; it is a δ^2-separated family of Heisenberg δ-tubes. Hence

$$\Big\| \sum_{T\in\mathcal{T}} \chi_T \Big\|_{L^1(\mathbb{R}^3)} \leq \Big(\Big| \bigcup_{T\in\mathcal{T}} T \Big| \Big)^{\frac{1}{3}} \Big\| \sum_{T\in\mathcal{T}} \chi_T \Big\|_{L^{3/2}(\mathbb{R}^3)} \overset{\text{Thm.1.4}}{\lesssim_\varepsilon} \delta^{-\varepsilon} \Big(\Big| \bigcup_{T\in\mathcal{T}} T \Big| \Big)^{\frac{1}{3}} \Big\| \sum_{T\in\mathcal{T}} \chi_T \Big\|_{L^1(\mathbb{R}^3)}^{\frac{2}{3}},$$

which finally implies

$$|E^\delta| \geq \Big| \bigcup_{T\in\mathcal{T}} T \Big| \gtrsim_\varepsilon \delta^\varepsilon \Big\| \sum_{T\in\mathcal{T}} \chi_T \Big\|_{L^1(\mathbb{R}^3)} \sim \delta^{3+\varepsilon}\, \mathrm{card}\,(\mathcal{T}) \sim \delta^{1+\varepsilon}. \tag{4.3}$$

We stress that had we proven Theorem 1.4 only for δ-separated δ-tubes, we would have the weaker $|E^\delta| \gtrsim_\varepsilon \delta^{2+\varepsilon} = \delta^{4-2+\varepsilon}$ in (4.3). This would only give the non-sharp lower bound $\underline{\dim}_M E \geq 2$.

We reprove Liu's result as a corollary of Theorem 1.2 for $p = 3$:

Proposition 4.4. *If $E \subseteq \mathbb{H}^1$ is a Kakeya set of horizontal unit line segments, then* $\dim_H E \geq 3$.

Proof. With (1.3) at hand, this follows by a standard argument (found e.g. in [26] or [18]). Let $E \subseteq \mathbb{H}^1$ be a Heisenberg Kakeya set and fix $0 < \alpha < 3$. Let $B_j = B(x_j, r_j)$ be an arbitrary cover of E by Korányi balls of radius $r_j \leq 1$. It suffices to show that $\sum_j r_j^\alpha \gtrsim 1$.

For each $e \in S^1$, we let $\ell_e = x_e \cdot I_e$ denote a line segment contained in E. As usual, we group the covering sets into families of balls of comparable size by defining

$$J_k := \{j : 2^{-k} \leq r_j < 2^{1-k}\}, \quad k \in \mathbb{N}^+,$$

and the set of directions in which a line segment is well covered by balls of comparable radius,

$$S_k := \left\{ e : \mathcal{H}^1(\ell_e \cap \bigcup_{j \in J_k} B_j) \geq \frac{1}{2k^2} \right\}.$$

The sets S_k, $k \in \mathbb{N}^+$, cover all relevant directions. If there was $e \in S^1 \setminus \bigcup_{k \in \mathbb{N}^+} S_k$, then

$$1 = \mathcal{H}^1(E \cap \ell_e) \leq \sum_{k=1}^{\infty} \frac{1}{2k^2} < 1,$$

which is impossible.

We will apply the Heisenberg Kakeya maximal operator to the function

$$f = \chi_{F_k}, \quad \text{where } F_k = \bigcup_{j \in J_k} B(x_j, 2r_j).$$

However, we firstly need to show that for $e \in S_k$

$$|T_{2^{-k}}(x_e, e) \cap F_k| \gtrsim \frac{1}{k^2} |T_{2^{-k}}(x_e, e)|. \tag{4.5}$$

Fix $e \in S_k$ and set $I_k(e) := \bigcup_{j \in J_k} B_j \cap \ell_e$. Note that for each $z \in I_k(e)$ there is a $j \in J_k$ such that $B(z, 2^{-k}) \subseteq B(x_j, 2r_j)$. Thus,

$$\bigcup_{z \in I_k(e)} B(z, 2^{-k}) \subseteq T_{2^{-k}}(x_e, e) \cap \bigcup_{j \in J_k} B(x_j, 2r_j) = T_{2^{-k}}(x_e, e) \cap F_k. \tag{4.6}$$

