The tapering length of needles in martensite/martensite macrotwins

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Abstract

We study needle formation at martensite/martensite macro interfaces in shape-memory alloys. We characterize the scaling of the energy in terms of the needle tapering length and the transformation strain, both in geometrically linear and in finite elasticity. We find that linearized elasticity is unable to predict the value of the tapering length, as the energy tends to zero with needle length tending to infinity. Finite elasticity shows that the optimal tapering length is inversely proportional to the order parameter, in agreement with previous numerical simulations. The upper bound in the scaling law is obtained by explicit constructions. The lower bound is obtained using rigidity arguments, and as an important intermediate step we show that the Friesecke-James-Müller geometric rigidity estimate holds with a uniform constant for uniformly Lipschitz domains.

Contents

1   Introduction

Shape-memory materials are special alloys that undergo a diffusionless solid-solid phase transformation upon a change of temperature or stress. During nucleation, complex microstructures emerge, and these microscopic patterns seem to be closely linked to macroscopic properties of the materials [Bha92, BJ05, Jam19]. The patterns are usually modeled in the framework of the phenomenological theory of martensite [BJ87], based on finite or linearized elasticity. The linearized theory is widely used and has been proven to arise naturally as Γ-limit of the nonlinear theory for small displacements [DNP02]. While the linearized theory often provides a good approximation to the physical (nonlinear) setting, several mathematical results indicate that in various situations the linear theory can lead to qualitatively different predictions, see e.g. [Bha92, Bha93, DM95, CK11, BKS19, CDPR$^+$20].

During nucleation of martensite in an austenitic matrix, various interfaces between austenite and martensite or between different martensitic variants are formed. We focus on interfaces between different regions of laminated martensites, so called martensite/martensite macrotwins. At such interfaces, one often observes needle-type microstructures, i.e., laminates where the minority variant drops out at the interface (see Figure 1 and [BSK01]). We point out that in many experiments, the needles show a more complex topological structure, and in particular branch close to the interface. This leads to different mathematical challenges [KM92, KM94, CDKZ21], but we will focus here on simple needle structures as sketched in Figure 2.

The mathematical treatment of such structures within the framework of the phenomenological theory started with the work in [BSK01, BS02, SBKB01, SBP$^+$02]. In [BSK01], the authors used a static variational model based on linearized elasticity and were able to predict the bending angle at which the needle meets the interface, in terms of the measured tapering length of the needle. We point out that the authors here did not intend to make predictions on the tapering length from the theory. Related problems of optimal needle shapes near martensite/martensite or martensite/austenite interfaces have since then been studied extensively, both in the analytical and the numerical literature, see e.g. [Sal91, BS02, Li01, SBKB01, SBP$^+$02, LRP13, LR15, BO09, FLBGS10, SMF19, CLL$^+$20]. In many numerical simulations, it has been observed that numerical schemes with geometrically linearized elastic energies are unstable or do not reproduce the experimental results while geometrically nonlinear models appear more appropriate.
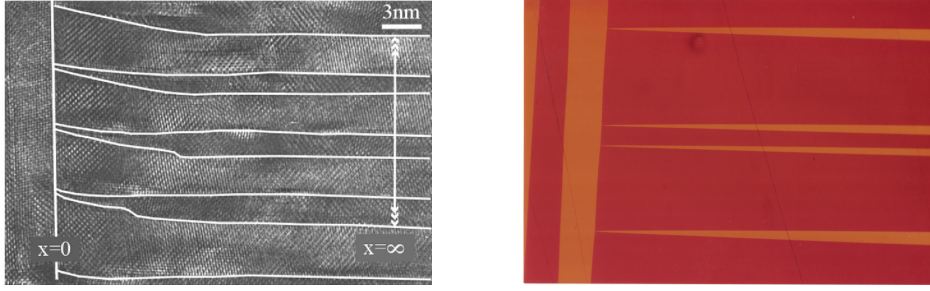
Figure 1: Left: Experimental (high-resolution transmission electron microscopy) image of needles in the $Ni_{65}Al_{35}$ shape-memory alloy, from Boullay, Schryvers and Kohn [BSK01, Fig. 5]. Right: Experimental (optical microscopy with polarized light) image of needles in the $Cu_{14}Al_{3.9}Ni$ shape-memory alloy, courtesy of Chu and James [Chu93, CJ95]. The two pictures show a similar phenomenon at very different length scales, of the order of a few nanometers on the left, and of a fraction of a millimeter on the right.

We follow here the recent approach from the numerical study in [CLL$^+$20]. We aim at a better understanding of length scale of such needle structures, in particular the tapering length. To determine this length in terms of the material parameters, a shape optimization problem for the parametrized needle shape is considered.

Let us finally comment on some simplifications that have been proposed in the literature. The most popular simplification is the linearization of the elastic energy (see, e.g., [BSK01]). In this note, we will underline the numerical findings from [CLL$^+$20] that the linearized theory is not appropriate to determine the tapering length. More precisely, we show (see Theorem 3.1) that for any energy-minimizing sequence in the geometrically linearized setting, the tapering lengths tend to infinity. This is in accordance with several other numerical findings in which it was observed that the geometrically linear model does not yield the expected results (see [BO09, FLBGS10, CLL$^+$20]).

In [ZJM09, Zwi14], a related problem of needle-type microstructures near austenite/martensite interfaces has been investigated. There, the situation is simplified by assuming constant gradient in the very thin needles. Although one does not expect a large contribution to the elastic energy from these small domains, our analysis here indicates that the optimal energy scaling is not preserved under this simplification. Indeed, a significant effect seems to come from rotations at rather large angles (comparable to the shear in the variants).

Let us start with a brief qualitative explanation of the relevant effects. We work in two dimensions, and denote the eigenstrains of the two martensitic variants by

$$A_\delta = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}, \qquad B_\delta = \begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

We start from the large-scale picture, as summarized in Figure 2 and 3. On the far right there
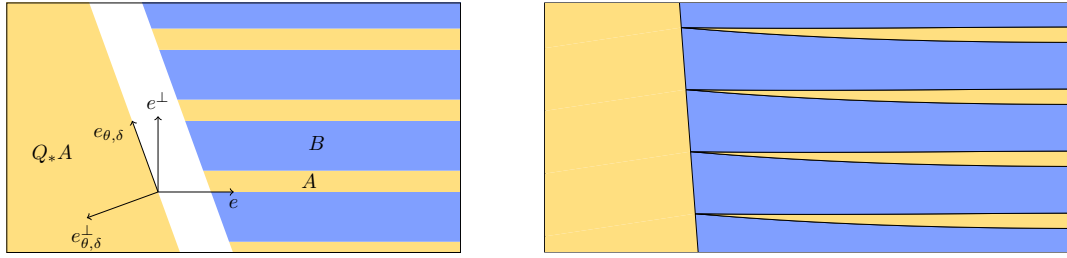
Figure 2: Sketch of the geometry. The right picture is based on data from the numerical simulation in [CLL$^+$20].
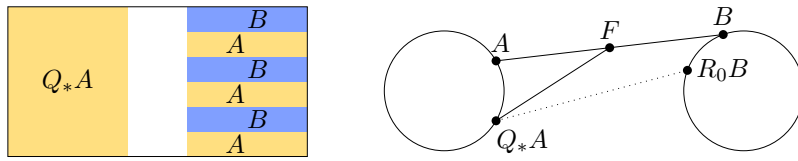

Figure 3: Sketch of the geometry around a martensite / twinned martensite interface. This geometry is often called a macrotwin.

is a laminate between A_δ and B_δ, with volume fraction $\theta \in (0,1)$; its average deformation is $F := \theta A_\delta + (1-\theta) B_\delta$. On the left of the macro interface we have only variant A_δ, but with a different rotation $Q_* \in \mathrm{SO}(2)$. Compatibility of the macro interface implies that $Q_* A_\delta - F$ is rank-one, and the orientation $e_{\theta,\delta}$ of the macro interface is characterized by the condition $(Q_* A_\delta - F) e_{\theta,\delta} = 0$. This fixes both $Q_* \in \mathrm{SO}(2)$ and $e_{\theta,\delta}$ as functions of θ and δ, details are given in Lemma 2.1 below (see also Figure 2). As $Q_* \neq \mathrm{Id}$, the two regions in the A_δ variant are not rank-one compatible, $\mathrm{rank}(A_\delta - Q_* A_\delta) = 2$.

In the geometry of Figure 2, needles are structures by which the volume fraction of the A_δ phase varies from 0 close to the macro interface to the asymptotic value θ at large x_1. For this qualitative discussion we use non-orthogonal coordinates and assume that $x_1 = 0$ corresponds to the macro interface. The entire construction is affine-periodic in the direction of the macro interface, with a periodicity condition given by the macroscopic deformation on the left, $u(x + e_{\theta,\delta}) = Q_* A_\delta e_{\theta,\delta} = F e_{\theta,\delta}$. For definiteness, let us assume that the period is 1, and denote by $\ell > 0$ the characteristic length scale in the x_1 direction. For any x_1, let $a(x_1)$ be the volume fraction (averaged over one period) of the A_δ phase at given x_1; correspondingly $b(x_1)$ for the B_δ phase. Obviously, $a(x_1) + b(x_1) = 1$ for all x_1, and by the boundary conditions $a(0) = 0$, whereas $a(x_1) \sim \theta$ and $b(x_1) \sim (1-\theta)$ for $x_1 \gg \ell$.

If no rotation is present, and the energy is exactly zero, necessarily $\partial_2 u_1 = \delta$ in phase A_δ, and $-\delta$ in phase B_δ. The vertical average of $\partial_2 u_1$ over one period is then $(a - b)\delta$, which matches the periodicity requirement only if $a = \theta$ and $b = 1 - \theta$. We next include infinitesimal rotations in the picture, assuming that the rotation angle depends on x_1 but not

on x_2. In particular, if $\beta(x_1)$ is the average lattice rotation angle at given x_1, then one has $\partial_1 u_2 = \delta + \beta$ in the A_δ and $\partial_1 u_2 = -\delta + \beta$ in the B_δ phase, with $\partial_1 u_2 = -\beta$ everywhere. Equating the vertical average to the one required by the periodicity leads to

$$(a(x_1) - b(x_1))\delta + \beta(x_1) = (2\theta - 1)\delta. \tag{1.2}$$

This relation between tapering profile and rotation was first studied in [BSK01], it permits in particular to express β in terms of a and b. Treating the individual layers as elastic plates, we see that the change in lattice rotation $\beta'(x_1)$ generates a bending energy depending on the curvature, and proportional to

$$\int_0^\ell (\frac{d}{dx_1}\beta(x_1))^2 dx_1. \tag{1.3}$$

Inserting the previous expression for β leads to minimizing

$$\delta^2 \int_0^\ell (a' - b')^2 dx_1, \tag{1.4}$$

which yields naturally $a(x_1) = 1 - b(x_1) \simeq \theta x_1/\ell$ for $x_1 \in (0, \ell)$. Therefore the total energy is proportional to $\delta^2\theta^2/\ell$, and the optimal value for ℓ is ∞ (see Theorem 3.1 below for a precise statement). The geometrically linear model is unable to predict a finite length scale. With finite elasticity a new term enters the picture. Indeed, in the matrix product

$$R_\beta A_\delta = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\beta & \delta\cos\beta + \sin\beta \\ -\sin\beta & \cos\beta - \delta\sin\beta \end{pmatrix} \tag{1.5}$$

(and the same with B_δ) there are more nontrivial entries. The $\partial_2 u_1$ term, similarly to (1.2) above, prescribes $\beta(x_1)$ in terms of $a(x_1) - b(x_1)$; the $\partial_1 u_1$ and $\partial_1 u_2$ terms are not important for the same reason as above. However, periodicity requires $\partial_2 u_2$ to have average 1, so that $|\partial_2 u_2 - (R_\beta A_\delta)_{22}| \sim \frac{1}{2}\beta^2 + \delta\beta$. The total energy density is then of order $\beta'^2 + \beta^2\delta^2$, and balancing terms one obtains that the characteristic length scale is $\ell \sim 1/\delta$. We refer to Proposition 4.3 below for a precise construction. As in (1.2) we have $\beta \sim \delta\theta$, and therefore this results in a total energy scaling as $\theta^2\delta^3$, as specified in Theorem 4.2 below.

These results are made precise below. We first formulate a precise mathematical model for the needle geometry following [ZJM09, CLL$^+$20], with boundary conditions that fix the long-range structure and the topology but leave the shape of the domain boundaries free. The key assumptions are affine-periodic boundary conditions along the macro interface, a boundary data corresponding to a laminate at large x_1, and the fact that the phase boundaries are Lipschitz functions. We then provide, separately in the linear and in the nonlinear case, explicit upper-bound constructions which make the above arguments rigorous. We finally show that these constructions are, up to universal factors, optimal, by providing corresponding lower bounds on the energy. This involves minimizing not only over the elastic deformation, but also over the phase interfaces, and hence over the shape of the domains of

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the different phases. We present in Section 2 the model, in Section 3 the linear results, and in Section 4 the nonlinear results.

One important difficulty in proving the lower bound is that one has to deal with Sobolev functions on varying domains. Our argument in particular uses a trace theorem, a Poincaré inequality, and a geometric rigidity inequality with constants which are uniform for uniformly Lipschitz domains. Whereas the first two are already present in the literature, the geometric rigidity estimate has, to the best of our knowledge, up to now only been proven with domain-dependent constants, even if uniformity of the constant for certain classes of domains has been formulated in the literature without explicit proof (for example, [BCDM02, Prop. 1]), and used in the study of microstructures near austenite/martensite interfaces (see e.g. [ZJM09, Zwi14]). We provide in Section 5 a full proof of the uniform geometric rigidity inequality for a general class of domains, which we believe to be of independent interest. The key ingredient is a uniform weighted Poincaré inequality, which also permits to easily obtain as byproducts uniform trace and Poincaré estimates.

2 Kinematics and reduction to a 2D problem

We describe the crystallographic situation under consideration (see also [BSK01]) building on the crystallographic theory of martensite (see [BJ87]). We consider a macrotwin using two martensitic variants, given by wells $SO(3)U_1$ and $SO(3)U_2$. We make the standard assumption that the transformation stretch matrices are positive definite with $\det U_1 = \det U_2$ and nontrivial in the sense that $U_1 \notin SO(3)U_2$. If laminates with gradients in the two wells are possible, then the wells are rank-one connected, i.e., there exist $\tilde{R} \in SO(3)$, and non-zero vectors $a, n \in \mathbb{R}^3$ such that

$$\tilde{R}U_2 - U_1 = \tilde{a} \otimes \tilde{n}.$$

It is shown in [BJ92] that under these assumptions, we may without loss of generality (up to a linear change of variables) restrict to matrices of the form

$$A_\delta := \mathrm{Id} + \delta e \otimes e^\perp, \qquad \text{and} \qquad B_\delta := \mathrm{Id} - \delta e \otimes e^\perp \qquad (2.1)$$

for some orthogonal unit vectors $e, e^\perp \in \mathbb{R}^3$ with $\delta > 0$. Here, $\delta > 0$ measures the shear, and we will focus on the case of small and moderate strains, $|\delta| \leq 1$, and in particular on the limit $\delta \to 0$. While we consider the three-dimensional setting, it turns out that in our analysis, we may restrict to a two-dimensional simplification since, as shown in [BJ91], the optimal strains are plane strains, cf. Remark 2.2(iii). Specifically, we shall work in the plane spanned by e and $e^\perp$, and denote by $f \in \mathbb{R}^3$ a unit vector perpendicular to that plane. Following [BJ87], the normals to the laminate and macrotwin interfaces can be determined from the crystallographic theory of martensite. For that, we collect the necessary linear algebra results in this section.

2.1 Nonlinear elasticity

In the setting of nonlinear elasticity, the direction of the macro interface is not orthogonal to the one of the laminate, as illustrated in Figure 2. We denote the orientation of the macro interface (in the plane orthogonal to f) by

$$e_{\delta,\theta} := \frac{1}{\sqrt{1+(\delta\theta)^2}}(e^\perp - \delta\theta e).\tag{2.2}$$

This expression arises as the unique (up to a sign) nontrivial solution of the rank-one compatibility condition between A_δ and the weighted average of $\theta A_\delta + (1-\theta)B_\delta$. We summarize the relevant algebraic conditions in the next Lemma. All assertions can be checked by direct computation, see for example [Bha03].

Lemma 2.1 *Let $(e, e^\perp, f)$ be an orthonormal basis of $\mathbb{R}^3$, $\delta \in \mathbb{R}\backslash\{0\}$, $\theta \in (0,1)$, and let A_δ and B_δ be the matrices given in (2.1). Then there holds:*

(i) $A_\delta - B_\delta = 2\delta e \otimes e^\perp$.

