Γ-limit of 2D traveling waves in the FitzHugh-Nagumo system

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Abstract: Traveling waves are commonly observed in evolution systems. Such waves are robust in the sense that they are stable and exist for a wide range of parameters. Through Γ-convergence analysis, a well-known tool for studying concentration phenomena, a geometric variational problem representing the Γ-limit of a FitzHugh-Nagumo system in two dimensional domains is studied; this yields both the wave speed and the structure of a minimizer. In particular we demonstrate that 1D traveling fronts can become unstable when subject to 2D perturbation. In suitable parameter regimes multiple traveling waves, including non-planar structures, can co-exist. Stationary waves have been studied using geometric variational problems; ours represent the first attempt to treat non-stationary wave problems in multi-dimensional domains.

Key words: Γ-convergence, traveling wave, FitzHugh-Nagumo system, geometric variational problem, wave stability.

AMS subject classification: 35B08, 35K40, 35K57, 49J40.

1 Introduction

Patterns and waves are fundamental subjects that have been extensively studied [3, 15, 16, 20, 22, 30, 34, 37, 45, 46] in evolution systems. For reaction-diffusion systems, regularly recurring patterns are frequently found when physical parameters lie in the vicinity of Turing's instability regime [42]. On the other hand recent advances [8, 9, 10, 11, 17, 18, 19, 21, 24, 35, 43, 44] demonstrate that certain patterns and waves may possess localized spatial or temporal structures. Such localized structures, far from trivial steady states, are robust and exist for a wide range of parameters.

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An interesting reaction-diffusion model is (a special form of) the FitzHugh-Nagumo system

$$\begin{cases} u_t &= \Delta u + \frac{1}{d}(f_\epsilon(u) - \epsilon\sigma v)\,, \\ \\ v_t &= \Delta v + u - \gamma v\,, \end{cases} \tag{1.1}$$

see [26] and [33], where

$$f_\epsilon(\xi) = -\xi(\xi - \beta_\epsilon)(\xi - 1), \qquad \beta_\epsilon = \frac{1}{2} - \frac{\alpha\epsilon}{\sqrt{2}} \tag{1.2}$$

and $d, \alpha, \gamma, \sigma, \epsilon$ are all positive parameters. Originally derived as an excitable system for modeling nerve impulse propagation (when the term Δv is absent), it is now of great interest to the scientific community as the breeding ground for pattern formation and wave propagation. The physical parameter $\epsilon\alpha$ measures the drive towards a non-trivial state while $\epsilon\sigma$ and $1/\gamma$ are stabilizing inhibition mechanisms that favor the opposite. Such a competition leads to interesting dynamics and the emergence of patterns. The parameter d, small in many applications, makes the patterns more pronounced as it results in sharp spatial transition zones.

The existence of a singular limit as $\epsilon \to 0$ will ease qualitative understanding of the self-organization mechanisms responsible for these pattern formations. The notion of Γ-convergence [7] is particularly useful in this regard: when $d = \epsilon^2$ singular limits of stationary solutions of (1.1) are governed by a geometric variational problem [1, 13, 14] associated with the action functional

$$\mathcal{J}_D(\Omega) = P(\Omega; D) - \alpha|\Omega| + \frac{\sigma}{2}\int_\Omega \mathcal{N}_D(\Omega)dx\,, \tag{1.3}$$

where $D \subset \mathbb{R}^N$ is a given domain. For any subset $\Omega \subset D$, $|\Omega|$ and $P(\Omega; D)$ denote the volume of Ω and its perimeter in D, respectively. Let χ_Ω denote the characteristic function of Ω. The integral term in (1.3) represents a nonlocal interaction energy and $\mathcal{N}_D(\Omega)$ is the solution of the modified Helmholtz equation

$$-\Delta\mathcal{N}_D(\Omega) + \gamma\mathcal{N}_D(\Omega) = \chi_\Omega$$

subject to prescribed boundary condition on ∂D. The stable and unstable ball shaped stationary sets in $\mathbb{R}^N$ have been completely classified in [13, 14]. Similar results for periodic lamellar structures in square tori have also been obtained in [1, 2]. Another model that gives rise to nonlocal geometric variational problems in the literature is the Ohta-Kawasaki model, see for example [3, 23, 36] and the references therein, for which a Γ-convergence analysis yields a limiting problem that involves the Laplace operator. However, while a (length) scaling argument is sometimes possible for the Laplace, it never works for the Helmholtz operator which appears in the model considered here.

The search for the Γ-limit of temporal patterns of reaction-diffusion systems is still in its infancy; the only known result seems to be the 1D case studied in [12]. Traveling waves, the most known temporal patterns, are ubiquitous in physical and biological systems. These waves appear stationary when viewed by an observer moving with the wave speed. (There is no known traveling waves in the Ohta-Kawasaki model). A traveling front connects 2 distinct stationary solutions while a pulse originates and ends at the same state. Front propagation is found in diverse fields such as phase transition, combustion and population dynamics. Pulses typically result from a delicate balance between gain and loss in reaction kinetics free of external input. Precise conditions on the parameters have been given in [12] for the existence of 1D traveling fronts and pulses of the Γ-limit formulation. The focus of this paper is the Γ-limit of traveling waves in a 2D domain given by an infinite rectangular strip with width T under a periodicity condition in the vertical direction. In particular, under the right parameter regime, we demonstrate that 1D traveling fronts are stable for small T and unstable for large T when subject to 2D perturbations.

Since throughout the paper we will work in a periodic setting, it is convenient to introduce the *flat torus* $\mathbb{T}^2_T$ defined as the set of equivalence classes of points in $\mathbb{R}^2$ under the equivalence relation

$$(x,y)\sim(x',y') \quad \text{if and only if} \quad x'=x,\ y'=y+hT \ \text{ for some } \ h\in\mathbb{Z}$$

and endowed with the metric and the differential structure inherited from $\mathbb{R}^2$. However, for the ease of presentation we sometimes identify the flat torus $\mathbb{T}^2_T$ with the infinite strip

$$\Omega_T = \mathbb{R}\times\Big(-\frac{T}{2},\frac{T}{2}\Big], \tag{1.4}$$

and denote an element of $\mathbb{T}^2_T$, which is an equivalence class, by $z=(x,y)$, where (x,y) is the unique representative of the class such that $(x,y)\in\Omega_T$.

If $1\le p<\infty$ we will denote by $L^p_e(\mathbb{T}^2_T)$ the set of functions $u\in L^1_{loc}(\mathbb{T}^2_T)$ such that $\|u\|_{L^p_e(\mathbb{T}^2_T)} = \big(\int_{\mathbb{T}^2_T} e^x|w|^p\,dz\big)^{1/p}<\infty$ and by $H^1_e(\mathbb{T}^2_T)$ the space of functions $u\in L^2_e(\mathbb{T}^2_T)$ with derivatives in $L^2_e(\mathbb{T}^2_T)$ equipped with the norm $\|u\|_{H^1_e(\mathbb{T}^2_T)}=\sqrt{\|u\|^2_{L^2_e(\mathbb{T}^2_T)}+\|Du\|^2_{L^2_e(\mathbb{T}^2_T)}}$. Clearly, $L^2_e(\mathbb{T}^2_T)$ and $H^1_e(\mathbb{T}^2_T)$ are Hilbert spaces with their inner products defined in the obvious way. The spaces of functions $C^k(\mathbb{T}^2_T)$ and $C^{k,\alpha}(\mathbb{T}^2_T)$, where $k\ge 0$ is an integer and $\alpha\in(0,1]$, are also defined as usual. The set of functions in $C^k(\mathbb{T}^2_T)$ with compact support in $\mathbb{T}^2_T$ will be denoted by $C^k_c(\mathbb{T}^2_T)$.

In deploying the variational approach to the existence of traveling waves of (1.1) which are periodic in the y direction, we assume the ansatz $(u(c(x-ct),y),v(c(x-ct),y))$ proposed in [28]. In turn this leads to proving the existence of a weak solution $(u,v)\in H^1_e(\mathbb{T}^2_T)\times H^1_e(\mathbb{T}^2_T)$ of the elliptic system

$$dc^2u_{xx}+du_{yy}+dc^2u_x+f_\epsilon(u)-\epsilon\sigma v=0\,, \tag{1.5}$$

$$c^2v_{xx}+v_{yy}+c^2v_x-\gamma v+u=0\,, \tag{1.6}$$

for some wave speed c to be determined. To this aim we set $F_\epsilon(w):=-\int_0^w f_\epsilon(\xi)\,d\xi$ so that

$$F_\epsilon=F_0+\alpha\epsilon G, \quad \text{where} \quad F_0(u):=\frac{1}{4}u^2(u-1)^2, \quad G(u):=\frac{1}{\sqrt{2}}\Big(\frac{u^3}{3}-\frac{u^2}{2}\Big)\,. \tag{1.7}$$

In this decomposition, F_0 is a balanced bistable nonlinearity in the sense that $F_0(0)=F_0(1)=\min F_0=0$; $G(0)=0$ is a local maximum and $G(1)=-1/6\sqrt{2}$ is a local minimum. As for their sum, when ϵ is small $F_\epsilon(0)=0$ is a local minimum, $F_\epsilon(1)=-\frac{1-2\beta_\epsilon}{12}=-\frac{1}{6\sqrt{2}}\alpha\epsilon$ is the global minimum, while $F_\epsilon(\beta_\epsilon)>0$ is the unique local maximum.

For any $u \in L^2_e(\mathbb{T}^2_T)$ let $v = \mathcal{L}_c u$ be the unique solution in $H^1_e(\mathbb{T}^2_T)$ of (1.6). Note that v is the minimizer in $H^1_e(\mathbb{T}^2_T)$ of the functional

$$v \mapsto \int_{\mathbb{T}^2_T} e^x \Big(\frac{c^2 v_x^2}{2} + \frac{v_y^2}{2} + \frac{\gamma v^2}{2} - vu \Big)\, dz\,. \tag{1.8}$$

It is easy to check that $\mathcal{L}_c : L^2_e(\mathbb{T}^2_T) \to H^1_e(\mathbb{T}^2_T)$ is a self-adjoint operator with respect to the inner product of $L^2_e(\mathbb{T}^2_T)$. Note that if $E \subset \mathbb{T}^2_T$ is measurable then $0 \le \mathcal{L}_c \chi_E \le 1/\gamma$. Indeed, if it were otherwise, the function $v_{cut} = (\mathcal{L}_c \chi_E \vee 0) \wedge \frac{1}{\gamma}$, which belongs also to $H^1_e(\mathbb{T}^2_T)$, would lower the functional (1.8), thus contradicting the minimality of $\mathcal{L}_c \chi_E$.

Given $c, \epsilon, d > 0$, let $\mathcal{I}_{c,d,\epsilon} : H^1_e(\mathbb{T}^2_T) \to \mathbb{R}$ be defined as

$$\mathcal{I}_{c,d,\epsilon}(w) = \int_{\mathbb{T}^2_T} e^x \left(\frac{dc^2}{2} w_x^2 + \frac{d}{2} w_y^2 + F_\epsilon(w) + \frac{\epsilon\sigma}{2} w \mathcal{L}_c w \right) dz\,. \tag{1.9}$$

A standard variational argument shows that (u, v, c) solves (1.5)-(1.6) provided u is a critical point of $\mathcal{I}_{c,d,\epsilon}$ and $v = \mathcal{L}_c u$. The last term in the integral above is referred to as the nonlocal energy. A simple calculation shows that $\int_{\mathbb{T}^2_T} e^x w\, \mathcal{L}_c w\, dx \ge 0$ for all $w \in L^2_e(\mathbb{T}^2_T)$.

A function in $H^1_e(\mathbb{T}^2_T)$ is not necessarily bounded on $\mathbb{T}^2_T$. To seek a (bounded) traveling wave solution when $\epsilon \le 1$, we will choose $\widetilde{M} > 2$ such that

$$(\widetilde{M} - 1)(\widetilde{M} - 2)^2 > \sigma \frac{\widetilde{M}}{\gamma}\,, \tag{1.10}$$

an assumption that will be used in Section 5, and restrict $\mathcal{I}_{c,d,\epsilon}$ to the domain

$$Y := \left\{ w \in H^1_e(\mathbb{T}^2_T) : \int_{\mathbb{T}^2_T} e^x w^2\, dz = 1,\ -\widetilde{M} \le w \le \widetilde{M} \right\}. \tag{1.11}$$

Note that the constraint $\|u\|_{L^2_e(\mathbb{T}^2_T)} = 1$ imposed in Y eliminates a continuum of minimizers due to translation in the x-direction. Assume that for some fixed σ, α, γ, ϵ and d, one can find a suitable value of c and a function $u \in Y$ such that $\mathcal{I}_{c,d,\epsilon}(u) = \inf_Y \mathcal{I}_{c,d,\epsilon} = 0$. Then one can show that the Lagrange multiplier associated with this integral constraint is zero. At this point one would like to prove that $||u||_{L^\infty} < \widetilde{M}$ so that the minimizer u is unconstrained and satisfies the Euler-Lagrange equations (1.5)-(1.6). In the 1D case (with ϵ not necessarily small), such an argument has been successfully carried out in [10]. Unfortunately a generalization to the multi-dimensional case seems to be technically challenging.

In this paper we present an alternative path to tackle this problem. We introduce a geometric variational functional $\mathcal{J}_c$ which turns out to be in a proper sense the Γ-limit of $\mathcal{I}_{c,d,\epsilon}$ as $\epsilon \to 0$. Under suitable restrictions on the parameters, we prove that there exist a speed $c_0 > 0$ and a minimizer set E_0 such that $\mathcal{J}_{c_0}(E_0) = \inf \mathcal{J}_{c_0} = 0$. From this, using the Γ-convergence result, we are able to deduce the existence of a traveling wave solution to (1.5)-(1.6) for small ϵ. We explain this procedure in details below.

Given $c, \epsilon > 0$, we define

$$J_{c,\epsilon}(w) = \begin{cases} \displaystyle\int_{\mathbb{T}^2_T} e^x \left\{ \frac{\epsilon w_x^2}{2} + \frac{\epsilon w_y^2}{2c^2} + \frac{F_0(w)}{\epsilon} + \alpha G(w) + \frac{\sigma}{2} w \mathcal{L}_c w \right\} dz\,, & \text{if } w \in Y, \\ \infty, & \text{if } w \in L^2_e(\mathbb{T}^2_T) \setminus Y. \end{cases} \tag{1.12}$$

Observe that $J_{c,\epsilon}(w) = \epsilon^{-1}\mathcal{I}_{c,d,\epsilon}(w)$ whenever $w \in Y$ and $d = \epsilon^2/c^2$. By the change of variables $\tilde{y} = cy$, setting $\tilde{w}(x,\tilde{y}) = w(x,\tilde{y}/c)$, we have

$$\int_{\mathbb{T}^2_T} e^x \left\{ \frac{\epsilon w_x^2}{2} + \frac{\epsilon}{2c^2} w_y^2 + \frac{F_0(w)}{\epsilon} \right\} dz = \frac{1}{c} \int_{\mathbb{T}^2_{cT}} e^x \left\{ \frac{\epsilon |\nabla \tilde{w}|^2}{2} + \frac{F_0(\tilde{w})}{\epsilon} \right\} dx\, d\tilde{y}\,.$$

If $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, it is well known, see [32], that the functionals

$$\int_\Omega \left\{ \frac{\epsilon |\nabla w|^2}{2} + \frac{F_0(w)}{\epsilon} \right\} dz$$

Γ-converge in $L^1(\Omega)$ as $\epsilon \to 0$ to the perimeter of a limit set $\widetilde{E} \subset \Omega$. A similar result holds for the functionals in (1.12). However in our case some technical difficulties arise due to the non-compactness of $\mathbb{T}^2_T$ and to the presence of the weight, while the nonlocal term is easy to handle. In order to give the representation formula for the Γ-limit of (1.12), we set

$$\mathcal{J}_c(E) = \frac{1}{c} \frac{\sqrt{2}}{12} \mathcal{P}_e(E_c; \mathbb{T}^2_{cT}) - \frac{\sqrt{2}}{12} \alpha \int_{\mathbb{T}^2_T} e^x \chi_E \, dz + \frac{\sigma}{2} \int_{\mathbb{T}^2_T} e^x \chi_E \mathcal{L}_c \chi_E \, dz\,, \tag{1.13}$$

where $E \subset \mathbb{T}^2_T$ is a measurable set,

$$E_c := \left\{ (x, \tilde{y}) \in \mathbb{T}^2_{cT} : (x, \tilde{y}/c) \in E \right\} \tag{1.14}$$

and $\mathcal{P}_e(E_c; \mathbb{T}^2_{cT})$ denotes the the *weighted perimeter* of E_c in $\mathbb{T}^2_{cT}$, see the definition in (2.4). Note that if $E \subset \mathbb{T}^2_T$ is a smooth open set then

$$\mathcal{P}_e(E; \mathbb{T}^2_T) = \int_{\partial E} e^x \, d\mathcal{H}^1\,,$$

where ∂E *is the boundary of* E *as a subset of* $\mathbb{T}^2_T$ and $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Recall also that the function $\mathcal{L}_c \chi_E$ in (1.13) is the unique solution in $H^1_e(\mathbb{T}^2_T)$ of the equation

$$-c^2 v_{xx} - v_{yy} - c^2 v_x + \gamma v = \chi_E\,. \tag{1.15}$$

Roughly speaking, it turns out that the Γ-limit of the functionals $J_{c,\epsilon}$ is $\mathcal{J}_c$. To be precise, let us define $J^*_c : L^2_e(\mathbb{T}^2_T) \to (-\infty, +\infty]$ as

$$J^*_c(w) = \begin{cases} \mathcal{J}_c(E) & \text{if } w = \chi_E \text{ for a measurable set } E \subset \mathbb{T}^2_T \text{ with } |E|_e = 1, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.16}$$

where $|E|_e$ stands for the *weighted volume of* E, defined as

$$|E|_e := \int_{\mathbb{T}^2_T} e^x \chi_E \, dz \, .$$

Then, we have that $J^*_c = \Gamma\text{-}\lim_{\epsilon\to 0} J_{c_\epsilon,\epsilon}$ in $L^2_e(\mathbb{T}^2_T)$, whenever $c_\epsilon \to c > 0$ as $\epsilon \to 0$, see Theorem 4.2.

We now state our main results, starting with the existence of traveling waves solutions for the limit problem and for the FitzHugh-Nagumo equation. To this aim we introduce the following conditions on the parameters α, σ, γ and c:

$$(TW1) \qquad \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0 \, , \tag{1.17}$$

$$(TW2) \qquad \inf\{\mathcal{J}_c(E) : \, E \subset \mathbb{T}^2_T \ \text{ and } \ |E|_e = 1\} = 0 \, . \tag{1.18}$$

Theorem 1.1 (Traveling waves of the limiting problem)**.**
(i) Let $c, \gamma > 0$*,* $\sigma \geq 0$ *and* $\alpha \in \mathbb{R}$*. The minimum problem*

$$\min\{\mathcal{J}_c(E) : \, E \subset \mathbb{T}^2_T \ \text{ and } \ |E|_e = 1\} \tag{1.19}$$

admits at least a solution E*. Any such minimizer is a connected open set with boundary of class* $C^{3,\beta}$ *for all* $0 < \beta < 1$*, such that* $E \subset \big\{(x,y) \in \mathbb{T}^2_T : \, x \leq M\big\}$ *for some* $M > 0$ *depending only on* c, σ, γ *and* T*.*

Moreover, if (TW1) holds there exists $c_0 > 0$ *such that (TW2) holds. In this case any absolute minimizer* E_0 *of* $\mathcal{J}_{c_0}$ *is an unconstrained critical point of* $\mathcal{J}_{c_0}$*. Therefore the Lagrange multiplier associated with the volume constraint* $|E|_e = 1$ *is zero and so* c_0 *is the wave speed associated with the limiting wave profile* E_0*.*

(ii) Let E *be an (unconstrained) critical point of* $\mathcal{J}_c$ *of class* C^2*. Then it satisfies the following Euler-Lagrange equation*

$$\frac{\sqrt{2}}{12}\frac{c\,\kappa}{(c^2\sin^2\theta + \cos^2\theta)^{3/2}} + \frac{\sqrt{2}}{12}\frac{c\sin\theta}{\sqrt{\cos^2\theta + c^2\sin^2\theta}} + \sigma\mathcal{L}_c\chi_E - \frac{\sqrt{2}}{12}\alpha = 0 \quad \textit{on } \partial E \, . \tag{1.20}$$

Here κ *is the signed curvature of* ∂E *(i.e.* $\kappa = \operatorname{div}_\tau \nu$ *where* ν *is the exterior unit normal) and* θ *is the signed angle made by the tangent vector with the positive* x*-axis.*

In the following, whenever E is a minimizer of problem (1.19) we shall say that E is a *constrained minimizer* of $\mathcal{J}_c$. Note that formula (1.20) has a simpler expression if one replaces the curvature of ∂E with that of ∂E_c, where E_c is the set defined in (1.14), see (6.20).

As a consequence of the previous result we are able to recover the existence of traveling waves for the FitzHugh-Nagumo equations provided ϵ is sufficiently small .

Theorem 1.2 (Traveling waves for FitzHugh-Nagumo equations)**.**

1. *Assume condition (TW1) holds. Then there exists* $\epsilon_1 > 0$ *with the property that for any* $0 < \epsilon < \epsilon_1$*, there is* $c_\epsilon > 0$ *such that, on setting* $d_\epsilon = \epsilon^2/c^2_\epsilon$ *and* $v_\epsilon = \mathcal{L}_{c_\epsilon} u_\epsilon$*, there exists a traveling wave solution* $(u_\epsilon, v_\epsilon, c_\epsilon)$ *of (1.5)-(1.6) with* $\|u_\epsilon\|_{L^2_e(\Omega_T)} = 1$*. Moreover* $\mathcal{I}_{c_\epsilon,d_\epsilon,\epsilon}(u_\epsilon) = 0$ *and* u_ϵ *is an unconstrained minimizer of* $\mathcal{I}_{c_\epsilon,d_\epsilon,\epsilon}$*.*

2. *If $c_0 > 0$ is an isolated root of the function $c \to \min_Y \mathcal{J}_c$ and E_0 is a strict minimizer of $\mathcal{J}_{c_0}$, then there exist a sequence $\epsilon_h \to 0^+$ and a corresponding sequence (u_h, v_h, c_h) of traveling wave solutions of* (1.5)-(1.6) *such that $c_h \to c_0$, $u_h \to \chi_{E_0}$ in $L^2_e(\mathbb{T}^2)$ and $v_h \to \mathcal{L}_{c_0}\chi_{E_0}$ in $H^1_e(\mathbb{T}^2_T)$.*

The proof of the above Theorem is based on the existence of a speed c_0 for which condition (TW2) holds, on the continuity of the function $c \to \inf_Y \mathcal{J}_c$ and on the already mentioned Γ-convergence result Theorem 4.2.

We now turn to the issue of the stability of traveling waves. Given a smooth vector field $X : \mathbb{T}^2_T \mapsto \mathbb{T}^2_T$ with compact support, we consider the associated flow $\Phi : \mathbb{T}^2_T \times (-\infty, \infty) \mapsto \mathbb{T}^2_T$ defined as the solution of the following equation

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(z,t) = X(\Phi(z,t)) \ , \\ \Phi(z,0) = z. \end{cases} \tag{1.21}$$

The global existence and uniqueness of Φ are a consequence of the fact that the vector field X is smooth and bounded. Note that $\Phi(\cdot, t)$ is a smooth diffeomorphism from $\mathbb{T}^2_T$ to $\mathbb{T}^2_T$ for all t. Let $E \subset \mathbb{T}^2_T$ be an open set of class C^2 and set $E_t := \Phi(\cdot, t)(E)$. The *first* and the *second variations of $\mathcal{J}_c$ at E with respect to the vector field X* are defined, respectively, as

$$\partial \mathcal{J}_c(E)[X] = \frac{d}{dt}\mathcal{J}_c(E_t)\Big|_{t=0}, \qquad \partial^2 \mathcal{J}_c(E)[X] = \frac{d^2}{dt^2}\mathcal{J}_c(E_t)\Big|_{t=0}.$$

The first and second variation formulae will be proved in Section 6. The first variation is given in (6.22). Setting the first variation to zero for all X leads to the Euler-Lagrange equation. The second variation formula is more involved and requires some additional notation. To this aim, let E and X be as above and denote by ν and ν_c the exterior unit normals to the sets E and E_c, respectively. Then, define a vector field $X_c : \mathbb{T}^2_{cT} \mapsto \mathbb{T}^2_{cT}$ as follows

$$X_c(x,y) = (X_{c,1}, X_{c,2}) := (X_1(x, y/c), cX_2(x, y/c)) \qquad \text{for } (x,y) \in \mathbb{T}^2_{cT}$$

and set $Z_c := DX_c[X_c] = \sum_{j=1}^2 X_{c,j}\, D_j X_c$. Finally, denote by D_{τ_c} and div_{τ_c} the tangential gradient and divergence, respectively, on ∂E_c and by G the Green's function associated with the operator $\mathcal{L}_c$, so that if $u \in L^2_e(\mathbb{T}^2_T)$ then $\mathcal{L}_c u\,(z) = \int_{\mathbb{T}^2_T} G(z,w)\, u(w)\, dw$ for all $z \in \mathbb{T}^2_T$.

Theorem 1.3 (Second variation formula)**.**
Let $E \subset \mathbb{T}^2_T$ be an open set of class C^2 and let $X : \mathbb{T}^2_T \mapsto \mathbb{T}^2_T$ be a smooth field with compact support. Setting $v = \mathcal{L}_c\chi_E$, we have

$$\begin{aligned} \partial^2 \mathcal{J}_c(E)[X] =& \frac{\sqrt{2}}{12\, c}\int_{\partial E_c} e^x \left(X_{c1}^2 + DX_{c1}\cdot X_c + 2X_{c1}\mathrm{div}_{\tau_c} X_c + \mathrm{div}_{\tau_c} Z_c + |(D_{\tau_c} X_c)\cdot \nu_c|^2 \right) d\mathcal{H}^1 \\ &+ \sigma \int_{\partial E} d\mathcal{H}^1_w \int_{\partial E} e^x G(z,w)\, X(z)\cdot\nu(z)\, X(w)\cdot \nu_E(w) d\mathcal{H}^1_z \\ &+ \sigma \int_{\partial E} \mathrm{div}(e^x v X)\, X\cdot\nu\, d\mathcal{H}^1 - \frac{\sqrt{2}}{12}\alpha \int_{\partial E} \mathrm{div}\,(e^x X)\, X\cdot \nu\, d\mathcal{H}^1 \ . \end{aligned} \tag{1.22}$$

In case X is weighted volume preserving, so that $\mathrm{div}\,(e^x X) = 0$ *on $\mathbb{T}^2_T$, the last integral on the right hand side is zero.*

We use this theorem to investigate the stability of a planar traveling front for the limit problem.

In the 1D counterpart of our problem the necessary and sufficient conditions for the existence of a traveling front turns out to be the same as condition (TW1), see [12]. In this case the wave speed c_f is uniquely given by

$$c_f = \frac{2h_*\sqrt{\gamma}}{\sqrt{1-h_*^2}}, \quad \text{where} \quad h_* := 1 - \frac{(\alpha-1)\gamma}{3\sqrt{2}\sigma} > 0\,. \tag{1.23}$$

From [12, Lemma 6.2] we know that

$$(A1)^* \qquad \alpha \geq \frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0$$

is a necessary and sufficient condition for the planar front $(-\infty, 0)$ to be the unique global minimizer of the 1D counterpart of problem (1.19) with $\inf J_{c_f} = 0$; therefore it is locally stable when subject to 1D perturbations.

At the same time the condition

$$(A2) \qquad \frac{3\sqrt{2}\sigma}{\gamma} > \alpha > \alpha - 1 > 0$$

is necessary and sufficient for a global 1D minimizer to be a planar pulse for some unique wave speed c_p, see [12, Lemma 7.3 and Remark 7.9]. Since c_f and c_p satisfy (6.4) and (7.4) of [12], respectively, one easily shows that $c_p < c_f$ whenever both planar waves coexist in the same parameter regimes.

A planar traveling wave which happens to be a global minimizer among 1D configurations needs to be a connected interval, see [12, Lemma 5.1]. Therefore from the above discussion it is clear that in the parameter range not considered in $(A1)^*$ and $(A2)$, that is when $0 < \frac{3\sqrt{2}\sigma}{\gamma} \leq \alpha - 1$, no such 1D traveling wave exists.

We now go back to our 2D analysis. Conditions $(A1)^*$ and $(A2)$ will continue to play crucial roles for the local stability of the planar front. If a traveling wave E is merely a critical point of $\mathcal{J}_c$ (and therefore satisfies the Euler-Lagrange equation) but not a global minimizer, there is a possibility that it is composed of disconnected sets. However the necessary condition $\mathcal{J}_c(E) = 0$ is always satisfied; this follows from setting $X = e_1 := (1, 0)$ in $0 = \partial\mathcal{J}_c(E)[X] = \frac{d}{dt}(e^t\mathcal{J}_c(E))\big|_{t=0} = \mathcal{J}_c(E)$.

Theorem 1.4 (Stability of Γ-limit traveling planar front)**.**
Suppose conditions (TW1) and (TW2) hold and W is a traveling planar front with speed c_f.

1. *Let condition $(A1)^*$ hold. Then the front W is a global minimizer among all 1D configurations, and is locally stable with respect to 2D perturbations for all strip width T.*

2. *Suppose condition (A2) holds and $h_+ := \frac{1}{2}\left(\sqrt{1+\frac{4}{\alpha-1}} - 1\right)$. Then*
   (a) If $1 < \alpha \leq 3/2$, the front W is stable for all T.
   (b) If $\alpha > 3/2$ and

$$(A2a) \qquad \frac{3\sqrt{2}\sigma}{\gamma} > \frac{\alpha-1}{1-h_+}\,,$$

there exists a unique $T_0 > 0$ such that whenever $T > T_0$, the front W is unstable, and for $T < T_0$, the front is stable.
(c) If $\alpha > 3/2$ and

$$\alpha < \frac{3\sqrt{2}\sigma}{\gamma} \leq \frac{\alpha - 1}{1 - h_+}, \tag{A2b}$$

then W is stable for all T.

In Theorem 1.1 we have established a traveling wave of the limiting problem in $\mathbb{T}^2_T$; this wave may be planar. With further restriction on the parameters, there are multiple co-existing planar and non-planar traveling waves.

