EFFECTIVE QUASISTATIC EVOLUTION MODELS FOR PERFECTLY PLASTIC PLATES WITH PERIODIC MICROSTRUCTURE

Abstract. An effective model is identified for thin perfectly plastic plates whose microstructure consists of the periodic assembling of two elastoplastic phases, as the periodicity parameter converges to zero. Assuming that the thickness of the plates and the periodicity of the microstructure are comparably small, a limiting description is obtained in the quasistatic regime via simultaneous homogenization and dimension reduction by means of evolutionary Γ-convergence, two-scale convergence, and periodic unfolding.

1. Introduction

With this paper, we begin the task of identifying reduced models for thin composite elastoplastic plates with periodic microstructure. We focus here on the case in which the thickness h of the plates and their microstructure width ε_h are asymptotically comparable, namely, we assume the existence of the limit

$$\lim_{h\to 0} \frac{h}{\varepsilon_h} =: \gamma \in (0, +\infty).$$

This corresponds, roughly speaking, to the situation in which homogenization and dimension reduction occur somewhat simultaneously and a strong interaction between vanishing thickness and periodicity comes into play. Different scalings of γ (i.e., $\gamma = 0$ and $\gamma = +\infty$) will be the subject of a forthcoming companion paper.

Finding lower dimensional models for thin three-dimensional structures is a classical task in the Mathematics of Continuum Mechanics. A rigorous identification of a reduced model for perfectly plastic plates in the quasistatic regime has been undertaken in [13]. An additional regularity result for the associated stress has been established in [19]. The case of dynamic perfect plasticity is the subject of [37, 27], whereas the setting of shallow shells has been tackled in [36]. A parallel analysis in the presence of hardening has been performed in [34, 35] We further mention the two works [14, 15] in the purview of finite plasticity.

The study of composite elastoplastic materials is a challenging endeavour. In the small strain regime, limit plasticity equations have been identified in [41, 31, 30] both in the periodic and in the aperiodic and stochastic settings. The Fleck and Willis model is the subject of [25, 26], whereas gradient plasticity has been studied in [29]. For completeness, we also mention [9, 10, 16, 18] for an analysis of large-strain stratified composites in crystal plasticity and [17] for a static result in the finitely plastic setting. The characterization of inhomogeneous perfectly plastic materials and a subsequent periodic homogenization have been undertaken in [24, 23].

The novelty of the present contribution consists in the fact that we combine both dimension reduction and periodic homogenization in order to deduce a limiting description, as the two smallness scales (thickness and width of the microstructure) converge to zero, for perfectly plastic thin plates.

To complete our literature overview, we briefly recall the main mathematical contributions on simultaneous homogenization and dimension reduction. In [6], the author derives a limiting plate model starting from 3d linearized elasticity, while assuming the material to be isotropic and the microstructure to be periodic. In [12], the case of linear elastic plates with possible aperiodic microstructure is tackled by relying on material (planar) symmetries of the elasticity tensor, and by introducing the notion of H-convergence adapted to dimension reduction. In [4] an effective plate model is identified in the general

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case (without further periodicity or material-symmetries assumptions) by means of Γ-convergence (the analysis presented there also covers some non-linear models). We also mention the book [40] where linear rod and plate models are obtained by simultaneous homogenization and dimension reduction, and appropriate estimates are also provided, as well as the recent work [5] on high-contrast elastic plates. Different non-linear elastic plate models obtained by Γ-convergence are discussed in [8, 39, 32, 3, 44].

To the Authors knowledge, this manuscript represents instead the first work on effective theories for plates undergoing inelastic deformations.

We conclude this introduction by briefly presenting our results. First, after establishing a general disintegration result for measures in the image of suitable first-order differential operators, cf. Proposition 4.2, and relying on an auxiliary result related to De Rham cohomology, cf. Proposition 4.11 , in Theorem 4.14, we identify two-scale limits of rescaled strains. We point out that the intermediate results in Proposition 4.2 are of independent interest and apply to a more general setting than that investigated in this contribution. We have chosen to pursue this avenue for those tools will be instrumental also for the analysis of further regimes of plastic thin-plates homogenization. We emphasize that for identifying two-scale limits of rescaled strains we could not rely on the results obtained in the context of elasticity (see, e.g. [4]), since these results relied on Korn inequalities which are not available in the plastic setting, hence a new approach needed to be developed.

For a given boundary datum w, the limiting model that we identify is finite on triples $(u, E, P) \in \mathcal{A}^{hom}_{\gamma}(w)$, where the latter denotes the set of limits of plastic triples given by displacements, elastic, and plastic strains in the sense of two-scale convergence for measures, cf. Definition 3.9. We refer to Definition 5.7 and to Subsection 5.2 for the precise definition and main disintegration properties of the class $\mathcal{A}^{hom}_{\gamma}(w)$. On such triples, the effective elastic energy and dissipation potential are homogenized densities depending only on the limiting two-scale elastic and plastic strain, respectively. Our analysis stems from adapting the approach of [23] to the setting of dimension reduction problems for composite plates. This is, however, a non-trivial task: a first hurdle consists in the already mentioned compactness result for rescaled strains, see Section 4.3. Further difficulties originate from the fact that the limit problem is of fourth order, see Section 5. Further, analogously to [13], the limiting description is truly three-dimensional. We refer to [19, Section 5] for a discussion of this issue and an example. Our effective model is completely characterized in Subsection 5.5. After introducing a suitable notion of stress-strain duality, in Theorem 5.15 we prove a two-scale limiting Hill's principle. The lower semicontinuity of the effective energy and dissipation functionals is proven in Theorem 5.17 Key tools are an adaptation of unfolding techniques for dimension reduction (see Proposition 4.17), as well as a technical rank-one decomposition characterization (see Lemma 4.18). Finally, with Theorem 6.2 we prove the main result of this contribution, showing via evolutionary Γ-convergence, cf. [38] the convergence of three-dimensional inhomogeneous quasistatic evolutions to energetic solutions for our two-scale reduced model.

The paper is organized as follows. Section 2 contains some preliminary results on two-scale convergence, disintegration of Radon measures, BD and BH functions, as well as some auxiliary claims about stress tensors. In Section 3 we specify the setting of the problem and the main assumptions. We additionally recall the existence results for quasistatic evolution for general multi-phase materials. The characterization of limiting triples in the sense of two-scale convergence for Radon measures is the focus of Section 4. The effective stress-strain duality is analyzed in Section 5, whereas the convergence of quasistatic evolutions is proven in Section 6.

2. Preliminaries

In this section we specify our notation and collect a few preliminary results.

2.1. **Notation.**

We will write any point $x \in \mathbb{R}^3$ as a pair (x', x_3), with $x' \in \mathbb{R}^2$ and $x_3 \in \mathbb{R}$, and we will use the notation $\nabla_{x'}$ to denote the gradient with respect to x'. We denote by $y \in \mathcal{Y}$ the points on a flat 2-dimensional torus. We denote by I the open interval $I := \left(-\frac{1}{2}, \frac{1}{2}\right)$. In what follows we will also adopt the following notation for scaled gradients and symmetrized scaled gradients:

$$\begin{aligned}\nabla_h v &:= \left[\ \nabla_{x'} v \ \middle|\ \tfrac{1}{h}\,\partial_{x_3} v\ \right], \quad E_h v := \operatorname{sym} \nabla_h v, \\ \widetilde{\nabla}_\gamma v &:= \left[\ \nabla_y v \ \middle|\ \tfrac{1}{\gamma}\partial_{x_3} v\ \right], \quad \widetilde{E}_\gamma v := \operatorname{sym} \widetilde{\nabla}_\gamma v,\end{aligned} \tag{2.1}$$

where $h, \gamma > 0$ and v is a function on the appropriate domain. The scaled divergence operators div_h and $\widetilde{\operatorname{div}}_\gamma$ are defined in the following way:

$$\operatorname{div}_h v := \partial_{x_1} v_1 + \partial_{x_2} v_2 + \frac{1}{h}\partial_{x_3} v_3, \quad \widetilde{\operatorname{div}}_\gamma v := \partial_{y_1} v_1 + \partial_{y_2} v_2 + \frac{1}{\gamma}\partial_{x_3} v_3.$$

Analogously, we define the operators curl and $\widetilde{\operatorname{curl}}_\gamma$, for functions taking values in $\mathbb{R}^3$. Note that the operators $\widetilde{\nabla}_\gamma$, $\widetilde{\operatorname{div}}_\gamma$, $\widetilde{\operatorname{curl}}_\gamma$ act on functions that have as (part of) their domain $I \times \mathcal{Y}$ (with a slight abuse of notation we write this domain with I on the first place, despite the fact that the associated differential operators are defined as above).

If $a, b \in \mathbb{R}^N$, we write $a \cdot b$ for the Euclidean scalar product, and we denote by $|a| := \sqrt{a \cdot a}$ the Euclidean norm. We write $\mathbb{M}^{N\times N}$ for the set of real $N \times N$ matrices. If $A, B \in \mathbb{M}^{N\times N}$, we use the Frobenius scalar product $A : B := \sum_{i,j} A_{ij}\, B_{ij}$ and the associated norm $|A| := \sqrt{A : A}$. We denote by $\mathbb{M}^{N\times N}_{\text{sym}}$ the space of real symmetric $N \times N$ matrices, and by $\mathbb{M}^{N\times N}_{\text{dev}}$ the set of real deviatoric matrices, respectively, i.e. the subset of $\mathbb{M}^{N\times N}_{\text{sym}}$ given by matrices having null trace. For every matrix $A \in \mathbb{M}^{N\times N}$ we denote its trace by $\operatorname{tr}A$, and its deviatoric part by A_{dev} will be given by

$$A_{\text{dev}} = A - \frac{1}{N}\operatorname{tr}A.$$

The *symmetrized tensor product* $a \odot b$ of two vector $a, b \in \mathbb{R}^N$ is the symmetric matrix with entries $(a \odot b)_{ij} := \frac{a_i b_j + a_j b_i}{2}$. Note that $\operatorname{tr}(a \odot b) = a \cdot b$, and that $|a \odot b|^2 = \frac{1}{2}|a|^2|b|^2 + \frac{1}{2}(a \cdot b)^2$, so that

$$\frac{1}{\sqrt{2}}|a||b| \le |a \odot b| \le |a||b|.$$

Given a vector $v \in \mathbb{R}^3$, we will use the notation v' to denote the vector

$$v' := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Analogously, given a matrix $A \in \mathbb{M}^{3\times 3}$, we will denote by A'' the minor

$$A'' := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The Lebesgue measure in $\mathbb{R}^N$ and the $(N-1)$-dimensional Hausdorff measure are denoted by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively. For $U \subset \mathbb{R}^N$, $\overline{U}$ denotes its closure. Given an open subset $U \subset \mathbb{R}^N$ and a finite dimensional Euclidean space E, we use standard notations for Lebesgue spaces $L^p(U; E)$ and Sobolev spaces $H^1(U; E)$ or $W^{1,p}(U; E)$. The characteristic function of U will be given by $\mathbb{1}_U$.

We will write $C^k(U; E)$ for the space of k-times continuously differentiable functions $\varphi : U \to E$ and $C^\infty(U; E) := \bigcap_{k=0}^\infty C^k(U; E)$ for the space of infinitely differentiable function. We will distinguish between the spaces $C^k_c(U; E)$ (C^k functions with compact support contained in U) and $C^k_0(U; E)$ (C^k functions "vanishing on ∂U"). We will write $C(\mathcal{Y}; E)$ to denote the space of all continuous functions which are $[0,1]^2$-periodic, and set $C^k(\mathcal{Y}; E) := C^k(\mathbb{R}^2; E) \cap C(\mathcal{Y}; E)$. We will identify $C^k(\mathcal{Y}; E)$ with the space of all C^k functions on the 2-dimensional torus.

We will frequently make use of the *standard mollfier* $\rho \in C^\infty(\mathbb{R}^N)$, defined by

$$\rho(x) := \begin{cases} C\, \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^N} \rho(x)\, dx = 1$, and the associated family $\{\rho_\epsilon\}_{\epsilon>0} \subset C^\infty(\mathbb{R}^N)$ with

$$\rho_\epsilon(x) := \frac{1}{\epsilon^N} \rho\left(\frac{x}{\epsilon}\right).$$

Throughout the text, the letter C stands for generic constants which may vary from line to line.

2.2. **Measures.** We first recall some basic notions from measure theory that we will use throughout the paper (see, e.g. [22]).

Given a Borel set $U \subset \mathbb{R}^N$ and a finite dimensional Hilbert space X, we denote by $\mathcal{M}_b(U; X)$ the space of bounded Borel measures on U taking values in X, and endowed with the norm $\|\mu\|_{\mathcal{M}_b(U;X)} := |\mu|(U)$, where $|\mu| \in \mathcal{M}_b(U; \mathbb{R})$ is the total variation of the measure μ. For every $\mu \in \mathcal{M}_b(U; X)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ and μ^s is singular with respect to $\mathcal{L}^N$. If $\mu^s = 0$, we always identify μ with its density with respect to $\mathcal{L}^N$, which is a function in $L^1(U; X)$. With a slight abuse of notation, we will write $\mathcal{M}_b(U; \mathbb{R}) = \mathcal{M}_b(U)$ and $\mathcal{M}_b(U; \mathbb{R}^+) = \mathcal{M}_b^+(U)$.

If the relative topology of U is locally compact, by Riesz representation theorem the space $\mathcal{M}_b(U; X)$ can be identified with the dual of $C_0(U; X)$, which is the space of all continuous functions $\varphi : U \to X$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(U; X)$ is defined using this duality.

The *restriction* of $\mu \in \mathcal{M}_b(U; X)$ to a subset $E \in U$ is the measure $\mu\lfloor E \in \mathcal{M}_b(E; X)$ defined by

$$\mu\lfloor E(B) := \mu(E \cap B), \qquad \text{for every Borel set } B \subset U.$$

Given two real-valued measures μ_1, $\mu_2 \in \mathcal{M}_b(U)$ we write $\mu_1 \geq \mu_2$ if $\mu_1(B) \geq \mu_2(B)$ for every Borel set $B \subset U$.

2.2.1. *Convex functions of measures.* Let U be an open set of $\mathbb{R}^N$. For every $\mu \in \mathcal{M}_b(U; X)$ let $\frac{d\mu}{d|\mu|}$ be the Radon-Nikodym derivative of μ with respect to its variation $|\mu|$. Let $H : X \to [0, +\infty)$ be a convex and positively one-homogeneous function such that

$$r|\xi| \leq H(\xi) \leq R|\xi| \quad \text{for every } \xi \in X, \tag{2.2}$$

where r and R are two constants, with $0 < r \leq R$.

Using the theory of convex functions of measures, developed in [28] and [21], we introduce the non-negative Radon measure $H(\mu) \in \mathcal{M}_b^+(U)$ defined by

$$H(\mu)(A) := \int_A H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|,$$

for every Borel set $A \subset U$. We also consider the functional $\mathcal{H} : \mathcal{M}_b(U; X) \to [0, +\infty)$ defined by

$$\mathcal{H}(\mu) := H(\mu)(U) = \int_U H\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

One can prove that $\mathcal{H}$ is lower semicontinuous on $\mathcal{M}_b(U; X)$ with respect to weak* convergence (see, e.g., [1, Theorem 2.38]).

Let a, $b \in [0, T]$ with $a \leq b$. The *total variation* of a function $\mu : [0, T] \to \mathcal{M}_b(U; X)$ on $[a, b]$ is defined by

$$\mathcal{V}(\mu; a, b) := \sup\left\{\sum_{i=1}^{n-1} \|\mu(t_{i+1}) - \mu(t_i)\|_{\mathcal{M}_b(U;X)} : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

Analogously, we define the *$\mathcal{H}$-variation* of a function $\mu : [0, T] \to \mathcal{M}_b(U; X)$ on $[a, b]$ as

$$\mathcal{D}_{\mathcal{H}}(\mu; a, b) := \sup\left\{\sum_{i=1}^{n-1} \mathcal{H}\left(\mu(t_{i+1}) - \mu(t_i)\right) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

From (2.2) it follows that

$$r\mathcal{V}(\mu; a, b) \leq \mathcal{D}_{\mathcal{H}}(\mu; a, b) \leq R\mathcal{V}(\mu; a, b). \tag{2.3}$$

2.2.2. *Disintegration of a measure.* Let S and T be measurable spaces and let μ be a measure on S. Given a measurable function $f : S \to T$, we denote by $f_\# \mu$ the *push-forward* of μ under the map f, defined by

$$f_\#\mu(B) := \mu\left(f^{-1}(B)\right), \quad \text{for every measurable set } B \subseteq T.$$

In particular, for any measurable function $g : T \to \overline{\mathbb{R}}$ we have

$$\int_S g \circ f \, d\mu = \int_T g \, d(f_\#\mu).$$

Note that in the previous formula $S = f^{-1}(T)$.

Let $S_1 \subset \mathbb{R}^{N_1}$, $S_2 \subset \mathbb{R}^{N_2}$, for some $N_1, N_2 \in \mathbb{N}$, be open sets, and let $\eta \in \mathcal{M}_b^+(S_1)$. We say that a function $x_1 \in S_1 \to \mu_{x_1} \in \mathcal{M}_b(S_2;\mathbb{R}^M)$ is η-measurable if $x_1 \in S_1 \to \mu_{x_1}(B)$ is η-measurable for every Borel set $B \subseteq S_2$.

Given a η-measurable function $x_1 \to \mu_{x_1}$ such that $\int_{S_1} |\mu_{x_1}| \, d\eta < +\infty$, then the *generalized product* $\eta \overset{\text{gen.}}{\otimes} \mu_{x_1}$ satisfies $\eta \overset{\text{gen.}}{\otimes} \mu_{x_1} \in \mathcal{M}_b(S_1 \times S_2; \mathbb{R}^M)$ and is such that

$$\langle \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}, \varphi \rangle := \int_{S_1} \left(\int_{S_2} \varphi(x_1, x_2) \, d\mu_{x_1}(x_2) \right) d\eta(x_1),$$

for every bounded Borel function $\varphi : S_1 \times S_2 \to \mathbb{R}$.

Moreover, the following disintegration result holds (c.f. [1, Theorem 2.28 and Corollary 2.29]):

Theorem 2.1. *Let $\mu \in \mathcal{M}_b(S_1 \times S_2; \mathbb{R}^M)$ and let proj : $S_1 \times S_2 \to S_1$ be the projection on the first factor. Denote by η the push-forward measure $\eta := proj_\# |\mu| \in \mathcal{M}_b^+(S_1)$. Then there exists a unique family of bounded Radon measures $\{\mu_{x_1}\}_{x_1 \in S_1} \subset \mathcal{M}_b(S_2; \mathbb{R}^M)$ such that $x_1 \to \mu_{x_1}$ is η-measurable, and*

$$\mu = \eta \overset{gen.}{\otimes} \mu_{x_1}.$$

For every $\varphi \in L^1(S_1 \times S_2, d|\mu|)$ we have

$$\varphi(x_1, \cdot) \in L^1(S_2, d|\mu_{x_1}|) \quad \textit{for } \eta\textit{-a.e. } x_1 \in S_1,$$

$$x_1 \to \int_{S_2} \varphi(x_1, x_2) \, d\mu_{x_1}(x_2) \ \in L^1(S_1, d\eta),$$

$$\int_{S_1 \times S_2} \varphi(x_1, x_2) \, d\mu(x_1, x_2) = \int_{S_1} \left(\int_{S_2} \varphi(x_1, x_2) \, d\mu_{x_1}(x_2) \right) d\eta(x_1).$$

Furthermore,

$$|\mu| = \eta \overset{gen.}{\otimes} |\mu_{x_1}|.$$

Arguing as in [23, Remark 5.5], we have the following:

Proposition 2.2. *With the same notation as in Theorem 2.1, for η-a.e. $x_1 \in S_1$*

$$\frac{d\mu}{d|\mu|}(x_1, \cdot) = \frac{d\mu_{x_1}}{d|\mu_{x_1}|} \quad |\mu_{x_1}|\textit{-a.e. on } S_2.$$

Proof. Since $\frac{d\mu}{d|\mu|} \in L^1(S_1 \times S_2, d|\mu|)$, from Theorem 2.1 we have $\frac{d\mu}{d|\mu|}(x_1, \cdot) \in L^1(S_2, d|\mu_{x_1}|)$ for η-a.e. $x_1 \in S_1$. Thus,

$$\eta \overset{\text{gen.}}{\otimes} \frac{d\mu_{x_1}}{d|\mu_{x_1}|} |\mu_{x_1}| = \eta \overset{\text{gen.}}{\otimes} \mu_{x_1} = \mu = \frac{d\mu}{d|\mu|} |\mu| = \eta \overset{\text{gen.}}{\otimes} \frac{d\mu}{d|\mu|}(x_1, \cdot) \, |\mu_{x_1}|,$$

from which we have the claim. □

2.3. BD and BH functions.

2.3.1. *Functions with bounded deformation.* Let U be an open set of $\mathbb{R}^N$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^N)$ whose symmetric gradient $Eu := \operatorname{sym} Du$ (in the sense of distributions) satisfies $Eu \in \mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})$. We point out that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1(U;\mathbb{R}^N)} + \|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})}.$$

We say that a sequence $\{u^k\}_k$ converges to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^N)$ and $Eu^k \rightharpoonup Eu$ weakly* in $\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})$. As a consequence of compactness, then necessarily $\{u^k\}_k$ converges to u strongly in L^1. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, $BD(U)$ can be embedded into $L^{N/(N-1)}(U;\mathbb{R}^N)$ (the embedding is compact in L^p, for $1 \le p < N/(N-1)$) and every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^N)$. If Γ is a nonempty open subset of ∂U, there exists a constant $C > 0$, depending on U and Γ, such that

$$\|u\|_{L^1(U;\mathbb{R}^N)} \le C\|u\|_{L^1(\Gamma)} + C\|Eu\|_{\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})}. \tag{2.4}$$

(see [42, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(U)$ we refer to [42].

2.3.2. *Functions with bounded Hessian.* The space $BH(U)$ of functions with *bounded Hessian* is the space of all functions $u \in W^{1,1}(U)$ whose Hessian D^2u (in the sense of distributions) belongs to $\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{\rm sym})$. It is a Banach space endowed with the norm

$$\|u\|_{L^1(U)} + \|\nabla u\|_{L^1(U;\mathbb{R}^N)} + \|D^2u\|_{\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})}.$$

If U has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $\mathcal{M}_b(U;\mathbb{M}^{N\times N}_{sym})$. If U is bounded and has a Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,N/(N-1)}(U)$. If U is bounded and has a C^2 boundary, then for every function $u \in BH(U)$ one can define the traces of u and of ∇u, still denoted by u and ∇u; they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U;\mathbb{R}^N)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$ in $L^1(\partial U)$, where τ is any tangent vector to ∂U. If, in addition, $N = 2$, then $BH(U)$ embeds into $C(\overline{U})$, which is the space of all continuous functions on $\overline{U}$. The general properties of the space $BH(U)$ can be found in [20].

2.4. **Auxiliary claims about stress tensors.**

2.4.1. *Traces of stresses.* We suppose here that U is an open bounded set of class C^2 in $\mathbb{R}^N$. If $\sigma \in L^2(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\operatorname{div}\sigma \in L^2(U;\mathbb{R}^N)$, then we can define a distribution $[\sigma\nu]$ on ∂U by

$$[\sigma\nu](\psi) := \int_U \psi \cdot \operatorname{div}\sigma \, dx + \int_U \sigma : E\psi \, dx, \tag{2.5}$$

for every $\psi \in H^1(U;\mathbb{R}^N)$. It follows that $[\sigma\nu] \in H^{-1/2}(\partial U;\mathbb{R}^N)$ (see, e.g., [43, Chapter 1, Theorem 1.2]). If, in addition, $\sigma \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\operatorname{div}\sigma \in L^N(U;\mathbb{R}^N)$, then (2.5) holds for $\psi \in W^{1,1}(U;\mathbb{R}^N)$. By Gagliardo's extension theorem, in this case we have $[\sigma\nu] \in L^\infty(\partial U;\mathbb{R}^N)$, and

$$[\sigma_k\nu] \xrightharpoonup{*} [\sigma\nu] \quad \text{weakly* in } L^\infty(\partial U;\mathbb{R}^N),$$

whenever $\sigma_k \xrightharpoonup{*} \sigma$ weakly* in $L^\infty(U;\mathbb{M}^{N\times N}_{\rm sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^N(U;\mathbb{R}^N)$.

We will consider the normal and tangential parts of $[\sigma\nu]$, defined by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu, \quad [\sigma\nu]_\nu^\perp := [\sigma\nu] - ([\sigma\nu]\cdot\nu)\nu.$$

Since $\nu \in C^1(\partial U;\mathbb{R}^N)$, we have that $[\sigma\nu]_\nu$, $[\sigma\nu]_\nu^\perp \in H^{-1/2}(\partial U;\mathbb{R}^N)$. If, in addition, $\sigma_{\rm dev} \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})$, then it was proved in [33, Lemma 2.4] that $[\sigma\nu]_\nu^\perp \in L^\infty(\partial U;\mathbb{R}^N)$ and

$$\|[\sigma\nu]_\nu^\perp\|_{L^\infty(\partial U;\mathbb{R}^N)} \le \frac{1}{\sqrt{2}}\|\sigma_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})}.$$

More generally, if U has Lipschitz boundary and is such that there exists a compact set $S \subset \partial U$ with $\mathcal{H}^{N-1}(S) = 0$ such that $\partial U \setminus S$ is a C^2-hypersurface, then arguing as in [24, Section 1.2] we can uniquely determine $[\sigma\nu]_\nu^\perp$ as an element of $L^\infty(\partial U;\mathbb{R}^N)$ through any approximating sequence $\{\sigma_n\} \subset C^\infty(\overline{U};\mathbb{M}^{N\times N}_{\rm sym})$ such that

$$\begin{aligned}
&\sigma_n \to \sigma \quad \text{strongly in } L^2(U;\mathbb{M}^{N\times N}_{\rm sym}),\\
&\mathrm{div}\sigma_n \to \mathrm{div}\sigma \quad \text{strongly in } L^2(U;\mathbb{R}^N),\\
&\|(\sigma_n)_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})} \leq \|\sigma_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})}.
\end{aligned}$$

2.4.2. *L^p regularity.* We recall the following proposition from [24] (see also [33]).