(ii) The equation

$$QA_\delta - (\theta A_\delta + (1-\theta)B_\delta) = b \otimes n, \qquad Q \in \mathrm{SO}(3), b \in \mathbb{R}^3, n \in S^2 \tag{2.3}$$

has four solutions. Two of them are $(\mathrm{Id}, \pm(1-\theta)2\delta e, \pm e^\perp)$, the other two have the form $(Q_*, \pm b_*, \pm e_{\delta,\theta})$, where $b_* \in \mathbb{R}^3$,

$$e_{\delta,\theta}^\perp := -\frac{e + \delta\theta e^\perp}{\sqrt{1+(\delta\theta)^2}}\tag{2.4}$$

and

$$
\begin{aligned}
Q_* :=& \frac{1-\delta^2(1-\theta)^2}{1+\delta^2(1-\theta)^2}\left(e \otimes e + e^\perp \otimes e^\perp\right)\\
&+ \frac{2\delta(1-\theta)}{1+\delta^2(1-\theta)^2}\left(e^\perp \otimes e - e \otimes e^\perp\right) + f \otimes f.
\end{aligned}\tag{2.5}
$$

In particular,

$$Q_* A_\delta e_{\delta,\theta} = (\theta A_\delta + (1-\theta)B_\delta)e_{\delta,\theta}.\tag{2.6}$$

(iii) There is a unique $R_0 \in \mathrm{SO}(3)$ such that

$$Q_* A_\delta - R_0 B_\delta = a \otimes e \qquad \text{for some} \quad a \in \mathbb{R}^3.\tag{2.7}$$

The rotation R_0 is given by

$$
\begin{aligned}
R_0 &= Q_* \left(\frac{1 - \delta^2}{1 + \delta^2} \left(e \otimes e + e^\perp \otimes e^\perp \right) + \frac{2\delta}{1 + \delta^2} \left(e \otimes e^\perp - e^\perp \otimes e \right) + f \otimes f \right) \\
&= \frac{1 + \delta^4 (1 - \theta)^2 - \delta^2 (-2 + 2\theta + \theta^2)}{(1 + \delta^2)(1 + \delta^2 (1 - \theta)^2)} (e \otimes e + e^\perp \otimes e^\perp) + \\
&\quad + \frac{2\delta(1 + \delta^2 (1 - \theta))\theta}{(1 + \delta^2)(1 + \delta^2 (1 - \theta)^2)} (e \otimes e^\perp - e^\perp \otimes e) + f \otimes f
\end{aligned}
\tag{2.8}
$$

(iv) All matrices act trivially on f, i.e.,

$$
A_\delta f = B_\delta f = Q_* f = R_0 f = f.
\tag{2.9}
$$

We shall in Section 4 consider a macrotwin with normal $e_{\delta,\theta}^\perp$, and twin plane normal $e^\perp$ deep in the laminate on the right hand side of the macrotwin, see Figure 3.

Remark 2.2   *(i) It follows from (2.2) that $e_{\delta,\theta} \to e^\perp$ as $\delta \to 0$, but $e_{\delta,\theta} \neq e^\perp$ for finite δ.*

(ii) If $|\delta|$ and θ are small, then the rotation Q_* as given in (2.5) is a rotation by roughly $2\delta(1 - \theta)$, while R_0 is a rotation by roughly $-2\delta\theta$. In particular, both rotations are of order δ.

(iii) Motivated by assertion (iv) of Lemma 2.1, we make the following simplification: We consider only deformations u satisfying $\nabla u f = f$, and, slightly abusing notation, we identify the matrices introduced above with their restrictions to the plane spanned by e and $e^\perp$.

2.2   Linearization

We now turn to the geometrically linearized setting. Precisely, for the small strain case $|\delta| \ll 1$, we linearize around the identity and define the strain matrices

$$
A^{\text{lin}} := \frac{A_\delta - \text{Id}}{\delta} = e \otimes e^\perp, \quad B^{\text{lin}} := \frac{B_\delta - \text{Id}}{\delta} = -e \otimes e^\perp.
\tag{2.10}
$$

We denote by $\xi_{\text{sym}} := \frac{1}{2}(\xi + \xi^T)$ the symmetric part of a matrix, so that

$$
A_{\text{sym}}^{\text{lin}} = \frac{1}{2}(e \otimes e^\perp + e^\perp \otimes e) \quad \text{and} \quad B_{\text{sym}}^{\text{lin}} = -\frac{1}{2}(e \otimes e^\perp + e^\perp \otimes e).
\tag{2.11}
$$

We note the geometrically linearized compatibility properties.

Lemma 2.3 *Suppose that $\theta \in (0, 1)$, and let $A_{\text{sym}}^{\text{lin}}$ and $B_{\text{sym}}^{\text{lin}}$ be given by (2.11). Then there holds:*

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(i) $A_{\text{sym}}^{\text{lin}} - B_{\text{sym}}^{\text{lin}} = e \otimes e^\perp + e^\perp \otimes e.$

(ii) $\theta A_{\text{sym}}^{\text{lin}} + (1 - \theta) B_{\text{sym}}^{\text{lin}} - A_{\text{sym}}^{\text{lin}} = (\theta - 1)(e \otimes e^\perp + e^\perp \otimes e).$

(iii) The equation $(A_{\text{sym}}^{\text{lin}} - B_{\text{sym}}^{\text{lin}} - S)v = 0$ has a solution $S \in \mathbb{R}_{\text{skw}}^{2 \times 2}$ if and only if v is parallel to either e or $e^\perp$.

Proof: The proof follows from a direct computation. $\qquad\square$

In particular, in the geometrically linearized setting, both, the macrotwin and the laminates can have normals e or $e^\perp$. The main difference to the geometrically nonlinear setting is that the compatibility plane between the two variants is aligned with the compatibility plane of the macrotwin.

3 Energy scaling in the geometrically linearized setting

Following [BSK01], we first consider the geometrically linearized setting. With the strain matrices $A_{\text{sym}}^{\text{lin}}$ and $B_{\text{sym}}^{\text{lin}}$ as defined in (2.11), we are led to study the geometrically linearized shape optimization problem involving the displacement. We consider a periodic cell of a macrotwin using linearized kinematics (see, e.g., [Bha92]) and briefly recall the setting. By Lemma 2.3(i), there are two possible directions for $A_{\text{sym}}^{\text{lin}}/B_{\text{sym}}^{\text{lin}}$ laminates, given by the normals e and $e^\perp$. Deep in the twinned region on the right-hand side of the macrotwin, we choose the twin planes to be parallel to e, and the macrotwin plane parallel to $e^\perp$. Since we work in an orthogonal coordinate system, for the ease of notation, we set $e_1 := e$ and $e_2 := e^\perp$, and for $x \in \text{span}\{e, e^\perp\} = \mathbb{R}^2$, we use the notation

$$x_1 := x \cdot e_1 = x \cdot e, \qquad x_2 := x \cdot e_2 = x \cdot e^\perp, \qquad \text{and} \qquad x = (x_1, x_2).$$

We describe the needle by the two confining curves $f^\pm : [0, \infty) \to \mathbb{R}$ which we assume to be measurable and satisfy

$$f^- \leq f^+ \leq f^- + 1 \quad \text{and} \quad f^+(0) = f^-(0) = 0, \tag{3.1}$$

see Figure 4. Note that we do not impose any regularity assumptions on $f^\pm$ but to get closer to the experimental results, we could also impose a length or Lipschitz condition without changing the results, see Remark 3.2(i).

We assume that the displacement $v \in W_{\text{loc}}^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$ obeys the periodicity condition

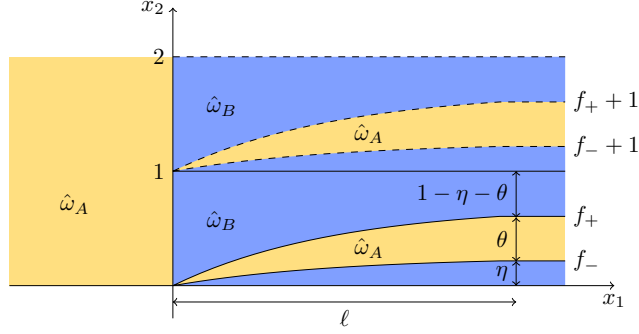$$v(x + e_2) - v(x) = \left(\theta A^{\text{lin}} + (1 - \theta) B^{\text{lin}}\right) e_2 = (-1 + 2\theta)e_1, \tag{3.2}$$

Figure 4: Sketch of the parametrization of the domain in the geometrically linear setting.

and we consider the two sets

$$\hat{\omega}_A := \{x \in \mathbb{R}^2 : x_1 \leq 0\} \cup \left\{ x \in \mathbb{R}^2 : x_1 > 0, x_2 \in \bigcup_{k \in \mathbb{Z}} \left[k + f^-(x_1), k + f^+(x_1) \right) \right\}$$
(3.3)

and its complement,

$$\hat{\omega}_B := \left\{ x \in \mathbb{R}^2 : x_1 > 0, x_2 \in \bigcup_{k \in \mathbb{Z}} \left[k + f^+(x_1), k + 1 + f^-(x_1) \right) \right\}.$$
(3.4)

We further assume that there is $\ell > 0$ such that for $x_1 \geq \ell$ the deformation coincides with a simple laminate, in the sense that there is $\eta \in \mathbb{R}$ such that

$$f^-(x_1) = \eta \quad \text{and} \quad f^+(x_1) = \eta + \theta \quad \text{for } x_1 \geq \ell,$$
(3.5)

and

$$\nabla v(x) = \begin{cases} A^{\text{lin}}, & \text{if } x \in (\ell, \infty) \times \cup_{k \in \mathbb{Z}}(k + \eta, k + \eta + \theta), \\ B^{\text{lin}}, & \text{if } x \in (\ell, \infty) \times \cup_{k \in \mathbb{Z}}(k + \eta + \theta, k + 1 + \eta). \end{cases}$$
(3.6)

These conditions characterize the class of admissible configurations

$$\mathcal{A}_{\text{lin}}^{(\ell)} := \left\{ (f^\pm, v) : \exists \eta \in \mathbb{R} \text{ s.t. } f^\pm \text{ satisfy (3.1) and (3.5)}, v \text{ satisfies (3.2) and (3.6)} \right\}. (3.7)$$

As the experimental results indicate that ℓ is large (see Figure 1), we restrict ourselves to the case $\ell \geq 1$. By periodicity, for the computation of the energy it suffices to integrate over one period, and therefore to consider

$$\omega_{A,B} := \hat{\omega}_{A,B} \cap (\mathbb{R} \times (0,1))$$
(3.8)

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(alternatively, one could take for $x_1 > 0$ only the $k = 0$ contribution in (3.3) and (3.4), by periodicity the two choices are equivalent). The elastic shape optimization problem is then to minimize

$$\mathcal{E}_{\text{lin}}[f^{\pm}, v] \quad := \quad \int_{\omega_A} \frac{1}{2} \mathbb{C}(e(v) - A_{\text{sym}}^{\text{lin}}) \cdot (e(v) - A_{\text{sym}}^{\text{lin}}) dx \tag{3.9}$$
$$+ \int_{\omega_B} \frac{1}{2} \mathbb{C}(e(v) - B_{\text{sym}}^{\text{lin}}) \cdot (e(v) - B_{\text{sym}}^{\text{lin}}) dx$$

over $\mathcal{A}_{\text{lin}}^{(\ell)}$. Note that we minimize with respect to both the configuration given by $f^{\pm}$ and the displacement v. Here we denote the symmetric part of the gradient by $e(v) := \frac{1}{2}(\nabla v + \nabla^T v)$, and $\mathbb{C}$ represents the elastic modulus which satisfies the standard boundedness and coercivity properties, i.e. $\mathbb{C}(\xi - \xi^T) = 0$ for all ξ and there exists $\alpha > 0$ such that

$$\alpha |\xi_{\text{sym}}|^2 \leq \mathbb{C}\xi \cdot \xi \leq \frac{1}{\alpha} |\xi_{\text{sym}}|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^{2 \times 2}. \tag{3.10}$$

The energy functional (3.9) does not contain any interfacial energy term penalizing the lengths of the interfaces parametrized by $f^{\pm}$. Typically such terms are necessary to identify the appropriate length scale on which the twin structures form. In our setting of a periodic cell, however, it is of higher order as long as the curves are sufficiently regular.
We find the following scaling law for the minimal energy.

Theorem 3.1 *There is a constant $c > 0$ such that for all $\theta \in (0, 1/2]$ and all $\ell \geq 1$, we have*

$$\frac{1}{c} \frac{\theta^2}{\ell} \leq \inf \left\{ \mathcal{E}_{\text{lin}}[f^{\pm}, v] : (f^{\pm}, v) \in \mathcal{A}_{\text{lin}}^{(\ell)} \right\} \leq c \frac{\theta^2}{\ell}.$$

Remark 3.2 *(i) We derive the scaling law for the minimal energy without regularity assumptions on $f^{\pm}$ but the upper bound uses only a Lipschitz profile with Lipschitz constant bounded by one.*

(ii) If on the right boundary we impose boundary conditions only on $f^{\pm}$ (see (3.1)) but not on the deformation (see (3.6)) then the infimum of the energy is zero. The reason for that is that in the linearized setting we can have a strain-free $A_{\text{sym}}^{\text{lin}}/B_{\text{sym}}^{\text{lin}}$ interface along the macrotwin plane $\{x_1 = 0\}$ (a fact that will also be used in the proof of the upper bound below). Precisely, consider for $n \in \mathbb{N}$ the configuration

$$f_n^- \equiv 0, \qquad f_n^+(x_1) = \begin{cases} 0, & \text{if } x_1 \leq \ell - \frac{1}{n}, \\ \theta n x_1 + \theta(1 - n\ell), & \text{if } \ell - \frac{1}{n} < x_1 \leq \ell, \\ \theta, & \text{if } x_1 > \ell, \end{cases}$$

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with displacement

$$v_n(x) = (2\theta - 1)x_2 e_1 + \begin{cases} 2(1-\theta)x_1 e_2, & \text{if } x_1 \leq 0, \\ -2\theta x_1 e_2, & \text{if } x_1 > 0. \end{cases}$$

Then $f_n^{\pm}$ satisfy (3.1) and (3.5) with $\eta = 0$, and v_n satisfies the periodicity condition (3.2). Further, $v \in W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ with

$$e(v_n) = \begin{cases} A_{\mathrm{sym}}^{\mathrm{lin}}, & \text{if } x_1 < 0, \\ B_{\mathrm{sym}}^{\mathrm{lin}}, & \text{if } x_1 > 0, \end{cases}$$

and therefore,

$$0 \leq \mathcal{E}_{\mathrm{lin}}[f_n^{\pm}, v_n] \leq \int_{\ell-1/n}^{\ell} \int_0^{f_n^+(x_1)} \frac{1}{\alpha} \left| A_{\mathrm{sym}}^{\mathrm{lin}} - B_{\mathrm{sym}}^{\mathrm{lin}} \right|^2 dx_2\, dx_1 \leq \frac{\theta}{\alpha n},$$

which implies that $\lim_{n \to \infty} \mathcal{E}_{\mathit{lin}}[f_n^{\pm}, v_n] = 0$.

(iii) *The scaling law in Theorem 3.1 implies that the tapering length of needles is not determined by linearized elasticity. Precisely, setting $\ell = \infty$, the infimum of the energy equals zero, which implies that the optimal tapering length of the needle in this linearized setting is infinite, contradicting the experimental findings.*

(iv) *If one interprets the linearized energy as an approximation to the nonlinear energy (cf. (2.10)), the resulting energy scaling is $\delta^2 \theta^2 / \ell$.*

Proof: *Upper bound.* To prove the upper bound, i.e., the second inequality in the assertion, we use a special case of the construction from [BSK01, Figure 4], which makes precise the sketch discussed in the introduction (see (1.2)–(1.4)). We set

$$f^-(x_1) := 0, \quad \eta := 0, \quad \text{and} \quad f^+(x_1) := \begin{cases} \frac{\theta}{\ell} x_1, & \text{if } x_1 \in [0, \ell), \\ \theta, & \text{if } x_1 \geq \ell. \end{cases} \tag{3.11}$$

Then $f^{\pm}$ satisfy (3.1) and (3.5). We first describe the associated displacement $v = (v_1, v_2)$ for $x_1 \geq 0$. Precisely, we set

$$v_1(x) := \begin{cases} x_2\left(1 - 2\theta\frac{x_1}{\ell} + 2\theta\right) - 2\theta\frac{x_1}{\ell} + 2\theta, & \text{if } 0 \leq x_1 \leq \ell \text{ and } 0 \leq x_2 \leq \frac{\theta}{\ell}x_1, \\ -\left(2\theta\left(\frac{x_1}{\ell} - 1\right)x_2 + x_2\right) + 2\theta, & \text{if } 0 \leq x_1 \leq \ell \text{ and } \frac{\theta x_1}{\ell} < x_2 \leq 1, \end{cases} \tag{3.12}$$

and

$$v_2(x) := \frac{\theta}{\ell}x_1^2 - 2\theta x_1 + \theta\ell \qquad \text{for } 0 \leq x_1 \leq \ell. \tag{3.13}$$

We then extend the displacement constantly in x_1, i.e., we set

$$v(x_1, x_2) = v(\ell, x_2) \quad \text{for } x_1 > \ell.$$

As $v(x_1, 1) - v(x_1, 0) = (2\theta - 1)e_1$ we can extend it affine-periodic to $\mathbb{R}^2$ using (3.2). Then $v \in W^{1,2}_{\text{loc}}((0, \infty) \times \mathbb{R}; \mathbb{R}^2)$ satisfies (3.2) and (3.6). For the gradients, we have inside the needle, i.e., for $0 \leq x_1 \leq \ell$ and $0 \leq x_2 \leq \theta x_1/\ell$

$$\nabla v(x) = -\frac{2\theta}{\ell}(x_2 + 1)e_1 \otimes e_1 + (1 - \frac{2\theta x_1}{\ell} + 2\theta)e_1 \otimes e_2 + \left(\frac{2\theta x_1}{\ell} - 2\theta\right)e_2 \otimes e_1$$

and thus

$$e(v)(x) = A^{\text{lin}}_{\text{sym}} - \frac{2\theta}{\ell}(x_2 + 1)e_1 \otimes e_1. \tag{3.14}$$

Outside the needle for $0 \leq x_1 \leq \ell$ and $\theta x_1/\ell \leq x_2 \leq 1$, we have

$$\nabla v(x) = -\frac{2\theta x_2}{\ell}e_1 \otimes e_1 - \left(\frac{2\theta x_1}{\ell} - 2\theta + 1\right)e_1 \otimes e_2 + \left(\frac{2\theta x_1}{\ell} - 2\theta\right)e_2 \otimes e_1,$$

and thus

$$e(v)(x) = B^{\text{lin}}_{\text{sym}} - \frac{2\theta x_2}{\ell}e_1 \otimes e_1. \tag{3.15}$$

For $x_1 < 0$, we extend v such that the elastic energy in $\{x_1 < 0\}$ vanishes. Precisely, we set

$$v_1(x) := v_1(0, x_2) = 2\theta x_2 - x_2 + 2\theta, \quad \text{and}$$
$$v_2(x) := 2(1 - \theta)x_1 + \theta\ell.$$

Then $e(v) = A^{\text{lin}}_{\text{sym}}$ in $\{x_1 < 0\}$, v is continuous in $\mathbb{R}^2$ and is admissible. By (3.14) and (3.15), we find with that there is a constant (not depending on ℓ or θ) such that by (3.10)

$$
\begin{aligned}
\mathcal{E}_{\text{lin}}[f^{\pm}, v] &= \int_{\omega_A \cap \{x_1 > 0\}} \frac{1}{2}\mathbb{C}(e(v) - A^{\text{lin}}_{\text{sym}}) \cdot (e(v) - A^{\text{lin}}_{\text{sym}})dx \\
&\quad + \int_{\omega_B} \frac{1}{2}\mathbb{C}(e(v) - B^{\text{lin}}_{\text{sym}}) \cdot (e(v) - B^{\text{lin}}_{\text{sym}})dx \\
&\leq c\int_{(0,\ell) \times (0,1)} \frac{\theta^2}{\ell^2} dx \leq c\frac{\theta^2}{\ell}.
\end{aligned}
\tag{3.16}
$$

This concludes the proof of the upper bound. For later reference we remark that f^+ and f^- are θ/ℓ-Lipschitz.