Theorem 1.5. *Fix $\gamma > 0$ and let condition (A2) hold with $\frac{3\sqrt{2}\sigma}{\gamma} = A\alpha$ for some $A > 1$. Then there exists $A_0 > 1$ such that for every $A \geq A_0$, there are two positive constants $T_0 = T_0(A)$ and $\alpha_* = \alpha_*(A)$ with the following property. Whenever the torus size $T \geq T_0$ and $\alpha \geq \alpha^*$, there co-exist at least 3 traveling waves: a non-planar global minimizer with speed c_*, a planar pulse with speed c_p and a planar front with speed c_f satisfying the inequalities $c_* < c_p < c_f$.*

The layout of this paper is as follows. In Section 2 we introduce the space $BV_e(\mathbb{T}^2_T)$ of functions of finite weighted total variation in $\mathbb{T}^2_T$. These functions will be used in the proof of the Γ-convergence worked out in Theorem 4.2. The section contains also the definition and the main properties of sets of finite weighted perimeter in $\mathbb{T}^2_T$. In Section 3 we prove the existence of a minimizer for problem (1.19). We will also show that any minimizer is connected and bounded from the right in the x-direction. Finally we will prove that condition (TW1) yields the existence of a speed c_0 for which (TW2) holds. These results will establish Statement (i) of Theorem 1.1. Section 4 starts with a compactness property for a sequence of functions $u_h \in Y$ such that $\sup_h J_{c_h,\epsilon_h}(u_h) < \infty$, with $\epsilon_h \to 0+$ and $c_h \to c > 0$. This is the key ingredient in the proof of the Γ-convergence of the functionals (1.13). The characterization of the Γ-limit is then used in Section 5 to deduce the existence and stability of traveling waves for the FitzHugh-Nagumo system stated in Theorem 1.2. In Section 6 we calculate the first and the second variations of the geometric functional $\mathcal{J}_c$ subject to a smooth vector field. The former leads to the Euler-Lagrange equation (Statement (ii) of Theorem 1.1), while the latter is the content of Theorem 1.3. The second variation at a critical point of $\mathcal{J}_c$ will allow us in Section 7 to study the stability and instability of a planar traveling front with respect to 2D perturbations. As stated in Theorem 1.4, depending on the values of mutual relations among the parameters α, γ and σ, this wave is always stable when T is small, but may be unstable when T is large. In Section 8 we give a proof of Theorem 1.5 using energy comparison. Finally the Appendix contains the proofs of some technical facts used in the paper, including a regularity result for minimizers of the functional $\mathcal{J}_c$.

2 Periodic BV functions and sets of finite perimeter

Given $u \in L^1_{loc}(\mathbb{T}^2_T)$, we define the *weighted total variation of u in $\mathbb{T}^2_T$* with respect to the measure $e^x dz$ as

$$\|Du\|_e(\mathbb{T}^2_T) := \sup\left\{\int_{\mathbb{T}^2_T} u \operatorname{div}(e^x\varphi)\, dz :\ \varphi \in C^1_c(\mathbb{T}^2_T;\mathbb{R}^2), |\varphi| \leq 1\right\}. \tag{2.1}$$

Note that if $\|Du\|_e(\mathbb{T}^2_T) < \infty$, for any bounded open set $U \subset \mathbb{T}^2$ the *total variation of u in U*

$$\|Du\|(U) := \sup\left\{ \int_U u \,\mathrm{div}\varphi \, dz : \ \varphi \in C^1_c(\mathbb{T}^2_T;\mathbb{R}^2), \ \mathrm{supp}\,\varphi \subset U, \ |\varphi| \le 1 \right\}$$

is also finite. Therefore by the Riesz representation theorem there exist a Radon measure μ in $\mathbb{T}^2_T$ and a μ-measurable function $\sigma : \mathbb{T}^2_T \to \mathbb{T}^2_T$ with $|\sigma| = 1$ μ-a.e., such that for any $\varphi \in C^1_c(\mathbb{T}^2_T;\mathbb{R}^2)$

$$\int_{\mathbb{T}^2_T} u \,\mathrm{div}\varphi \, dz = - \int_{\mathbb{T}^2_T} \varphi \cdot \sigma \, d\mu \,. \tag{2.2}$$

It follows from (2.2) that the measure $\sigma d\mu$ coincides with the distributional derivative Du, which is a vector-valued measure. In particular if u is $C^1(\mathbb{T}^2_T)$ then $d\mu = |Du| \, dz$ and $\sigma = Du/|Du|$.

We denote by $BV_e(\mathbb{T}^2_T)$ the space of the functions $u \in L^1_e(\mathbb{T}^2_T)$ such that $\|Du\|_e(\mathbb{T}^2_T) < \infty$. Then $BV_e(\mathbb{T}^2_T)$ is a Banach space when equipped with the norm

$$\|u\|_{BV_e(\mathbb{T}^2_T)} = \|u\|_{L^1_e(\mathbb{T}^2_T)} + \|Du\|_e(\mathbb{T}^2_T) \,.$$

Note that (2.2) implies that if $u \in BV_e(\mathbb{T}^2_T)$ then for any $\varphi \in C^1_c(\mathbb{T}^2_T;\mathbb{R}^2)$

$$\int_{\mathbb{T}^2_T} u \,\mathrm{div}(e^x \varphi) \, dz = - \int_{\mathbb{T}^2_T} e^x \, \varphi \cdot \, \sigma \, d\mu \,.$$

Observe also that if $u : \mathbb{T}^2_T \to \mathbb{R}$ is locally Lipschitz, from (2.1) we have immediately that

$$\|Du\|_e(\mathbb{T}^2_T) = \int_{\mathbb{T}^2_T} e^x |Du(z)| \, dz \,. \tag{2.3}$$

The following lemma is a straightforward consequence of the definition (2.1) for weighted total variation.

Lemma 2.1 (Lower semicontinuity of the total variation)**.**
Let $\{u_k\} \subset L^1_{loc}(\mathbb{T}^2_T)$. If $u_k \to u_0$ in $L^1_{loc}(\mathbb{T}^2_T)$, then $\|Du_0\|_e(\mathbb{T}^2_T) \le \liminf_{k\to\infty} \|Du_k\|_e(\mathbb{T}^2_T)$.

The next lemma is proved as in the standard case of BV functions, see [25, Theorem 2, p.172]. To this aim, given $z \in \mathbb{R}^2$ we will denote by $B_r(z)$ the open ball of radius $r > 0$ centered at z. When $z = 0$ this ball will be simply denoted by B_r.

Lemma 2.2 (Approximation by smooth functions)**.**
Let $u \in BV_e(\mathbb{T}^2_T)$. There exists a sequence $u_k \in C^\infty(\mathbb{T}^2_T)$ such that
(i) $u_k \to u$ in $L^1_e(\mathbb{T}^2_T)$,
(ii) $\|Du_k\|_e(\mathbb{T}^2_T) \to \|Du\|_e(\mathbb{T}^2_T)$.

Proof. Throughout the proof of this lemma, in order to simplify the notation, given a function $v : \mathbb{T}^2_T \to \mathbb{R}$, we will denote with the same symbol also its T-periodic extension to $\mathbb{R}^2$.

Let $u \in BV_e(\mathbb{T}^2_T)$. Fix a standard mollifier $\varrho \ge 0$ with $\mathrm{supp}\,\varrho = \overline{B}_1$, $\int_{\mathbb{R}^2} \varrho \, dz = 1$ and for every $\varepsilon > 0$ and $z \in \mathbb{R}^2$ set $\varrho_\varepsilon(z) = \frac{1}{\varepsilon^2} \varrho\big(\frac{z}{\varepsilon}\big)$ and $u_\varepsilon = \varrho_\varepsilon * u$. Then u_ε is a smooth T-periodic function in the y-direction and $u_\varepsilon \to u$ in $L^1_e(\mathbb{T}^2_T)$.

To prove (ii), fix $\varphi \in C^1_c(\mathbb{T}^2_T, \mathbb{R}^2)$ with $|\varphi| \le 1$. By changing variable and then using Fubini's theorem, we get

$$\begin{aligned}\int_{\mathbb{T}^2_T} u_\varepsilon(z)\mathrm{div}(e^x\varphi(z))\,dz &= \int_{\Omega_T} dz \int_{\mathbb{R}^2} \varrho_\varepsilon(z-w)u(w)\mathrm{div}(e^x\varphi(z))\,dw \\ &= \int_{\mathbb{R}^2} \varrho_\varepsilon(v)\,dv \int_{\Omega_T} u(z-v)\mathrm{div}(e^x\varphi(z))\,dz\,.\end{aligned}$$

Denoting by (v_1, v_2) the components of v and setting $z' = (x', y') = z - v$, from the definition of periodic weighted total variation in (2.1) we obtain

$$\begin{aligned}\int_{\mathbb{T}^2_T} u_\varepsilon(z)\mathrm{div}(e^x\varphi(z))\,dz &= \int_{\mathbb{R}^2} \varrho_\varepsilon(v)\,dv \int_{\Omega_T - v_2 e_2} u(z')\mathrm{div}_{z'}(e^{x'+v_1}\varphi(z'+v))\,dz' \\ &= \int_{\mathbb{R}^2} \varrho_\varepsilon(v)e^{v_1}\,dv \int_{\mathbb{T}^2_T} u(z')\mathrm{div}_{z'}(e^{x'}\varphi(z'+v))\,dz' \\ &\le \|Du\|_e \int_{B_\varepsilon} \varrho_\varepsilon(v)e^{v_1}\,dv \le e^\varepsilon \|Du\|_e\,.\end{aligned}$$

From this inequality, passing to the supremum with respect to φ in the left hand side, and letting $\varepsilon \to 0^+$, we get

$$\limsup_{\varepsilon\to 0^+} \|Du_\varepsilon\|_e \le \|Du\|_e\,.$$

The conclusion then follows on combining this inequality with the one provided by Lemma 2.1.

□

Lemma 2.3. *Let $u \in BV_e(\mathbb{T}^2_T)$ and $h \in \mathbb{R}$. Then*

(i) $\|u(\cdot + he_1)\|_{L^1_e(\mathbb{T}^2_T)} = e^{-h}\|u\|_{L^1_e(\mathbb{T}^2_T)}$ *;*

(ii) $\|D(u(\cdot + he_1))\|_e(\mathbb{T}^2_T) = e^{-h}\|Du\|_e(\mathbb{T}^2_T)$ *;*

(iii) u^+ *and* u^- *are in* $BV_e(\mathbb{T}^2_T)$ *and* $\|Du^\pm\|_e(\mathbb{T}^2_T) \le \|Du\|_e(\mathbb{T}^2_T)$ *;*

(iv) $\|u\|_{L^1_e(\mathbb{T}^2_T)} \le \|Du\|_e(\mathbb{T}^2_T)$ *;*

(v) There exists a constant $C > 0$ such that for all $T > 0$

$$\int_{\mathbb{T}^2_T} e^{2x}u(z)^2\,dz \le C\max\{1, \frac{1}{T^2}\}\big(\|Du\|_e(\mathbb{T}^2_T)\big)^2\,.$$

Proof. (i) and (ii) follow directly from the definitions.

In order to prove (iii)-(v) we assume that u is smooth. The general case will follow using Lemmas 2.2 and 2.1. If u is smooth, from (2.3) we have

$$\|Du^+\|_e = \int_{\mathbb{T}^2_T \cap \{u>0\}} e^x|Du|\,dz \le \|Du\|_e\,.$$

A similar estimate holds for u^-, hence (iii) follows.

To prove (iv), fix $\delta > 0$ and choose $R > 0$ such that $0 \leq \int_{\mathbb{T}^2_T \cap \{|x|>R\}} e^x |u|\, dz \leq \delta$. Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi(x) = 1$ on $[-R, R]$, $\psi(x) = 0$ for $|x| \geq R+2$, $0 \leq \psi \leq 1$ when $R \leq |x| \leq R+2$ and $|\psi'| \leq 1$. Integrating by parts, we have

$$
\begin{aligned}
\|u\|_{L^1_e} - \delta \leq \int_{\mathbb{T}^2_T} e^x |u| \psi\, dz &= -\int_{\mathbb{T}^2_T \cap \{u \neq 0\}} e^x \psi(x) \frac{u}{|u|} u_x\, dz - \int_{\mathbb{T}^2_T} e^x |u| \psi'(x)\, dz \\
&\leq \|Du\|_e + \int_{\mathbb{T}^2_T \cap \{R \leq |x| \leq R+2\}} e^x |u|\, dz \leq \|Du\|_e + \delta\, .
\end{aligned}
$$

Hence (iv) follows by letting $\delta \to 0^+$.

In order to prove (v) we fix a function $\eta \in C^\infty(\mathbb{R})$ with compact support in $(-T, T)$, $0 \leq \eta \leq 1$, $\eta = 1$ in $[-T/2, T/2]$, $|\eta'| \leq 3/T$ and, still denoting by u its T-periodic extension to $\mathbb{R}^2$, set $v(x, y) = u(x, y)\eta(y)$. Observe that $e^x v(x, y) \in W^{1,1}(\mathbb{R}^2)$. Therefore from the Sobolev inequality and (iv) we have

$$
\begin{aligned}
\int_{\mathbb{T}^2} e^{2x} u(z)^2\, dz &\leq \int_{\mathbb{R}^2} e^{2x} v(z)^2\, dz \leq C \|D(e^x v)\|^2_{L^1(\mathbb{R}^2)} \\
&\leq C \left(\int_{\mathbb{R}^2} e^x (|Du|\eta + (\eta + |\eta'|)\, |u|)\, dz \right)^2 \leq C \max\{1, \frac{1}{T^2}\} \|Du\|^2_{L^1_e(\mathbb{T}^2_T)}\, .
\end{aligned}
$$

□

Let $E \subset \mathbb{T}^2_T$ be a measurable set. The *weighted perimeter of E in $\mathbb{T}^2_T$* is defined by setting

$$
\mathcal{P}_e(E; \mathbb{T}^2_T) = \sup \left\{ \int_{\mathbb{T}^2_T} \chi_E \operatorname{div}(e^x \varphi)\, dz : \ \varphi \in C^1_c(\mathbb{T}^2; \mathbb{R}^2), |\varphi| \leq 1 \right\}. \tag{2.4}
$$

If $\mathcal{P}_e(E; \mathbb{T}^2_T) < \infty$ we say that E has *finite weighted perimeter*. In this case $\mathcal{P}_e(E; \mathbb{T}^2_T) = \|D\chi_E\|_e(\mathbb{T}^2_T)$.

Recall that if $F \subset \mathbb{R}^2$ is measurable and $U \subset \mathbb{R}^2$ is an open set, the *perimeter of F in U* is defined as

$$
P(F; U) = \sup \left\{ \int_U \chi_F \operatorname{div}\varphi\, dz : \ \ \varphi \in C^1_c(U; \mathbb{R}^2), \ |\varphi| \leq 1 \right\}. \tag{2.5}
$$

The perimeter of F in $\mathbb{R}^2$ will be simply denoted by $P(F)$.

Remark 2.4. *Note that if E has finite weighted perimeter in $\mathbb{T}^2_T$, then its T-periodic extension to $\mathbb{R}^2$, denoted by $\widehat{E}$, is a set of locally finite perimeter in $\mathbb{R}^2$, i.e., $P(\widehat{E}; U) < \infty$ for every bounded open set $U \subset \mathbb{R}^2$.*

We recall a few important facts from the theory of sets of (locally) finite perimeter. As a reference, the reader may consult the books [4, 31]. We start with De Giorgi's structure theorem, see [4, Theorem 3.59] which in the 2-dimensional case reads as follows (recall that $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^2$).

Theorem 2.5. *Let $F \subset \mathbb{R}^2$ be a set of locally finite perimeter. There exist a Borel set $\partial^* F \subset \partial F$ and a Borel measurable map $\nu_F : \partial^* F \to \mathbb{S}^1$ such that for any $\varphi \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$*

$$\int_F \mathrm{div}\varphi \, dz = \int_{\partial^* F} \varphi \cdot \nu_F \, d\mathcal{H}^1 \,. \tag{2.6}$$

The set $\partial^* F$ is a 1-rectifiable subset in $\mathbb{R}^2$, i.e., $\partial^* F$ coincides, up to a set of zero $\mathcal{H}^1$ measure, with an at most countable union of pairwise disjoint compact sets $\{K_i\}_{i\in I}$ with $K_i \subset M_i$, where each M_i is a 1-dimensional manifold of class C^1. Moreover if $x \in K_i$ for some $i \in I$, the unit vector $\nu_F(x)$ is orthogonal to the tangent line to M_i at x.

We refer to the set $\partial^* F$ as the *reduced boundary* of F, while ν_F is the *generalized exterior unit normal to* F. In the following we will denote by τ_F the unit vector field obtained by rotating ν_F counterclockwise by $\pi/2$. When F is an open set with a C^1 boundary, then $\partial^* F = \partial F$ and ν_F is the usual exterior unit normal to ∂F while τ_F is a unit tangent vector. Note also that from the definition (2.5) and the generalized divergence formula (2.6), we have that for any open set $U \subset \mathbb{R}^2$

$$P(F;U) = \mathcal{H}^1(\partial^* F \cap U) \,.$$

Thanks to Remark 2.4 it is clear that De Giorgi's structure theorem applies to a set $E \subset \mathbb{T}^2_T$ of finite weighted perimeter with the obvious changes due to periodicity. From (2.6) we have

$$\int_E \mathrm{div}(e^x \varphi) \, dz = \int_{\partial^* E} e^x \varphi \cdot \nu_E \, d\mathcal{H}^1 \qquad \text{for all } \varphi \in C^1_c(\mathbb{T}^2_T;\mathbb{R}^2) \,.$$

Then, from the above formula, recalling the definition (2.4), we get

$$\mathcal{P}_e(E;\mathbb{T}^2_T) = \int_{\partial^* E} e^x \, d\mathcal{H}^1 \,. \tag{2.7}$$

It is well known that sets of locally finite perimeter can be approximated by smooth sets, see [4, Theorem 3.42]. Here we need a weighted version of this approximation result.

Theorem 2.6. *Let $E \subset \mathbb{T}^2_T$ be a set with finite weighted perimeter and volume. Then there exists a sequence of smooth bounded open sets $E_h \subset \mathbb{T}^2_T$ such that $\chi_{E_h} \to \chi_E$ in $L^1_e(\mathbb{T}^2_T)$, $|E_h|_e = |E|_e$ and $\mathcal{P}_e(E_h;\mathbb{T}^2_T) \to \mathcal{P}_e(E;\mathbb{T}^2_T)$.*

The proof of this theorem requires some technical facts from geometric measure theory and will be given in the Appendix. Next we recall a simple consequence of the area formula, see [4, Theorem 2.91], that we will use later. Let $E \subset \mathbb{R}^2$ be a set of locally finite perimeter and $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ a C^1 diffeomorphism. Then $\Phi(E)$ is a set of locally finite perimeter and for any Borel function $g : \mathbb{R}^2 \to [0,\infty)$

$$\int_{\partial^* \Phi(E)} g(w) \, d\mathcal{H}^1 = \int_{\partial^* E} g(\Phi(z)) |D\Phi(z)\tau_E(z)| \, d\mathcal{H}^1 \tag{2.8}$$

where τ_E is the unit tangent vector defined above. As an immediate consequence of this formula, we have

Corollary 2.7. *Let $E \subset \mathbb{T}^2_T$ be a set of finite weighted perimeter. Then $c \to \frac{1}{c}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT})$ is decreasing for $c > 0$.*

Proof. From (2.7), by applying (2.8) to the map $\Phi(x, y) = (x, cy)$, we get, on setting the unit tangent vector $\tau_E = (\tau_1, \tau_2)$,

$$\frac{1}{c}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) = \frac{1}{c}\int_{\partial^* E_c} e^x \, d\mathcal{H}^1 = \int_{\partial^* E} e^x \sqrt{\frac{\tau_1^2}{c^2} + \tau_2^2}\, d\mathcal{H}^1 \tag{2.9}$$

which is clearly a decreasing function in c. □

Note that (2.9) implies that

$$\max\{1, c\}\mathcal{P}_e(E;\mathbb{T}^2_T) \geq \mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) \geq \min\{1, c\}\mathcal{P}_e(E;\mathbb{T}^2_T)\,. \tag{2.10}$$

3 Existence of minimizers

We now prove that problem (1.19) has a solution. To this end we define the functional $\mathcal{K}_c$ setting for every measurable $E \subset \mathbb{T}^2_T$

$$\mathcal{K}_c(E) := \frac{\sqrt{2}}{12\,c}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) + \frac{\sigma}{2}\int_{\mathbb{T}^2_T} e^x \chi_E\, \mathcal{L}_c\chi_E\, dz \tag{3.1}$$

and consider the following minimum problem

$$\min\{\mathcal{K}_c(E) :\ E \subset \mathbb{T}^2_T \ \text{ is measurable and } \ |E|_e = 1\} \tag{3.2}$$

which is equivalent to problem (1.19), since $\mathcal{J}_c(E)$ and $\mathcal{K}_c(E)$ differ by a constant if $|E|_e = 1$.

Let $|E|_e = 1$. Multiplying both sides of (1.15) by $e^x\mathcal{L}\chi_E$ and integrating by parts, we get

$$\|\mathcal{L}_c\chi_E\|_{L^2_e} \leq \frac{1}{\gamma}\|\chi_E\|_{L^2_e} = \frac{1}{\gamma} \tag{3.3}$$

and

$$0 \leq \int_{\mathbb{T}^2_T} e^x \chi_E \mathcal{L}_c\chi_E\, dz \leq \|\chi_E\|_{L^2_e}\, \|\mathcal{L}_c\chi_E\|_{L^2_e} \leq \frac{1}{\gamma}\,.$$

The inequality (3.3) leads to the observation that

$$\chi_{E_h} \to \chi_E \quad \text{in} \quad L^1_e \qquad \Longrightarrow \qquad \mathcal{L}_c\chi_{E_h} \to \mathcal{L}_c\chi_E \quad \text{in} \quad L^2_e\,. \tag{3.4}$$

Let $W := \{(x, y) \in \mathbb{T}^2_T : x < \log 1/T\}$ be a front satisfying $|W|_e = 1$. Using (2.9) and the explicit calculation of $\mathcal{L}_c\chi_E$ made in [12, Section 5], a direct computation gives

$$\mathcal{K}_c(W) = \frac{\sqrt{2}}{12} + \frac{\sigma}{2\gamma}\,\frac{(\sqrt{c^2 + 4\gamma} - c)}{\sqrt{c^2 + 4\gamma}} \leq m_W := \frac{\sqrt{2}}{12} + \frac{\sigma}{2\gamma}\,. \tag{3.5}$$

Note that the constant m_W does not depend on T or c.

We recall a well known property of Hausdorff measures, see for instance [4, Proposition 2.49]. Let π denote the projection of $\mathbb{R}^2$ onto a straight line L. Then for every Borel set $S \subset \mathbb{R}^2$

$$\mathcal{H}^1(\pi(S)) \leq \mathcal{H}^1(S). \tag{3.6}$$

In the following we shall denote by π_x and π_y the projections on the x and y axis, respectively.

Lemma 3.1. *Let $E \subset\subset (0,\infty)\times(-T/2,T/2)$ be a connected open set such that $\mathcal{P}_e(E;\mathbb{T}^2_T) \leq C_0$. Set $m = \inf\, \pi_x(E)$ and assume that $\partial E \cap \{x > m\}$ is of class C^1. Then*

$$|E|_e \leq \frac{e^{C_0-m}}{4\pi}(\mathcal{P}_e(E;\mathbb{T}^2_T))^2\,.$$

Proof. Set $M = \sup \pi_x(E)$ and observe that $(m, M] \subset \pi_x(\partial E \cap \{x > m\})$, since E is connected. Therefore, by (3.6) and (2.7) we have

$$M - m \leq \mathcal{H}^1(\partial E \cap \{x > m\}) \leq \int_{\partial E\cap\{x>m\}} e^x \, d\mathcal{H}^1 \leq \mathcal{P}_e(E;\mathbb{T}^2_T) \leq C_0\,.$$

Recall that by the isoperimetric inequality $4\pi|E| \leq (P(E))^2$. Then the conclusion follows from the previous estimate, since

$$|E|_e \leq e^M|E| \leq \frac{e^M}{4\pi}(P(E))^2 \leq \frac{e^{(M-2m)}}{4\pi}(\mathcal{P}_e(E;\mathbb{T}^2_T))^2 \leq \frac{e^{C_0-m}}{4\pi}(\mathcal{P}_e(E;\mathbb{T}^2_T))^2\,.$$

□

Lemma 3.2. *Let $C_0 > 0$. If $E \subset \mathbb{T}^2_T$ is a measurable set such that $|E|_e < \infty$ and $\mathcal{P}_e(E;\mathbb{T}^2_T) \leq C_0$. Then for all $m > \max\{\log \frac{C_0}{T}, 0\}$*

$$\int_{E\cap\{x>m\}} e^x \, dz \leq \frac{e^{C_0-m}}{\pi}(\mathcal{P}_e(E;\mathbb{T}^2_T))^2\,. \tag{3.7}$$

Proof. Thanks to Theorem 2.6 it suffices to prove (3.7) when E is a smooth, bounded open set. Take $m > \max\{\log \frac{C_0}{T}, 0\}$. From (3.6) we have

$$\mathcal{H}^1(\pi_y(\partial E \cap \{x \geq m\})) \leq \mathcal{H}^1(\partial E \cap \{x \geq m\}) \leq \frac{1}{e^m}\mathcal{P}_e(E;\mathbb{T}^2_T) < T\,, \tag{3.8}$$

where the last inequality follows from the choice of m. Therefore, there exists an interval $(t_0, t_1) \subset (-T/2; T/2)$ such that $E\cap(\{x \geq m\}\times(t_0, t_1)) = \emptyset$. By translating E in the y direction if necessary, we may assume that $E^+_m \subset\subset (0,\infty) \times (-T/2, T/2)$, where $E^+_m = E \cap \{x > m\}$. Let $\{F_i\}_{i\in I}$ be the connected components of E^+_m and for every $i \in I$ set $m_i = \inf \pi_x(F_i) \geq m > 0$. Observe that the sets F_i satisfy the assumptions of Lemma 3.1. Therefore,

$$|F_i|_e \leq \frac{e^{C_0-m_i}}{4\pi}(\mathcal{P}_e(F_i;\mathbb{T}^2_T))^2 \leq \frac{e^{C_0-m}}{4\pi}(\mathcal{P}_e(F_i;\mathbb{T}^2_T))^2\,.$$

From this inequality we have

$$\int_{E\cap\{x>m\}} e^x\,dz = \sum_{i\in I}|F_i|_e \le \sum_{i\in I}\frac{e^{C_0-m}}{4\pi}(\mathcal{P}_e(F_i;\mathbb{T}^2_T))^2 \le \frac{e^{C_0-m}}{4\pi}\Big(\sum_{i\in I}\mathcal{P}_e(F_i;\mathbb{T}^2_T)\Big)^2. \tag{3.9}$$

Observe now that from (2.7)

$$\sum_{i\in I}\mathcal{P}_e(F_i;\mathbb{T}^2_T) \le \sum_{i\in I}\int_{\partial F_i} e^x\,d\mathcal{H}^1 \le \int_{\partial E\cap\{x\ge m\}} e^x\,dx + e^m\mathcal{H}^1(E\cap\{x=m\}). \tag{3.10}$$

Note that if $(m,y)\in E$ then there exists $x>m$ such that $(x,y)\in\partial E$. Thus $E\cap\{x=m\}\subset \pi_y(\partial E\cap\{x>m\})$ and from (3.6) we have

$$e^m\mathcal{H}^1(E\cap\{x=m\}) \le e^m\mathcal{H}^1(\partial E\cap\{x>m\}) \le \int_{\partial E\cap\{x\ge m\}} e^x\,dx.$$

From this inequality and (3.10) we then have

$$\sum_{i\in I}\mathcal{P}_e(F_i;\mathbb{T}^2_T) \le 2\int_{\partial E\cap\{x\ge m\}} e^x\,dx$$

and the conclusion follows thanks to (3.9). □

Let us proceed to the proof of the existence of minimizers for problem (3.2).

Theorem 3.3. *Let $\sigma\ge 0$. Problem* (3.2) *admits always a minimizer.*

Proof. Assume that $\{E_h\}\subset\mathbb{T}^2_T$ is a minimizing sequence , i.e., a sequence such that $|E_h|_e=1$ for all h and $\mathcal{K}_c(E_h)\to\inf\{\mathcal{K}_c(F): |F|_e=1\}$. Since the sequence $\{\mathcal{P}_e((E_h)_c;\mathbb{T}^2_{cT})\}$ is bounded, from (2.10) we infer that there exists $C_0>0$ such that

$$\mathcal{P}_e(E_h;\mathbb{T}^2_T)\le C_0 \qquad \text{for all } h.$$

From this inequality it follows that for all $k\in\mathbb{N}$ the sets E_h have equibounded perimeters in $Q_k=\mathbb{T}^2_T\cap\{|x|<k\}$. Thus, by a well known compactness result, see [4, Theorem 3.39], we get a subsequence $\{E_{h_i}\}$ and a measurable set $G_k\subset\mathbb{T}^2_T$ such that $\chi_{E_{h_i}}\to\chi_{G_k}$ in $L^1(Q_k)$. Therefore a standard diagonalization argument yields that there exist a measurable set $E\subset\mathbb{T}^2_T$ and a subsequence E_{h_r} such that $\chi_{E_{h_r}}\to\chi_E$ in $L^1_{loc}(\mathbb{T}^2_T)$ and a.e. in $\mathbb{T}^2_T$.

We claim that $\chi_{E_{h_r}}\to\chi_E$ in $L^1_e(\mathbb{T}^2_T)$. Note that if this claim is true, we have $|E|_e=1$. Moreover, since $\chi_{(E_{h_r})_c}\to\chi_{E_c}$ in $L^1_e(\mathbb{T}^2_{cT})$, by the lower semicontinuity of the perimeter, see Lemma 2.1, we have $\mathcal{P}_e(E_c;\mathbb{T}^2_{cT})\le\liminf_{r\to\infty}\mathcal{P}_e((E_{h_r})_c;\mathbb{T}^2_{cT})$, thus proving the existence of a minimizer when $\sigma=0$.

When $\sigma>0$, observe that (3.4) yields that $\mathcal{L}_c\chi_{E_{h_r}}\to\mathcal{L}_c\chi_E$ in L^2_e. In turn, this implies the convergence of the nonlocal term and we conclude again that E is a minimizer of problem (3.2).

It remains to prove the claim. Take $\varepsilon\in(0,1)$ and fix $m>\max\{\log\frac{1}{\varepsilon},\log\frac{C_0}{T}\}$. By Lemma 3.2 we have that for all $r\in\mathbb{N}$

$$\int_{E_{h_r}\cap\{x>m\}} e^x\,dz \le \frac{e^{C_0-m}}{\pi}C_0^2 = \frac{C}{e^m}\le C\varepsilon. \tag{3.11}$$

At the same time the $L^1_{loc}(\mathbb{T}^2_T)$ convergence of χ_{E_h} yields that there exists $r_\varepsilon \in \mathbb{N}$, depending on ε as well as m, such that

$$\int_{\mathbb{T}^2_T \cap \{|x|\le m\}} |\chi_{E_{h_r}} - \chi_{E_{h_s}}|\, dz \le \varepsilon \qquad \text{for all } \; r, s \ge r_\varepsilon \,.$$

Therefore when $r, s \ge r_\varepsilon$, from this inequality and (3.11)

$$\begin{aligned} \|\chi_{E_{h_r}} - \chi_{E_{h_s}}\|_{L^1_e(\mathbb{T}^2_T)} &\le 2T \int_{-\infty}^{-m} e^x\, dx + \int_{\mathbb{T}^2_T \cap \{|x|\le m\}} e^x |\chi_{E_{h_r}} - \chi_{E_{h_s}}|\, dz + 2C\varepsilon \\ &\le 2Te^{-m} + \varepsilon + 2C\varepsilon \le (2T + 1 + 2C)\varepsilon. \end{aligned}$$

This shows that the sequence $\chi_{E_{h_r}}$ is a Cauchy sequence in $L^1_e(\mathbb{T}^2_T)$. This proves the claim, thus concluding the proof. □

We will prove later, see Theorem 9.2 and Remark 9.3 that a minimizer of problem (3.2) is an open set with boundary of class $C^{3,\beta}$ for any $0 < \beta < 1$.

