AN EXAMPLE IN DIMENSION FOUR OF A RIEMANNIAN MANIFOLD SUCH THAT ITS SECTIONAL CURVATURE IS POSITIVE AND ITS CURVATURE OPERATOR IS NOT NONNEGATIVE

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ABSTRACT. In this short note we exhibit a manifold of dimension 4 such that its section curvature is positive and its curvature operator is not nonnegative. This example is based on a discussion in [2], therefore the author thanks all participants in this discussion. *Not meant for publication and the comments are welcome.*

Consider the complex projective space $\mathbb{CP}^2$ with the Fubini–Study metric g. On the open set

$$U := \big\{ \big[(w_1, w_2, w_3)\big] \in \mathbb{CP}^2 : w_1 \neq 0 \big\} ,$$

there are holomorphic coordinates, given by

$$z^1 = \frac{w_2}{w_1} , \qquad z^2 = \frac{w_3}{w_1} ,$$

and the associated usual coordinates, defined by

$$x^1 = \mathrm{Re}\Big(\frac{w_2}{w_1}\Big) , \quad y^1 = \mathrm{Im}\Big(\frac{w_2}{w_1}\Big) , \quad x^2 = \mathrm{Re}\Big(\frac{w_3}{w_1}\Big) , \quad y^2 = \mathrm{Im}\Big(\frac{w_3}{w_1}\Big) .$$

Let us now consider the function

$$f := \big[\chi\big(|z^1|^2\big) + \chi\big(|z^2|^2\big)\big] \, \mathrm{I}_U ,$$

where $\chi : \mathbb{R} \to [0, 1]$ is a smooth function that is 1 on the open interval $(-1/2, 1/2)$ and 0 outside the interval $[-3/2, 3/2]$, and I_U is the characteristic function of the set U. Observe that the function f is smooth, nonconstant and nonnegative. Then, for every $C_1, C_2 > 0$, we can define the Riemannian metric

$$\widetilde{g}_{C_1,C_2} = e^{f/M(C_1,C_2)} \, g \tag{0.1}$$

conformal to the background metric g, where we set

$$M(C_1, C_2) = \sup\{C_1 \, |\nabla f|; C_2 \, |\nabla^2 f|\} ,$$

which belongs to $(0, \infty)$, by the compactness of $\mathbb{CP}^2$. The metric $\widetilde{g}_{C_1,C_2}$ is compatible with the complex structure J of $\mathbb{CP}^2$, as it is conformal to g, but it is not Kählerian, since

$$\frac{\partial(\widetilde{g}_{C_1,C_2})_{1\overline{k}}}{\partial z^2} \neq \frac{\partial(\widetilde{g}_{C_1,C_2})_{2\overline{k}}}{\partial z^1} \tag{0.2}$$

on U, by construction and by recalling that the Hermitian matrix $(g_{k\bar{l}})$ of the Fubini–Study metric g with respect to the initial holomorphic coordinate chart $(U, (z^1, z^2))$ is

$$\frac{1}{\big[1 + |z^1|^2 + |z^2|^2\big]^2} \begin{pmatrix} 1 + |z^2|^2 & -\overline{z}^1 z^2 \\ -\overline{z}^2 z^1 & 1 + |z^1|^2 \end{pmatrix} .$$

The next step is to find some constants $C_1, C_2 > 0$ in a way such that $\widetilde{g}_{C_1,C_2}$ has a positive sectional curvature. For clarity, we specify that we adopt the following conventions

$$\begin{aligned}\mathrm{Riem}(X,Y,Z,W) &= h(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z, W)\,,\\ k_1 \owedge k_2(X,Y,Z,W) &= k_1(X,Z)k_2(Y,W) + k_1(Y,W)k_2(X,Z)\\ &\quad - k_1(X,W)k_2(Y,Z) - k_1(Y,Z)k_2(X,W)\,,\end{aligned}$$

for every $X, Y, Z, W \in \Gamma(TN)$ and for every couple (k_1, k_2) of $(0,2)$–symmetric tensors on a Riemannian manifold (N,h), being ∇ the Levi–Civita connection with respect to the metric h. Then, the transformation law for the Riemannian tensor of type $(0,4)$ under conformal change of metric defined by formula (0.1) is

$$\widetilde{\mathrm{Riem}}_{C_1,C_2} = e^{2\varphi}\Big\{\mathrm{Riem} - \big[\nabla^2\varphi - d\varphi\otimes d\varphi + |\nabla\varphi|^2 g/2\big] \owedge g\Big\}\,,$$