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Lower bound. Let $\tilde{c} > 0$ be a fixed (small) constant chosen below. Let $(f^{\pm}, v) \in \mathcal{A}_{\text{lin}}^{(\ell)}$ be an arbitrary admissible configuration. If the elastic energy on the left hand side of the interface is large, i.e.,

$$\int_{(-1,0)\times(0,1)} |e(v) - A_{\text{sym}}^{\text{lin}}|^2 dx \geq \tilde{c}\frac{\theta^2}{\ell}$$

then the assertion follows. Hence, from now on, we assume that

$$\int_{(-1,0)\times(0,1)} |e(v) - A_{\text{sym}}^{\text{lin}}|^2 dx < \tilde{c}\frac{\theta^2}{\ell}.$$

Thus, by Korn's inequality, there exists a constant $c_K > 0$ and an infinitesimal rotation $W := w(e_2 \otimes e_1 - e_1 \otimes e_2) \in \mathbb{R}_{\text{skew}}^{2\times2}$ with some $w \in \mathbb{R}$ such that

$$\int_{(-1,0)\times(0,1)} \left|\nabla v(x) - A^{\text{lin}} - W\right|^2 dx \leq \tilde{c}c_K\frac{\theta^2}{\ell},$$

and hence in particular,

$$\int_{(-1,0)\times(0,1)} |\partial_2 v_1 - 1 + w|^2 dx \leq \tilde{c}c_K\frac{\theta^2}{\ell}.$$

By Fubini's theorem and Hölder's inequality, there exists $x_1^* \in (-1, 0)$ such that

$$\int_0^1 |\partial_2 v_1(x_1^*, x_2) - 1 + w|dx_2 \leq \tilde{c}^{1/2}c_K^{1/2}\frac{\theta}{\ell^{1/2}}. \tag{3.17}$$

By (3.2), this implies that

$$|(-1 + 2\theta) - (1 - w)| = \left|\int_0^1 \partial_2 v_1(x_1^*, x_2)dx_2 - 1 + w\right| \leq \tilde{c}^{1/2}c_K^{1/2}\frac{\theta}{\ell^{1/2}}. \tag{3.18}$$

We finally note that using Poincaré's inequality and (3.17), there exists $b_1 \in \mathbb{R}$ such that

$$\int_0^1 |v_1(x_1^*, x_2) - (1 - w)x_2 - b_1| \, dx_2 \leq \tilde{c}^{1/2}c_K^{1/2}\frac{\theta}{\ell^{1/2}}.$$

With (3.18) we can eliminate w and obtain

$$\int_0^1 |v_1(x_1^*, x_2) - (2\theta - 1)x_2 - b_1| \, dx_2 \leq 2\tilde{c}^{1/2}c_K^{1/2}\frac{\theta}{\ell^{1/2}}. \tag{3.19}$$

We now consider the slice at $x_1 = \ell$. By (3.6) and the condition $\theta \leq 1/2$, there exists an interval $(t, t+1/4) \subset (0, 1)$ (depending on η) of length $1/4$ such that $\partial_2 v_1(\ell, \cdot) = (B^{\text{lin}})_{12} =$

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-1 and therefore $v_1(\ell, x_2) = b_2 - x_2$ on this interval. However by (3.19) $v_1(x_1^*, x_2)$ is close in L^1 to a different affine function than $v_1(\ell, x_2)$. We thus estimate the energy from below with this difference using Hölder's and triangle inequality,

$$
\begin{aligned}
\mathcal{E}_{\text{lin}}[f^\pm, v] &\geq \int_{(-1,\ell)\times(0,1)} |\partial_1 v_1|^2 dx \geq \int_{(x_1^*,\ell)\times(t,t+\frac{1}{4})} |\partial_1 v_1|^2 dx \geq \frac{4}{\ell} \left(\int_{(x_1^*,\ell)\times(t,t+\frac{1}{4})} |\partial_1 v_1| dx \right)^2 \\
&\geq \frac{4}{\ell} \left(\int_t^{t+1/4} \left| \int_{x_1^*}^{\ell} \partial_1 v_1 \, dx_1 \right| dx_2 \right)^2 \\
&= \frac{4}{\ell} \left(\int_t^{t+1/4} |v_1(\ell, x_2) - v_1(x_1^*, x_2)| \, dx_2 \right)^2.
\end{aligned}
$$

If $\tilde{c} > 0$ is chosen small enough such that $\tilde{c}^{1/2} \leq \frac{1}{128 c_K^{1/2}}$, then for all $\ell \geq 1$, by $v_1(\ell, x_2) = b_2 - x_2$ and (3.19)

$$
\begin{aligned}
&\int_t^{t+1/4} |v_1(\ell, x_2) - v_1(x_1^*, x_2)| \, dx_2 \\
&\geq \int_t^{t+1/4} |b_2 - b_1 - 2\theta x_2| dx_2 - \int_t^{t+1/4} |v_1(x_1^*, x_2) - (2\theta - 1)x_2 - b_1| dx_2 \\
&\geq \frac{1}{32}\theta - 2(\tilde{c} c_K)^{1/2} \frac{\theta}{\ell^{1/2}} \geq \frac{1}{64}\theta,
\end{aligned}
$$

and hence

$$
\mathcal{E}_{\text{lin}}[f^\pm, v] \geq \frac{\theta^2}{1024 \ell}.
$$

This concludes the proof of the lower bound. $\qquad\square$

The fact that the minimal energy tends to zero as $\ell \to \infty$ indicates that we cannot existence of minimizers for the problem on the infinite domain. We show that this is indeed the case, at least if we prescribe that the phase boundaries are uniformly Lipschitz.

Proposition 3.3 *Let $L \geq 1$.*

(i) *For any $\ell \geq 1$ there exists a minimizer $(f^\pm, v)$ of $\mathcal{E}_{\text{lin}}$ in*

$$
\mathcal{B}_{\text{lin}}^{(L,\ell)} := \left\{ (f^\pm, v) \in \mathcal{A}_{\text{lin}}^{(\ell)} : f^\pm \text{ L-Lipschitz} \right\}.
$$

(ii) *For $\mathcal{B}_{\text{lin}}^{(L)} := \bigcup_{\ell > 0} \mathcal{B}_{\text{lin}}^{(L,\ell)}$, there holds*

$$
\inf_{\mathcal{B}_{\text{lin}}^{(L)}} \mathcal{E}_{\text{lin}} = 0,
$$

and there exists no minimizer.

Proof: (i). Let $\ell \geq 1$. By (3.16) we have $\inf_{\mathcal{B}_{\mathrm{lin}}^{(L,\ell)}} \mathcal{E}_{\mathrm{lin}} < \infty$. Let $(f_n^\pm, v_n)$ be a minimizing sequence. Then by the Lipschitz condition and (3.1), we have a uniform bound $\sup_n \|f_n^\pm\|_{C^{0,1}([0,\ell])} < \infty$. By Arzelà-Ascoli, there exists a subsequence (not relabeled, the same subsequence for f^+ and f^-) such that $f_n^\pm \to f^\pm$ uniformly on $[0, \ell]$, which implies $f^\pm(0) = 0$, $(f^+ - f^-)(\ell) = \theta$, $f^- \leq f^+ \leq f^- + 1$, and $f^\pm \in C^{0,1}([0, \ell])$ with $\mathrm{Lip}(f^\pm) \leq L$. From boundedness of the energy, we get that $\sup_n \|e(v_n)\|_{L^2((-\infty,\ell)\times(0,1))} < \infty$. Since the periodicity condition (3.2) fixes that the average of $(\nabla v_n)_{12}$ is $-1 + 2\theta$, this implies a bound on the full gradients, $\sup_n \|\nabla v_n\|_{L^2((a,\ell)\times(0,1))} < \infty$ for all $a < 0$. By adding a constant, we can assume without restriction that all v_n have mean zero over $(0,1)^2$, and thus by Poincare's inequality, we obtain a subsequence that converges weakly in $W_{\mathrm{loc}}^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$ to an admissible function $v \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$. The boundary condition and the periodicity condition immediately pass to the limit. It remains to estimate the energy of the limit. Let $\varepsilon > 0$. Then, by uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\mathrm{graph}(f_n^\pm) \subset B_\varepsilon(f^\pm) := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathrm{graph}(f^+) \cup \mathrm{graph}(f^-)) < \varepsilon\}$. Then by lower semicontinuity,

$$\int_{\omega_A^{(n)}} \frac{1}{2}\mathbb{C}(e(v_n) - A_{\mathrm{sym}}^{\mathrm{lin}}) \cdot (e(v_n) - A_{\mathrm{sym}}^{\mathrm{lin}})dx$$
$$\geq \int_{\omega_A^{(n)} \setminus B_\varepsilon(f^\pm)} \frac{1}{2}\mathbb{C}(e(v_n) - A_{\mathrm{sym}}^{\mathrm{lin}}) \cdot (e(v_n) - A_{\mathrm{sym}}^{\mathrm{lin}})dx$$
$$\geq \int_{\omega_A \setminus B_\varepsilon(f^\pm)} \frac{1}{2}\mathbb{C}(e(v) - A_{\mathrm{sym}}^{\mathrm{lin}}) \cdot (e(v) - A_{\mathrm{sym}}^{\mathrm{lin}})dx,$$

and analogously in $\omega_B^{(n)}$. Taking $\varepsilon \to 0$, the assertion follows.

(ii). By the upper bound of Theorem 3.1, we obtain for $\ell \to \infty$ a sequence $(f_\ell^\pm, v_\ell) \in \mathcal{B}_{\mathrm{lin}}^{(L,\ell)}$ such that $\lim_{\ell \to \infty} \mathcal{E}_{\mathrm{lin}}[f_\ell^\pm, v_\ell] = 0$. On the other hand, let $(f^\pm, v) \in \mathcal{B}_{\mathrm{lin}}^{(L)}$. Then there exists $\ell > 0$ such that $(f^\pm, v) \in \mathcal{B}_{\mathrm{lin}}^{(L,\ell)}$, and by the lower bound of Theorem 3.1, $\mathcal{E}_{\mathrm{lin}}[f^\pm, v] > 0$. $\square$

4 Energy scaling in the geometrically nonlinear setting

It appears that in the geometrically nonlinear setting, the qualitative behavior of the minimal energy is rather different. On a technical level, the main difference seems to be that the macrotwin habit plane $e_{\delta,\theta}$ is not parallel to a plane of compatibility of the two wells $e^\perp$. Recall that this property was in particular used to extend the test function in the upper bound of Theorem 3.1 with vanishing energy to the left-hand side of the interface.

We first introduce the setting, using the notation from Lemma 2.1. As in Section 3, we

assume without loss of generality $e := e_1$ and $e^\perp = e_2$. We recall the definitions

$$e_{\delta,\theta} := \frac{1}{\sqrt{1 + (\delta\theta)^2}}(e_2 - \delta\theta e_1), \qquad e_{\delta,\theta}^\perp := -\frac{1}{\sqrt{1 + (\delta\theta)^2}}(e_1 + \delta\theta e_2). \qquad (4.1)$$

We shall impose the following periodicity condition on admissible deformations

$$u(x + e_{\delta,\theta}) = u(x) + (\theta A_\delta + (1 - \theta)B_\delta)e_{\delta,\theta} \qquad \text{for all } x \in \mathbb{R}^2. \qquad (4.2)$$

To parametrize the needle shapes, let $f^\pm : [0, \infty) \to \mathbb{R}$ be measurable and such that

$$f^- \leq f^+ \leq f^- + 1 \qquad \text{and} \qquad f^+(0) = f^-(0) = 0. \qquad (4.3)$$

(Later on, we will assume that they are L-Lipschitz.)

The periodicity condition (4.2) suggests that we use the non-orthogonal coordinates introduced by the macrotwin, see Figure 2. Precisely, we define the linear map $T_{\delta,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_{\delta,\theta} := (e_{\delta,\theta} \cdot e_2)e_1 \otimes e_1 + e_{\delta,\theta} \otimes e_2. \qquad (4.4)$$

We also define $d, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x) := \frac{x \cdot e_2}{e_{\delta,\theta} \cdot e_2} = e_2 \cdot T_{\delta,\theta}^{-1}(x), \qquad g(x) := -\frac{x \cdot e_{\delta,\theta}^\perp}{(e_{\delta,\theta} \cdot e_2)^2} = e_1 \cdot T_{\delta,\theta}^{-1}(x), \qquad (4.5)$$

which is equivalent to

$$T_{\delta,\theta}^{-1}(x) = g(x)e_1 + d(x)e_2. \qquad (4.6)$$

Note that $d(e_1) = 0$ and $d(e_{\delta,\theta}) = 1$, so that $\lfloor d(x) \rfloor \in \mathbb{Z}$ is the index of the period x is in. In turn, $g(x)$ denotes the coordinate along the needle, which corresponds to x_1 in the geometrically linear setting. We set

$$\hat{\omega}_A := \{x \in \mathbb{R}^2 : x \cdot e_{\delta,\theta}^\perp \geq 0\} \cup$$
$$\cup \left\{ x \in \mathbb{R}^2 : x \cdot e_{\delta,\theta}^\perp < 0, \ d(x) \in \bigcup_{k \in \mathbb{Z}} \left[k + f^-(g(x)), k + f^+(g(x))\right) \right\} \qquad (4.7)$$
$$= T_{\delta,\theta} \left\{ y \in \mathbb{R}^2 : y_1 \leq 0 \text{ or } y_1 > 0 \text{ and } y_2 \in \bigcup_{k \in \mathbb{Z}} \left[k + f^-(y_1), k + f^+(y_1)\right) \right\}$$

and

$$\hat{\omega}_B := \mathbb{R}^2 \setminus \hat{\omega}_A. \qquad (4.8)$$

By periodicity, for the computation of the energy it suffices to integrate over one period, and therefore to consider

$$\omega_{A,B} := \hat{\omega}_{A,B} \cap \left\{x \in \mathbb{R}^2 : d(x) \in (0, 1)\right\} = T_{\delta,\theta}(\mathbb{R} \times (0, 1)). \qquad (4.9)$$

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The class of admissible configurations is given by

$$\mathcal{A}_{\mathrm{nl}} := \left\{ (f^{\pm}, u) : f^{\pm} : \mathbb{R} \to \mathbb{R} \text{ satisfy (4.3)}, u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2) \text{ satisfies (4.2)} \right\}, \quad (4.10)$$

it depends implicitly on δ and θ via (4.2). For $L > 0$ we further set

$$\mathcal{A}_{\mathrm{nl}}^L := \{ (f^{\pm}, u) \in \mathcal{A}_{\mathrm{nl}} : f^{\pm} \text{ are } L\text{-Lipschitz} \}. \quad (4.11)$$

The resulting variational problem then is to minimize over this set the functional

$$\mathcal{E}_{\mathrm{nl}}[f^{\pm}, u] := \int_{\omega_A} W(\nabla u A_\delta^{-1}) dx + \int_{\omega_B} W(\nabla u B_\delta^{-1}) dx. \quad (4.12)$$

Here, $W : \mathbb{R}^{3 \times 3} \to [0, \infty)$ is a typical nonlinear elastic energy density satisfying

$$\frac{1}{c_W} \operatorname{dist}^2(F, \mathrm{SO}(2)) \leq W(F) = W(RF) \leq c_W \operatorname{dist}^2(F, \mathrm{SO}(2))$$
$$\text{for all } R \in \mathrm{SO}(2) \text{ and } F \in \mathbb{R}^{2 \times 2} \quad (4.13)$$

with some constant $c_W > 0$.

