A NOTE ON THE ONE-DIMENSIONAL CRITICAL POINTS OF THE AMBROSIO-TORTORELLI FUNCTIONAL

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Abstract. This note addresses the question of convergence of critical points of the Ambrosio-Tortorelli functional in the one-dimensional case under pure Dirichlet boundary conditions. An asymptotic analysis argument shows the convergence to two possible limits points: either a globally affine function or a step function with a single jump at the middle point of the space interval, which are both critical points of the one-dimensional Mumford-Shah functional under a Dirichlet boundary condition. As a byproduct, non minimizing critical points of the Ambrosio-Tortorelli functional satisfying the energy convergence assumption as in [2] are proved to exist.

1. Introduction

This note can be seen as a companion to our paper [2] in which we address the convergence of critical points of the Ambrosio-Tortorelli functional. We refer to [2] and references therein for motivation on this topic related to image segmentation [6] or fracture mechanics [3]. We focus here on the one-dimensional case where the Ambrosio-Tortorelli functional is defined by

$$AT_\varepsilon(u,v) := \int_0^L (\eta_\varepsilon + v^2)|u'|^2 \,\mathrm{d}x + \int_0^L \left(\varepsilon|v'|^2 + \frac{(v-1)^2}{4\varepsilon}\right) \mathrm{d}x \qquad \text{for all } (u,v) \in [H^1(0,L)]^2, \tag{1.1}$$

where $L > 0$ and $\varepsilon \to 0$, $\eta_\varepsilon \to 0$ are infinitesimal parameter satisfying $0 < \eta_\varepsilon \ll \varepsilon$. This functional, originally introduced in [1] can be interpreted as a *phase-field regularization* of the Mumford-Shah functional (see [6])

$$(u,v) \mapsto \begin{cases} MS(u) := \displaystyle\int_0^L |u'|^2 \,\mathrm{d}x + \#(J_u) & \text{if } \begin{cases} u \in SBV^2(0,L)\,, \\ v = 1 \text{ in } (0,L)\,, \end{cases} \\ +\infty & \text{otherwise}\,. \end{cases} \tag{1.2}$$

More precisely, it has been proved in [1] that AT_ε Γ-converges in the $[L^2(0,L)]^2$-topology as $\varepsilon \to 0$ to the Mumford-Shah functional. As a consequence, the fundamental theorem of Γ-convergence ensures the convergence of global minimizers $(u_\varepsilon, v_\varepsilon)$ of AT_ε (under suitable boundary conditions) to $(u,1)$ as $\varepsilon \to 0$ where $u \in SBV^2(0,L)$ is a global minimizer of MS.

In the present work we address the asymptotic analysis of critical points of the Ambrosio-Tortorelli functional, i.e. is a solution $(u_\varepsilon, v_\varepsilon)$ of the ordinary differential equation

$$\begin{cases} \left[(\eta_\varepsilon + v_\varepsilon^2)u_\varepsilon'\right]' = 0 & \text{in } (0,L)\,, \\ -\varepsilon v_\varepsilon'' + v_\varepsilon|u_\varepsilon'|^2 + \dfrac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0,L)\,, \\ u_\varepsilon(0) = 0\,,\ u_\varepsilon(L) = a\,, \\ v_\varepsilon(0) = v_\varepsilon(L) = 1\,. \end{cases} \tag{1.3}$$

If global minimizers always define critical points, the converse might fail due to the non convexity of AT_ε. Note also that, in contrast with [4, 5], we consider a pure Dirichlet problem. As in [4] there is a selection phenomenon of possible accumulation points of $(u_\varepsilon, v_\varepsilon)$.

According to [2, Remark 1.1], in this setting, a function $u \in SBV^2(0,L)$ is a critical point of the Mumford-Shah functional in $SBV^2(0,L)$ if and only if either $u(x) = \frac{ax}{L}$ for $x \in [0,L]$ or u is piecewise constant with a finite number of jumps $\widehat{J}_u = \{x_1, \dots, x_m\}$ with $x_i \in [0,L]$ for all $i = 1, \dots, m$. In the first case the energy of u is

$$MS(u) = \int_0^L \left|\frac{a}{L}\right|^2 \,\mathrm{d}x = \frac{a^2}{L}\,,$$

while in the second case

$$MS(u) = \#(\widehat{J}_u \cap [0,L]) = m\,.$$

Thus, if $a^2 < L$ then $u(x) = ax/L$ is the global minimizer whereas if $a^2 > L$ then a constant function with exactly one jump anywhere in the closed interval $[0,L]$ is a global minimizer (if $a^2 = L$ then all previous functions are global minimizers). In particular, we have

$$\min_{SBV^2(0,L)} MS = \min\left\{\frac{a^2}{L}, 1\right\}. \tag{1.4}$$

We now consider a family of critical points of AT_ε, i.e., a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ in $[H^1(0,L)]^2$ satisfying (1.3) together with the uniform energy bound

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x + \int_0^L \left(\varepsilon|v_\varepsilon'|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon}\right) \mathrm{d}x \le C\,. \tag{1.5}$$

The following result extends [4, Theorem 2.2] to the case of Dirichlet boundary conditions for the phase-field variable v. It states that only two critical points of the Mumford-Shah functional are attainable through this asymptotic analysis procedure: either the affine solution (with no jumps) or the step function with a single jump at the middle point of the interval $(0,L)$. It also improves [2, Theorem 1.2] in the one-dimensional case since the energy convergence assumption is no longer required.