Proposition 2.3. *Let $U \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary. The set*

$$\mathcal{S}(U) := \left\{\sigma \in L^2(U;\mathbb{M}^{N\times N}_{\rm sym}) : \mathrm{div}\,\sigma \in L^N(U;\mathbb{R}^N),\ \sigma_{\rm dev} \in L^\infty(U;\mathbb{M}^{N\times N}_{\rm dev})\right\},$$

is a subset of $L^p(U;\mathbb{M}^{N\times N}_{\rm sym})$ for every $1 \leq p < \infty$, and

$$\|\sigma\|_{L^p(U;\mathbb{M}^{N\times N}_{\rm sym})} \leq C_p\left(\|\sigma\|_{L^2(U;\mathbb{M}^{N\times N}_{\rm sym})} + \|\mathrm{div}\,\sigma\|_{L^N(U;\mathbb{R}^N)} + \|\sigma_{\rm dev}\|_{L^\infty(U;\mathbb{M}^{N\times N}_{dev})}\right).$$

3. Setting of the problem

We describe here our modeling assumptions and recall a few associated instrumental results. Unless otherwise stated, $\omega \subset \mathbb{R}^2$ is a bounded, connected, and open set with C^2 boundary. Given a small positive number $h > 0$, we assume that the set

$$\Omega^h := \omega \times (hI),$$

is the reference configuration of a linearly elastic and perfectly plastic plate.

We consider a non-zero Dirichlet boundary condition on the whole lateral surface, i.e. the Dirichlet boundary of Ω^h is given by $\Gamma_D^h := \partial\omega \times (hI)$.

We work under the assumption that the body is only submitted to a hard device on Γ_D^h and that there are no applied loads, i.e. the evolution is only driven by time-dependent boundary conditions. More general boundary conditions, together with volume and surfaces forces have been considered, e.g., in [11, 24, 13] but will, for simplicity of exposition, be neglected in this analysis.

3.1. Phase decomposition.

We recall here some basic notation and assumptions from [23].

Recall that $\mathcal{Y} = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-dimensional torus, let $Y := [0,1)^2$ be its associated periodicity cell, and denote by $\mathcal{I} : \mathcal{Y} \to Y$ their canonical identification. We denote by $\mathcal{C}$ the set

$$\mathcal{C} := \mathcal{I}^{-1}(\partial Y).$$

For any $\mathcal{Z} \subset \mathcal{Y}$, we define

$$\mathcal{Z}_\varepsilon := \left\{x \in \mathbb{R}^2 : \frac{x}{\varepsilon} \in \mathbb{Z}^2 + \mathcal{I}(\mathcal{Z})\right\}, \tag{3.1}$$

and to every function $F : \mathcal{Y} \to X$ we associate the ε-periodic function $F_\varepsilon : \mathbb{R}^2 \to X$, given by

$$F_\varepsilon(x) := F(y_\varepsilon), \quad \text{for} \quad \frac{x}{\varepsilon} - \left\lfloor \frac{x}{\varepsilon} \right\rfloor = \mathcal{I}(y_\varepsilon) \in Y.$$

With a slight abuse of notation we will also write $F_\varepsilon(x) = F\left(\frac{x}{\varepsilon}\right)$.

The torus $\mathcal{Y}$ is assumed to be made up of finitely many phases $\mathcal{Y}_i$ together with their interfaces. We assume that those phases are pairwise disjoint open sets with Lipschitz boundary. Then we have $\mathcal{Y} = \bigcup_i \overline{\mathcal{Y}}_i$ and we denote the interfaces by

$$\Gamma := \bigcup_{i,j} \partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j.$$

Furthermore, the interfaces are assumed to have a negligible intersection with the set $\mathcal{C}$, i.e. for every i

$$\mathcal{H}^1(\partial\mathcal{Y}_i \cap \mathcal{C}) = 0. \tag{3.2}$$

We will write

$$\Gamma := \bigcup_{i \neq j} \Gamma_{ij},$$

where Γ_{ij} stands for the interface between $\mathcal{Y}_i$ and $\mathcal{Y}_j$.

We assume that ω is composed of the finitely many phases $(\mathcal{Y}_i)_\varepsilon$, and that $\Omega^h \cup \Gamma_D^h$ is a geometrically admissible multi-phase domain in the sense of [24, Subsection 1.2]. Additionally, we assume that Ω^h is a specimen of an elasto-perfectly plastic material having periodic elasticity tensor and dissipation potential.

We are interested in the situation when the period ε is a function of the thickness h, i.e. $\varepsilon = \varepsilon_h$, and we assume that the limit

$$\gamma := \lim_{h \to 0} \frac{h}{\varepsilon_h}.$$

exists in $(0, +\infty)$. We additionally require that Γ satisfies the following: there exists a compact set $S \subset \Gamma$ with $\mathcal{H}^1(S) = 0$ such that $\Gamma \setminus S$ is a C^2-hypersurface.

We say that a multi-phase torus $\mathcal{Y}$ is *geometrically admissible* if it satisfies the above assumptions.

Remark 3.1. *We point out that we assume greater regularity than that in [23], where the interface $\Gamma \setminus S$ was allowed to be a C^1-hypersurface. Under such weaker assumptions, in fact, the tangential part of the trace of an admissible stress $[\sigma\nu]_\nu^\perp$ at a point x on $\Gamma \setminus S$ would not be defined independently of the considered approximating sequence. By requiring a higher regularity of $\Gamma \setminus S$, we will avoid dealing with this situation.*

The set of admissible stresses.

We assume there exist convex compact sets $K_i \in \mathbb{M}^{3\times3}_{\mathrm{dev}}$ associated to each phase $\mathcal{Y}_i$. We work under the assumption that there exist two constants r_K and R_K, with $0 < r_K \leq R_K$, such that for every i

$$\{\xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : |\xi| \leq r_K\} \subseteq K_i \subseteq \{\xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}} : |\xi| \leq R_K\}.$$

Finally, we define

$$K(y) := K_i, \qquad \text{for } y \in \mathcal{Y}_i.$$

The elasticity tensor.

For every i, let $(\mathbb{C}_{\mathrm{dev}})_i$ and k_i be a symmetric positive definite tensor on $\mathbb{M}^{3\times3}_{\mathrm{dev}}$ and a positive constant, respectively, such that there exist two constants r_c and R_c, with $0 < r_c \leq R_c$, satisfying

$$r_c|\xi|^2 \leq (\mathbb{C}_{\mathrm{dev}})_i\xi : \xi \leq R_c|\xi|^2 \qquad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\mathrm{dev}}, \tag{3.3}$$

$$r_c \leq k_i \leq R_c. \tag{3.4}$$

Let $\mathbb{C}$ be the *elasticity tensor*, considered as a map from $\mathcal{Y}$ taking values in the set of symmetric positive definite linear operators, $\mathbb{C} : \mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to \mathbb{M}^{3\times3}_{\mathrm{sym}}$, defined as

$$\mathbb{C}(y)\xi := \mathbb{C}_{\mathrm{dev}}(y)\,\xi_{\mathrm{dev}} + (k(y)\,\mathrm{tr}\xi)\ I_{3\times3} \qquad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3},$$

where $\mathbb{C}_{\mathrm{dev}}(y) = (\mathbb{C}_{\mathrm{dev}})_i$ and $k(y) = k_i$ for every $y \in \mathcal{Y}_i$.

Let $Q : \mathcal{Y} \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, and given by

$$Q(y, \xi) := \frac{1}{2}\mathbb{C}(y)\xi : \xi \qquad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}}.$$

It follows that Q satisfies

$$r_c|\xi|^2 \leq Q(y, \xi) \leq R_c|\xi|^2 \quad \text{for every } y \in \mathcal{Y} \text{ and } \xi \in \mathbb{M}^{3\times3}_{\mathrm{sym}}. \tag{3.5}$$

The dissipation potential.

For each i, let $H_i : \mathbb{M}^{3\times3}_{\mathrm{dev}} \to [0, +\infty)$ be the support function of the set K_i, i.e

$$H_i(\xi) = \sup_{\tau \in K_i} \tau : \xi.$$

It follows that H_i is convex, positively 1-homogeneous, and satisfies

$$r_k|\xi| \leq H_i(\xi) \leq R_k|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}_{\text{dev}}. \tag{3.6}$$

Then we define the dissipation potential $H : \mathcal{Y} \times \mathbb{M}^{3\times 3}_{\text{dev}} \to [0, +\infty]$ as follows:

(i) For every $y \in \mathcal{Y}_i$, we take

$$H(y, \xi) := H_i(\xi).$$

(ii) For a point $y \in \Gamma \setminus S$ on the interface between $\mathcal{Y}_i$ and $\mathcal{Y}_j$, such that the associated normal $\nu(y)$ points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$, we set

$$H(y, \xi) := \begin{cases} H_{ij}(a, \nu(y)) & \text{if } \xi = a \odot \nu(y) \in \mathbb{M}^{3\times 3}_{\text{dev}}, \\ +\infty & \text{otherwise on } \mathbb{M}^{3\times 3}_{\text{dev}}, \end{cases}$$

where for $a \in \mathbb{R}^3$ and $\nu \perp a \in \mathbb{S}^2$,

$$H_{ij}(a, \nu) := \inf \Big\{ H_i(a_i \odot \nu) + H_j(-a_j \odot \nu) : \\ a = a_i - a_j,\ a_i \perp \nu,\ a_j \perp \nu \Big\}.$$

(iii) For $y \in S$, we define H arbitrarily (e.g. $H(y, \xi) := r_k\,|\xi|$).

Remark 3.2. *We point out that H is a Borel function on $\mathcal{Y} \times \mathbb{M}^{3\times 3}_{\text{dev}}$. Furthermore, for each $y \in \mathcal{Y}$, the function $\xi \mapsto H(y, \xi)$ is positively 1-homogeneous and convex. However, the function $(y, \xi) \mapsto H(y, \xi)$ is not necessarily lower semicontinous. This creates additional difficulties in proving lower semicontinuity of dissipation functional given in Theorem 5.17, see also [23, Theorem 5.7].*

Admissible triples and energy.

On Γ^h_D we prescribe a boundary datum being the trace of a map $w^h \in H^1(\Omega^h; \mathbb{R}^3)$ of the following form:

$$w^h(z) := \left(\bar{w}_1(z') - \frac{z_3}{h} \partial_1 \bar{w}_3(z'),\ \bar{w}_2(z') - \frac{z_3}{h} \partial_2 \bar{w}_3(z'),\ \frac{1}{h} \bar{w}_3(z') \right) \text{ for a.e. } z = (z', z_3) \in \Omega^h, \tag{3.7}$$

where $\bar{w}_\alpha \in H^1(\omega)$, $\alpha = 1, 2$, and $\bar{w}_3 \in H^2(\omega)$. The *set of admissible displacements and strains* for the boundary datum w^h is denoted by $\mathcal{A}(\Omega^h, w^h)$ and is defined as the class of all triples $(v, f, q) \in BD(\Omega^h) \times L^2(\Omega^h; \mathbb{M}^{3\times 3}_{\text{sym}}) \times \mathcal{M}_b(\Omega^h; \mathbb{M}^{3\times 3}_{\text{dev}})$ satisfying

$$Ev = f + q \quad \text{in } \Omega^h,$$
$$q = (w^h - v) \odot \nu_{\partial\Omega^h} \mathcal{H}^2 \quad \text{on } \Gamma^h_D.$$

The function v represents the *displacement* of the plate, while f and q are called the *elastic* and *plastic strain*, respectively.

For every admissible triple $(v, f, q) \in \mathcal{A}(\Omega^h, w^h)$ we define the associated *energy* as

$$\mathcal{E}_h(v, f, q) := \int_{\Omega^h} Q\left(\frac{z'}{\varepsilon_h}, f(z) \right) dz + \int_{\Omega^h \cup \Gamma^h_D} H\left(\frac{z'}{\varepsilon_h}, \frac{dq}{d|q|} \right) d|q|.$$

The first term represents the elastic energy, while the second term accounts for plastic dissipation.

3.2. **The rescaled problem.** As usual in dimension reduction problems, it is convenient to perform a change of variables in such a way to rewrite the system on a fixed domain independent of h. To this purpose, we consider the open interval $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and set

$$\Omega := \omega \times I, \qquad \Gamma_D := \partial\omega \times I.$$

We consider the change of variables $\psi_h : \overline{\Omega} \to \overline{\Omega^h}$, defined as

$$\psi_h(x', x_3) := (x', hx_3) \quad \text{for every } (x', x_3) \in \overline{\Omega}, \tag{3.8}$$

and the linear operator $\Lambda_h : \mathbb{M}^{3\times3}_{\rm sym} \to \mathbb{M}^{3\times3}_{\rm sym}$ given by

$$\Lambda_h \xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{h}\xi_{13} \\ \xi_{21} & \xi_{22} & \frac{1}{h}\xi_{23} \\ \frac{1}{h}\xi_{31} & \frac{1}{h}\xi_{32} & \frac{1}{h^2}\xi_{33} \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{\rm sym}. \tag{3.9}$$

To any triple $(v, f, q) \in \mathcal{A}(\Omega^h, w^h)$ we associate a triple $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\rm sym})$ defined as follows:

$$u := (v_1, v_2, hv_3) \circ \psi_h, \qquad e := \Lambda_h^{-1} f \circ \psi_h, \qquad p := \tfrac{1}{h}\Lambda_h^{-1} \psi_h^{\#}(q).$$

Here the measure $\psi_h^{\#}(q) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times3})$ is the pull-back measure of q, satisfying

$$\int_{\Omega\cup\Gamma_D} \varphi : d\psi_h^{\#}(q) = \int_{\Omega^h\cup\Gamma_D^h} (\varphi \circ \psi_h^{-1}) : dq \quad \text{for every } \varphi \in C_0(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}).$$

According to this change of variable we have

$$\mathcal{E}_h(v, f, q) = h\mathcal{Q}_h(\Lambda_h e) + h\mathcal{H}_h(\Lambda_h p),$$

where

$$\mathcal{Q}_h(\Lambda_h e) = \int_\Omega Q\left(\frac{x'}{\varepsilon_h}, \Lambda_h e\right) dx \tag{3.10}$$

and

$$\mathcal{H}_h(\Lambda_h p) = \int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p}{d|\Lambda_h p|}\right) d|\Lambda_h p|. \tag{3.11}$$

We also introduce the scaled Dirichlet boundary datum $w \in H^1(\Omega; \mathbb{R}^3)$, given by

$$w(x) := (\bar{w}_1(x') - x_3\partial_1 w_3(x'), \bar{w}_2(x') - x_3\partial_2 w_3(x'), w_3(x')) \quad \text{for a.e. } x \in \Omega.$$

By the definition of the class $\mathcal{A}(\Omega^h, w^h)$ it follows that the scaled triple (u, e, p) satisfies the equalities

$$Eu = e + p \quad \text{in } \Omega, \tag{3.12}$$

$$p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_D, \tag{3.13}$$

$$p_{11} + p_{22} + \tfrac{1}{h^2}p_{33} = 0 \quad \text{in } \Omega \cup \Gamma_D. \tag{3.14}$$

We are thus led to introduce the class $\mathcal{A}_h(w)$ of all triples $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\rm sym})$ satisfying (3.12)–(3.14), and to define the functional

$$\mathcal{J}_h(u, e, p) := \mathcal{Q}_h(\Lambda_h e) + \mathcal{H}_h(\Lambda_h p) \tag{3.15}$$

for every $(u, e, p) \in \mathcal{A}_h(w)$. In the following we will study the asymptotic behaviour of the quasistatic evolution associated with $\mathcal{J}_h$, as $h \to 0$ and $\varepsilon_h \to 0$.

Notice that if $\bar{w}_\alpha \in H^1(\widetilde{\omega})$, $\alpha = 1, 2$, and $\bar{w}_3 \in H^2(\widetilde{\omega})$, where $\omega \subset \widetilde{\omega}$, then we can trivially extend the triple (u, e, p) to $\widetilde{\Omega} := \widetilde{\omega} \times I$ by

$$u = w, \qquad e = Ew, \qquad p = 0 \qquad \text{on } \widetilde{\Omega} \setminus \overline{\Omega}.$$

In the following we will always denote this extension also by (u, e, p), whenever such an extension procedure is needed.

Kirchhoff-Love admissible triples and limit energy.

We consider the set of *Kirchhoff-Love displacements*, defined as

$$KL(\Omega) := \big\{u \in BD(\Omega) : \ (Eu)_{i3} = 0 \quad \text{for } i = 1, 2, 3\big\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha = \bar{u}_\alpha - x_3\partial_{x_\alpha} u_3, \quad \alpha = 1, 2. \tag{3.16}$$

In particular, if $u \in KL(\Omega)$, then

$$Eu = \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & & 0 \\ & & 0 \\ 0 \quad 0 & & 0 \end{pmatrix}. \tag{3.17}$$

If, in addition, $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ for some $1 \le p \le \infty$, then $\bar{u} \in W^{1,p}(\omega;\mathbb{R}^2)$ and $u_3 \in W^{2,p}(\omega)$. We call $\bar{u}, u_3$ the *Kirchhoff-Love components* of u.

For every $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ we define the class $\mathcal{A}_{KL}(w)$ of *Kirchhoff-Love admissible triples* for the boundary datum w as the set of all triples $(u, e, p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{3\times3}_{\rm sym})$ satisfying

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_D, \tag{3.18}$$

$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_D, \quad i = 1, 2, 3. \tag{3.19}$$

Note that the space

$$\left\{\xi \in \mathbb{M}^{3\times3}_{\rm sym} : \ \xi_{i3} = 0 \text{ for } i = 1, 2, 3\right\}$$

is canonically isomorphic to $\mathbb{M}^{2\times2}_{\rm sym}$. Therefore, in the following, given a triple $(u, e, p) \in \mathcal{A}_{KL}(w)$ we will usually identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times2}_{\rm sym})$ and p with a measure in $\mathcal{M}_b(\Omega\cup\Gamma_D;\mathbb{M}^{2\times2}_{\rm sym})$. Note also that the class $\mathcal{A}_{KL}(w)$ is always nonempty as it contains the triple $(w, Ew, 0)$.

To provide a useful characterisation of admissible triplets in $\mathcal{A}_{KL}(w)$, let us first recall the definition of zeroth and first order moments of functions.

Definition 3.3. *For $f \in L^2(\Omega;\mathbb{M}^{2\times2}_{\rm sym})$ we denote by $\bar{f}$, $\hat{f} \in L^2(\omega;\mathbb{M}^{2\times2}_{\rm sym})$ and $f^\perp \in L^2(\Omega;\mathbb{M}^{2\times2}_{\rm sym})$ the following orthogonal components (with respect to the scalar product of $L^2(\Omega;\mathbb{M}^{2\times2}_{\rm sym})$) of f:*

$$\bar{f}(x') := \int_{-\frac12}^{\frac12} f(x', x_3)\,dx_3, \qquad \hat{f}(x') := 12\int_{-\frac12}^{\frac12} x_3 f(x', x_3)\,dx_3$$

for a.e. $x' \in \omega$, and

$$f^\perp(x) := f(x) - \bar{f}(x') - x_3\hat{f}(x')$$

for a.e. $x \in \Omega$. We name $\bar{f}$ the zero-th order moment *of f and $\hat{f}$ the* first order moment *of f.*

The coefficient in the definition of $\hat{f}$ is chosen from the computation $\int_I x_3^2\,dx_3 = \frac{1}{12}$. It ensures that if f is of the form $f(x) = x_3 g(x')$, for some $g \in L^2(\omega;\mathbb{M}^{2\times2}_{\rm sym})$, then $\hat{f} = g$.

Analogously, we have the following definition of zeroth and first order moments of measures.

Definition 3.4. *For $\mu \in M_b(\Omega\cup\Gamma_D;\mathbb{M}^{2\times2}_{\rm sym})$ we define $\bar{\mu}$, $\hat{\mu} \in M_b(\omega\cup\gamma_D;\mathbb{M}^{2\times2}_{\rm sym})$ and $\mu^\perp \in M_b(\Omega\cup\Gamma_D;\mathbb{M}^{2\times2}_{\rm sym})$ as follows:*

$$\int_{\omega\cup\gamma_D} \varphi : d\bar{\mu} := \int_{\Omega\cup\Gamma_D} \varphi : d\mu, \qquad \int_{\omega\cup\gamma_D} \varphi : d\hat{\mu} := 12\int_{\Omega\cup\Gamma_D} x_3\varphi : d\mu$$

for every $\varphi \in C_0(\omega\cup\gamma_D;\mathbb{M}^{2\times2}_{\rm sym})$, and

$$\mu^\perp := \mu - \bar{\mu}\otimes\mathcal{L}^1_{x_3} - \hat{\mu}\otimes x_3\mathcal{L}^1_{x_3},$$

where $\otimes$ is the usual product of measures, and $\mathcal{L}^1_{x_3}$ is the Lebesgue measure restricted to the third component of $\mathbb{R}^3$. We name $\bar{\mu}$ the zero-th order moment *of μ and $\hat{\mu}$ the* first order moment *of μ.*

Remark 3.5. *More generally, for any function f which is integrable over I, we will use the short-hand notation*

$$\bar{f} := \int_I f(\cdot, x_3)\,dx_3, \qquad \hat{f} := 12\int_I x_3\, f(\cdot, x_3)\,dx_3.$$

We are now ready to recall the following characterisation of $\mathcal{A}_{KL}(w)$, given in [13, Proposition 4.3].

Proposition 3.6. *Let $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym}) \times \mathcal{M}_b(\Omega\cup\Gamma_D;\mathbb{M}^{3\times3}_{\rm dev})$. Then $(u, e, p) \in \mathcal{A}_{KL}(w)$ if and only if the following three conditions are satisfied:*

(i) $E\bar{u} = \bar{e} + \bar{p}$ *in* ω *and* $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* γ_D*;*
(ii) $D^2 u_3 = -(\hat{e} + \hat{p})$ *in* ω*,* $u_3 = w_3$ *on* γ_D*, and* $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* γ_D*;*
(iii) $p^\perp = -e^\perp$ *in* Ω *and* $p^\perp = 0$ *on* Γ_D*.*

3.3. **Definition of quasistatic evolutions.** Recalling Section 2.2, the $\mathcal{H}_h$-variation of a map $p^h : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{dev}})$ on $[a,b]$ is defined as

$$\mathcal{D}_{\mathcal{H}_h}(P; a, b) := \sup\left\{\sum_{i=1}^{n-1} \mathcal{H}\left(P(t_{i+1}) - P(t_i)\right) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

For every $t \in [0,T]$ we prescribe a boundary datum $w(t) \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and we assume the map $t \mapsto w(t)$ to be absolutely continuous from $[0,T]$ into $H^1(\Omega;\mathbb{R}^3)$.

Definition 3.7. *Let* $h > 0$*. An* h-quasistatic evolution *for the boundary datum* $w(t)$ *is a function* $t \mapsto (u^h(t), e^h(t), p^h(t))$ *from* $[0,T]$ *into* $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\text{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{dev}})$ *that satisfies the following conditions:*

(qs1)$_h$ *for every* $t \in [0,T]$ *we have* $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(w(t))$ *and*

$$\mathcal{Q}_h(\Lambda_h e^h(t)) \le \mathcal{Q}_h(\Lambda_h \eta) + \mathcal{H}_h(\Lambda_h \pi - \Lambda_h p^h(t)),$$

for every $(\upsilon, \eta, \pi) \in \mathcal{A}_h(w(t))$*.*

(qs2)$_h$ *the function* $t \mapsto p^h(t)$ *from* $[0,T]$ *into* $\mathcal{M}_b(\Omega \cup \Gamma_D; \mathbb{M}^{3\times3}_{\text{dev}})$ *has bounded variation and for every* $t \in [0,T]$

$$\mathcal{Q}_h(\Lambda_h e^h(t)) + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) = \mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right) \Lambda_h e^h(s) : E\dot{w}(s)\, dx ds.$$

The following existence result of a quasistatic evolution for a general multi-phase material can be found in [24, Theorem 2.7].

Theorem 3.8. *Assume* (3.3) *and* (3.5)*. Let* $h > 0$ *and let* $(u^h_0, e^h_0, p^h_0) \in \mathcal{A}_h(w(0))$ *satisfy the global stability condition (qs1)*$_h$*. Then, there exists a two-scale quasistatic evolution* $t \mapsto (u^h(t), e^h(t), p^h(t))$ *for the boundary datum* $w(t)$ *such that* $u^h(0) = u_0$*,* $e^h(0) = e^h_0$*, and* $p^h(0) = p^h_0$*.*

Our goal is to study the asymptotics of the quasistatic evolution when h goes to zero. The main result is given by Theorem 6.2.

3.4. **Two-scale convergence adapted to dimension reduction.** We briefly recall some results and definitions from [23].