Remark 4.1 *Note that in contrast to the geometrically linearized setting, we do not assign boundary conditions for the deformation deep in the laminate.*

Theorem 4.2 *For every $L \geq 1$ there are constants $c_L > 0$ and $\delta_0 > 0$ such that for all $\theta \in (0, 1/2]$ and all $\delta \in (-\delta_0, \delta_0)$, we have*

$$\frac{1}{c_L} |\delta|^3 \theta^2 \leq \inf\{\mathcal{E}_{\mathrm{nl}}[f^{\pm}, u] : (f^{\pm}, u) \in \mathcal{A}_{\mathrm{nl}}^L\} \leq c_L |\delta|^3 \theta^2.$$

There is $c_f > 0$ such that, if $\delta \neq 0$, the same holds if one imposes that for $x_1 \geq c_f |\delta|^{-1}$

$$f^-(x_1) = 0, \quad f^+(x_1) = \theta, \quad \nabla u(x) = \begin{cases} A_\delta, & \text{if } d(x) \in (0, \theta), \\ B_\delta, & \text{if } d(x) \in (\theta, 1). \end{cases} \quad (4.14)$$

Proof: The upper bound follows from Proposition 4.3, the lower bound from Proposition 4.9. □

4.1 Upper bound

Proposition 4.3 *There exists a constant $c > 0$ such that for every $\delta \in [-1, 1]$ and $\theta \in (0, \frac{1}{2}]$ there are $(f^{\pm}, u) \in \mathcal{A}_{\mathrm{nl}}^{1}$ such that*

$$\mathcal{E}_{\mathrm{nl}}[f^{\pm}, u] \leq c|\delta|^3 \theta^2.$$

The functions $(f^{\pm}, u)$ obey (4.14) for $x_1 \geq c_f |\delta|^{-1}$ for some universal $c_f > 0$ (provided $\delta \neq 0$).

Proof: For $\delta = 0$, we have $A_\delta = B_\delta = \mathrm{Id}$, and an affine function $u(x) = x$ has vanishing energy, with $f^+ = f^- = 0$. Consider now $\delta \neq 0$. Let Q_* be as in Lemma 2.1. Left of the interface, in $\{g(x) \leq 0\} = \{x \cdot e_{\delta,\theta}^{\perp} \geq 0\}$, we set

$$u(x) := Q_* A_\delta x.$$

Note that by (2.6), this definition satisfies the periodicity condition (4.2). Set

$$f^-(t) := 0 \text{ for all } t \geq 0, \tag{4.15}$$

and let $f^+ : [0, \infty) \to [0, 1)$ be a 1-Lipschitz function with $f(0) = 0$, to be determined later. We now describe the deformation. By periodicity, we consider for the ease of notation a shifted cell of periodicity, i.e.,

$$\begin{aligned}
\omega_A^* &:= \{x \in \mathbb{R}^2 : g(x) \geq 0, \ 0 \leq d(x) \leq f^+(g(x))\}, \\
\omega_B^* &:= \{x \in \mathbb{R}^2 : g(x) \geq 0, \ f^+(g(x)) - 1 \leq d(x) < 0\}.
\end{aligned}$$

To make an ansatz for the deformation on the right-hand side of the interface, we consider a rotation and a shift as independent parameters. Precisely, let $R : [0, \infty) \to \mathrm{SO}(2)$ and $w : [0, \infty) \to \mathbb{R}^2$ be differentiable functions to be determined later, and set

$$\phi(t) := (e_2 \cdot e_{\delta,\theta}) \int_0^t R(s) e_1 ds. \tag{4.16}$$

The reason for this choice will become clear in (4.23) below. With these quantities, we define the deformation $u : \omega_A^* \cup \omega_B^* \to \mathbb{R}^2$ as

$$u(x) := \begin{cases} u_A(x), & \text{if } x \in \omega_A^*, \\ u_B(x), & \text{if } x \in \omega_B^*, \end{cases} \tag{4.17}$$

where

$$\begin{aligned}
u_A(x) &:= \phi(g(x)) + R(g(x)) A_\delta e_{\delta,\theta} d(x) + w(g(x)) d(x), \\
u_B(x) &:= \phi(g(x)) + R(g(x)) B_\delta e_{\delta,\theta} d(x) + w(g(x)) d(x).
\end{aligned} \tag{4.18}$$

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We note that this yields a continuous function in $\omega_A^* \cup \omega_B^*$ since for $d(x) = 0$ we have $u_A(x) = u_B(x)$. To obtain an admissible configuration, the functions R, f^+, and w have to satisfy the following properties. First, f^+ should be 1-Lipschitz with

$$f^+(0) = 0 \quad \text{and} \quad f^+(t) \in [0,1) \qquad \text{for all } t > 0. \tag{4.19}$$

Second, this definition, when extended with the periodicity condition (4.2), should generate a continuous function on $\{g(x) \geq 0\} = \{x \cdot e_{\delta,\theta}^{\perp} \leq 0\}$. This requires that

$$u_A(x + e_{\delta,\theta}) - u_B(x) = (\theta A_\delta + (1-\theta)B_\delta)e_{\delta,\theta} \quad \text{if } d(x) = f^+(g(x)) - 1.$$

Using $d(x + e_{\delta,\theta}) = d(x) + 1$ and $d(x) + 1 = f^+(g(x))$ we obtain

$$u_A(x + e_{\delta,\theta}) - u_B(x) = R(g(x))B_\delta e_{\delta,\theta} \\ + f^+(g(x))R(g(x))(A_\delta - B_\delta)e_{\delta,\theta} + w(g(x)), \tag{4.20}$$

so that the periodicity condition is equivalent to

$$\left(f^+(g(x))R(g(x)) - \theta\,\mathrm{Id}\right)(A_\delta - B_\delta)e_{\delta,\theta} \\ + (R(g(x)) - \mathrm{Id})B_\delta e_{\delta,\theta} + w(g(x)) = 0. \tag{4.21}$$

Note that on the line $\{g(x) = 0\} = \mathbb{R}e_{\delta,\theta}$, as $f^+(0) = 0$ the function $t \mapsto u(te_{\delta,\theta})$ is affine. By (2.6), the periodicity condition and $u(0) = 0$ it coincides with the expression $Q_* A_\delta x$ that we used to define u on $\{g(x) \leq 0\}$. Equivalently, one can see from (4.21) that $(R(0) - \mathrm{Id})B_\delta e_{\delta,\theta} + w(0) = \theta(A_\delta - B_\delta)e_{\delta,\theta}$, so that (4.17)-(4.18) give $u(te_{\delta,\theta}) = u_B(te_{\delta,\theta}) = R(0)B_\delta te_{\delta,\theta} + w(0)t = (\theta A_\delta + (1-\theta)B_\delta)e_{\delta,\theta}t$, which by (2.6) equals $Q_* A_\delta e_{\delta,\theta}t$.

From now on, we restrict to $\{g(x) > 0\}$. Before we give the explicit constructions, we provide an estimate for the energy within this ansatz. We observe that $\nabla d = \frac{1}{e_{\delta,\theta} \cdot e_2}e_2$, $\nabla g = -(e_{\delta,\theta} \cdot e_2)^{-2}e_{\delta,\theta}^{\perp}$, and

$$-e_1 \otimes e_{\delta,\theta}^{\perp} + e_{\delta,\theta} \otimes e_2 = (e_2 \cdot e_{\delta,\theta})\,\mathrm{Id}. \tag{4.22}$$

The definition of ϕ (see (4.16)) was chosen so that $\phi' = (e_2 \cdot e_{\delta,\theta})Re_1 = (e_2 \cdot e_{\delta,\theta})RA_\delta e_1$, which – using (4.22) – implies

$$\phi'(g(x)) \otimes \nabla g + R(g(x))A_\delta e_{\delta,\theta} \otimes \nabla d = \frac{1}{e_2 \cdot e_{\delta,\theta}}R(g(x))A_\delta(-e_1 \otimes e_{\delta,\theta}^{\perp} + e_{\delta,\theta} \otimes e_2) \\ = R(g(x))A_\delta. \tag{4.23}$$

Therefore

$$\nabla u_A(x) = \phi'(g(x)) \otimes \nabla g + R(g(x))A_\delta e_{\delta,\theta} \otimes \nabla d + d(x)R'(g(x))A_\delta e_{\delta,\theta} \otimes \nabla g \\ + d(x)w'(g(x)) \otimes \nabla g + w(g(x)) \otimes \nabla d \\ = R(g(x))A_\delta - \frac{1}{(e_2 \cdot e_{\delta,\theta})^2}d(x)\left(R'(g(x))A_\delta e_{\delta,\theta} + w'(g(x))\right) \otimes e_{\delta,\theta}^{\perp} \\ + \frac{1}{e_{\delta,\theta} \cdot e_2}w(g(x)) \otimes e_2,$$

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and similarly

$$\nabla u_B(x) = R(g(x))B_\delta - \frac{1}{(e_2 \cdot e_{\delta,\theta})^2}d(x)\left(R'(g(x))B_\delta e_{\delta,\theta} + w'(g(x))\right) \otimes e_{\delta,\theta}^\perp$$
$$+ \frac{1}{e_{\delta,\theta} \cdot e_2}w(g(x)) \otimes e_2.$$

Hence, the elastic energy of such an admissible test function is estimated by

$$\mathcal{E}_{\mathrm{nl}}[f^\pm, u] \leq c_W \left(\int_{\omega_A^*} |\nabla u_A(x) - RA_\delta|^2\,dx + \int_{\omega_B^*} |\nabla u_B(x) - RB_\delta|^2\,dx\right)$$
$$\leq c \int_0^\infty \left(|R'(t)|^2 + |w'(t)|^2 + |w(t)|^2\right)\,dt. \tag{4.24}$$

Finally, we specify how to choose R, w and f^+. We consider the periodicity condition (4.21) and divide it into two equations, testing with e_1 and e_2. First, we set

$$w \cdot e_1 = 0, \tag{4.25}$$

and take the scalar product of (4.21) with e_1. Using that $(A_\delta - B_\delta)e_{\delta,\theta} = \frac{2\delta}{\sqrt{1+(\delta\theta)^2}}e_1$ and $B_\delta e_{\delta,\theta} = \frac{e_2-\delta(1+\theta)e_1}{\sqrt{1+(\delta\theta)^2}}$, we obtain, multiplying by $\sqrt{1+(\delta\theta)^2}$ and skipping the arguments $g(x)$ everywhere,

$$2\delta\left(f^+e_1 \cdot Re_1 - \theta\right) + e_1 \cdot Re_2 - \delta(1+\theta)e_1 \cdot Re_1 + \delta(1+\theta) = 0.$$

We let $\alpha := e_1 \cdot Re_1$, $\beta := e_1 \cdot Re_2$ and solve this equation for f^+, which leads to the definition

$$f^+ := \frac{\theta}{\alpha} - \frac{\beta}{2\delta\alpha} + \frac{1+\theta}{2} - \frac{1+\theta}{2\alpha}$$
$$= -\frac{1-\theta}{2\alpha} - \frac{\beta}{2\delta\alpha} + \frac{1+\theta}{2}. \tag{4.26}$$

Since R is a rotation, we have $|\alpha| = \sqrt{1-\beta^2}$, and we choose $\alpha = \sqrt{1-\beta^2}$. Roughly speaking, we expect that for large arguments approximately $\beta = 0$ and $\alpha = 1$, which correspond to $f^+ = \theta$. On the other hand, in view of Remark 2.2(ii) since R_0 is a rotation by roughly $-2\delta\theta$, we expect that for small arguments, $\beta \approx 2\delta\theta$ which is positive for $\delta > 0$ and negative for $\delta < 0$. Hence, we assume that $\frac{\beta}{\delta}$ is monotonically decreasing and α is monotonically increasing, so that by (4.26) also f^+ is monotonically increasing. We shall fix the value $\beta(0)$ from the condition $f^+(0) = 0$, so that monotonicity of f^+ implies $0 \leq f^+ \leq \theta$ everywhere and in particular (4.19).

Rearranging terms, the condition $f^+(0) = 0$ is (for $\alpha(0) \neq 0$) the same as

$$(1 - \theta) + \frac{\beta(0)}{\delta} = (1 + \theta)\alpha(0). \tag{4.27}$$

We choose $\beta(0)$ such that $\beta(0)/\delta > 0$, hence squaring and inserting $\alpha^2 = 1 - \beta^2$, this is equivalent to

$$\left(1 + \delta^2(1 + \theta)^2\right)\left(\frac{\beta(0)}{\delta}\right)^2 + 2(1 - \theta)\frac{\beta(0)}{\delta} - 4\theta = 0. \tag{4.28}$$

As $\theta \in (0, 1/2]$, this quadratic equation has a unique solution with $\beta(0)/\delta > 0$. Since the left-hand side is larger than

$$\left(\frac{\beta(0)}{\delta}\right)^2 + \frac{\beta(0)}{\delta} - 2,$$

we find that $\beta(0)/\delta \leq 1$, i.e., $|\beta(0)| \leq |\delta|$. Analogously, since the left-hand side of (4.28) is larger than

$$2|\beta(0)|^2 + |\beta(0)| - 2,$$

we obtain $|\beta(0)| \leq \frac{4}{5}$. Further, as the first term in (4.28) is positive we have $2(1 - \theta)\frac{\beta(0)}{\delta} < 4\theta$, and with $2(1 - \theta) \geq 1$ this leads to $|\beta(0)| \leq 4\theta\,|\delta|$. Summarizing,

$$|\beta(0)| \leq \min\{|\delta|, 4|\delta|\,\theta, \frac{4}{5}\}. \tag{4.29}$$

We then set, for some $\ell > 0$ to be chosen later,

$$\beta(s) := \begin{cases} \frac{\ell - s}{\ell}\beta(0) & \text{if } s \leq \ell, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \alpha(s) := \sqrt{1 - \beta^2(s)}. \tag{4.30}$$

Finally, $w \cdot e_2$ is determined from (4.21) by testing with e_2. Since $R \in \mathrm{SO}(2)$ the definitions of α and β imply $e_2 \cdot Re_2 = \alpha$, $e_2 \cdot Re_1 = -\beta$. A similar computation as above leads to

$$e_2 \cdot w = \frac{1}{\sqrt{1 + (\delta\theta)^2}}\left(1 - \alpha + \left(2\delta f^+ - \delta(1 + \theta)\right)\beta\right), \tag{4.31}$$

which together with (4.25) defines w. Note that for $s \geq \ell$, we have $\alpha = 1$ and $\beta = 0$ which implies that $w(s) = 0$. To estimate the energy, we observe that $|\beta(s)| \leq |\beta(0)| \leq \frac{4}{5}$, which implies $\alpha \geq \frac{1}{5}$ and $|\alpha'| = \frac{|\beta\beta'|}{\sqrt{1 - \beta^2}} \leq 5|\beta'|$. As $|\beta'|(s) = \frac{|\beta(0)|}{\ell} \leq \frac{4|\delta|\theta}{\ell}$ for $s \in (0, \ell)$, from (4.26) we have

$$|(f^+)'| \leq \left|\frac{\alpha'}{\alpha^2}\right| + \left|\frac{\beta'}{2\delta\alpha}\right| + \left|\frac{\beta\alpha'}{2\delta\alpha^2}\right| \leq c\frac{|\beta'|}{|\delta|} \leq c_f\frac{\theta}{\ell}, \tag{4.32}$$

for some universal constant $c_f > 0$, and thus from (4.31)

$$|w'| \leq |\alpha'| + 3|\delta|\,|\beta'| + 2|\delta|\,|\beta|\,\left|(f^+)'\right| \leq c|\beta'| \leq c\frac{|\delta|\theta}{\ell}.$$

Finally, with $1 - \alpha = \frac{\beta^2}{1+\alpha} \leq \beta^2 \leq |\delta| \, |\beta|$ and (4.31) we obtain

$$|w| \leq |1 - \alpha| + 2|\delta||\beta| \leq 3|\delta| \, |\beta| \leq c\delta^2\theta.$$

Hence, using (4.24) we can estimate the energy by

$$\mathcal{E}_{\mathrm{nl}}[f^{\pm}, u] \leq c \int_0^{\ell} \left(|R'|^2 + |w'|^2 + |w|^2 \right) \, dt \leq c \left(\frac{\delta^2\theta^2}{\ell} + \ell\delta^4\theta^2 \right).$$

Recalling (4.32), if we choose $\ell := c_f |\delta|^{-1}$ we obtain that f^+ is 1-Lipschitz and $\mathcal{E}_{\mathrm{nl}}[f^{\pm}, u] \leq c|\delta|^3\theta^2$. $\qquad \square$

4.2 Lower bound

For the lower bound, we need some auxiliary statements.

4.2.1 Technical preliminaries

For $v \in \mathbb{R}^2$ we write $v^{\perp} := (-v_2, v_1)$.

Lemma 4.4 *There is $c > 0$ such that if $\alpha \in \mathbb{R}$, $Q \in \mathrm{SO}(2)$ and $v \in \mathbb{R}^2 \backslash \{0\}$ are such that*

$$|Q(v - \alpha v^{\perp}) - v| \leq \eta |v|$$

for some $\eta \geq 0$, then

$$|Q - (\mathrm{Id} + \alpha J)| \leq \sqrt{2}\,\eta$$

where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof: This follows immediately by the fact that all matrices considered are conformal. For clarity we present a short explicit computation. By scaling we can assume $|v| = 1$. Let $\phi \in (-\pi, \pi]$ be such that $Q = \cos\phi\,\mathrm{Id} + \sin\phi J$. Then

$$\eta^2 \geq |Q(v - \alpha v^{\perp}) - v|^2 = |v - \alpha v^{\perp} - Q^T v|^2 = |v - \alpha v^{\perp} - \cos\phi v + \sin\phi v^{\perp}|^2$$
$$= (1 - \cos\phi)^2 + (\alpha - \sin\phi)^2.$$

Then

$$|Q - (\mathrm{Id} + \alpha J)|^2 = |(\cos\phi - 1)\,\mathrm{Id} + (\sin\phi - \alpha)J|^2 = 2(1 - \cos\phi)^2 + 2(\sin\phi - \alpha)^2 \leq 2\eta^2$$

concludes the proof. $\qquad \square$

The next lemma concerns stability of the rank-one directions.