Theorem 1.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset [H^1(0,L)]^2$ be a family satisfying* (1.3) *and* (1.5)*. Then, up to a subsequence $\varepsilon \to 0$,*

(i) $(u_\varepsilon, v_\varepsilon) \to (u_*, 1)$ *in* $[L^2(0,L)]^2$ *where* $u_* \in \{u_{\text{jump}}, u_{\text{aff}}\}$ *with*

$$u_{\text{jump}} = a\mathbf{1}_{[L/2,L]}\,, \qquad u_{\text{aff}}(x) = \frac{ax}{L} \quad \text{for all } x \in [0,L]\,. \tag{1.6}$$

(ii) $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2\mathcal{L}^1 \llcorner (0,L) \overset{*}{\rightharpoonup} |u_*'|^2\mathcal{L}^1 \llcorner (0,L)$ *weakly* in* $\mathcal{M}([0,L])$,

(iii) $\left|\varepsilon|v_\varepsilon'|^2 - \dfrac{(1-v_\varepsilon)^2}{4\varepsilon}\right| \to 0$ *strongly in* $L^1(0,L)$,

(iv) $\varepsilon|v_\varepsilon'|^2\mathcal{L}^1 \llcorner (0,L) \overset{*}{\rightharpoonup} \alpha\delta_{\frac{L}{2}}$ *weakly* in* $\mathcal{M}([0,L])$, *with* $\alpha = 0$ *or* $\alpha = 1/2$. *Moreover, if* $u_* = u_{\text{jump}}$, *then* $\alpha = 1/2$.

Remark 1.1. We emphasize that we must have $\alpha = 1/2$ for $u_* = u_{\text{jump}}$. However, $\alpha = 0$ does not necessarily implies that $u_* = u_{\text{aff}}$. Indeed $\alpha = 1/2$ provided v_ε has a v-jump at $L/2$ in the terminology of [4], i.e., as soon as $v_\varepsilon(L/2) \le \sqrt{C_*\varepsilon}$ for some constant $C_* > 0$. However it could happen that this v-jump disappears in the limit and does not create a discontinuity for u_*.

As in [4] we are in presence of a selection phenomenon since critical points of AT_ε cannot approximate any critical points of MS but only specific ones. Here the selection phenomenon is much stronger than in [4] in the sense that only two critical points u_{jump} and u_{aff} of MS can be reached as limits of critical points of AT_ε.

We also show the existence of a family of critical points of AT_ε approximating u_{jump}.

Theorem 1.2. *There exists a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset [H^1(0,L)]^2$ satisfying* (1.3) *and* (1.5) *such that*

$$(u_\varepsilon, v_\varepsilon) \to (u_{\text{jump}}, 1) \text{ strongly in } [L^2(0,L)]^2 \text{ as } \varepsilon \to 0\,.$$

Remark 1.2. According to Theorem 1.1, we obtain that the family in Theorem 1.2 satisfies the energy convergence $AT_\varepsilon(u_\varepsilon, v_\varepsilon) \to MS(u_{\rm jump})$. Furthermore, if $a^2 < L$, then (1.4) ensures that $u_{\rm jump}$ is not a global minimizer of MS. This shows the existence of non minimizing critical points of AT_ε satisfying the assumption of convergence of energy in [2, Theorem 1.2].

2. Preliminary estimates

We start by using the first equation in (1.3) to find a constant $c_\varepsilon \in \mathbf{R}$ such that

$$(\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon = c_\varepsilon\,, \tag{2.1}$$

which implies that u'_ε has a constant sign. Since we assume $u_\varepsilon(0) = 0$ and $u_\varepsilon(L) = a > 0$, we deduce that $u'_\varepsilon \geq 0$ and $c_\varepsilon \geq 0$. Then the second equation in (1.3) can be rewritten as

$$-\varepsilon v''_\varepsilon + \frac{c_\varepsilon^2 v_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{4\varepsilon} = 0\,. \tag{2.2}$$

We observe that, thanks to the energy bound (1.5),

$$ac_\varepsilon = \int_0^L u'_\varepsilon c_\varepsilon \,{\rm d}x = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \,{\rm d}x \leq AT_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C\,,$$

hence $\{c_\varepsilon\}_{\varepsilon>0}$ is bounded and, up to a subsequence, we can assume that

$$c_\varepsilon \to c_0\,. \tag{2.3}$$

As in [4, Lemma 3.2] (and using [2, Proposition 4.1]), we have the following result.

Lemma 2.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset [H^1(0,L)]^2$ satisfying (1.3) and (1.5). Then, up to a subsequence, $(u_\varepsilon, v_\varepsilon) \to (u_*, 1)$ strongly in $[L^2(0,L)]^2$ with $u_* \in SBV^2(0,L)$. Furthermore, $u'_\varepsilon \to c_0$ a.e. in $(0,L)$, $|Du_*|((0,L)) \leq a$ and $c_0 \leq a/L$.*

In the one-dimensional setting, the Noether type conservation law of [2, Proposition 4.2] reads as

$$\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\right)' = 0 \text{ in } (0,L)\,,$$

and it implies the existence of a constant $d_\varepsilon \in \mathbf{R}$, sometimes called discrepancy, such that

$$\frac{(1-v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 = d_\varepsilon\,. \tag{2.4}$$

Thanks to the energy bound (1.5), it is easy to see that $\{d_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence, and thus (up to a further subsequence)

$$d_\varepsilon \to d_0\,. \tag{2.5}$$

It also ensures the following uniform bounds (see [4, Lemma 3.4]).

Lemma 2.2. *For $\varepsilon > 0$ small enough,*

$$\|u'_\varepsilon\|_{L^\infty(0,L)} \leq \frac{2}{\sqrt{\varepsilon\eta_\varepsilon}}\,, \quad \|v'_\varepsilon\|_{L^\infty(0,L)} \leq \frac{2}{\varepsilon}\,.$$

Moreover, if $c_0 > 0$, then the following refined estimates hold

$$\|u'_\varepsilon\|_{L^\infty(0,L)} \leq \frac{2}{c_0\varepsilon}\,, \quad \min_{[0,L]} v_\varepsilon \geq \frac{c_0\sqrt{\varepsilon}}{2}\,.$$

We next show the following strong maximum principle.

