MULTIPLE SOLUTIONS FOR DOUBLE PHASE PROBLEMS IN $\mathbb{R}^n$ VIA RICCERI'S PRINCIPLE

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Abstract. In this paper we establish some multiplicity results for double phase problems in $\mathbb{R}^n$ involving different types of nonlinearities. Our approach is based on Ricceri's principle, suitable truncation arguments and Moser iterations.

1. Introduction

In this paper we focus on the following class of nonlinear double phase problems:

$$
\begin{cases} -\operatorname{div}(a_1(x)|\nabla u|^{p_1-2}\nabla u) - \operatorname{div}(a_2(x)|\nabla u|^{p_2-2}\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}
$$

where $1 < p_1 < p_2 < n$, $\lambda, \mu > 0$ are parameters, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $a_1 : \mathbb{R}^n \to \mathbb{R}$ and $a_2 : \mathbb{R}^n \to \mathbb{R}$ are measurable functions such that

$$
a_i \in L^\infty(\mathbb{R}^n) \text{ with } \operatorname{essinf}_{x\in\mathbb{R}^n} a_i(x) > 0, \text{ for } i = 1, 2. \tag{1.2}
$$

When $a_1 = a_2 = 1$, equation in (1.1) is related to the search of stationary solutions for the following reaction diffusion system

$$
u_t = \operatorname{div}[D(u)\nabla u] + c(x,u), \quad D(u) := |\nabla u|^{p_1-2} + |\nabla u|^{p_2-2} \tag{1.3}
$$

which finds application in physics and related sciences, such as, for instance, biophysics, plasma physics, and chemical reaction design; see [4]. In these contexts, the function u in (1.3) represents a concentration, the term $\operatorname{div}[D(u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $D(u)$, and the reaction term $c(x,u)$ relates to source and loss processes. Usually, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration u. For some existence and multiplicity results for (p_1,p_2)-Laplacian problems in bounded or unbounded domains, the interested reader may

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consult [1, 2, 5, 9, 11, 12, 16, 17, 19]. On the other hand, the functional associated with the (p_1, p_2)-Laplacian operator falls in the realm of the following double-phase functional

$$\mathcal{F}_{p_1,p_2}(u;\Omega) \coloneqq \int_\Omega (|\nabla u|^{p_1} + b(x)|\nabla u|^{p_2})\,\mathrm{dx},$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $0 \leq b(x) \in L^\infty(\Omega)$, originally studied by Zhikov [24] to provide models for strongly anisotropic materials in the context of homogenization phenomena. We refer to [13, 14] for more details about the regularity of functionals with non-standard growth of (p_1, p_2)-type.

When $p_1 = p_2 = p$ and $a_1 = a_2 = a$, then (1.1) boils down to the following p-Laplacian problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.4}$$

which has been widely investigated by several authors; see for instance [6, 7, 8, 10, 22, 23]. In particular, Drabek [6] proved that there exist $\lambda > 0$ and a positive C^1 function satisfying (1.4) with $f(x,t)$ having a subcritical growth with respect to t and $g \equiv 0$. Figueiredo and Furtado [10] applied minimax theorems and Ljusternik-Schnirelmann theory to obtain the existence of a positive ground state solution of (1.4) with $\lambda = \mu = 1$, $f(x,t) = f(t)$ is a superlinear function with subcritical growth at infinity, $g(x,t) = |t|^{p^\star-2}t$ where $p^\star \coloneqq \frac{np}{n-p}$ is the critical Sobolev exponent, and they related the number of positive solutions with the topology of the set where the function $a(x)$ attains its minimum. They also provided a multiplicity result for a supercritical version of the problem under consideration. El Manouni [7] applied the Ricceri principle to deduce the existence of multiple solutions for (1.4) when f and g are subcritical nonlinearities. Zhao and Yan [22] extended the result in [7] when g has a supercritical or exponential growth in $\mathbb{R}^n$ (see also [8, 23] for related results in bounded domains). Motivated by the previous papers and the interest in double phase problems, in this work we establish some multiplicity results for (1.1) by considering different types on nonlinearities. With respect to the above mentioned papers dealing with the p-Laplacian case, the main difficulty to attack (1.1) is related to the presence of the operator $\operatorname{div}(a_1(x)|\nabla u|^{p_1-2}\nabla u) + \operatorname{div}(a_2(x)|\nabla u|^{p_2-2}\nabla u)$ which is non-homogeneous in scaling. For this reason, some accurate estimates and suitable tricks will be needed to achieve our main results.