Lemma 3.4. *Let $\sigma > 0$ and let E be a minimizer of* (3.2)*. Then E is a connected, open set.*

Proof. We argue by contradiction assuming that E is not connected. If this is the case, since by Theorem 9.2 ∂E is of class C^1, we have that $E = E_1 \cup E_2$, with E_1, E_2 disjoint, nonempty, open sets such that $\overline{E}_1 \cap \overline{E}_2 = \emptyset$. Then, recalling that $\mathcal{L}_c$ is self-adjoint,

$$\mathcal{K}_c(E) = \mathcal{K}_c(E_1) + \mathcal{K}_c(E_2) + \sigma \int_{\mathbb{T}^2_T} e^x \chi_{E_1}\, \mathcal{L}_c \chi_{E_2}\, dz \,. \tag{3.12}$$

Since $|E_1|_e = e^{-h_1}$ and $|E_2|_e = e^{-h_2}$ for some $h_1, h_2 > 0$ satisfying $e^{-h_1} + e^{-h_2} = 1$, we have $|E_i + h_i e_1|_e = 1$ for $i = 1, 2$. From the minimality of E it follows that

$$\mathcal{K}_c(E) \le \mathcal{K}_c(E_i + h_i e_1) = e^{h_i} \mathcal{K}_c(E_i)\,, \quad i = 1, 2.$$

Inserting these inequalities in (3.12) and using that $e^{-h_1} + e^{-h_2} = 1$,

$$\mathcal{K}_c(E) \ge (e^{-h_1} + e^{-h_2}) \mathcal{K}_c(E) + \sigma \int_{\mathbb{T}^2_T} e^x \chi_{E_1}\, \mathcal{L}_c \chi_{E_2}\, dz \,,$$

which implies $0 \ge \int_{\mathbb{T}^2_T} e^x \chi_{E_1}\, \mathcal{L}_c \chi_{E_2}\, dz$. But this leads to a contradiction since E_1 has positive measure and $\mathcal{L}_c \chi_{E_2} > 0$ in $\mathbb{T}^2_T$, hence $\int_{\mathbb{T}^2_T} e^x \chi_{E_1}\, \mathcal{L}_c \chi_{E_2}\, dz > 0$. This contradiction concludes the proof. □

Proposition 3.5. *Given $c > 0$, there exists a constant $M = M(c, T, \sigma, \gamma) > 0$ such that if E is a minimizer of* (3.2)*, then $E \subset \{(x, y) \in \mathbb{T}^2_T : x \le M\}$. Moreover $M_1 := \sup_{0<c\le 1} M(c, T, \sigma, \gamma)$ is bounded for $T > 0$ fixed.*

Proof. Let $E \subset \mathbb{T}^2_T$ be a minimizer of (3.2). From (2.10) and (3.5) we have that

$$\mathcal{P}_e(E;\mathbb{T}^2_T) \leq \max\Big\{1,\frac{1}{c}\Big\}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) \leq 6\sqrt{2}\max\{c,1\}\mathcal{K}_c(W) \leq 6\sqrt{2}\max\{c,1\}m_W := C_0\,.$$

Let $m_1 = 1+\max\{\log\frac{C_0}{T},0\}$. Then, (3.8) holds with m replaced by m_1. Therefore, arguing as in the proof of Lemma 3.2, we may assume that, up to a vertical translation if needed, $E\cap\{x>m_1\} \subset\subset \mathbb{R}\times(-T/2,T/2)$. Since E is a connected open set, $\pi_x(E\cap\{x>m_1\}) = \pi_x(E_c\cap\{x>m_1\})$ is a bounded open interval. If this interval is empty, then we are done choosing $M=m_1$. Otherwise let us denote this interval by (a,b) with $m_1\leq a\leq b<+\infty$. We have

$$b-a = \mathcal{H}^1((a,b)) \leq \mathcal{P}_e(E_c;\Omega_{cT}) \leq 6c\sqrt{2}\,m_W < \infty\,. \tag{3.13}$$

Let us now fix $m>m_1$, depending only on c and T, so that

$$\frac{e^{C_0-m}}{\pi}C_0^2 < 1\,.$$

Then from (3.7) we have that

$$\int_{E\cap\{x>m\}} e^x\,dz < 1$$

which in particular yields $m_1 \leq a < m$. Therefore, from (3.13) we conclude that

$$b < m + 6c\sqrt{2}\,m_W := M(c,T,\sigma,\gamma)\,.$$

We now fix T. Both C_0 and m_1 are clearly uniformly bounded when $0<c\leq 1$, thus $\sup_{0<c\leq 1} M$ is bounded as well. □

As observed at the beginning of this section, Theorem 3.3 shows that for any $c>0$ there exists a minimizer $\mathbb{E}(c)$ of the volume constrained problem (1.19).

In the following we shall say that a set $E\subset\mathbb{T}^2_T$ of weighted finite perimeter is a *traveling wave of the limiting problem* if it is a critical point of $\mathcal{J}_c$ for some $c>0$. In order to show that such a traveling wave exists, it is enough to prove that there exists $c_0>0$ such that the set $\mathbb{E}(c_0)$ is a critical point of the volume constrained problem (1.19) with $\mathcal{J}_{c_0}(\mathbb{E}(c_0))=0$, see Lemma 6.6. As a reminder, we have already shown that $\mathcal{J}_{c_0}(\mathbb{E}(c_0))=0$ is a necessary condition for $\mathbb{E}(c_0)$ be a traveling wave, see the discussion just before Theorem 1.4.

The existence of a speed $c_0>0$ such that the minimum of the problem (1.19) is zero will be proved under the assumptions that the parameters α,σ and γ satisfy condition $(TW1)$ in (1.17). To this aim we begin with some preliminary lemmas.

Lemma 3.6. *Let $|E|_e=1$. For any $T,\gamma>0$, $\mathcal{L}_c\chi_E\to 0$ in $L^2_e(\mathbb{T}^2_T)$ and $\int_{\mathbb{T}^2_T} e^x\chi_E\,\mathcal{L}_c\chi_E\,dz\to 0$ as $c\to\infty$.*

Proof. Integrating the 1-dimensional Poincaré type inequality stated in [30, Corollary 4.2] we have that for any function $w\in H^1_e(\mathbb{T}^2_T)$

$$\frac{1}{4}\int_{\mathbb{T}^2_T} e^x w^2\,dz \leq \int_{\mathbb{T}^2_T} e^x w_x^2\,dz\,. \tag{3.14}$$

Let $v = \mathcal{L}_c\chi_E$. From (1.15) we have

$$\int_{\mathbb{T}^2_T} e^x(v_x^2 + \frac{1}{c^2}v_y^2 + \frac{\gamma}{c^2}v^2)\,dz = \frac{1}{c^2}\int_{\mathbb{T}^2_T} e^x v\chi_E\,dz\,.$$

Combining this equation with (3.14) we immediately get that $\|v\|_{L^2_e} \le 4/c^2$. Consequently $0 \le \int_{\mathbb{T}^2_T} e^x\chi_E\,\mathcal{L}_c\chi_E\,dz \le \|v\|_{L^2_e} \le 4/c^2$. Hence, the result follows. □

Lemma 3.7. *Let $\alpha > 1$. Then $\mathcal{J}_c(\mathbb{E}(c)) < 0$ for c sufficiently large.*

Proof. Recall that $\mathcal{P}_e(W_c;\mathbb{T}^2_{cT}) = c$, where $W := \{(x,y) \in \mathbb{T}^2_T : x < \log 1/T\}$. Thus, from Lemma 3.6 if c sufficiently large we have

$$\begin{aligned}\mathcal{J}_c(\mathbb{E}(c)) \le \mathcal{J}_c(W) &= \frac{\sqrt{2}}{12}\left(\frac{1}{c}\mathcal{P}_e(W_c;\Omega_{cT}) - \alpha\right) + \frac{\sigma}{2}\int_W e^x\mathcal{L}_c\chi_W\,dz\\ &= \frac{\sqrt{2}}{12}\,(1-\alpha) + o(1) < 0\,.\end{aligned}$$

□

Next we study the behavior of $\mathcal{J}_c(\mathbb{E}(c))$ as $c \to 0$. To this aim we first prove

Lemma 3.8. *Let $\mathbb{E}(c)$ be a minimizer of (3.2). Then* $\liminf_{c\to 0} \frac{1}{c}\mathcal{P}_e((\mathbb{E}(c))_c;\mathbb{T}^2_{cT}) \ge 1$.

Proof. For $c > 0$ set

$$\mathcal{B}(c) := \pi_x(\partial\,\mathbb{E}(c)) = \pi_x(\partial\,(\mathbb{E}(c))_c)\,.$$

Observe that $\mathcal{B}(c)$ is closed. Let $\{c_k\} \subset (0,1)$ be a sequence converging to 0 such that

$$\liminf_{c\to 0}\frac{1}{c}\mathcal{P}_e((\mathbb{E}(c))_c;\mathbb{T}^2_{cT}) = \lim_{k\to\infty}\frac{1}{c_k}\mathcal{P}_e((\mathbb{E}(c_k))_{c_k};\mathbb{T}^2_{c_kT})\,. \tag{3.15}$$

From (2.10) and (3.5) we have that

$$\frac{\sqrt{2}}{12}\mathcal{P}_e(\mathbb{E}(c_k);\mathbb{T}^2_T) \le \frac{\sqrt{2}}{12c_k}\mathcal{P}_e((\mathbb{E}(c_k))_{c_k};\mathbb{T}^2_{c_kT}) \le \mathcal{K}_{c_k}(\mathbb{E}(c_k)) \le m_W = \frac{\sqrt{2}}{12} + \frac{\sigma}{2\gamma}\,. \tag{3.16}$$

Thus the sets $\mathbb{E}(c_k)$ have equibounded weighted perimeters $\mathcal{P}_e(\mathbb{E}(c_k);\mathbb{T}^2_T)$. Therefore, arguing as in the proof of Theorem 3.3 we may conclude that there exists a measurable set $E_0 \subset \mathbb{T}^2_T$ such that, up to a (not relabelled subsequence), $\chi_{\mathbb{E}(c_k)} \to \chi_{E_0}$ in $L^1_{loc}(\mathbb{T}^2_T)$. Note also that by Proposition 3.5 there exists $M_1 > 0$ such that $\mathbb{E}(c_k) \subset \{(x,y) \in \mathbb{T}^2_T : x \le M_1\}$. Therefore we may conclude that to $\chi_{\mathbb{E}(c)} \to \chi_{E_0}$ in $L^1_e(\mathbb{T}^2_T)$ and thus $|E_0|_e = 1$.

Let $A < 0$. Since by Theorem 9.2 $\partial\mathbb{E}(c_k)$ is of class C^1, from (3.6), (2.7) and (3.16) we have

$$\begin{aligned}e^A\mathcal{H}^1\big(\mathcal{B}(c_k)\cap\{x > A\}\big) \le e^A\mathcal{H}^1\big(\partial(\mathbb{E}(c_k))_{c_k}\cap\{x>A\}\big) &\le \int_{\partial(\mathbb{E}(c_k))_{c_k}\cap\{x>A\}} e^x\,dz\\ &\le \mathcal{P}_e((\mathbb{E}(c_k))_{c_k};\mathbb{T}^2_{c_kT}) \le 6c_k\sqrt{2}\,m_W\,.\end{aligned}$$

Thus $\mathcal{H}^1\big(\mathcal{B}(c_k)\cap\{x>A\}\big)\to 0$ as $k\to\infty$. By Lemma 3.4 the sets $\mathbb{E}(c_k)$ are connected open sets. Therefore the projection $\pi_x(\mathbb{E}(c_k))$ is an open interval (a_k,b_k), with $b_k\le M_1$. Thus, up to another not relabelled subsequence k, we may assume that $a_k\to a_0$ and $b_k\to b_0$ for some $-\infty\le a_0\le b_0\le M_1$.

We claim that $E_0=\{(x,y)\in\mathbb{T}^2_T:\, a_0<x<b_0\}$. In fact, observe that for every k

$$\Big(\big((a_k,b_k)\cap(A,M_1)\big)\setminus\mathcal{B}(c_k)\Big)\times\Big(-\frac{T}{2},\frac{T}{2}\Big)\subset\mathbb{E}(c_k)\cap\{x>A\}\subset\{(x,y)\in\mathbb{T}^2_T:\, a_k<x<b_k\}\,.$$

Therefore, passing to the limit as $k\to\infty$ and recalling that $\mathcal{H}^1\big(\mathcal{B}(c_k)\cap\{x>A\}\big)\to 0$ we have that for every $A<0$, up to possibly removing from E_0 a set of Lebesgue measure zero, the following inclusions hold

$$\big((a_0,b_0)\cap(A,M_1)\big)\times\Big(-\frac{T}{2},\frac{T}{2}\Big)\subset E_0\cap\{x>A\}\subset\{(x,y)\in\mathbb{T}^2_T:\, a_0<x<b_0\}\,.$$

Hence, the claim follows by letting $A\to-\infty$. The claim leads immediately to the conclusion of the proof since from (3.15) and (2.10) we have, by the lower semicontinuity of the perimeter,

$$\begin{aligned}\liminf_{c\to0}\ \frac{1}{c}\mathcal{P}_e((\mathbb{E}(c))_c;\mathbb{T}^2_{cT}) &\ge \liminf_{k\to\infty}\mathcal{P}_e(\mathbb{E}(c_k);\mathbb{T}^2_T)\ge\mathcal{P}_e(E_0;\mathbb{T}^2_T)\\ &= T(e^{b_0}+e^{a_0})\ge\ T(e^{b_0}-e^{a_0})=\int_{\mathbb{T}^2_T}e^x\chi_{E_0}\,dz=1\,.\end{aligned}$$

□

Next we analyze the nonlocal term. First we need the following lemma.

Lemma 3.9. *Let G_c be the fundamental solution of the 1D operator $-c^2\frac{d^2}{dx^2}-c^2\frac{d}{dx}+\gamma$ for $c>0$. Then*
(i) $G_c\to\frac{1}{\gamma}\delta$ in distribution sense as $c\to0$, where δ is the Dirac delta distribution;
(ii) Let $v\in H^1_{loc}(\mathbb{R})$ be the weak solution of the equation $-c^2v''-c^2v'+\gamma v=\chi_F$, where $F=[a,b]$. Then $v=G_c*\chi_F$ and $v(x)\to\frac{1}{\gamma}\chi_F(x)$ for all $x\in\mathbb{R}\setminus\{a,b\}$ as $c\to0$.

Proof. A simple computation shows that if $c>0$ the fundamental solution of the 1D operator $-c^2\frac{d^2}{dx^2}-c^2\frac{d}{dx}+\gamma$ is

$$G_c(\xi)=\begin{cases}\frac{1}{c\sqrt{c^2+4\gamma}}e^{r_2\xi}\,, & \text{if } \xi<0,\\[2ex] \frac{1}{c\sqrt{c^2+4\gamma}}e^{r_1\xi}\,, & \text{if } \xi>0.\end{cases}$$

Here $r_1<-1<0<r_2$ are the roots of the quadratic equation $c^2r^2+c^2r-\gamma=0$, i.e.,

$$r=\frac{1}{2c}(-c\pm\sqrt{c^2+4\gamma})\,.$$

By direct computation it can be readily checked that

(a) $\gamma \int_{\mathbb{R}} G_c(\xi)\, d\xi = 1$ for all $c > 0$;

(b) $\gamma \int_{\mathbb{R}} e^{\xi} G_c(\xi)\, d\xi = 1$ for all $c > 0$;

(c) for any $\varepsilon > 0$ we have $\displaystyle\int_{|\xi|\geq\varepsilon} G_c(\xi)\, d\xi \to 0$ as $c \to 0$;

(d) for any $\varepsilon > 0$ we have $\displaystyle\int_{|\xi|\geq\varepsilon} e^{\xi} G_c(\xi)\, d\xi \to 0$ as $c \to 0$.

Statements (a) and (c) imply that $\gamma G_c * \varphi \to \varphi$ pointwise as $c \to 0$ for all $\varphi \in C_c(\mathbb{R})$. From this convergence, if $v = G_c * \chi_F$ is the weak solution of the equation $-c^2 v'' - c^2 v' + \gamma v = \chi_F$ with $F = [a, b]$, we immediately get that for all $x \in \mathbb{R} \setminus \{a, b\}$

$$v(x) = (G_c * \chi_F)(x) \to \frac{1}{\gamma}(\delta * \chi_F)(x) = \frac{1}{\gamma}\chi_F(x)$$

as $c \to 0$. □

Lemma 3.10. *Let $\mathbb{E}(c)$ be a minimizer of* (3.2)*. Then* $\displaystyle\liminf_{c\to 0} \int_{\mathbb{T}^2_T} e^x\, \chi_{\mathbb{E}(c)}\, \mathcal{L}_c \chi_{\mathbb{E}(c)}\, dz \geq 1/\gamma$.

Proof. Let $c_k \to 0^+$ be a sequence such that

$$\liminf_{c\to 0} \int_{\mathbb{T}^2_T} e^x\, \chi_{\mathbb{E}(c)}\, \mathcal{L}_c \chi_{\mathbb{E}(c)}\, dz = \lim_{k\to\infty} \int_{\mathbb{T}^2_T} e^x\, \chi_{\mathbb{E}(c_k)}\, \mathcal{L}_{c_k} \chi_{\mathbb{E}(c_k)}\, dz\,.$$

Arguing as in the proof of Lemma 3.8 we may assume that $\chi_{\mathbb{E}(c_k)} \to \chi_{E_0}$ in $L^1_e(\mathbb{T}^2_T)$, where $E_0 = \mathbb{T}^2_T \cap \{a_0 < x < b_0\}$ for some $-\infty \leq a_0 < b_0 < +\infty$. As the non-local energy is always non-negative, it follows that $\int_{\mathbb{T}^2_T} e^x (\chi_{\mathbb{E}(c_k)} - \chi_{E_0})\, \mathcal{L}_{c_k} (\chi_{\mathbb{E}(c_k)} - \chi_{E_0})\, dz \geq 0$ for all k. Combining this inequality with the self-adjointness of the operators $\mathcal{L}_{c_k}$ with respect to the weighted L^2_e inner product, we have

$$\int_{\mathbb{T}^2_T} e^x \chi_{\mathbb{E}(c_k)}\, \mathcal{L}_{c_k} \chi_{\mathbb{E}(c_k)}\, dz + \int_{\mathbb{T}^2_T} e^x \chi_{E_0}\, \mathcal{L}_{c_k} \chi_{E_0}\, dz \geq 2 \int_{\mathbb{T}^2_T} e^x \chi_{\mathbb{E}(c_k)}\, \mathcal{L}_{c_k} \chi_{E_0}\, dz\,. \tag{3.17}$$

Note that the function $\mathcal{L}_{c_k} \chi_{E_0}$ depends only on the x variable. On using Lemma 3.9 we have

$$\mathcal{L}_{c_k} \chi_{E_0} = G_{c_k} * \chi_{(a_0,b_0)} \to \frac{1}{\gamma}\chi_{E_0} \quad \text{pointwise in } \mathbb{T}^2_T\,.$$

Using the dominated convergence theorem to pass to the limit in the second and third integral in (3.17) we conclude that

$$\liminf_{c\to 0} \int_{\mathbb{T}^2_T} e^x \chi_{\mathbb{E}(c)}\, \mathcal{L}_c \chi_{\mathbb{E}(c)}\, dz = \lim_{k\to\infty} \int_{\mathbb{T}^2_T} e^x\, \chi_{\mathbb{E}(c_k)}\, \mathcal{L}_{c_k} \chi_{\mathbb{E}(c_k)}\, dz \geq \frac{1}{\gamma} \int_{\mathbb{T}^2_T} e^x (\chi_{E_0})^2\, dz = \frac{1}{\gamma}\,.$$

□

We conclude this section with the following crucial

Lemma 3.11. *Suppose condition (TW1) holds, i.e.,* $\frac{3\sqrt{2}\sigma}{\gamma} > \alpha - 1 > 0$*. Then there exist* $c_0 > 0$ *and a minimizer* $\mathbb{E}(c_0)$ *of* (1.19) *such that* $\mathcal{J}_{c_0}(\mathbb{E}(c_0)) = 0$*. In other words* $\mathbb{E}(c_0)$ *is a traveling wave of the limiting problem with speed* c_0*.*

Proof. From Lemmas 3.8 and 3.10 we have

$$\begin{aligned}\liminf_{c\to 0} \mathcal{J}_c(\mathbb{E}(c)) &= \liminf_{c\to 0} \Big(\frac{\sqrt{2}}{12c}\mathcal{P}_e((\mathbb{E}(c))_c; \mathbb{T}^2_{cT}) - \frac{\sqrt{2}}{12}\alpha + \frac{\sigma}{2}\int_{\mathbb{T}^2_T} e^x \chi_{\mathbb{E}(c)}\, \mathcal{L}_c \chi_{\mathbb{E}(c)}\, dz\Big) \\ &\geq -\frac{\sqrt{2}}{12}(\alpha - 1) + \frac{\sigma}{2\gamma} > 0\ . \end{aligned} \tag{3.18}$$

Therefore, the assumption $(TW1)$ implies that the right hand side of this inequality is positive for $c > 0$ small. Then, recalling Lemma 3.7, the conclusion follows from the intermediate value theorem and the continuity of the function $c \mapsto \min_Y \mathcal{J}_c$ on $(0, +\infty)$, a property that will be proved in Lemma 4.4 in the next section. □

Note that the assumption $\alpha \geq 1$ is a necessary condition for Lemma 3.11 to hold.

Lemma 3.12. *Suppose* $\alpha < 1$*. Then for any* $c > 0$ *and any minimizer* $\mathbb{E}(c)$ *of* (1.19) *one has* $\mathcal{J}_c(\mathbb{E}(c)) > 0$*.*

Proof. Fix $c > 0$. Recalling (2.9) and applying the divergence theorem to the smooth minimizer $\mathbb{E}(c)$, one gets, denoting by $\nu = (\nu_1, \nu_2)$ the exterior normal to $\partial\mathbb{E}(c)$,

$$\begin{aligned}\mathcal{J}_c(\mathbb{E}(c)) &> \frac{\sqrt{2}}{12c}\mathcal{P}_e((\mathbb{E}(c))_c; \mathbb{T}^2_{cT}) - \frac{\sqrt{2}}{12}\alpha \\ &\geq \frac{\sqrt{2}}{12}\Big(\int_{\partial\mathbb{E}(c)} e^x|\tau_2|\, d\mathcal{H}^1 - \alpha\Big) \geq \frac{\sqrt{2}}{12}\Big(\int_{\partial\mathbb{E}(c)} e^x \nu_2\, d\mathcal{H}^1 - \alpha\Big) \\ &= \frac{\sqrt{2}}{12}\Big(\int_{\mathbb{E}(c)} e^x\, dz - \alpha\Big) = \frac{\sqrt{2}}{12}(1 - \alpha) > 0\ .\end{aligned}$$

□

As we already mentioned in the Introduction, in [12] we proved that (A1)* and (A2), are necessary and sufficient conditions for the existence and uniqueness of planar traveling front and traveling pulse, respectively. Both (A1)* and (A2) imply (TW1). Hence in both cases Lemma 3.11 ensures the existence in the 2-dimensional case of a traveling wave global minimizer in 2D.

4 Γ-convergence

Let us now define the function $\phi(\xi) = \int_0^\xi \sqrt{2F_0(\eta)}\, d\eta$, where F_0 is as in (1.7), which is commonly used in the framework of phase transition problems, see for instance [29, 32]. Although the situation considered in these papers is similar to ours, in dealing with traveling waves instead of stationary solutions additional complications arise: an unbounded domain, a nonlocal term and the weight e^x. Observe that ϕ is a strictly increasing function with $\phi(0)=0$ and $\phi(1)=\frac{\sqrt{2}}{12}$. A direct calculation yields

$$\phi(\xi)=\begin{cases} \frac{1}{\sqrt{2}}(\frac{\xi^3}{3}-\frac{\xi^2}{2}) & \text{if } \xi<0\,,\\ \frac{1}{\sqrt{2}}(-\frac{\xi^3}{3}+\frac{\xi^2}{2}) & \text{if } 0<\xi<1\,,\\ \frac{1}{\sqrt{2}}\left(\frac{\xi^3}{3}-\frac{\xi^2}{2}+\frac{1}{3}\right), & \text{if } \xi\geq 1\,. \end{cases}$$

Let $\widetilde{M}>2$ be the number appearing in the definition (1.11). Note that there exists $k_0>0$ such that for all $|\xi|\leq\widetilde{M}$

$$|\phi(\xi)|\geq k_0\xi^2\,. \tag{4.1}$$

Lemma 4.1 (Compactness and lower bound)**.**
Let $\{\epsilon_h\},\{c_h\}\subset(0,\infty)$ be two sequences such that $\epsilon_h\to 0$, $c_h\to c>0$ and let $\{w_h\}\subset Y$ be such that $\liminf_{h\to\infty} J_{c_h,\epsilon_h}(w_h)\leq C_0$ for some $C_0>0$. Then there exists a measurable set $E\subset\mathbb{T}^2_T$ such that, up to a not relabelled subsequence, $w_h\to\chi_E$ in $L^2_e(\mathbb{T}^2_T)$. Moreover,

$$\frac{\phi(1)}{c}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT})\leq\liminf_{h\to\infty}\int_{\mathbb{T}^2_T} e^x\left(\epsilon_h\frac{w_{h,x}^2}{2}+\frac{\epsilon_h}{c_h^2}\frac{w_{h,y}^2}{2}+\frac{F_0(w_h)}{\epsilon_h}\right)dz\,. \tag{4.2}$$

Proof. Throughout the proof of this lemma, in order to ease the presentation, we will denote with the same symbol a function $u:\mathbb{T}^2_T\to\mathbb{R}$ and its T-periodic extension to $\mathbb{R}^2$.

From the definition of Y we have $\|w_h\|_{L^\infty}\leq\widetilde{M}$, hence $\|w_h\|_{L^p_e}\leq\|w_h\|^{2/p}_{L^2_e}\,\|w_h\|^{(p-2)/p}_{L^\infty}\leq\widetilde{M}^{(p-2)/p}$ for all $p>2$. For every $(x,y)\in\mathbb{R}^2$ we set $\widetilde{w}_h(x,y)=w_h(x,y/c_h)$. Note that $\widetilde{w}_h\in H^1_{loc}(\mathbb{R}^2)$ and is c_hT-periodic in y. Recalling the definition of Ω_T in (1.4), and (1.7),

$$\begin{aligned} \frac{1}{c_h}\int_{\Omega_{c_hT}} e^x|D(\phi(\widetilde{w}_h))|\,dz &= \int_{\mathbb{T}^2_T} e^x\sqrt{2F_0(w_h)}\,\sqrt{w_{h,x}^2+\frac{1}{c_h^2}w_{h,y}^2}\,dz\\ &\leq\int_{\mathbb{T}^2_T} e^x\left(\epsilon_h\frac{w_{h,x}^2}{2}+\frac{\epsilon_h}{c_h^2}\frac{w_{h,y}^2}{2}+\frac{F_0(w_h)}{\epsilon_h}\right)dz\\ &\leq J_{c_h,\epsilon_h}(w_h)+\alpha\int_{\Omega_T} e^x|G(w_h)|\,dz\\ &\leq J_{c_h,\epsilon_h}(w_h)+\frac{\alpha}{\sqrt{2}}\Big(\frac{1}{3}\int_{\Omega_T} e^x|w_h|^3\,dz+\frac{1}{2}\int_{\Omega_T} e^x|w_h|^2\,dz\Big)\leq C_1\,, \end{aligned} \tag{4.3}$$

for some constant C_1 depending only on α, C_0 and $\widetilde{M}$.

Since the functions $\phi(\widetilde{w}_h)$ are c_hT-periodic in y, they are equibounded in $W^{1,1}(B_k)$ for all integers $k\geq 1$. Therefore by the compactness theorem for $W^{1,1}$ functions on bounded, smooth domains and a standard diagonalization argument, we may find a (not relabeled) subsequence $\phi(\widetilde{w}_h)$ converging in $L^1_{loc}(\mathbb{R}^2)$ and a.e. to a function $\Phi_0\in L^1_{loc}(\mathbb{R}^2)$. We may also assume that along the subsequence $\widetilde{w}_h$ there exists the limit

$$L=\lim_{h\to\infty}\int_{\Omega_{c_hT}} e^x|D(\phi(\widetilde{w}_h))|\,dz<\infty\,. \tag{4.4}$$

We claim that Φ_0 is cT-periodic in y. Indeed using (4.4), for h large we have,

$$\begin{aligned}
&\|\phi(\widetilde{w}_h(x,y)) - \phi(\widetilde{w}_h(x,y+cT))\|_{L^1_e(\Omega_{cT})} = \|\phi(\widetilde{w}_h(x,y+c_hT)) - \phi(\widetilde{w}_h(x,y+cT))\|_{L^1_e(\Omega_{cT})} \\
&\le \int_{\Omega_{cT}} e^x\,dz \int_0^1 \Big|\frac{d}{d\theta}\left(\phi(\widetilde{w}_h(x,y+cT+\theta(c_h-c)T)\right)\,d\theta\Big| \\
&\le T|c_h-c| \int_{\Omega_{cT}} e^x\,dz \int_0^1 |D_y(\phi(\widetilde{w}_h))(x,y+cT+\theta(c_h-c)T))|\,d\theta \\
&\le 2LT\,|c_h-c| \to 0\,.
\end{aligned}$$

Since ϕ is a strictly increasing function, setting $\widetilde{w}_0 := \phi^{-1}(\Phi_0)$, we have that $\widetilde{w}_h \to \widetilde{w}_0$ a.e. and thus $-\widetilde{M} \le \widetilde{w}_0 \le \widetilde{M}$ a.e.. Setting $w_0(x,y) = \widetilde{w}_0(x,cy)$, w_0 is T-periodic in y and $w_h \to w_0$ in $L^1_{loc}(\mathbb{R}^2)$ and a.e.. Observe that from (4.3) we have

$$\int_{\Omega_T} e^x F_0(w_h)\,dz \le \epsilon_h \left\{ J_{c_h,\epsilon_h}(w_h) + \alpha \int_{\Omega_T} e^x |G(w_h)|\,dz \right\} \le \epsilon_h C_1\,.$$

An application of the Fatou's lemma gives

$$0 \le \int_{\Omega_T} e^x F_0(w_0)\,dz \le \liminf_{h\to\infty} \int_{\Omega_T} e^x F_0(w_h)\,dz = 0\,,$$

which forces $F_0(w_0) = 0$ a.e.. In turn this implies that $w_0(z) \in \{0,1\}$ for all $z \in \mathbb{R}^2$. Therefore, there exists a measurable set $E \subset \mathbb{R}^2$ such that $w_0 = \chi_E$ and $\widetilde{w}_0 = \chi_{E_c}$. Note that $E + Te_2 = E$.