where the function φ is given by

$$\varphi = \frac{f}{2M(C_1,C_2)}\,.$$

Accordingly, at a generic $p \in U$ there holds

$$\begin{aligned}&\widetilde{\mathrm{Sec}}_{C_1,C_2}(\langle u,v\rangle) =\\ &\quad = e^{-f/M(C_1,C_2)}\Big\{\mathrm{Riem} - \Big[\frac{\nabla^2 f}{2M(C_1,C_2)} - \frac{df\otimes df}{4M^2(C_1,C_2)} + \frac{|\nabla f|^2 g}{8M^2(C_1,C_2)}\Big] \owedge g\Big\}(u,v,u,v)\,,\end{aligned}$$

for every $u, v \in T_p\mathbb{CP}^2$ such that

$$g(u,v) = 0\,, \qquad |u|_g^2 = |v|_g^2 = 1\,. \tag{0.3}$$

Since we obtain the inequalities

$$\begin{aligned}\Big|\big[\nabla^2 f \owedge g\big](u,v,u,v)\Big| &= \Big|\nabla^2 f(u,u)g(v,v) + \nabla^2 f(v,v)g(u,u)\Big| \leq 2\,|\nabla^2 f|\\ \Big|\big[df\otimes df \owedge g\big](u,v,u,v)\Big| &\leq 2\,|df\otimes df| = 2\,|\nabla f|^2\end{aligned}$$

as a consequence of the conditions (0.3), it follows

$$\widetilde{\mathrm{Sec}}_{C_1,C_2}(\langle u,v\rangle) \geq e^{-f/M(C_1,C_2)}\Big\{\mathrm{Sec}(\langle u,v\rangle) - \frac{|\nabla^2 f|}{M(C_1,C_2)} - \frac{3\,|\nabla f|^2}{4M^2(C_1,C_2)}\Big\}\,,$$

then, choosing $C_1 = 1$ and $C_2 = 8$, we have

$$\widetilde{\mathrm{Sec}}_{1,8}(\langle u,v\rangle) \geq e^{-f/M(1,8)}\Big(1 - \frac{1}{8} - \frac{3}{4}\Big) \geq \frac{1}{8}\,e^{-2/M(1,8)} > 0\,.$$

Thus, the Riemannian metric $\widetilde{g}_{1,8}$ has positive sectional curvature, but its curvature operator has some negative eigenvalues, by the following proposition (see [1, Theorem 2.1] and references therein).

Proposition 0.1. *A closed simply connected manifold with nonnegative curvature operator is isometric to a Riemannian product of*

1. *standard spheres with metrics of nonnegative curvature operator;*
2. *closed Kähler manifolds biholomorphic to complex projective spaces whose Kähler metric has nonnegative curvature operator on real (1,1)-forms;*
3. *compact irreducible Riemannian symmetric spaces with their natural metrics of nonnegative curvature operator.*

Indeed, the first case is not possible by a topological reason, while the second one is excluded as $(\mathbb{CP}^2, \widetilde{g}_{1,8})$ is not a Kähler manifold, by formula (0.2). Concerning the third case, we recall that the only simply connected symmetric spaces having positive sectional curvature are the spheres, complex and quaternionic projective spaces, and the Cayley plane $\mathrm{F}_4/\mathrm{Spin}(9)$, all endowed with their standard metric, in particular, those of dimension 4 among them are $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{HP}^1$, as the Cayley plane has dimension 16. Now, $(\mathbb{CP}^2, \widetilde{g}_{1,8})$ can not be isometric to $\mathbb{S}^4$ and $\mathbb{HP}^1$, since the second homology groups with integer coefficients of $\mathbb{CP}^2$ and $\mathbb{S}^4$ are different and the quaternionic projective line $\mathbb{HP}^1$ is homeomorphic to the 4–sphere. Trivially, $(\mathbb{CP}^2, \widetilde{g}_{1,8})$ and $(\mathbb{CP}^2, g)$ are not isometric. Finally, $(\mathbb{CP}^2, \widetilde{g}_{1,8})$ can not be isometric to a Riemannian product, since it has positive sectional curvature and the sectional curvatures of the "mixed" planes in a Riemannian product are equal to zero.

REFERENCES

1. A. Derdziński, F. Mercuri, and M. H. Noronha, *Manifolds with pure nonnegative curvature operator*, Bol. Soc. Brasil. Mat. **18** (1987), 13–22.
2. MathOverflow, *Positive sectional curvature does not imply positive definite curvature operator?*, https://mathoverflow.net/q/264896, 2017.

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