Before stating our theorems, we introduce the assumptions on f and g. First we consider the case when f and g are subcritical. More precisely, we suppose

$$|f(x,t)| \le m(x)|t|^\gamma \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } t \in \mathbb{R}, \tag{1.5}$$

where

$$m \in L^{\frac{p_2^\star}{p_2^\star-1}}(\mathbb{R}^n) \cap L^{\frac{\nu}{\nu-1}\left(\frac{p_2^\star}{p_2^\star-(\gamma+1)}\right)}(\mathbb{R}^n), \tag{1.6}$$

with

$$p_2 < \gamma + 1 < \nu < p_2^\star,$$

where $p_i^\star := \frac{np_i}{n-p_i}$ with $i = 1, 2$. Regarding the function g, it satisfies the condition $g(x, 0) = 0$ and one of the following assumptions:

there exists a positive function $h \in L^{\frac{p_2^\star}{p_2^\star-r}}(\mathbb{R}^n) \cap L^{\frac{p_2^\star}{p_2^\star-r}+\eta}(\mathbb{R}^n)$ and $0 < \eta < 1$ such that

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}\setminus\{0\}} \frac{|g(x,t)|}{h(x)|t|^r} < +\infty \quad \text{for some } 0 < r < p_2^\star - 1; \tag{1.7}$$

there exists a function $h \in L^\infty(\mathbb{R}^n)$ such that

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}\setminus\{0\}} \frac{|g(x,t)|}{h(x)|t|^{p_2^\star}} < +\infty. \tag{1.8}$$

In order to state the main results of this paper, it will be fundamental to recall the following definition and the abstract variational principle due to Ricceri [20].

Definition 1.1. *Let X be a real Banach space. We denote by W_X the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if (u_k) is a sequence in X converging weakly to $u \in X$ and $\liminf_{k\to+\infty} \Phi(u_k) \le \Phi(u)$, then (u_k) has a subsequence converging strongly to u.*

Theorem 1.2 ([20]). *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional in W_X, bounded on each bounded subset of X and whose derivative admits a continuous inverse on $X^\star$; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, setting*

$$\begin{aligned} \alpha &:= \max\left\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\right\}, \\ \beta &:= \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)}, \end{aligned} \tag{1.9}$$

assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subset]\frac{1}{\beta}, \frac{1}{\alpha}[$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and *every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation*

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

has at least three solutions in X whose norms are less than r.

The first result of this paper can be stated as follows.

Theorem 1.3. *Let $1 < p_1 < p_2 < n$ and assume that (1.2), (1.5) and (1.6) hold. Moreover, we assume that there exists a positive function $\alpha \in L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)$, with $1 < \tau < p_1$, such that*

$$\limsup_{|t|\to+\infty} \frac{F(x,t)}{\alpha(x)|t|^\tau} \leq M < +\infty \quad \text{uniformly in } x \in \mathbb{R}^n, \tag{1.10}$$

and

$$\sup_{u\in D_{p_1,p_2}(\mathbb{R}^n)} \int_{\mathbb{R}^n} F(x,u)\,\mathrm{dx} > 0 \quad \text{where } F(x,t) = \int_0^t f(x,s)\,\mathrm{ds}\,. \tag{1.11}$$

Define

$$\omega := \inf\left\{ \frac{\frac{1}{p_1}\int_{\mathbb{R}^n} a_1(x)|\nabla u|^{p_1}\,\mathrm{dx} + \frac{1}{p_2}\int_{\mathbb{R}^n} a_2(x)|\nabla u|^{p_2}\,\mathrm{dx}}{\int_{\mathbb{R}^n} F(x,u)\,\mathrm{dx}} \;:\; u \in D_{p_1,p_2}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} F(x,u)\,\mathrm{dx} > 0 \right\}.$$

Then, for each compact $[a, b] \subset]\omega, +\infty[$, there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$, and every Carathéodory function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, satisfying (1.7) or (1.8), there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem (1.1) has at least two nontrivial solutions in $D_{p_1,p_2}(\mathbb{R}^n)$ whose norms are less than r.