Definition 3.9. *Let* $\Omega \subset \mathbb{R}^3$ *be an open set. Let* $\{\mu^h\}_{h>0}$ *be a family in* $\mathcal{M}_b(\Omega)$ *and consider* $\mu \in \mathcal{M}_b(\Omega \times \mathcal{Y})$*. We say that*

$$\mu^h \xrightharpoonup{2-*} \mu \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}),$$

if for every $\chi \in C_0(\Omega \times \mathcal{Y})$

$$\lim_{h\to0} \int_\Omega \chi\left(x, \frac{x'}{\varepsilon_h}\right) d\mu^h(x) = \int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\mu(x,y).$$

The convergence above is called two-scale weak* convergence*.*

Remark 3.10. *Notice that the family* $\{\mu^h\}_{h>0}$ *determines the family of measures* $\{\nu^h\}_{h>0} \subset \mathcal{M}_b(\Omega \times \mathcal{Y})$ *obtained by setting*

$$\int_{\Omega\times\mathcal{Y}} \chi(x,y)\, d\nu^h = \int_\Omega \chi\left(x, \frac{x'}{h}\right) d\mu^h(x)$$

for every $\chi \in C^0_0(\Omega \times \mathcal{Y})$*. Thus* μ *is simply the weak* limit in* $\mathcal{M}_b(\Omega \times \mathcal{Y})$ *of a suitable subsequence of* $\{\nu^h\}_{h>0}$*.*

We collect some basic properties of two-scale convergence below:

Proposition 3.11. *(i) Any sequence that is bounded in $\mathcal{M}_b(\Omega)$ admits a two-scale weakly* convergent subsequence.*

(ii) Let $\mathcal{D} \subset \mathcal{Y}$ and assume that $\operatorname{supp}(\mu^h) \subset \Omega \cap (\mathcal{D}_{\varepsilon_h} \times I)$. If $\mu^h \xrightarrow{2-*} \mu$ two-scale weakly in $\mathcal{M}_b(\Omega \times \mathcal{Y})$, then $\operatorname{supp}(\mu) \subset \Omega \times \overline{\mathcal{D}}$.*

4. Compactness results

In this section, we provide a characterization of two-scale limits of symmetrized scaled gradients. We will consider sequences of deformations $\{v^h\}$ such that $v^h \in BD(\Omega^h)$ for every $h > 0$, their L^1-norms are uniformly bounded (up to rescaling), and their symmetrized gradients Ev^h form a sequence of uniformly bounded Radon measures (again, up to rescaling). As already explained in Section 3.2, we associate to the sequence $\{v^h\}$ above a rescaled sequence of maps $\{u^h\} \subset BD(\Omega)$, defined as

$$u^h := (v_1^h, v_2^h, hv_3^h) \circ \psi_h,$$

where ψ_h is defined in (3.8). The symmetric gradients of the maps $\{v^h\}$ and $\{u^h\}$ are related as follows

$$\frac{1}{h} Ev^h = (\psi_h)_\# (\Lambda_h Eu^h). \tag{4.1}$$

The boundedness of $\frac{1}{h}\|Ev^h\|_{\mathcal{M}_b(\Omega^h;\mathbb{M}^{3\times 3}_{sym})}$ is equivalent to the boundedness of $\|\Lambda_h Eu^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times 3}_{sym})}$. We will express our compactness result with respect to the sequence $\{u^h\}_{h>0}$.

We first recall a compactness result for sequences of non-oscillating fields (see [13]).

Proposition 4.1. *Let $\{u^h\}_{h>0} \subset BD(\Omega)$ be a sequence such that there exists a constant $C > 0$ for which*

$$\|u^h\|_{L^1(\Omega;\mathbb{R}^3)} + \|\Lambda_h Eu^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times 3}_{sym})} \le C.$$

Then, there exist functions $\bar{u} = (\bar{u}_1, \bar{u}_2) \in BD(\omega)$ and $u_3 \in BH(\omega)$ such that, up to subsequences, there holds

$$\begin{aligned}
&u_\alpha^h \to \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \text{strongly in } L^1(\Omega), \quad \alpha \in \{1,2\},\\
&u_3^h \to u_3, \quad \text{strongly in } L^1(\Omega),\\
&Eu^h \xrightarrow{*} \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{weakly* in } \mathcal{M}_b(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}).
\end{aligned}$$

Now we turn to identifying the two-scale limits of the sequence $\Lambda_h Eu^h$.

4.1. Corrector properties and duality results.

In order to define and analyze the space of measures which arise as two-scale limits of scaled symmetrized gradients of BD functions, we will consider the following general framework (see also [2]).

Let V and W be finite-dimensional Euclidean spaces of dimensions N and M, respectively. We will consider k^{th} order linear homogeneous partial differential operators with constant coefficients $\mathcal{A} : C_c^\infty(\mathbb{R}^n;V) \to C_c^\infty(\mathbb{R}^n;W)$. More precisely, the operator $\mathcal{A}$ acts on functions $u : \mathbb{R}^n \to V$ as

$$\mathcal{A}u := \sum_{|\alpha|=k} A_\alpha \partial^\alpha u.$$

where the coefficients $A_\alpha \in W \otimes V^* \cong \operatorname{Lin}(V;W)$ are constant tensors, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ is a multi-index and $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ denotes the distributional partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We define the space

$$BV^{\mathcal{A}}(U) = \Big\{ u \in L^1(U;V) : \mathcal{A}u \in \mathcal{M}_b(U;W) \Big\}$$

of *functions with bounded $\mathcal{A}$-variations* on an open subset U of $\mathbb{R}^n$. This is a Banach space endowed with the norm

$$\|u\|_{BV^{\mathcal{A}}(U)} := \|u\|_{L^1(U;V)} + |\mathcal{A}u|(U).$$

Here, the distributional $\mathcal{A}$-gradient is defined and extended to distributions via the duality

$$\int_U \varphi \cdot d\mathcal{A}u := \int_U \mathcal{A}^*\varphi \cdot u\, dx, \quad \varphi \in C_c^\infty(U; W^*),$$

where $\mathcal{A}^* : C_c^\infty(\mathbb{R}^n; W^*) \to C_c^\infty(\mathbb{R}^n; V^*)$ is the formal L^2-adjoint operator of $\mathcal{A}$

$$\mathcal{A}^* := (-1)^k \sum_{|\alpha|=k} A_\alpha^* \partial^\alpha.$$

The *total $\mathcal{A}$-variation* of $u \in L^1_{loc}(U; V)$ is defined as

$$|\mathcal{A}u|(U) := \sup\left\{ \int_U \mathcal{A}^*\varphi \cdot u\, dx : \varphi \in C_c^k(U; W^*),\ |\varphi| \leq 1 \right\}.$$

Let $\{u_n\} \subset BV^{\mathcal{A}}(U)$ and $u \in BV^{\mathcal{A}}(U)$. We say that $\{u_n\}$ converges weakly* to u in $BV^{\mathcal{A}}$ if $u_n \to u$ in $L^1(U; V)$ and $\mathcal{A}u_n \overset{*}{\rightharpoonup} \mathcal{A}u$ in $\mathcal{M}_b(U; W)$.

In order to characterize the two-scale weak* limit of scaled symmetrized gradients, we will generally consider two domains $\Omega_1 \subset \mathbb{R}^{N_1}$, $\Omega_2 \subset \mathbb{R}^{N_2}$, for some $N_1, N_2 \in \mathbb{N}$ and assume that the operator $\mathcal{A}_{x_2}$ is defined through partial derivatives only with respect to the entries of the n_2-tuple x_2. In the spirit of [23, Section 4.2], we will define the space

$$\mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1) := \Big\{ \mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; V) : \mathcal{A}_{x_2}\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W), \\ \mu(F \times \Omega_2) = 0 \text{ for every Borel set } S \subseteq \Omega_1 \Big\}.$$

We will assume that $BV^{\mathcal{A}_{x_2}}(\Omega_2)$ satisfies the following weak* compactness property:

Assumption 1. *If $\{u_n\} \subset BV^{\mathcal{A}_{x_2}}(\Omega_2)$ is uniformly bounded in the $BV^{\mathcal{A}_{x_2}}$-norm, then there exists a subsequence $\{u_m\} \subseteq \{u_n\}$ and a function $u \in BV^{\mathcal{A}_{x_2}}(\Omega_2)$ such that $\{u_m\}$ converges weakly* to u in $BV^{\mathcal{A}_{x_2}}(\Omega_2)$, i.e.*

$$u_m \to u \ \text{ in } L^1(\Omega_2; V) \ \text{ and } \ \mathcal{A}_{x_2}u_m \overset{*}{\rightharpoonup} \mathcal{A}_{x_2}u \ \text{ in } \mathcal{M}_b(\Omega_2; W).$$

Furthermore, there exists a countable collection $\{U^k\}$ of open subsets of $\mathbb{R}^{n_2}$ that increases to Ω_2 (i.e. $\overline{U^k} \subset U^{k+1}$ for every $k \in \mathbb{N}$, and $\Omega_2 = \bigcup_k U^k$) such that $BV^{\mathcal{A}_{x_2}}(U^k)$ satisfies the weak compactness property above for every $k \in \mathbb{N}$.*

The following theorem is our main disintegration result for measures in $\mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$, which will be instrumental to define a notion of duality for admissible two-scale configurations. The proof is an adaptation of the arguments in [23, Proposition 4.7].

Proposition 4.2. *Let Assumption 1 be satisfied. Let $\mu \in \mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$. Then there exist $\eta \in \mathcal{M}_b^+(\Omega_1)$ and a Borel map $(x_1, x_2) \in \Omega_1 \times \Omega_2 \mapsto \mu_{x_1}(x_2) \in V$ such that, for η-a.e. $x_1 \in \Omega_1$,*

$$\mu_{x_1} \in BV^{\mathcal{A}_{x_2}}(\Omega_2), \qquad \int_{\Omega_2} \mu_{x_1}(x_2)\, dx_2 = 0, \qquad |\mathcal{A}_{x_2}\mu_{x_1}|(\Omega_2) \neq 0, \tag{4.2}$$

and

$$\mu = \mu_{x_1}(x_2)\, \eta \otimes \mathcal{L}^{n_2}_{x_2}. \tag{4.3}$$

Moreover, the map $x_1 \mapsto \mathcal{A}_{x_2}\mu_{x_1} \in \mathcal{M}_b(\Omega_2; W)$ is η-measurable and

$$\mathcal{A}_{x_2}\mu = \eta \overset{gen.}{\otimes} \mathcal{A}_{x_2}\mu_{x_1}.$$

Proof. By assumption, we have $\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; V)$ and $\lambda := \mathcal{A}_{x_2}\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W)$. Setting

$$\eta := proj_\#|\mu| + proj_\#|\lambda| \in \mathcal{M}_b^+(\Omega_1),$$

where $proj_\#$ is the push-forward by the projection of $\Omega_1 \times \Omega_2$ on Ω_1, we obtain as a consequence of Theorem 2.1:

$$\mu = \eta \overset{\text{gen.}}{\otimes} \mu_{x_1} \ \text{ and } \ \lambda = \eta \overset{\text{gen.}}{\otimes} \lambda_{x_1}, \tag{4.4}$$

with $\mu_{x_1} \in \mathcal{M}_b(\Omega_2; V)$ and $\lambda_{x_1} \in \mathcal{M}_b(\Omega_2; W)$. Further, if we set $S := \{x_1 \in \Omega_1 : |\lambda_{x_1}|(\Omega_2) \neq 0\}$, then $\lambda = \eta \lfloor S \overset{\text{gen.}}{\otimes} \lambda_{x_1}$.

For every $\varphi^{(1)} \in C_c^\infty(\Omega_1)$ and $\varphi^{(2)} \in C_c^\infty(\Omega_2; W^*)$ we have

$$\begin{aligned}
\int_{\Omega_1} \varphi^{(1)}(x_1) \left\langle \mu_{x_1}, \mathcal{A}^*_{x_2} \varphi^{(2)} \right\rangle \cdot d\eta(x_1) &= \int_{\Omega_1} \left(\int_{\Omega_2} \varphi^{(1)}(x_1) \mathcal{A}^*_{x_2} \varphi^{(2)}(x_2) \cdot d\mu_{x_1}(x_2) \right) \cdot d\eta(x_1) \\
&= \left\langle \eta \overset{\text{gen.}}{\otimes} \mu_{x_1}, \varphi^{(1)} \mathcal{A}^*_{x_2} \varphi^{(2)} \right\rangle = \left\langle \mu, \mathcal{A}^*_{x_2} \left(\varphi^{(1)} \varphi^{(2)} \right) \right\rangle \\
&= \left\langle \mathcal{A}_{x_2} \mu, \varphi^{(1)} \varphi^{(2)} \right\rangle = \left\langle \eta \lfloor S \overset{\text{gen.}}{\otimes} \lambda_{x_1}, \varphi^{(1)} \varphi^{(2)} \right\rangle \\
&= \int_{\Omega_1} \left(\int_{\Omega_2} \varphi^{(1)}(x_1) \varphi^{(2)}(x_2) \cdot d\lambda_{x_1}(x_2) \right) \mathbb{1}_S(x_1) \cdot d\eta(x_1) \\
&= \int_{\Omega_1} \varphi^{(1)}(x_1) \left\langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi^{(2)} \right\rangle \cdot d\eta(x_1).
\end{aligned}$$

From this we infer that for η-a.e. $x_1 \in \Omega_1$ and for every $\varphi \in C_c^\infty(\Omega_2; W^*)$

$$\left\langle \mu_{x_1}, \mathcal{A}^*_{x_2} \varphi \right\rangle = \left\langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi \right\rangle. \tag{4.5}$$

We can consider μ_{x_1} and λ_{x_1} as measures on $\mathbb{R}^{n_2}$ if we extend the measure μ by zero on the complement of Ω_1. Then, using the standard mollifiers $\{\rho_\epsilon\}_{\epsilon>0}$ on $\mathbb{R}^{n_2}$, we define the functions $\mu^\epsilon_{x_1} := \mu_{x_1} * \rho_\epsilon$ and $\lambda^\epsilon_{x_1} := \lambda_{x_1} * \rho_\epsilon$, which are smooth and uniformly bounded in $L^1(\Omega_2; V)$ and $L^1(\Omega_2; W)$, respectively. For every $\varphi \in C_c^k(\Omega_2; W^*)$, $\text{supp}(\varphi) \subset U^k$ for k large enough. Furthermore, the support of $\varphi * \rho_\epsilon$ is contained in Ω_2 provided ϵ is sufficiently small (smallness depending only on k), and thus from (4.5) we have

$$\begin{aligned}
\langle \mu^\epsilon_{x_1}, \mathcal{A}^*_{x_2} \varphi \rangle &= \int_{\mathbb{R}^{n_2}} (\mu_{x_1} * \rho_\epsilon) \cdot \mathcal{A}^*_{x_2} \varphi \, dx_2 = \int_{\mathbb{R}^{n_2}} \left(\mathcal{A}^*_{x_2} \varphi * \rho_\epsilon \right) \cdot d\mu_{x_1} \\
&= \int_{\mathbb{R}^{n_2}} \mathcal{A}^*_{x_2} (\varphi * \rho_\epsilon) \cdot d\mu_{x_1} = \langle \mu_{x_1}, \mathcal{A}^*_{x_2} (\varphi * \rho_\epsilon) \rangle \\
&= \langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi * \rho_\epsilon \rangle = \int_{\mathbb{R}^{n_2}} (\varphi * \rho_\epsilon) \cdot \mathbb{1}_S(x_1) \, d\lambda_{x_1} \\
&= \int_{\mathbb{R}^{n_2}} \mathbb{1}_S(x_1) \, (\lambda_{x_1} * \rho_\epsilon) \cdot \varphi \, dx_2 \\
&= \langle \mathbb{1}_S(x_1) \lambda^\epsilon_{x_1}, \varphi \rangle.
\end{aligned}$$

Hence, for η-a.e. $x_1 \in \Omega_1$ the sequence $\{\mu^\epsilon_{x_1}\}$ is eventually bounded in $BV^{\mathcal{A}_{x_2}}(U^k)$. By Assumption 1, this implies strong convergence in $L^1(U^k; V)$ up to a subsequence. As $\epsilon \to 0$, we have both $\varphi * \rho_\epsilon \to \varphi$ and $\mathcal{A}^*_{x_2} \varphi * \rho_\epsilon \to \mathcal{A}^*_{x_2} \varphi$ uniformly, so by the Lebesgue's dominated convergence theorem we obtain, for η-a.e. $x_1 \in \Omega_1$,

$$\langle \mu^\epsilon_{x_1}, \mathcal{A}^*_{x_2} \varphi \rangle \to \langle \mu_{x_1}, \mathcal{A}^*_{x_2} \varphi \rangle \quad \text{and} \quad \langle \mathbb{1}_S(x_1) \lambda^\epsilon_{x_1}, \varphi \rangle \to \langle \mathbb{1}_S(x_1) \lambda_{x_1}, \varphi \rangle.$$

From the convergence above, we conclude for η-a.e. $x_1 \in \Omega_1$ that $\mu^\epsilon_{x_1} \to \mu_{x_1}$ strongly in $L^1(U^k; V)$. Since μ_{x_1} has bounded total variation, we have that $\mu_{x_1} \in L^1(\Omega_2; V)$ for η-a.e. $x_1 \in \Omega_1$. This, together with (4.5), implies

$$\mu_{x_1} \in BV^{\mathcal{A}_{x_2}}(\Omega_2) \quad \text{and} \quad \mathcal{A}_{x_2} \mu_{x_1} = \mathbb{1}_S(x_1) \lambda_{x_1}.$$

From (4.4) we now have that μ is absolutely continuous with respect to $\eta \otimes \mathcal{L}^{n_2}_{x_2}$. Consequently, for η-a.e. $x_1 \in \Omega_1$ there exists a Borel measurable function which is equal to μ_{x_1} for $\mathcal{L}^{n_2}_{x_2}$-a.e. $x_2 \in \Omega_2$, so that (4.3) immediately follows.

Finally, since $\mu(F \times \Omega_2) = 0$ for every Borel set $F \subseteq \Omega_1$, we have

$$\int_{\Omega_1} f(x_1) \left(\int_{\Omega_2} \mu_{x_1}(x_2) \, dx_2 \right) d\eta(x_1) = \int_{\Omega_1 \times \Omega_2} f(x_1) \, d\mu(x_1, x_2) = 0$$

for every $f \in C_c(\Omega_1)$, from which we obtain the second claim in (4.2). This concludes the proof. □

Lastly, we give a necessary and sufficient condition with which we can characterize the $\mathcal{A}_{x_2}$-gradient of a measure, under the following two assumptions.

Assumption 2. *For every $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathcal{A}_{x_2}^* \chi = 0$ (in the sense of distributions), there exists a sequence of smooth functions $\{\chi_n\} \subset C_c^\infty(\Omega_1 \times \Omega_2; W)$ such that $\mathcal{A}_{x_2}^* \chi_n = 0$ for every n, and $\chi_n \to \chi$ in $L^\infty(\Omega_1 \times \Omega_2; W)$.*

Assumption 3. *The following Poincaré-Korn type inequality holds in $BV^{\mathcal{A}_{x_2}}(\Omega_2)$:*

$$\left\| u - \fint_{\Omega_2} u \, dx_2 \right\|_{L^1(\Omega_2;V)} \le C|\mathcal{A}_{x_2} u|(\Omega_2), \quad \forall u \in BV^{\mathcal{A}_{x_2}}(\Omega_2).$$

Proposition 4.3. *Let Assumption 1, 2 and 3 be satisfied. Let $\lambda \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W)$. Then, the following items are equivalent:*

(i) For every $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathcal{A}_{x_2}^* \chi = 0$ (in the sense of distributions) we have

$$\int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\lambda(x_1, x_2) = 0.$$

(ii) There exists $\mu \in \mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1)$ such that $\lambda = \mathcal{A}_{x_2}\mu$.

Proof. Let $\chi \in C_0(\Omega_1 \times \Omega_2; W)$ with $\mathcal{A}_{x_2}^* \chi = 0$ (in the sense of distributions) and let $\{\chi_n\}$ be an approximating sequence of χ as in Assumption 2. Assume that (ii) holds. Then, we have

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\lambda(x_1, x_2) &= \int_{\Omega_1 \times \Omega_2} \chi(x_1, x_2) \cdot d\mathcal{A}_{x_2}\mu(x_1, x_2) \\ &= \lim_n \int_{\Omega_1 \times \Omega_2} \chi_n(x_1, x_2) \cdot d\mathcal{A}_{x_2}\mu(x_1, x_2) \\ &= \lim_n \int_{\Omega_1 \times \Omega_2} \mathcal{A}_{x_2}^* \chi_n(x_1, x_2) \, d\mu(x_1, x_2) = 0.\end{aligned}$$

So we have (i) .

Let us prove that the space

$$\mathcal{E}^{\mathcal{A}_{x_2}} = \left\{ \mathcal{A}_{x_2}\mu : \mu \in \mathcal{X}^{\mathcal{A}_{x_2}}(\Omega_1) \right\}$$

is weakly* closed in $\mathcal{M}_b(\Omega_1 \times \Omega_2; W)$. By the Krein-Šmulian theorem it is enough to show that the intersection of $\mathcal{E}^{\mathcal{A}_{x_2}}$ with every closed ball in $\mathcal{M}_b(\Omega_1 \times \Omega_2; W)$ is weakly* closed. This implies, since the weak* topology is metrizable on any closed ball of $\mathcal{M}_b(\Omega_1 \times \Omega_2; W)$, that it is enough to prove that $\mathcal{E}^{\mathcal{A}_{x_2}}$ is sequentially weakly* closed.

Let $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathcal{E}^{\mathcal{A}_{x_2}}$ and $\lambda \in \mathcal{M}_b(\Omega_1 \times \Omega_2; W)$ be such that

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \text{ in } \mathcal{M}_b(\Omega_1 \times \Omega_2; W).$$

By the definition of the space $\mathcal{E}^{\mathcal{A}_{x_2}}$, there exist measures $\mu_n \in \mathcal{M}_b(\Omega_1 \times \Omega_2; V)$ such that $\lambda_n = \mathcal{A}_{x_2}\mu_n$. By Proposition 4.2, for every $n \in \mathbb{N}$ we have that there exist $\eta_n \in \mathcal{M}_b^+(\Omega_1)$ and $\mu_{x_1}^n \in BV^{\mathcal{A}_{x_2}}(\Omega_2)$ such that, for η_n-a.e. $x_1 \in \Omega_1$,

$$\mu_n = \mu_{x_1}^n(x_2)\, \eta_n \otimes \mathcal{L}_{x_2}^{n_2}, \qquad \mathcal{A}_{x_2}\mu_n = \eta_n \overset{\text{gen.}}{\otimes} \mathcal{A}_{x_2}\mu_{x_1}^n.$$

Additionally, $\mu_{x_1}^n$ satisfies $\int_{\Omega_2} \mu_{x_1}^n(x_2)\, dx_2 = 0$ for every $n \in \mathbb{N}$. Then, by Assumption 3, there is a constant C independent of n such that

$$\begin{aligned}|\mu_n|(\Omega_1 \times \Omega_2) &= \int_{\Omega_1 \times \Omega_2} |\mu_n(x_1, x_2)|\, dx_1 dx_2 = \int_{\Omega_1} \left(\int_{\Omega_2} |\mu_{x_1}^n(x_2)|\, dx_2 \right) d\eta_n(x_1) \\ &\le C \int_{\Omega_1} |\mathcal{A}_{x_2}\mu_{x_1}^n|(\Omega_2)\, d\eta_n(x_1) = C \int_{\Omega_1} \left(\int_{\Omega_2} d|\mathcal{A}_{x_2}\mu_{x_1}^n|(x_2) \right) d\eta_n(x_1) \\ &= C \int_{\Omega_1 \times \Omega_2} d\left(\eta_n \overset{\text{gen.}}{\otimes} |\mathcal{A}_{x_2}\mu_{x_1}^n| \right) = C|\mathcal{A}_{x_2}\mu_n|(\Omega_1 \times \Omega_2) \le C.\end{aligned}$$

Hence there exists a subsequence of $\{\mu_n\}$, not relabeled, and an element $\mu \in \mathcal{M}_b(\Omega_1 \times \Omega_2; V)$ such that

$$\mu_n \overset{*}{\rightharpoonup} \mu \ \text{ in } \mathcal{M}_b(\Omega_1 \times \Omega_2; V).$$

Then, for every $\varphi \in C_c^\infty(\Omega_1 \times \Omega_2; W^*)$ we have

$$\begin{aligned}\langle \lambda, \varphi \rangle &= \lim_n \langle \lambda_n, \varphi \rangle = \lim_n \langle \mathcal{A}_{x_2} \mu_n, \varphi \rangle \\ &= \lim_n \langle \mu_n, \mathcal{A}_{x_2}^* \varphi \rangle = \langle \mu, \mathcal{A}_{x_2}^* \varphi \rangle.\end{aligned}$$

From the convergence above we deduce that $\lambda = \mathcal{A}_{x_2}\mu \in \mathcal{E}^{\mathcal{A}_{x_2}}$. This implies that $\mathcal{E}^{\mathcal{A}_{x_2}}$ is weakly* closed in $\mathcal{M}_b(\Omega_1 \times \Omega_2; W) = (C_0(\Omega_1 \times \Omega_2; W^*))'$.

Assume now that (i) holds. If $\lambda \notin \mathcal{E}^{\mathcal{A}_{x_2}}$, by Hahn-Banach's theorem, there exists $\chi \in C_0(\Omega_1 \times \Omega_2; W^*)$ such that

$$\int_{\Omega_1 \times \Omega_2} \chi \cdot d\lambda = 1, \tag{4.6}$$

and, for every $u \in BV^{\mathcal{A}_{x_2}}(\Omega_1 \times \Omega_2)$,

$$\int_{\Omega_1 \times \Omega_2} \chi \cdot d\mathcal{A}_{x_2} u = 0. \tag{4.7}$$

In particular, choosing u to be a smooth function, (4.7) implies that $\mathcal{A}_{x_2}^* \chi = 0$ (in the sense of distributions). As a consequence, (4.6) contradicts (i). Thus, $\lambda \in \mathcal{E}^{\mathcal{A}_{x_2}}$. □

4.1.1. *Compactness result for scaled maps with finite energy.* If we consider $\mathcal{A}_{x_2} = \widetilde{E}_\gamma$, $\mathcal{A}_{x_2}^* = \widetilde{\operatorname{div}}_\gamma$, $\Omega_1 = \omega$ with points $x_1 = x'$, and $\Omega_2 = I \times \mathcal{Y}$ with points $x_2 = (x_3, y)$, then we denote the associated spaces from the previous section by:

$$BD_\gamma(I \times \mathcal{Y}) := \Big\{ u \in L^1(I \times \mathcal{Y}; \mathbb{R}^3) : \widetilde{E}_\gamma u \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}}) \Big\},$$

$$\begin{aligned}\mathcal{X}_\gamma(\omega) := \Big\{ \mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{R}^3) : \widetilde{E}_\gamma \mu \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}}),\\ \mu(F \times I \times \mathcal{Y}) = 0 \text{ for every Borel set } F \subseteq \omega \Big\}.\end{aligned}$$

Despite the fact that $\mathcal{Y}$ is a flat torus, Proposition 4.2 and Proposition 4.3 are satisfied if we establish the validity of Assumption 1, 2 and 3, which will be done below.