We now estimate $|T_{2^{-k}}(x_e, e) \cap F_k|$ in terms of $\mathcal{H}^1(I_k(e))$. Let $P \subseteq I_k(e)$ be a maximal $2^{-(k-1)}$-separated set in $I_k(e)$ (with respect to $d_{\mathbb{H}}$), then $\{B(y, 2^{-k}) : y \in P\}$ is a pairwise disjoint family of sets and so

$$|\bigcup_{z \in I_k(e)} B(z, 2^{-k})| \geq |\bigcup_{y \in P} B(y, 2^{-k})| \sim (2^{-k})^4 \text{card}(P). \tag{4.7}$$

On the other hand, $\{B(y, 2^{-(k-1)}) \cap (x_e \cdot \text{span}\,(e)) : y \in P\}$ covers $I_k(e)$ and so

$$2^{-(k-1)} \text{card}(P) \gtrsim \mathcal{H}^1(I_k(e)). \tag{4.8}$$

Combining the inclusion (4.6) and the inequalities (4.7), (4.8), we have

$$|T_{2^{-k}}(x_e, e) \cap F_k| \gtrsim (2^{-k})^4 \text{card}(P) \gtrsim (2^{-k})^3 \mathcal{H}^1(I_k(e)) \gtrsim \frac{1}{k^2} |T_{2^{-k}}(x_e, e)|,$$

as desired. This then implies that $M_{2^{-k}} f \gtrsim k^{-2}$ on S_k and so

$$\|M_{2^{-k}} f\|_{L^3} \gtrsim k^{-2} \sigma(S_k)^{1/3}. \tag{4.9}$$

On the other hand, Theorem 1.2 (for $p = 3$ and suitably chosen $\varepsilon > 0$ to be determined) implies that

$$\|M_{2^{-k}} f\|_{L^3} \leq C_\varepsilon \, 2^{k\varepsilon} \, 2^{k/3} \|f\|_{L^3(\mathbb{R}^3)} \lesssim_\varepsilon 2^{k\varepsilon} \, 2^{k/3} \left(\text{card}(J_k) \, 2^{(1-k)4} \right)^{1/3} \lesssim_\varepsilon 2^{k(\varepsilon-1)} \text{card}(J_k)^{1/3}. \tag{4.10}$$

Combining (4.9) and (4.10) yields

$$\sigma(S_k) \lesssim k^6\, 2^{3k(\varepsilon-1)}\, \mathrm{card}(J_k) \lesssim_\varepsilon 2^{3k(2\varepsilon-1)}\, \mathrm{card}(J_k).$$

This shows that

$$\sum_j r_j^{3-6\varepsilon} \gtrsim \sum_{k=1}^{\infty} \mathrm{card}(J_k) 2^{3k(2\varepsilon-1)} \gtrsim_\varepsilon \sum_{k=1}^{\infty} \sigma(S_k) \gtrsim 1.$$

Hence, if $3 - 6\varepsilon > \alpha$, we get the desired lower bound $\sum_j r_j^\alpha \gtrsim 1$. □

Remark 4.11. Slight changes to the proof of Proposition 4.4 show that $\|M_\delta\|_{p\to p} \lesssim_\varepsilon \delta^{-\alpha-\varepsilon}$ for all $\delta \in (0,1)$ and $\varepsilon > 0$ implies $\dim_H E \geq 4 - \alpha p$ for all Heisenberg Kakeya sets E. Since $\dim_H E \geq 3$ is sharp, it must be $\alpha \geq 1/p$, limiting the possible $L^p \to L^p$ bounds on M_δ.

Remark 4.12. With the same exponent of δ, Theorem 1.4 with the exponent $3/2$ replaced by any (finite) exponent strictly larger than 1 would still yield the sharp lower bound of Proposition 4.4 (and, of course, of Remark 4.2).

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