Lemma 2.3. *Let $(u_\varepsilon, v_\varepsilon) \in [H^1(0,L)]^2$ satisfying (1.3), then $0 < v_\varepsilon < 1$ in $(0,L)$.*

Proof. Let x_0 be a minimum point of v_ε in $[0, L]$. If $v_\varepsilon(x_0) = 1$, using that $0 \leq v_\varepsilon \leq 1$, we deduce that $v_\varepsilon \equiv 1$ in $[0, L]$. Inserting into the second equation of (1.3), we find that u_ε is a constant function in $[0, L]$ which is in contradiction with $u_\varepsilon(0) = 0$ and $u_\varepsilon(L) = a > 0$. As a consequence of the Dirichlet boundary condition for v_ε, we have $x_0 \in (0, L)$ and thus $v_\varepsilon''(x_0) \geq 0$. If $v_\varepsilon(x_0) = 0$, using again the second equation in (1.3) we find that $-\varepsilon v_\varepsilon''(x_0) = \frac{1}{4\varepsilon} > 0$ which is a contradiction. Therefore, $v_\varepsilon \geq v_\varepsilon(x_0) > 0$ in $(0, L)$.

Likewise, let x_1 be a maximum point of v_ε in $[0, L]$. If $x_1 \in (0, L)$ and $v_\varepsilon(x_1) = 1$, then we use that $v_\varepsilon''(x_1) \leq 0$ together with (2.2) to obtain that $c_\varepsilon = 0$. This implies by (2.1) that $u_\varepsilon' = 0$ which is a contradiction since $u_\varepsilon(0) = 0$ and $u_\varepsilon(L) = a > 0$. It shows again that $v_\varepsilon < 1$ in $(0, L)$. □

The selection phenomenon already observed in [4] is due to the following symmetry property which is similar to [4, Lemma 4.1].

Proposition 2.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family in $[H^1(0, L)]^2$ satisfying (1.3) and (1.5). Then v_ε possesses a unique critical point in $(0, L)$ located at $L/2$, which is a minimum of v_ε on $[0, L]$. Moreover, v_ε is decreasing in $(0, L/2)$, increasing in $(L/2, L)$ and the graph of v_ε is symmetric with respect to the vertical line $x = L/2$.*

Proof. From Lemma 2.3, v_ε cannot be identically constant equal to 1. Thus by Rolle's theorem, v_ε admits critical points in $(0, L)$.

Let $x_0 \in (0, L)$ be an arbitrary critical point of v_ε in $(0, L)$. If $x_0 \in (0, L/2)$ then the function

$$\tilde{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in (0, x_0] \\ v_\varepsilon(2x_0 - x) & \text{if } x \in (x_0, 2x_0) \end{cases}$$

is a solution of (2.2) in the interval $(0, 2x_0)$. In particular, v_ε and $\tilde{v}_\varepsilon$ are two solutions of an ODE of the form $v_\varepsilon'' = f_\varepsilon(x, v_\varepsilon)$ in $(x_0, 2x_0)$ for some function f_ε of class $\mathscr{C}^2$ with $v_\varepsilon(x_0) = \tilde{v}_\varepsilon(x_0)$ and $v_\varepsilon'(x_0) = \tilde{v}_\varepsilon'(x_0) = 0$. Cauchy-Lipschitz Theorem yields in turn that $v_\varepsilon = \tilde{v}_\varepsilon$ in $(x_0, 2x_0)$. In particular, $v_\varepsilon(2x_0) = v_\varepsilon(0) = 1$ which contradicts Lemma 2.3 since $2x_0 \in (0, L)$. Thus $x_0 \in [L/2, L)$ and a symmetric argument shows that $x_0 \in (0, L/2]$. Finally, the only possibility left is $x_0 = L/2$.

In particular, v_ε admits a unique critical point in $(0, L)$ at the point $L/2$, which must be a minimum of v_ε on $[0, L]$. Moreover, the graph of v_ε is symmetric with respect to the vertical line $\{x = L/2\}$. Since v_ε is a smooth function satisfying $v_\varepsilon(0) = 1$, $v_\varepsilon(L/2) < 1$ and $v_\varepsilon' \neq 0$ in $(0, L/2)$, we deduce that v_ε is decreasing in $(0, L/2)$. By symmetry v_ε is increasing in $(L/2, L)$. □

A crucial step in the proof of Theorem 1.1 is the following characterization of possible limiting slopes c_0 in (2.3), which strongly rests on the symmetry property of v_ε. We refer to [4, Lemma 4.4] for the proof.

Lemma 2.4. *The limiting slope c_0 in (2.3) satisfies that either $c_0 = 0$ or $c_0 = a/L$.*

Using that $u_\varepsilon' = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}$ in (2.4), we find that

$$\frac{(1 - v_\varepsilon)^2}{4\varepsilon} - \varepsilon |v_\varepsilon'|^2 - \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2} = d_\varepsilon \,.$$

Thus, since $v_\varepsilon \leq 1$ and c_ε and d_ε are bounded, we obtain that, for some constant $C_* > 0$ independent of ε,

$$v_\varepsilon^2(L/2)\Big(1 - v_\varepsilon(L/2)\Big)^2 \leq \Big(\eta_\varepsilon + v_\varepsilon^2(L/2)\Big)\Big(1 - v_\varepsilon(L/2)\Big)^2 \leq C_* \varepsilon \,.$$

This implies, thanks to the study of the function $X \mapsto X^2(1 - X)^2$ on $[0, 1]$, that

$$\text{either} \quad v_\varepsilon\left(\frac{L}{2}\right) \geq 1 - \sqrt{C_* \varepsilon} \quad \text{or} \quad v_\varepsilon\left(\frac{L}{2}\right) \leq \sqrt{C_* \varepsilon} \,.$$

In the latter case, $L/2$ corresponds to a so-called v-jump according to the terminology of [4]. The previous dichotomy implies that either v_ε converges uniformly to 1 or there exists exactly one v-jump which is a minimum of v_ε located at $L/2$.