Remark 4.4. *To each $u \in BD_\gamma(I \times \mathcal{Y})$, we can associate a function $v := \Big(\frac{1}{\gamma}u_1, \frac{1}{\gamma}u_2, u_3\Big)$. Then*

$$Ev = \begin{pmatrix} \frac{1}{\gamma} E_y u' & \frac{1}{2}\Big(D_y u_3 + \frac{1}{\gamma}\,\partial_{x_3} u'\Big) \\ \frac{1}{2}\Big(D_y u_3 + \frac{1}{\gamma}\,\partial_{x_3} u'\Big)^T & \partial_{x_3} u_3 \end{pmatrix},$$

from which we can see that $v \in BD(I \times \mathcal{Y})$. Here $E_y u'$ denotes the symmetrized gradient in y of the field u', which is a 2×2 matrix. Alternatively, we can define the change of variables $\psi : (\gamma I) \times \mathcal{Y} \to I \times \mathcal{Y}$ given by $\psi(x_3, y) := \Big(\frac{1}{\gamma}x_3, y\Big)$ and consider the function $w := u \circ \psi$. Then $w \in BD((\gamma I) \times \mathcal{Y})$ and we have

$$\widetilde{E}_\gamma u = \frac{1}{\gamma}\psi_\#(\widetilde{E}_1 w).$$

Using any one of these scalings, we obtain that $BD_\gamma(I \times \mathcal{Y})$ satisfies the weak compactness property Assumption 1.*

The following lemma establishes the validity of Assumption 2.

Lemma 4.5. *For any $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$ with $\widetilde{\operatorname{div}}_\gamma \chi(x, y) = 0$ (in the sense of distributions), we can construct an approximating sequence which satisfies Assumption 2.*

Proof. We take $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$, extend it by zero outside Ω and define

$$\tilde{\chi}^\epsilon(x,y) := \Lambda_{1+\epsilon}\chi\left(\varphi^\epsilon(x')x', (1+\epsilon)x_3, y\right),$$

where $\Lambda_{1+\epsilon}$ is the linear operator described in (3.9), and $\varphi^\epsilon : \omega \to [0,1]$ is a continuous function that is zero in a neighbourhood of $\partial\omega$ and equal to 1 for $x' \in \omega$ such that $\text{dist}(x', \partial\omega) \geq \epsilon$. Notice that $\tilde{\chi}^\epsilon \in C_c(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$, $\tilde{\chi}^\epsilon \to \chi$ as $\epsilon \to 0$ in L^∞ and $\widetilde{\text{div}}_\gamma \tilde{\chi}^\epsilon = 0$ (in the sense of distributions). The C^∞-regularity of the approximating sequence follows by convolving $\{\tilde{\chi}^\epsilon\}$ with a standard sequence of mollifiers. □

The following claim establishes the validity of Assumption 3.

Theorem 4.6. *There exists a constant $C > 0$ such that*

$$\left\| u - \int_{I\times\mathcal{Y}} u \right\|_{L^1(I\times\mathcal{Y};\mathbb{R}^3)} \leq C|\widetilde{E}_\gamma u|(I \times \mathcal{Y})$$

for each function $u \in BD_\gamma(I \times \mathcal{Y})$. The constant C can be chosen independently of γ in a fixed interval $[\gamma_1, \gamma_2]$, for $0 < \gamma_1 < \gamma_2 < \infty$.

Proof. In view of Remark 4.4, it is enough to show the claim for the case $\gamma = 1$. We argue by contradiction. If the thesis does not hold, then there exists a sequence $\{u_n\}_n \subset BD(I \times \mathcal{Y})$ such that

$$\int_{I\times\mathcal{Y}} |u_n|\, dx_3 dy > n|\widetilde{E}_1 u_n|(I \times \mathcal{Y}), \quad \text{with} \quad \int_{I\times\mathcal{Y}} u_n\, dx_3 dy = 0.$$

We can normalize the sequence such that

$$\int_{I\times\mathcal{Y}} |u_n|\, dx_3 dy = 1, \quad \text{and} \quad |\widetilde{E}_1 u_n|(I \times \mathcal{Y}) < \frac{1}{n}.$$

In particular the sequence $\{u_n\}$ is bounded in $BD(I \times \mathcal{Y})$.

By Assumption 1, there exists a subsequence $\{u_m\} \subseteq \{u_n\}$ and a function $u \in BD(I \times \mathcal{Y})$ such that $\{u_m\}$ converges weakly* to u in $BD(I \times \mathcal{Y})$, i.e.

$$u_m \to u \ \text{ in } L^1(I \times \mathcal{Y}; \mathbb{R}^3), \quad \text{and} \quad \widetilde{E}_1 u_m \xrightarrow{*} \widetilde{E}_1 u \ \text{ in } \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}}).$$

It's clear that the limit satisfies

$$\int_{I\times\mathcal{Y}} |u|\, dx_3 dy = 1, \quad \text{with} \quad \int_{I\times\mathcal{Y}} u\, dx_3 dy = 0. \tag{4.8}$$

Also, by the weak* lower semicontinuity of the total variation of measures, we have

$$|\widetilde{E}_1 u|(I \times \mathcal{Y}) = 0, \tag{4.9}$$

which implies $\widetilde{E}_1 u = 0$. As a result, the limit u is a rigid deformation, i.e. is of the form

$$u(x_3, y) = A \begin{pmatrix} y_1 \\ y_2 \\ x_3 \end{pmatrix} + b, \quad \text{where} \ \ A \in \mathbb{M}^{3\times 3}_{\text{skew}}, b \in \mathbb{R}^3.$$

Further, (4.9) implies that u has no jumps along C^1 hypersurfaces contained in $I \times \mathcal{Y}$. Hence, due to the structure of skew-symmetric matrices, u must be a constant vector. However, this contradicts with (4.8). □

Remark 4.7. *If one doesn't assume periodicity, then the following version of the Poincaré-Korn inequality can be proved, using the arguments in the proof of Theorem 4.6: There exists a constant $C > 0$ such that*

$$\left\| u - A \begin{pmatrix} x_1 \\ x_2 \\ \gamma x_3 \end{pmatrix} - b \right\|_{L^1((0,1)^2\times I;\mathbb{R}^3)} \leq C|E_\gamma u|((0,1)^2 \times I)$$

for each function $u \in BD_\gamma((0,1)^2 \times I)$ and suitably chosen $A \in \mathbb{M}^{3\times 3}_{\text{skew}}$, $b \in \mathbb{R}^3$, depending on u. Again, the constant C can be chosen independently of γ in a fixed interval $[\gamma_1, \gamma_2]$, for $0 < \gamma_1 < \gamma_2 < \infty$.

The following two propositions are now a consequence of Proposition 4.2 and Proposition 4.3, respectively.

Proposition 4.8. *Let $\mu \in \mathcal{X}_\gamma(\omega)$. Then there exist $\eta \in \mathcal{M}_b^+(\omega)$ and a Borel map $(x', x_3, y) \in \Omega \times \mathcal{Y} \mapsto \mu_{x'}(x_3, y) \in \mathbb{R}^3$ such that, for η-a.e. $x' \in \omega$,*

$$\mu_{x'} \in BD_\gamma(I \times \mathcal{Y}), \qquad \int_{I\times\mathcal{Y}} \mu_{x'}(x_3, y)\, dx_3 dy = 0, \qquad |\widetilde{E}_\gamma \mu_{x'}|(I \times \mathcal{Y}) \neq 0, \tag{4.10}$$

and

$$\mu = \mu_{x'}(x_3, y)\, \eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y. \tag{4.11}$$

Moreover, the map $x' \mapsto \widetilde{E}_\gamma \mu_{x'} \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ is η-measurable and

$$\widetilde{E}_\gamma \mu = \eta \overset{gen.}{\otimes} \widetilde{E}_\gamma \mu_{x'}.$$

Proposition 4.9. *Let $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$. The following items are equivalent:*

(i) For every $\chi \in C_0(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\text{sym}})$ with $\widetilde{\operatorname{div}}_\gamma \chi(x, y) = 0$ (in the sense of distributions) we have

$$\int_{\Omega\times\mathcal{Y}} \chi(x, y) : d\lambda(x, y) = 0.$$

(ii) There exists $\mu \in \mathcal{X}_\gamma(\omega)$ such that $\lambda = \widetilde{E}_\gamma \mu$.

Additionally, we state the following property, which will be used in the proof of Lemma 4.18. The proof is analogous to [23, Proposition 4.7. item (b)].

Proposition 4.10. *Let $\mu \in \mathcal{X}_\gamma(\omega)$. For any C^1-hypersurface $\mathcal{D} \subseteq \mathcal{Y}$, if ν denotes a continuous unit normal vector field to $\mathcal{D}$, then*

$$\widetilde{E}_\gamma \mu \lfloor \Omega \times \mathcal{D} = a(x, y) \odot \nu(y)\, \eta \otimes (\mathcal{H}^2_{x_3,y} \lfloor I \times \mathcal{D}),$$

where $a : \Omega \times \mathcal{D} \mapsto \mathbb{R}^3$ is a Borel function.

4.2. **Auxiliary results.** We will need the following result, which is connected with the compactly supported De Rham cohomology. Recall the definitions of $\widetilde{\operatorname{curl}}_\gamma$, $\widetilde{\nabla}_\gamma$, and $\widetilde{\operatorname{div}}_\gamma$. In the next proposition, we will consider the case $\gamma = 1$.

Proposition 4.11. *(a) Let $\mathcal{Y}^{(3)}$ be a flat torus in $\mathbb{R}^3$ and let $\chi \in C^\infty(\mathcal{Y}^{(3)}; \mathbb{R}^3)$ be such that $\operatorname{div}\chi = 0$ and $\int_{\mathcal{Y}^{(3)}} \chi = 0$. Then there exists $F \in C^\infty(\mathcal{Y}^{(3)}; \mathbb{R}^3)$ such that $\operatorname{curl} F = \chi$.*

(b) Let $\mathcal{Y}$ be a flat torus in $\mathbb{R}^2$ and let $\chi \in C_c^\infty(I \times \mathcal{Y}; \mathbb{R}^3)$ be such that $\widetilde{div}_1 \chi = 0$ and $\int_{I\times\mathcal{Y}} \chi = 0$. Then there exists $F \in C_c^\infty(I \times \mathcal{Y}; \mathbb{R}^3)$ such that

$$\widetilde{\operatorname{curl}}_1 F = \chi.$$

Proof. The first claim is standard and can be easily proved by, e.g, Fourier transforms. For the second claim, observing that χ is also periodic on $\mathcal{Y}^{(3)}$, by the first part of the statement we obtain $\tilde{F} \in C^\infty(\mathcal{Y}^{(3)}; \mathbb{R}^3)$ such that $\operatorname{curl} \tilde{F} = \chi$ on $\mathcal{Y}^{(3)}$. Since χ has compact support in $I \times \mathcal{Y}$, there exists $0 < \delta < \frac{1}{2}$ such that $\widetilde{\operatorname{curl}}_1 \tilde{F} = 0$ on $\tilde{I}_\delta \times \mathcal{Y}$, where $\tilde{I}_\delta = \{(\frac{1}{2} - \delta, \frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{2} + \delta)\}$. Let now $\tilde{\varphi} \in C^\infty(S_\delta)$, where $S_\delta = \tilde{I}_\delta \times (0, 1)^2$, be such that $\tilde{F} = \widetilde{\nabla}_1 \tilde{\varphi}$ on S_δ. For $\alpha \in \{1, 2\}$, let

$$\sum_{k\in\mathbb{Z}} a_k^\alpha(x_3, y_2) e^{2\pi i k y_1}$$

be the exponential Fourier series of $\tilde{F}_\alpha = \partial_{y_\alpha} \tilde{\varphi}$ with respect to the variable y_1. Note that the coefficients $\{a_k^\alpha(x_3, y_2)\}_{k\in\mathbb{Z}}$ are smooth functions and periodic with respect to the variable y_2 and x_3. Additionally, the Fourier series of smooth functions converges uniformly, and the result of differentiating or integrating the series term by term will converge to the derivative or integral of the original series. Hence, we infer that

$$\tilde{\varphi}(x_3, y) = a_0^1(x_3, y_2)y_1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k^1(x_3, y_2)}{2\pi ik} e^{2\pi iky_1} + b^1(x_3, y_2) \text{ on } S_\delta, \tag{4.12}$$

for a suitable smooth function $b^1(x_3, y_2)$. Then, differentiating with respect to y_1 and y_2, we have that

$$\partial_{y_1y_2}\tilde{\varphi}(x_3, y) = \partial_{y_2}a_0^1(x_3, y_2) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \partial_{y_2}a_k^1(x_3, y_2)e^{2\pi iky_1} \text{ on } S_\delta.$$

However, since

$$\partial_{y_1y_2}\tilde{\varphi}(x_3, y) = \partial_{y_1}\tilde{F}_2(x_3, y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} 2\pi ika_k^2(x_3, y_2)e^{2\pi iky_1} \text{ on } S_\delta,$$

by the uniqueness of the Fourier expansion we have that $\partial_{y_2}a_0^1(x_3, y_2) = 0$, i.e.

$$a_0^1(x_3, y_2) = c_1(x_3), \tag{4.13}$$

for some $c_1 \in C^\infty(\tilde{I}_\delta)$. Further, differentiating (4.12) with respect to y_2, we have that

$$\partial_{y_2}\tilde{\varphi}(x_3, y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\partial_{y_2}a_k^1(x_3, y_2)}{2\pi ik} e^{2\pi iky_1} + \partial_{y_2}b^1(x_3, y_2) \text{ on } S_\delta.$$

Since $\partial_{y_2}\tilde{\varphi} = \tilde{F}_2$ is periodic, we conclude that $\partial_{y_2}b^1$ is also periodic with respect to the variable y_2 and we can consider its Fourier series. Let $c_2 \in C^\infty(\tilde{I}_\delta)$ be the corresponding zero-th term. Then the antiderivative of $\partial_{y_2}b^1 - c_2$ with respect to y_2 is a periodic function. Combining this fact with (4.12) and (4.13), we deduce that there exists a smooth function $\hat{\varphi} \in C^\infty(\tilde{I}_\delta; C^\infty(\mathcal{Y}))$ such that $\tilde{\varphi}$ can be rewritten as

$$\tilde{\varphi}(x_3, y) = \hat{\varphi}(x_3, y) + c_1(x_3)\, y_1 + c_2(x_3)\, y_2 \text{ on } \tilde{I}_\delta \times \mathcal{Y}.$$

From this, differentiating with respect to x_3, we have that

$$\tilde{F}_3(x_3, y) = \partial_{x_3}\hat{\varphi}(x_3, y) + c_1'(x_3)\, y_1 + c_2'(x_3)\, y_2 \text{ on } \tilde{I}_\delta \times \mathcal{Y}.$$

As a consequence of the periodicity of $\tilde{F}_3$ and $\partial_{x_3}\hat{\varphi}$ in the variables y_1 and y_2, we conclude that $c_1' = 0$ and $c_2' = 0$. Since $\tilde{I}_\delta \times \mathcal{Y}$ is a union of two disjoint open sets, we have that c_1, c_2 are constant on each connected component. Using the fact that, for $\alpha \in \{1, 2\}$,

$$\partial_{y_\alpha}\tilde{\varphi}(x_3, y) = \partial_{y_\alpha}\hat{\varphi}(x_3, y) + c_\alpha(x_3) \text{ on } \tilde{I}_\delta \times \mathcal{Y}, \tag{4.14}$$

the periodicity of $\tilde{F}_\alpha = \partial_{y_\alpha}\tilde{\varphi}$ implies that c_1, c_2 are in fact constant. This can be seen by integrating the equation (4.14) over the plane $x_3 = -\frac{1}{2}$ and $x_3 = \frac{1}{2}$. Thus we conclude that

$$\tilde{F}(x_3, y) = \widetilde{\nabla}_1\hat{\varphi}(x_3, y) + \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \text{ on } \tilde{I}_\delta \times \mathcal{Y}. \tag{4.15}$$

Consider now the exponential Fourier series of $\tilde{F}_3$ with respect to the x_3 variable, such that

$$\tilde{F}_3(x_3, y) = \sum_{k\in\mathbb{Z}} a_k^3(y)e^{2\pi ikx_3} \text{ on } \tilde{I}_\delta \times \mathcal{Y}.$$

Integrating the third component in (4.15) with respect to x_3, we have that there exists a smooth function $b^3(x_3, y)$, which has values $b_+^3(y)$ and $b_-^3(y)$ on each of the two parts of $\tilde{I}_\delta \times \mathcal{Y}$, such that

$$\hat{\varphi}(x_3, y) = a_0^3(y)x_3 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{a_k^3(y)}{2\pi ik} e^{2\pi ikx_3} + b^3(x_3, y) \text{ on } \tilde{I}_\delta \times \mathcal{Y}.$$

From this and (4.14) we have, for $\alpha \in \{1, 2\}$,

$$\tilde{F}_\alpha(x_3, y) - c_\alpha = \partial_{y_\alpha}a_0^3(y)x_3 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\partial_{y_\alpha}a_k^3(y)}{2\pi ik} e^{2\pi ikx_3} + \partial_{y_\alpha}b^3(x_3, y) \text{ on } \tilde{I}_\delta \times \mathcal{Y}.$$

Considering the continuity and periodicity in x_3 of the above terms, evaluating in $x_3 = -\frac{1}{2}$ and $x_3 = \frac{1}{2}$ gives $\partial_{y_\alpha} a^3_0(y) = \partial_{y_\alpha} b^3_-(y) - \partial_{y_\alpha} b^3_+(y)$. From this we have that there exists a constant c_3 and a map $\varphi \in C^\infty(\mathcal{Y} \times \tilde{I}_\delta)$ such that φ and all its derivatives are periodic in the x_3 variable, and for which

$$\hat{\varphi}(x_3, y) = \varphi(x_3, y) + c_3 x_3 \quad \text{on } \tilde{I}_\delta \times \mathcal{Y}.$$

From this and (4.15) we conclude that

$$\tilde{F}(x_3, y) = \widetilde{\nabla}_1 \varphi(x_3, y) + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{on } \tilde{I}_\delta \times \mathcal{Y}.$$

Finally, we consider a smooth function $k : I \to \mathbb{R}$ that is zero on the set $\left[-\frac{1}{2} + \delta, \frac{1}{2} - \delta\right]$ and one in a neighbourhood of $x_3 = -\frac{1}{2}$, $x_3 = \frac{1}{2}$. By taking

$$F := \tilde{F} - \widetilde{\nabla}_1 (k\,\varphi) - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{on } I \times \mathcal{Y}.$$

we have the claim. □

Remark 4.12. *By considering functions scaled by γ in the third component and by $\frac{1}{\gamma}$ in the direction x_3, one can apply the proof item (b) in Proposition 4.11 so that the statement is valid for maps in the space $C^\infty_c((\gamma I) \times \mathcal{Y}; \mathbb{R}^3)$.*

Consequently, for $\chi \in C^\infty_c(I \times \mathcal{Y}; \mathbb{R}^3)$ such that $\widetilde{\operatorname{div}}_\gamma \chi = 0$ and $\int_{I \times \mathcal{Y}} \chi = 0$ there exists $F \in C^\infty_c(I \times \mathcal{Y}; \mathbb{R}^3)$ such that $\widetilde{\operatorname{curl}}_\gamma F = \chi$, which can be easily seen by rescaling in the direction x_3.

Remark 4.13. *If $\chi \in C^\infty_c(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}})$ is such that $\widetilde{\operatorname{div}}_\gamma \chi = 0$, then for a.e. $x' \in \omega$*

$$\int_{I \times \mathcal{Y}} \chi_{3i}(x, y)\, dx_3 dy = 0, \quad i = 1, 2, 3.$$

Indeed, by putting

$$\varphi(x) = \begin{pmatrix} 2\gamma\, x_3\, c_1(x') \\ 2\gamma\, x_3\, c_2(x') \\ \gamma\, x_3\, c_3(x') \end{pmatrix},$$

for $c \in C^\infty_c(\omega; \mathbb{R}^3)$, we infer that

$$\widetilde{E}_\gamma \varphi = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

and the conclusion results from testing χ with $\widetilde{E}_\gamma \varphi$ on $I \times \mathcal{Y}$, and by the arbitrariness of the maps c_i, $i = 1, 2, 3$.

4.3. **Two-scale limits of scaled symmetrized gradients.** We are now ready to prove the first main result of this section.

Theorem 4.14. *Let $\{u^h\}_{h>0} \subset BD(\Omega)$ be a sequence such that there exists a constant $C > 0$ for which*

$$\|u^h\|_{L^1(\Omega;\mathbb{R}^3)} + \|\Lambda_h E u^h\|_{\mathcal{M}_b(\Omega;\mathbb{M}^{3\times3}_{sym})} \leq C.$$

Then there exist $\bar{u} = (\bar{u}_1, \bar{u}_2) \in BD(\omega)$, $u_3 \in BH(\omega)$ and $\mu \in \mathcal{X}_\gamma(\omega)$, and a subsequence of $\{u^h\}_{h>0}$, not relabeled, which satisfy:

$$\Lambda_h E u^h \xrightarrow{2-*} \begin{pmatrix} E\bar{u} - x_3 D^2 u_3 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma \mu \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}}).$$

Proof. Owing to [42, Chapter II, Remark 3.3], we can assume without loss of generality that the maps u^h are smooth functions for every $h > 0$. Further, the uniform boundedness of the sequence $\{Ev^h\}$ implies that

$$\int_\Omega |\partial_{x_\alpha} u_3^h + \partial_{x_3} u_\alpha^h| \, dx \leq Ch, \qquad \text{for } \alpha = 1,2, \tag{4.16}$$

$$\int_\Omega |\partial_{x_3} u_3^h| \, dx \leq Ch^2. \tag{4.17}$$

In the following, we will consider $\lambda \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that

$$\Lambda_h E u^h \xrightarrow{2-*} \lambda \quad \text{two-scale weakly* in } \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

By using Proposition 4.1 we have that there exist $(\bar{u}_1, \bar{u}_2) \in BD(\omega)$, $u_3 \in BH(\omega)$ such that

$$(Eu^h)''(x) \xrightarrow{*} E\bar{u}(x') - x_3 D^2 u_3(x') \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Let $\chi \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ be such that $\widetilde{\mathrm{div}}_\gamma \chi = 0$. We have

$$\begin{aligned}
&\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\lambda(x,y) \\
&= \lim_{h\to 0} \int_\Omega \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big) : d\left(\Lambda_h E u^h(x)\right) = -\lim_{h\to 0} \int_\Omega u^h(x) \cdot \mathrm{div}\left(\Lambda_h \chi\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\right) dx \\
&= -\lim_{h\to 0} \sum_{\alpha=1,2} \int_\Omega u_\alpha^h(x) \left(\partial_{x_1}\chi_{\alpha 1} + \partial_{x_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx - \lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h(x) \left(\partial_{x_1}\chi_{31} + \partial_{x_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx \\
&\quad - \lim_{h\to 0} \sum_{\alpha=1,2} \frac{1}{\varepsilon_h} \int_\Omega u_\alpha^h(x) \left(\partial_{y_1}\chi_{\alpha 1} + \partial_{y_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx - \lim_{h\to 0} \frac{1}{h\varepsilon_h} \int_\Omega u_3^h(x) \left(\partial_{y_1}\chi_{31} + \partial_{y_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx \\
&\quad - \lim_{h\to 0} \sum_{\alpha=1,2} \frac{1}{h} \int_\Omega u_\alpha^h(x) \, \partial_{x_3}\chi_{\alpha 3}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx - \lim_{h\to 0} \frac{1}{h^2} \int_\Omega u_3^h(x) \, \partial_{x_3}\chi_{33}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx \\
&= -\lim_{h\to 0} \sum_{\alpha=1,2} \int_\Omega u_\alpha^h \cdot \left(\partial_{x_1}\chi_{\alpha 1} + \partial_{x_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx - \lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h \cdot \left(\partial_{x_1}\chi_{31} + \partial_{x_2}\chi_{32}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx \\
&\quad + \lim_{h\to 0} \left(\frac{h}{\varepsilon_h \gamma} - 1\right) \left(\sum_{\alpha=1,2} \frac{1}{h} \int_\Omega u_\alpha^h \cdot \partial_{x_3}\chi_{\alpha 3}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx + \frac{1}{h^2} \int_\Omega u_3^h \cdot \partial_{x_3}\chi_{33}\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx\right),
\end{aligned} \tag{4.18}$$

where in the last equality we used that $\frac{1}{\varepsilon_h}\partial_{y_1}\chi_{i1} + \frac{1}{\varepsilon_h}\partial_{y_2}\chi_{i2} + \frac{1}{h}\partial_{x_3}\chi_{i3} = \left(\frac{1}{h} - \frac{1}{\varepsilon_h \gamma}\right)\partial_{x_3}\chi_{i3}$.