To prove Theorem 1.3, we will show that the conditions imposed on f and g are suited to study (1.1) via variational methods and apply Theorem 1.2. We stress that some auxiliary results established for the subcritical case will be also useful to treat the next cases.

Secondly, we study the following supercritical problem:

$$\begin{cases} -\operatorname{div}(a_1(x)|\nabla u|^{p_1-2}\nabla u) - \operatorname{div}(a_2(x)|\nabla u|^{p_2-2}\nabla u) = \lambda f(x,u) + \mu h(x)|u|^{r-2}u & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.12}$$

where $1 < p_1 < p_2 < n$, $\lambda, \mu > 0$ are parameters, a_1 and a_2 satisfy (1.2), f is Carathéodory function verifying (1.5) and (1.6), $r > p_2^\star$ and h fulfills

$$h \in L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n) \cap L^{\frac{p_2^\star}{p_2^\star-p_2}}(\mathbb{R}^n). \tag{1.13}$$

Our second main result is the following one.

Theorem 1.4. *Let us suppose that (1.2), (1.5), (1.6), (1.11) and (1.13) hold. Assume that there exists a positive function* $\xi \in L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)$, *for some* $1 < \tau < p_1$, *such that*

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{\xi(x)|t|^\tau} \le M < \infty \quad \textit{uniformly in } x \in \mathbb{R}^n. \tag{1.14}$$

Then, for each compact interval $[a,b] \subset]\theta,+\infty[$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$, *there exists* $\delta > 0$ *such that, for all* $\mu \in [0,\delta]$, *problem (1.12) has at least three solutions (two nontrivial) in* $D_{p_1,p_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *whose* $D_{p_1,p_2}(\mathbb{R}^n)$ *norms are less than* ρ.

The main difficulty in studying (1.12) by means of variational methods is that the associated energy functional is not well defined on $D_{p_1,p_2}(\mathbb{R}^n)$ for $r > p_2^\star$. For this purpose, we use a truncation argument inspired by [3, 18] which consists in considering an auxiliary problem with subcritical nonlinearities and ultimately relies on a Moser iteration argument [15].

Finally, we deal with the following nonlinear problem involving an exponential nonlinearity:

$$\begin{cases} -\operatorname{div}(a_1(x)|\nabla u|^{p_1-2}\nabla u) - \operatorname{div}(a_2(x)|\nabla u|^{p_2-2}\nabla u) = \lambda f(x,u) + \mu h(x)e^u & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.15}$$

where $1 < p_1 < p_2 < n$, $\lambda, \mu > 0$ are parameters, a_1 and a_2 fulfill (1.2), f is Carathéodory function satisfying (1.5) and (1.6), and h verifies

$$h \in L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n) \cap L^{\frac{p_2^\star}{p_2^\star-1}}(\mathbb{R}^n). \tag{1.16}$$

Our last result is the following one.

Theorem 1.5. *Let us assume that (1.2), (1.5), (1.6), (1.11) and (1.16) holds. Suppose that for some* $1 < \tau < p_1$ *and some positive function* $\xi \in L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)$ *it holds*

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{\xi(x)|t|^\tau} \le M < \infty \quad \textit{uniformly in } x \in \mathbb{R}^n. \tag{1.17}$$

Then, for each compact interval $[a,b] \subset]\theta,+\infty[$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$, *there exists* $\delta > 0$ *such that, for all* $\mu \in [0,\delta]$, *problem (1.15) has at least three nontrivial solutions in* $D_{p_1,p_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *whose* $D_{p_1,p_2}(\mathbb{R}^n)$ *norms are less than* ρ.

As for the supercritical case, the proof of Theorem 1.5 will be obtained by combining a suitable truncation argument which allows us to consider an auxiliary subcritical problem, a Moser iteration argument and an application of the Ricceri principle.

The paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the proof of Theorem 1.3. In Section 4 we give the proof of Theorem 1.4. In Section 5 we provide the proof of Theorem 1.5.

2. Preliminaries

For $i = 1, 2$, let us define the following functional space

$$D^{1,p_i}(\mathbb{R}^n) := \{u \in L^{p_i^\star}(\mathbb{R}^n) \, : \, \nabla u \in (L^{p_i}(\mathbb{R}^n))^n\}$$

equipped with the norm

$$\|u\|_{D^{1,p_i}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} a_i(x) |\nabla u|^{p_i} \, \mathrm{dx} \right)^{\frac{1}{p_i}}.$$

In order to study (1.1), we consider the functional space

$$D_{p_1,p_2}(\mathbb{R}^n) := D^{1,p_1}(\mathbb{R}^n) \cap D^{1,p_2}(\mathbb{R}^n),$$

endowed with the norm

$$\|u\|_{p_1,p_2} := \|u\|_{D^{1,p_1}(\mathbb{R}^n) \cap D^{1,p_2}(\mathbb{R}^n)} = \|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}.$$

Remark 1. *By the Sobolev inequality and (1.2), we can deduce that $D^{1,p_i}(\mathbb{R}^n)$ can be embedded continuosly in $L^{p_i^\star}(\mathbb{R}^n)$ and that there exists a constant $\tilde{C} > 0$ such that*

$$\|u\|_{L^{p_i^\star}(\mathbb{R}^n)} \leq \tilde{C} \, \|u\|_{D^{1,p_i}(\mathbb{R}^n)} \qquad \textit{for all } u \in D^{1,p_i}(\mathbb{R}^n).$$

In particular, since $\|u\|_{D^{1,p_i}(\mathbb{R}^n)} \leq \|u\|_{p_1,p_2}$ we can infer that, for some $C_2 > 0$ we have

$$\|u\|_{L^{p_2^\star}(\mathbb{R}^n)} \leq C_2 \, \|u\|_{p_1,p_2} \qquad \textit{for all } u \in D_{p_1,p_2}(\mathbb{R}^n).$$

This means that $D_{p_1,p_2}(\mathbb{R}^n)$ is continuously embedded in $L^{p_2^\star}(\mathbb{R}^n)$. Moreover, $D_{p_1,p_2}(\mathbb{R}^n)$ is a reflexive Banach space.

Let us introduce the functional $\Phi : D_{p_1,p_2}(\mathbb{R}^n) \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{p_1} \|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \frac{1}{p_2} \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}.$$

It is easy to show that Φ is well defined and continuously Gâteaux differentiable with

$$\Phi'(u)v = \int_{\mathbb{R}^n} a_1(x) |\nabla u|^{p_1 - 2} \nabla u \nabla v \, \mathrm{dx} + \int_{\mathbb{R}^n} a_2(x) |\nabla u|^{p_2 - 2} \nabla u \nabla v \, \mathrm{dx} \qquad \text{for all } u, v \in D_{p_1,p_2}(\mathbb{R}^n).$$

Let us also notice that Φ is weakly lower semicontinuous and bounded on each bounded subset of $D_{p_1,p_2}(\mathbb{R}^n)$. Moreover, Φ is a uniformly monotone operator in $D_{p_1,p_2}(\mathbb{R}^n)$ by means of Simon inequalities [21]. Next we prove that Φ is coercive on $D_{p_1,p_2}(\mathbb{R}^n)$.