3. Proof of Theorem 1.1

We are now ready to prove items i) and ii) of Theorem 1.1, i.e., the selection principle for limit of critical points of AT_ε and the convergence of the bulk energy.

Proof of i) and ii) in Theorem 1.1. Step 1. Assume first that $v_\varepsilon(L/2) \geq 1 - \sqrt{C_*\varepsilon}$. Then we have that $0 \leq 1 - v_\varepsilon \leq \sqrt{C_*\varepsilon} \to 0$ uniformly in $[0, L]$. For ε small enough we also have that $v_\varepsilon \geq 1/2$ so that the energy bound (1.5) yields

$$\frac{1}{2}\int_0^L |u_\varepsilon'|^2 \,\mathrm{d}x \leq \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x \leq C\,.$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(0,L)$ up to a subsequence we have that $u_\varepsilon \rightharpoonup u_*$ weakly in $H^1(0,L)$ with $u_* \in H^1(0,L)$. We can pass to the limit in the first equation of (2.1) using (2.3) to obtain that $u_*' = c_0$ in $(0,L)$. Since $u_*(0) = 0$ and $u_*(L) = a$ we find that $u_* = u_{\rm aff}$ and $c_0 = a/L$. Moreover, thanks to the uniform convergence of v_ε,

$$\lim_{\varepsilon\to 0}\int_0^L |u_\varepsilon'|^2 \,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x = \lim_{\varepsilon\to 0} c_\varepsilon \int_0^L u_\varepsilon' \,\mathrm{d}x = c_0\int_0^L u_{\rm aff}' \,\mathrm{d}x = \int_0^L |u_{\rm aff}'|^2 \,\mathrm{d}x\,.$$

It yields $u_\varepsilon \to u_{\rm aff}$ strongly in $H^1(0,L)$ and, in particular $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \to |u_{\rm aff}'|^2$ strongly in $L^1(0,L)$, hence also weakly* in $\mathcal{M}([0,L])$.

Step 2. Assume now that $v_\varepsilon\left(\frac{L}{2}\right) < \sqrt{C_*\varepsilon}$. We first notice that $u_\varepsilon(L/2) = a/2$. Indeed, thanks to the symmetry property of v_ε and a change of variables, we find that

$$a = \int_0^L u_\varepsilon' \,\mathrm{d}x = \int_0^L \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}\,\mathrm{d}x = 2\int_0^{L/2} \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}\,\mathrm{d}x = 2u_\varepsilon\left(\frac{L}{2}\right). \tag{3.1}$$

We next claim that for each $0 < \delta < L/2$, the function $v_\varepsilon \to 1$ uniformly on $[0,\delta]$. To this purpose, define $A_\varepsilon := \{x \in (0,L) : \ v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$. By the monotonicity properties of v_ε, the set A_ε is a closed interval centered in $L/2$. Thanks to the energy bound (1.5),

$$C \geq \int_0^L \frac{(1-v_\varepsilon)^2}{\varepsilon}\,\mathrm{d}x \geq \frac{\mathcal{L}^1(A_\varepsilon)}{\varepsilon^{1/2}}\,,$$

which implies that

$$\mathrm{diam}(A_\varepsilon) \leq C\varepsilon^{1/2} < \frac{L}{2} - \delta$$

for ε small enough. Hence $A_\varepsilon \cap [0,\delta] = \emptyset$ for ε small. In particular $v_\varepsilon \to 1$ uniformly on $[0,\delta]$, and then $u_\varepsilon' = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} \to c_0$ uniformly on $[0,\delta]$. We deduce that

$$u_\varepsilon(x) = \int_0^x u_\varepsilon'(t)\,\mathrm{d}t \to c_0 x \text{ uniformly with respect to } x \in [0,\delta]\,.$$

Thus $u_\varepsilon(x) \to c_0 x$ for a.e. $x \in (0, \frac{L}{2})$, and we prove in the same way that $u_\varepsilon(x) \to a - c_0(L-x)$ for a.e. $x \in (\frac{L}{2}, L)$. Since $c_0 = 0$ or $c_0 = a/L$ by Lemma 2.4, then we find that either $u_* = u_{\rm jump}$ or $u_* = u_{\rm aff}$ (see (1.6)). Observe that the case $u_* = u_{\rm jump}$ only occurs in the case $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$.

We finally show the convergence of the bulk energy. From the first equation in (1.3) we can write that $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 = c_\varepsilon u_\varepsilon'$. Thus, for all $\varphi \in \mathscr{C}_c^\infty(\mathbf{R})$,

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2\varphi\,\mathrm{d}x = c_\varepsilon\int_0^L u_\varepsilon'\varphi\,\mathrm{d}x = -c_\varepsilon\int_0^L u_\varepsilon\varphi'\,\mathrm{d}x + c_\varepsilon a\varphi(L) \to -c_0\int_0^L u_*\varphi'\,\mathrm{d}x + c_0 a\varphi(L)\,.$$

If $c_0 = 0$, then $u_* = u_{\rm jump}$ and $u'_{\rm jump} = 0$. We thus get in that case,

$$-c_0 \int_0^L u_{\rm jump}\varphi' \,{\rm d}x + c_0 a\varphi(L) = 0 = \int_0^L |u'_{\rm jump}|^2\varphi \,{\rm d}x\,.$$

If $c_0 = a/L$, then $u_* = u_{\rm aff}$ and thus

$$-c_0 \int_0^L u_{\rm aff}\varphi' \,{\rm d}x + c_0 a\varphi(L) = \frac{a}{L}\int_0^L u'_{\rm aff}\varphi \,{\rm d}x = \int_0^L |u'_{\rm aff}|^2\varphi \,{\rm d}x\,.$$

In any case, we obtain

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\varphi \,{\rm d}x \to \int_0^L |u'_*|^2\varphi \,{\rm d}x\,,$$

which proves the announced items i) and ii). □

From now on, the function u stands for either $u_{\rm aff}$ or $u_{\rm jump}$. The argument in the previous proof actually shows that $u = u_{\rm aff}$ if $c_0 = a/L$, while $u = u_{\rm jump}$ if $c_0 = 0$.