Lemma 4.5 *Suppose that $\delta \in [-1, 1]$ and that $Q_A, Q_B \in \mathrm{SO}(2)$, $t \in S^1$ satisfy $t \cdot e_1 > 0$ and, for some $\eta > 0$,*

$$|(Q_A A_\delta - Q_B B_\delta)t| \leq \eta. \tag{4.33}$$

Then

$$|Q_A - Q_B| \leq \frac{3}{t \cdot e_1}\eta. \tag{4.34}$$

Proof: We write $t = t_1 e_1 + t_2 e_2$ and find that $|A_\delta t| + |B_\delta t| \leq 2(1 + |\delta|) \leq 4$ since $|\delta| \leq 1$. Assumption (4.33) gives $||A_\delta t| - |B_\delta t|| = ||Q_A A_\delta t| - |Q_B B_\delta t|| \leq \eta$ and therefore

$$4\eta \geq ||A_\delta t| - |B_\delta t|| \cdot ||A_\delta t| + |B_\delta t|| = \left||A_\delta t|^2 - |B_\delta t|^2\right| = 4|\delta||t_1 t_2| \tag{4.35}$$

which implies that

$$|t_2| \leq \frac{\eta}{|\delta| t_1}. \tag{4.36}$$

From $A_\delta = \mathrm{Id} + \delta e_1 \otimes e_2$ and $B_\delta = \mathrm{Id} - \delta e_1 \otimes e_2$ we deduce

$$|A_\delta t - t| = |B_\delta t - t| = |\delta||e_2 \cdot t| \leq \frac{\eta}{t_1},$$

so that with (4.33)

$$
\begin{aligned}
\frac{|Q_A - Q_B|}{\sqrt{2}} &= |Q_A t - Q_B t| \leq |Q_A t - Q_A A_\delta t| + |Q_A A_\delta t - Q_B B_\delta t| + |Q_B B_\delta t - Q_B t| \\
&= |t - A_\delta t| + |Q_A A_\delta t - Q_B B_\delta t| + |B_\delta t - t| \\
&\leq 2\frac{\eta}{t_1} + \eta \leq \frac{3}{t_1}\eta
\end{aligned}
$$
$$\tag{4.37}$$

concludes the proof. $\qquad\square$

The next two statements are uniform geometric rigidity and trace statements on domains which are appropriate sections of the sets $\tilde{\omega}_A$, $\tilde{\omega}_B$ defined in (4.7)-(4.8). For clarity we present here the specific assertion used in the lower bound, postponing to Section 5 the proof in a more general context and the specific definition of (L, R)-Lipschitz sets.

Proposition 4.6 *For any $L, M > 0$ there are constants $\hat{L}, \hat{R}, c_{L,M} > 0$ with the following property. Let $\ell > 0$, $f, g : [0, \ell] \to \mathbb{R}$ be L-Lipschitz functions. Assume that*

$$\frac{L\ell}{M} \leq g(t) - f(t) \leq ML\ell \quad \text{for all } t \in [0, \ell] \tag{4.38}$$

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and define

$$\omega^{f,g} := \{x : x_1 \in (0, \ell), f(x_1) < x_2 < g(x_1)\}, \tag{4.39}$$

fix any $\theta \in (0, \frac{1}{2}]$ and any $\delta \in [-1, 1]$. Then the sets $\omega^{f,g}$ and $T_{\delta,\theta}(\omega^{f,g})$ are $(\hat{L}, \hat{R})$-Lipschitz. Further, for any $u \in W^{1,2}(\omega^{f,g}; \mathbb{R}^2)$, and any $F \in \{A_\delta, B_\delta, \mathrm{Id}\}$, there is $Q_u^F \in \mathrm{SO}(2)$ such that

$$\int_{T_{\delta,\theta}(\omega^{f,g})} |\nabla u - Q_u^F F|^2 dx \leq c_{L,M} \int_{T_{\delta,\theta}(\omega^{f,g})} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)F) dx \tag{4.40}$$

and, for some $d_u \in \mathbb{R}^2$,

$$\int_{T_{\delta,\theta}(\omega^{f,g})} |u(x) - d_u - Q_u^F F x|^2 dx \leq c_{L,M} \int_{T_{\delta,\theta}(\omega^{f,g})} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)F) dx. \tag{4.41}$$

Proof: By Lemma 5.4 the sets $\omega^{f,g}$ are $(2L + 1, R)$-Lipschitz (see Definition 5.1), for some R which depends only on L and M. We observe that the definition (4.4) implies $T_{\delta,\theta}^{-1} = \sqrt{1 + (\delta\theta)^2}(\mathrm{Id} + \delta\theta e_1 \otimes e_2)$, and therefore

$$|T_{\delta,\theta}| \leq \sqrt{2}, \qquad |T_{\delta,\theta}^{-1}| \leq \sqrt{2}(\sqrt{2} + \frac{1}{2}) \leq 3. \tag{4.42}$$

By Lemma 5.3 we obtain that the sets $T_{\delta,\theta}(\omega^{f,g})$ are $(6(2L + 1), 6R)$-Lipschitz. The result for $F = \mathrm{Id}$ follows then immediately from Theorem 5.10.

Consider now $F = A_\delta$. For notational simplicity we prove the statement for $\omega^{f,g}$, the argument for $T_{\delta,\theta}(\omega^{f,g})$ is identical. We define $v \in W^{1,2}(A_\delta \omega^{f,g}; \mathbb{R}^2)$ by $v(x) := u(A_\delta^{-1}x)$, so that $\nabla v(x) = \nabla u(A_\delta^{-1}x)A_\delta^{-1}$, which implies

$$\mathrm{dist}(\nabla v, \mathrm{SO}(2))(x) = \mathrm{dist}(\nabla u A_\delta^{-1}, \mathrm{SO}(2))(A_\delta^{-1}x) \leq |A_\delta^{-1}| \, \mathrm{dist}(\nabla u, \mathrm{SO}(2)A_\delta)(A_\delta^{-1}x). \tag{4.43}$$

By Lemma 5.3, using that $|A_\delta|, |A_\delta^{-1}| \leq 3$, we obtain that the sets $A_\delta^{-1}\omega^{f,g}$ are $(c(2L + 1), cR)$-Lipschitz. Therefore Theorem 5.10 implies that there is $Q_u^{A_\delta} \in \mathrm{SO}(2)$ such that

$$\int_{A_\delta^{-1}\omega^{f,g}} |\nabla v - Q_u^{A_\delta}|^2 dx \leq c_{L,M} \int_{A_\delta^{-1}\omega^{f,g}} \mathrm{dist}^2(\nabla v, \mathrm{SO}(2)) dx. \tag{4.44}$$

Using (4.43) and a change of variables, this implies

$$\int_{\omega^{f,g}} |\nabla u - Q_u^{A_\delta} A_\delta|^2 dx \leq c'_{L,M} \int_{\omega^{f,g}} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)A_\delta) dx, \tag{4.45}$$

and concludes the proof. The case $F = B_\delta$ is identical.
The second bound follows immediately from Theorem 5.8. $\qquad\square$

For completeness, we finally note a rescaling property of the trace norm.

Corollary 4.7 (Trace estimate) *Let $M_0, L > 0$. There exists a constant C_T with the following property: Let $\ell > 0$, and let $f, g : [0, \ell] \to \mathbb{R}$ be L-Lipschitz continuous with $\frac{L\ell}{M_0} < g(t) - f(t) < M_0 L\ell$ for all $t \in [0, \ell]$. Then, setting $\omega_{f,g} := \{x \in \mathbb{R}^2 : x_1 \in (0, \ell), f(x_1) < x_2 < g(x_1)\}$, for every $u \in W^{1,2}(\omega_{f,g})$ there exists $d_u \in \mathbb{R}$ such that*

$$\|Tu - d_u\|^2_{L^2(\partial \omega_{f,g})} \leq C_T \ell \|\nabla u\|^2_{L^2(\omega_{f,g})}.$$

Proof: For $\ell = 1$, this follows from Lemma 5.4, Theorem 5.8, and Theorem 5.9. The general case follows from rescaling $f_\ell : (0, \ell) \to \mathbb{R}$, $f_\ell(t) := \ell f_1(\frac{t}{\ell})$ (similarly for g_ℓ) and $u_\ell \in W^{1,2}(\omega_{f_\ell, g_\ell})$ given by $u_\ell(x_1, x_2) := u(x_1/\ell, x_2/\ell)$. $\qquad\square$

4.2.2 Proof of the lower bound

We start introducing some notation. Recall that the periodicity condition (4.2) is the same as $u(T_{\delta,\theta}(y + e_2)) = u(T_{\delta,\theta}(y)) + (\theta A_\delta + (1 - \theta) B_\delta) T_{\delta,\theta}(e_2)$. Let $f^\pm : [0, \infty) \to \mathbb{R}$ be L-Lipschitz with $f^- \leq f^+ \leq f^- + 1$. Given $I \subseteq (0, \infty)$ we set

$$\begin{aligned}
\omega_A^I &:= T_{\delta,\theta}(\{y : y_1 \in I, f^-(y_1) < y_2 < f^+(y_1)\}), \\
\omega_B^I &:= T_{\delta,\theta}(\{y : y_1 \in I, f^+(y_1) < y_2 < f^-(y_1) + 1\})
\end{aligned} \tag{4.46}$$

and (for $I \subseteq (0, \infty)$ Borel measurable)

$$\mathcal{E}[I; (f^\pm, u)] := \int_{\omega_A^I} \operatorname{dist}^2(\nabla u, \mathrm{SO}(2) A_\delta) dx + \int_{\omega_B^I} \operatorname{dist}^2(\nabla u, \mathrm{SO}(2) B_\delta) dx. \tag{4.47}$$

Proposition 4.8 *Let $L > 0$, assume that $f^\pm$ are L-Lipschitz with $f^- \leq f^+ \leq f^- + 1$, and let $I_* \subseteq (0, \infty)$ be an interval of length $1/4L$. For any $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ there is $Q \in \mathrm{SO}(2)$ such that*

$$\int_{\omega_A^{I_*}} |\nabla u - Q A_\delta|^2 dx + \int_{\omega_B^{I_*}} |\nabla u - Q B_\delta|^2 dx \leq c \mathcal{E}[I_*; (f^\pm, u)]. \tag{4.48}$$

The constant may depend on L.

Proof: For brevity in this proof we write $\mathcal{E}(I)$ for $\mathcal{E}[I; (f^\pm, u)]$. We can assume $L \geq 1$ in the proof (otherwise we cover I_* with c_L subintervals of length $1/4$ and use the result for $L = 1$ in each of them).

Step 1: Estimate on $\omega_A^{I_*}$. Let t_* be the midpoint of I_* and ℓ_* its length. We assume that $(f^+ - f^-)(t_*) \geq \frac{1}{2}$. If not, then $f^- + 1 - f^+ \geq \frac{1}{2}$, and the same argument can be used swapping A_δ with B_δ and $(f^+, f^- + 1)$ with (f^-, f^+). This implies

$$1 \geq (f^+ - f^-)(t) \geq \frac{1}{2} - 2L \frac{\ell_*}{2} \geq \frac{1}{4} \qquad \text{for all } t \in I_*. \tag{4.49}$$
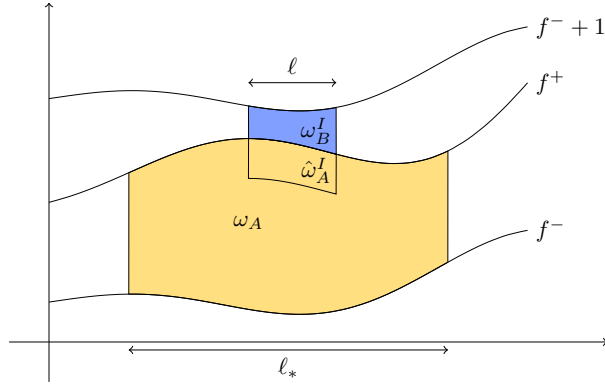
Figure 5: Sketch of the sets entering the proof of Proposition 4.8.

We write $\omega_A := \omega_A^{I_*}$, and c_L for a generic constant that may change from line to line but depends only on L. Proposition 4.6 can be applied (with $M = 4$) to the set ω_A, and there is $Q \in \mathrm{SO}(2)$ such that

$$\int_{\omega_A} |\nabla u - QA_\delta|^2 dx \leq c_L \int_{\omega_A} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)A_\delta)dx \leq c_L \mathcal{E}(I_*). \tag{4.50}$$

Note that this concludes the proof in the degenerate case $\omega_B^{I_*} = \emptyset$, i.e., if $I_* \cap \{f^- + 1 > f^+\} = \emptyset$. In the other case, we note that here is $d_A \in \mathbb{R}^2$ such that

$$\int_{\omega_A} |u(x) - QA_\delta x - d_A|^2 dx \leq c_L \mathcal{E}(I_*). \tag{4.51}$$

Step 2: Estimate on $\omega_B^{I_*}$. For any Borel set $I \subseteq I_*$ we write, recalling (4.47) and $\mathcal{E}(I) := \mathcal{E}[I; (f^\pm, u)]$,

$$\hat{\mathcal{E}}(I) := \mathcal{E}(I) + \int_{\omega_A^I} |u(x) - QA_\delta x - d_A|^2 dx. \tag{4.52}$$

It is clear that $\mathcal{E}$ and $\hat{\mathcal{E}}$ are measures on I_*, and that $\hat{\mathcal{E}}(I_*) \leq c_L \mathcal{E}(I_*)$. We shall first obtain estimates on suitable subintervals of I_*, and then cover I_* by countably many such subintervals. Let $M > 0$ be a fixed number, we shall choose $M = 16$ below.
Assume that $I \subseteq I_*$ is an interval of length $\ell \in (0, \ell_*]$ such that

$$\frac{L\ell}{M} \leq (f^- + 1) - f^+ \leq ML\ell \qquad \text{pointwise on } I, \tag{4.53}$$

see Figure 5. Then by Proposition 4.6 there is $Q_B^I \in \mathrm{SO}(2)$ such that

$$\int_{\omega_B^I} |\nabla u - Q_B^I B_\delta|^2 dx \leq c_L \int_{\omega_B^I} \mathrm{dist}^2(\nabla u, \mathrm{SO}(2)B_\delta)dx \leq c_L \mathcal{E}(I). \tag{4.54}$$

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By Poincaré and the trace theorem (see Corollary 4.7), there is $d_B^I \in \mathbb{R}^2$ such that, setting $\gamma_+^I := T_{\delta,\theta}(\{(t, f^+(t)) : t \in I\})$,

$$\int_{\gamma_+^I} |u(x) - Q_B^I B_\delta x - d_B^I|^2 d\mathcal{H}^1 \leq c_L \ell \|\nabla u - Q_B^I B_\delta\|_{L^2(\omega_B^I)}^2 \leq c_L \ell \mathcal{E}(I). \tag{4.55}$$

Analogously, the trace theorem on $\hat{\omega}_A^I := T_{\delta,\theta}(\{y : y_1 \in I, f^+(y_1) - \frac{\ell}{4} < y_2 < f^+(y_1)\})$, which by (4.49) and $\ell \leq \ell_* \leq 1/4$ is contained in ω_A^I, gives

$$\int_{\gamma_+^I} |u(x) - QA_\delta x - d_A|^2 d\mathcal{H}^1 \leq c_L \ell \hat{\mathcal{E}}(I) \tag{4.56}$$

so that with a triangular inequality and $\mathcal{E} \leq \hat{\mathcal{E}}$ we obtain

$$\int_{\gamma_+^I} |QA_\delta x + d_A - Q_B^I B_\delta x - d_B^I|^2 d\mathcal{H}^1 \leq c_L \ell \hat{\mathcal{E}}(I). \tag{4.57}$$

Therefore there is $v \in \mathbb{R}^2$ with $v_1 \geq \frac{1}{3}\ell$ and $|v_2| \leq Lv_1$ such that

$$|(QA_\delta - Q_B^I B_\delta)v|^2 \leq c_L \hat{\mathcal{E}}(I). \tag{4.58}$$

We apply Lemma 4.5 with $t := v/|v|$, $Q_A := Q$, $Q_B := Q_B^I$, and $\eta := (c_L \hat{\mathcal{E}}(I))^{1/2}/|v| \leq 3(c_L \hat{\mathcal{E}}(I))^{1/2}/\ell$. Since

$$t_1 = \frac{v_1}{|v|} \geq \frac{1}{\sqrt{1 + L^2}},$$

we obtain $|Q - Q_B^I| \leq 3\sqrt{1 + L^2}\eta$ and therefore

$$\ell^2 |Q - Q_B^I|^2 \leq c_L \hat{\mathcal{E}}(I). \tag{4.59}$$

Combining (4.54) and (4.59) we conclude that

$$\int_{\omega_B^I} |\nabla u - QB_\delta|^2 dx \leq c_L \hat{\mathcal{E}}(I). \tag{4.60}$$

As $\mathcal{L}^2\left(\omega_B^{I_* \cap \{f^- + 1 = f^+\}}\right) = 0$, it remains to show that $I_* \cap \{f^- + 1 > f^+\}$ can be covered (up to null sets) by countably many intervals I with the property (4.53) and finite overlap. For any $t \in I_* \cap \{f^- + 1 > f^+\}$, the interval

$$\left(t - \frac{f^-(t) + 1 - f^+(t)}{4L}, t + \frac{f^-(t) + 1 - f^+(t)}{4L}\right) \tag{4.61}$$

contains t and obeys the property (4.53) with $M = 3$. By the Besicovitch covering theorem this family contains a countable set of intervals $(I_k)_k$ which covers $I_* \cap \{f^- + 1 > f^+\}$

and has finite overlap. The intervals $(I_k \cap I_*)_k$ obey property (4.53) with $M = 6$, since $\frac{f^-(t)+1-f^+(t)}{4L} \le \frac{1}{4L} = \ell^*$ implies $\mathcal{L}^1(I_k \cap I_*) \ge \frac{1}{2}\mathcal{L}^1(I_k)$. Therefore by (4.60)

$$\int_{\omega_B^{I_*}} |\nabla u - QB_\delta|^2 dx = \int_{\omega_B^{I_* \cap \{f^-+1>f^+\}}} |\nabla u - QB_\delta|^2 dx \le c_L \sum_k \int_{\omega_B^{I_* \cap I_k}} |\nabla u - QB_\delta|^2 dx$$

$$\le c_L \sum_k \hat{\mathcal{E}}(I_* \cap I_k) \le c_L \hat{\mathcal{E}}(I_*)$$