Lemma 2.1. *Φ is coercive on $D_{p_1,p_2}(\mathbb{R}^n)$.*

Proof. We claim that

$$\lim_{\|u\|_{p_1,p_2}\to+\infty} \frac{\Phi(u)}{\|u\|_{p_1,p_2}} = +\infty.$$

Let $\|u\|_{p_1,p_2} \to +\infty$. We distinguish three cases. If both the terms $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ go to infinity, then we get

$$\begin{aligned}\frac{\Phi(u)}{\|u\|_{p_1,p_2}} &\geq \frac{C\big(\,\|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}\,\big)^{p_1}}{\|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}} \\ &= C\,\|u\|_{p_1,p_2}^{p_1-1} \longrightarrow +\infty.\end{aligned}$$

On the other hand, if $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ tends to infinity and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ has a finite limit, we obtain

$$\frac{\Phi(u)}{\|u\|_{p_1,p_2}} \geq \frac{C\,\|u\|_{D^{1,p_1}(\mathbb{R}^n)}^{p_1}}{\|u\|_{D^{1,p_1}(\mathbb{R}^n)}} = C\,\|u\|_{D^{1,p_1}(\mathbb{R}^n)}^{p_1-1} \longrightarrow +\infty.$$

In a similar way, if $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ has a finite limit and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ tends to infinity,

$$\frac{\Phi(u)}{\|u\|_{p_1,p_2}} \geq \frac{C\,\|u\|_{D^{1,p_2}(\mathbb{R}^n)}^{p_2}}{\|u\|_{D^{1,p_2}(\mathbb{R}^n)}} = C\,\|u\|_{D^{1,p_2}(\mathbb{R}^n)}^{p_2-1} \longrightarrow +\infty.$$

The proof of lemma is now complete. □

Remark 2. *Using the Browder-Minty theorem we can see that Φ' admits a continuous inverse on $(D_{p_1,p_2}(\mathbb{R}^n))^\star$, that is $\Phi \in W_{D_{p_1,p_2}(\mathbb{R}^n)}$.*

3. Proof of Theorem 1.3

Let us consider the functionals $J : D_{p_1,p_2}(\mathbb{R}^n) \to \mathbb{R}$ and $\Psi : D_{p_1,p_2}(\mathbb{R}^n) \to \mathbb{R}$ defined respectively as

$$J(u) \coloneqq \int_{\mathbb{R}^n} F(x,u)\,\mathrm{d}x \quad \text{and} \quad \Psi(u) \coloneqq \int_{\mathbb{R}^n} G(x,u)\,\mathrm{d}x\,.$$

Next, we show that the hypotheses of Theorem 1.2 are fulfilled.

Lemma 3.1. *J' is a compact operator from $D_{p_1,p_2}(\mathbb{R}^n)$ to $(D_{p_1,p_2}(\mathbb{R}^n))^*$.*

Proof. Let q be such that $\frac{1}{q} + \frac{\gamma}{p_2^\star} + \frac{1}{p_2^\star} = 1$, and notice that

$$q = \frac{p_2^\star}{p_2^\star - (\gamma+1)} \in \left[\frac{p_2^\star}{p_2^\star - 1}, \frac{\nu}{\nu-1}\left(\frac{p_2^\star}{p_2^\star-(\gamma+1)}\right)\right].$$

Thanks to Hölder's inequality and Sobolev embedding, for every $R > 0$ we get

$$\begin{aligned}\int_{|x|\geq R} f(x,u)v\,\mathrm{dx} &\leq \left(\int_{|x|\geq R}|m|^q\,\mathrm{dx}\right)^{\frac{1}{q}}\left(\int_{|x|\geq R}|u|^{p_2^\star}\,\mathrm{dx}\right)^{\frac{\gamma}{p_2^\star}}\left(\int_{|x|\geq R}|v|^{p_2^\star}\,\mathrm{dx}\right)^{\frac{1}{p_2^\star}}\\ &\leq C\left(\int_{|x|\geq R}|m|^q\,\mathrm{dx}\right)^{\frac{1}{q}}\|u\|^{\gamma}_{L^{p_2^\star}(\mathbb{R}^n)}\|v\|^{p_2^\star}_{D^{1,p_2}(\mathbb{R}^n)}\\ &\leq C\left(\int_{|x|\geq R}|m|^q\,\mathrm{dx}\right)^{\frac{1}{q}}\|u\|^{\gamma}_{p_1,p_2}\|v\|^{p_2^\star}_{p_1,p_2}\end{aligned}$$

for all $u, v \in D_{p_1,p_2}(\mathbb{R}^n)$.