At this stage, it remains to show the equipartition of energy and the convergence of the diffuse surface energy (points iii) and iv) in Theorem 1.1). The key argument is the following result stating that there is very few diffuse surface energy far way from $L/2$, the only possible limit jump point. The proof is an adaptation of [4, Lemma 6.1].

Lemma 3.1. *For every compact set $K \subset [0, L] \setminus \{L/2\}$, there exists a constant $C_K > 0$ such that*

$$\int_K \left(\varepsilon|v'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon}\right) {\rm d}x \le C_K\varepsilon^{1/4}\,.$$

Proof. We already know that the set $A_\varepsilon := \{x \in (0, L) :\ v_\varepsilon(x) \le 1-\varepsilon^{1/4}\}$ is a closed interval centred at $L/2$ with $\mathrm{diam}(A_\varepsilon) \le C\sqrt{\varepsilon}$. Let $\delta < \frac{1}{2}\mathrm{dist}(L/2, K)$. If $\varepsilon > 0$ is small enough then $A_\varepsilon \subset [L/2-\delta, L/2+\delta]$, hence $K \cap A_\varepsilon = \emptyset$. Since $K \subset V_\delta := [0, L] \setminus [L/2-\delta, L/2+\delta]$, it suffices to show that

$$\int_{V_\delta} \left(\varepsilon|v'_\varepsilon|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon}\right) {\rm d}x \le C_K\varepsilon^{1/4}\,.$$

We multiply the second equation in (1.3) by $v_\varepsilon - 1 \in H^1_0(0, L)$ and we integrate by parts to obtain

$$\int_{V_\delta} \left(\varepsilon|v'_\varepsilon|^2 + \frac{(v_\varepsilon-1)^2}{4\varepsilon}\right) {\rm d}x = \varepsilon v'_\varepsilon(L/2-\delta)(v_\varepsilon(L/2-\delta)-1) - \varepsilon v'_\varepsilon(L/2+\delta)(v_\varepsilon(L/2+\delta)-1) + \int_{V_\delta} \frac{c_\varepsilon^2 v_\varepsilon(1-v_\varepsilon)}{(\eta_\varepsilon+v_\varepsilon^2)^2}\,{\rm d}x\,.$$

By definition of A_ε and V_δ, we have $|1 - v_\varepsilon| \le \varepsilon^{1/4}$ on V_δ. Using further the gradient bound for v_ε in Lemma 2.2, we find that

$$\int_{V_\delta} \left(\varepsilon|v'_\varepsilon|^2 + \frac{(v_\varepsilon-1)^2}{4\varepsilon}\right) {\rm d}x \le C_K\varepsilon^{1/4}\,,$$

which completes the proof of the lemma. □

Arguing as in [4, Lemma 6.3], we also have the following result which relates the limit slope c_0 to the limit d_0 (respectively defined in (2.3) and (2.5)).

Lemma 3.2. *The real numbers c_0 and d_0 satisfy $d_0 + c_0^2 = 0$.*

We are now in position to complete the proof of Theorem 1.1.

Proof of iii) and iv) in Theorem 1.1. Step 1. Let us consider the function

$$f_\varepsilon := \frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2 = d_\varepsilon + (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 = d_\varepsilon + c_\varepsilon u_\varepsilon' = d_\varepsilon + \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2}\,.$$

If $u_* = u_{\rm jump}$, then

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x \to 0\,,$$

hence $c_\varepsilon a = c_\varepsilon \int_0^L u_\varepsilon' \,\mathrm{d}x = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x \to 0$. It shows that $c_0 = 0$ and thus $d_0 = 0$ owing to Lemma 3.2. We thus infer that

$$\int_0^L \left|\frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2\right| \mathrm{d}x \to 0\,.$$

Assume next that $u_* = u_{\rm aff}$. In that case, we have

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \,\mathrm{d}x \not\to 0\,,$$

and the same argument as before shows that $c_0 \neq 0$. Then $d_0 \neq 0$ by Lemma 3.2. In particular $d_\varepsilon \neq 0$ for $\varepsilon > 0$ small enough. The function f_ε reaches its maximum when v_ε is minimal, i.e., at the point $L/2$. Since $L/2$ is a critical point of v_ε, we have

$$\max_{[0,L]} f_\varepsilon = f_\varepsilon(L/2) = \frac{(v_\varepsilon(L/2) - 1)^2}{4\varepsilon} \geq 0\,.$$

Similarly, f_ε reaches its minimum when v_ε attains its maximum. Since v_ε is maximal on the boundary with $v_\varepsilon(0) = v_\varepsilon(L) = 1$ we find that

$$\min_{[0,L]} f_\varepsilon = f_\varepsilon(0) = -\varepsilon|v_\varepsilon'(0)|^2 \leq 0\,.$$

As a consequence, there exists $s_\varepsilon \in (0, L/2)$ such that $f_\varepsilon(s_\varepsilon) = 0$. From Lemma 3.2, it follows that

$$v_\varepsilon^2(s_\varepsilon) = -\eta_\varepsilon - \frac{c_\varepsilon^2}{d_\varepsilon} \to -\frac{c_0^2}{d_0} = 1\,.$$