Let us now prove that w_h converge to χ_E in $L^2_e(\Omega_T)$. To this end fix $M > 0$ large. We are now going to estimate

$$\int_{\Omega_T\cap\{x>M\}} e^x w_h^2\,dz = \int_{\Omega_T\cap\{x>M\}\cap\{|w_h|\le 1/2\}} e^x w_h^2\,dz + \int_{\Omega_T\cap\{x>M\}\cap\{|w_h|>1/2\}} e^x w_h^2\,dz\,.$$

The first integral on the right hand side is easily controlled as follows:

$$\int_{\Omega_T\cap\{x>M\}\cap\{|w_h|\le 1/2\}} e^x w_h^2\,dz \le C \int_{\Omega_T\cap\{|w_h|\le 1/2\}} e^x F_0(w_h)\,dz \le C \int_{\Omega_T} e^x F_0(w_h)\,dz \le C_2\epsilon_h\,.$$

To control the second integral on the right hand side we use Lemma 2.3 and (4.1):

$$\begin{aligned}
\int_{\Omega_T\cap\{x>M\}\cap\{|w_h|>1/2\}} e^x w_h^2\,dz &= \frac{1}{c_h} \int_{\Omega_{c_hT}\cap\{x>M\}\cap\{|\widetilde{w}_h|>1/2\}} e^x \widetilde{w}_h^2\,dz \\
&\le \frac{1}{c_h k_0} \int_{\Omega_{c_hT}\cap\{x>M\}\cap\{|\widetilde{w}_h|>1/2\}} e^x |\phi(\widetilde{w}_h|\,dz \\
&\le \frac{1}{\phi(1/2)c_h k_0} \int_{\Omega_{c_hT}\cap\{x>M\}\cap\{|\widetilde{w}_h|>1/2\}} e^x \phi(\widetilde{w}_h)^2\,dz \\
&\le Ce^{-M} \int_{\Omega_{c_hT}\cap\{x>M\}} e^{2x} \phi(\widetilde{w}_h)^2\,dz \\
&\le Ce^{-M}\big(\|D\phi(\widetilde{w}_h)\|_{L^1_e(\Omega_{c_hT})}\big)^2 \le C_3 e^{-M}
\end{aligned}$$

for some constant C_3 independent of h. Summing the above estimates, we obtain

$$\int_{\Omega_T\cap\{x>M\}} e^x w_h^2\,dz \le C_2\epsilon_h + C_3 e^{-M}\,.$$

We can now conclude the proof of the L^2_e convergence. Indeed, for any h,k we have

$$\begin{aligned}
&\int_{\Omega_T} e^x|w_h-w_k|^2\,dz\\
\le&\int_{\Omega_T\cap\{x\le -M\}} e^x|w_h-w_k|^2\,dz+\int_{\Omega_T\cap\{-M\le x\le M\}} e^x|w_h-w_k|^2\,dz+\int_{\Omega_T\cap\{x\ge M\}} e^x|w_h-w_k|^2\,dz\\
\le&4\widetilde{M}^2e^{-M}T+\int_{\Omega_T\cap\{-M\le x\le M\}} e^x|w_h-w_k|^2\,dz+2C_2(\epsilon_h+\epsilon_k)+4C_3e^{-M}\,.
\end{aligned}$$

Let $\delta>0$. First, choose $M>0$ so that both the first and the fourth term in the last line of the above formula are smaller than δ. Next pick an h_0 so that the third addend is smaller than δ when $h,k>h_0$. Finally, the conclusion follows by observing that since the functions w_h are uniformly bounded and converge to χ_E pointwise, they also converge in $L^2(\Omega_T\cap\{|x|\le M\})$.

Let us now show (4.2). Observe that if c_h were a decreasing sequence, then the sets Ω_{c_hT} would also decrease with respect to inclusion and they would all contain Ω_{cT}. In this case, since $\widetilde{w}_h\to\chi_{E_c}$ in $L^1_{loc}(\mathbb{R}^2)$, the proof of (4.2) would be an immediate consequence of the Lemma 2.1 and of the first two lines of (4.3). However, since we have no such information on the sequence c_h we need to prove an apriori estimate on the weighted total variation of $D(\phi(\widetilde{w}_h))$ in a strip slightly larger than Ω_{c_hT}.

To this end, we fix an integer $N>1$ and subdivide the interval $(-T/2,T/2)$ in N intervals of equal length with endpoints $t_i=T\big(-\frac12+\frac{i}{N}\big)$, for $i=0,\dots,N$. Recall (4.4) and observe that, passing possibly to a further (not relabelled) subsequence, we may always assume that for every $i=1,\dots,N$ there exists also the limit

$$\lim_{h\to\infty}\int_{c_ht_{i-1}}^{c_ht_i} dy\int_{\mathbb{R}} e^x|D(\phi(\widetilde{w}_h))|\,dx\,.$$

Then, by (4.4) it is clear that there exists $j\in\{1,\dots,N\}$ such that

$$\lim_{h\to\infty}\int_{c_ht_{j-1}}^{c_ht_j} dy\int_{\mathbb{R}} e^x|D(\phi(\widetilde{w}_h))|\,dx\le\frac{L}{N}\,.$$

Set now $\widetilde{v}_h(x,y)=\widetilde{w}_h\big(x,y+c_hT\big(\frac{j}{N}-\frac{1}{2N}\big)\big)$ and $\widetilde{v}_0(x,y)=\widetilde{w}_0\big(x,y+cT\big(\frac{j}{N}-\frac{1}{2N}\big)\big)$ for all $(x,y)\in\mathbb{R}^2$ and observe that the functions $\widetilde{v}_h$ converge in $L^1_{loc}(\mathbb{R}^2)$ to $\widetilde{v}_0$, which in turn is equal to the characteristic function of $E_c-cT\big(\frac{j}{N}-\frac{1}{2N}\big)e_2$. Moreover from (4.4) and the inequality above, we have that

$$\lim_{h\to\infty}\int_{\Omega_{c_h(T+T/N)}} e^x|D(\phi(\widetilde{v}_h))|\,dz\le L\Big(1+\frac{1}{N}\Big)\,.$$

For h sufficiently large, we have $\Omega_{cT} \subset \Omega_{c_h(T+T/N)}$. Recalling Lemma 2.1 and the first inequality in (4.3), we conclude that

$$\begin{aligned}\phi(1)\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) &= \phi(1)\mathcal{P}_e\Big(E_c - cT\Big(\frac{j}{N}-\frac{1}{2N}\Big)e_2;\mathbb{T}^2_{cT}\Big) = \|D(\phi(\widetilde{v}_0))\|_e(\mathbb{T}^2_{cT})\\ &\le \lim_{h\to\infty}\int_{\Omega_{c_h(T+T/N)}} e^x|D(\phi(\widetilde{v}_h))|\,dz \le L\Big(1+\frac{1}{N}\Big)\\ &= \frac{N+1}{N}\lim_{h\to\infty}\int_{\Omega_{c_hT}} e^x|D(\phi(\widetilde{w}_h))|\,dz\\ &\le \frac{(N+1)c}{N}\liminf_{h\to\infty}\int_{\Omega_T} e^x\Big(\epsilon_h\frac{w^2_{h,x}}{2}+\frac{\epsilon_h}{c_h^2}\frac{w^2_{h,y}}{2}+\frac{F_0(w_h)}{\epsilon_h}\Big)\,dz\,.\end{aligned}$$

Then (4.2) follows by letting $N\to\infty$. □

We are now going to prove that the functional J^*_c defined in (1.16) is the Γ-limit of $J_{c,\epsilon}$ with respect to the L^2_e convergence, the main result in this section.

Theorem 4.2. *Let $\epsilon_1 > 0$ and let $c : (0,\epsilon_1) \to (0,\infty)$ such that $\lim_{\epsilon\to 0} c(\epsilon) = \hat{c} > 0$. Then the following two properties hold:*

(i) if $\{w_h\} \subset L^2_e(\mathbb{T}^2_T)$ is a sequence converging to w_0 in $L^2_e(\mathbb{T}^2_T)$ and $\epsilon_h \to 0$, then

$$J^*_{\hat{c}}(w_0) \le \liminf_{h\to\infty} J_{c(\epsilon_h),\epsilon_h}(w_h)\,; \tag{4.5}$$

(ii) for any $w_0 \in L^2_e(\mathbb{T}^2_T)$ and any sequence $\epsilon_h \to 0$ there exists a sequence $\{w_h\} \subset L^2_e(\mathbb{T}^2_T)$ converging in $L^2_e(\mathbb{T}^2_T)$ to w_0 such that

$$J^*_{\hat{c}}(w_0) \ge \limsup_{h\to\infty} J_{c(\epsilon_h),\epsilon_h}(w_h)\,. \tag{4.6}$$

When both conditions (i) and (ii) in the above Theorem are satisfied, we say that the functionals $J_{c(\epsilon),\epsilon}$ Γ-*converge* to $J^*_{\hat{c}}$ in $L^2_e(\mathbb{T}^2_T)$ and write $J^*_{\hat{c}} = \Gamma\text{-}\lim_{\epsilon\to 0} J_{c(\epsilon),\epsilon}$.

Before proving Theorem 4.2, recall that if $u \in L^2_e(\mathbb{T}^2_T)$, from (1.6), setting $v = \mathcal{L}_c u$,

$$\int_{\mathbb{T}^2_T} e^x\big\{c^2((\mathcal{L}_c u)_x)^2 + ((\mathcal{L}_c u)_y)^2 + \gamma(\mathcal{L}_c u)^2\big\}\,dz = \int_{\mathbb{T}^2_T} e^x u\,\mathcal{L}_c u\,dz\,.$$

From this inequality it follows that when the function $c(\epsilon)$ satisfies the assumption of Theorem 4.2 then $\|\mathcal{L}_{c(\epsilon)}u\|_{H^1_e} \le C\|u\|_{L^2_e}$ for some positive constant C, independent of ϵ. Observe also that if $w_h \to w_0$ in L^2_e and $\{c_h\}$ is a sequence of positive numbers converging to $c > 0$ then $\mathcal{L}_{c(\epsilon_h)}w_h \to \mathcal{L}_c w_0$ in H^1_e and thus

$$\int_{\mathbb{T}^2_T} e^x w_h\,\mathcal{L}_{c_h} w_h\,dz \to \int_{\mathbb{T}^2_T} e^x w_0\,\mathcal{L}_c w_0\,dz \qquad \text{as } \epsilon \to 0\,. \tag{4.7}$$

The proof of Theorem 4.2 will be achieved by proving the two conditions (i) and (ii) separately. Let us start with the

Proof of (4.5). Without loss of generality and passing possibly to a not relabelled subsequence, we may assume that the lim inf in (4.5) is a limit and that it is finite. Thus for h large $w_h \in Y$; hence $\|w_h\|_{L^2_e} = 1$ and $-\widetilde{M} \leq w_h \leq \widetilde{M}$. By assumption $w_h \to w_0$ in $L^2_e(\mathbb{T}^2_T)$ and thus from Lemma 4.1 we have $w_0 = \chi_E$, where $E \subset \mathbb{T}^2_T$ is a measurable set such that $|E|_e = 1$.

Since the sequence w_h is bounded in L^∞ and converges to χ_E in L^2_e, we have also $w_h \to \chi_E$ in L^3_e. As a consequence,

$$\alpha \int_{\mathbb{T}^2_T} e^x G(w_\epsilon)\, dz \to \alpha \int_{\mathbb{T}^2_T} e^x G(\chi_E)\, dz = \alpha G(1) = -\frac{\sqrt{2}}{12}\alpha\,.$$

The conclusion then follows at once on recalling (4.2) and (4.7). □

The proof of the limsup inequality relies on the following proposition whose proof is contained in the Appendix.

Proposition 4.3. *$E \subset \mathbb{T}^2_T$ be a bounded, smooth, open set. Then, there exists a family of Lipschitz functions $v_\epsilon : \mathbb{T}^2_T \to [0,1]$, $\epsilon \in (0,1)$, satisfying the following conditions:*
- for all ϵ

$$\int_{\mathbb{T}^2_T} e^x v_\epsilon^2\, dz = \int_{\mathbb{T}^2_T} e^x \chi_E\, dz\,,$$

- there exists $R > 0$ such that $v_\epsilon(x,y) = 0$ whenever $|x| > R$,
- $v_\epsilon \to \chi_E$ in $L^1_e(\mathbb{T}^2_T)$ and

$$\limsup_{\epsilon\to 0} \int_{\mathbb{T}^2_T} e^x \left\{ \frac{\epsilon}{2}|Dv_\epsilon(z)|^2 + \frac{F_0(v_\epsilon)}{\epsilon} \right\} dz \leq \phi(1)\mathcal{P}_e(E;\mathbb{T}^2_T)\,. \tag{4.8}$$

We complete the proof of Theorem 4.2 by giving the

Proof of (4.6). It suffices to assume $J^*_{\hat{c}}(w_0) < \infty$. From (1.16) it follows that $w_0 = \chi_E$, where $E \subset \mathbb{T}^2_T$, $|E|_e = 1$ and $\mathcal{P}_e(E_{\hat{c}};\mathbb{T}^2_{\hat{c}T}) < \infty$.

Assume first that E is a bounded, smooth open set. Let $\{v_\epsilon\} : \mathbb{T}^2_{\hat{c}T} \to [0,1]$ be the family of Lipschitz functions obtained by applying Proposition 4.3 with E and $\mathbb{T}^2_T$ replaced by $E_{\hat{c}}$ and $\mathbb{T}^2_{\hat{c}T}$, respectively. In particular we have $\|v_\epsilon\|^2_{L^2_\epsilon(\mathbb{T}^2_{\hat{c}T})} = \int_{\mathbb{T}^2_{\hat{c}T}} e^x \chi_{E_{\hat{c}}} dz = \hat{c}$. Then, given a sequence $\epsilon_h \to 0$, on setting $w_h(x,y) = v_{\epsilon_h}(x,\hat{c}y)$ we get $\|w_h\|_{L^2_e(\mathbb{T}^2_T)} = 1$ for all h, $w_h \to \chi_E$ in $L^2_e(\mathbb{T}^2_T)$ and, recalling (4.8),

$$\begin{aligned}\limsup_{h\to\infty} \int_{\mathbb{T}^2_T} e^x \left(\epsilon_h \frac{w^2_{h,x}}{2} + \frac{\epsilon_h}{\hat{c}^2}\frac{w^2_{h,y}}{2} + \frac{F_0(w_h)}{\epsilon_h} \right) dz \\ = \frac{1}{\hat{c}} \limsup_{h\to 0\infty} \int_{\mathbb{T}^2_{\hat{c}T}} e^x \left\{ \frac{\epsilon_h}{2}|Dv_{\epsilon_h}(z)|^2 + \frac{F_0(v_{\epsilon_h})}{\epsilon_h} \right\} dz \leq \frac{\phi(1)}{\hat{c}}\mathcal{P}_e(E_{\hat{c}};\Omega_{\hat{c}T})\,.\end{aligned}$$

It is immediate from this inequality that

$$\limsup_{h\to\infty} \int_{\mathbb{T}^2_T} e^x \left(\epsilon_h \frac{w^2_{h,x}}{2} + \frac{\epsilon_h}{c(\epsilon_h)^2}\frac{w^2_{h,y}}{2} + \frac{F_0(w_h)}{\epsilon_h} \right) dz \leq \frac{\phi(1)}{\hat{c}}\mathcal{P}_e(E_{\hat{c}};\Omega_{\hat{c}T})\,.$$

Then (4.6) follows at once from the above inequality, (4.7) and the fact that $w_h \to \chi_E$ in $L^2_e(\mathbb{T}^2_T)$.

Now let $E \subset \mathbb{T}^2_T$ be such that $J^*_{\hat{c}}(\chi_E) < \infty$. Take a sequence of bounded, smooth, open sets E_j as in Theorem 2.6 and observe that $\lim_{j\to\infty} J^*_{\hat{c}}(\chi_{E_j}) = J^*_{\hat{c}}(\chi_E)$. Fix a sequence $\epsilon_h \to 0$. Then, for each j there exists a sequence $\{w_h^{(j)}\}_{h=1}^{\infty}$ converging to χ_{E_j} in $L^2_e(\mathbb{T}^2_T)$ and such that $J^*_{\hat{c}}(\chi_{E_j}) \geq \limsup_{h\to\infty} J_{c(\epsilon_h),\epsilon_h}(w_h^{(j)})$. Thus we can find a strictly increasing sequence $\{h_j\}$ such that $J^*_{c_0}(\chi_{E_j}) + \frac{1}{j} \geq J_{c_{\epsilon_{h_j}},\epsilon_{h_j}}(w_{h_j}^{(j)})$ and $\|\chi_E - w_{h_j}^{(j)}\|_{L^2_e} \leq 1/j$ for all j. Then the result follows by taking the sequences $\epsilon_j = \epsilon_{h_j}$ and $w_j = w_{h_j}^{(j)}$. □

We conclude this section by showing that the minimum value of problem (1.19) is a continuous function of the parameter c. Recall that $\mathbb{E}(c)$ denotes any minimizer of of $\mathcal{J}_c(E)$ under the volume constraint $|E|_e = 1$.

Lemma 4.4. *(i) The function $c \mapsto \mathcal{J}_c(\mathbb{E}(c))$ is continuous for $c \in (0,\infty)$;*
(ii) If $\epsilon_h \to 0^+$ and $c_h \to c$, for some $c > 0$, then $\lim_{h\to\infty} \big(\inf_{L^2_e} J_{c_h,\epsilon_h}\big) = \mathcal{J}_c(\mathbb{E}(c))$.

Proof. Part (ii) is a standard result in Γ-convergence. We supply the proof for completeness. Using Theorem 4.2 we may find a sequence $\{w_h\} \subset Y$ converging to $\chi_{\mathbb{E}(c)}$ in $L^2(\mathbb{T}^2_T)$ such that (4.6) holds. Then

$$\mathcal{J}_c(\mathbb{E}(c)) = J^*_c(\chi_{\mathbb{E}(c)}) \;\geq\; \limsup_{h\to\infty} J_{c_h,\epsilon_h}(w_h) \geq \limsup_{h\to\infty} \big(\inf_{L^2_e} J_{c_h,\epsilon_h}\big)\,.$$

To prove the opposite inequality, passing possibly to a not relabelled subsequence, we may assume that the $\liminf_{h\to\infty}\big(\inf_{L^2_e} J_{c_h,\epsilon_h}\big)$ is indeed a limit. Then for every h we may choose a function $w_h \in Y$ such that $\inf_{L^2_e} J_{c_h,\epsilon_h} > J_{c_h,\epsilon_h}(w_h) - 1/h$. Using compactness Lemma 4.1 and passing to a further subsequence if needed, we have that there exists $E \subset \mathbb{T}^2_T$ such that $w_h \to \chi_E$ in $L^2_e(\mathbb{T}^2_T)$. Therefore, by (4.5) we have

$$\liminf_{h\to\infty} \big(\inf_{L^2_e} J_{c_h,\epsilon_h}\big) \;\geq\; \liminf_{h\to\infty} J_{c_h,\epsilon_h}(w_h) \geq J^*_c(\chi_E) \geq \mathcal{J}_c(\mathbb{E}(c))\,,$$

thus concluding the proof of (ii).

Let $\{c_h\} \subset (0,\infty)$ be a sequence converging to some $c > 0$. Observe that for any set $E \subset \mathbb{T}^2_T$ of finite weighted measure and weighted perimeter, from (2.9) and (4.7) we have

$$\lim_{h\to\infty} \mathcal{P}_e(E_{c_h};\mathbb{T}^2_{c_hT}) = \mathcal{P}_e(E_c;\mathbb{T}^2_{cT}), \qquad \mathcal{L}_{c_h}\chi_E \to \mathcal{L}_c\chi_E \;\text{ in }\; H^1_e(\mathbb{T}^2_T)\,.$$

Therefore we immediately have

$$\begin{aligned}\mathcal{J}_c(\mathbb{E}(c)) &= \lim_{h\to\infty}\left\{\frac{1}{c_h}\frac{\sqrt{2}}{12}\mathcal{P}_e((\mathbb{E}(c))_{c_h};\mathbb{T}^2_{c_hT}) - \frac{\sqrt{2}}{12}\alpha + \frac{\sigma}{2}\int_{\mathbb{T}^2_T} e^x \chi_{\mathbb{E}(c)}\mathcal{L}_{c_h}\chi_{\mathbb{E}(c)}\,dz\right\}\\ &\geq \limsup_{h\to\infty} \mathcal{J}_{c_h}(\mathbb{E}(c_h))\,.\end{aligned}$$

Fix now a sequence $\epsilon_j \to 0$. For any h, thanks to (ii) we have in particular that $\lim_{j\to\infty}\big(\inf_{L^2_e} J_{c_h,\epsilon_j}\big) = \mathcal{J}_{c_h}(\mathbb{E}(c_h))$. Therefore we may find a strictly increasing sequence of integers $\{j_h\}$ and a sequence $\{w_h\} \subset Y$ such that $\mathcal{J}_{c_h}(\mathbb{E}(c_h)) > J_{c_h,\epsilon_{j_h}}(w_h) - 1/h$ for all h. Then, by Lemma 4.1 and passing possibly to a not relabelled subsequence, we have that $w_h \to \chi_E$ for some $E \subset \mathbb{T}^2_T$. Hence, recalling (4.5), we get

$$\liminf_{h\to\infty} \mathcal{J}_{c_h}(\mathbb{E}(c_h)) \geq \liminf_{h\to\infty} J_{c_h,\epsilon_{j_h}}(w_h) \geq \mathcal{J}_c(E) \geq \mathcal{J}_c(\mathbb{E}(c)) .$$

Then the conclusion follows. □

5 Minimizer for the FitzHugh-Nagumo equations

The two lemmas in this section enable us to recover traveling wave solutions for the FitzHugh-Nagumo equations (1.5)-(1.6) from the minimizers of the limit functional $\mathcal{J}_c$.

Lemma 5.1. *Assume condition (TW1) in* (1.17) *holds. There exists $\epsilon_1 > 0$ with the property that for any $\epsilon \in (0, \epsilon_1)$, there exists $c > 0$ and $u \in Y$ such that, taking $d = \epsilon^2/c^2$,*

$$\mathcal{I}_{c,d,\epsilon}(u) = \min_Y \mathcal{I}_{c,d,\epsilon} = 0 ,$$

where $\mathcal{I}_{c,d,\epsilon}$ is defined as in (1.9).

Proof. We first show that if $\inf_Y \mathcal{I}_{c,d,\epsilon} = 0$ for some $c, d, \epsilon > 0$ with ϵ sufficiently small, then $\mathcal{I}_{c,d,\epsilon}$ has a minimizer $u \in Y$.

To see this observe that there exists a constant $M_2 > 0$, depending only on α, such that $-M_2\xi^2 \leq F_\epsilon(\xi)$ for all $\xi \in \mathbb{R}$ and $0 < \epsilon \leq 1$. Therefore for any $c, d > 0$, $\epsilon \in (0, 1)$ and any $w \in Y$ we have

$$\mathcal{I}_{c,d,\epsilon}(w) \geq \int_{\mathbb{T}^2_T} e^x F_\epsilon(w)\, dz \geq -M_2 \int_{\mathbb{T}^2_T} e^x w^2\, dz = -M_2$$

so that $\inf_Y \mathcal{I}_{c,d,\epsilon}$ is bounded from below. Let $\{w_n\} \subset Y$ be a minimizing sequence such that $\mathcal{I}_{c,d,\epsilon}(w_n) \leq \inf_Y \mathcal{I}_{c,d,\epsilon} + 1$. Then

$$\begin{aligned}\frac{d}{2}\min\{c^2, 1\} \int_{\mathbb{T}^2_T} e^x |\nabla w_n|^2\, dz &\leq \mathcal{I}_{c,d,\epsilon}(w_n) - \int_{\mathbb{T}^2_T} e^x F_\epsilon(w_n)\, dz \\ &\leq \inf_Y \mathcal{I}_{c,d,\epsilon} + 1 + M_2 .\end{aligned}$$

Using the Poincaré inequality (3.14) we get that the sequence $\{w_n\}$ is bounded in $H^1_e(\mathbb{T}^2_T)$. Therefore we may assume that there exists $w \in H^1_e(\mathbb{T}^2_T)$ such that, up to a (not relabelled) subsequence, $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{T}^2_T)$, strongly in $L^2_{loc}(\mathbb{T}^2_T)$ and pointwise a.e.. Thus $|w| \leq \widetilde{M}$ and

$$\int_{\mathbb{T}^2_T} e^x \left(\frac{dc^2}{2} w_x^2 + \frac{d}{2} w_y^2\right) dz \leq \liminf_{n\to\infty} \int_{\mathbb{T}^2_T} e^x \left(\frac{dc^2}{2}(w_n)_x^2 + \frac{d}{2}(w_n)_y^2\right) dz .$$

From the inequality $\int_{\mathbb{T}^2_T} e^x (w_n - w)\, \mathcal{L}_c(w_n - w)\, dz \geq 0$, we easily obtain

$$\int_{\mathbb{T}^2_T} e^x w \mathcal{L}_c w\, dz \leq \liminf_{n\to\infty} \int_{\mathbb{T}^2_T} e^x w_n \mathcal{L}_c w_n\, dz\ .$$

It remains to control the integrals of $e^x F_\epsilon(w_n)$. Fix $M > 0$ and split

$$\int_{\mathbb{T}^2_T \cap \{x>M\}} e^x F_\epsilon(w_n)\, dz = \int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w_n<1/4\}} e^x F_\epsilon(w_n)\, dz + \int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w_n\geq 1/4\}} e^x F_\epsilon(w_n)\, dz\, .$$

A similar splitting for the integral of $e^x F_\epsilon(w)$ into the sets $\{w < 1/4\}$ and $\{w \geq 1/4\}$ is also performed. Observe that if $\epsilon > 0$ is sufficiently small, depending on α, then $F_\epsilon(\xi) \geq 0$ for all $\xi \leq 1/4$. Thus the first integral on the right hand side can be easily estimated by Fatou's lemma

$$\int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w<1/4\}} e^x F_\epsilon(w)\, dz \leq \liminf_{n\to\infty} \int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w_n<1/4\}} e^x F_\epsilon(w_n)\, dz\, ,$$

observing that $\chi_{\{w<1/4\}}(z) \leq \liminf_{n\to\infty} \chi_{\{w_n<1/4\}}(z)$ for a.e. z.

To control the second integral on the right hand side, we first fix a constant $k_1 > 0$ so that $|F_\epsilon(\xi)| \leq k_1 \xi^2$ for all $1/4 \leq \xi \leq \widetilde{M}$ and all $\epsilon \in (0, 1)$. Setting $w_n(x, y) = \widetilde{w}_n(x, cy)$ and $w(x, y) = \widetilde{w}(x, cy)$, we use Lemma 2.3 and (4.1) to obtain, arguing as in the proof of (4.3),

$$\begin{aligned}
\int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w_n\geq 1/4\}} e^x |F_\epsilon(w_n)|\, dz &= \frac{1}{c} \int_{\mathbb{T}^2_{cT} \cap \{x>M\}\cap\{\widetilde{w}_n\geq 1/4\}} e^x |F_\epsilon(\widetilde{w}_n)|\, dz \\
&\leq \frac{k_1}{ck_0} \int_{\mathbb{T}^2_{cT} \cap \{x>M\}\cap\{\widetilde{w}_n\geq 1/4\}} e^x \phi(\widetilde{w}_n)\, dz \\
&\leq \frac{k_1}{ck_0\, \phi(1/4)} \int_{\mathbb{T}^2_{cT} \cap \{x>M\}\cap\{\widetilde{w}_n\geq 1/4\}} e^x \phi(\widetilde{w}_n)^2\, dz \\
&\leq Ce^{-M} \int_{\mathbb{T}^2_{cT} \cap \{x>M\}} e^{2x} \phi(\widetilde{w}_n)^2\, dz \leq Ce^{-M} \big(\|D\phi(\widetilde{w}_n)\|_{L^1_e(\mathbb{T}^2_{cT})}\big)^2 \\
&\leq Ce^{-M} \left(\frac{1}{\epsilon} \mathcal{I}_{c,d,\epsilon}(w_n) + \int_{\mathbb{T}^2_T} e^x |G(w_n)|\, dz\right)^2 \\
&\leq Ce^{-M} \left(\inf_Y \mathcal{I}_{c,d,e} + 1 + \|w_n\|^3_{L^3_e} + \|w_n\|^2_{L^2_e}\right)^2 \leq Ce^{-M}\, ,
\end{aligned}$$

for some constant C (which may change from line to line) depending on ϵ and on the uniform bounds on w_n, but independent of n and M. A similar calculation yields

$$\int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w\geq 1/4\}} e^x |F_\epsilon(w)|\, dz \leq Ce^{-M}\ .$$

The above inequalities lead to

$$\begin{aligned}
\int_{\mathbb{T}^2_T \cap \{x>M\}} e^x F_\epsilon(w)\, dz &\leq \liminf_{n\to\infty} \int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w_n\leq 1/4\}} e^x F_\epsilon(w_n)\, dz + \int_{\mathbb{T}^2_T \cap \{x>M\}\cap\{w\geq 1/4\}} e^x |F_\epsilon(w)|\, dz \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{T}^2_T \cap \{x>M\}} e^x F_\epsilon(w_n)\, dz + 2Ce^{-M}\ .
\end{aligned}$$

Then, from this inequality, observing that $\int_{\mathbb{T}^2_T\cap\{x\le M\}} e^x F_\epsilon(w)\,dz = \lim_{n\to\infty}\int_{\mathbb{T}^2_T\cap\{x\le M\}} e^x F_\epsilon(w_n)\,dz$ by the dominated convergence and letting $M\to\infty$, we get

$$\int_{\mathbb{T}^2_T} e^x F_\epsilon(w)\,dz \le \liminf_{n\to\infty}\int_{\mathbb{T}^2_T} e^x F_\epsilon(w_n)\,dz$$

and thus

$$\mathcal{I}_{c,d,\epsilon}(w) \le \liminf_{n\to\infty}\mathcal{I}_{c,d,\epsilon}(w_n) = \inf_Y \mathcal{I}_{c,d,\epsilon} = 0\ .$$

Note that $\|w\|_{L^2_e} \le \liminf_{n\to\infty}\|w_n\|_{L^2_e} = 1$. If $\|w\|_{L^2_e}=1$, then $w\in Y$ and it is a minimizer of $\mathcal{I}_{c,d,\epsilon}$ in Y. Otherwise, if $\|w\|_{L^2_e}<1$, we first observe that w cannot be identically equal to zero because otherwise from the above estimates we would have, recalling (3.14),

$$\begin{aligned}0=\liminf_{n\to\infty}\mathcal{I}_{c,d,\epsilon}(w_n) &\ge \frac{dc^2}{2}\liminf_{n\to\infty}\|(w_n)_x\|^2_{L^2_e} + \liminf_{n\to\infty}\int_{\mathbb{T}^2_T} e^x\left(F_\epsilon(w_n)+\epsilon\frac{\sigma}{2}w_n\,\mathcal{L}_c w_n\right)dz\\ &\ge \frac{dc^2}{8}\liminf_{n\to\infty}\|w_n\|^2_{L^2_e} + \int_{\mathbb{T}^2_T} e^x\left(F_\epsilon(w)+\epsilon\frac{\sigma}{2}w\,\mathcal{L}_c w\right)dz = \frac{dc^2}{8} > 0\ .\end{aligned}$$

Thus $\|w\|_{L^2_e}>0$. Therefore we can shift w to the right by a distance $a>0$ so that, setting $u(x,y):=w(x-a,y)$, we have $\|u\|_{L^2_e} = e^{a/2}\|w\|_{L^2_e}=1$. Thus $u\in Y$ and $\mathcal{I}_{c,d,\epsilon}(u) = e^a\mathcal{I}_{c,d,\epsilon}(w) \le \mathcal{I}_{c,d,\epsilon}(w)\le 0$. Hence $u\in Y$ is a minimizer of $\mathcal{I}_{c,d,\epsilon}$ with $\mathcal{I}_{c,d,\epsilon}(u)=0$.