From Proposition 4.1 we know that we have the following convergences:

$$\begin{aligned}
&u_\alpha^h \to \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3, \quad \text{strongly in } L^1(\Omega), \quad \alpha = 1,2, \\
&u_3^h \to u_3, \quad \text{strongly in } L^1(\Omega).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\lim_{h\to 0} \sum_{\alpha=1,2} \int_\Omega u_\alpha^h(x) \left(\partial_{x_1}\chi_{\alpha 1} + \partial_{x_2}\chi_{\alpha 2}\right)\Big(x, \tfrac{x'}{\varepsilon_h}\Big) \, dx \\
&= \sum_{\alpha=1,2} \int_\Omega \left(\bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3\right) \left(\partial_{x_1} \int_{\mathcal{Y}} \chi_{\alpha 1}(x,y) \, dy + \partial_{x_2} \int_{\mathcal{Y}} \chi_{\alpha 2}(x,y) \, dy\right) dx \\
&= -\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\left(\begin{pmatrix} E\bar{u}(x') - x_3 D^2 u_3(x') & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathcal{L}_y^2\right).
\end{aligned} \tag{4.19}$$

Next, in view of Remark 4.13, we can use item (b) in Proposition 4.11, i.e. Remark 4.12 to conclude that there exists $F \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{R}^3)$ such that $\widetilde{\operatorname{curl}}_\gamma F = (\chi_{3i})_{i=1,2,3}$. Thus we have

$$\chi_{31} = \partial_{y_2} F_3 - \frac{1}{\gamma} \partial_{x_3} F_2, \tag{4.20}$$

$$\chi_{32} = \frac{1}{\gamma} \partial_{x_3} F_1 - \partial_{y_1} F_3. \tag{4.21}$$

Next we compute

$$\begin{aligned} \lim_{h\to 0} \frac{1}{\varepsilon_h} \int_\Omega u_3^h(x)\, \partial_{x_1 y_2} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx &= \lim_{h\to 0} \int_\Omega u_3^h(x)\, \partial_{x_2} \Big(\partial_{x_1} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\Big)\, dx \\ &\quad - \lim_{h\to 0} \int_\Omega u_3^h(x)\, \partial_{x_1 x_2} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx. \end{aligned} \tag{4.22}$$

Notice that

$$\lim_{h\to 0} \int_\Omega u_3^h(x)\, \partial_{x_1 x_2} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big) = \int_{\Omega\times\mathcal{Y}} u_3\, \partial_{x_1 x_2} F_3(x,y) dx dy = \int_\Omega \partial_{x_1 x_2} u_3 \int_{\mathcal{Y}} F_3(x,y) dy dx. \tag{4.23}$$

Recalling (4.16), we find

$$\begin{aligned} \lim_{h\to 0} \int_\Omega u_3^h(x)\, \partial_{x_2} \Big(\partial_{x_1} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\Big)\, dx &= -\lim_{h\to 0} \int_\Omega \partial_{x_2} u_3^h(x)\, \partial_{x_1} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\ &= \lim_{h\to 0} \int_\Omega \partial_{x_3} u_2^h\, \partial_{x_1} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\ &= -\lim_{h\to 0} \int_\Omega u_2^h\, \partial_{x_1 x_3} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\ &= -\int_{\Omega\times\mathcal{Y}} (\bar{u}_2 - x_3 \partial_{x_2} u_3)\, \partial_{x_1 x_3} F_3(x,y) dx dy \\ &= \int_\Omega \partial_{x_1 x_2} u_3 \int_{\mathcal{Y}} F_3(x,y) dy dx. \end{aligned} \tag{4.24}$$

From (4.22), (4.23), (4.24) we infer

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h(x)\, \partial_{x_1 y_2} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx &= \lim_{h\to 0} \frac{1}{\varepsilon_h \gamma} \int_\Omega u_3^h(x)\, \partial_{x_1 y_2} F_3\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\ &= 0. \end{aligned} \tag{4.25}$$

In a similar way for u_3^h (recalling (4.17)), we deduce

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h(x)\, \partial_{x_1 x_3} F_2\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx &= -\lim_{h\to 0} \frac{1}{h} \int_\Omega \partial_{x_3} u_3^h(x)\, \partial_{x_1} F_2\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx \\ &= 0. \end{aligned} \tag{4.26}$$

From (4.20), (4.25), (4.26) we conclude that

$$\lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h(x)\, \partial_{x_1} \chi_{31}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx = 0. \tag{4.27}$$

Analogously, we obtain

$$\lim_{h\to 0} \frac{1}{h} \int_\Omega u_3^h(x)\, \partial_{x_2} \chi_{32}\Big(x, \tfrac{x'}{\varepsilon_h}\Big)\, dx = 0. \tag{4.28}$$

Lastly, using similar arguments as above, we compute

$$
\begin{aligned}
&\lim_{h\to 0}\left(\frac{h}{\varepsilon_h\gamma}-1\right)\left(\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega u^h_\alpha(x)\,\partial_{x_3}\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u^h_3(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\right)\\
&=\lim_{h\to 0}\left(\frac{h}{\varepsilon_h\gamma}-1\right)\left(-\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega \partial_{x_3}u^h_\alpha(x)\,\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u^h_3(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\right)\\
&=\lim_{h\to 0}\left(\frac{h}{\varepsilon_h\gamma}-1\right)\left(\sum_{\alpha=1,2}\frac{1}{h}\int_\Omega \partial_{x_\alpha}u^h_3(x)\,\chi_{\alpha 3}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\frac{1}{h^2}\int_\Omega u^h_3(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\right)\\
&=\lim_{h\to 0}\left(\frac{h}{\varepsilon_h\gamma}-1\right)\left(-\frac{1}{h}\int_\Omega u^h_3(x)\,(\partial_{x_1}\chi_{31}+\partial_{x_2}\chi_{32})\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx+\left(\frac{h}{\varepsilon_h\gamma}+1\right)\frac{1}{h^2}\int_\Omega u^h_3(x)\,\partial_{x_3}\chi_{33}\Big(x,\tfrac{x'}{\varepsilon_h}\Big)\,dx\right)\\
&=0. \qquad (4.29)
\end{aligned}
$$

From (4.18), (4.19), (4.27), (4.28), (4.29) we have that

$$\int_{\Omega\times\mathcal{Y}}\chi(x,y):d\left(\lambda(x,y)-\begin{pmatrix}E\bar u(x')-x_3D^2u_3(x') & 0\\ 0 & 0\end{pmatrix}\otimes\mathcal{L}^2_y\right)=0.$$

From this and Proposition 4.9 we find that there exists $\mu\in\mathcal{X}_\gamma(\omega)$ such that

$$\lambda-\begin{pmatrix}E\bar u-x_3D^2u_3 & 0\\ 0 & 0\end{pmatrix}\otimes\mathcal{L}^2_y=\widetilde{E}_\gamma\mu.$$

This, in turn, yields the claim. □

4.4. **Unfolding adapted to dimension reduction.** We proceed along the lines of [23, Section 4.3].

For every $\varepsilon>0$ and $i\in\mathbb{Z}^2$, let

$$Q^i_\varepsilon:=\left\{x\in\mathbb{R}^2:\frac{x-\varepsilon i}{\varepsilon}\in Y\right\}.$$

Given an open set $\omega\subseteq\mathbb{R}^2$, we will set

$$I_\varepsilon(\omega):=\left\{i\in\mathbb{Z}^2:Q^i_\varepsilon\subset\omega\right\}.$$

Given $\mu_\varepsilon\in\mathcal{M}_b(\omega\times I)$ and $Q^i_\varepsilon\subset\omega$, we define $\mu^i_\varepsilon\in\mathcal{M}_b(I\times\mathcal{Y})$ such that

$$\int_{I\times\mathcal{Y}}\psi(x_3,y)\,d\mu^i_\varepsilon(x_3,y)=\frac{1}{\varepsilon^2}\int_{Q^i_\varepsilon\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right)d\mu_\varepsilon(x),\quad \psi\in C(I\times\mathcal{Y}).$$

Definition 4.15. *For every $\varepsilon>0$, the* unfolding measure *associated with μ_ε is the measure $\tilde\lambda_\varepsilon\in\mathcal{M}_b(\omega\times I\times\mathcal{Y})$ defined by*

$$\tilde\lambda_\varepsilon:=\sum_{i\in I_\varepsilon(\omega)}\left(\mathcal{L}^2_{x'}\lfloor Q^i_\varepsilon\right)\otimes\mu^i_\varepsilon.$$

The following proposition provides the relationship between the two-scale weak* convergence and unfolding measures. The proof is analogous to [23, Proposition 4.11.].

Proposition 4.16. *Let $\omega\subseteq\mathbb{R}^2$ be an open set and let $\{\mu_\varepsilon\}\subset\mathcal{M}_b(\omega\times I)$ be a bounded family such that*

$$\mu_\varepsilon\xrightarrow{2-*}\mu_0\quad\textit{two-scale weakly* in }\mathcal{M}_b(\omega\times I\times\mathcal{Y}).$$

Let $\{\tilde\lambda_\varepsilon\}\subset\mathcal{M}_b(\omega\times I\times\mathcal{Y})$ be the family of unfolding measures associated with $\{\mu_\varepsilon\}$. Then

$$\tilde\lambda_\varepsilon\xrightarrow{*}\mu_0\quad\textit{weakly* in }\mathcal{M}_b(\omega\times I\times\mathcal{Y}).$$

To analyze the sequences of symmetrized scaled gradients of BD function in the context of unfolding, we will need to consider the following auxiliary spaces

$$BD_{\frac{h}{\varepsilon}}(I\times\mathcal{Y}) := \Big\{u \in L^1(I\times\mathcal{Y};\mathbb{R}^3) : \widetilde{E}_{\frac{h}{\varepsilon}} u \in \mathcal{M}_b(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})\Big\},$$
$$BD_{\frac{h}{\varepsilon}}\left((0,1)^2\times I\right) := \Big\{u \in L^1\left((0,1)^2\times I;\mathbb{R}^3\right) : E_{\frac{h}{\varepsilon}} u \in \mathcal{M}_b\left((0,1)^2\times I;\mathbb{M}^{3\times3}_{\mathrm{sym}}\right)\Big\},$$

where $\widetilde{E}_{\frac{h}{\varepsilon}}$ and $E_{\frac{h}{\varepsilon}}$ denote the distributional symmetrized scaled gradients, cf. (2.1). Similarly as in Remark 4.4, scaling in the the first two components shows that these auxiliary spaces are equivalent to the usual BD space on the appropriate domain.

Proposition 4.17. *Let $\omega \subseteq \mathbb{R}^2$ be an open set and let $\mathcal{B} \subseteq \mathcal{Y}$ be an open set with Lipschitz boundary. Let $\gamma_0 \in (0,1]$ and let h, $\varepsilon > 0$ be such that*

$$\gamma_0 \leq \frac{h}{\varepsilon} \leq \frac{1}{\gamma_0}.$$

If $u_\varepsilon \in BD(\omega\times I)$, the unfolding measure associated with $\Lambda_h E u_\varepsilon\lfloor(\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon)\times I$ is given by

$$\sum_{i\in I_\varepsilon(\omega)} \left(\mathcal{L}^2_{x'}\lfloor Q^i_\varepsilon\right)\otimes \widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}\lfloor I\times(\mathcal{B}\setminus\mathcal{C}), \tag{4.30}$$

where $\hat{u}^i_{h,\varepsilon} \in BD_{\frac{h}{\varepsilon}}(I\times\mathcal{Y})$ is such that

$$\int_{I\times\mathcal{B}} |\hat{u}^i_{h,\varepsilon}|\,dx_3\,dy + \int_{I\times\partial\mathcal{B}} |\hat{u}^i_{h,\varepsilon}|\,d\mathcal{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\left(I\times(\mathcal{B}\cap\mathcal{C})\right) \leq \frac{C}{\varepsilon^2}|\Lambda_h E u_\varepsilon|\left(int(Q^i_\varepsilon)\times I\right), \tag{4.31}$$

for some constant C independent of i, h and ε.

Proof. Since $\mathcal{B}_\varepsilon$ has Lipschitz boundary, $u_\varepsilon \mathbb{1}_{\mathcal{B}_\varepsilon\times I} \in BD_{loc}(\omega\times I)$ with

$$Eu_\varepsilon\lfloor\mathcal{B}_\varepsilon\times I = E\left(u_\varepsilon\mathbb{1}_{\mathcal{B}_\varepsilon\times I}\right) + \left[u_\varepsilon\lfloor\partial\mathcal{B}_\varepsilon\times I\odot\nu\right]\mathcal{H}^2\lfloor\partial\mathcal{B}_\varepsilon\times I,$$

where $u_\varepsilon\lfloor\partial\mathcal{B}_\varepsilon\times I$ denotes the trace of $u_\varepsilon\mathbb{1}_{\mathcal{B}_\varepsilon\times I}$ on $\partial\mathcal{B}_\varepsilon\times I$, while ν is the exterior normal to $\partial\mathcal{B}_\varepsilon\times I$. We note that the third component of ν is equal to zero.

Remark that $\mathcal{C}_\varepsilon = \left(\cup_i\partial Q^i_\varepsilon\right)\cap\omega$. Accordingly, for $i\in I_\varepsilon(\omega)$ and $\psi\in C^1(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})$,

$$\begin{aligned}
&\int_{Q^i_\varepsilon\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : d\left(\Lambda_h Eu_\varepsilon\lfloor(\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon)\times I\right)(x) = \int_{int(Q^i_\varepsilon)\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : d\left(\Lambda_h Eu_\varepsilon\lfloor\mathcal{B}_\varepsilon\times I\right)(x)\\
&= \int_{int(Q^i_\varepsilon)\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : d\Lambda_h E\left(u_\varepsilon\mathbb{1}_{\mathcal{B}_\varepsilon\times I}\right)(x)\\
&\quad + \int_{int(Q^i_\varepsilon)\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : \Lambda_h\left[u_\varepsilon\lfloor\partial\mathcal{B}_\varepsilon\times I\odot\nu\right]\,d\mathcal{H}^2\lfloor\partial\mathcal{B}_\varepsilon\times I(x).
\end{aligned}$$

We set $v^i_{h,\varepsilon}(x) := \mathrm{diag}\left(1,1,\frac{1}{h}\right)\,u_\varepsilon(\varepsilon i+\varepsilon x',x_3)$ for $x\in(0,1)^2\times I$. Then $v^i_{h,\varepsilon}\in BD_{\frac{h}{\varepsilon}}\left((0,1)^2\times I\right)$, and $E_{\frac{h}{\varepsilon}}v^i_{h,\varepsilon}(x) = \varepsilon\Lambda_h Eu_\varepsilon(\varepsilon i+\varepsilon x',x_3)$. Performing a change of variables, we find

$$\begin{aligned}
&\int_{Q^i_\varepsilon\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : d\left(\Lambda_h Eu_\varepsilon\lfloor(\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon)\times I\right)(x)\\
&= \varepsilon\int_{(0,1)^2\times I}\psi\left(x_3,x'\right) : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon}\mathbb{1}_{\mathcal{I}(\mathcal{B})\times I}\right)(x)\\
&\quad + \varepsilon\int_{(0,1)^2\times I}\psi\left(x_3,x'\right) : \Lambda_h\left[\mathrm{diag}(1,1,h)\,v^i_{h,\varepsilon}\lfloor\mathcal{I}(\partial\mathcal{B})\times I\odot\nu\right]\,d\mathcal{H}^2(x)\\
&= \varepsilon\int_{(0,1)^2\times I}\psi\left(x_3,x'\right) : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon}\mathbb{1}_{\mathcal{I}(\mathcal{B})\times I}\right)(x) + \varepsilon\int_{(0,1)^2\times I}\psi\left(x_3,x'\right) : \left[v^i_{h,\varepsilon}\lfloor\mathcal{I}(\partial\mathcal{B})\times I\odot\nu\right]\,d\mathcal{H}^2(x).
\end{aligned}$$

Notice that we can assume that

$$\int_{(0,1)^2\times I} |v^i_{h,\varepsilon}|dx+\int_{\partial(0,1)^2\times I} |v^i_{h,\varepsilon}\lfloor\partial(0,1)^2\times I|\,d\mathcal{H}^2 \le C|E_{\frac{h}{\varepsilon}} v^i_{h,\varepsilon}|\left((0,1)^2\times I\right) = \frac{C}{\varepsilon}|\Lambda_h Eu_\varepsilon|\left(int(Q^i_\varepsilon)\times I\right),$$

for some constant C independent of i, h and ε. This can be achieved by using Remark 4.7 since subtracting a rigid deformation to u_ε on $Q^i_\varepsilon \times I$ corresponds to subtracting an element of the kernel of $E_{\frac{h}{\varepsilon}}$ to $v^i_{h,\varepsilon}$, which does not modify the calculations done thus far. Hence, by the trace theorem and Poincaré-Korn's inequality in $BD\left((0,1)^2\times I\right)$, we get the desired inequality.

Defining $\hat{u}^i_{h,\varepsilon}(x_3,y) := \frac{1}{\varepsilon} v^i_{h,\varepsilon}\left(\mathcal{I}(y),x_3\right)$, we obtain

$$\begin{aligned}
|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times\mathcal{Y}) &\le \int_{I\times\mathcal{C}} |\hat{u}^i_{h,\varepsilon}\lfloor I\times\mathcal{C}|\,d\mathcal{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times(\mathcal{Y}\setminus\mathcal{C}))\\
&= \frac{1}{\varepsilon}\int_{\partial(0,1)^2\times I} |v^i_{h,\varepsilon}\lfloor\partial(0,1)^2\times I|\,d\mathcal{H}^2 + \frac{1}{\varepsilon}|E_{\frac{h}{\varepsilon}}v^i_{h,\varepsilon}|\left((0,1)^2\times I\right)\\
&\le \frac{C+1}{\varepsilon}|E_{\frac{h}{\varepsilon}}v^i_{h,\varepsilon}|\left((0,1)^2\times I\right) = \frac{C+1}{\varepsilon^2}|\Lambda_h Eu_\varepsilon|\left(int(Q^i_\varepsilon)\times I\right).
\end{aligned}$$

Furthermore,

$$\varepsilon\int_{(0,1)^2\times I}\psi : dE_{\frac{h}{\varepsilon}}\left(v^i_{h,\varepsilon}\mathbb{1}_{\mathcal{I}(\mathcal{B})\times I}\right) = \varepsilon^2\int_{I\times(\mathcal{Y}\setminus\mathcal{C})}\psi : d\widetilde{E}_{\frac{h}{\varepsilon}}\left(\hat{u}^i_{h,\varepsilon}\mathbb{1}_{\mathcal{B}\times I}\right)$$

and

$$\varepsilon\int_{(0,1)^2\times I}\psi : \left[v^i_{h,\varepsilon}\lfloor\mathcal{I}(\partial\mathcal{B})\times I\odot\nu\right]\,d\mathcal{H}^2 = \varepsilon^2\int_{I\times(\mathcal{Y}\setminus\mathcal{C})}\psi : \left[\hat{u}^i_{h,\varepsilon}\lfloor I\times(\partial\mathcal{B}\setminus\mathcal{C})\odot\nu\right]\,d\mathcal{H}^2.$$

So we have

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{Q^i_\varepsilon\times I}\psi\left(x_3,\frac{x'}{\varepsilon}\right) : d\left(\Lambda_h Eu_\varepsilon\lfloor(\mathcal{B}_\varepsilon\setminus\mathcal{C}_\varepsilon)\times I\right)(x)\\
&= \int_{I\times(\mathcal{Y}\setminus\mathcal{C})}\psi(x_3,y) : d\widetilde{E}_{\frac{h}{\varepsilon}}\left(\hat{u}^i_{h,\varepsilon}\mathbb{1}_{\mathcal{B}\times I}\right)(y,x_3)\\
&\quad + \int_{I\times(\mathcal{Y}\setminus\mathcal{C})}\psi(x_3,y) : \left[\hat{u}^i_{h,\varepsilon}\lfloor I\times(\partial\mathcal{B}\setminus\mathcal{C})\odot\nu\right]\,d\mathcal{H}^2(x_3,y)\\
&= \int_{I\times\mathcal{Y}}\psi(x_3,y) : d\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}\lfloor I\times(\mathcal{B}\setminus\mathcal{C})(x_3,y),
\end{aligned}$$

from which (4.30) follows. It remains to prove (4.31). Again, up to adding an affine transformation to $\hat{u}^i_{h,\varepsilon}$ (cf. Remark 4.7) on $I\times\mathcal{B}$, we can assume

$$\begin{aligned}
&\int_{I\times\mathcal{B}}|\hat{u}^i_{h,\varepsilon}|\,dx_3dy + \int_{I\times\partial\mathcal{B}}|\hat{u}^i_{h,\varepsilon}|\,d\mathcal{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times(\mathcal{B}\cap\mathcal{C}))\\
&\le C|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times\mathcal{B}) + |\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times(\mathcal{B}\cap\mathcal{C})) \le C|\widetilde{E}_{\frac{h}{\varepsilon}}\hat{u}^i_{h,\varepsilon}|\,(I\times\mathcal{Y})\\
&\le \frac{C}{\varepsilon^2}|\Lambda_h Eu_\varepsilon|\left(int(Q^i_\varepsilon)\times I\right).
\end{aligned}$$

This concludes the proof of the theorem. □

As a consequence of Proposition 4.17, we deduce the following lemma, which in turn will be used in the proof of the lower semicontinuity of $\mathcal{H}^{hom}$ in Section 5.5.

Lemma 4.18. *Let $\mathcal{B}\subseteq\mathcal{Y}$ be an open set with Lipschitz boundary, such that $\partial\mathcal{B}\setminus\mathcal{T}$ is a C^1-hypersurface, for some compact set $\mathcal{T}$ with $\mathcal{H}^1(\mathcal{T})=0$. Additionally, assume that $\partial\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{T}$. Let $v^h\in BD(\Omega)$ be such that*

$$v^h \xrightharpoonup{*} v \quad \textit{weakly* in } BD(\Omega)$$

and

$$\Lambda_h Ev^h\lfloor\Omega\cap(\mathcal{B}_{\varepsilon_h}\times I) \xrightharpoonup{2-*} \pi \quad \textit{two-scale weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}).$$

Then π is supported in $\Omega \times \bar{\mathcal{B}}$ and

$$\pi\lfloor \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}) = a(x,y) \odot \nu(y)\,\zeta, \tag{4.32}$$

where $\zeta \in \mathcal{M}_b^+(\Omega \times (\partial\mathcal{B} \setminus \mathcal{T}))$, $a : \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}) \to \mathbb{R}^3$ is a Borel map, and ν is the exterior normal to $\partial\mathcal{B}$.

Proof. Denote by $\tilde{\pi} \in \mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ the two-scale weak* limit (up to a subsequence) of

$$\Lambda_h E v^h \lfloor \Omega \cap ((\mathcal{B}_{\varepsilon_h} \setminus \mathcal{C}_{\varepsilon_h}) \times I) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

Then it is enough to prove the analogue of (4.32) for $\tilde{\pi}$. Indeed, the two-scale weak* limit (up to a subsequence) of

$$\Lambda_h E v^h \lfloor \Omega \cap ((\mathcal{B}_{\varepsilon_h} \cap \mathcal{C}_{\varepsilon_h}) \times I) \in \mathcal{M}_b(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$$

is supported on $\Omega \times \overline{\mathcal{B} \cap \mathcal{C}}$. Since by assumption $\partial\mathcal{B} \cap \mathcal{C} \subseteq \mathcal{T}$, we have that $\partial\mathcal{B} \setminus \mathcal{T}$ and $\overline{\mathcal{B} \cap \mathcal{C}}$ are disjoint sets, which implies

$$\pi\lfloor \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}) = \tilde{\pi}\lfloor \Omega \times (\partial\mathcal{B} \setminus \mathcal{T}).$$

By Theorem 4.17, the unfolding measure associated with $\Lambda_h E v^h \lfloor (\mathcal{B}_{\varepsilon_h} \setminus \mathcal{C}_{\varepsilon_h}) \times I$ is given by

$$\sum_{i \in I_{\varepsilon_h}(\omega)} \left(\mathcal{L}^2_{x'} \lfloor Q^i_{\varepsilon_h}\right) \otimes \widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \lfloor I \times (\mathcal{B} \setminus \mathcal{C}), \tag{4.33}$$

where $\hat{v}^i_{\varepsilon_h} \in BD(I \times \mathcal{Y})$ is such that

$$\int_{I \times \mathcal{B}} |\hat{v}^i_{\varepsilon_h}|\, dx_3 dy + \int_{I \times \partial\mathcal{B}} |\hat{v}^i_{\varepsilon_h}|\, d\mathcal{H}^2 + |\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h}| \left(I \times (\mathcal{B} \cap \mathcal{C})\right) \leq \frac{C}{{\varepsilon_h}^2} |\Lambda_h E v^h| \left(int(Q^i_{\varepsilon_h}) \times I\right). \tag{4.34}$$

Further, by Theorem 4.16, the family of associated measures in (4.33) converge weakly* to $\tilde{\pi}$ in $\mathcal{M}_b(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Then, for every $\chi \in C_c^\infty(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ with $\widetilde{\operatorname{div}}_\gamma \chi(x,y) = 0$, we get

$$\begin{aligned}
&\int_{\Omega \times \mathcal{Y}} \chi(x,y) : d\tilde{\pi}(x,y) \\
&= \lim_h \int_{\Omega \times \mathcal{Y}} \chi(x,y) : d\left(\sum_{i \in I_{\varepsilon_h}(\omega)} \left(\mathcal{L}^2_{x'} \lfloor Q^i_{\varepsilon_h}\right) \otimes \widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \lfloor I \times (\mathcal{B} \setminus \mathcal{C}) \right) \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \left(\int_{I \times (\mathcal{B} \setminus \mathcal{C})} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \right) dx' \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \left(\int_{I \times \mathcal{B}} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} - \int_{I \times (\mathcal{B} \cap \mathcal{C})} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \right) dx'.
\end{aligned}$$

By the integration by parts formula for BD functions over $I \times \mathcal{B}$ we have

$$\begin{aligned}
&\int_{\Omega \times \mathcal{Y}} \chi(x,y) : d\tilde{\pi}(x,y) \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \Bigg(- \int_{I \times \mathcal{B}} \widetilde{\operatorname{div}}_{\frac{h}{\varepsilon_h}} \chi(x,y) \cdot \hat{v}^i_{\varepsilon_h}(x_3,y)\, dx_3 dy + \int_{I \times \partial\mathcal{B}} \chi(x,y) : \left[\hat{v}^i_{\varepsilon_h}(x_3,y) \odot \nu\right] d\mathcal{H}^2(x_3,y) \\
&\qquad - \int_{I \times (\mathcal{B} \cap \mathcal{C})} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \Bigg) dx' \\
&= \lim_h \sum_{i \in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \Bigg(- \left(\frac{\varepsilon_h}{h} - \frac{1}{\gamma}\right) \int_{I \times \mathcal{B}} \partial_{x_3} \chi(x,y) \cdot \hat{v}^i_{\varepsilon_h}(y,x_3)\, dx_3 dy \\
&\qquad + \int_{I \times \partial\mathcal{B}} \chi(x,y) : \left[\hat{v}^i_{\varepsilon_h}(y,x_3) \odot \nu\right] d\mathcal{H}^2(y,x_3) - \int_{(\mathcal{B} \cap \mathcal{C}) \times I} \chi(x,y) : d\widetilde{E}_{\frac{h}{\varepsilon_h}} \hat{v}^i_{\varepsilon_h} \Bigg) dx'.
\end{aligned}$$

Owing to (4.34), we conclude that the the sum

$$\sum_{i\in I_{\varepsilon_h}(\omega)} \int_{Q^i_{\varepsilon_h}} \int_{I\times\mathcal{B}} \partial_{x_3}\chi(x,y)\cdot \hat{v}^i_{\varepsilon_h}(y,x_3)\,dx_3dy$$

is finite. Further, in view of (4.34) we can rewrite the above limit as

$$\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\tilde{\pi}(x,y) = \lim_h \left(\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\lambda^h_1(x,y) + \int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\lambda^h_2(x,y) \right), \tag{4.35}$$

with λ^h_1, $\lambda^h_2 \in \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$, such that (up to a subsequence)

$$\lambda^h_1 \xrightharpoonup{*} \lambda_1 \ \text{ and } \ \lambda^h_2 \xrightharpoonup{*} \lambda_2 \quad \text{weakly* in } \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$$

for suitable $\lambda_1,\lambda_2 \in \mathcal{M}_b(\Omega\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$. Then, we have $\text{supp}(\lambda_1) \subseteq \Omega\times\partial\mathcal{B}$ and $\text{supp}(\lambda_2) \subseteq \Omega\times(\overline{\mathcal{B}\cap\mathcal{C}})$.