Up to a subsequence, there exists $s_0 \in [0, L/2]$ such that $s_\varepsilon \to s_0$. By monotonicity of v_ε, we get that $v_\varepsilon \mathbf{1}_{[0,s_\varepsilon]} \to \mathbf{1}_{[0,s_0]}$ for a.e. $s \in (0, L/2)$. Hence using again Lemma 3.2,

$$d_\varepsilon s_\varepsilon + c_\varepsilon u_\varepsilon(s_\varepsilon) = \int_0^{s_\varepsilon} \left(d_\varepsilon + \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2}\right) \mathrm{d}x \to \int_0^{s_0} (d_0 + c_0^2) \,\mathrm{d}x = 0\,. \tag{3.2}$$

Using the symmetry of v_ε, (3.1) and (3.2), we compute

$$\begin{aligned}
\int_0^L \left|\frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v_\varepsilon'|^2\right| \mathrm{d}x &= \int_0^L |f_\varepsilon| \,\mathrm{d}x = 2\int_0^{L/2} |f_\varepsilon| \,\mathrm{d}x \\
&= -2\int_0^{s_\varepsilon} f_\varepsilon \,\mathrm{d}x + 2\int_{s_\varepsilon}^{L/2} f_\varepsilon \,\mathrm{d}x \\
&= 2d_\varepsilon\left(\frac{L}{2} - 2s_\varepsilon\right) + 2c_\varepsilon\left(u_\varepsilon\left(\frac{L}{2}\right) - 2u_\varepsilon(s_\varepsilon)\right) \\
&\to Ld_0 + c_0 a = 0\,.
\end{aligned}$$

It completes the proof of the equipartition of energy.

Step 2. We finally show the convergence of the diffuse surface energy. According to Lemma 3.1, we have

$$\left(\varepsilon|v_\varepsilon'|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon}\right) \mathcal{L}^1 \llcorner (0, L) \overset{*}{\rightharpoonup} \mu \quad \text{weakly* in } \mathcal{M}([0, L])\,,$$

for some nonnegative measure $\mu \in \mathcal{M}([0,L])$ supported on $\{L/2\}$, and thus of the form $\mu = c\delta_{L/2}$ with $c \geq 0$. On the one hand, since μ is concentrated at $L/2$, we have

$$c = \mu((0,L)) = \lim_{\varepsilon\to 0}\int_0^L \left(\varepsilon|v_\varepsilon'|^2 + \frac{(v_\varepsilon-1)^2}{4\varepsilon}\right) \mathrm{d}x\,.$$

On the other hand, the equipartition of energy ensures that

$$\int_0^L \left(\varepsilon|v_\varepsilon'|^2 + \frac{(v_\varepsilon-1)^2}{4\varepsilon} - |v_\varepsilon'|(1-v_\varepsilon)\right) \mathrm{d}x = \int_0^L \left(\sqrt{\varepsilon}|v_\varepsilon'| - \frac{1-v_\varepsilon}{2\sqrt{\varepsilon}}\right)^2 \mathrm{d}x$$
$$\leq \int_0^L \left|\varepsilon|v_\varepsilon'|^2 - \frac{|1-v_\varepsilon|^2}{4\varepsilon}\right| \mathrm{d}x \to 0\,.$$

Since $v_\varepsilon' \leq 0$ on $(0,L/2)$ and $v_\varepsilon' \geq 0$ on $(L/2,L)$, by symmetry of v_ε with respect to the vertical axis $\{x = L/2\}$, we have

$$\int_0^L |v_\varepsilon'|(1-v_\varepsilon)\,\mathrm{d}x = -2\int_0^{L/2} v_\varepsilon'(1-v_\varepsilon)\,\mathrm{d}x = (1-v_\varepsilon(L/2))^2\,.$$

If $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$, then $c = 1$ and $\mu = \delta_{L/2}$, while if $v_\varepsilon(L/2) \geq 1 - \sqrt{C_*\varepsilon}$, then $c = 0$ and $\mu = 0$. Using again the equipartition of energy, we infer that

$$\varepsilon|v_\varepsilon'|^2\mathcal{L}^1 \llcorner (0,L) \xrightharpoonup{*} \frac{c}{2}\delta_{L/2} \quad \text{weakly* in } \mathcal{M}([0,L])\,,$$

so that the desired convergence holds with $\alpha = \frac{c}{2} \in \{0, \frac{1}{2}\}$. If $u_* = u_{\text{jump}}$, then we must have $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$, and it follows that $\alpha = 1/2$ in that case. □

4. Proof of Theorem 1.2

This section is devoted to prove Theorem 1.2, following again arguments similar to those of [4, Section 5]. By the symmetry properties of Theorem 1.1 it suffices to construct a critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε in $(0,L/2)$ such that $v_\varepsilon(0) = 1$, $v_\varepsilon'(L/2) = 0$, $u_\varepsilon(0) = 0$ and $u_\varepsilon(L/2) = a/2$.

To this aim, let $\alpha \in (0,1)$ independent of ε, and set

$$\mathcal{V} := \left\{v \in H^1(0,L/2) :\ v(0) = 1,\ v\left(\frac{L}{2}\right) \leq \alpha\right\},$$

$$\mathcal{B} := \left\{(u,v) \in H^1(0,L/2) \times \mathcal{V} :\ u(0) = 0,\ u\left(\frac{L}{2}\right) = \frac{a}{2}\right\}.$$

For $(u,v) \in \mathcal{B}$ and $0 \leq r \leq s \leq L/2$, we define the localized bulk and diffuse surface energies by

$$\mathcal{E}_\varepsilon(u,v;r,s) = \int_r^s (\eta_\varepsilon + v^2)|u'|^2\,\mathrm{d}x, \quad \mathcal{F}_\varepsilon(v;r,s) := \int_r^s \left(\varepsilon|v'|^2 + \frac{(1-v)^2}{4\varepsilon}\right) \mathrm{d}x\,,$$

and the Ambrosio-Tortorelli energy localized on $(0,L/2)$ by

$$\widetilde{AT}_\varepsilon(u,v) := \mathcal{E}_\varepsilon(u,v;0,L/2) + \mathcal{F}_\varepsilon(v;0,L/2)\,.$$

The following result has been established in [4, Section 5].