To conclude the proof we need to show that for $\epsilon>0$ small there exist $c,d>0$ such that $\inf_Y\mathcal{I}_{c,d,\epsilon}=0$. From (3.18) and Lemma 3.7 we know that there exist $0<c_+<c_-$ such that $\mathcal{J}_{c_+}(\mathbb{E}(c_+))>0$ and $\mathcal{J}_{c_-}(\mathbb{E}(c_-))<0$. As a consequence of statement (ii) of Lemma 4.4, by choosing ϵ_1 small enough we can ensure that $\inf_{L^2_e} J_{c_+,\epsilon}>0$ and $\inf_{L^2_e} J_{c_-,\epsilon}<0$ for all $0<\epsilon\le\epsilon_1$. Set now for $0<\epsilon<\epsilon_1$

$$\widehat{c} = \inf\left\{c\in(c_+,c_-):\ \inf_{L^2_e} J_{c,\epsilon}<0\right\}.$$

We claim that $\inf_{L^2_e} J_{\widehat{c},\epsilon}=0$. Indeed fix a sequence $c_n\in[\widehat{c},c_-]$ such that $\inf_{L^2_e} J_{c_n,\epsilon}<0$ for all n and $c_n\to\widehat{c}$. For every n take a function $w_n\in Y$ such that $J_{c_n,\epsilon}(w_n)<\inf_{L^2_e} J_{c_n,\epsilon}+1/n$. Then, with the same argument used above, we get that there exists $w\in H^1_e(\mathbb{T}^2_T)$ such that, up to a not relabelled subsequence, $J_{\widehat{c},\epsilon}(w)\le\liminf_{n\to\infty} J_{c_n,\epsilon}(w_n)\le 0$. Again, the same argument as above shows that w is not identically zero and that, possibly shifting a bit w to the right, one can find $u\in Y$ such that $J_{\widehat{c},\epsilon}(u)\le 0$. Hence, $\inf_{L^2_e} J_{\widehat{c},\epsilon}\le 0$.

On the other hand this infimum cannot be strictly negative because otherwise $\widehat{c}>c+$ and we could take $\overline{w}\in Y$ such that $J_{\widehat{c},\epsilon}(\overline{w})<0$. But then there would exists $c\in(c_+,\widehat{c})$ such that also $J_{c,\epsilon}(\overline{w})<0$, which is impossible by the definition of $\widehat{c}$. Thus $\inf_{L^2_e} J_{\widehat{c},\epsilon}=0$. This concludes the proof since, taking $d=\epsilon^2/\widehat{c}^2$, we have $J_{\widehat{c},\epsilon}=\mathcal{I}_{\widehat{c},d,\epsilon}$. □

Lemma 5.2. *There exists $\epsilon_0>0$ with the following property. Let u be a global minimizer of $\mathcal{I}_{c,d,\epsilon}$ in Y such that $\mathcal{I}_{c,d,\epsilon}(u)=0$ for some $0<\epsilon<\epsilon_0$ and let $v=\mathcal{L}_c u$. Then (u,v,c) satisfies (1.5)-(1.6); i.e., it is a traveling wave solution of (1.1) with speed c.*

Proof. We claim that if $|u|\le\widetilde{M}-1$, then the Lagrange multiplier associated with the remaining constraint $\|w\|_{L^2_e}=1$ in Y is zero. Note that this will immediately imply that u is a traveling wave of (1.1) with speed c. To prove the claim we set $L(u) = \mathcal{I}_{c,d,\epsilon}(u) + \Lambda \int_{\mathbb{T}_T^2} e^x u^2\, dz$, where Λ is the Lagrange multiplier. Then u satisfies the following Euler-Lagrange equation

$$L'(u)w = \mathcal{I}'_{c,d,\epsilon}(u)w + 2\Lambda \int_{\mathbb{T}_T^2} e^x uw\, dz = 0 \qquad \text{for all } w \in H_e^1(\Omega_T)\ ,$$

where L' and $\mathcal{I}'_{c,d,\epsilon}$ stand for the Fréchet derivative of L and $\mathcal{I}_{c,d,\epsilon}$, respectively. Note that u is a smooth function whose first derivatives belong to $H_e^1(\mathbb{T}_T^2)$. To see this set $\tilde{u} := e^{x/2}u$. Then, taking into account the additional Lagrange multiplier term $2\Lambda u$, (1.5) becomes the linear equation $dc^2\tilde{u}_{xx} + d\tilde{u}_{yy} + e^{x/2}g = 0$ for a suitable function g such that $e^{x/2}g \in L^2(\mathbb{T}_T^2)$. Applying local $W^{2,2}$ estimate yields that there exists a constant C such that $\|\tilde{u}\|_{W^{2,2}(k,k+1)} \leq C(\|u\|_{L_e^2(k-1,k+2)} + \|g\|_{L_e^2(k-1,k+2)})$ for any integer $k \in \mathbb{Z}$. Then, summing over k, one gets that $u \in W_e^{2,2}(\mathbb{T}_T^2)$. Therefore we may plug $w = u_x$ in the above equation, thus getting

$$\begin{aligned} 0 = L'(u)u_x &= \int_{\mathbb{T}_T^2} e^x (dc^2 u_x u_{xx} + d u_y u_{xy} - f_\epsilon(u)u_x + \epsilon\sigma u_x \mathcal{L}_c u)\, dz + 2\Lambda \int_{\mathbb{T}_T^2} e^x u u_x\, dz \\ &= \int_{\mathbb{T}_T^2} e^x \left(\frac{dc^2}{2}(u_x^2)_x + \frac{d}{2}(u_y^2)_x + (F_\epsilon(u))_x + \frac{\epsilon\sigma}{2}(u\mathcal{L}_c u)_x \right) dz + \Lambda \int_{\mathbb{T}_T^2} e^x (u^2)_x\, dz \\ &= -\mathcal{I}_{c,d,\epsilon}(u) - \Lambda \int_{\mathbb{T}_T^2} e^x u^2\, dz = -\mathcal{I}_{c,d,\epsilon}(u) - \Lambda \end{aligned}$$

so that $\Lambda = 0$, since $\mathcal{I}_{c,d,\epsilon}(u) = 0$.

We now prove that $|u| \leq \widetilde{M} - 1$. First, comparing the energy of $\mathcal{L}_c u$ with the one of the truncated function $\big(\mathcal{L}_c u \vee -\widetilde{M}/\gamma\big) \wedge (\widetilde{M}/\gamma)$, we get $\|\mathcal{L}_c u\|_{L^\infty} \leq \frac{\widetilde{M}}{\gamma}$. Suppose $\widetilde{M} - 1 < u \leq \widetilde{M}$ on a set S with positive measure. We define a cut-off function setting $u_{cut} := u \wedge (\widetilde{M} - 1)$. As before, we get $\|\mathcal{L}_c u_{cut}\|_{L^\infty} \leq \frac{\widetilde{M}}{\gamma}$. By the convexity of F_ϵ on $[\widetilde{M} - 1, \infty)$ for $\epsilon \leq \epsilon_0$ sufficiently small, we have

$$\begin{aligned} \mathcal{I}_{c,d,\epsilon}(u_{cut}) - \mathcal{I}_{c,d,\epsilon}(u) &\leq \int_{\Omega_T} e^x \Big(F_\epsilon(u_{cut}) - F_\epsilon(u) + \frac{\epsilon\sigma}{2}(u_{cut} - u)\, \mathcal{L}_c(u_{cut} + u) \Big)\, dz \\ &\leq \int_S e^x (u - u_{cut}) \Big(f_\epsilon(\widetilde{M} - 1) - \frac{\epsilon\sigma}{2}(\mathcal{L}_c u + \mathcal{L}_c u_{cut}) \Big)\, dz \\ &\leq \int_S e^x (u - u_{cut}) \bigg(-(\widetilde{M} - 1)\,(\widetilde{M} - 2)^2 + \sigma \frac{\widetilde{M}}{\gamma} \bigg)\, dz < 0 \end{aligned}$$

by the choice of $\widetilde{M}$ in (1.10) and the definition of f_ϵ in (1.2). Hence $\mathcal{I}_{c,d,\epsilon}(u_{cut}) < 0$. Since $0 < \|u_{cut}\|_{L_e^2} < \|u\|_{L_e^2} = 1$, we shift u_{cut} to the right by a distance $a > 0$, so that the function $U_{cut}(x, y) := u_{cut}(x, y - a)$ satisfies the constraint $\|U_{cut}\|_{L_e^2} = e^{a/2}\|u_{cut}\|_{L_e^2} = 1$. Thus $\mathcal{I}_{c,d,\epsilon}(U_{cut}) = e^a \mathcal{I}_{c,d,\epsilon}(u_{cut}) < 0$, a contradiction to the minimality of u in Y. Hence $u \leq \widetilde{M} - 1$. A similar argument gives $u \geq -\widetilde{M} + 1$. □

Remark 5.3. *The above lemmas and their proofs establish Statement 1 in Theorem 1.2. Statement 2 in the same Theorem can be deduced using the same argument in proving Lemma 5.1 with the new c_+ and c_- being taken in a small neighborhood of c_0 and [7, Theorem 5.1].*

6 First and second variation formulae

In this section we shall denote by $X : \mathbb{T}^2_T \mapsto \mathbb{T}^2_T$ a *smooth vector field with compact support* and by $\Phi : \mathbb{T}^2_T \times (-\infty, \infty) \mapsto \mathbb{T}^2_T$ *the associated vector field* defined as in (1.21). Note that $\Phi(\cdot, t)$ is a smooth diffeomorphism for all t. Therefore if $E \subset \mathbb{T}^2_T$ is a *smooth open set* the same is true also for $E_t := \Phi(\cdot, t)(E)$. We will consider a weight (or density) of the type e^{ψ} where $\psi : \mathbb{T}^2_T \to \mathbb{R}$ is a smooth function. Though we are interested in the case when $\psi(x, y) = x$ to treat traveling waves, we will derive the first and the second variation formulae for such general weight and we shall always assume that E has *finite weighted measure and finite weighted perimeter* with respect to the weight under consideration. Let us denote the weighted volume of E_t by

$$\mathcal{V}(t) = \int_{E_t} e^{\psi(z)}\, dz\, .$$

We calculate the derivatives of $\mathcal{V}$. To this end we recall that

$$D\Phi(\cdot, t) = I + t\, DX + \frac{t^2}{2} DZ + o(t^2), \tag{6.1}$$

where $Z = DX[X]$, that is $Z_i = \sum_{j=1}^2 X_j D_j X_i$ for $i = 1, 2$. Denoting by $J\Phi$ the Jacobian of the diffeomorphism $\Phi(\cdot, t)$, from the above formula we have (see [23, (2.28) and (2.30)])

$$J\Phi = 1 + t\, \mathrm{div} X + \frac{t^2}{2} \mathrm{div}((\mathrm{div} X)X) + o(t^2), \tag{6.2}$$

Proposition 6.1. *If X, E and ψ are as above, then*

$$\mathcal{V}'(t) = \int_{\partial E_t} e^{\psi} X \cdot \nu_{E_t}\, d\mathcal{H}^1, \tag{6.3}$$

$$\mathcal{V}''(t) = \int_{\partial E_t} \mathrm{div}(e^{\psi} X)\, X \cdot \nu_{E_t}\, d\mathcal{H}^1. \tag{6.4}$$

Proof. For any t

$$\mathcal{V}(t) = \int_{E_t} e^{\psi}\, dz = \int_E e^{\psi(\Phi)} J\Phi\, dw\, .$$

From (6.1) we immediately get

$$e^{\psi(\Phi(z,t))} = e^{\psi(z)} \Big\{ 1 + tX \cdot D\psi + \frac{t^2}{2} \big((X \cdot D\psi)^2 + X \cdot D(X \cdot D\psi) \big) \Big\} + o(t^2)\, . \tag{6.5}$$

Thus, recalling (6.2), we have

$$\begin{aligned} \mathcal{V}(t) = \int_E e^{\psi} \Big\{ & 1 + t\big(X \cdot D\psi + \mathrm{div} X \big) \\ & + \frac{t^2}{2} \big(2(X \cdot D\psi)\mathrm{div} X + (X \cdot D\psi)^2 + X \cdot D(X \cdot D\psi) + \mathrm{div}((\mathrm{div} X)X \big) \Big\}\, dx + o(t^2)\, . \end{aligned}$$

From this formula, using the divergence theorem, we obtain

$$\mathcal{V}'(0) = \int_E \big(e^\psi (X \cdot D\psi) + e^\psi \mathrm{div} X\big)\, dz = \int_E \mathrm{div}(e^\psi X)\, dz = \int_{\partial E} e^\psi X \cdot \nu_E \, d\mathcal{H}^1 ,$$

$$\begin{aligned}\mathcal{V}''(0) &= \int_E e^\psi \Big((X\cdot D\psi)^2 + X \cdot D(X\cdot D\psi) + 2(X\cdot D\psi)\mathrm{div}X + \mathrm{div}((\mathrm{div}X)X)\Big)\, dz \\ &= \int_E \Big(\mathrm{div}(e^\psi (X\cdot D\psi)X) + \mathrm{div}(e^\psi(\mathrm{div}X)X)\Big)\, dz = \int_{\partial E} e^\psi X\cdot \nu_E (X\cdot D\psi + \mathrm{div}X)\, d\mathcal{H}^1 \\ &= \int_{\partial E} \mathrm{div}(e^\psi X)\, X\cdot\nu_E \, d\mathcal{H}^1.\end{aligned}$$

Thus we have proved (6.3) and (6.4) for $t = 0$. To get the corresponding formulae for any t it is enough to recall the semigroup property of Φ, that is

$$\Phi(\Phi(z,t), s) = \Phi(z, t+s),$$

for any s. □

Now we establish the first and second variation formulae for the weighted perimeter. To this end, given a set E and a smooth vector field X as above, we recall that the tangential divergence of X along ∂E is defined as

$$\mathrm{div}_\tau X = \mathrm{div} X - \sum_{i,j=1}^{2} D_j X_i \, \nu_i \nu_j,$$

where $\nu = (\nu_1, \nu_2)$ is the exterior unit normal to ∂E. We also recall that the tangential Jacobian of $J_\tau \Phi$ on ∂E can be computed as follows, see [39, p.63],

$$\begin{aligned} J_\tau \Phi &= 1 + t\, \mathrm{div}_\tau X + \frac{t^2}{2}\Big(\mathrm{div}_\tau Z + (\mathrm{div}_\tau X)^2 + |(D_\tau X)\cdot \nu|^2 - |(D_\tau X)\cdot\tau|^2\Big) + o(t^2) \\ &= 1 + t\, \mathrm{div}_\tau X + \frac{t^2}{2}\big(\mathrm{div}_\tau Z + |(D_\tau X)\cdot\nu|^2\big) + o(t^2)\ . \end{aligned} \tag{6.6}$$

Set

$$P(t) = \int_{\partial E_t} e^\psi \, d\mathcal{H}^1 .$$

Clearly $P(t) = \mathcal{P}_e(E_t; \mathbb{T}^2_T)$ if $\psi(x,y) = x$.

Proposition 6.2. *If X, E and ψ are as above, then*

$$\begin{aligned} P'(t) &= \int_{\partial E_t} e^\psi \kappa_\psi \, X\cdot \nu \, d\mathcal{H}^1 , \\ P''(t) &= \int_{\partial E_t} e^\psi \big((X\cdot D\psi)^2 + X\cdot D(X\cdot D\psi) + 2(X\cdot D\psi)\mathrm{div}_\tau X + \mathrm{div}_\tau Z + |(D_\tau X)\cdot \nu_{E_t}|^2\big)\, d\mathcal{H}^1 \end{aligned}$$

where $\kappa_\psi := \kappa + D\psi\cdot\nu$ and κ is the curvature of E_t. In particular when $\psi(x,y)=x$ we have

$$P'(t) = \int_{\partial E_t} e^x(\kappa+\nu_1)\, X\cdot\nu\, d\mathcal{H}^1\ , \tag{6.7}$$

$$P''(t) = \int_{\partial E_t} e^x\big(X_1^2 + DX_1\cdot X + 2X_1\mathrm{div}_\tau X + \mathrm{div}_\tau Z + |(D_\tau X)\cdot\nu_{E_t}|^2\big)\, d\mathcal{H}^1\ . \tag{6.8}$$

Proof. By a change of variable, we have

$$P(t) = \int_{\partial E_t} e^\psi d\mathcal{H}^1 = \int_{\partial E} e^{\psi(\Phi)} J_\tau\Phi\, d\mathcal{H}^1\,. \tag{6.9}$$

Then one may argue as in the proof of Proposition 6.1 using (6.6) instead of (6.2) to obtain

$$P'(0) = \int_{\partial E} e^\psi(X\cdot D\psi + \mathrm{div}_\tau X)\, d\mathcal{H}^1\ .$$

Since

$$\mathrm{div}_\tau(e^\psi X) = e^\psi\mathrm{div}_\tau X + e^\psi\, X\cdot\nabla_\tau\psi\ ,$$

it follows that

$$e^\psi(X\cdot D\psi + \mathrm{div}_\tau X) = \mathrm{div}_\tau(e^\psi X) + e^\psi(X\cdot\nu)\,(D\psi\cdot\nu)\ ,$$

resulting in

$$P'(0) = \int_{\partial E_t}\Big(\mathrm{div}_\tau(e^\psi X) + e^\psi(X\cdot\nu)(D\psi\cdot\nu)\Big)\, d\mathcal{H}^1 = \int_{\partial E_t} e^\psi(\kappa + D\psi\cdot\nu)\, X\cdot\nu\, d\mathcal{H}^1\,.$$

The sum $\kappa_\psi := \kappa + D\psi\cdot\nu$ is known as the generalized mean curvature, see [40]. Note that $P'(0)$ depends only on the normal component $X\cdot\nu$.

Similar calculations as in the proof of Proposition 6.1 yield $P''(0)$, while the semigroup property of Φ yields the formulae for $P'(t)$ and $P''(t)$. □

We now calculate the first and the second variation of the nonlocal energy.

Lemma 6.3. *Let X be as above and $E\subset\mathbb{T}^2_T$ a smooth open set with finite volume and perimeter with respect to the weight e^x. Then*

1. *for all $\varphi\in H^1_e(\mathbb{T}^2_T)$*
$$\frac{d}{dt}\int_{E_t} e^x\varphi\, dz = \int_{\partial E_t} e^x\varphi X\cdot\nu_{E_t}\, d\mathcal{H}^1\,. \tag{6.10}$$

2. *Let $v:\mathbb{T}^2_T\times\mathbb{R}\to[0,1/\gamma]$ be the function defined for every $t\in\mathbb{R}$ by setting $v(\cdot,t)=\mathcal{L}_c(\chi_{E_t})$. Then, denoting by v_t the distributional derivative of v with respect to the parameter t, we have that for every $t\in\mathbb{R}$*
$$v_t(z,t) = \int_{\partial E_t} G(z,w)\, X(w)\cdot\nu_{E_t}(w)\, d\mathcal{H}^1_w \qquad \text{for a.e. } \ z\in\mathbb{T}^2_T\ , \tag{6.11}$$
where G is the Green's function on $\mathbb{T}^2_T$ for the operator $\mathcal{L}_c = -c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - c^2\frac{\partial}{\partial x} + \gamma$. Moreover, $v_t(\cdot,t)\in H^1_e(\mathbb{T}^2_T)$ for all t.

Proof. Fix $\varphi \in H^1_e(\mathbb{T}^2_T)$. Arguing as in the proof of (6.3) we have

$$\begin{aligned}\frac{d}{dt}\Big(\int_{E_t} e^x\varphi\,dz\Big)\Big|_{t=0} &= \frac{d}{dt}\Big(\int_E e^{\Phi_1(z,t)}\varphi(\Phi(z,t))J\Phi(z,t)\,dz\Big)\Big|_{t=0}\\ &= \int_E \operatorname{div}(e^x\varphi X)\,dz = \int_{\partial E} e^x\varphi\,X\cdot\nu_E\,d\mathcal{H}^1\,.\end{aligned}$$

This proves (6.10) for $t = 0$. As in the proof of Lemma 6.1 the general case follows from the semigroup property of the flow Φ.

We now recall a result in distribution theory, see [38, Example 5.59, p.148]. Let $u(\cdot,t)$ be a distribution for each value of the parameter $t \in \mathbb{R}$ such that $\frac{\partial}{\partial t}\int_{\mathbb{T}^2_T} u(x,t)\phi(x)\,dx$ exists for all $\phi \in C^\infty_c(\mathbb{T}^2_T)$. Then $\frac{\partial u}{\partial t}$ exists in distribution sense. Indeed $\frac{u(\cdot,t+h)-u(\cdot,t)}{h}$ is a distribution. Since the limit of $\int_{\mathbb{T}^2_T}\frac{u(x,t+h)-u(x,t)}{h}\phi(x)\,dx$ exists as $h \to 0$, by [38, Theorem 5.31] we have that $\frac{u(\cdot,t+h)-u(\cdot,t)}{h}$ converges in the distribution sense to a distribution that we denote by $\frac{\partial u}{\partial t}$.

From (6.10) we have $(\chi_{E_t})_t = (X\cdot\nu_{E_t})\mathcal{H}^1\lfloor\partial E_t := \mu_t$. We claim $\mu_t \in H^{-1}_e(\mathbb{T}^2_T)$. Indeed for any $\varphi \in H^1_e(\mathbb{T}^2_T)$

$$\Big|\int_{\mathbb{T}^2_T} e^x\varphi\,d\mu_t\Big| = \Big|\int_{\partial E_t} e^x\varphi\,X\cdot\nu_{E_t}\,d\mathcal{H}^1\Big| = \Big|\int_{E_t}\operatorname{div}(e^x\varphi X)\,dz\Big| \le \int_{\mathbb{T}^2_T}|\operatorname{div}(e^x\varphi X)|\,dz \le C\|\varphi\|_{H^1_e(\mathbb{T}^2_T)}$$

for some constant C depending only on $\|X\|_{H^1_e(\mathbb{T}^2_T)}$.

Let $v(\cdot,t) = \mathcal{L}_c\chi_{E_t}$. Now

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{T}^2_T} e^x v\phi\,dz &= \frac{\partial}{\partial t}\int_{\mathbb{T}^2_T} e^x\phi\,\mathcal{L}_c\chi_{E_t}\,dz = \frac{\partial}{\partial t}\int_{\mathbb{T}^2_T} e^x\chi_{E_t}\,\mathcal{L}_c\phi\,dz\\ &= \int_{\mathbb{T}^2_T} e^x(\chi_{E_t})_t\,\mathcal{L}_c\phi\,dz = \int_{\mathbb{T}^2_T} e^x\phi\,\mathcal{L}_c\mu_t\,dz\end{aligned}$$

so that $v_t = \mathcal{L}_c\,\mu_t$. This proves (6.11). Then standard regularity estimates imply $v_t \in H^1_e(\mathbb{T}^2_T)$. □

Set for all t

$$N(t) = \frac{1}{2}\int_{E_t} e^x\mathcal{L}_c\chi_{E_t}\,dz = \frac{1}{2}\int_{\mathbb{T}^2_T} e^x\left(c^2v_x^2(z,t) + v_y^2(z,t) + \gamma v^2(z,t)\right)\,dz\,. \tag{6.12}$$

Proposition 6.4. *Let X and E be as in Lemma 6.3. Then we have*

$$N'(t) = \int_{\partial E_t} e^x v(z,t)\,X(z)\cdot\nu_{E_t}(z)\,d\mathcal{H}^1 = \int_{E_t} dw\int_{\partial E_t} e^x G(z,w)X(z)\cdot\nu_{E_t}(z)\,d\mathcal{H}^1_z\,, \tag{6.13}$$

$$\begin{aligned}N''(t) = &\int_{\partial E_t} d\mathcal{H}^1_w\int_{\partial E_t} e^x G(z,w)X(z)\cdot\nu_{E_t}(z)X(w)\cdot\nu_{E_t}(w)d\mathcal{H}^1_z\\ &+\int_{\partial E_t}\operatorname{div}(e^x v(z,t)X(z))X(z)\cdot\nu_{E_t}(z)d\mathcal{H}^1_z.\end{aligned} \tag{6.14}$$

Remark 6.5. *Observe that the first summand in* (6.14) *can be written as*

$$\int_{\partial E_t} d\mathcal{H}^1_w \int_{\partial E_t} e^x G(z,w) X(z)\cdot\nu_{E_t}(z) X(w)\cdot\nu_{E_t}(w) d\mathcal{H}^1_z = \int_{\mathbb{T}^2_T} e^x \mathcal{L}_c(\mu_t)\, d\mu_t,$$

where μ_t is the singular measure $(X\cdot\nu_{E_t})\mathcal{H}^1\lfloor\partial E_t$.

Proof of Proposition 6.4. We differentiate (6.12) with respect to t and use the fact that by (6.11) $v_t = \mathcal{L}_c(\mu_t)$ to get

$$N'(t) = \int_{\mathbb{T}^2_T} e^x(c^2 v_{tx}v_x + v_{ty}v_y + \gamma v_t v)\, dz = \int_{\mathbb{T}^2_T} e^x v\, d\mu_t = \int_{\partial E_t} e^x v(z,t)\, X(z)\cdot\nu_{E_t}(z)\, d\mathcal{H}^1 . \quad (6.15)$$

Expressing v by means of the Green's function G gives (6.13).

Observe that the same argument used in the proof of (6.1) yields the following general formula

$$\frac{d}{dt}\int_{E_t} f(x,t)\, dz = \int_{E_t} \frac{\partial f}{\partial t} dz + \int_{\partial E_t} f X\cdot\nu\, d\mathcal{H}^1 .$$

Writing the last integral in (6.15) as a volume integral and using the above formula we have

$$\begin{aligned} N''(t) &= \frac{d}{dt}\int_{E_t} \operatorname{div}(e^x v\, X)\, dz \\ &= \int_{E_t} \operatorname{div}(e^x \frac{\partial v}{\partial t} X)\, dz + \int_{\partial E_t} \operatorname{div}(e^x v X)\ (X\cdot\nu)\, d\mathcal{H}^1 \\ &= \int_{\partial E_t} e^x v_t(z,t)\, X(z)\cdot\nu_{E_t}(z)\, d\mathcal{H}^1_z + \int_{\partial E_t} \operatorname{div}(e^x v X)\ (X\cdot\nu)\, d\mathcal{H}^1 = I + II . \end{aligned}$$

On substituting v_t in I by (6.11) we obtain (6.14). □

To derive the Euler-Lagranage equation that governs a critical point E of $\mathcal{K}_c$ or $\mathcal{J}_c$, we need to calculate how the perimeter of the stretched set $(E_t)_c$ evolves under a given velocity field. We achieve this by computing $(E_c)_t$ under a modified velocity field X_c such that $(E_t)_c = (E_c)_t$. The coordinate transformations $\tilde{x} = x$ and $\tilde{y} = cy$ map a point $(x,y)\in E$ to $(\tilde{x},\tilde{y})\in E_c$. In a different notation the flow (1.21) is written as $dx/dt = X_1(x,y)$ and $dy/dt = X_2(x,y)$. Evolution of E_c is then governed by

$$\frac{d\tilde{x}}{dt} = X_1(\tilde{x},\tilde{y}/c)\ , \qquad \frac{d\tilde{y}}{dt} = cX_2(\tilde{x},\tilde{y}/c)$$

with the initial condition $(\tilde{x}(0),\tilde{y}(0)) = (x(0), cy(0))$. The modified velocity field is therefore given by

$$X_c(\tilde{x},\tilde{y}) = (X_1(\tilde{x},\tilde{y}/c), cX_2(\tilde{x},\tilde{y}/c)) . \quad (6.16)$$

Assume E is of class C^2. If ν is the exterior unit normal to E we denote by τ the tangent vector obtained by rotating ν counter-clockwise by $\pi/2$ and by θ the signed angle made by τ with the positive x-axis. Thus, $\tau = (\cos\theta, \sin\theta)$ and $\nu = (\sin\theta, -\cos\theta)$. The signed curvature of ∂E

is defined as $\kappa = \mathrm{div}_\tau \nu$. Note that if s is an arc length along ∂E inducing the same orientation of τ, then $\kappa = d\theta/ds$. Observe also that if ν_c is the exterior unit normal to E_c, τ_c is the tangent vector oriented as above and θ_c is the corresponding signed angle, then $\tan\theta_c = c\tan\theta$, $\tau_c \parallel (\cos\theta, c\sin\theta)$ and $\nu_c \parallel (c\sin\theta, -\cos\theta)$. We can therefore infer that

$$X_c \cdot \nu_c\Big|_{(\tilde{x},\tilde{y})} = \frac{c}{\sqrt{c^2\sin^2\theta + \cos^2\theta}}\, X\cdot\nu\Big|_{(x,y)} . \tag{6.17}$$

Let $\mathcal{K}_c$ be the functional defined as in (3.1). From (6.7) and (6.13) we get

$$\begin{aligned}\partial\mathcal{K}_c(E)[X] &:= \frac{\sqrt{2}}{12c}\frac{d}{dt}\mathcal{P}_e((E_t)_c;\mathbb{T}^2_{cT})\Big|_{t=0} + \sigma N'(0) \\ &= \frac{\sqrt{2}}{12c}\int_{\partial E_c} e^x(\kappa_c + \nu_{c1})\, X_c\cdot\nu_c\, d\mathcal{H}^1_c + \sigma\int_{\partial E} e^x\,\mathcal{L}_c\chi_E\, X\cdot\nu\, d\mathcal{H}^1 ,\end{aligned} \tag{6.18}$$

where $\nu_{c1} := \frac{c\sin\theta}{\sqrt{c^2\sin^2\theta+\cos^2\theta}}$ and, for the sake of clarity, we have denoted by $\mathcal{H}^1_c$ the one-dimensional Hausdorff measure in the $(x,\tilde{y})$-plane. Note that under the change of variable $(\tilde{x},\tilde{y}) = (x, cy)$ one has, see (2.8), $d\mathcal{H}^1_c = \sqrt{\cos^2\theta + c^2\sin^2\theta}\, d\mathcal{H}^1$. Let s_c and s are arc lengths along ∂E_c and ∂E, respectively. With the above orientation choice, we have the signed curvature $\kappa_c = d\theta_c/ds_c = \frac{d\theta_c}{d\theta}\frac{d\theta}{ds}\frac{ds}{ds_c} = \kappa\frac{d\theta_c}{d\theta}\frac{ds}{ds_c}$; a direct calculation gives

$$\begin{aligned}\partial\mathcal{K}_c(E)[X] = {} & \frac{\sqrt{2}}{12c}\int_{\partial E} e^x \frac{c\kappa}{\sqrt{c^2\sin^2\theta+\cos^2\theta}}\frac{d\theta_c}{d\theta}\, X\cdot\nu\, d\mathcal{H}^1 \\ & + \frac{\sqrt{2}}{12c}\int_{\partial E} e^x \frac{c^2\sin\theta\,(X\cdot\nu)}{\sqrt{\cos^2\theta + c^2\sin^2\theta}} d\mathcal{H}^1 + \sigma\int_{\partial E} e^x\,\mathcal{L}_c\chi_E\, X\cdot\nu\, d\mathcal{H}^1 \\ = {} & \int_{\partial E} e^x\left(\frac{\sqrt{2}}{12}\,\frac{c\kappa}{(c^2\sin^2\theta+\cos^2\theta)^{3/2}} + \frac{\sqrt{2}}{12}\frac{c\sin\theta}{\sqrt{\cos^2\theta+c^2\sin^2\theta}} + \sigma\,\mathcal{L}_c\chi_E\right) X\cdot\nu\, d\mathcal{H}^1 ,\end{aligned} \tag{6.19}$$

where the last equality follows from $\tan\theta_c = c\tan\theta$, which in turn gives $\frac{d\theta_c}{d\theta} = \frac{c}{c^2\sin^2\theta+\cos^2\theta}$. Observe that $\partial\mathcal{K}_c(E)[X]$ depends only on $X\cdot\nu$, but not on the tangential component of X.