By the density argument described in Lemma 4.5, we conclude that (4.35) holds for every $\chi \in C_0(\Omega\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$ with $\widetilde{\text{div}}_\gamma\chi = 0$. The definition of λ_1 and λ_2 then yields

$$\int_{\Omega\times\mathcal{Y}} \chi(x,y) : d\left(\tilde{\pi}-\lambda_1-\lambda_2\right)(x,y) = 0.$$

Thus, from Proposition 4.9 we conclude that there exists $\mu \in \mathcal{X}_\gamma(\omega)$ such that

$$\tilde{\pi}-\lambda_1-\lambda_2 = \widetilde{E}_\gamma\mu.$$

Recalling the assumption that $\partial\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{T}$ and using the same argument as above, we obtain

$$\tilde{\pi}\lfloor\Omega\times(\partial\mathcal{B}\setminus\mathcal{T}) = \lambda_1\lfloor\Omega\times(\partial\mathcal{B}\setminus\mathcal{T}) + \widetilde{E}_\gamma\mu\lfloor\Omega\times(\partial\mathcal{B}\setminus\mathcal{T})$$

In view of Proposition 4.10 and recalling the assumption that $\partial\mathcal{B}\setminus\mathcal{T}$ is a C^1-hypersurface, we are left to prove the analogue of (4.32) for λ_1.

We consider

$$\hat{v}^h(x,y) = \sum_{i\in I_{\varepsilon_h}(\omega)} \mathbb{1}_{Q^i_{\varepsilon_h}}(x')\,\hat{v}^i_{\varepsilon_h}(x_3,y),$$

so that $\lambda^h_1(x,y) = \left[\hat{v}^h(x_3,y)\odot\nu\right]\mathcal{L}^2_{x'}\otimes(\mathcal{H}^2_{x_3,y}\lfloor I\times\partial\mathcal{B})$. Then $\{\hat{v}^h\}$ is bounded in $L^1(\Omega\times\partial\mathcal{B};\mathbb{R}^3)$ by (4.34). Up to a subsequence,

$$\hat{v}^h\,\mathcal{L}^2_{x'}\otimes(\mathcal{H}^2_{x_3,y}\lfloor I\times\partial\mathcal{B}) \xrightharpoonup{*} \eta \quad \text{weakly* in } \mathcal{M}_b(\Omega\times\partial\mathcal{B};\mathbb{R}^3)$$

for a suitable $\eta\in\mathcal{M}_b(\Omega\times\partial\mathcal{B};\mathbb{R}^3)$. Since ν is continuous on $\partial\mathcal{B}\setminus\mathcal{T}$, we infer

$$\lambda_1\lfloor\Omega\times(\partial\mathcal{B}\setminus\mathcal{T}) = \frac{\eta}{|\eta|}(x,y)\odot\nu(y)\,|\eta|\lfloor\Omega\times(\partial\mathcal{B}\setminus\mathcal{T}),$$

which concludes the proof, since $\frac{\eta}{|\eta|}$ is a Borel function. □

5. Two-scale statics and duality

In this section we define a notion of stress-strain duality and analyze the two-scale behavior of our functionals.

5.1. Stress-plastic strain duality on the cell.

Definition 5.1. *Let $\gamma\in(0,+\infty)$. The set $\mathcal{K}_\gamma$ of admissible stresses is defined as the set of all elements $\Sigma\in L^2(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$ satisfying:*

(i) $\widetilde{\text{div}}_\gamma\Sigma = 0$ *in* $I\times\mathcal{Y}$,
(ii) $\Sigma\,\vec{e}_3 = 0$ *on* $\partial I\times\mathcal{Y}$,
(iii) $\Sigma_{\text{dev}}(x_3,y)\in K(y)$ *for* $\mathcal{L}^1_{x_3}\otimes\mathcal{L}^2_y$*-a.e.* $(x_3,y)\in I\times\mathcal{Y}$.

Since condition (iii) implies that $\Sigma_{\text{dev}}\in L^\infty(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$, for every $\Sigma\in\mathcal{K}_\gamma$ we deduce from Proposition 2.3 that $\Sigma\in L^p(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})$ for every $1\le p<\infty$.

Definition 5.2. *Let $\gamma \in (0, +\infty)$. The family $\mathcal{A}_\gamma$ of admissible configurations is given by the set of triplets*

$$u \in BD_\gamma(I \times \mathcal{Y}), \qquad E \in L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}), \qquad P \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}),$$

such that

$$\widetilde{E}_\gamma u = E\,\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P \quad \text{ in } I \times \mathcal{Y}.$$

Definition 5.3. *Let $\Sigma \in \mathcal{K}_\gamma$ and let $(u, E, P) \in \mathcal{A}_\gamma$. We define the distribution $[\Sigma_{\mathrm{dev}} : P]$ on $\mathbb{R} \times \mathcal{Y}$ by*

$$[\Sigma_{\mathrm{dev}} : P](\varphi) := -\int_{I\times\mathcal{Y}} \varphi\,\Sigma : E\,dx_3dy - \int_{I\times\mathcal{Y}} \Sigma : \big(u \odot \widetilde{\nabla}_\gamma\varphi\big)\,dx_3dy, \tag{5.1}$$

for every $\varphi \in C_c^\infty(\mathbb{R} \times \mathcal{Y})$.

Remark 5.4. *Note that the second integral in (5.1) is well defined since $BD(I \times \mathcal{Y})$ is embedded into $L^{3/2}(I \times \mathcal{Y}; \mathbb{R}^3)$. Moreover, the definition of $[\Sigma_{\mathrm{dev}} : P]$ is independent of the choice of (u, E), so (5.1) defines a meaningful distribution on $\mathbb{R} \times \mathcal{Y}$.*

The following results can be established from the proofs of [24, Theorem 6.2] and [24, Proposition 3.9] respectively, by treating the relative boundary of the "Dirichlet" part as empty, the "Neumann" part as $\partial I \times \mathcal{Y}$, and considering approximating sequences which must be periodic in $\mathcal{Y}$.

Proposition 5.5. *Let $\Sigma \in \mathcal{K}_\gamma$ and $(u, E, P) \in \mathcal{A}_\gamma$. Then $[\Sigma_{\mathrm{dev}} : P]$ can be extended to a bounded Radon measure on $\mathbb{R} \times \mathcal{Y}$, whose variation satisfies*

$$|[\Sigma_{\mathrm{dev}} : P]| \le \|\Sigma_{\mathrm{dev}}\|_{L^\infty(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}})}|P| \quad \text{ in } \mathcal{M}_b(\mathbb{R} \times \mathcal{Y}).$$

Proposition 5.6. *Let $\Sigma \in \mathcal{K}_\gamma$ and $(u, E, P) \in \mathcal{A}_\gamma$. If $\mathcal{Y}$ is a geometrically admissible multi-phase torus, then*

$$H\left(y, \frac{dP}{d|P|}\right)|P| \ge [\Sigma_{\mathrm{dev}} : P] \quad \text{ in } \mathcal{M}_b(I \times \mathcal{Y}).$$

5.2. **Disintegration of admissible configurations.** Let $\widetilde{\omega} \subseteq \mathbb{R}^2$ be an open and bounded set such that $\omega \subset \widetilde{\omega}$ and $\widetilde{\omega} \cap \partial\omega = \gamma_D$. We also denote by $\widetilde{\Omega} = \widetilde{\omega} \times I$ the associated reference domain.

In order to make sense of the duality between the two-scale limits of stresses and plastic strains, we will need to disintegrate the two-scale limits of the kinematically admissible fields in such a way to obtain elements of $\mathcal{A}_\gamma$, for $\gamma \in (0, +\infty)$.

Definition 5.7. *Let $w \in H^1(\widetilde{\Omega}; \mathbb{R}^3) \cap KL(\widetilde{\Omega})$. We define the class $\mathcal{A}^{hom}_\gamma(w)$ of admissible two-scale configurations relative to the boundary datum w as the set of triplets (u, E, P) with*

$$u \in KL(\widetilde{\Omega}), \qquad E \in L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}), \qquad P \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}),$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \qquad \text{ on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y},$$

and also such that there exists $\mu \in \mathcal{X}_\gamma(\widetilde{\omega})$ with

$$Eu \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma\mu = E\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P \qquad \text{ in } \widetilde{\Omega} \times \mathcal{Y}. \tag{5.2}$$

Lemma 5.8. *Let $(u, E, P) \in \mathcal{A}^{hom}_\gamma(w)$ with the associated $\mu \in \mathcal{X}_\gamma(\widetilde{\omega})$, and let $\bar{u} \in BD(\widetilde{\omega})$ and $u_3 \in BH(\widetilde{\omega})$ be the Kirchhoff-Love components of u. Set*

$$\eta := \mathcal{L}^2_{x'} + (proj_\#|P|)^s \in \mathcal{M}^+_b(\widetilde{\omega}).$$

Then the following disintegrations hold true:

$$Eu \otimes \mathcal{L}^2_y = \begin{pmatrix} A_1(x') + x_3A_2(x') & 0 \\ 0 & 0 \end{pmatrix} \eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y, \tag{5.3}$$

$$E\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y = C(x')E(x, y)\,\eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y \tag{5.4}$$

$$P = \eta \overset{gen.}{\otimes} P_{x'}. \tag{5.5}$$

Above, $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\rm sym}$ and $C : \widetilde{\omega} \to [0, +\infty]$ are Radon-Nikodym derivatives of $E\bar{u}$, $-D^2u_3$ and $\mathcal{L}^2_{x'}$ with respect to η, $E(x, y)$ is a Borel representative of E, and $P_{x'} \in \mathcal{M}_b(I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\rm dev})$ for η-a.e. $x' \in \widetilde{\omega}$.

Furthermore, we can choose a Borel map $(x', x_3, y) \in \widetilde{\Omega} \times \mathcal{Y} \mapsto \mu_{x'}(x_3, y) \in \mathbb{R}^3$ such that, for η-a.e. $x' \in \widetilde{\omega}$,

$$\mu = \mu_{x'}(x_3, y)\, \eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y, \quad \widetilde{E}_\gamma \mu = \eta \overset{gen.}{\otimes} \widetilde{E}_\gamma \mu_{x'}, \tag{5.6}$$

where $\mu_{x'} \in BD_\gamma(I \times \mathcal{Y})$, $\int_{I\times\mathcal{Y}} \mu_{x'}(x_3, y)\, dx_3 dy = 0$.

Proof. The proof is analogous to [23, Lemma 5.4]. The only difference is the statement and argument for the disintregration of $Eu \otimes \mathcal{L}^2_y$, that we detail below.

First we note that $proj_\# \left(\widetilde{E}_\gamma \mu\right)_{\alpha\beta} = proj_\# \left(E_y \mu\right)_{\alpha\beta} = 0$ for $\alpha, \beta = 1, 2$. Then, from (5.2) we get

$$\begin{aligned}(E\bar{u})_{\alpha\beta} = proj_\# \left(Eu \otimes \mathcal{L}^2_y\right)_{\alpha\beta} &= \left(\int_{I\times\mathcal{Y}} E_{\alpha\beta}(x, y)\, dx_3 dy\right) \mathcal{L}^2_{x'} + proj_\#(P)_{\alpha\beta} \\ &\leq e^{(1)}_{\alpha\beta}(x')\, \mathcal{L}^2_{x'} + (proj_\# |P|)^s_{\alpha\beta},\end{aligned}$$

where we set $e^{(1)}(x') := \int_{I\times\mathcal{Y}} |E(x, y)|\, dx_3 dy + (proj_\#|P|)^a \in L^2(\widetilde{\omega}; \mathbb{M}^{3\times 3}_{\rm sym})$. Similarly, after multipliying equation (5.2) by x_3, we have that

$$\begin{aligned}\left(-D^2u_3\right)_{\alpha\beta} = \frac{1}{12}\, proj_\# \left(x_3 Eu \otimes \mathcal{L}^2_y\right)_{\alpha\beta} &= \frac{1}{12}\left(\int_{I\times\mathcal{Y}} x_3 E_{\alpha\beta}(x, y)\, dx_3 dy\right) \mathcal{L}^2_{x'} + \frac{1}{12}\, proj_\#(x_3 P) \\ &\leq e^{(2)}_{\alpha\beta}(x')\, \mathcal{L}^2_{x'} + \frac{1}{12}\, (proj_\# |x_3 P|)^s_{\alpha\beta},\end{aligned}$$

where we set $e^{(2)}(x') := \frac{1}{12} \int_{I\times\mathcal{Y}} |x_3 E(x, y)|\, dx_3 dy + \frac{1}{12}\, (proj_\#|x_3 P|)^a \in L^2(\widetilde{\omega}; \mathbb{M}^{3\times 3}_{\rm sym})$. Consequently, the measures $E\bar{u}$ and $-D^2u_3$ are absolutely continuous with respect to η, so we find

$$\begin{aligned}E\bar{u} \otimes \mathcal{L}^2_y &= A_1(x')\, \eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y, \\ -D^2u_3 \otimes \mathcal{L}^2_y &= A_2(x')\, \eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y,\end{aligned}$$

for suitable $A_1, A_2 : \widetilde{\omega} \to \mathbb{M}^{2\times 2}_{\rm sym}$ such that (5.3) hold true. ☐

Remark 5.9. *From the above disintegration, we have that, for η-a.e. $x' \in \widetilde{\omega}$,*

$$\widetilde{E}_\gamma \mu_{x'} = \left[C(x')E(x, y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right] \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y + P_{x'} \quad \text{in } I \times \mathcal{Y}.$$

Thus, the triple

$$\left(\mu_{x'}, \left[C(x')E(x, y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right], P_{x'}\right)$$

is an element of $\mathcal{A}_\gamma$.

5.3. **Admissible stress configurations and approximations.** For every $e^h \in L^2(\Omega; \mathbb{M}^{3\times 3}_{\rm sym})$ we define $\sigma^h(x) := \mathbb{C}\left(\frac{x'}{\varepsilon_h}\right) \Lambda_h e^h(x)$. Then, in view of [24, Theorem 3.6], we introduce the set

$$\begin{aligned}\mathcal{K}_h = \Bigg\{ &\sigma^h \in L^2(\Omega; \mathbb{M}^{3\times 3}_{\rm sym}) : {\rm div}_h \sigma^h = 0 \text{ in } \Omega,\ \sigma^h \nu = 0 \text{ in } \partial\Omega \setminus \overline{\Gamma}_D, \\ &\sigma^h_{\rm dev}(x', x_3) \in K\left(\frac{x'}{\varepsilon_h}\right) \text{ for a.e. } x' \in \omega,\ x_3 \in I \Bigg\},\end{aligned}$$

which is the set of stresses for the rescaled h problems. Next we introduce the set of two-scale limiting stresses.

Definition 5.10. *The set $\mathcal{K}^{hom}_\gamma$ is the set of all elements $\Sigma \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\rm sym})$ satisfying:*

(i) $\widetilde{\rm div}_\gamma \Sigma(x', \cdot) = 0$ *in* $I \times \mathcal{Y}$ *for a.e.* $x' \in \omega$,
(ii) $\Sigma(x', \cdot)\, \vec{e}_3 = 0$ *on* $\partial I \times \mathcal{Y}$ *for a.e.* $x' \in \omega$,

(iii) $\Sigma_{\rm dev}(x,y) \in K(y)$ *for* $\mathcal{L}^3_x \otimes \mathcal{L}^2_y$*-a.e.* $(x,y) \in \Omega \times \mathcal{Y}$,
(iv) $\sigma_{i3}(x) = 0$ *for* $i = 1,2,3$,
(v) $\mathrm{div}_{x'}\bar{\sigma} = 0$ *in* ω,
(vi) $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = 0$ *in* ω,

where $\sigma := \int_{\mathcal{Y}} \Sigma(\cdot,y)\,dy$, *and* $\bar{\sigma}$, $\hat{\sigma} \in L^2(\omega;\mathbb{M}^{2\times 2}_{\rm sym})$ *are the zero-th and first order moments of the* 2×2 *minor of* σ.

Remark 5.11. *Notice that as a consequence of the properties (iii) and (iv) in the Definition 5.10 we can actually conclude that* $\bar{\sigma}$, $\hat{\sigma} \in L^\infty(\omega;\mathbb{M}^{2\times 2}_{\rm sym})$. *Namely, the uniform boundedness of sets* $K(y)$ *implies that the deviatoric part of the weak limit, i.e.* $\sigma_{\rm dev} = \sigma - \frac{1}{3}\operatorname{tr}\sigma I_{3\times 3}$, *is bounded in* $L^\infty(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$. *Thus we have that*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} \sigma_{11}+\sigma_{22} & 0 & 0 \\ 0 & \sigma_{11}+\sigma_{22} & 0 \\ 0 & 0 & \sigma_{11}+\sigma_{22} \end{pmatrix} \text{ is bounded in } L^\infty(\Omega;\mathbb{M}^{3\times 3}_{\rm sym}).$$

Hence, the components $\sigma_{\alpha\beta}$ *are all bounded in* $L^\infty(\Omega)$.

In the following proposition we show that the set $\mathcal{K}^{\rm hom}_\gamma$ characterizes weak two-scale limits of sequences of elastic stresses $\{\sigma^h\}$.

Proposition 5.12. *Let* $\{\sigma^h\}$ *be a bounded family in* $L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ *such that* $\sigma^h \in \mathcal{K}_h$ *for every* h, *and*

$$\sigma^h \xrightharpoonup{2} \Sigma \quad \text{two-scale weakly in } L^2(\Omega\times\mathcal{Y};\mathbb{M}^{3\times 3}_{\rm sym}).$$

Then $\Sigma \in \mathcal{K}^{hom}_\gamma$.

Proof. Consider a sequence $\{\sigma^h\} \subset L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$ such that $\sigma_h \in \mathcal{K}_h$ for every h, and assume that $\sigma^h \rightharpoonup \sigma$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm sym})$. We first establish the macroscopic properties (iv), (v), (vi). To obtain (iv), let $v \in C^\infty_c(\Omega;\mathbb{R}^3)$ and $V \in C^\infty(\overline{\Omega};\mathbb{R}^3)$ be defined by

$$V(x',x_3) := \int_{-\frac{1}{2}}^{x_3} v(x',\zeta)\,d\zeta.$$

From the condition $\mathrm{div}_h\sigma^h = 0$ in Ω, for every $\varphi \in H^1(\Omega;\mathbb{R}^3)$ with $\varphi = 0$ on Γ_D we have

$$\int_\Omega \sigma^h(x) : E_h\varphi(x)\,dx = 0. \tag{5.7}$$

Setting

$$\varphi(x) = \begin{pmatrix} 2h\,V_1(x) \\ 2h\,V_2(x) \\ h\,V_3(x) \end{pmatrix},$$

and passing to the limit as $h \to 0$, we find

$$\int_\Omega \sigma(x) : \begin{pmatrix} 0 & 0 & v_1(x) \\ 0 & 0 & v_2(x) \\ v_1(x) & v_2(x) & v_3(x) \end{pmatrix} dx = \int_\Omega \sigma(x) : \begin{pmatrix} 0 & 0 & \partial_{x_3}V_1(x) \\ 0 & 0 & \partial_{x_3}V_2(x) \\ \partial_{x_3}V_1(x) & \partial_{x_3}V_2(x) & \partial_{x_3}V_3(x) \end{pmatrix} dx = 0.$$

Consequently, from the arbitrariness of v, we infer that $\sigma_{i3} = 0$.

To obtain (iv) and (v) let $\bar{\varphi} \in C^\infty_c(\omega;\mathbb{R}^3)$ and choose the test function

$$\varphi(x) = \begin{pmatrix} \bar{\varphi}_1(x') - x_3\,\partial_{x_1}\bar{\varphi}_3(x') \\ \bar{\varphi}_2(x') - x_3\,\partial_{x_2}\bar{\varphi}_3(x') \\ \frac{1}{h}\,\bar{\varphi}_3(x') \end{pmatrix}.$$

We deduce from (5.7) that

$$\int_\Omega \sigma^h(x) : \begin{pmatrix} E\bar{\varphi}(x') - x_3 D^2\bar{\varphi}_3(x') & 0 \\ 0 & 0 \end{pmatrix} dx = 0.$$

Passing to the limit, we conclude that

$$\mathrm{div}_{x'}\bar{\sigma} = 0 \text{ in } \omega, \quad \text{and} \quad \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = 0 \text{ in } \omega.$$

Next we prove the microscopic properties (i), (ii) and (iii). Consider test functions $\varepsilon_h\,\phi\left(x, \frac{x'}{\varepsilon_h}\right)$, for $\phi \in C_c^\infty(\omega; C^\infty(\overline{I} \times \mathcal{Y}; \mathbb{R}^3))$ in (5.7). We first observe that the sequence

$$\nabla_h\left(\varepsilon_h\,\phi\left(x, \frac{x'}{\varepsilon_h}\right)\right) = \left[\,\varepsilon_h\,\nabla_{x'}\phi\left(x, \frac{x'}{\varepsilon_h}\right) + \nabla_y\phi\left(x, \frac{x'}{\varepsilon_h}\right)\;\middle|\; \frac{\varepsilon_h}{h}\,\partial_{x_3}\phi\left(x, \frac{x'}{\varepsilon_h}\right)\right]$$

converges strongly two-scale in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3})$. Hence, passing to the limit as $h \to 0$, we find

$$\int_{\Omega\times\mathcal{Y}} \Sigma(x,y) : \widetilde{E}_\gamma\phi\,(x,y)\;dxdy = 0.$$

Suppose now that $\phi\,(x,y) = \psi^{(1)}(x')\,\psi^{(2)}(x_3,y)$ for $\psi^{(1)} \in C_c^\infty(\omega)$ and $\psi^{(2)} \in C^\infty(\overline{I} \times \mathcal{Y}; \mathbb{R}^3)$. Then

$$\int_\omega \psi^{(1)}(x')\left(\int_{I\times\mathcal{Y}} \Sigma(x,y) : \widetilde{E}_\gamma\psi^{(2)}(x_3,y)\,dx_3dy\right)dx' = 0.$$

Thus, for a.e. $x' \in \omega$,

$$\begin{aligned}
0 &= \int_{I\times\mathcal{Y}} \Sigma(x,y) : \widetilde{E}_\gamma\psi^{(2)}(x_3,y)\,dx_3dy \\
&= -\int_{I\times\mathcal{Y}} \widetilde{\mathrm{div}}_\gamma\Sigma(x,y)\cdot\psi^{(2)}(x_3,y)\,dx_3dy + \int_{\partial(I\times\mathcal{Y})} \Sigma(x,y)\,\nu\cdot\psi^{(2)}(x_3,y)\,d\mathcal{H}^2(x_3,y) \\
&= -\int_{I\times\mathcal{Y}} \widetilde{\mathrm{div}}_\gamma\Sigma(x,y)\cdot\psi^{(2)}(x_3,y)\,dx_3dy + \int_{\partial I\times\mathcal{Y}} \Sigma(x,y)\,\vec{e}_3\cdot\psi^{(2)}(x_3,y)\,d\mathcal{H}^2(x_3,y),
\end{aligned}$$

from which we infer $\widetilde{\mathrm{div}}_\gamma\Sigma(x',\cdot) = 0$ in $I \times \mathcal{Y}$ and $\Sigma(x',\cdot)\,\vec{e}_3 = 0$ on $\partial I \times \mathcal{Y}$.

Finally, we define

$$\Sigma^h(x,y) = \sum_{i\in I_{\varepsilon_h}(\widetilde{\omega})} \mathbb{1}_{Q^i_{\varepsilon_h}}(x')\,\sigma^h(\varepsilon_h i + \varepsilon_h\mathcal{I}(y), x_3), \tag{5.8}$$

and consider the set

$$S = \{\Xi \in L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}) : \Xi_{\mathrm{dev}}(x,y) \in K(y) \;\text{ for } \mathcal{L}^3_x \otimes \mathcal{L}^2_y\text{-a.e. } (x,y) \in \Omega\times\mathcal{Y}\}.$$

The construction of Σ^h from $\sigma^h \in \mathcal{K}_h$ ensures that $\Sigma^h \in S$ and that $\Sigma^h \rightharpoonup \Sigma$ weakly in $L^2(\Omega\times\mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Since the compactness of $K(y)$ implies that S is is convex and weakly closed in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$, we have that $\Sigma \in S$, which concludes the proof. □

Conversely, under additional star-shapedness assumptions on ω, we now provide an approximation result for elements of $\mathcal{K}^{\mathrm{hom}}_\gamma$.