Lemma 4.1. *For all x_1, x_2 and $x_3 \in (0,L/2)$ we have*

$$\mathcal{F}_\varepsilon(v;x_1,x_3) \geq \left|\Phi(v(x_1)) + \Phi((v(x_3)) - 2\Phi(v(x_2))\right|,$$

with $\Phi(t) = t - t^2/2$.

Using the monotone increasing character of Φ on $[0,1]$ and choosing $x_1 = x_2 = 0$ and $x_3 = L/2$, we get for all $v \in \mathcal{V}$

$$\mathcal{F}_\varepsilon\left(v; 0, \frac{L}{2}\right) \geq \Phi(1) - \Phi\left(v\left(\frac{L}{2}\right)\right) \geq \frac{1}{2} - \Phi(\alpha)\,. \tag{4.1}$$

Moreover, arguing as in [4, Section 5], we can show the existence of a minimizer $(u_\varepsilon, v_\varepsilon)$ over $\mathcal{B}$ of $\widetilde{AT}_\varepsilon$ such that

$$\limsup_{\varepsilon\to 0} \widetilde{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \frac{1}{2}. \tag{4.2}$$

We will show that, for a convenient choice of $\alpha \in (0,1)$, such a minimizer is a critical point of $\widetilde{AT}_\varepsilon$ with the desired boundary conditions. The proof of the following result is similar to that of [4, Lemma 5.1 & Proposition 5.2].

Lemma 4.2. *There exists $\alpha \in (0,1)$ independent of $\varepsilon > 0$ such that if $(u_\varepsilon, v_\varepsilon)$ is a minimizer over $\mathcal{B}$ of $\widetilde{AT}_\varepsilon$, then it is a critical point of $\widetilde{AT}_\varepsilon$ with $v_\varepsilon'(L/2) = 0$ for ε small enough.*

Proof. It is sufficient to show the existence of $\alpha \in (0,1)$, independent of ε, such that if $(u_\varepsilon, v_\varepsilon)$ is a minimizer over $\mathcal{B}$ of $\widetilde{AT}_\varepsilon$, then $v_\varepsilon(L/2) < \alpha$. Indeed, in that case the minimizer $(u_\varepsilon, v_\varepsilon)$ belongs to the interior of $\mathcal{B}$ and variations of the form $(u_\varepsilon + t\phi, v_\varepsilon + t\psi)$ with $\phi \in \mathscr{C}_c^\infty((0, L/2))$ and $\psi \in \mathscr{C}_c^\infty((0, L/2])$ are allowed. Let $\alpha \in (0,1)$ small enough so that

$$\frac{a^2}{L} - 2\Phi(\alpha) > 0\,. \tag{4.3}$$

Assume by contradiction that there exists $\varepsilon_j \to 0$ such that $v_{\varepsilon_j}(L/2) = \alpha$ for all $j \in \mathbf{N}$. Then consider the sequence

$$\alpha_j^* := \min_{x\in[0,L/2]} v_{\varepsilon_j}(x) \leq \alpha\,.$$

Applying Lemma 4.1 with $x_1 = 0, x_3 = L/2$ and $x_2 = y_j$ where $v_{\varepsilon_j}(y_j) = \alpha_j^*$ leads to

$$\widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{1}{2} + \Phi(\alpha) - 2\Phi(\alpha_j^*)\,.$$

We claim that for all $j \in \mathbf{N}$,

$$\Phi(\alpha) - 2\Phi(\alpha_j^*) \geq \frac{\Phi(\alpha)}{2} > 0\,. \tag{4.4}$$

Provided the claim is proved, we infer from (4.2) that

$$\frac{1}{2} \geq \limsup_{j\to\infty} \widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{1}{2} + \frac{\Phi(\alpha)}{2}\,,$$

which is a contradiction since $\Phi(\alpha) > 0$. We are now reduced to show (4.4).

Proof of the Claim. Let $\alpha_1 \in (0, \alpha)$ be such that $\Phi(\alpha) \geq 4\Phi(\alpha_1)$ and assume by contradiction that $v_{\varepsilon_j}(x) \geq \alpha_1$ for all $x \in [0, L/2]$. Using variations with compact support in $(0, L/2)$, we get that $(u_{\varepsilon_j}, v_{\varepsilon_j})$ solves

$$\begin{cases} \Big((\eta_{\varepsilon_j} + v_{\varepsilon_j}^2)u_{\varepsilon_j}'\Big)' = 0 & \text{in } (0, L/2), \\ -\varepsilon_j v_{\varepsilon_j}'' + v_{\varepsilon_j}|u_{\varepsilon_j}'|^2 + \dfrac{v_{\varepsilon_j} - 1}{\varepsilon_j} = 0 & \text{in } (0, L/2)\,. \end{cases}$$

From the first equation we obtain that $u_{\varepsilon_j}'(\eta_{\varepsilon_j} + v_{\varepsilon_j}^2) = c_j$ a.e. in $(0, L/2)$, for some constant $c_j \in \mathbf{R}$. The upper bound (4.2) shows that $v_{\varepsilon_j} \to 1$ in $L^2(0, L/2)$ and

$$\widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \int_0^{L/2} (\eta_{\varepsilon_j} + v_{\varepsilon_j}^2)|u_{\varepsilon_j}'|^2 \,\mathrm{d}x = \int_0^{L/2} c_j u_{\varepsilon_j}' \,\mathrm{d}x = \frac{ac_j}{2}\,.$$