Let $\mathcal{E}(c)$ be a critical point of $\mathcal{J}_c(E)$ under the constraint $|E|_e = 1$. Recall that if $\mathcal{E}(c)$ is a minimizer, we have denoted it by $\mathbb{E}(c)$.

Lemma 6.6. *Let $\mathcal{E}(c)$ be a critical point of $\mathcal{J}_c$ under the constraint $|E|_e = 1$, for some $c > 0$. If, in addition, $\mathcal{J}_c(\mathcal{E}(c)) = 0$, then $\mathcal{E}(c)$ is also an unconstrained critical point of $\mathcal{J}_c$.*

Moreover, if E is any unconstrained critical point of $\mathcal{J}_c$ of class C^2, then it satisfies the following Euler-Lagrange equation

$$\frac{\sqrt{2}}{12}(\kappa_c + \nu_{c1}) + \sigma V - \frac{\sqrt{2}}{12}\alpha = 0 \quad \text{on } \partial E_c . \tag{6.20}$$

where κ_c and ν_c are defined as above and V is the unique solution in $H^1_e(\mathbb{T}^2_{cT})$of the equation

$$-c^2V_{xx} - c^2V_{yy} - c^2V_x + \gamma V = \chi_{E_c} \quad \text{on } \mathbb{T}^2_{cT} . \tag{6.21}$$

Finally, equation (6.20) *can be equivalently rewritten on ∂E as* (1.20).

Proof. Let $\mathcal{E}(c)$ be a critical point of $\mathcal{J}_c$ under the volume constraint $|E|_e = 1$, such that $\mathcal{J}_c(\mathcal{E}(c)) = 0$ and let Λ be the Lagrange multiplier associated with the constraint. Set $E_t = \mathcal{E}(c) + te_1$, and

$$L(t) := \mathcal{J}_c(E_t) - \Lambda \int_{E_t} e^x\, dz = e^t\Big(\mathcal{J}_c(\mathcal{E}(c)) - \Lambda \int_{\mathcal{E}(c)} e^x\, dz\Big) = -\Lambda e^t\,.$$

From the criticality assumption on $\mathcal{E}(c)$ we then get $L'(0) = 0$, hence $\Lambda = 0$. This proves that $\mathcal{E}(c)$ is also an unconstrained critical point of $\mathcal{J}_c$.

Assume now that E is an unconstrained critical point of $\mathcal{J}_c$ of class C^2. From (6.18) and (6.3) we have for any smooth vector field X with compact support in $\mathbb{T}^2_T$

$$\begin{aligned}\partial\mathcal{J}_c(E)[X] = {} & \frac{\sqrt{2}}{12c}\int_{\partial E_c} e^x(\kappa_c{+}\nu_{c_1})\, X_c\cdot\nu_c\, d\mathcal{H}^1_c \\ & + \sigma\int_{\partial E} e^x\,\mathcal{L}_c\chi_E\, X\cdot\nu\, d\mathcal{H}^1 - \frac{\sqrt{2}}{12}\alpha\int_{\partial E} e^x X\cdot\nu\, d\mathcal{H}^1 \\ = {} & \frac{1}{c}\int_{\partial E_c} e^x\left\{\frac{\sqrt{2}}{12}(\kappa_c+\nu_{c_1}) + \sigma V - \frac{\sqrt{2}}{12}\alpha\right\} X_c\cdot\nu_c\, d\mathcal{H}^1_c\,,\end{aligned} \tag{6.22}$$

where the last equality follows by a change of variables in the integrals on ∂E, recalling (6.17) and that $d\mathcal{H}^1_c = \sqrt{\cos^2\theta + c^2\sin^2\theta}\, d\mathcal{H}^1$ and observing that $V(x,y) = \mathcal{L}_c\chi_E(x, y/c)$ so that V is the solution of equation (6.21). The above formula immediately yields (6.20) by the arbitrariness of X. Finally the equivalence between (6.20) and (1.20) follows immediately from (6.22), rewriting the integral on the first line on ∂E as we did in (6.19). □

Note that the above Lemma, together with Theorem 3.3, Lemma 3.11 and Remark 9.3 completes the proof of Theorem 1.1. At the same time Theorem 1.3 follows at once from (6.4), (6.8) applied to ∂E_c and (6.14).

7 Stability analysis of a planar traveling front

In this section we discuss the local stability of a volume constrained critical point $\mathcal{E}(c)$ of $\mathcal{J}_c$. To this aim we study its second variation $\partial^2\mathcal{J}_c(\mathcal{E}(c))[X]$ where X is a smooth velocity field with compact support such that the associated flow Φ is volume preserving. By setting $\mathcal{V}' = 0$ and expressing the right side on (6.3) as a volume integral, this amounts to requiring that

$$\operatorname{div}(e^x X) = X_1 + \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} = 0 \qquad \text{on } \mathbb{T}^2_T\,. \tag{7.1}$$

Thus in the following we say that $\mathcal{E}(c)$ is *locally stable* if

$$\partial^2\mathcal{J}_c(\mathcal{E}(c))[X] > 0$$

for all smooth vector fields with compact support, not identically zero, satisfying (7.1). If instead $\partial^2\mathcal{J}_c(\mathcal{E}(c))[X] < 0$ for some vector field X we say that $\mathcal{E}(c)$ is *unstable*. Note that the weighted volume of a set increases strictly when translating it to the right along the x-direction. Therefore condition (7.1) eliminates a pure translation mode. The actual computation of the second variation requires a detailed knowledge of the shape of $\mathcal{E}(c)$. We therefore study the simplest case: a planar traveling front.

Suppose $\mathcal{E}(c) = W = \mathbb{T}^2_T \cap \{x < a\}$ is a planar front traveling in the x-direction and that W is an unconstrained critical point of $\mathcal{J}_c$. Since the curvature κ is identically zero and the angle θ in (1.20) is constantly equal to $\pi/2$, the the Euler-Lagrange equation becomes

$$\frac{\sqrt{2}}{12}(1-\alpha) + \sigma \mathcal{L}_c \chi_W = 0 \quad \text{at } \partial W\ . \tag{7.2}$$

The function $\mathcal{L}_c\chi_W$ depends only on the variable x and it coincides with the unique solution $v \in H^1_e(\mathbb{R})$ of the ODE (7.7) below, whose explicit expression is given in (7.8). An elementary calculations shows that $\mathcal{L}_c\chi_W\big|_{\partial W} = \frac{1}{2\gamma}(1-H(c))$, where $H(c) := \frac{c}{\sqrt{c^2+4\gamma}}$, a quantity independent of a. Thus we have

$$\frac{\sqrt{2}}{12}(1-\alpha) + \frac{\sigma}{2\gamma}(1-H(c)) = 0\ . \tag{7.3}$$

Note that this equation turns out to be the same as $\mathcal{J}_c(\mathcal{E}(c)) = 0$. Hence the existence of a positive root c for (7.3) is equivalent to saying that W is an unconstrained critical point of $\mathcal{J}_c$, hence a planar traveling wave. The proof in [12, Lemma 6.2] indicates that condition (TW1) in (1.17) is both necessary and sufficient for the existence of such positive root whose unique value c_f is given in (1.23).

We now study the local stability of this planar front, which in principle may depend also on the other parameters. Imposing the volume constraint $|W|_e = 1$, this uniquely yields $W = \mathbb{T}^2_T \cap \{x < \log\frac{1}{T}\}$. Note that both ∂W and ∂W_c lie on the line $x = \log\frac{1}{T}$.

Let $X = (X_1, X_2) : \mathbb{T}^2_T \mapsto \mathbb{T}^2_T$ be a smooth vector field with compact support satisfying (7.1). Thus the weighted volume W_t stays constant under the corresponding flow. We now calculate the second variation $\partial^2\mathcal{J}_c(W)[X]$ using Theorem 1.3. There are altogether 4 terms on the right in (1.22), which we label as I to IV. The first term I is the perimeter term, II and III come from the nonlocal term and the volumetric term IV is zero, due to (7.1). Recall that $X_c(x,y) = (X_1(x, y/c), cX_2(x, y/c))$ for $(x,y) \in \mathbb{T}^2_{cT}$, see (6.16). It is easy to check that also X_c satisfies (7.1); hence the modified velocity field X_c also preserves the weighted volume $|E_c|_e$.

Let us first examine the first term. A direct computation, using periodicity, yields

$$\begin{aligned}
I =& \frac{\sqrt{2}}{12\,c}\int_{-cT/2}^{cT/2} e^x \left\{ \left(X_{c1}^2 + DX_{c1}\cdot X_c + 2X_{c1}\,\mathrm{div}_{\tau_c} X_c\right) + \frac{\partial Z_{c2}}{\partial y} + \left(\frac{\partial X_{c1}}{\partial y}\right)^2 \right\} dy \\
=& \frac{\sqrt{2}}{12\,cT}\int_{-cT/2}^{cT/2} \left\{ (X_{c1}^2 + X_{c1}\frac{\partial X_{c1}}{\partial x} + X_{c2}\frac{\partial X_{c1}}{\partial y} + 2X_{c1}\frac{\partial X_{c2}}{\partial y}) + \frac{1}{c^2}\left(\frac{\partial X_1}{\partial y}\right)^2\Big|_{(x,y/c)} \right\} dy \\
=& \frac{\sqrt{2}}{12\,cT}\left\{ \int_{-cT/2}^{cT/2} \left(X_{c1}(X_{c1} + \frac{\partial X_{c1}}{\partial x} + \frac{\partial X_{c2}}{\partial y}) + \frac{\partial}{\partial y}(X_{c1}X_{c2}) \right) dy + \frac{1}{c}\int_{-T/2}^{T/2}\left(\frac{\partial X_1}{\partial y}\right)^2 dy \right\} \\
=& \frac{\sqrt{2}}{12\,c^2 T}\int_{-T/2}^{T/2}\left(\frac{\partial X_1}{\partial y}\right)^2 dy
\end{aligned}$$

which depends only on the value of X_1 along $x = \log\frac{1}{T}$. Set $\varphi(y) := X_1(\log 1/T, y)$ for some smooth T-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$, and decompose $\varphi(y)$ using the Fourier modes $\{\cos(2j\pi y/T) : j = 0, 1, 2, \dots\} \cup \{\sin(2j\pi y/T) :\ j = 1, 2, \dots\}$. Since X is a volume preserving vector field we have $\int_{-T/2}^{T/2} \varphi(y)dy = 0.$[1] Hence $\varphi(y) = \sum_{j=1}^{\infty}(a_j \cos(2j\pi y/T) + b_j \sin(2j\pi y/T))$ for some constants a_j, b_j. Thus

$$I = \frac{\sqrt{2}}{12\,c^2 T}\int_{-T/2}^{T/2} \varphi'^2\,dy = \frac{\sqrt{2}}{12\,c^2 T}\sum_{j=1}^{\infty}\frac{2j^2\pi^2}{T}(a_j^2 + b_j^2)\,. \tag{7.4}$$

Next we compute the nonlocal term II. To this aim, see Remark 6.5, we consider the measure $\mu = (X\cdot\nu)\,\mathcal{H}^1\lfloor\partial W$ which turns out to be the product measure $\delta_a \times (\varphi(y)dy)$, where δ_a is the 1D delta distribution at $a := \log\frac{1}{T}$. To compute $\hat{v} = \mathcal{L}_c(\mu)$ by separation of variable note that

$$\hat{v}(x,y) = \sum_{j=1}^{\infty}\hat{v}_j(x)(a_j\cos\frac{2j\pi y}{T} + b_j\sin\frac{2j\pi y}{T})\,.$$

where $\hat{v}_j : \mathbb{R}\to\mathbb{R}$ is the unique solution of the ODE

$$-c^2(\hat{v}_j)_{xx} - c^2(\hat{v}_j)_x + (\gamma + \frac{4j^2\pi^2}{T^2})\,\hat{v}_j = \delta_a\,. \tag{7.5}$$

Consequently

$$II = \sigma\int_{\mathbb{T}_T^2} e^x\mathcal{L}_c(\mu)\,d\mu = \frac{\sigma}{T}\int_{-T/2}^{T/2}\hat{v}(a,y)\varphi(y)\,dy = \frac{\sigma}{2}\sum_{j=1}^{\infty}\hat{v}_j(a)(a_j^2 + b_j^2)\,. \tag{7.6}$$

To calculate III, observe that $v = \mathcal{L}_c\chi_W$ solves the ODE

$$-c^2 v_{xx} - c^2 v_x + \gamma v = \chi_{(-\infty,a)}. \tag{7.7}$$

Hence, recalling (7.1) and using the fact that v is a function of the variable x only,

$$\begin{aligned} III &= \sigma\int_{\partial W}\operatorname{div}(e^x vX)\,X_1\,d\mathcal{H}^1 = \sigma\int_{\partial W}e^x(Dv\cdot X)\,X_1\,d\mathcal{H}^1 \\ &= \frac{\sigma}{T}\int_{\partial W}\frac{\partial v}{\partial x}\,X_1^2\,d\mathcal{H}^1 = \frac{\sigma v'(a)}{T}\int_{-T/2}^{T/2}\varphi^2\,dy = \frac{\sigma v'(a)}{2}\sum_{j=1}^{\infty}(a_j^2 + b_j^2)\,, \end{aligned}$$

where, we recall, we have set $a = \log 1/T$. Thus, from the above expression of III, (7.6) and (7.4)

$$\partial^2\mathcal{J}_c(W)[X] = I + II + III = \frac{1}{2}\sum_{j=1}^{\infty}\left\{\frac{\sqrt{2}}{12c^2}\frac{4j^2\pi^2}{T^2} + \sigma(\hat{v}_j(a) + v'(a))\right\}(a_j^2 + b_j^2)\,.$$

[1] Note that if $\varphi : \mathbb{R}\to\mathbb{R}$ is a smooth T-periodic function with zero average on the interval $(-T/2, T/2)$, then the vector field $X(x,y) = \big(\psi(x)\varphi(y), -(\psi(x) + \psi'(x))\int_0^y\varphi(t)dt\big)$, where $\psi : \mathbb{R}\to\mathbb{R}$ is a smooth function with compact support such that $\psi(\log 1/T) = 1$, clearly satisfies (7.1) and the additional condition $X_1(\log 1/T, y) := \varphi(y)$.

We need to evaluate $\hat{v}_j(a)$ and $v'(a)$. The solution of (7.5) is

$$\hat{v}_j(x) = \begin{cases} A_j e^{\beta_{j,+}(x-a)} & \text{if } x \le a \;, \\ A_j e^{\beta_{j,-}(x-a)} & \text{if } x \ge a \;, \end{cases}$$

where

$$\beta_{j,\pm} = \frac{1}{2}\Big(-1 \pm \sqrt{1 + \frac{4}{c^2}\Big(\gamma + \frac{4j^2\pi^2}{T^2}\Big)} \Big)$$

and

$$A_j = \frac{1}{c^2\sqrt{1 + \frac{4}{c^2}\Big(\gamma + \frac{4j^2\pi^2}{T^2}\Big)}} \;.$$

In particular $\hat{v}_j(a) = A_j$. At the same time

$$v(x) = \begin{cases} \frac{1}{\gamma} + A_- e^{\alpha_+(x-a)} & \text{if } x \le a \;, \\ A_+ e^{\alpha_-(x-a)} & \text{if } x \ge a \;, \end{cases} \tag{7.8}$$

where

$$\alpha_{\pm} = \frac{1}{2}\Big(-1 \pm \sqrt{1 + \frac{4\gamma}{c^2}} \Big)$$

and

$$A_+ = \frac{\alpha_+}{\gamma(\alpha_+ - \alpha_-)} \;, \qquad A_- = \frac{\alpha_-}{\gamma(\alpha_+ - \alpha_-)} \;.$$

Thus

$$v'(a) = A_+\alpha_- = -\frac{1}{c\sqrt{c^2 + 4\gamma}} \;.$$

To simplify the notation we set $B := \sqrt{1 + \frac{4\gamma}{c^2}}$ and $B_j := \sqrt{1 + \frac{4}{c^2}\Big(\gamma + \frac{4j^2\pi^2}{T^2}\Big)}$. Therefore

$$\begin{aligned} \partial^2 \mathcal{J}_c(W)[X] &= \sum_{j=1}^{\infty} \frac{1}{2c^2}\left(\frac{\sqrt{2}}{12}\,\frac{4j^2\pi^2}{T^2} + \sigma\Big(\frac{1}{B_j} - \frac{1}{B}\Big) \right) (a_j^2 + b_j^2) \\ &= \sum_{j=1}^{\infty} \frac{2j^2\pi^2}{c^2T^2}\left(\frac{\sqrt{2}}{12} - \frac{4\sigma}{c^2}\,\frac{1}{BB_j(B+B_j)} \right) (a_j^2 + b_j^2) \\ &:= \sum_{j=1}^{\infty} \frac{2j^2\pi^2}{c^2T^2} g(j,c,T)\,(a_j^2 + b_j^2) \;. \end{aligned}$$

A positive sign for g for all weighted volume preserving X gives local stability, while a negative sign for some X indicates instability. Note that $\frac{\partial g}{\partial j} > 0$ and $\frac{\partial g}{\partial T} < 0$. Thus $j = 1$ and large T is the most unstable scenario. Therefore in order to carry on our stability analysis it suffices to examine the sign of g for the mode $j = 1$ and for $c = c_f$, which is the unique solution of (7.3).

Recall that the planar front W is a travelling wave solution with unique speed (1.23) if and only if condition (TW1) in (1.17) holds. In turn, assuming the validity of this condition, there are essentially two different scenarios depending on which of the two conditions (A1)* and (A2) applies. As we recalled in Section 1, under condition (A1)* the front is a global minimizer among all 1D configuration, therefore it is locally stable when subject to a 1D perturbation. On the other hand, under condition (A2) the global minimizer among all 1D configurations is a 1D pulse. However we would like to understand in both cases whether or not the front is locally stable with respect to 2D perturbations.

Suppose condition (TW1) holds, so that $0 < h_* < 1$, where h_* is as in (1.23). Setting $j = 1$,

$$\begin{aligned}\lim_{T\to\infty} g(1, c_f, T) &= \frac{\sqrt{2}}{12} - \frac{2\sigma}{c_f^2 B^3} = \frac{\sqrt{2}}{12} - \frac{2\sigma c_f}{(c_f^2 + 4\gamma)^{3/2}} \\ &= \frac{\sqrt{2}}{12} - \frac{\sigma h_*(1-h_*^2)}{2\gamma} = \frac{\sqrt{2}}{12}\Big(1 - (\alpha-1)h_*(1+h_*)\Big) .\end{aligned}$$

Therefore a necessary and sufficient condition for W to be stable for all $T > 0$ is

$$Q(\alpha, h_*) := 1 - (\alpha - 1)h_*(1 + h_*) \geq 0 ,$$

because the function $T \to g(1, c_f, T)$ is strictly decreasing. Note that the graph of $Q(\alpha, \cdot)$ is a (concave) parabola with a positive root h_+ and a negative root h_- with

$$h_+ = \frac{1}{2}\left(\sqrt{1 + \frac{4}{\alpha - 1}} - 1\right) .$$

It is readily seen that $h_+ > 1/\alpha$. Moreover $h_+ \leq 1$ iff $\alpha \geq 3/2$. Stability of the front amounts to $0 < h_* \leq h_+$. We are now in position to give the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume first that condition (A1)* holds. By (1.23) we have $\frac{3\sqrt{2}\sigma}{\gamma} = \frac{\alpha-1}{1-h_*}$. Thus condition (A1)* is equivalent to saying that $\alpha \geq \frac{\alpha-1}{1-h_*} > \alpha - 1 > 0$. In turn this implies that $h_* \leq 1/\alpha < h_+$ so that $Q(\alpha, h_*) > 0$. This proves Statement 1.

We next assume that condition (A2) holds. Note that (A2) is equivalent to $\frac{\alpha-1}{1-h_*} > \alpha > 1$. If $1 < \alpha \leq 3/2$, then $h_+ \geq 1 > h_*$. Hence $Q(\alpha, h_*) > 0$ and Statement 2(a) holds.

Now assume $\alpha > 3/2$ so that $1 > h_+$. Suppose the more stringent condition (A2a) holds. Then $h_* > h_+$, which leads to $Q(\alpha, h_*) < 0$. This implies that W is unstable for sufficiently large T. On the other hand as $T \to 0$, it is immediate that $B_1 \to \infty$ so that $g(1, c_f, T) \to \frac{\sqrt{2}}{12} > 0$. As a result the front W is stable in such cases. Statement 2(b) is now clear because $\frac{\partial g}{\partial T} < 0$.

Finally Statement 2(c) results from the fact that in this case $h_* \leq h_+$. □

Remark 7.1. *Theorem 1.4 gives a precise description of all cases in which W is a a strictly stable critical point for $\mathcal{J}_{c_f}$, depending on the the appropriate parameter constraints. Although we will not do it here, with the techniques introduced in [3] one could prove that all these case W is also a local minimizer with respect to all variations E satisfying the weighted volume constraint $|E|_e = 1$ such that $|E \Delta W|_e$ is sufficiently small.*

8 Non-planar wave

In this section we show the existence of non-planar traveling waves. Throughout the section we assume Condition (A2); this is the necessary and sufficient condition for the existence and uniqueness of a planar traveling pulse P, obtained as a global unconstrained minimizer of $\mathcal{J}_c$ among all 1D profiles. The analysis yields $P = \mathbb{T}^2_T \cap \{a < x < b\}$ with length $\ell_c := b - a$. Both the speed c_p and the length ℓ_c are uniquely determined (see [12, (7.3), (7.4) and Remark 7.9]). A key step in proving the existence of a non-planar traveling wave is to show that $\inf \mathcal{J}_c < 0$ when $c = c_p$. Such requirement imposes further conditions on α and σ, while γ is fixed.

We write Condition (A2) as $\sigma_1 > \alpha > 1$ where $\sigma_1 := \frac{3\sqrt{2}\sigma}{\gamma}$. Let $\sigma_1 = A\alpha$ for a fixed $A > 1$. Then [12, (7.3) and (7.4)] can be cast as

$$A(1 + H(c))\,(1 - e^{-r_2\ell_c}) = 1 + 1/\alpha\ , \tag{8.1}$$

$$A(1 - H(c))\,(1 - e^{r_1\ell_c}) = 1 - 1/\alpha\ , \tag{8.2}$$

where $r_1 < 0 < r_2$ are given by $\frac{1}{2c}(-c \pm \sqrt{c^2 + 4\gamma})$ and $H(c) := \frac{c}{\sqrt{c^2+4\gamma}}$. The next lemma gives the precise rate of convergence of c_p to 0 when $\alpha \to \infty$. To simplify the notation we set $c = c_p$ for the rest of this section.

Lemma 8.1. *Let $\alpha \geq 2$, $\sigma_1 = A\alpha$ for a fixed $A > 1$ and let $B := \big(1 + (A-1)\log\frac{A-1}{A}\big)^{-1}$. Then (i) B is a strictly increasing function of A such that $B \to 1$ as $A \to 1^+$, and $B = 2A + O(1)$ as $A \to \infty$;*
(ii) $\alpha c_p = B\sqrt{4\gamma} + O(1/\alpha)$ as $\alpha \to \infty$.

Proof. Define $g(x) := 1 + (x-1)\log\frac{x-1}{x}$ for $x > 1$. Then $g'(x) = \frac{1}{x} + \log\frac{x-1}{x}$ and $g''(x) = \frac{1}{x^2(x-1)} > 0$. With $\lim_{x\to\infty} g'(x) = 0$, it is clear that $g' < 0$ for all $x > 1$. We observe that $g(1^+) = 1$, at the same time $g(x) = \frac{1}{2x} + O(\frac{1}{x^2})$ as $x \to \infty$; Statement (i) is now clear.

Assume by contradiction that there exists a sequence $\{(\alpha_n, c_n)\}_{n=1}^\infty$ such that $\alpha_n \to \infty$ and $c_n \to \infty$. Then, the left hand side of (8.2) converges to 0 while the right hand one tends to 1. This contradiction shows that c is bounded when $\alpha \to \infty$.

Next we eliminate ℓ_c from (8.1) and (8.2) to obtain

$$\left(1 - \frac{1 + 1/\alpha}{A(1 + H(c))}\right)^{r_1} \left(1 - \frac{1 - 1/\alpha}{A(1 - H(c))}\right)^{r_2} = 1\ . \tag{8.3}$$

First we claim that $c \to 0$ as $\alpha \to \infty$. Indeed if otherwise, we can find a sequence $\{(\alpha_n, c_n)\}_{n=1}^\infty$ such that $\alpha_n \to \infty$ and $c_n \to c_0$ for some $c_0 > 0$. Correspondingly $H(c_n) \to H_0 := H(c_0) > 0$. Rewrite (8.3) as

$$-\frac{r_2}{r_1} = \frac{\log\left(1 - \frac{1+1/\alpha}{A(1+H(c))}\right)}{\log\left(1 - \frac{1-1/\alpha}{A(1-H(c))}\right)} \tag{8.4}$$

and take the limit as $n \to \infty$. This gives

$$\frac{1 - H_0}{1 + H_0} = \frac{\log\left(1 - \frac{1}{A(1+H_0)}\right)}{\log\left(1 - \frac{1}{A(1-H_0)}\right)}$$

which is equivalent to

$$\left(1-\frac{1}{A(1-H_0)}\right)^{A(1-H_0)} = \left(1-\frac{1}{A(1+H_0)}\right)^{A(1+H_0)}. \tag{8.5}$$

For $x > 1$ set $h(x) := (1-\frac{1}{x})^x$ and $h_1 := \log h$. Therefore $h_1' = \frac{1}{x-1} + \log(1-\frac{1}{x})$ and $h_1'' = -\frac{1}{x(x-1)^2} < 0$. Since $\lim_{x\to\infty} h_1'(x) = 0$, it follows that h_1, and hence h, are strictly increasing on the interval $(1,\infty)$. Thus (8.5) gives a contradiction.

Using the established claim, we expand (8.4) up to the second order, thus obtaining

$$1-\frac{2c}{\sqrt{4\gamma}}+O(c^2) = \frac{\log(1-\frac{1}{A}) + \log\left(1-\frac{1}{A-1}(\frac{1}{\alpha}-\frac{c}{\sqrt{4\gamma}}+O(c^2)\right)}{\log(1-\frac{1}{A}) + \log\left(1+\frac{1}{A-1}(\frac{1}{\alpha}-\frac{c}{\sqrt{4\gamma}}+O(c^2)\right)},$$

which further simplifies to

$$\frac{c}{\sqrt{4\gamma}}(A-1)\log\frac{A-1}{A} = \frac{1}{\alpha} - \frac{c}{\sqrt{4\gamma}} + O(c^2) + O(\frac{1}{\alpha^2}).$$

This shows Statement (ii). □

Note that

$$6\sqrt{2}\,\mathcal{J}_c(E) = \frac{1}{c}\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) - \alpha\int_{\mathbb{T}^2_T} e^x\chi_E\,dz + \gamma\alpha A\int_{\mathbb{T}^2_T} e^x\chi_E\mathcal{L}_c\chi_E\,dz. \tag{8.6}$$

We want to show that $\mathcal{J}_c(E) < 0$ for some ellipse E when appropriate conditions are imposed on α, A and T, in particular when α becomes large. However by Lemma 8.1 this implies that $c \to 0$, which turns into a loss of uniform ellipticity in (1.6). To obtain tight estimates when c is small, we note that the case $c = 0$ corresponding to stationary waves may help. Some results in the latter case can be found in [13, 14].

Lemma 8.2. *Let $t_0 > 0$ and consider the equation*

$$t^2\frac{d^2w}{dt^2} + t\frac{dw}{dt} - \gamma t^2 w = -t^2\chi_{[0,t_0]} \tag{8.7}$$

on the interval $(0,\infty)$. Then

$$w = \begin{cases} \dfrac{1}{\gamma} - \dfrac{t_0}{\sqrt{\gamma}}K_1(t_0\sqrt{\gamma})\,I_0(t\sqrt{\gamma}), & \text{if } t < t_0, \\ \dfrac{t_0}{\sqrt{\gamma}}I_1(t_0\sqrt{\gamma})\,K_0(t\sqrt{\gamma}), & \text{if } t > t_0, \end{cases} \tag{8.8}$$

where I_j and K_j are the j^{th} order modified Bessel functions of the first and second kind, respectively.

Proof. Let $\tau := t\sqrt{\gamma}$, $\tau_0 := t_0\sqrt{\gamma}$ and $W(\tau) := w(t)$. Then (8.7) becomes

$$\tau^2 \frac{d^2W}{d\tau^2} + \tau \frac{dW}{d\tau} - \tau^2 W = -\frac{\tau^2}{\gamma}\chi_{[0,\tau_0]}\ .$$

Setting the above left hand side to 0 yields the zeroth order modified Bessel equation which has I_0 and K_0 as its independent solutions. Thus

$$W = \begin{cases} \dfrac{1}{\gamma} - \dfrac{\tau_0}{\gamma} K_1(\tau_0)\, I_0(\tau)\,, & \text{if } \tau < \tau_0\,, \\ \dfrac{\tau_0}{\gamma} I_1(\tau_0)\, K_0(\tau)\,, & \text{if } \tau > \tau_0\,, \end{cases}$$

which is equivalent to (8.8). Indeed it suffices to check the continuity of W and W' at $\tau = \tau_0$. The first condition amounts to $\tau\,(K_1(\tau)I_0(\tau) + I_1(\tau)K_0(\tau)) = 1$ at $\tau = \tau_0$, which is valid by [5, formula (9.6.15)]. The continuity of W' requires $-K_1(\tau)I_0'(\tau) = I_1(\tau)K_0'(\tau)$ at $\tau = \tau_0$. Since $I_0' = I_1$ and $K_0' = -K_1$ (see [5, (9.6.27)]), this is true as well. □

From now on in this section we adopt the notations $X = x/c$, $Y = y$ and $Z = (X, Y)$.