Lemma 5.13. *Let $\omega \subset \mathbb{R}^2$ be an open bounded set that is star-shaped with respect to one of its points and let $\Sigma \in \mathcal{K}^{hom}_\gamma$. Then, there exists a sequence $\Sigma_n \in L^2(\mathbb{R}^2 \times I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ such that the following holds:*

- *(a) $\Sigma_n \in C^\infty(\mathbb{R}^2; L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}}))$ and $\Sigma_n \to \Sigma$ strongly in $L^2(\omega \times I \times \mathcal{Y}; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$,*
- *(b) $\widetilde{\mathrm{div}}_\gamma\Sigma_n(x',\cdot) = 0$ on $I \times \mathcal{Y}$ for every $x' \in \mathbb{R}^2$,*
- *(c) $\Sigma_n(x',\cdot)\,\vec{e}_3 = 0$ on $\partial I \times \mathcal{Y}$ for every $x' \in \mathbb{R}^2$,*
- *(d) $(\Sigma_n(x,y))_{\mathrm{dev}} \in K(y)$ for every $x' \in \mathbb{R}^2$ and $\mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y$-a.e. $(x_3,y) \in I \times \mathcal{Y}$.*

Further, if we set $\sigma_n(x) := \int_\mathcal{Y} \Sigma_n(x,y)\,dy$, and $\bar{\sigma}_n$, $\hat{\sigma}_n \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ are the zero-th and first order moments of the 2×2 minor of σ_n, then:

- *(e) $\sigma_n \in C^\infty(\mathbb{R}^2 \times I; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$ and $\sigma_n \to \sigma$ strongly in $L^2(\omega \times I; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$,*
- *(f) $\mathrm{div}_{x'}\bar{\sigma}_n = 0$ in ω,*
- *(g) $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_n = 0$ in ω.*

Proof. After a translation we may assume that ω is star-shaped with respect to the origin.

Thus, in particular,

$$\omega \subseteq \alpha\omega, \quad \text{for all } \alpha \geq 1. \tag{5.9}$$

We extend Σ to $\mathbb{R}^2 \times I \times \mathcal{Y}$ by setting $\Sigma = 0$ outside $\Omega \times \mathcal{Y}$. Let ρ be the standard mollifier on $\mathbb{R}^2$ and define the planar dilation $d_n(x') = \left(\frac{n}{n+1}x'\right)$, for every $n \in \mathbb{N}$. Owing to (5.9), there exists a vanishing sequence $\epsilon_n > 0$ such that for every map $\varphi \in C_c^\infty(\omega;\mathbb{R}^2)$

$$\operatorname{supp}(\rho_{\epsilon_n} * \varphi) \subset\subset \tfrac{n+1}{n}\omega = d_n^{-1}(\omega) \text{ implies } \operatorname{supp}\left((\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}\right) \subset\subset \omega. \tag{5.10}$$

We then set

$$\Sigma_n(x', x_3, y) := \big(\,(\Sigma \circ d_n)\,(\cdot, x_3, y) * \rho_{\epsilon_n}\big)(x'). \tag{5.11}$$

With a slight abuse of notation, we have

$$\begin{aligned}
\sigma_n(x', x_3) &= \big(\,(\sigma \circ d_n)\,(\cdot, x_3) * \rho_{\epsilon_n}\big)(x'),\\
\bar{\sigma}_n(x') &= \big(\,(\bar{\sigma} \circ d_n) * \rho_{\epsilon_n}\big)(x'),\\
\hat{\sigma}_n(x') &= \big(\,(\hat{\sigma} \circ d_n) * \rho_{\epsilon_n}\big)(x').
\end{aligned}$$

Items (a) and (e) are immediate consequences of the above construction, while item (d) follows from Jensen's inequality since $K(y)$ is convex. Next, for $x' \in \mathbb{R}^2$

$$\widetilde{\operatorname{div}}_\gamma \Sigma_n(x', \cdot) = \widetilde{\operatorname{div}}_\gamma\,(\Sigma \circ d_n) * \rho_{\epsilon_n} = 0 \text{ in } I \times \mathcal{Y},$$

which proves item (b).

To prove item (f), we observe that, for every map $\varphi \in C_c^\infty(\omega;\mathbb{R}^2)$ there holds

$$\begin{aligned}
\langle \operatorname{div}_{x'}\bar{\sigma}_n, \varphi\rangle &= -\int_{\mathbb{R}^2} \bar{\sigma}_n : \nabla_{x'}\varphi\, dx' = -\int_{\mathbb{R}^2} (\bar{\sigma} \circ d_n) : (\rho_{\epsilon_n} * \nabla_{x'}\varphi)\, dx'\\
&= -\int_{\mathbb{R}^2} (\bar{\sigma} \circ d_n) : \nabla_{x'}(\rho_{\epsilon_n} * \varphi)\, dx' = -(\tfrac{n+1}{n})^2 \int_{\mathbb{R}^2} \bar{\sigma} : [\nabla_{x'}(\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}]\, dx'\\
&= -(\tfrac{n+1}{n}) \int_{\mathbb{R}^2} \bar{\sigma} : \nabla_{x'}[(\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}]\, dx' = (\tfrac{n+1}{n})\langle \operatorname{div}_{x'}\bar{\sigma}, (\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}\rangle = 0,
\end{aligned}$$

where in last equation we used that $\operatorname{div}_{x'}\bar{\sigma} = 0$ in ω and (5.10).

Similarly for item (g), for every map $\varphi \in C_c^\infty(\omega)$ we have

$$\begin{aligned}
\langle \operatorname{div}_{x'}\operatorname{div}_{x'}\hat{\sigma}_n, \varphi\rangle &= \int_{\mathbb{R}^2} \bar{\sigma}_n : \nabla^2_{x'}\varphi\, dx' = \int_{\mathbb{R}^2} (\hat{\sigma} \circ d_n) : (\rho_{\epsilon_n} * \nabla^2_{x'}\varphi)\, dx'\\
&= \int_{\mathbb{R}^2} (\hat{\sigma} \circ d_n) : \nabla^2_{x'}(\rho_{\epsilon_n} * \varphi)\, dx' = (\tfrac{n+1}{n})^2 \int_{\mathbb{R}^2} \hat{\sigma} : [\nabla^2_{x'}(\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}]\, dx'\\
&= \int_{\mathbb{R}^2} \hat{\sigma} : \nabla^2_{x'}[(\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}]\, dx' = \langle \operatorname{div}_{x'}\operatorname{div}_{x'}\hat{\sigma}, (\rho_{\epsilon_n} * \varphi) \circ d_n^{-1}\rangle = 0,
\end{aligned}$$

where in last equation we used that $\operatorname{div}_{x'}\operatorname{div}_{x'}\hat{\sigma} = 0$ in ω and (5.10). □

5.4. **The principle of maximum plastic work.** The aim of this subsection is to prove an inequality between two-scale dissipation and plastic work, which in turn will be essential to prove the global stability condition of two-scale quasistatic evolutions. The claim is given in Corollary 5.16 below.

The proof of the following proposition and consequently Theorem 5.15 relies on the approximation argument given in Lemma 5.13 and on two-scale duality, which can be established only for smooth stresses by disintegration and Definition 5.3, see also[23, Proposition 5.11]. The problem is that the measure η defined in Lemma 5.8 can concentrate on the points where the stress (which is only in L^2) is not well-defined. The difference with respect to [23, Proposition 5.11] is that one can rely only on the approximation given by Lemma 5.13 which is given for star-shaped domains. To prove it for general domains we use the localization argument (see the proof of Step 2 of Proposition 5.14 and the proof of Theorem 5.15).

Proposition 5.14. *Let $\Sigma \in \mathcal{K}_\gamma^{hom}$ and $(u, E, P) \in \mathcal{A}_\gamma^{hom}(w)$ with the associated $\mu \in \mathcal{X}_\gamma(\widetilde{\omega})$. There exists an element $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ such that for every $\varphi \in C_c^2(\widetilde{\omega})$*

$$\langle \lambda, \varphi \rangle = -\int_{\Omega\times\mathcal{Y}} \varphi(x')\, \Sigma : E\, dxdy + \int_\omega \varphi\, \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12}\int_\omega \varphi\, \hat{\sigma} : D^2 w_3\, dx'$$
$$- \int_\omega \bar{\sigma} : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\, dx' - \frac{1}{6}\int_\omega \hat{\sigma} : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big)\, dx'$$
$$- \frac{1}{12}\int_\omega (u_3 - w_3)\, \hat{\sigma} : \nabla^2\varphi\, dx'.$$

Furthermore, the mass of λ is given by

$$\lambda(\widetilde{\Omega} \times \mathcal{Y}) = -\int_{\Omega\times\mathcal{Y}} \Sigma : E\, dxdy + \int_\omega \bar{\sigma} : E\bar{w}\, dx' - \frac{1}{12}\int_\omega \hat{\sigma} : D^2 w_3\, dx'. \tag{5.12}$$

Proof. The proof is subdivided into two steps.

Step 1. Suppose that ω is star-shaped with respect to one of its points.

Let $\{\Sigma_n\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}}))$ be the sequence given by Lemma 5.13. We set

$$\lambda_n := \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}] \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}),$$

where the duality $[(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}]$ is a well defined bounded measure on $I \times \mathcal{Y}$ for η-a.e. $x' \in \widetilde{\omega}$ and η is defined in Lemma 5.8. Further, in view of Remark 5.9, Definition 5.3 gives

$$\int_{\mathbb{R}\times\mathcal{Y}} \psi\, d[(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}]$$
$$= -\int_{I\times\mathcal{Y}} \psi(x_3, y)\, \Sigma_n(x, y) : \left[C(x')E(x, y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right] dx_3 dy$$
$$- \int_{I\times\mathcal{Y}} \Sigma_n(x, y) : \big(\mu_{x'}(x_3, y) \odot \widetilde{\nabla}_\gamma \psi(x_3, y)\big)\, dx_3 dy,$$

for every $\psi \in C^1(\mathbb{R} \times \mathcal{Y})$, and

$$|[(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}]| \leq \|(\Sigma_n)_{\text{dev}}(x', \cdot)\|_{L^\infty(I\times\mathcal{Y};\mathbb{M}^{3\times3}_{\text{sym}})} |P_{x'}| \leq C\, |P_{x'}|,$$

where the last inequality stems from item (d) in Lemma 5.13. This in turn implies that

$$|\lambda_n| = \eta \overset{\text{gen.}}{\otimes} |[(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}]| \leq C\, \eta \overset{\text{gen.}}{\otimes} |P_{x'}| = C\, |P|,$$

from which we conclude that $\{\lambda_n\}$ is a bounded sequence.

Let now $\widetilde{I} \supset I$ be an open set which compactly contains I. Let ξ be a smooth cut-off function with $\xi \equiv 1$ on I, and with support contained in $\widetilde{I}$. Finally, consider a test function $\phi(x, y) := \varphi(x')\xi(x_3)$, for $\varphi \in C_c^\infty(\widetilde{\omega})$. Since $\widetilde{\nabla}_\gamma \phi(x, y) = 0$, we have

$$\langle \lambda_n, \phi \rangle = \int_{\widetilde{\omega}} \left(\int_{I\times\mathcal{Y}} \phi(x, y)\, d[(\Sigma_n)_{\text{dev}}(x', \cdot) : P_{x'}]\right) d\eta(x')$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\, \Sigma_n(x, y) : \left[C(x')E(x, y) - \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix}\right] d\left(\eta \otimes \mathcal{L}^1_{x_3} \otimes \mathcal{L}^2_y\right)$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\, \Sigma_n(x, y) : E(x, y)\, dxdy + \int_{\widetilde{\Omega}} \varphi(x')\, \sigma_n(x) : \begin{pmatrix} A_1(x') + x_3 A_2(x') & 0 \\ 0 & 0 \end{pmatrix} d\left(\eta \otimes \mathcal{L}^1_{x_3}\right)$$
$$= -\int_{\widetilde{\Omega}\times\mathcal{Y}} \varphi(x')\, \Sigma_n(x, y) : E(x, y)\, dxdy + \int_{\widetilde{\Omega}} \varphi(x')\, \sigma_n(x) : dEu(x) \tag{5.13}$$

Since $u \in KL(\widetilde{\Omega})$, we infer

$$\int_{\widetilde{\Omega}} \varphi(x')\, \sigma_n(x) : dEu(x) = \int_{\widetilde{\omega}} \varphi(x')\, \bar{\sigma}_n(x') : dE\bar{u}(x') - \frac{1}{12}\int_{\widetilde{\omega}} \varphi(x')\, \hat{\sigma}_n(x') : dD^2 u_3(x'), \tag{5.14}$$

where $\bar{u} \in BD(\widetilde{\omega})$ and $u_3 \in BH(\widetilde{\omega})$ are the Kirchhoff-Love components of u. From the characterization given in Proposition 3.6, we can thus conclude that

$$\begin{aligned}\int_{\widetilde{\Omega}} \varphi(x')\, \sigma_n(x) : dEu(x) &= \int_{\widetilde{\omega}} \varphi(x')\, \bar{\sigma}_n(x') : \bar{e}(x')\, dx' + \int_{\widetilde{\omega}} \varphi(x')\, \bar{\sigma}_n(x') : d\bar{p}(x') \\ &\quad + \frac{1}{12} \int_{\widetilde{\omega}} \varphi(x')\, \hat{\sigma}_n(x') : \hat{e}(x')\, dx' + \frac{1}{12} \int_{\widetilde{\omega}} \varphi(x')\, \hat{\sigma}_n(x') : d\hat{p}(x') \\ &= \int_{\widetilde{\omega}} \varphi(x')\, \bar{\sigma}_n(x') : \bar{e}(x')\, dx' + \int_{\widetilde{\omega}} \varphi(x')\, d[\bar{\sigma}_n : \bar{p}](x') \\ &\quad + \frac{1}{12} \int_{\widetilde{\omega}} \varphi(x')\, \hat{\sigma}_n(x') : \hat{e}(x')\, dx' + \frac{1}{12} \int_{\widetilde{\omega}} \varphi(x')\, d[\hat{\sigma}_n : \hat{p}](x'), \end{aligned} \tag{5.15}$$

where in the last equality we used that $\bar{\sigma}_n$ and $\hat{\sigma}_n$ are smooth functions. Notice that, since $\bar{p} \equiv 0$ and $\hat{p} \equiv 0$ outside of $\omega \cup \gamma_D$, there holds

$$\int_{\widetilde{\omega}} \varphi\, d[\bar{\sigma}_n : \bar{p}] = \int_{\omega \cup \gamma_D} \varphi\, d[\bar{\sigma}_n : \bar{p}], \quad \int_{\widetilde{\omega}} \varphi\, d[\hat{\sigma}_n : \hat{p}] = \int_{\omega \cup \gamma_D} \varphi\, d[\hat{\sigma}_n : \hat{p}].$$

Since $e = E = E\bar{w} - x_3 D^2 w_3$ on $\widetilde{\Omega} \setminus \Omega$, we deduce, using (5.13)-(5.15),that

$$\begin{aligned}\langle \lambda_n, \phi \rangle &= -\int_{\widetilde{\Omega} \times \mathcal{Y}} \varphi(x')\, \Sigma_n : E\, dxdy + \int_{\widetilde{\omega}} \varphi\, \bar{\sigma}_n : \bar{e}\, dx' + \frac{1}{12} \int_{\widetilde{\omega}} \varphi\, \hat{\sigma}_n : \hat{e}\, dx' \\ &\quad + \int_{\omega \cup \gamma_D} \varphi\, d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12} \int_{\omega \cup \gamma_D} \varphi\, d[\hat{\sigma}_n : \hat{p}] \\ &= -\int_{\Omega \times \mathcal{Y}} \varphi(x')\, \Sigma_n : E\, dxdy + \int_{\omega} \varphi\, \bar{\sigma}_n : \bar{e}\, dx' + \frac{1}{12} \int_{\omega} \varphi\, \hat{\sigma}_n : \hat{e}\, dx' \\ &\quad + \int_{\omega \cup \gamma_D} \varphi\, d[\bar{\sigma}_n : \bar{p}] + \frac{1}{12} \int_{\omega \cup \gamma_D} \varphi\, d[\hat{\sigma}_n : \hat{p}]. \end{aligned} \tag{5.16}$$

Using that $\mathrm{div}_{x'} \bar{\sigma}_n = 0$ in ω, by applying an integration by parts (see also [13, Proposition 7.2]) we obtain for every $\varphi \in C^1(\overline{\omega})$

$$\int_{\omega \cup \gamma_D} \varphi\, d[\bar{\sigma}_n : \bar{p}] + \int_{\omega} \varphi\, \bar{\sigma}_n : (\bar{e} - E\bar{w})\, dx' + \int_{\omega} \bar{\sigma}_n : ((\bar{u} - \bar{w}) \odot \nabla\varphi)\, dx' = 0. \tag{5.17}$$

Likewise in view of the fact that $\mathrm{div}_{x'}\mathrm{div}_{x'} \hat{\sigma}_n = 0$ in ω and $u_3 = w_3$ on γ_D, by integration by parts (see also [13, Proposition 7.6]) we find that for every $\varphi \in C^2(\overline{\omega})$

$$\begin{aligned}&\int_{\omega \cup \gamma_D} \varphi\, d[\hat{\sigma}_n : \hat{p}] + \int_{\omega} \varphi\, \hat{\sigma}_n : (\hat{e} + D^2 w_3)\, dx' \\ &+ 2 \int_{\omega} \hat{\sigma}_n : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big)\, dx' + \int_{\omega} (u_3 - w_3)\, \hat{\sigma}_n : \nabla^2\varphi\, dx' = 0. \end{aligned} \tag{5.18}$$

Let now $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ be such that (up to a subsequence)

$$\lambda_n \xrightharpoonup{*} \lambda \quad \text{weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}).$$

By items (a) and (e) in Lemma 5.13, owing to (5.16)-(5.18) we obtain

$$
\begin{aligned}
\langle \lambda, \phi \rangle &= \lim_n \langle \lambda_n, \phi \rangle \\
&= \lim_n \Big[- \int_{\Omega \times \mathcal{Y}} \varphi(x') \, \Sigma_n : E \, dxdy + \int_\omega \varphi \, \bar\sigma_n : E\bar w \, dx' - \frac{1}{12} \int_\omega \varphi \, \hat\sigma_n : D^2 w_3 \, dx' \\
&\qquad - \int_\omega \bar\sigma_n : ((\bar u - \bar w) \odot \nabla\varphi) \, dx' - \frac{1}{6} \int_\omega \hat\sigma_n : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big) \, dx' \\
&\qquad - \frac{1}{12} \int_\omega (u_3 - w_3) \, \hat\sigma_n : \nabla^2 \varphi \, dx' \Big] \\
&= - \int_{\Omega \times \mathcal{Y}} \varphi(x') \, \Sigma : E \, dxdy + \int_\omega \varphi \, \bar\sigma : E\bar w \, dx' - \frac{1}{12} \int_\omega \varphi \, \hat\sigma : D^2 w_3 \, dx' \\
&\quad - \int_\omega \bar\sigma : ((\bar u - \bar w) \odot \nabla\varphi) \, dx' - \frac{1}{6} \int_\omega \hat\sigma : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big) \, dx' \\
&\quad - \frac{1}{12} \int_\omega (u_3 - w_3) \, \hat\sigma : \nabla^2 \varphi \, dx'.
\end{aligned}
$$

Taking $\varphi \nearrow \mathbb{1}_{\widetilde\omega}$, we deduce (5.12).

Step 2. If ω is not star-shaped, then since ω is a bounded C^2 domain (in particular, with Lipschitz boundary) by [7, Proposition 2.5.4] there exists a finite open covering $\{U_i\}$ of $\overline\omega$ such that $\omega \cap U_i$ is (strongly) star-shaped with Lipschitz boundary.

Let $\{\psi_i\}$ be a smooth partition of unity subordinate to the covering $\{U_i\}$, i.e. $\psi_i \in C^\infty(\overline\omega)$, with $0 \le \psi_i \le 1$, such that $\mathrm{supp}(\psi_i) \subset U_i$ and $\sum_i \psi_i = 1$ on $\overline\omega$.

For each i, let

$$
\Sigma^i(x,y) := \begin{cases} \Sigma(x,y) & \text{if } x' \in \omega \cap U_i, \\ 0 & \text{otherwise.} \end{cases}
$$

Since $\Sigma^i \in \mathcal{K}^{hom}_\gamma$, the construction in Step 1 yields that there exist sequences $\{\Sigma^i_n\} \subset C^\infty(\mathbb{R}^2; L^2(I \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}))$ and

$$
\lambda^i_n := \eta \overset{\text{gen.}}{\otimes} [(\Sigma^i_n)_{\mathrm{dev}}(x', \cdot) : P_{x'}] \in \mathcal{M}_b((\omega \cap U_i) \times I \times \mathcal{Y}),
$$

where again η is defined in Lemma 5.8 such that

$$
\lambda^i_n \xrightharpoonup{*} \lambda^i \quad \text{weakly* in } \mathcal{M}_b((\omega \cap U_i) \times I \times \mathcal{Y}),
$$

with

$$
\begin{aligned}
\langle \lambda^i, \varphi \rangle = &- \int_{(\omega\cap U_i) \times I \times \mathcal{Y}} \varphi(x') \, \Sigma : E \, dxdy + \int_{\omega\cap U_i} \varphi \, \bar\sigma : E\bar w \, dx' - \frac{1}{12} \int_{\omega\cap U_i} \varphi \, \hat\sigma : D^2 w_3 \, dx' \\
&- \int_{\omega\cap U_i} \bar\sigma : ((\bar u - \bar w) \odot \nabla\varphi) \, dx' - \frac{1}{6} \int_{\omega\cap U_i} \hat\sigma : \big(\nabla(u_3 - w_3) \odot \nabla\varphi\big) \, dx' \\
&- \frac{1}{12} \int_{\omega\cap U_i} (u_3 - w_3) \, \hat\sigma : \nabla^2 \varphi \, dx'.
\end{aligned}
$$

for every $\varphi \in C^2_c(\overline\omega \cap U_i)$. This allows us to define measures on $\widetilde\Omega \times \mathcal{Y}$ by letting, for every $\phi \in C_0(\widetilde\Omega \times \mathcal{Y})$,

$$
\langle \lambda_n, \phi \rangle := \sum_i \langle \lambda^i_n, \psi_i(x') \, \phi \rangle,
$$

and

$$
\langle \lambda, \phi \rangle := \sum_i \langle \lambda^i, \psi_i(x') \, \phi \rangle.
$$

From the above computations, $\lambda_n \xrightharpoonup{*} \lambda$ weakly* in $\mathcal{M}_b(\widetilde\Omega \times \mathcal{Y})$, and λ satisfies all the required properties. □

The next theorem allows us to compare the density of the dissipation due to the limiting two-scale plastic strain and that of the measure λ.

Theorem 5.15. *Let $\Sigma \in \mathcal{K}_\gamma^{hom}$ and $(u, E, P) \in \mathcal{A}_\gamma^{hom}(w)$ with the associated $\mu \in \mathcal{X}_\gamma(\widetilde{\omega})$. Then*

$$H\left(y, \frac{dP}{d|P|}\right) |P| \geq \lambda,$$

where $\lambda \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y})$ is given by Proposition 5.14.

Proof. Let $\{\Sigma_n^i\}$, $\{\lambda_n^i\}$ and λ^i be defined as in Step 2 of the proof of Proposition 5.14. Item (d) in Lemma 5.13 implies that

$$(\Sigma_n^i)_{\text{dev}}(x, y) \in K(y) \text{ for every } x' \in \omega \text{ and } \mathcal{L}_{x_3}^1 \otimes \mathcal{L}_y^2\text{-a.e. } (x_3, y) \in I \times \mathcal{Y}.$$

By Proposition 5.6, we have for η-a.e. $x' \in \widetilde{\omega}$

$$H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \geq [(\Sigma_n^i)_{\text{dev}}(x', \cdot) : P_{x'}] \text{ as measures on } I \times \mathcal{Y}.$$

Since $\frac{dP}{d|P|}(x, y) = \frac{dP_{x'}}{d|P_{x'}|}(x_3, y)$ for $|P_{x'}|$-a.e. $(x_3, y) \in I \times \mathcal{Y}$ by Proposition 2.2, we can conclude that

$$\begin{aligned} H\left(y, \frac{dP}{d|P|}\right) |P| &= \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP}{d|P|}\right) |P_{x'}| = \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \\ &= \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} H\left(y, \frac{dP_{x'}}{d|P_{x'}|}\right) |P_{x'}| \\ &\geq \sum_i \psi_i \eta \overset{\text{gen.}}{\otimes} [(\Sigma_n^i)_{\text{dev}}(x', \cdot) : P_{x'}] \\ &= \sum_i \psi_i \lambda_n^i = \lambda_n. \end{aligned}$$

By passing to the limit, we have the desired inequality. ☐

As a directconsequence of the previous theorem and (5.12), we are now in a position to statea principle of maximum plastic work in our setting.

Corollary 5.16. *Let $\gamma \in (0, +\infty)$. Then*

$$\mathcal{H}^{hom}(P) \geq -\int_{\Omega \times \mathcal{Y}} \Sigma : E \, dxdy + \int_\omega \bar{\sigma} : E\bar{w} \, dx' - \frac{1}{12} \int_\omega \hat{\sigma} : D^2 w_3 \, dx',$$

for every $\Sigma \in \mathcal{K}_\gamma^{hom}$ and $(u, E, P) \in \mathcal{A}_\gamma^{hom}(w)$.