It implies that $\{c_j\}_{j\in\mathbf{N}}$ is bounded so that, up to a subsequence, $c_j \to c_0$ for some $c_0 \in \mathbf{R}$. Since $v_{\varepsilon_j} \geq \alpha_1$, we deduce that $\{u'_{\varepsilon_j}\}_{j\in\mathbf{N}}$ is bounded in $L^\infty(0,L/2)$. Then Lebesgue's dominated convergence yields

$$\frac{a}{2} = \int_0^{L/2} u'_{\varepsilon_j}\,\mathrm{d}x = \int_0^{L/2} \frac{c_j}{\eta_{\varepsilon_j}+v^2_{\varepsilon_j}}\,\mathrm{d}x \to \int_0^{L/2} c_0\,\mathrm{d}x\,,$$

so that $c_0 = a/L$. Now,

$$\liminf_{j\to\infty} \int_0^{L/2} (\eta_{\varepsilon_j}+v^2_{\varepsilon_j})|u'_{\varepsilon_j}|^2\,\mathrm{d}x = \liminf_{j\to\infty}\int_0^{L/2} c_j u'_{\varepsilon_j}\,\mathrm{d}x = c_0 a = \frac{a^2}{L}\,,$$

and thus, by (4.1) and (4.2),

$$\frac{1}{2} \geq \limsup_{j\to\infty} \widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{a^2}{L} + \frac{1}{2} - \Phi(\alpha)\,,$$

which is in contradiction with (4.3).

We have thus proved by contradiction that $\min_{[0,L/2]} v_{\varepsilon_j} = \alpha_j^* \leq \alpha_1$. Since we assumed that $\Phi(\alpha) \geq 4\Phi(\alpha_1) \geq 4\Phi(\alpha_j^*)$, we infer that (4.4) is satisfied. □

We can now conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $(u_\varepsilon, v_\varepsilon) \in \mathcal{B}$ and $\alpha \in (0,1)$ given by Lemma 4.2 (see (4.3)) so that $(u_\varepsilon, v_\varepsilon)$ satisfies

$$\begin{cases} \Big((\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon\Big)' = 0 & \text{in } (0,L/2)\,,\\ -\varepsilon v''_\varepsilon + v_\varepsilon |u'_\varepsilon|^2 + \dfrac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0,L/2)\,,\\ u_\varepsilon(0) = 0\,,\ u_\varepsilon(L/2) = a/2\,,\\ v_\varepsilon(0) = 0\,,\ v'_\varepsilon(L/2) = 0\,. \end{cases}$$

By the first equation, there exists $c_\varepsilon \in \mathbf{R}$ such that $u'_\varepsilon = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}$. Extending v_ε to $(0,L)$ by symmetry with respect to the vertical axis $\{x = L/2\}$, we obtain a function (still denoted by v_ε) which belongs to $H^1(0,L)$ with $v_\varepsilon(0) = v_\varepsilon(L) = 1$ (this reflexion argument is possible since $v'_\varepsilon(L/2) = 0$). Note that the boundary conditions satisfied by u_ε implies

$$c_\varepsilon = \frac{a}{2}\left(\int_0^{L/2} \frac{\mathrm{d}x}{\eta_\varepsilon + v_\varepsilon^2}\right)^{-1} = a\left(\int_0^{L} \frac{\mathrm{d}x}{\eta_\varepsilon + v_\varepsilon^2}\right)^{-1},$$

where the last equality holds because v_ε is symmetric with respect to the vertical axis $\{x = L/2\}$. The function u_ε is extended to $(0,L)$ (into a function still denoted by u_ε) by setting

$$u_\varepsilon(x) = \int_0^x \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2(t)}\,\mathrm{d}t\,.$$

By construction, $(u_\varepsilon, v_\varepsilon)$ solves

$$\begin{cases} \Big((\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon\Big)' = 0 & \text{in } (0,L)\,,\\ -\varepsilon v''_\varepsilon + v_\varepsilon |u'_\varepsilon|^2 + \dfrac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0,L)\,,\\ u_\varepsilon(0) = 0\,,\ u_\varepsilon(L) = a\,,\\ v_\varepsilon(0) = v_\varepsilon(L) = 0\,. \end{cases}$$

Moreover, the symmetry properties of u_ε and v_ε together with a change of variable yield

$$\limsup_{\varepsilon\to 0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) = 2\limsup_{\varepsilon\to 0} \widetilde{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq 1\,. \tag{4.5}$$

By Theorem 1.1, up to a subsequence, $(u_\varepsilon, v_\varepsilon) \to (u_*, 1)$ in $[L^2(0,L)]^2$ where $u_* \in \{u_{\rm jump}, u_{\rm aff}\}$. Assume by contraction that $u_* = u_{\rm aff}$. According to [2, Proposition 4.1], we have

$$\frac{a^2}{L} \leq \liminf_{\varepsilon\to 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \, {\rm d}x \,. \tag{4.6}$$

By (4.1) together with a change of variable and the symmetry property of v_ε, we obtain

$$\liminf_{\varepsilon\to 0} \int_0^L \left(\varepsilon |v_\varepsilon'|^2 + \frac{(1-v_\varepsilon)^2}{4\varepsilon} \right) {\rm d}x = 2 \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon \left(v_\varepsilon; 0, \frac{L}{2} \right) \geq 1 - 2\Phi(\alpha) \,. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) leads to $a^2/L - \Phi(\alpha) \leq 0$, which is in contradiction with our choice of α in (4.3). Therefore $u_* = u_{\rm jump}$, and the proof is complete. □

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