Lemma 8.3. *Let* $E := \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{c^2r^2} + \frac{y^2}{r^2} \le 1\}$ *and let* $\mathcal{N} \in W^{1,2}(\mathbb{R}^2)$ *be the weak solution of the equation* $c^2\mathcal{N}_{xx} + \mathcal{N}_{yy} - \gamma\mathcal{N} = -\chi_E$ *in the whole* $\mathbb{R}^2$*. Then* E *is mapped into the ball* $\hat{E}$ *of the* (X, Y) *plane of radius* r *centered at the origin and the function* $\hat{\mathcal{N}}(X, Y) := \mathcal{N}(cX, Y)$ *is radially symmetric. Precisely,* $\hat{\mathcal{N}}(X, Y) = \widetilde{\mathcal{N}}(\sqrt{X^2 + Y^2})$ *where*

$$\widetilde{\mathcal{N}}(t) = \begin{cases} \dfrac{1}{\gamma} - \dfrac{r}{\sqrt{\gamma}} K_1(r\sqrt{\gamma})\, I_0(t\sqrt{\gamma})\,, & \text{if } 0 < t < r\,, \\ \dfrac{r}{\sqrt{\gamma}} I_1(r\sqrt{\gamma})\, K_0(t\sqrt{\gamma})\,, & \text{if } t > r\,. \end{cases} \tag{8.9}$$

Moreover

$$\int_{\mathbb{R}^2} \chi_E\, \mathcal{N}\, dz = \frac{2\pi c r^2}{\gamma}\left(\frac{1}{2} - I_1(r\sqrt{\gamma})K_1(r\sqrt{\gamma})\right). \tag{8.10}$$

Proof. Note that under the change of variable $X = x/c$, $Y = y$ the ellipse E is mapped into the ball $\hat{E}$ of radius r centered at the origin, while $\hat{\mathcal{N}}$ is the unique solution of $\Delta\hat{\mathcal{N}} - \gamma\hat{\mathcal{N}} = -\chi_{\hat{E}}$ in $\mathbb{R}^2$. Therefore $\hat{\mathcal{N}}$ is radially symmetric and thus $\hat{\mathcal{N}}(X, Y) = \widetilde{\mathcal{N}}(\sqrt{X^2 + Y^2})$, where $\widetilde{\mathcal{N}}(t)$ is the unique solution in $(0, \infty)$ of (8.7) with $t_0 = r$. Then (8.9) follows from Lemma 8.2.

Now we directly compute the unweighted nonlocal term using (8.9):

$$\begin{aligned} \int_{\mathbb{R}^2} \chi_E\, \mathcal{N}\, dz &= c\int_{\mathbb{R}^2} \chi_{\hat{E}}\, \hat{\mathcal{N}}\, dZ = 2\pi c \int_0^r \widetilde{\mathcal{N}}(t)\, t\, dt = 2\pi c\, \{\frac{r^2}{2\gamma} - \frac{r}{\sqrt{\gamma}} K_1(r\sqrt{\gamma}) \int_0^r I_0(t\sqrt{\gamma})\, t\, dt\} \\ &= 2\pi c \left\{\frac{r^2}{2\gamma} - \frac{r}{\gamma\sqrt{\gamma}} K_1(r\sqrt{\gamma}) \int_0^{r\sqrt{\gamma}} I_0(t)\, t\, dt\right\}\,. \end{aligned}$$

As I_0 satisfies the equation $(tI_0')' - tI_0 = 0$, on integrating we have $\int_0^{r\sqrt{\gamma}} I_0(t)\, t\, dt = r\sqrt{\gamma}\, I_0'(r\sqrt{\gamma}) = r\sqrt{\gamma}\, I_1(r\sqrt{\gamma})$. Putting this integral in the above equality yields (8.10). □

Lemma 8.4. *Define $h:(0,\infty)\to\mathbb{R}$ such that $h(t)=\frac{1}{2}-I_1(t)K_1(t)$. Then h is a positive strictly increasing function and*
(i) $h(0^+)=0$ and $\lim_{t\to\infty}h(t)=1/2$;
(ii) When t is small, we have $h(t)=-\frac{t^2}{4}\log t\,(1+o(1))$, resulting in $h(t)\le\frac{t^2}{2}|\log t\,|$.

Proof. The product I_1K_1 is strictly decreasing, see [6, Theorem 1]. Using [5, (9.6.7), (9.6.9)] we obtain the limit of h as $t\to 0$, while [5, (9.7.1), (9.7.2)] yields the limit of h as $t\to\infty$ in Statement (i).

To obtain a more precise estimate of h for small argument, we recall [5, (9.6.10), (9.6.11)]:

$$I_1(t)=\frac{t}{2}(1+O(t^2))\ ,$$
$$K_1(t)=\frac{1}{t}+I_1(t)\,\log\frac{t}{2}+O(t)=\frac{1}{t}+\frac{t\log t}{2}+O(t)\ .$$

Simple algebraic manipulation gives Statement (ii). □

The function $\mathcal{N}$ is not T-periodic in y-direction. With a slight abuse of notation we still denote $\int_{\mathbb{R}}\int_{-T/2}^{T/2}\dots dydx$ by $\int_{\mathbb{T}_T^2}\cdots dz$ when the integrand involves $\mathcal{N}$. Note that $\int_{\mathbb{T}_T^2}\chi_E\mathcal{N}\,dz=\int_{\mathbb{R}^2}\chi_E\mathcal{N}\,dz$ when $E\subset\mathbb{T}_T^2$.

Let $A_0>1$ and T_0 be sufficiently large; both to be chosen later. Fix $A\ge A_0$ and $r=\frac{2}{B\sqrt{\gamma}}$. Even if $\alpha\to\infty$, the size r stays fixed at $O(1)$. A first requirement on T_0 is that the torus is wide enough to accommodate the ellipse; there will be other additional constraints. Recall that E and $\hat{E}$ denote the ellipse and the ball, respectively, as stated in the Lemma 8.3.

Lemma 8.5. *Let $c^2\mathcal{P}_{xx}+\mathcal{P}_{yy}-\gamma\mathcal{P}=-\chi_E$ in $\mathbb{T}_T^2$. Then there exists $T_0\ge 4r$, depending on A, such that $|\int_{\mathbb{T}_T^2}\chi_E\,(\mathcal{N}-\mathcal{P})\,dz|\le\frac{c\pi r^2}{\sqrt{T}}$ whenever $T\ge T_0$. Define $\hat{\mathcal{P}}(X,Y)=\mathcal{P}(x,y)$ so that $\Delta\hat{\mathcal{P}}-\gamma\hat{\mathcal{P}}=-\chi_{\hat{E}}$. Then both $\hat{\mathcal{P}}$ and $\nabla\hat{\mathcal{P}}$ are $O(e^{-\sqrt{\gamma}|X|})$ as $|X|\to\infty$. Let $c_0:=\min\{1,\gamma,\sqrt{\gamma}\}$. Then there exists a constant $M>0$ such that $\int_{\mathbb{T}_T^2}e^{cX}|\nabla\hat{\mathcal{P}}|^2\,dZ\le M$ for all $c\le c_0$.*

Remark 8.6. *We may need to further increase T_0 in the proof of Theorem 1.5 below to satisfy additional requirements.*

Proof. Using the variational functional $w\mapsto\int_{\mathbb{T}_T^2}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}\gamma w^2-\chi_{\hat{E}}w)\,dZ$ we infer the existence and uniqueness of $\hat{\mathcal{P}}$ (and hence those of $\mathcal{P}$) with $\|\hat{\mathcal{P}}\|_{H^1(\mathbb{T}_T^2)}\le\frac{1}{\min\{1,\gamma\}}\|\chi_{\hat{E}}\|_{L^2(\mathbb{T}_T^2)}=\frac{r\sqrt{\pi}}{\min\{1,\gamma\}}$ and $0\le\hat{\mathcal{P}}\le 1/\gamma$ on $\mathbb{T}_T^2$. Hence for any $\epsilon>0$, we have $\|\hat{\mathcal{P}}\|_{L^2(\{|X|\ge\ell\})}\le\epsilon$ if ℓ is sufficiently large. Regularity estimates ensure that $\hat{\mathcal{P}}$ is C^∞ on $\{|X|>r\}$. Consider 2 concentric balls $\mathcal{B}_1,\mathcal{B}_2$ of radius 1 and 2, respectively, and move their common center in the set $\{|X|\ge\ell+2\}$. Applying local H^2 estimate to these balls, we see that

$$\|\hat{\mathcal{P}}\|_{C(\overline{\mathcal{B}_1})}\le C\|\hat{\mathcal{P}}\|_{H^2(\mathcal{B}_1)}\le C\|\hat{\mathcal{P}}\|_{L^2(\mathcal{B}_2)}\le C\epsilon$$

so that $\hat{\mathcal{P}}\to 0$ uniformly as $|X|\to\infty$.

Let $\hat{E}_j=\hat{E}+jTe_2$, $j\in\mathbb{Z}$, by translating the ball $\hat{E}$ in the Y-direction. Using the method of images, and still denoting with the same symbol the T-periodic extension of $\hat{\mathcal{P}}$ in y-direction, we have $\Delta\hat{\mathcal{P}} - \gamma\hat{\mathcal{P}} = -\chi_F$ in $\mathbb{R}^2$ where $F = \cup_{j=-\infty}^{\infty}\hat{E}_j$. The fundamental solution associated with the differential operator $(-\Delta+\gamma)$ is $\frac{1}{2\pi}K_0(\sqrt{\gamma}\,|z|)$. It solves the equation $-\Delta w+\gamma w = 0$ in $\mathbb{R}^2$ minus the origin near which it behaves like $-\frac{1}{2\pi}\log|z|$. Note that K_0 is a strictly decreasing function. When $T \geq 4r$ is sufficiently large, for any $Z = (X,Y) \in \hat{E} \subset \mathbb{T}_T^2$

$$\begin{aligned}\hat{\mathcal{P}}(Z) &= \hat{\mathcal{N}}(Z) + \frac{1}{2\pi}\sum_{j\neq 0, j=-\infty}^{\infty}\int_{\mathbb{R}^2} K_0(\sqrt{\gamma}\,|Z-\eta|)\,\chi_{\hat{E}_j}(\eta)\,d\eta \leq \hat{\mathcal{N}}(Z) + r^2\sum_{j=1}^{\infty}K_0(\sqrt{\gamma}(jT-2r))\\ &\leq \hat{\mathcal{N}}(Z) + r^2\sum_{j=1}^{\infty}K_0(\frac{(2j-1)\sqrt{\gamma}\,T}{2})\\ &\leq \hat{\mathcal{N}}(Z) + 2r^2\sum_{j=1}^{\infty}\sqrt{\frac{\pi}{(2j-1)\sqrt{\gamma}\,T}}\;e^{-\frac{(2j-1)\sqrt{\gamma}\,T}{2}} \quad \text{by twice the leading order term in [5,(9.7.2)]}\\ &\leq \hat{\mathcal{N}}(Z) + \frac{1}{\sqrt{T}}\;,\end{aligned}$$

leading to

$$0 \leq \int_{\mathbb{T}_T^2}\chi_E\,(\mathcal{P}-\mathcal{N})\,dz = \;c\int_{\mathbb{T}_T^2}\chi_{\hat{E}}\,(\hat{\mathcal{P}}-\hat{\mathcal{N}})\,dZ \leq c\pi r^2\|\hat{\mathcal{P}}-\hat{\mathcal{N}}\|_{L^\infty(E)} \leq \frac{c\pi r^2}{\sqrt{T}}\;.$$

Next consider the function $q(X,Y) = \frac{1}{\gamma}e^{-\sqrt{\gamma}(X-r-1)}$ on the set $\{X \geq r+1\}$. Clearly $\Delta q - \gamma q = 0$ with $q = 1/\gamma \geq \hat{\mathcal{P}}$ at $X = r+1$. With $\hat{\mathcal{P}}$ going to 0 for large $|X|$, we can employ the maximum principle. In fact q serves as an upper barrier function for $\hat{\mathcal{P}}$ so that $0 \leq \hat{\mathcal{P}} \leq q$ on $\{X \geq r+1\}$. Thus $\hat{\mathcal{P}}$ decays at or faster than the rate $e^{-\sqrt{\gamma}X}$. Now employ local $W^{2,p}$ estimate for a fixed $p > 2$,

$$\|\hat{\mathcal{P}}\|_{C^1(\overline{\mathcal{B}_1})} \leq C\|\hat{\mathcal{P}}\|_{W^{2,p}(\mathcal{B}_1)} \leq C\|\hat{\mathcal{P}}\|_{L^\infty(\mathcal{B}_2)} \leq Ce^{-\sqrt{\gamma}X}$$

so that $|\nabla\hat{\mathcal{P}}| = O(e^{-\sqrt{\gamma}X})$ as $X \to \infty$. The same is true for large negative X. Thus there exists a positive constant M such that $\int_{\mathbb{T}_T^2} e^{cX}|\nabla\hat{\mathcal{P}}|^2\,dz < M$ for all $c < 2\sqrt{\gamma}$.

We claim that M can be chosen to be T-independent if $c \leq c_0 = \min\{1,\gamma,\sqrt{\gamma}\}$. Write the equation on $\hat{\mathcal{P}}$ as

$$(e^{cX}\hat{\mathcal{P}}_X)_X + e^{cX}\hat{\mathcal{P}}_{YY} - \gamma e^{cX}\hat{\mathcal{P}} = -e^{cX}\chi_{\hat{E}} + ce^{cX}\hat{\mathcal{P}}_X\;,$$

which leads to

$$\int_{\mathbb{T}_T^2} e^{cX}(|\nabla\hat{\mathcal{P}}|^2 + \gamma\hat{\mathcal{P}}^2)\,dz = \int_{\mathbb{T}_T^2} e^{cX}\chi_{\hat{E}}\hat{\mathcal{P}}\,dz - c\int_{\mathbb{T}_T^2} e^{cX}\hat{\mathcal{P}}\,\hat{\mathcal{P}}_X\,dz$$

so that

$$\int_{\mathbb{T}_T^2} e^{cX}(|\nabla\hat{\mathcal{P}}|^2 + \gamma\hat{\mathcal{P}}^2)\,dz \leq 2\int_{\mathbb{T}_T^2} e^{cX}\chi_{\hat{E}}\hat{\mathcal{P}}\,dz\;.$$

This immediately gives

$$\sqrt{\int_{\mathbb{T}^2_T} e^{cX}\hat{\mathcal{P}}^2\,dz}\ \le \frac{2}{\gamma}\sqrt{\int_{\mathbb{T}^2_T} e^{cX}\chi^2_{\hat{E}}\,dz} \le \frac{2re^{cr/2}\sqrt{\pi}}{\gamma}\ ,$$

$$\sqrt{\int_{\mathbb{T}^2_T} e^{cX}|\nabla\hat{\mathcal{P}}|^2\,dz}\ \le \frac{2re^{cr/2}\sqrt{\pi}}{\sqrt{\gamma}}$$

and establishes the above claim. □

Lemma 8.7. *Suppose* $v = \mathcal{L}_c\chi_E$, *i.e.* $c^2 v_{xx} + v_{yy} + c^2 v_x - \gamma v = -\chi_E$ *on* $\mathbb{T}^2_T$, *then*

$$\Big|\int_{\mathbb{T}^2_T} e^x \chi_E\,\mathcal{L}_c\chi_E\,dz - \int_{\mathbb{T}^2_T}\chi_E\,\mathcal{P}\,dz\Big| \le c\pi r A\,O(\frac{1}{\alpha})\ . \tag{8.11}$$

Proof. Denote the left hand side of (8.11) by LHS. Then

$$LHS \le \Big|\int_{\mathbb{T}^2_T}(e^x-1)\,\chi_E\,\mathcal{L}_c\chi_E\,dz\Big| + \Big|\int_{\mathbb{T}^2_T}\chi_E\,(\mathcal{L}_c\chi_E - \mathcal{P})\,dz\Big| := I + II\ .$$

We need to control both terms on the right. For the first one,

$$\begin{aligned} I &\le (e^{cr}-1)\,\|\chi_E\|_{L^2(E)}\,\|\mathcal{L}_c\chi_E\|_{L^2(E)} \le e^{cr/2}\,(e^{cr}-1)\,\|\chi_E\|_{L^2(\mathbb{T}^2_T)}\,\|\mathcal{L}_c\chi_E\|_{L^2_e(E)} \\ &\le \frac{e^{cr/2}(e^{cr}-1)}{\gamma}\,\|\chi_E\|_{L^2(\mathbb{T}^2_T)}\,\|\chi_E\|_{L^2_e(\mathbb{T}^2_T)} \le \frac{e^{cr}(e^{cr}-1)}{\gamma}\,c\pi r^2 = c\pi r^2\,O(\frac{1}{\alpha})\ , \end{aligned} \tag{8.12}$$

where in the last equality we used the fact that, using (ii) of Lemma 8.1 and the definition of r, one has $\alpha c r = 4 + O(\frac{1}{\alpha})$. Next let $\hat{v}(X,Y) = v(x,y)$ so that $\Delta\hat{v} + c\hat{v}_X - \gamma\hat{v} = -\chi_{\hat{E}}$. Multiplying $\Delta\hat{\mathcal{P}} - \gamma\hat{\mathcal{P}} = -\chi_{\hat{E}}$ by $\hat{\mathcal{P}}$ and integrating over $\mathbb{T}^2_T$, such energy estimate yields $\|\hat{\mathcal{P}}\|_{L^2(\mathbb{T}^2_T)} \le \frac{1}{\gamma}\|\chi_{\hat{E}}\|_{L^2(\mathbb{T}^2_T)} = \frac{r\sqrt{\pi}}{\gamma}$ and $\|\nabla\hat{\mathcal{P}}\|_{L^2(\mathbb{T}^2_T)} \le \frac{1}{\sqrt{\gamma}}\|\chi_{\hat{E}}\|_{L^2(\mathbb{T}^2_T)} = r\sqrt{\frac{\pi}{\gamma}}$. Since

$$\Delta(\hat{v}-\hat{\mathcal{P}}) + c(\hat{v}-\hat{\mathcal{P}})_X - \gamma(\hat{v}-\hat{\mathcal{P}}) = -c\hat{\mathcal{P}}_X\ ,$$

this leads to

$$\int_{\mathbb{T}^2_T} e^{cX}\left(|\nabla(\hat{v}-\hat{\mathcal{P}})|^2 + \gamma(\hat{v}-\hat{\mathcal{P}})^2\right)dZ = c\int_{T^2_T} e^{cX}\hat{\mathcal{P}}_X(\hat{v}-\hat{\mathcal{P}})\,dZ$$

and therefore by Lemma 8.5

$$\sqrt{\int_{\mathbb{T}^2_T} e^{cX}|\hat{v}-\hat{\mathcal{P}}|^2\,dZ} \le \frac{c}{\gamma}\sqrt{\int_{\mathbb{T}^2_T} e^{cX}|\hat{\mathcal{P}}_X|^2\,dZ} \le \frac{c\sqrt{M}}{\gamma}\ .$$

Consequently, recalling (i) and (ii) of Lemma 8.1,

$$\begin{aligned} II &= \Big|c\int_{\mathbb{T}^2_T}\chi_{\hat{E}}\,(\hat{v}-\hat{\mathcal{P}})\,dZ\Big| \le c\|\chi_{\hat{E}}\|_{L^2(\hat{E})}\|\hat{v}-\hat{\mathcal{P}}\|_{L^2(\hat{E})} \le cre^{cr/2}\sqrt{\pi}\sqrt{\int_{\hat{E}} e^{cX}|\hat{v}-\hat{\mathcal{P}}|^2\,dZ} \\ &\le \frac{c^2 re^{cr/2}\sqrt{\pi M}}{\gamma} = c\pi r A\,O(\frac{1}{\alpha})\ . \end{aligned}$$

□

Proof of Theorem 1.5. Using Lemmas 8.3 to 8.7, we can deduce from (8.6) that

$$\begin{aligned}
6\sqrt{2}\,\mathcal{J}_c(E) &\le \frac{2\pi cre^{cr}}{c} - \alpha\,\pi cr^2e^{-cr} + \gamma\alpha A\left\{\int_{\mathbb{R}^2}\chi_E\mathcal{N}\,dz + c\pi r\left(A\,O(\frac{1}{\alpha}) + \frac{r}{\sqrt{T}}\right)\right\}\\
&= 2\pi re^{cr} - \alpha\,\pi cr^2e^{-cr}\\
&\quad + \gamma\alpha A\left\{\frac{2\pi cr^2}{\gamma}\left(\frac{1}{2} - I_1(r\sqrt{\gamma})K_1(r\sqrt{\gamma})\right) + c\pi r\left(A\,O(\frac{1}{\alpha}) + \frac{r}{\sqrt{T}}\right)\right\}\\
&\le 2\pi re^{cr} - \alpha\,\pi cr^2e^{-cr} + \gamma\alpha A\left\{\pi cr^4|\log(r\sqrt{\gamma})| + c\pi r\left(A\,O(\frac{1}{\alpha}) + \frac{r}{\sqrt{T}}\right)\right\}.
\end{aligned}$$

Recall $r = \frac{2}{B\sqrt{\gamma}}$. When A is sufficiently large, we have $\frac{3}{4\sqrt{\gamma}} \le Ar = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{A}) \le \frac{2}{\sqrt{\gamma}}$ and thus

$$\gamma\alpha A\pi cr^4|\log(r\sqrt{\gamma})| \le 2\alpha\pi cr^3\,|\log(r\sqrt{\gamma})|\,\sqrt{\gamma}\,.$$

Choose $A_0 > 1$ sufficiently large and restrict to $A \ge A_0$. This ensures that

$$2r\,|\log(r\sqrt{\gamma})|\,\sqrt{\gamma} \le \frac{1}{8}\,.$$

Hence

$$6\sqrt{2}\,\mathcal{J}_c(E) \le \pi r\left\{2e^{cr} - \alpha cre^{-cr} + \frac{1}{8}\alpha cr + \alpha c\left(\gamma A^2\,O(\frac{1}{\alpha}) + \frac{Ar\gamma}{\sqrt{T}}\right)\right\}.$$

Now choose $T_0(A)$ so large that for $T \ge T_0$ one has $8A\gamma < \sqrt{T}$ and $\alpha_*(A)$ so large that if $\alpha > \alpha_*$, then

$$\gamma A^2\,O(\frac{1}{\alpha}) < \frac{1}{4B\sqrt{\gamma}} = \frac{r}{8}\,.$$

Thus we get

$$6\sqrt{2}\,\mathcal{J}_c(E) \le \pi r(2e^{cr} - \alpha cre^{-cr} + \frac{3}{8}\alpha cr)$$

Finally, recalling that $\alpha cr = 4+O(\frac{1}{\alpha})$, by taking α_* larger if needed, we conclude that $\mathcal{J}_c(E) < 0$.

Let $E^a := E - ae_1$ with $a \in \mathbb{R}$ chosen so that $|E^a|_e = 1$. It follows that

$$\mathcal{J}_c(E^a) = e^{-a}\mathcal{J}_c(E) < 0. \tag{8.13}$$

Recall $\mathbb{E}(c)$ represents a global minimizer of $\mathcal{J}_c$ when subject to the constraint $|E|_e = 1$. From (3.18) we infer that $\liminf_{c\to 0}\mathcal{J}_c(\mathbb{E}(c)) > 0$. By (8.13)

$$\mathcal{J}_{c_p}(\mathbb{E}(c_p)) \le \mathcal{J}_{c_p}(E^a) < 0 = \mathcal{J}_{c_p}(P),$$

where P is the planar pulse of speed c_p. Hence there exists $c_* < c_p$ such that $\mathcal{J}_{c_*}(\mathbb{E}(c_*)) = 0$; this implies that $\mathbb{E}(c_*)$ is an unconstrained minimizer of $\mathcal{J}_{c_*}$. The minimizer $\mathbb{E}(c_*)$ cannot be planar, as it will violate the known uniqueness results [12] of speed for planar pulse as well as planar front. In fact from Statement 2 of Theorem 1.4, we know the existence of a planar traveling front moving with the speed c_f; moreover $c_* < c_p < c_f$. These make up a total of at least 3 co-existing waves for the same parameters in the specified range. □

9 Appendix

In this section we are giving the proofs of some technical auxiliary results we used in this paper, all of them being the counterpart in our periodic and weighted setting of results which in the standard Euclidean setting are well known to the experts in Calculus of Variations or in the theory of sets of finite perimeter. However we believe that those who are not experts in these fields may find useful to have the proofs of these results available here. We start with the

Proof of Theorem 2.6. **Step 1.** We start by assuming that $E \subset \mathbb{T}^2_T$ is bounded. Since $\chi_E \in BV_e(\mathbb{T}^2_T)$ by Lemma 2.2 we have that there exists a sequence of functions of $u_j \in C^\infty(\mathbb{T}^2_T)$ such that

$$u_j \to \chi_E \quad \text{in } L^1_e(\mathbb{T}^2_T), \qquad \|Du_j\|_e(\mathbb{T}^2_T) \to \mathcal{P}_e(E;\mathbb{T}^2_T)\,. \tag{9.1}$$

Note also that since E is bounded and χ_E takes only the values 0 and 1, from the proof of Lemma 2.2 we have that the functions u_j have equibounded supports and that $0 \le u_j \le 1$ for all j. From (2.3) and the coarea formula, see [4, Theorem 3.40], we get

$$\|Du_j\|_e(\mathbb{T}^2_T) = \int_{\mathbb{T}^2_T} e^x |Du_j(z)|\,dz = \int_0^1 ds \int_{\{u_j=s\}} e^x\,d\mathcal{H}^1\,. \tag{9.2}$$

For every $j \in \mathbb{N}$ and $s \in (0,1)$, set $F_{j,s} = \{u_j > s\}$ and observe that the sets $F_{j,s}$ are equibounded. Moreover, since each u_j is a smooth function, by Sard's theorem there exists a set $N_j \subset (0,1)$ with zero Lebesgue measure such that for all $s \in (0,1) \setminus N_j$ the level set $\{u_j = s\}$ contains no critical points of u_j. Thus, setting $N = \cup_{j\in\mathbb{N}} N_j$, N has zero measure and, for every $j \in \mathbb{N}$ and every $s \in (0,1) \setminus N$, $F_{j,s}$ is a smooth open set with $\partial F_{j,s} = \{u_j = s\}$.

From (9.1), (9.2) and (2.7), by Fatou lemma we have

$$\begin{aligned}\mathcal{P}_e(E;\mathbb{T}^2_T) &= \lim_{j\to\infty} \int_0^1 ds \int_{\{u_j=s\}} e^x\,d\mathcal{H}^1 \\ &= \lim_{j\to\infty} \int_0^1 \mathcal{P}_e(F_{j,s};\mathbb{T}^2_T)\,ds \ge \int_0^1 \liminf_{j\to\infty} \mathcal{P}_e(F_{j,s};\mathbb{T}^2_T)\,ds\,.\end{aligned}$$

Note that it is always possible to choose $t \in (0,1) \setminus N$ such that

$$\int_0^1 \liminf_{j\to\infty} \mathcal{P}_e(F_{j,s};\mathbb{T}^2_T)\,ds \ge \liminf_{j\to\infty} \mathcal{P}_e(F_{j,t};\mathbb{T}^2_T) = \lim_{h\to\infty} \mathcal{P}_e(F_{j_h,t};\mathbb{T}^2_T)$$

for a suitable strictly increasing sequence j_h.

We claim that the sets $\widetilde{E}_h = F_{j_h,t}$ approximate E in (weighted) area and in perimeter. First, observe that by the above inequalities

$$\mathcal{P}_e(E;\mathbb{T}^2_T) \ge \lim_{h\to\infty} \mathcal{P}_e(\widetilde{E}_h;\mathbb{T}^2_T)\,.$$

Moreover $\chi_{\widetilde{E}_h} \to \chi_E$ in $L^1_e(\mathbb{T}^2_T)$. Indeed

$$\begin{aligned}\int_{\widetilde{E}_h\setminus E} e^x\,dz &\le \frac{1}{t}\int_{\widetilde{E}_h\setminus E} e^x |u_{j_h}(z) - \chi_E(z)|\,dz\,,\\ \int_{E\setminus\widetilde{E}_h} e^x\,dz &\le \frac{1}{1-t}\int_{E\setminus\widetilde{E}_h} e^x |u_{j_h}(z) - \chi_E(z)|\,dz\,.\end{aligned}$$

Therefore, thanks to (9.1), we have

$$\int_{\mathbb{T}^2_T} e^x|\chi_{\widetilde{E}_h}-\chi_E|\,dz \le \max\left\{\frac{1}{t},\frac{1}{1-t}\right\}\int_{\mathbb{T}^2_T} e^x|u_{j_h}-\chi_E|\,dz \to 0 \qquad \text{as } h\to\infty.$$

Finally, since $\chi_{\widetilde{E}_h}\to\chi_E$ in $L^1_{loc}(\mathbb{T}^2_T)$, by the lower semicontinuity of the perimeter we have

$$\lim_{h\to\infty}\mathcal{P}_e(\widetilde{E}_h;\mathbb{T}^2_T)\ge\mathcal{P}_e(E;\mathbb{T}^2_T)\,.$$

This proves the convergence of the weighted perimeters. To conclude the proof of the theorem in this case we have to satisfy the area constraint. To this aim we set $E_h=x_he_1+\widetilde{E}_h$, where x_h is chosen so that $|E_h|_e=|E|_e$. Since $\chi_{\widetilde{E}_h}\to\chi_E$ in $L^1_e(\Omega_T)$, it follows that $x_h\to 0$ and $\chi_{E_h}\to\chi_E$ in $L^1_e(\mathbb{T}^2_T)$. Moreover $\mathcal{P}_e(E_h;\mathbb{T}^2_T)=e^{x_h}\mathcal{P}_e(\widetilde{E}_h;\mathbb{T}^2_T)\to\mathcal{P}_e(E;\mathbb{T}^2_T)$.

Step 2. We now remove the assumption that E is bounded. Since E is a set of locally finite perimeter, there exists a countable set $Z\subset\mathbb{R}$ such that $\mathcal{H}^1(\partial^*E\cap\{x=t\})=0$ for all $t\in\mathbb{R}\setminus Z$. Let $E^{(1)}$ be the *set of points of density* 1 *at* E which is defined as

$$E^{(1)}:=\left\{z\in\mathbb{T}^2_T:\,\lim_{r\to 0}\frac{|E\cap B_r(z)|}{\pi r^2}=1\right\}.$$

Recall, see [31, (5.19)], that $E^{(1)}$ and E differ by as set of zero measure. By Fubini's theorem there exists an increasing sequence of positive numbers $t_h\to+\infty$ such that $t_h\in[0,\infty)\setminus Z$ and

$$\lim_{h\to\infty}\int_{-T/2}^{T/2}e^{t_h}\chi_{E^{(1)}}(\pm t_h,y)\,dy=0\,. \tag{9.3}$$

Set $F_h=E\cap\{|x|<t_h\}$. Then, since $t\notin Z$, $\partial^*F_h=\big(\partial^*E\cap\{|x|<t_h\}\big)\cup\big(E^{(1)}\cap\{|x|=t_h\}\big)$, up to a set of $\mathcal{H}^1$ zero measure, see [31, Theorem 16.3]. Thus, recalling (2.7), we have

$$\mathcal{P}_e(F_h;\mathbb{T}^2_T)=\int_{\partial^*E\cap\{|x|<t_h\}}e^x\,d\mathcal{H}^1+\int_{-T/2}^{T/2}e^{-t_h}\chi_{E^{(i)}}(-t_h,y)\,dy+\int_{-T/2}^{T/2}e^{t_h}\chi_{E^{(i)}}(t_h,y)\,dy\,.$$

In view of (9.3) we have that $\mathcal{P}_e(F_h;\mathbb{T}^2_T)\to\mathcal{P}_e(E;\mathbb{T}^2_T)$ as $h\to\infty$. Moreover, we have also that $\chi_{F_h}\to\chi_E$ in $L^1_e(\mathbb{T}^2_T)$. The conclusion then follows by a standard diagonalization argument, applying to each F_h the approximation result proved in Step 1 and then adjusting the weighted area as before. □

We now give the proof of Proposition 4.3. The proof is based on the following lemma, proved in [32, Lemma 4] in any dimension. In the 2-dimensional case, which is of interest here, it reads as follows.