(4.62)

and with (4.51) and $\hat{\mathcal{E}}(I_*) \le c_L \mathcal{E}(I_*)$ the proof is concluded. $\qquad\square$

Proposition 4.9 *Let $L > 0$. There are $c_L > 0$, $\delta_L > 0$ such that for all $\theta \in (0, \frac{1}{2}]$, all $\delta \in [-\delta_L, \delta_L]$, all $(f^\pm, u) \in \mathcal{A}_{\mathrm{nl}}^L$ there holds*

$$\mathcal{E}_{\mathrm{nl}}[f^\pm, u] \ge c_L |\delta|^3 \theta^2.$$

Proof: For brevity in this proof we write $\mathcal{E}(I) := \mathcal{E}[I; (f^\pm, u)]$ and $E := \mathcal{E}_{\mathrm{nl}}[f^\pm, u]$.
Step 1. Piecewise affine approximation. Consider the intervals

$$I_j := \left(\frac{j}{8L}, \frac{j+2}{8L} \right).$$

By Proposition 4.8 there are rotations $Q_j \in \mathrm{SO}(2)$ such that for any $j \in \mathbb{N}$ one has

$$\int_{\omega_A^{I_j}} |\nabla u - Q_j A_\delta|^2 dx + \int_{\omega_B^{I_j}} |\nabla u - Q_j B_\delta|^2 dx \le c_L \mathcal{E}(I_j) \qquad (4.63)$$

with the constant c_L (here and in all following estimates) depending only on L.
We shall use this estimate and the periodicity condition to obtain four different bounds, which are stated in (4.64), (4.65), (4.67) and (4.68).
Step 2. Continuity term. With a triangular inequality, using $\mathcal{L}^1(I_j \cap I_{j+1}) = 1/(8L)$ from (4.63) and $|A_\delta^{-1}| \le 3$ we obtain

$$\sum_{j \in \mathbb{N}} |Q_j - Q_{j+1}|^2 \le c_L \mathcal{E}((0, \infty)). \qquad (4.64)$$

Step 3. Left boundary term. By the trace theorem used in $\omega_B^{I_0}$, recalling that $f^-(0) = f^+(0)$, from (4.63) we obtain that for some $d_0 \in \mathbb{R}^2$

$$\int_{(0,1)} |u(te_{\delta,\theta}) - Q_0 B_\delta e_{\delta,\theta} t - d_0|^2 d\mathcal{H}^1 \le c_L \mathcal{E}(I_0) \le c_L E.$$

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By geometric rigidity and the trace theorem used in $T_{\delta,\theta}((-1,0) \times (0,2))$ there are $Q_- \in$ SO(2) and $d_- \in \mathbb{R}^2$ such that

$$\int_{(0,2)} |u(te_{\delta,\theta}) - Q_- A_\delta e_{\delta,\theta} t - d_-|^2 d\mathcal{H}^1 \leq c_L \int_{T_{\delta,\theta}((-1,0)\times(0,2))} W(\nabla u A_\delta^{-1}) dx \leq c_L E$$

which, with the periodicity condition, implies

$$|Q_- A_\delta e_{\delta,\theta} - v_\theta| \leq c_L E^{1/2}$$

where $v_\theta := (\theta A_\delta + (1-\theta)B_\delta)e_{\delta,\theta} = (\mathrm{Id} + \delta(2\theta-1)e_1 \otimes e_2)e_{\delta,\theta}$. With a triangular inequality,

$$|Q_0 B_\delta e_{\delta,\theta} - v_\theta| \leq c_L E^{1/2}.$$

We observe that $B_\delta e_{\delta,\theta} - v_\theta = -\theta(A_\delta - B_\delta)e_{\delta,\theta} = -2\delta\theta e_1/\sqrt{1 + \delta^2\theta^2}$. As $|v_\theta - e_2| \leq c|\delta|$, $B_\delta e_{\delta,\theta} = v_\theta + 2\delta\theta v_\theta^\perp + O(\delta^2\theta)$ we have

$$|Q_0(v_\theta + 2\delta\theta v_\theta^\perp) - v_\theta| \leq c_L \left(E^{1/2} + \delta^2\theta \right)$$

and from Lemma 4.4 we obtain

$$|Q_0 - \mathrm{Id} + 2\delta\theta J| \leq c_L E^{1/2} + c_L \delta^2\theta. \tag{4.65}$$

Step 4. Volume term. The periodicity condition (4.2) implies that the average of $\nabla u e_{\delta,\theta}$ over $\omega_A^{I_j} \cup \omega_B^{I_j}$ coincides with v_θ,

$$\frac{1}{\mathcal{L}^2(\omega_A^{I_j} \cup \omega_B^{I_j})} \int_{\omega_A^{I_j} \cup \omega_B^{I_j}} \nabla u e_{\delta,\theta} dx = (\theta A_\delta + (1-\theta)B_\delta)e_{\delta,\theta} = v_\theta.$$

From (4.63) we obtain

$$\left| \int_{\omega_A^{I_j}} \nabla u e_{\delta,\theta} dx - \mathcal{L}^2(\omega_A^{I_j}) Q_j A_\delta e_{\delta,\theta} \right| + \left| \int_{\omega_B^{I_j}} \nabla u e_{\delta,\theta} dx - \mathcal{L}^2(\omega_B^{I_j}) Q_j B_\delta e_{\delta,\theta} \right| \leq c_L \mathcal{E}^{1/2}(I_j).$$

Therefore, setting $\lambda_j := \mathcal{L}^2(\omega_A^{I_j})/\mathcal{L}^2(\omega_A^{I_j} \cup \omega_B^{I_j})$ and $w_j := (\lambda_j A_\delta + (1-\lambda_j)B_\delta)e_{\delta,\theta} = (\mathrm{Id} + \delta(2\lambda_j - 1)e_1 \otimes e_2)e_{\delta,\theta}$, we obtain

$$|Q_j w_j - v_\theta| \leq c_L \mathcal{E}^{1/2}(I_j). \tag{4.66}$$

Since

$$w_j - v_\theta = 2\delta(\lambda_j - \theta)(e_1 \otimes e_2)e_{\delta,\theta} = \frac{2\delta(\lambda_j - \theta)}{\sqrt{1 + \delta^2\theta^2}} e_1$$

and $|e_1 + v_\theta^\perp| \leq c|\delta|$, we have $|Q_j(v_\theta + 2\delta(\theta - \lambda_j)v_\theta^\perp) - v_\theta| \leq c\mathcal{E}^{1/2}(I_j) + c\delta^2|\lambda_j - \theta|$ and again by Lemma 4.4 we obtain our first volume estimate

$$|Q_j - \mathrm{Id} + 2\delta(\theta - \lambda_j)J| \leq c_L \left(\mathcal{E}^{1/2}(I_j) + \delta^2|\lambda_j - \theta| \right). \tag{4.67}$$

At the same time,

$$|v_\theta|^2 = \frac{1 + \delta^2(\theta - 1)^2}{1 + \delta^2\theta^2}$$

and

$$|Q_j w_j|^2 = |w_j|^2 = \frac{1 + \delta^2(2\lambda_j - \theta - 1)^2}{1 + \delta^2\theta^2}$$

so that from (4.66) and $|w_j| + |v_\theta| \leq C$ we obtain the second volume estimate,

$$c_L \mathcal{E}(I_j)^{1/2} \geq |Q_j w_j - v_\theta| \cdot |Q_j w_j + v_\theta| \geq \big||w_j|^2 - |v_\theta|^2\big| = \frac{4\delta^2(1 - \lambda_j)|\theta - \lambda_j|}{1 + \delta^2\theta^2}. \quad (4.68)$$

Step 5. Conclusion of the proof. Using (4.65) and (4.67) for $j = 0$, there is $c_L > 0$ (fixed for the rest of the proof) such that

$$|\delta|\,|\lambda_0| \leq c_L E^{1/2} + c_L \delta^2 \theta + c_L \delta^2 |\lambda_0 - \theta| \leq c_L E^{1/2} + 2c_L \delta^2 \theta + c_L \delta^2 |\lambda_0|.$$

Choose $\delta_0 > 0$ such that $\delta_0 \leq 1/(6c_L)$. Then for $|\delta| \leq \delta_0$, we have $c_L \delta^2 |\lambda_0| \leq \frac{1}{6}|\delta|\,|\lambda_0|$, the last term can be absorbed in the left-hand side, and we obtain

$$\frac{5}{6}|\delta|\,|\lambda_0| \leq c_L E^{1/2} + 2c_L \delta^2 \theta.$$

If $|\lambda_0| \geq \frac{1}{2}\theta$ then, using $2c_L \delta^2 \theta \leq \frac{1}{3}|\delta|\theta$, we obtain

$$\frac{5}{12}|\delta|\theta \leq c_L E^{1/2} + \frac{1}{3}|\delta|\theta$$

so that $E \geq C\delta^2\theta^2$ and we are done.

Assume now that $|\lambda_0| \leq \frac{1}{2}\theta$. By (4.68), for any $j \in \mathbb{N}$ such that $|\lambda_j| \leq \frac{1}{2}\theta$ we have $\mathcal{E}(I_j) \geq c'_L \delta^4 \theta^2$. Therefore there are at most finitely many such j. Let $\ell := \min\{j \in \mathbb{N} : |\lambda_j| > \frac{1}{2}\theta\}$, so that (4.68) gives

$$E \geq c'_L \ell \delta^4 \theta^2. \quad (4.69)$$

Using again (4.65) and (4.67) for $j = \ell$,

$$|Q_\ell - Q_0 - 2\delta\lambda_\ell J| \leq c''_L E^{1/2} + c''_L \delta^2 \theta + c''_L \delta^2 |\lambda_\ell|.$$

As $|2\delta\lambda_\ell J| = 2\sqrt{2}|\delta|\,|\lambda_\ell|$, if δ_0 is chosen so that $c''_L \delta_0 \leq \frac{1}{2}$ then

$$\begin{aligned}
|Q_\ell - Q_0| &\geq 2\sqrt{2}|\delta|\,|\lambda_\ell| - c''_L E^{1/2} - c''_L \delta^2 \theta - c''_L \delta^2 |\lambda_\ell| \\
&\geq 2|\delta|\,|\lambda_\ell| - c''_L E^{1/2} - c''_L \delta^2 \theta \\
&\geq |\delta|\,\theta - c''_L E^{1/2} - c''_L \delta^2 \theta \geq \frac{1}{2}|\delta|\theta - c''_L E^{1/2},
\end{aligned}$$

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where in the third step we used $|\lambda_\ell| \geq \frac{1}{2}\theta$. As above, if $E \geq (4c_L'')^{-2}\delta^2\theta^2$ then we are done. Otherwise $|Q_\ell - Q_0| \geq \frac{1}{4}|\delta|\theta$. With (4.64) and Cauchy-Schwarz we conclude

$$c_L''' E \geq \sum_{j=0}^{\ell-1} |Q_j - Q_{j+1}|^2 \geq \frac{1}{\ell}|Q_0 - Q_\ell|^2 \geq \frac{1}{16}\frac{\delta^2\theta^2}{\ell}$$

and therefore, recalling (4.69),

$$E \geq C_L \left(\frac{\delta^2\theta^2}{\ell} + \ell\delta^4\theta^2 \right) \geq 2C_L|\delta|^3\theta^2$$

which concludes the proof. $\qquad\square$

4.3 Existence of minimizers

Proposition 4.10 *Let W obey (4.13) and be quasiconvex. Let $L \geq 1$, $\theta \in (0, \frac{1}{2}]$, $\delta \in [-1, 1]$. Then the functional $\mathcal{E}_{\mathrm{nl}}$ defined in (4.12) has a minimizer in the set $\mathcal{A}_{\mathrm{nl}}^L$ defined in (4.11).*

Proof: We proceed along the lines of the proof of Proposition 3.3(i). Let $(f_j^\pm, u_j) \subset \mathcal{A}_{\mathrm{nl}}^L$ be a minimizing sequence. By subtracting constants, we can assume without loss of generality that

$$\int_{(0,1)\times(0,1)} u_j(x)\,dx = 0. \tag{4.70}$$

After passing to a subsequence, the functions $f_j^\pm$ converge locally uniformly to L-Lipschitz functions $f_*^\pm$ which by uniform convergence satisfy (4.3). For every $m \in \mathbb{N}$, by the lower bound in (4.13), there is a uniform bound on the L^2-norms

$$\|\nabla u_j\|_{L^2(T_{\delta,\theta}((-m,m)^2))} \leq Cm \left(\mathcal{E}((-m,m), (f_j^\pm, u_j)) + m|A_\delta|^2 \right),$$

and hence, by (4.70), there is a subsequence that converges locally weakly in $W^{1,2}$ to $u_* \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^2; \mathbb{R}^2)$. By Rellich's theorem, the limiting function u^* satsifies the periodicity condition (4.2). Let ω_A^* and ω_B^* denote the respective domains induced by $f_*^\pm$, which are defined as in (4.7)–(4.9). By quasiconvexity of W and the growth condition (4.13), we get lower semi-continuity of the energy restricted to compact sets in $(\omega_A^* \cup \omega_B^*) \setminus \mathrm{graph}(f_*^\pm) \setminus (\mathbb{R}e_{\delta,\theta})$, and hence, choosing a diagonal sequence as in the proof of Proposition 3.3, we find for every $m > 0$,

$$\mathcal{E}((-m,m), (f_*^\pm, u_*)) \leq \liminf_{j\to\infty} \mathcal{E}((-m,m), (f_j^\pm, u_j)). \tag{4.71}$$

Then

$$\mathcal{E}((-m, m), (f_*^\pm, u_*)) \leq \liminf_{j \to \infty} \mathcal{E}(\mathbb{R}, (f_j^\pm, u_j)) = \liminf_{j \to \infty} \mathcal{E}_{\mathrm{nl}}[f_j^\pm, u_j] = \inf_{\mathcal{A}_{\mathrm{nl}}^L} \mathcal{E}_{\mathrm{nl}}. \quad (4.72)$$

As $\mathcal{E}_{\mathrm{nl}}[f_*^\pm, u_*] = \mathcal{E}(\mathbb{R}, (f_*^\pm, u_*)) = \sup\{\mathcal{E}((-m, m), (f_*^\pm, u_*)) : m > 0\}$, this concludes the proof. $\qquad\square$

It would be natural to ask if for a minimizer $(f_*^\pm, u_*)$ the functions $f_*^\pm(x_1)$ have a finite limit as $x_1 \to \infty$. This does not follow from the above proof. Another open question is whether the condition that $f^\pm$ are Lipschitz is needed, or if it can be replaced by a term of the form

$$\sigma \int_0^\infty (\sqrt{1 + (f_+'(x_1))^2} + \sqrt{1 + (f_-'(x_1))^2} - 2)dx_1, \quad (4.73)$$

which represents the additional length of the interfaces with respect to the "flat" situation.

5 Korn's inequality and geometric rigidity for uniformly Lipschitz domains

5.1 Uniformly Lipschitz domains

We show that certain estimates on Sobolev functions hold uniformly for a class of bounded open sets which are uniformly Lipschitz. We focus on bounded sets and use the following definition, which is a variant of the one in [Leo17, Def. 13.11]. For $x \in \mathbb{R}^n$, we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Definition 5.1 *Let $L, R > 0$. An open set $\Omega \subseteq \mathbb{R}^n$ is (L, R)-Lipschitz if there is $\varepsilon > 0$ such that:*

(i) $|x - y| < R\varepsilon$ for all $x, y \in \Omega$;

(ii) For each $x \in \partial\Omega$ there are $f_x \in \mathrm{Lip}(\mathbb{R}^{n-1}; \mathbb{R})$ with $\mathrm{Lip}(f_x) \leq L$ and an isometry $A_x : \mathbb{R}^n \to \mathbb{R}^n$ such that $B_\varepsilon(x) \cap \Omega = B_\varepsilon(x) \cap V_x$, where

$$V_x := A_x\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n < f_x(y')\}. \quad (5.1)$$

This definition ensures on the one hand uniformity of the Lipschitz constant, on the other hand uniform size of the neighbourhoods in which (5.1) holds with respect to the size of Ω.