5.5. **Liminf inequalities under weak two-scale convergence.** For $(u, e, p) \in \mathcal{A}_h(w)$, we recall the definition of energy functionals $\mathcal{Q}_h$ and $\mathcal{H}_h$ given in (3.10) and (3.11). For $(u, E, P) \in \mathcal{A}_\gamma^{hom}(w)$ we now define

$$\mathcal{Q}^{hom}(E) := \int_{\Omega \times \mathcal{Y}} Q\,(y, E)\, dxdy \tag{5.19}$$

and

$$\mathcal{H}^{hom}(P) := \int_{\overline{\Omega} \times \mathcal{Y}} H\left(y, \frac{dP}{d|P|}\right) d|P|. \tag{5.20}$$

The next result shows that $\mathcal{Q}^{hom}$ and $\mathcal{H}^{hom}$ provide lower bounds for the asymptotic behavior of our elastic energies and dissipation potential with respect to weak two-scale convergence of elastic and plastic stresses.

Theorem 5.17. *Let $\gamma \in (0,+\infty)$. Let $(u^h, e^h, p^h) \in \mathcal{A}_h(w)$ be such that*

$$u^h \xrightharpoonup{*} u \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{5.21}$$

$$\Lambda_h e^h \xrightharpoonup{2} E \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}), \tag{5.22}$$

$$\Lambda_h p^h \xrightharpoonup{2-*} P \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}), \tag{5.23}$$

with $(u, E, P) \in \mathcal{A}^{hom}_\gamma(w)$. Then,

$$\mathcal{Q}^{hom}(E) \leq \liminf_h \mathcal{Q}_h(\Lambda_h e^h) \tag{5.24}$$

and

$$\mathcal{H}^{hom}(P) \leq \liminf_h \mathcal{H}_h(\Lambda_h p^h). \tag{5.25}$$

Proof. Let $\varphi \in C^\infty_c(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$. From the coercivity condition on the quadratic form $\mathcal{Q}_h$ we obtain

$$0 \leq \frac{1}{2}\int_\Omega \mathbb{C}\left(\frac{x'}{\varepsilon_h}\right)\left(\Lambda_h e^h(x) - \varphi\left(x, \frac{x'}{\varepsilon_h}\right)\right) : \left(\Lambda_h e^h(x) - \varphi\left(x, \frac{x'}{\varepsilon_h}\right)\right) dx.$$

Since $\mathbb{C}\left(\frac{x'}{\varepsilon_h}\right)\Lambda_h e^h(x) \xrightharpoonup{2} \mathbb{C}(y)E(x,y)$ weakly two-scale in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$, we can apply the $\liminf$ to the above inequality and we find

$$\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E(x,y) : \varphi(x,y)\, dxdy - \frac{1}{2}\int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\varphi(x,y) \ : \varphi(x,y)\, dx \leq \liminf_h \mathcal{Q}_h(\Lambda_h e).$$

Choosing φ such that $\varphi \to E$ strongly in $L^2(\Omega \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$ yields (5.24).

To prove (5.25), we can assume without loss of generality that

$$\liminf_h \mathcal{H}_h(\Lambda_h p^h) < \infty. \tag{5.26}$$

We write

$$p^h = \sum_i p^h_i + \sum_{i\neq j} p^h_{ij} \tag{5.27}$$

where $p^h_i := p^h \lfloor \Omega \cap ((\mathcal{Y}_i)_{\varepsilon_h} \times I)$ and $p^h_{ij} := p^h \lfloor \widetilde{\Omega} \cap ((\Gamma_{ij} \setminus S)_{\varepsilon_h} \times I)$. Up to a subsequence,

$$\Lambda_h p^h_i \xrightharpoonup{2-*} P_i \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}),$$

$$\Lambda_h p^h_{ij} \xrightharpoonup{2-*} P_{ij} \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}).$$

Clearly,

$$P = \sum_i P_i + \sum_{i\neq j} P_{ij},$$

with $\mathrm{supp}(P_i) \subseteq \widetilde{\Omega} \times \overline{\mathcal{Y}}_i$ and $\mathrm{supp}(P_{ij}) \subseteq \widetilde{\Omega} \times \Gamma_{ij}$. In view of (5.22), we infer

$$\Lambda_h E u^h \lfloor \widetilde{\Omega} \cap ((\mathcal{Y}_i)_{\varepsilon_h} \times I) \xrightharpoonup{2-*} E\, \mathbb{1}_{\widetilde{\Omega}\times\mathcal{Y}_i}\, \mathcal{L}^3_x \otimes \mathcal{L}^2_y + P_i \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}})$$

Recalling (3.2), we can additionally assume that $\Gamma_{ij} \cap \mathcal{C} \subseteq S$. Then, with a normal ν on Γ_{ij} that points from $\mathcal{Y}_j$ to $\mathcal{Y}_i$ for every $j \neq i$, Lemma 4.18 implies that

$$P_i \lfloor \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) = a_{ij}(x,y) \odot \nu(y)\, \eta_{ij} \tag{5.28}$$

for suitable $\eta_{ij} \in \mathcal{M}^+_b(\widetilde{\Omega} \times (\Gamma_{ij} \setminus S))$ and a Borel map $a_{ij} : \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) \to \mathbb{R}^3$ such that $a_{ij} \perp \nu$ for η_{ij}-a.e. $(x,y) \in \widetilde{\Omega} \times (\Gamma_{ij} \setminus S)$.

Using a version of Reshetnyak's lower semicontinuity theorem adapted for two-scale convergence (see [23, Lemma 4.6]), we deduce

$$
\begin{aligned}
&\liminf_h \int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_i^h}{d|\Lambda_h p_i^h|}\right) d|\Lambda_h p_i^h| \\
&= \liminf_h \int_{\widetilde{\Omega}} H_i\left(\frac{d\Lambda_h p_i^h}{d|\Lambda_h p_i^h|}\right) d|\Lambda_h p_i^h| \geq \int_{\widetilde{\Omega}\times\mathcal{Y}} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&= \int_{\widetilde{\Omega}\times\mathcal{Y}_i} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| + \int_{\widetilde{\Omega}\times\Gamma} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&\geq \int_{\widetilde{\Omega}\times\mathcal{Y}_i} H\left(y, \frac{dP_i}{d|P_i|}\right) d|P_i| + \sum_{j\neq i}\int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(\frac{dP_i}{d|P_i|}\right) d|P_i| \\
&\geq \int_{\widetilde{\Omega}\times\mathcal{Y}_i} H\left(y, \frac{dP_i}{d|P_i|}\right) d|P_i| + \sum_{j\neq i}\int_{\widetilde{\Omega}\times(\Gamma_{ij}\setminus S)} H_i\left(-a_{ij}(x,y)\odot\nu(y)\right) d\eta_{ij}.
\end{aligned} \tag{5.29}
$$

Next, we have

$$
\begin{aligned}
\Lambda_h p_{ij}^h &= \Lambda_h\left[(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right]\mathcal{H}^2\lfloor\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I) \\
&= \left[\mathrm{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right]\mathcal{H}^2\lfloor\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I),
\end{aligned}
$$

where u_i^h and u_j^h are the traces on $\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)$ of the restrictions of u^h to $(\mathcal{Y}_i)_{\varepsilon_h}\times I$ and $(\mathcal{Y}_j)_{\varepsilon_h}\times I$ respectively, such that $u_i^h - u_j^h$ is perpendicular to ν. Then, since the infimum in the inf-convolution definition of H on $\Gamma\setminus S$ is actually a minimum, we obtain

$$
\begin{aligned}
&\int_{\Omega\cup\Gamma_D} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H\left(\frac{x'}{\varepsilon_h}, \frac{d\Lambda_h p_{ij}^h}{d|\Lambda_h p_{ij}^h|}\right) d|\Lambda_h p_{ij}^h| \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H\left(\frac{x'}{\varepsilon_h}, \left[\mathrm{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right]\right) d\mathcal{H}^2(x) \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} H_{ij}\left(\mathrm{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h), \nu\left(\frac{x'}{\varepsilon_h}\right)\right) d\mathcal{H}^2(x) \\
&= \int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} \left[H_i\left(b_i^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right) + H_j\left(-b_j^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right)\right] d\mathcal{H}^2(x)
\end{aligned} \tag{5.30}
$$

for suitable Borel functions $b_i^{h,ij}, b_j^{h,ij} : \widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)\to\mathbb{R}^3$ which are orthogonal to ν for $\mathcal{H}^2$-a.e. $x\in(\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I$ and such that

$$
b_i^{h,ij} - b_j^{h,ij} = \mathrm{diag}\left(1,1,\frac{1}{h}\right)(u_i^h - u_j^h) \qquad \text{for } \mathcal{H}^2\text{-a.e. } x\in(\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I.
$$

From the coercivity condition of the dissipation potential H and (5.26), we conclude that

$$
\int_{\widetilde{\Omega}\cap((\Gamma_{ij}\setminus S)_{\varepsilon_h}\times I)} \left[\left|b_i^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right| + \left|b_j^{h,ij}(x)\odot\nu\left(\frac{x'}{\varepsilon_h}\right)\right|\right] d\mathcal{H}^2(x) \leq C,
$$

for some constant $C>0$, which implies the boundedness of $b_i^{h,ij}$ and $b_j^{h,ij}$ in L^1. We can now argue as in Step 2 of the proof of [23, Theorem 5.7] or [24, Proposition 2.3], using also (5.28), and infer that the existence of suitable measures $\zeta_{ij}\in\mathcal{M}_b^+(\widetilde{\Omega}\times(\Gamma_{ij}\setminus S))$, and Borel functions $c^i, c^j : \widetilde{\Omega}\times(\Gamma_{ij}\setminus S)\to\mathbb{R}^3$

which are orthogonal to ν for ζ_{ij}-a.e. $(x,y) \in \widetilde{\Omega} \times (\Gamma_{ij} \setminus S)$, and such that

$$P\lfloor \widetilde{\Omega} \times (\Gamma_{ij} \setminus S) = \big(c^i(x,y) - c^j(x,y)\big) \odot \nu(y)\, \zeta_{ij}.$$

Thus, by (5.29), we have

$$\begin{aligned}
&\liminf_h \mathcal{H}_h(\Lambda_h p^h) \\
&\geq \int_{\widetilde{\Omega} \times (\cup_i \mathcal{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| \\
&\quad + \sum_{i \neq j} \int_{\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)} \left[H_i\left(c^i(x,y) \odot \nu(y)\right) + H_j\left(-c^j(x,y) \odot \nu(y)\right)\right] d\zeta_{ij} \\
&\geq \int_{\widetilde{\Omega} \times (\cup_i \mathcal{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| + \sum_{i \neq j} \int_{\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)} H\left(y, \left(c^i(x,y) - c^j(x,y)\right) \odot \nu(y)\right) d\zeta_{ij} \\
&= \int_{\widetilde{\Omega} \times (\cup_i \mathcal{Y}_i)} H\left(y, \frac{dP}{d|P|}\right) d|P| + \sum_{i \neq j} \int_{\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)} H\left(y, \frac{dP}{d|P|}\right) d|P| \\
&= \mathcal{H}^{hom}(P),
\end{aligned}$$

which in turn concludes the proof. □

6. Two-scale quasistatic evolutions

We recall the definition of energy functionals $\mathcal{Q}^{hom}$ and $\mathcal{H}^{hom}$ given in (5.19) and (5.20). The associated $\mathcal{H}^{hom}$-variation of a function $P : [0,T] \to \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}})$ on $[a,b]$ is then defined as

$$\mathcal{D}_{\mathcal{H}^{hom}}(P; a, b) := \sup\left\{\sum_{i=1}^{n-1} \mathcal{H}^{hom}\left(P(t_{i+1}) - P(t_i)\right) : a = t_1 < t_2 < \ldots < t_n = b,\ n \in \mathbb{N}\right\}.$$

In this section we prescribe for every $t \in [0,T]$ a boundary datum $w(t) \in H^1(\widetilde{\Omega}; \mathbb{R}^3) \cap KL(\widetilde{\Omega})$ and we assume the map $t \mapsto w(t)$ to be absolutely continuous from $[0,T]$ into $H^1(\widetilde{\Omega}; \mathbb{R}^3)$.

We now give the notion of the limiting quasistatic elasto-plastic evolution.

Definition 6.1. *A* two-scale quasistatic evolution *for the boundary datum w is a function $t \mapsto (u(t), E(t), P(t))$ from $[0,T]$ into $KL(\widetilde{\Omega}) \times L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}) \times \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}})$ which satisfies the following conditions:*

(qs1)$^{hom}_\gamma$ for every $t \in [0,T]$ we have $(u(t), E(t), P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$ and

$$\mathcal{Q}^{hom}(E(t)) \leq \mathcal{Q}^{hom}(H) + \mathcal{H}^{hom}(\Pi - P(t)),$$

for every $(\upsilon, H, \Pi) \in \mathcal{A}^{hom}_\gamma(w(t))$.

(qs2)$^{hom}_\gamma$ the function $t \mapsto P(t)$ from $[0,T]$ into $\mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}})$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}^{hom}(E(t)) + \mathcal{D}_{\mathcal{H}^{hom}}(P; 0, t) = \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega \times \mathcal{Y}} \mathbb{C}(y) E(s) : E\dot{w}(s)\, dx dy ds.$$

Recalling the definition of h-quasistatic evolution for the boundary datum $w(t)$ given in Definition 3.7, we are in a position to formulate the main result of the paper.

Theorem 6.2. *Let $t \mapsto w(t)$ be absolutely continuous from $[0,T]$ into $H^1(\widetilde{\Omega}; \mathbb{R}^3) \cap KL(\widetilde{\Omega})$. Assume (3.3) and (3.5) and that there exists a sequence of triples $(u_0^h, e_0^h, p_0^h) \in \mathcal{A}_h(w(0))$ such that*

$$u_0^h \xrightharpoonup{*} u_0 \quad \textit{weakly* in } BD(\widetilde{\Omega}), \tag{6.1}$$

$$\Lambda_h e_0^h \xrightarrow{2} E_0 \quad \textit{two-scale strongly in } L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{sym}}), \tag{6.2}$$

$$\Lambda_h p_0^h \xrightharpoonup{2-*} P_0 \quad \textit{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\mathrm{dev}}), \tag{6.3}$$

for $(u_0, E_0, P_0) \in \mathcal{A}_\gamma^{hom}(w(0))$*. For every* $h > 0$*, let*

$$t \mapsto (u^h(t), e^h(t), p^h(t))$$

be a h*-quasistatic evolution in the sense of Definition 3.7 for the boundary datum* w *such that* $u^h(0) = u_0^h$*,* $e^h(0) = e_0^h$*, and* $p^h(0) = p_0^h$*. Then, there exists a two-scale quasistatic evolution*

$$t \mapsto (u(t), E(t), P(t))$$

for the boundary datum w *such that* $u(0) = u_0$*,* $E(0) = E_0$*, and* $P(0) = P_0$*, and such that (up to subsequences) for every* $t \in [0, T]$

$$u^h(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}), \tag{6.4}$$

$$\Lambda_h e^h(t) \xrightharpoonup{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{sym}}), \tag{6.5}$$

$$\Lambda_h p^h(t) \xrightharpoonup{2-*} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{3\times3}_{\text{dev}}). \tag{6.6}$$

Proof. The proof is subdivided into three steps, in the spirit of evolutionary Γ-convergence.

Step 1: *Compactness.*

We first prove that that there exists a constant C, depending only on the initial and boundary data, such that

$$\sup_{t\in[0,T]} \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} \leq C \;\text{ and }\; \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, T) \leq C, \tag{6.7}$$

for every $h > 0$. Indeed, the energy balance of the h-quasistatic evolution $(\text{qs2})_h$ and (3.5) imply

$$\begin{aligned} &r_c \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) \\ &\leq R_c \left\|\Lambda_h e^h(0)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + 2R_c \sup_{t\in[0,T]} \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} \int_0^T \|E\dot{w}(s)\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} \, ds, \end{aligned}$$

where the last integral is well defined as $t \mapsto E\dot{w}(t)$ belongs to $L^1([0,T]; L^2(\widetilde{\Omega}; \mathbb{M}^{3\times3}_{\text{sym}}))$. In view of the boundedness of $\Lambda_h e_0^h$ that is implied by (6.2), property (6.7) now follows by the Cauchy-Schwarz inequality.

From (6.7) and (3.6), we infer that

$$r_k \left\|\Lambda_h p^h(t) - \Lambda_h p_0^h\right\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})} \leq \mathcal{H}_h\left(\Lambda_h p^h(t) - \Lambda_h p_0^h\right) \leq \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h; 0, t) \leq C,$$

for every $t \in [0, T]$, which together with (6.3) implies

$$\sup_{t\in[0,T]} \left\|\Lambda_h p^h(t)\right\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})} \leq C. \tag{6.8}$$

Next, we note that $\|\cdot\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})}$ is a continuous seminorm on $BD(\widetilde{\Omega})$ which is also a norm on the set of rigid motions. Then, using a variant of Poincaré-Korn's inequality (see [42, Chapter II, Proposition 2.4]) and the fact that $(u^h(t), e^h(t), p^h(t)) \in \mathcal{A}_h(w(t))$, we conclude that, for every $h > 0$ and $t \in [0, T]$,

$$\begin{aligned} \left\|u^h(t)\right\|_{BD(\widetilde{\Omega})} &\leq C\left(\left\|u^h(t)\right\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \left\|Eu^h(t)\right\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})}\right) \\ &\leq C\left(\|w(t)\|_{L^1(\widetilde{\Omega}\setminus\overline{\Omega};\mathbb{R}^3)} + \left\|e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + \left\|p^h(t)\right\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})}\right) \\ &\leq C\left(\|w(t)\|_{L^2(\widetilde{\Omega};\mathbb{R}^3)} + \left\|\Lambda_h e^h(t)\right\|_{L^2(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{sym}})} + \left\|\Lambda_h p^h(t)\right\|_{\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{3\times3}_{\text{dev}})}\right). \end{aligned}$$

In view of the assumption on w , from (6.8) and the former inequality in (6.7) it follows that the sequences $\{u^h(t)\}$ are bounded in $BD(\widetilde{\Omega})$ uniformly with respect to t.

Owing to (2.3), we obtain that $\mathcal{D}_{\mathcal{H}_h}$ and $\mathcal{V}$ are equivalent norms, which immediately implies

$$\mathcal{V}(\Lambda_h p^h; 0, T) \leq C, \tag{6.9}$$

for every $h > 0$. Hence, by a generalized version of Helly's selection theorem (see [11, Lemma 7.2]) and Remark 3.10, there exists a (not relabeled) subsequence, independent of t, and $P \in BV(0,T;\mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\rm dev}))$ such that

$$\Lambda_h p^h(t) \xrightarrow{2-*} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\rm dev}),$$

for every $t \in [0,T]$, and $\mathcal{V}(P;0,T) \leq C$. By extracting a further subsequence (possibly depending on t),

$$u^{h_t}(t) \xrightarrow{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}),$$

$$\Lambda_{h_t} e^{h_t}(t) \xrightarrow{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\rm sym}),$$

for every $t \in [0,T]$. From Proposition 4.1, we conclude that $u(t) \in KL(\widetilde{\Omega})$ for every $t \in [0,T]$. According to Theorem 4.14, the above subsequence can be chosen so that there exists $\mu(t) \in \mathcal{X}_\gamma(\widetilde{\omega})$ for which

$$\Lambda_h E u^{h_t}(t) \xrightarrow{2-*} Eu(t) \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma \mu(t).$$

Since, $\Lambda_{h_t} E u^{h_t}(t) = \Lambda_{h_t} e^{h_t}(t) + \Lambda_{h_t} p^{h_t}(t)$ in $\widetilde{\Omega}$ for every $h > 0$ and $t \in [0,T]$, we deduce that $(u(t),E(t),P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$.

Consider now for every $t \in [0,T]$ the maps

$$\sigma^{h_t}(t) := \mathbb{C}\left(\tfrac{x'}{\varepsilon_{h_t}}\right) \Lambda_{h_t} e^{h_t}(t).$$

For a (not relabeled) subsequence, we have

$$\sigma^{h_t}(t) \xrightarrow{2} \Sigma(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y};\mathbb{M}^{3\times3}_{\rm sym}), \tag{6.10}$$

where $\Sigma(t) := \mathbb{C}(y)E(t)$. Since $\sigma^{h_t}(t) \in \mathcal{K}_{h_t}$ for every $t \in [0,T]$, by Proposition 5.12 we obtain that $\Sigma(t) \in \mathcal{K}^{hom}_\gamma$ for every $t \in [0,T]$.

Step 2: *Global stability.*

Since from Step 1 we have $(u(t),E(t),P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$ with the associated $\mu(t) \in \mathcal{X}_\gamma(\widetilde{\omega})$, then for every $(\upsilon,H,\Pi) \in \mathcal{A}^{hom}_\gamma(w(t))$ with the associated $\nu \in \mathcal{X}_\gamma(\widetilde{\omega})$ we have

$$(\upsilon - u(t), H - E(t), \Pi - P(t)) \in \mathcal{A}^{hom}_\gamma(0).$$

From the inclusion $\mathbb{C}(y)E(t) \in \mathcal{K}^{hom}_\gamma$, by Corollary 5.16 we infer

$$\begin{aligned}\mathcal{H}^{hom}(\Pi - P(t)) &\geq -\int_{\omega \times I \times \mathcal{Y}} \mathbb{C}(y)E(t) : (H - E(t))\,dxdy \\ &= \mathcal{Q}^{hom}(E(t)) + \mathcal{Q}^{hom}(H - E(t)) - \mathcal{Q}^{hom}(H).\end{aligned}$$

Thus,

$$\mathcal{H}^{hom}(\Pi - P(t)) + \mathcal{Q}^{hom}(H) \geq \mathcal{Q}^{hom}(E(t)) + \mathcal{Q}^{hom}(H - E(t)) \geq \mathcal{Q}^{hom}(E(t)),$$

hence we deduce $(\text{qs1})^{hom}_\gamma$.

Now we can prove that limit functions $u(t)$ and $E(t)$ do not depend on the subsequence. Assume that $(\upsilon(t),H(t),P(t)) \in \mathcal{A}^{hom}_\gamma(w(t))$ with the associated $\nu(t) \in \mathcal{X}_\gamma(\widetilde{\omega})$ also satisfy the global stability condition in the definition of the two-scale quasistatic evolution. By the strict convexity of $\mathcal{Q}^{hom}$, we find

$$H(t) = E(t).$$

Then, by (5.2),

$$\begin{aligned}E\upsilon(t) \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma \nu(t) &= H(t)\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P(t) \\ &= E(t)\,\mathcal{L}^3_x \otimes \mathcal{L}^2_y + P(t) \\ &= Eu(t) \otimes \mathcal{L}^2_y + \widetilde{E}_\gamma \mu(t).\end{aligned}$$

Identifing $Eu(t)$ and $E\upsilon(t)$ with elements of $\mathcal{M}_b(\widetilde{\Omega};\mathbb{M}^{2\times2}_{\rm sym})$ and integrating over $\mathcal{Y}$, we obtain

$$E\upsilon(t) = Eu(t).$$

Using the variant of Poincaré-Korn inequality in Step 1, we infer that $\upsilon(t) = u(t)$ on $\widetilde{\Omega}$.

This implies that the whole sequences converges without need to extract further t-dependent subsequences, i.e.

$$\begin{aligned} &u^h(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}), \\ &\Lambda_h e^h(t) \xrightharpoonup{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega}\times\mathcal{Y};\mathbb{M}^{3\times3}_{\mathrm{sym}}). \end{aligned}$$

Step 3: ***Energy balance.***

In order to prove $(\mathrm{qs2})^{hom}_{\gamma}$, it is enough (by arguing as in, e.g. [11, Theorem 4.7] and [24, Theorem 2.7]) to prove the energy inequality

$$\begin{aligned} &\mathcal{Q}^{hom}(E(t)) + \mathcal{D}_{\mathcal{H}^{hom}}(P;0,t) \\ &\leq \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)E(s) : E\dot{w}(s)\,dxdyds. \end{aligned} \tag{6.11}$$

For a fixed $t \in [0,T]$, consider a subdivision $0 = t_1 < t_2 < \ldots < t_n = t$ of $[0,t]$. In view of the lower semicontinuity of $\mathcal{Q}^{hom}$ and $\mathcal{H}^{hom}$ (see (5.24) and (5.25)), from $(\mathrm{qs2})_h$ we have

$$\begin{aligned} &\mathcal{Q}^{hom}(E(t)) + \sum_{i=1}^n \mathcal{H}^{hom}\left(P(t_{i+1}) - P(t_i)\right) \\ &\leq \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(t)) + \sum_{i=1}^n \mathcal{H}_h\left(\Lambda_h p^h(t_{i+1}) - \Lambda_h p^h(t_i)\right)\right) \\ &\leq \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(t)) + \mathcal{D}_{\mathcal{H}_h}(\Lambda_h p^h;0,t)\right) \\ &= \liminf_h \left(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right)\Lambda_h e^h(s) : E\dot{w}(s)\,dxds\right). \end{aligned}$$

By the strong convergence assumed in (6.2) and (6.10), owing to the Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned} &\lim_h \left(\mathcal{Q}_h(\Lambda_h e^h(0)) + \int_0^t \int_\Omega \mathbb{C}\left(\tfrac{x'}{\varepsilon_h}\right)\Lambda_h e^h(s) : E\dot{w}(s)\,dxds\right) \\ &= \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\Lambda_h E(s) : E\dot{w}(s)\,dxdyds. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\mathcal{Q}^{hom}(E(t)) + \sum_{i=1}^n \mathcal{H}^{hom}\left(P(t_{i+1}) - P(t_i)\right) \\ &\leq \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega\times\mathcal{Y}} \mathbb{C}(y)\Lambda_h E(s) : E\dot{w}(s)\,dxdyds \end{aligned}$$

Taking the supremum over all partitions of $[0,t]$ yields (6.11), which concludes the proof. ☐

Remark 6.3. *We point out that as a Corollary of Theorem 6.2 and of the fact that the limiting model satisfies an energy equality, we find that strong two-scale convergence in the L^2-topology of the scaled initial elastic strains and weak two-scale convergence in measure of the scaled initial plastic strains are enough to guarantee the strong two-scale convergence of the rescaled elastic strains in the L^2-topology to the effective one, as well as the convergence of rescaled dissipations to the limiting one.*

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