Lemma 9.1. *Let A and Ω be two smooth bounded open sets of $\mathbb{R}^2$ and let $d_A:\mathbb{R}^2\to\mathbb{R}$ be the signed distance from the boundary of A defined by setting*

$$d_A(z)=\begin{cases}\operatorname{dist}(z,\partial A) & \textit{if } z\in A\,,\\ -\operatorname{dist}(z,\partial A) & \textit{if } z\notin A\,.\end{cases}$$

Then d_A *is Lipschitz continuous and* $|Dd_A(z)| = 1$ *for a.e.* $z \in \mathbb{R}^2$*. Moreover, if* $\mathcal{H}^1(\partial A \cap \partial\Omega) = 0$*, then*

$$\lim_{t\to 0} \mathcal{H}^1(\{z :\, d_A(z) = t\} \cap \Omega) = \mathcal{H}^1(\partial A \cap \Omega)\,. \tag{9.4}$$

Following [32] we now define an auxiliary function, by considering the unique solution U_ϵ of the differential equation

$$U'_\epsilon = \frac{\sqrt{\epsilon + 2F_0(U_\epsilon)}}{\epsilon} \tag{9.5}$$

satisfying the condition $U_\epsilon(0) = 0$. Note that U_ϵ is strictly increasing and for all $t \in \mathbb{R}$

$$\int_0^{U_\epsilon(t)} \frac{\epsilon}{\sqrt{\epsilon + 2F_0(s)}}\, ds = t\,.$$

So there exists a unique $\rho_\epsilon > 0$ such that $U_\epsilon(\rho_\epsilon) = 1$. Moreover,

$$\rho_\epsilon = \int_0^1 \frac{\epsilon}{\sqrt{\epsilon + 2F_0(s)}}\, ds \le \sqrt{\epsilon}\,.$$

Having constructed U_ϵ as above, we define a Lipschitz increasing function $\chi_\epsilon : \mathbb{R} \to [0,1]$, by setting

$$\chi_\epsilon(t) = \begin{cases} 0 & \text{if } t \le 0\,, \\ U_\epsilon(t) & \text{if } 0 \le t \le \rho_\epsilon\,, \\ 1 & \text{if } t \ge \rho_\epsilon\,. \end{cases}$$

Let us now give the

Proof of Proposition 4.3. Given a smooth bounded open set $E \subset \mathbb{T}^2 \cap \{|x| < R\}$ for some $R > 0$, we denote by $\widehat{E}$ its T-periodic extension to $\mathbb{R}^2$. We set for all $\epsilon \in (0,1)$ and $z \in \mathbb{R}^2$, $v_\epsilon(z) = \chi_\epsilon(d_{\widehat{E}}(z) + \eta_\epsilon)$, where $d_{\widehat{E}}$ is defined as in Lemma 9.1 and $\eta_\epsilon \in [0, \rho_\epsilon]$ is chosen so that

$$\int_{\mathbb{T}^2_T} e^x \chi_\epsilon^2(d_{\widehat{E}}(z) + \eta_\epsilon)\, dz = \int_E e^x\, dz\,.$$

Such a choice is always possible since

$$\int_{\mathbb{T}^2_T} e^x \chi_\epsilon^2(d_{\widehat{E}}(z))\, dz \le \int_E e^x\, dz \le \int_{\mathbb{T}^2_T} e^x \chi_\epsilon^2(d_{\widehat{E}}(z) + \rho_\epsilon)\, dz\,.$$

Note that the periodicity of $\widehat{E}$ yields that $d_{\widehat{E}}$ and v_ϵ are T-periodic in the y-direction. Moreover, since $\widehat{E} \subset (-R, R) \times \mathbb{R}$ and $\rho_\epsilon \le \sqrt{\epsilon} \le 1$, we have that $v_\epsilon(x, y) = 0$ if $|x| > R + 1$. Note also, that, up to translating a bit $\widehat{E}$ in the y direction, we may always assume that

$$\mathcal{H}^1\big(\partial\widehat{E} \cap \{(x, y) \in \mathbb{R}^2 :\, |y| = T/2\}\big) = 0\,. \tag{9.6}$$

By the coarea formula, we have, recalling that $|Dd_{\widehat{E}}| = 1$ a.e. and setting $\chi_0 = \chi_{(0,\infty)}$,

$$\begin{aligned}\int_{\mathbb{T}^2_T} |v_\epsilon - \chi_E|\,dz &= \int_{\{|y|<T/2\}} |\chi_\epsilon(d_{\widehat{E}}(z)+\eta_\epsilon) - \chi_0(d_{\widehat{E}}(z))|\ \ |Dd_{\widehat{E}}(z)|\,dz \\ &= \int_{-\infty}^{\infty} dt \int_{\{d_{\widehat{E}}=t\}\cap\{|y|<T/2\}} |\chi_\epsilon(d_{\widehat{E}}(z)+\eta_\epsilon) - \chi_0(d_{\widehat{E}}(z))|\,d\mathcal{H}^1 \\ &= \int_{-\eta_\epsilon}^{\rho_\epsilon-\eta_\epsilon} |\chi_\epsilon(t+\eta_\epsilon) - \chi_0(t)|\,\mathcal{H}^1(\{d_{\widehat{E}}=t\}\cap\{|y|<T/2\})\,dt \\ &\le 2\rho_\epsilon \sup_{|t|\le\rho_\epsilon} \mathcal{H}^1(\{d_{\widehat{E}}=t\}\cap\{|y|<T/2\})\,.\end{aligned}$$

Thus the convergence of v_ϵ to χ_E in $L^1(\mathbb{T}^2_T)$ follows at once from (9.4), thanks to (9.6). Given a positive integer n we subdivide the interval $[-R-1, R+1]$ in n subintervals whose endpoints we denote by $-R-1 = \alpha_{0,n} < \alpha_{1,n}, \cdots < \alpha_{n,n} = R+1$, such that

$$\max_{j=1,\dots,n} (\alpha_{j,n} - \alpha_{j-1,n}) \le \frac{2R+3}{n}\,. \tag{9.7}$$

Moroever, since $\mathcal{H}^1(\partial\widehat{E}\cap\{x=t\}) = 0$ for all but countably many $t\in\mathbb{R}$, we may always choose the endpoints $\alpha_{j,n}$ so that

$$\mathcal{H}^1(\partial\widehat{E}\cap\{x=\alpha_{j,n}\}) = 0 \qquad \text{for all}\ \ n\ \ \text{and}\ \ j=0,1,\dots,n\,. \tag{9.8}$$

Let us denote by $R_{j,n}$ the open rectangle $R_{j,n} = (\alpha_{j-1,n}, \alpha_{j,n}) \times (-T/2, T/2)$. We use the coarea formula again to get

$$\begin{aligned}\int_{\mathbb{T}^2_T} e^x\Big(\frac{\epsilon|Dv_\epsilon|^2}{2} + \frac{F_0(v_\epsilon)}{\epsilon}\Big)\,dz &\le \sum_{j=1}^n e^{\alpha_{j,n}} \int_{R_{j,n}} \Big(\frac{\epsilon{\chi'_\epsilon}^2(d_{\widehat{E}}(z)+\eta_\epsilon)}{2} + \frac{F_0(\chi_\epsilon(d_{\widehat{E}}(z)+\eta_\epsilon))}{\epsilon}\Big)\,dz \\ &= \sum_{j=1}^n e^{\alpha_{j,n}} \int_{-\eta_\epsilon}^{\rho_\epsilon-\eta_\epsilon} \Big(\frac{\epsilon{U'_\epsilon}^2(t+\eta_\epsilon)}{2} + \frac{F_0(U_\epsilon(t+\eta_e))}{\epsilon}\Big)\mathcal{H}^1(\{d_{\widehat{E}}=t\}\cap R_{j,n})\,dt \\ &\le \sum_{j=1}^n e^{\alpha_{j,n}} S_{\epsilon,j,n} \int_0^{\rho_\epsilon} \Big(\frac{\epsilon{U'_\epsilon}^2(t)}{2} + \frac{F_0(U_\epsilon(t))}{\epsilon}\Big)\,dt\,,\end{aligned} \tag{9.9}$$

where we have set

$$S_{\epsilon,j,n} = \sup_{|t|\le\rho_\epsilon} \mathcal{H}^1(\{d_{\widehat{E}}=t\}\cap R_{j,n})\,.$$

On the other hand, recalling (9.5), we have

$$\begin{aligned}\int_0^{\rho_\epsilon} \Big(\frac{\epsilon{U'_\epsilon}^2}{2} + \frac{F_0(U_\epsilon)}{\epsilon}\Big)\,dt &\le \int_0^{\rho_\epsilon} \frac{\epsilon + 2F_0(U_\epsilon)}{\epsilon}\,dt \\ &= \int_0^{\rho_\epsilon} \sqrt{\epsilon + 2F_0(U_\epsilon)}\,U'_\epsilon\,dt = \int_0^1 \sqrt{\epsilon + 2F_0(s)}\,ds\,.\end{aligned}$$

Observe that from (9.6) and (9.8) we have that $\mathcal{H}^1(\partial\widehat{E}\cap\partial R_{j,n})=0$. Therefore, thanks to Lemma 9.1, passing to the limit as $\epsilon\to 0$ in (9.9), we get

$$\begin{aligned}\limsup_{\epsilon\to0^+}\int_{\mathbb{T}^2_T} e^x\left(\frac{\epsilon|Dv_\epsilon|^2}{2}+\frac{F_0(v_\epsilon)}{\epsilon}\right)dz &\le \sum_{j=1}^n e^{\alpha_{j,n}}\lim_{\epsilon\to0^+} S_{\epsilon,j,n}\int_0^1\sqrt{\epsilon+2F_0(s)}\,ds\\ &=\phi(1)\sum_{j=1}^n e^{\alpha_{j,n}}\mathcal{H}^1(\partial E\cap R_{j,n})\le\phi(1)\sum_{j=1}^n e^{\alpha_{j,n}-\alpha_{j-1,n}}\int_{\partial E\cap R_{j,n}} e^x\,d\mathcal{H}^1\,.\end{aligned}$$

The conclusion then follows recalling (9.7), letting $n\to\infty$ in the previous inequality. □

In the remaining part of this Appendix we are going to prove the following regularity result for the volume constrained minimizers of the functional $\mathcal{K}_c$.

Theorem 9.2. *Let $E\subset\mathbb{T}^2_T$ be a minimizer of the problem* (3.2). *Then E is an open set of class $C^{1,\alpha}$ for all $\alpha\in(0,\frac{1}{2})$.*

Remark 9.3. *The above regularity theorem actually holds in a stronger form. Indeed, take a point $z_0=(x_0,y_0)\in\partial E$. From Theorem 9.2 we have that in a neighborhood U of z_0 the boundary of E is the graph of a $C^{1,\alpha}$ function. Let us assume, without loss of generality, that $E\cap U=\{(x,y)\in U : y>f(x)\}$ where $f\in C^{1,\alpha}(I)$ for some open interval I containing x_0. Then, using the first variation formula* (6.18)*, one can see that f satisfies the following Euler-Lagrange equation*

$$-\frac{d}{dx}\big[F_z(x,f(x),f'(x))\big]=\sigma\,\mathcal{L}_c\chi_E(x,f(x))\,, \tag{9.10}$$

where $F=F(x,u,z): I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a smooth function such that $F_{zz}>0$. Therefore, if $\sigma=0$ then $f\in C^\infty(I)$, hence E is a smooth open set.

If $\sigma>0$, $\mathcal{L}_c\chi_E\in W^{2,p}_{loc}(\mathbb{T}^2_T)$ for all $p\ge1$ (see [27, Theorem 9.11]), hence $\mathcal{L}_c\chi_E\in C^{1,\alpha}(\mathbb{T}^2_T)$ for all $\alpha\in(0,1)$. Therefore the function $x\to\mathcal{L}_c\chi_E(x,f(x))$ is in $C^{1,\alpha}(I)$ for all $\alpha\in(0,1/2)$ and from (9.10) *we get that $f\in C^{3,\alpha}(I)$, hence ∂E is of class $C^{3,\alpha}$, for all $\alpha\in(0,1/2)$. Observe now that in particular the function $x\to\mathcal{L}_c\chi_E(x,f(x))$ is in $C^{1,\alpha}(I)$ for all $\alpha\in(0,1)$. Therefore, arguing as above we conclude that ∂E is of class $C^{3,\alpha}$, for all $\alpha\in(0,1)$*

The proof of Theorem 9.2 will be a consequence of the general theory of *perimeter almost minimizers*. We start by giving the definition of perimeter almost minimizer. Since we are only dealing with planar sets we give all the relevant definitions and results of the theory only in this setting. The standard notation $E\Delta F:=(E\setminus F)\cup(F\setminus E)$ will be employed for any two sets E and F.

Definition 9.4. *Given a set of locally finite perimeter $E\subset\mathbb{R}^2$, we say that E is an almost minimizer of the perimeter in an open set $U\subset\mathbb{R}^2$ if there exist a radius $r_0>0$ and a constant $\omega>0$ such that for every disk $B_r(z)\subset U$ with $0<r<r_0$ and any measurable set $F\subset\mathbb{R}^2$ such that $E\Delta F\subset\subset B_r(z)$ then*

$$P(E;B_r(z))\le P(F;B_r(z))+\omega r^2\,.$$

If the above inequality holds with $\omega=0$ we say that E is a local minimizer of the perimeter.

Thus an almost minimizer locally minimizes the perimeter in small balls with an error of the order of the area of the ball. Our main tool to prove Theorem 9.2 will be the following regularity theorem which is a variant of the celebrated regularity result of De Giorgi for perimeter minimizers, see for instance [41, Theorem 1.9].

Theorem 9.5. *Let $E \subset \mathbb{R}^2$ be an almost minimizer of the perimeter in an open set U. Then $E \cap U$ is open and $\partial E \cap U$ is of class $C^{1,\alpha}$ for all $\alpha \in \left(0, \frac{1}{2}\right)$. Moreover, if E is a local minimizer of the perimeter in U, then $\partial E \cap U$ is analytic.*

Before giving the proof of our Theorem 9.2 we need a couple of preliminary lemmas. The first one is an immediate consequence of the facts that $\mathcal{L}_c$ is self-adjoint with respect to the L^2_e inner product and of the inequality $0 \le \mathcal{L}_c \chi_E \le \frac{1}{\gamma}$ for a measurable set E.

Lemma 9.6. *Let $E, F \subset \mathbb{R}^2$ be two measurable sets with finite weighted measure in Ω_T. Then*

$$\left| \int_{\Omega_T} e^x (\chi_E \mathcal{L}_c \chi_E - \chi_F \mathcal{L}_c \chi_F)\, dz \right| \le \frac{2}{\gamma} \int_{\Omega_T} e^x |\chi_E - \chi_F|\, dz\,. \tag{9.11}$$

Lemma 9.7. *Let $E \subset \mathbb{T}^2_T$ be a minimizer of problem (3.2). For every $R > 0$ there exists a constant $C(R)$, depending only on R, c, σ, T and γ, such that whenever $B_r(z_0) \subset (-R, R) \times \mathbb{R}$ and $0 < r < cT/2$, we have*

$$P(\widehat{E}_c; B_r(z_0)) \le C(R) r\,, \tag{9.12}$$

where $\widehat{E}_c$ is the T-periodic extension of E_c to $\mathbb{R}^2$.

Proof. Let $B_r(z_0) \subset (-R, R) \times \mathbb{R}$, $0 < r < cT/2$. Since $\widehat{E}_c$ is cT-periodic in the y direction, up to translating it in the vertical direction, we may assume that $B_r(z_0) \subset\subset (-R, R) \times (-cT/2, cT2)$. Let $F := \left\{(x, y) \in \mathbb{T}^2_T : (x, cy) \in E_c \cup B_r(z_0)\right\}$. Note that $F_c = E_c \cup B_r(z_0)$. Moreover,

$$\begin{aligned} 0 \le |F|_e - 1 &= \left| \int_{\mathbb{T}^2_T} e^x (\chi_F - \chi_E)\, dz \right| \\ &= \frac{1}{c} \left| \int_{\mathbb{T}^2_{cT}} e^x (\chi_{F_c} - \chi_{E_c})\, dw \right| \le \frac{1}{c} \int_{B_r(z_0)} e^x\, dw \le \frac{\pi}{c} e^R r^2\,. \end{aligned} \tag{9.13}$$

As $|F|_e \ge 1$, there exists $h \ge 0$ such that $|F|_e = e^h$. Setting $\widetilde{F} = F - h e_1$, then $|\widetilde{F}|_e = 1$ and by the minimality of E we have

$$\mathcal{K}_c(E) \le \mathcal{K}_c(\widetilde{F}) = e^{-h} \mathcal{K}_c(F) \le \mathcal{K}_c(F)\,.$$

From this inequality we get, using (9.11) and (9.13),

$$\begin{aligned} \frac{\sqrt{2}}{12c} \mathcal{P}_e(E_c; \mathbb{T}^2_{cT}) &\le \frac{\sqrt{2}}{12c} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) + \frac{\sigma}{2} \int_{\mathbb{T}^2_T} e^x (\chi_F \mathcal{L}_c \chi_F - \chi_E \mathcal{L}_c \chi_E)\, dz \\ &\le \frac{\sqrt{2}}{12c} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) + \frac{\sigma}{\gamma} \int_{\mathbb{T}^2_T} e^x |\chi_E - \chi_F|\, dz \le \frac{\sqrt{2}}{12c} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) + \widetilde{C} r^2\,, \end{aligned}$$

where the constant $\widetilde{C}$ depends only on R, c, γ and σ. Therefore, recalling (2.7), from the above inequality we obtain

$$\int_{\partial^* E_c \cap \mathbb{T}^2_{cT}} e^x \, d\mathcal{H}^1 \leq \int_{\partial^*(E_c \cup B_r(z_0)) \cap \mathbb{T}^2_{cT}} e^x \, d\mathcal{H}^1 + \frac{12c}{\sqrt{2}} \widetilde{C} r^2 .$$

In turn, if $\varrho > r$ is a radius such that $B_\varrho(z_0) \subset\subset (-R, R) \times (-cT/2, cT2)$, this last inequality yields that

$$\begin{aligned}\int_{\partial^* E_c \cap B_\varrho(z_0)} e^x \, d\mathcal{H}^1 &\leq \int_{\partial^*(E_c \cup B_r(z_0)) \cap B_\varrho(z_0)} e^x \, d\mathcal{H}^1 + \frac{12c}{\sqrt{2}} \widetilde{C} r^2 \\ &\leq \int_{\partial^* E_c \cap (B_\varrho(z_0) \setminus \overline{B}_r(z_0))} e^x \, d\mathcal{H}^1 + \int_{\partial B_r(z_0)} e^x \, d\mathcal{H}^1 + \frac{12c}{\sqrt{2}} \widetilde{C} r^2 .\end{aligned}$$

Thus, letting $\varrho \downarrow r$ we obtain

$$\int_{\partial^* E_c \cap B_r(z_0)} e^x \, d\mathcal{H}^1 \leq C(R) r ,$$

that is (9.12). □

With this lemma in hands we can now give the

Proof of Theorem 9.2. Let $E \subset \mathbb{T}^2_T$ be a minimizer of the problem (3.2). We claim that for every $R > 0$ the set $\widehat{E}_c$ is an almost minimizer of the perimeter in $(-R, R) \times \mathbb{R}$. Note that, by Theorem 9.5, this claim implies that $\widehat{E}_c$ is an open set of class $C^{1,\alpha}$ for all $\alpha \in (0, \frac{1}{2})$ and thus that the same is true for E.

To this end we fix a ball $B_r(z_0) \subset (-R, R) \times \mathbb{R}$ with $0 < r < cT/2$ and $z_0 = (x_0, y_0)$. Up to a vertical translation of the set we may assume that $B_r(z_0) \subset\subset (-R, R) \times (-cT/2, cT2)$. We denote by G a set of locally finite perimeter in $\mathbb{R}^2$ such that $\widehat{E}_c \Delta G \subset\subset B_r(z_0)$. Let $F = \{(x, y) \in \mathbb{T}^2_T : (x, cy) \in G\}$, so that $F_c = G \cap (\mathbb{R} \times (-cT/2, cT/2))$. Arguing as in the proof of (9.13) we have

$$\big| |F|_e - 1 \big| \leq \frac{\pi}{c} e^R r^2 .$$

Therefore there exists $h \in \mathbb{R}$ such that $e^h = |F|_e$ and

$$|h| \leq C r^2 , \tag{9.14}$$

for some constant C, depending only on R, c and T. Setting $\widetilde{F} = e^{-h} F$, from the minimality of E we have that $\mathcal{K}_c(E) \leq \mathcal{K}_c(\widetilde{F}_c) = e^{-h} \mathcal{K}_c(F)$. Thus, from Lemma 9.6 and (9.14) we obtain

$$\begin{aligned}\mathcal{P}_e(E_c; \mathbb{T}^2_{cT}) &\leq e^{-h} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) - \frac{6\sigma c}{\sqrt{2}} \int_{\mathbb{T}^2_T} e^x (\chi_E \mathcal{L}_c \chi_E - e^{-h} \chi_F \mathcal{L}_c \chi_F) \, dz \\ &= e^{-h} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) - \frac{6\sigma c}{\sqrt{2}} \int_{\mathbb{T}^2_T} e^x (\chi_E \mathcal{L}_c \chi_E - \chi_F \mathcal{L}_c \chi_F) \, dz - (1 - e^{-h}) \frac{6\sigma c}{\sqrt{2}} \int_{\mathbb{T}^2_T} e^x \chi_F \mathcal{L}_c \chi_F \, dz \\ &\leq e^{-h} \mathcal{P}_e(F_c; \mathbb{T}^2_{cT}) + C r^2 ,\end{aligned}$$

for some constant C depending only on R, c, T, γ and σ. Multiplying both sides of the above inequality by e^h we get

$$e^h\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) \le \mathcal{P}_e(F_c;\mathbb{T}^2_{cT}) + e^h Cr^2\,.$$

Since $\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) \le C(c,T,\gamma,\sigma)$, this last inequality together with (9.14) yields that $\mathcal{P}_e(E_c;\mathbb{T}^2_{cT}) \le \mathcal{P}_e(G;\mathbb{T}^2_{cT}) + Cr^2$ for a possibly larger constant C, and in particular

$$\int_{\partial^* E_c\cap B_r(z_0)} e^x\,d\mathcal{H}^1 \le \int_{\partial^* G\cap B_r(z_0)} e^x\,d\mathcal{H}^1 + Cr^2,$$

from which, since for $(x,y)\in B_r(z_0)$, we have $e^{x_0-r} < e^x < e^{x_0+r}$,

$$e^{x_0-r}P(E_c;B_r(z_0)) \le e^{x_0+r}P(G;B_r(z_0)) + Cr^2\,.$$

Multiplying both sides by e^{-x_0-r} we have $e^{-2r}P(E_c;B_r(z_0)) \le P(G;B_r(z_0)) + e^{-x_0-r}Cr^2$, from which, recalling Lemma 9.7 we finally obtain, still denoting by C a possibly larger constant,

$$\begin{aligned} P(E_c;B_r(z_0)) &\le P(G;B_r(z_0)) + (1-e^{-2r})P(E_c;B_r(z_0)) + Cr^2 \\ &\le P(G;B_r(z_0)) + CrP(E_c;B_r(z_0)) + Cr^2 \le P(G;B_r(z_0)) + \omega r^2\,, \end{aligned}$$

for some constant ω depending only on c, R, T, γ and σ. This proves that $\widehat{E}_c$ is a perimeter almost minimizer in $(-R,R)\times\mathbb{R}$ for all $R>0$ and thus that $\widehat{E}_c$ is of class $C^{1,\alpha}$ for all $\alpha\in\big(0,\frac{1}{2}\big)$. □

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References

[1] E. Acerbi, C.-N. Chen and Y.S. Choi, Minimal lamellar structures in a periodic FitzHugh-Nagumo system. Nonlinear Analysis 194 (2020), 1-13.

[2] E. Acerbi, C.-N. Chen and Y.S. Choi, Stability of lamellar configurations in a nonlocal sharp interface model. SIAM J. Math. Anal. 54 (2022), 558-594.

[3] E. Acerbi, N. Fusco and M. Morini, Minimality via second variation for a nonlocal isoperimetric problem. Comm. Math. Phys. 322 (2013), 515-557.

[4] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variations and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, revised ed. Dover Publication, 1965.

[6] A. Baricz, On a product of modified Bessel functions, Proceeding of AMS, 137 (2009), 189-193.

[7] A. Braides, Local Minimization, Variational Evolution and Γ-convergence. Lecture Notes in Mathematics 2094. Springer, Cham, 2014.

[8] C.-N. Chen, C.-C. Chen and C.-C. Huang, Traveling waves for the FitzHugh-Nagumo system on an infinite channel. J. Differential Equations 261 (2016), 3010-3041.

[9] C.-N. Chen and Y.-S. Choi, Standing pulse solutions to FitzHugh-Nagumo equations. Arch. Rational Mech. Anal. 206 (2012), 741-777.

[10] C.-N. Chen and Y.-S. Choi, Traveling pulse solutions to FitzHugh-Nagumo equations. Calc. Var. Partial Differential Equations 54 (2015), 1-45.

[11] C.-N. Chen and Y.-S. Choi, Front propagation in both directions and co-existence of distinct traveling waves. Calc. Var. Partial Differential Equations, accepted. ArXiv:1807.01832, 2018.

[12] C.-N. Chen, Y.-S. Choi and N. Fusco, The Γ-limit of traveling waves in the FitzHugh-Nagumo system, J. Differential Equations 267 (2019), 1805-1835.

[13] C.-N. Chen, Y.-S. Choi and X. Ren, Bubbles and droplets in a singular limit of the FitzHugh-Nagumo system. Interfaces and Free Boundaries, 20 (2018), 165-210.

[14] C.-N. Chen, Y.-S. Choi, Y. Hu and X. Ren, Higher dimensional bubble profiles in a singular limit of the FitzHugh-Nagumo system. SIAM J. Math. Anal. 50 (2018), 5072-5095.

[15] C.-N. Chen, S.-I. Ei, Y.-P. Lin and S.-Y. Kung, Standing waves joining with Turing patterns in FitzHugh-Nagumo type systems. Comm. Partial Differential Equations 36 (2011), 998-1015.

[16] C.-N. Chen, S.-I. Ei and S.-Y. Tzeng, Heterogeneity-induced effects for pulse dynamics in FitzHugh-Nagumo type systems, Physica D: Nonlinear Phenomena, https://doi.org/10.1016/j.physd.2018.07.001.

[17] C.-N. Chen and X. Hu, Stability criteria for reaction-diffusion systems with skew-gradient structure. Comm. Partial Differential Equations 33 (2008), 189-208.

[18] C.-N. Chen and X. Hu, Stability analysis for standing pulse solutions to FitzHugh-Nagumo equations. Calc. Var. Partial Differential Equations 49 (2014), 827-845.

[19] C.-N. Chen, S.-Y. Kung and Y. Morita, Planar standing wavefronts in the FitzHugh-Nagumo equations. SIAM J. Math. Anal. 46 (2014), 657-690.

[20] C.-N. Chen and É. Séré, Multiple front standing waves in the FitzHugh-Nagumo equations. J. Differential Equations 302 (2021), 895-925.

[21] C.-N. Chen and K. Tanaka, A variational approach for standing waves of FitzHugh-Nagumo type systems. J. Differential Equations 257 (2014), 109-144.

[22] Y.-S. Choi and J. Lee, Existence of standing pulse solutions to a skew-gradient system. J. Differential Equations 302 (2021), 185-221.

[23] R. Choksi R. and P. Sternberg, On the first and second variations of a nonlocal isoperimetric problem. J. reine angew. Math. 611 (2007), 75-108.

[24] A. Doelman, P. van Heijster and T.J. Kaper, Pulse dynamics in a three-component system: existence analysis. J. Dynam. Differential Equations 21 (2008), 73-115.

[25] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[26] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane. Biophys. J. 1 (1961), 445-466.

[27] D. Gilbarg and N.S. Trudinger. Elliptic Partial Differential Equations of Second Order. Springer, Berlin, 1977.

[28] S. Heinze, A variational approach to traveling waves. Preprint 85, Max Planck Institute for Mathematics in Sciences, 2001.

[29] R. Kohn and P. Sternberg, Local minimizers and singular perturbations. Proc. Roy. Soc. Edinburgh Sect. A 111 (1989) 69-84.

[30] M. Lucia, C. Muratov and M. Novaga, Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium. Comm. Pure and Appl. Math. 57 (2004), 616-636.

[31] F. Maggi, Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory. Cambridge Studies in Advanced Mathematics, 135. Cambridge University Press, Cambridge, 2012.

[32] L. Modica, The gradient theory of phase transitions and the minimal interface criterion. Arch. Rat. Mech. Anal. 98 (1987),123-142.

[33] J. Nagumo, S. Arimoto, and S. Yoshizawa, An active pulse transmission line simulating nerve axon. Proc. IRE 50 (1962), 2061-2070.

[34] Y. Nishiura, T. Teramoto, X. Yuan and K.-I. Udea, Dynamics of traveling pulses in heterogeneous media. Chaos 17, 037104 (2007).

[35] C. Reinecke and G. Sweers, A positive solution on $\mathbb{R}^n$ to a equations of FitzHugh-Nagumo type. J. Differential Equations 153 (1999), 292-312.

[36] X. Ren and J. Wei, Single Droplet Pattern in the cylindrical phase of diblock copolymer morphology, .J. Nonlinear Sci, 17 (2007), 471-503.

[37] X. Ren and J. Wei, Nucleation in the FitzHugh-Nagumo system: Interface-spike solutions. J. Differential Equations 209 (2005), 266-301.

[38] M. Renardy and R. Rogers, An Introduction to Partial Differential Equations, 2nd edition, Springer-Verlag, 2004.

[39] L. Simon, Introduction to Geometric Measure Theory, 2014.

[40] C. Rosales, C. Antonio, B. Vincent and F. Morgan, On the isoperimetric problem in Euclidean space with density, Calculus of Variations and PDE 31 (2008), 27-46.

[41] I. Tamanini, Regularity results for almost minimal oriented hypersurfaces in $\mathbb{R}^n$. Quaderni del Dipartimento di Matematica dell Università di Lecce 1 (1984), 1-92. Available at http://siba-ese.unisalento.it/index.php/quadmat/issue/view/1073

[42] A.M. Turing, The chemical basis of morphogenesis. Phil. Trans. R. Soc. Lond. B 237 (1952), 37-72.

[43] P. van Heijster, C.-N. Chen, Y. Nishiura and T. Teramoto, Localized patterns in a three-component FitzHugh-Nagumo model revisited via an action functional. J. Dyn. Differ. Equ. 30 (2018), 521-555.

[44] P. van Heijster, C.-N. Chen, Y. Nishiura and T. Teramoto, Pinned solutions in a heterogeneous three-component FitzHugh-Nagumo model. J. Dyn. Differ. Equ. 31 (2019), 153-203.

[45] J. Wei and M. Winter, Clustered spots in the FitzHugh-Nagumo system. J. Differential Equations 213 (2005), 121-145.

[46] E. Yanagida, Standing pulse solutions in reaction-diffusion systems with skew-gradient structure. J. Dynam. Differential Equations 14 (2002), 189-205.