Remark 5.2 *(i) The definition is monotonous, in the sense that if Ω is (L, R)-Lipschitz then it is also (L', R')-Lipschitz for any $L' \geq L$, $R' \geq R$.*

(ii) *From $x \in \partial\Omega$ one immediately obtains that $\hat{y} := A_x^{-1}x$ obeys $\hat{y}_n = f(\hat{y}')$; one can assume without loss of generality that $\hat{y} = 0$.*

(iii) *Condition (ii) implies that the open segment joining $x = A_x\hat{y}$ with $A_x(\hat{y} - \varepsilon e_n)$ belongs to Ω; in particular $R \geq 1$.*

(iv) *Suppose $\Omega \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ satisfy property (i) of Definition 5.1 and property (ii) with $B_\varepsilon(x)$ replaced by $x + ((-\varepsilon, \varepsilon)^{n-1} \times (-2\varepsilon L, 2\varepsilon L))$. Then Ω is (L, R_0)-Lipschitz with $R_0 = R\max\{1, 1/(2L)\}$. On the other hand, if Ω is (L, R)-Lipschitz then Ω satisfies property (ii) of Definition 5.1 with $B_\varepsilon(x)$ replaced by $x + ((-\varepsilon_0, \varepsilon_0)^{n-1} \times (-2\varepsilon_0 L, 2\varepsilon_0 L))$ with $\varepsilon_0 := \frac{\varepsilon}{\sqrt{n}}\min\{1, 2L\}$. Similar statements holds for other sets whose size is uniformly controlled by ε.*

(v) *One can reduce to the case that only finitely many functions f_x appear, after changing R to $2R$. Indeed, if $\Omega \subset \mathbb{R}^n$ is (L, R)-Lipschitz then (using the notation of Definition 5.1) the balls $B_{\varepsilon/2}(x)$, $x \in \partial\Omega$, cover $\partial\Omega$. By Vitali's covering theorem there is a subset $x_1, \ldots, x_M$ such that $\partial\Omega \subseteq \cup_{i=1}^M B_{\varepsilon/2}(x_i)$ and $B_{\varepsilon/10}(x_i) \cap B_{\varepsilon/10}(x_j) = \emptyset$ for $i \neq j$. Since $\partial\Omega$ is contained in a ball of radius $R\varepsilon$, we have $M \leq (10 \cdot R + 1)^n$. As for every $x \in \partial\Omega$ there is $i \in \{1, \ldots, M\}$ such that $B_{\varepsilon/2}(x) \subseteq B_\varepsilon(x_i)$, only the M functions $f_{x_1}, \ldots, f_{x_M}$ are relevant.*

(vi) *Let Ω be such that there are open sets ω_i, $i = 1, \ldots, M$, Lipschitz functions f_i with $\mathrm{Lip}(f_i) \leq L$ and isometries A_i such that $\omega_i \cap \Omega = \omega_i \cap V_i$, $V_i := A_i\{y : y_n < f_i(y')\}$, and assume that there is $\varepsilon > 0$ such that for all $x \in \partial\Omega$ there is i with $B_\varepsilon(x) \subseteq \omega_i$. Then property (ii) in Definition 5.1 holds and Ω is $(L, \mathrm{diam}(\Omega)/\varepsilon)$-Lipschitz.*

Lemma 5.3 *Let $\Omega \subseteq \mathbb{R}^n$ be (L, R)-Lipschitz, $F \in \mathbb{R}^{n \times n}$ an invertible matrix. Then the set $F\Omega \subseteq \mathbb{R}^n$ is $(c_F(L + 1), c_F R)$ Lipschitz, with $c_F := |F| \cdot |F^{-1}|$.*

Proof: Let $\varepsilon > 0$ as in Definition 5.1 for Ω and set $\varepsilon' := \varepsilon/|F^{-1}|$. Pick $y \in \partial(F\Omega)$ and let $x := F^{-1}y \in \partial\Omega$. We first show that

$$B_{\varepsilon'}(y) \subseteq FB_\varepsilon(x).$$

To see this, we pick $z \in B_{\varepsilon'}(y)$, consider $z' := F^{-1}z$, and compute $|z' - x| = |F^{-1}(z - y)| \leq |F^{-1}||z - y| < \varepsilon$. Therefore $z' \in B_\varepsilon(x)$ and $z \in FB_\varepsilon(x)$. Further, $\mathrm{diam}(F\Omega) \leq |F|\,\mathrm{diam}\,\Omega$, hence setting $R' := |F|\,|F^{-1}|R$ we have $\mathrm{diam}(F\Omega) \leq R'\varepsilon'$.

Fix again $y \in \partial(F\Omega)$. We need to show that

$$(F\Omega) \cap B_{\varepsilon'}(y) = I_y\{w_n < g(w')\} \cap B_{\varepsilon'}(y) \tag{5.2}$$

for some isometry I_y and some $g \in \mathrm{Lip}(\mathbb{R}^{n-1}; \mathbb{R})$ with $\mathrm{Lip}(g) \leq L'$; the precise value of L' is given below.

We let $x := F^{-1}y$ as above. By property (ii) in Definition 5.1 there are an isometry A_x and an L-Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B_\varepsilon(x) = A_x\{z_n < f(z')\} \cap B_\varepsilon(x). \tag{5.3}$$

Then $B_{\varepsilon'}(y) \subseteq FB_\varepsilon(x)$ implies

$$(F\Omega) \cap B_{\varepsilon'}(y) = (F(\Omega \cap B_\varepsilon(x)) \cap B_{\varepsilon'}(y) = (FA_x\{z_n < f(z')\}) \cap B_{\varepsilon'}(y). \tag{5.4}$$

The isometry A_x can be written as $A_x z = b + Rz$ for some $R \in \mathrm{O}(n)$. We set $\eta := |FRe_n| \in (0, |F|]$, pick a rotation $Q \in \mathrm{SO}(n)$ such that $FRe_n = \eta Qe_n$, and let $T := Q^T FR \in \mathbb{R}^{n \times n}$. Then $FR = QT$ and $QTe_n = \eta Qe_n$, which implies $Te_n = \eta e_n$. We shall show below that

$$T\{z_n < f(z')\} = \{w_n < g(w')\} \tag{5.5}$$

with g as stated after (5.2). This implies that

$$\begin{aligned}
FA_x\{z_n < f(z')\} &= Fb + FR\{z_n < f(z')\} = Fb + QT\{z_n < f(z')\} \\
&= Fb + Q\{w_n < g(w')\} = I_y\{w_n < g(w')\}
\end{aligned} \tag{5.6}$$

where $I_y w := Fb + Qw$, which together with (5.4) concludes the proof of (5.2) and therefore of the Lemma.

It remains to construct $g \in \mathrm{Lip}(\mathbb{R}^{n-1}; \mathbb{R})$ such that (5.5) holds. We observe that T is invertible, with $|T^{-1}| = |F^{-1}|$ and $T^{-1}e_n = \frac{1}{\eta}e_n$, and write

$$T^{-1}(w', w_n) = \left(Sw', \frac{1}{\eta}w_n + s \cdot w'\right) \tag{5.7}$$

for some $s \in \mathbb{R}^{n-1}$ and $S \in \mathbb{R}^{(n-1)\times(n-1)}$, invertible. They obey $\max\{|S|, |s|\} \leq |T^{-1}| = |F^{-1}|$. Then

$$\frac{1}{\eta}w_n + s \cdot w' < f(Sw') \tag{5.8}$$

is the same as

$$w_n < g(w') := \eta \cdot (f(Sw') - s \cdot w') \tag{5.9}$$

so that

$$\{z_n < f(z')\} = T^{-1}\{w_n < g(w')\}, \tag{5.10}$$

which is the same as (5.5). The function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ constructed above is Lipschitz, with $\mathrm{Lip}(g) \leq \eta(|S|\,\mathrm{Lip}(f) + |s|) \leq |F|(|F^{-1}|L + |F^{-1}|)$. This concludes the proof. $\qquad\square$

Lemma 5.4 *Let* $f, g : [0, \ell] \to \mathbb{R}$, $\ell > 0$, $L > 0$, *and set*

$$\omega_{f,g} := \{x \in \mathbb{R}^2 : x_1 \in (0, \ell), \ f(x_1) < x_2 < g(x_1)\}.$$

Assume that $\alpha\ell \leq g - f \leq \beta\ell$ *for some* $\alpha, \beta > 0$, *and that* $\mathrm{Lip}(f), \mathrm{Lip}(g) \leq L$. *Then* $\omega_{f,g}$ *is* $(2L + 1, R)$*-Lipschitz for some* R *which depends only on* α, β *and* L.
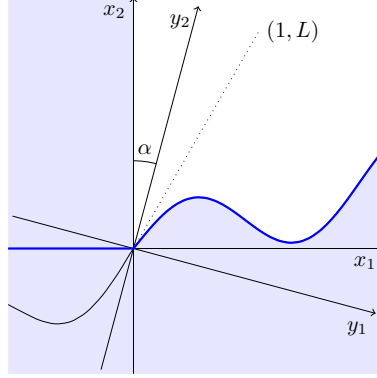
Figure 6: Sketch of the rotation in the proof of Lemma 5.4. The shaded area is the one where $y_2 < F(y_1)$, the dotted line shows the direction $(1, L)$.

Proof: Let $\varepsilon := \frac{\ell}{2(1+L)} \min\{\alpha, 1\}$, $R := (\beta + 1 + L)\ell/\varepsilon$. Clearly R depends only on α, β, and L, and its definition ensures that condition (i) in Definition 5.1 holds. The choice of ε ensures that $|(x_1, f(x_1)) - (x_1', g(x_1'))| \geq \varepsilon$ for all x_1, $x_1' \in [0, \ell]$, so that the top and bottom boundaries can be treated separately.

We consider points close to the lower-left corner, in the sense that we show property (ii) of Definition 5.1 for x in (see Figure 6)

$$A_{LL} := \{0\} \times [f(0), \frac{f(0) + g(0)}{2}] \cup \{(x_1, f(x_1)) : x_1 \in [0, \frac{1}{2}\ell]\} \subseteq \partial\omega_{f,g}; \qquad (5.11)$$

the other three corners can be treated analogously. We extend f to $\mathbb{R}$, setting $f(t) = f(0)$ for $t < 0$ and $f(t) = f(\ell)$ for $t > \ell$. By the choice of ε we see that for all $x \in A_{LL}$ the ball $B_\varepsilon(x)$ does not intersect $\{z : z_2 = g(z_1)\}$ and $\{z : z_1 = \ell\}$, in the sense that

$$B_\varepsilon(x) \cap \omega_{f,g} = B_\varepsilon(x) \cap \{z \in \mathbb{R}^2 : z_1 > 0, z_2 > f(z_1)\}. \qquad (5.12)$$

After a translation, we can assume $f(0) = 0$. In order to make the mentioned boundary the graph of a Lipschitz function we need a nontrivial rotation, as illustrated in Figure 6. Let $Q \in \mathrm{SO}(2)$ be such that Qe_2 bisects the angle between e_2 and $(1, L)$, which means that Q is a clockwise rotation by $\alpha := \frac{1}{2}(\frac{\pi}{2} - \arctan L)$. Then there is a unique function $F : \mathbb{R} \to \mathbb{R}$ such that

$$\{z \in \mathbb{R}^2 : z_2 > f(z_1)\} = Q\{y \in \mathbb{R}^2 : y_2 > F(y_1)\}, \qquad (5.13)$$

obviously $F(0) = 0$. One can check that F is $L' := \tan(\frac{\pi}{2} - \alpha)$-Lipschitz, and that $L' = L + \sqrt{1 + L^2} \leq 1 + 2L$. At the same time, by the definition of Q and L'

$$\{z \in \mathbb{R}^2 : z_1 > 0\} = Q\{y \in \mathbb{R}^2 : y_2 > -L'y_1\}. \qquad (5.14)$$

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Recalling (5.12), we see that it suffices to intersect these two sets. We define $\tilde{F}(t) := \max\{-L'y_1, F(y_1)\}$, which is also L'-Lipschitz. Then for every $x \in A_{LL}$ we have

$$B_\varepsilon(x) \cap \omega_{f,g} = B_\varepsilon(x) \cap Q\{y : y_2 > \tilde{F}(y_1)\}. \tag{5.15}$$

This concludes the proof. $\qquad\square$

5.2 Weighted Poincaré inequality

The next result is a quantitative version of the estimate in [Kuf80, Theorem 8.8].

Theorem 5.5 (Weighted Poincaré) *Let $\Omega \subseteq \mathbb{R}^n$ be a connected, bounded (L, R)-Lipschitz set, $p \in [1, \infty)$. Then for any $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^k)$ there is $a \in \mathbb{R}^k$ such that*

$$\|u - a\|_{L^p(\Omega)} \leq c_{\mathrm{WP}} \| \operatorname{dist}(\cdot, \partial\Omega)\nabla u\|_{L^p(\Omega)}. \tag{5.16}$$

In particular, $u \in L^p(\Omega; \mathbb{R}^k)$. The constant c_{WP} depends only on n, p, L and R.

We first prove the result in one dimension, by an explicit computation.

Lemma 5.6 *Let $I = (a, b) \subseteq \mathbb{R}$ be a bounded interval, $\varphi \in C^1(I)$, $E \subseteq I$ with positive measure, $\alpha \in \mathbb{R}$, $p \in [1, \infty)$. Then*

$$\int_I |\varphi - \alpha|^p dt \leq c_p \frac{\mathcal{L}^1(I)}{\mathcal{L}^1(E)} \left[\int_E |\varphi - \alpha|^p dt + \int_I (t - a)^p |\varphi'|^p(t) dt \right]. \tag{5.17}$$

The constant c_p depends only on p.

Proof: If the right-hand side is finite, then the function φ can be extended continuously to $(a, b]$. Let $\beta := \varphi(b)$. For any $x \in I$ by the fundamental theorem of calculus applied to $|\varphi - \beta|^p$ we have

$$|\varphi - \beta|^p(x) \leq p \int_x^b |\varphi(t) - \beta|^{p-1} |\varphi'(t)| dt. \tag{5.18}$$

We integrate over all $x \in I$ and use Fubini's theorem,

$$\int_a^b |\varphi - \beta|^p(x) dx \leq p \int_a^b \int_x^b |\varphi(t) - \beta|^{p-1} |\varphi'(t)| dt \, dx = p \int_a^b (t - a) |\varphi(t) - \beta|^{p-1} |\varphi'(t)| dt. \tag{5.19}$$

With Hölder's inequality,

$$\int_a^b |\varphi - \beta|^p(x) dx \leq p \| |\varphi - \beta|^{p-1} \|_{L^{p'}(I)} \| (\cdot - a)\varphi' \|_{L^p(I)} \tag{5.20}$$

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so that, as $\||\varphi - \beta|^{p-1}\|_{L^{p'}} = \|\varphi - \beta\|_{L^p}^{p-1}$,

$$\|\varphi - \beta\|_{L^p(I)} \leq p\|(\cdot - a)\varphi'\|_{L^p(I)}. \tag{5.21}$$

By a triangular inequality,

$$|\alpha - \beta|(\mathcal{L}^1(E))^{1/p} \leq \|\varphi - \alpha\|_{L^p(E)} + \|\varphi - \beta\|_{L^p(I)} \tag{5.22}$$

so that, with a further triangular inequality,

$$\begin{aligned}
\|\varphi - \alpha\|_{L^p(I)} &\leq |\alpha - \beta|(\mathcal{L}^1(I))^{1/p} + \|\varphi - \beta\|_{L^p(I)} \\
&\leq 2p\frac{(\mathcal{L}^1(I))^{1/p}}{(\mathcal{L}^1(E))^{1/p}} \left[\|\varphi - \alpha\|_{L^p(E)} + \|(t - a)\varphi'\|_{L^p(I)}\right]
\end{aligned} \tag{5.23}$$

concludes the proof. $\qquad\square$

Lemma 5.7 *Let $\Omega \subset \mathbb{R}^n$ be (L, R)-Lipschitz, ε as in Definition 5.1, $x_* \in \partial\Omega$, $r \in (0, \varepsilon/(4 + 4L)]$, $p \in [1, \infty)$. For any $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ there is $\alpha \in \mathbb{R}$ such that*

$$\int_{\Omega \cap B_r(x_*)} |u - \alpha|^p dx \leq c \int_\Omega \mathrm{dist}^p(x, \partial\Omega)|\nabla u|^p(x)dx. \tag{5.24}$$

The constant c depends only on n, p, and L.

Proof: By Definition 5.1 we have $B_\varepsilon(x_*) \cap \Omega = B_\varepsilon(x_*) \cap V$, where (as in (5.1))

$$V := A\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n < f(y')\} \tag{5.25}$$

for some isometry A and L-Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. As the assertion is invariant under rotations and translations we can assume that A is the identity and that $x_* = 0$; from $x_* \in \partial\Omega$ we then obtain $f(0) = 0$.

Let $h := rL \in (0, \varepsilon/4)$ and consider the cylinder $T := B_r' \times (-3h, -2h)$ (see Figure 7). For any $x' \in B_r'$ we have $f(x') \geq -rL = -h$, and therefore $f(x') - x_n \geq h$ for all $(x', x_n) \in T$. Further, from $(3h)^2 + r^2 \leq \frac{9}{16}\varepsilon^2 + \frac{1}{16}\varepsilon^2 = \frac{5}{8}\varepsilon^2 \leq \varepsilon^2$ we obtain that $T \subseteq B_\varepsilon \cap V$. As the shape of T (up to uniform rescaling) depends only on L, by the usual Poincaré inequality (for a fixed domain) there is $\alpha \in \mathbb{R}$ with

$$\int_T |u - \alpha|^p dx \leq cr^p \int_T |\nabla u|^p dx \tag{5.26}$$

with c depending only on n, p, L. For every $x' \in B_r'$ we apply Lemma 5.6 to $u(x', \cdot)$ with $I = (-3h, f(x'))$ and $E = (-3h, -2h)$, and obtain, using $h = \mathcal{L}^1(E) \leq \mathcal{L}^1(I) \leq 4h$,

$$\int_I |u(x) - \alpha|^p dx_n \leq c \int_I |f(x') - x_n|^p |\nabla u|^p(x)dx_n + c \int_E |u(x) - \alpha|^p dx_n. \tag{5.27}$$

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Let $U := (B'_r \times (-3h, \infty)) \cap V$, so that $B_r \cap \Omega = B_r \cap U$ and $U \subseteq B_\varepsilon \cap V = B_\varepsilon \cap \Omega$. We integrate over $x' \in B'_r$, use (5.26) and $rL = h \leq f(x') - x_n$ for all $(x', x_n) \in T$ to conclude

$$\int_U |u - \alpha|^p dx \leq c \int_{B'_r} \int_{(-3h, f(x'))} |f(x') - x_n|^p |\nabla u|^p dx_n dx' + c \int_T |f(x') - x_n|^p |\nabla u|^p dx$$

$$\leq c \int_U |f(x') - x_n|^p |\nabla u|^p dx,$$

(5.28)

where the constant c depends only on L, p, and n. We finally show that there is $c_L > 0$, depending only on L, such that

$$|f(x') - x_n| \leq c_L \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in U. \tag{5.29}$$

Indeed, for any $x \in U$ let $y \in \partial\Omega$ be such that $d := \operatorname{dist}(x, \partial\Omega) = |x - y|$. We know that $|f(x')| \leq h = rL$ and that $x_n \in (-3h, h)$, which imply $|f(x') - x_n| \leq 4h \leq \varepsilon$, and with $|x'| < r$ we obtain $|x| \leq \sqrt{r^2 + (3h)^2} \leq \sqrt{\frac{5}{8}}\varepsilon$. We distinguish two cases. If $y \notin B_\varepsilon$ then $|x-y| \geq (1-\sqrt{\frac{5}{8}})\varepsilon$ and the proof of (5.29) is concluded (if c_L is sufficiently large). If instead $y \in B_\varepsilon$ then necessarily $y \in \partial V$ and $y_n = f(y')$, which implies $d^2 = |y'-x'|^2 + |f(y')-x_n|^2$. As f is L-Lipschitz, $f(y') \geq f(x') - L|x' - y'|$, so that

$$d \geq \max\{|x' - y'|, f(x') - L|x' - y'| - x_n\}. \tag{5.30}$$

If $d \geq \frac{1}{2}(f(x') - x_n)$, then we are done (with any $c_L \geq 2$). Otherwise, $\frac{1}{2}(f(x') - x_n) > d \geq (f(x') - x_n) - L|x' - y'|$ implies $Ld \geq L|x' - y'| \geq \frac{1}{2}(f(x') - x_n)$ which concludes the proof of (5.29) for any $c_L \geq 2L$.

Inserting (5.29) and $B_r \cap V \subseteq U$ in (5.28) concludes the proof. $\qquad\square$

Proof:[Proof of Theorem 5.5] It suffices to consider the scalar case; by density it suffices to prove the inequality if u is $C^1(\Omega)$. For brevity, let $A := \|\operatorname{dist}(\cdot, \partial\Omega)\nabla u\|_{L^p(\Omega)}$. Let ε be as in Definition 5.1 and fix $r_B := \varepsilon/(12(L+1))$ (the reason will become clear below). We first show that for every $x \in \overline{\Omega}$ there is $\alpha(x) \in \mathbb{R}$ such that

$$\|u - \alpha(x)\|_{L^p(B_{r_B}(x) \cap \Omega)} \leq cA \tag{5.31}$$

with c depending only on n, p, L and R. To see this, we distinguish two cases. If $\operatorname{dist}(x, \partial\Omega) \geq 2r_B$ this follows from the usual Poincaré inequality applied to the ball $B_{r_B}(x) \subset \Omega$, with $\operatorname{dist}(\cdot, \partial\Omega) \geq r_B$ in $B_{r_B}(x)$. If not, we fix $x_* \in \partial\Omega$ with $|x_* - x| < 2r_B$ and use Lemma 5.7 to $B_{3r_B}(x_*)$ (this is the point where the size of r_B is fixed). As $B_{r_B}(x) \subseteq B_{3r_B}(x_*)$, this concludes the proof of (5.31).
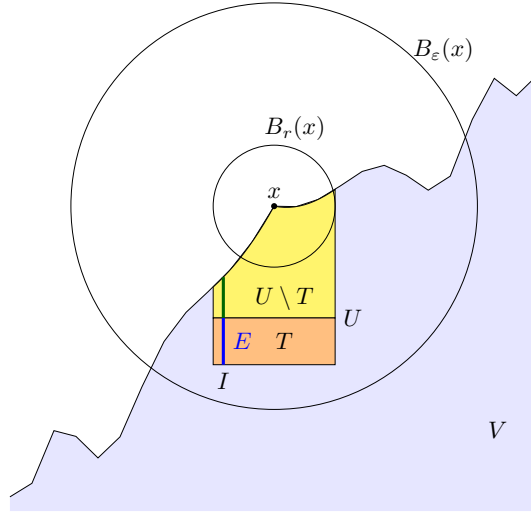
Figure 7: Sketch of the geometry in the construction of Lemma 5.7.

By Vitali's covering theorem, there is a finite set $x_0, \ldots, x_K \in \overline{\Omega}$ such that $\overline{\Omega} \subset \cup_{k=0}^K B_k$, $B_k := B_{r_B/2}(x_k)$, with the smaller balls $B_{r_B/10}(x_k)$ pairwise disjoint. In particular, since they are all centered in $\overline{\Omega}$ and the diameter of Ω is bounded by $R\varepsilon$, we obtain $K \leq (1 + 10R\varepsilon/r_B)^n \leq cR^n L^n$. Let $\alpha_k := \alpha(x_k)$. We claim that for every $k = 1, \ldots, K$ one has

$$r_B^{n/p}|\alpha_0 - \alpha_k| \leq cKA. \tag{5.32}$$

To see this, fix k, and let $j_0 := 0, j_1, \ldots, j_H := k$ be finitely many indices in $\{0, K\}$ such that $B_{j_h} \cap B_{j_{h+1}} \cap \Omega \neq \emptyset$ for all h. They exist since Ω is connected, which means that there is a continuous curve in Ω which joins a point of $B_0 \cap \Omega$ with a point of $B_k \cap \Omega$; as the curve is compact it is covered by finitely many of the balls. We can further assume the indices j_h to be distinct. Indeed, if $j_h = j_{h'}$ for some $h < h'$, we can remove $h, h+1, \ldots, h'-1$ from the set. As there are at most K balls, we obtain $H \leq K$.

In turn, $B_{j_h} \cap B_{j_{h+1}} \cap \Omega \neq \emptyset$ implies that the larger balls have significant overlap. Indeed, for each $x \in \overline{\Omega}$ one has $\mathcal{L}^n(\Omega \cap B_{r_B/2}(x)) \geq c_L r_B^n$, and recalling that the radius of the balls B_k is $r_B/2$ we obtain

$$c_L r_B^n \leq \mathcal{L}^n(B_{r_B}(x_{j_h}) \cap B_{r_B}(x_{j_{h+1}}) \cap \Omega). \tag{5.33}$$

Using (5.31) on these two balls and then a triangular inequality,

$$r_B^{n/p}|\alpha_{j_h} - \alpha_{j_{h+1}}| \leq cA, \tag{5.34}$$

which, as $H \leq K$, implies (5.32). Finally, using that the balls $B_0, \ldots, B_K$ cover Ω,

$$\|u - \alpha_0\|_{L^p(\Omega)} \leq \sum_{k=0}^K \left[\|u - \alpha_k\|_{L^p(B_k \cap \Omega)} + (\mathcal{L}^n(B_k))^{1/p}|\alpha_k - \alpha_0| \right] \leq cK^2 A. \tag{5.35}$$

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$\square$

Many well-known results from the literature follow easily from the weighted Poincaré inequality and its proof above. We start with a Poincaré inequality as follows e.g. also from [Rui12, Theorem 1.2].

Theorem 5.8 (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded (L, R)-Lipschitz set and $p \in [1, \infty)$. Then there exists a constant $c_{\mathrm{Po}} > 0$ depending only on n, p, L, and R such that for all $u \in W^{1,p}(\Omega; \mathbb{R}^k)$ there exists $d_u \in \mathbb{R}^k$ with*

$$\|u - d_u\|_{L^p(\Omega)} \leq c_{\mathrm{Po}} \operatorname{diam}(\Omega) \|\nabla u\|_{L^p(\Omega)}.$$

Proof: This follows from Theorem 5.5, using that $\operatorname{dist}(x, \partial\Omega) \leq \operatorname{diam}(\Omega)$ for all $x \in \Omega$. $\square$

With the same strategy as above we can obtain a uniform estimate on the trace operator, as a map from $W^{1,p}$ to L^p of the boundary. Also this result is well-known in the literature, see e.g. [Leo17, Theorem 18.40].

Theorem 5.9 (Trace) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (L, R)-Lipschitz set, $p \in [1, \infty)$. Then the trace operator $T : W^{1,p}(\Omega; \mathbb{R}^k) \to L^p((\partial\Omega, \mathcal{H}^{n-1}); \mathbb{R}^k)$ obeys*

$$\|Tu\|_{L^p(\partial\Omega)} \leq c_{\mathrm{Tr}}(d^{1-1/p}\|\nabla u\|_{L^p(\Omega)} + \frac{1}{d^{1/p}}\|u\|_{L^p(\Omega)}) \tag{5.36}$$

where $d := \operatorname{diam}(\Omega)$. The constant c_{Tr} depends only on n, p, L and R.

Proof: This can be proven along the same lines as Theorem 5.5. One starts showing that, in the setting of Lemma 5.6, one has

$$|\varphi(a) - \alpha|^p \leq c_p \frac{1}{\mathcal{L}^1(E)} \int_E |\varphi - \alpha|^p dt + c_p(\mathcal{L}^1(I))^{p-1} \int_I |\varphi'|^p(t)dt. \tag{5.37}$$

To see this, it suffices to observe that for any $t' \in E$ the fundamental theorem of calculus in $(a, t') \subseteq I$ gives

$$|\varphi - \alpha|(a) \leq |\varphi - \alpha|(t') + \int_I |\varphi'|(t)dt. \tag{5.38}$$

We take the p-th power, average over all $t' \in E$, and use Hölder's inequality in the last term to obtain (5.37).
In the next step we observe that, with r as in Lemma 5.7, for any $x \in \partial\Omega$

$$\int_{B_r(x)\cap\partial\Omega} |u - \alpha|^p d\mathcal{H}^{n-1} \leq cr^{p-1} \int_\Omega |\nabla u|^p dx + \frac{c}{r} \int_\Omega |u - \alpha|^p dx, \tag{5.39}$$

with c depending only on n, p, L. This follows from (5.37) with a change of variables, integrating over the same domain as in Lemma 5.7. The passage to the integral in $\mathcal{H}^{n-1}$ follows observing that $B_r(x) \cap \partial\Omega$ is the graph of an L-Lipschitz function. The coefficients can be replaced by the corresponding powers of d (by property (iii) in Remark (iii) $\varepsilon \leq \operatorname{diam}\Omega$, by property (i) in Definition (i) $\operatorname{diam}\Omega \leq R\varepsilon$). Finally, we cover $\partial\Omega$ with no more than $cR^n(\varepsilon/r)^n$ such balls, as in the proof of Theorem 5.5, and conclude the proof. $\qquad\square$

5.3 Rigidity

We prove a version of geometric rigidity from [FJM02] and of Korn's inequality with uniform constant on all (L, R)-Lipschitz sets. Instead of repeating the entire proof, and checking that the variuos constants depend on the domain only through the parameters L and R, we show that the estimate for a general domain can be reduced to the one for a cube. We refer to [CDM14] for the proof for general p and a review of the literature on Korn's inequality and rigidity. The explicit dependence of the constant on the shape of the domain was analyzed, in the specific case of long and thin domains, in [Har18, Har20]. Korn's inequality was derived for John domains and related classes using different techniques, see [DM04, ADM06, JK17] and references therein.

Theorem 5.10 (Uniform rigidity) *Let $\Omega \subseteq \mathbb{R}^n$ be a connected, bounded (L, R)-Lipschitz set, $p \in (1, \infty)$. Then for any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ there are $R \in \mathrm{SO}(n)$ and $S \in \mathbb{R}^{n\times n}_{\mathrm{skw}}$ such that*

$$\|\nabla u - R\|_{L^p(\Omega)} \leq c_{\mathrm{Rig}}\|\operatorname{dist}(\nabla u, \mathrm{SO}(n))\|_{L^p(\Omega)} \tag{5.40}$$

and

$$\|\nabla u - S\|_{L^p(\Omega)} \leq c_{\mathrm{Rig}}\|\nabla u + \nabla u^T\|_{L^p(\Omega)}. \tag{5.41}$$

The constant c_{Rig} depends only on n, p, L and R.

We recall that both estimates do not hold for $p = 1$ and $p = \infty$.

Proof: For the case that Ω is a cube both results are well known. We prove the first inequality, the proof of the second one is almost identical.

The key observation is that the constant depends on the domain only via the weighted Poincaré estimate. The general structure of the argument, using a Whitney decomposition of the domain, is similar to the one used in [FJM02] for the harmonic part. Indeed, there rigidity was first proven in small cubes, and then the second derivative was estimated using harmonic estimates. Here we instead construct a new function (called β below) which interpolates between the values of the rotation in each cube, using a partition of unity. This permits to avoid discussing the dependence on the domain of the constants involved in the initial truncation in the proof of [FJM02]. This argument is almost identical to the one used in [CG21], we repeat it here for completeness.

We intend to construct a partition of unity subordinated to a Whitney covering of Ω, done as in [Ste70, Section VI.1] or [EG92, Sect. 6.5]. Precisely, we select countably many cubes $Q^j := x_j + (-r_j, r_j)^n$ such that, letting $\hat{Q}^j := x_j + (-\frac{1}{2}r_j, \frac{1}{2}r_j)^n$ denote cubes with the same center and half the size,

$$\chi_\Omega \leq \sum_j \chi_{\hat{Q}^j} \leq \sum_j \chi_{Q^j} \leq c\chi_\Omega, \tag{5.42}$$

and

$$r_j \leq \operatorname{dist}(Q^j, \partial\Omega) \leq cr_j. \tag{5.43}$$

Both constants depend only on the dimension n. These conditions imply that the cubes have finite overlap, and that if $Q^j \cap Q^k \neq \emptyset$ then $1/c \leq r_j/r_k \leq c$.

We fix $\varphi^* \in C_c^\infty((-1,1)^n; [0,1])$ with $\varphi^* = 1$ on $(-\frac{1}{2}, \frac{1}{2})^n$, let $\hat{\varphi}_j(x) := \varphi^*((x - x_j)/r_j)$ and $\varphi_j := \hat{\varphi}_j/\sum_k \hat{\varphi}_k$. Using (5.42) one obtains $\varphi_j \in C_c^\infty(Q^j)$, $\sum_j \varphi_j = 1$ in Ω, and $|\nabla\varphi_j| \leq c/r_j$. By the estimate for the cube, for each j there is $R_j \in \operatorname{SO}(n)$ such that

$$\|\nabla u - R_j\|_{L^p(Q^j)} \leq c_{n,p}\|\operatorname{dist}(\nabla u, \operatorname{SO}(n))\|_{L^p(Q^j)}. \tag{5.44}$$

We define $\beta : \Omega \to \mathbb{R}^{n \times n}$ as a smooth interpolation between the R_j, $\beta := \sum_j \varphi_j R_j$. Using $\sum_j \varphi_j = 1$ in Ω, $\varphi_j \leq 1$ and the finite overlap,

$$\|\nabla u - \beta\|_{L^p(\Omega)}^p = \|\sum_j \varphi_j(\nabla u - R_j)\|_{L^p(\Omega)}^p \leq c\sum_j \|\nabla u - R_j\|_{L^p(Q_j)}^p. \tag{5.45}$$

Using first (5.44) in each cube Q^j and then summing via (5.42) leads to

$$\|\nabla u - \beta\|_{L^p(\Omega)}^p \leq c\sum_j \int_{Q_j} |\nabla u - R_j|^p dx \leq c\int_\Omega \operatorname{dist}^p(\nabla u, \operatorname{SO}(n))dx. \tag{5.46}$$

At the same time, the distance between ∇u and the R_j controls the derivative of β. Indeed, from $\sum_j \varphi_j = 1$ we obtain $\sum_j \nabla\varphi_j = 0$ on Ω, so that

$$\nabla\beta = \sum_j \nabla\varphi_j R_j = \sum_j \nabla\varphi_j(R_j - \nabla u). \tag{5.47}$$

At this point we recall that $|\nabla\varphi_j| \leq c/r_j$, and that $\operatorname{dist}(Q_j, \partial\Omega) \leq cr_j$, which implies that

$$\operatorname{dist}(x, \partial\Omega)|\nabla\varphi_j|(x) \leq c\chi_{Q_j}(x) \quad \text{for all } x \in \Omega. \tag{5.48}$$

Therefore

$$\int_\Omega \operatorname{dist}^p(x, \partial\Omega)|\nabla\beta|^p dx \leq c\sum_j \int_{Q_j} \operatorname{dist}^p(x, \partial\Omega)|\nabla\varphi_j|^p|\nabla u - R_j|^p dx$$
$$\leq c\sum_j \int_{Q_j} |\nabla u - R_j|^p dx. \tag{5.49}$$

We then apply the weighted Poincaré inequality in Theorem 5.5 and obtain that there is $R_* \in \mathbb{R}^{n \times n}$ with

$$\|\beta - R_*\|_{L^p(\Omega)}^p \le c \int_\Omega \operatorname{dist}^p(x, \partial\Omega)|\nabla\beta|^p dx \le c \int_\Omega \operatorname{dist}^p(\nabla u, \operatorname{SO}(n)) dx. \qquad (5.50)$$

Finally, we let R be the matrix in $\operatorname{SO}(n)$ closest to R_*. Then, using that $|R - R_*| = \operatorname{dist}(R_*, \operatorname{SO}(n)) \le |R_* - \nabla u|(x) + \operatorname{dist}(\nabla u(x), \operatorname{SO}(n))$ pointwise we obtain

$$|R - R_*|(\mathcal{L}^n(\Omega))^{1/p} \le c \| \operatorname{dist}(\nabla u, \operatorname{SO}(n))\|_{L^p(\Omega)} \qquad (5.51)$$

which, together with (5.46) and (5.50), concludes the proof. $\qquad \square$

Acknowledgments

Data Availability

No data has been produced in the original research reported in this manuscript.

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