Let now $(u_k) \subset D_{p_1,p_2}(\mathbb{R}^n)$ be such that $u_k \rightharpoonup u$ in $D_{p_1,p_2}(\mathbb{R}^n)$. From $m \in L^q(\mathbb{R}^n)$, we derive that

$$\lim_{R\to+\infty}\int_{|x|\geq R}|m|^q\,\mathrm{dx} = 0.$$

Since (u_k) is a bounded sequence in $D_{p_1,p_2}(\mathbb{R}^n)$, fixed $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{|x|\geq R_\varepsilon} f(x,u_k)v\,\mathrm{dx} \leq \varepsilon \quad \text{and} \quad \int_{|x|\geq R_\varepsilon} f(x,u)v\,\mathrm{dx} \leq \varepsilon \tag{3.1}$$

holds for every $k \in \mathbb{N}$ and $v \in D_{p_1,p_2}(\mathbb{R}^n)$. On the other hand, applying (1.5), (1.6) and using Young's inequality, we obtain

$$\begin{aligned}f^{\frac{\nu}{\nu-1}}(x,t) &\leq m(x)^{\frac{\nu}{\nu-1}}t^{\frac{\gamma\nu}{\nu-1}}\\ &\leq \frac{p_2^\star-(\gamma+1)}{p_2^\star}m(x)^{\frac{\nu}{\nu-1}\left(\frac{p_2^\star}{p_2^\star-(\gamma+1)}\right)} + \frac{\gamma+1}{p_2^\star}t^{\frac{\gamma\nu}{\nu-1}\frac{p_2^\star}{\gamma+1}},\end{aligned}$$

for a.e. $x \in B_{R_\varepsilon}$ and for all $t \in \mathbb{R}$. Since $\gamma + 1 < \nu$, we have that

$$\frac{\gamma\nu}{\nu-1}\frac{p_2^\star}{\gamma+1} < p_2^\star.$$

Therefore, the Nemytskii operator $N_{f^{\frac{\nu}{\nu-1}}}$ associated with $f^{\frac{\nu}{\nu-1}}$ is continuous from $L^{\frac{\gamma\nu}{\nu-1}\frac{p_2^\star}{\gamma+1}}(B_\varepsilon)$ in $L^1(B_\varepsilon)$. Then we can infer that

$$\int_{|x|<R_\varepsilon} f(x,u_k)^{\frac{\nu}{\nu-1}}\,\mathrm{dx} \longrightarrow \int_{|x|<R_\varepsilon} f(x,u)^{\frac{\nu}{\nu-1}}\,\mathrm{dx} \qquad \text{as } k \to +\infty,$$

and this implies that $f(x,u_k)$ converges to $f(x,u)$ in $L^{\frac{\nu}{\nu-1}}(B_\varepsilon)$. But $L^{p_2^\star}(B_\varepsilon) \subset L^\nu(B_\varepsilon)$ and hence $f(x,u_k)v$ converges to $f(x,u)v$ in $L^1(B_\varepsilon)$, that is

$$\int_{B_\varepsilon}(f(x,u_k)-f(x,u))v\,\mathrm{dx} \longrightarrow \ \text{as } k \to +\infty \tag{3.2}$$

for all $v \in L^\nu(B_\varepsilon)$. Combining (3.1) and (3.2), we get

$$\int_{\mathbb{R}^n} f(x,u_k)v\,\mathrm{dx} \longrightarrow \int_{\mathbb{R}^n} f(x,u)v\,\mathrm{dx} \qquad \text{as } k \to +\infty,$$

for all $v \in D_{p_1,p_2}(\mathbb{R}^n)$. This proves that J' is a compact operator. $\square$

Proposition 3.2. *It holds $\alpha < \beta$, where α and β are defined as in (1.9) with $x_0 = 0$.*

Proof. Using (1.5), we see that

$$|F(x,t)| \le \frac{1}{\gamma+1} m(x)|t|^{\gamma+1} \quad \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

which combined with the Hölder inequality, (1.6) and Sobolev embedding gives

$$\int_{\mathbb{R}^n} F(x,u)\,\mathrm{dx} \le \frac{1}{\gamma+1}\left(\int_{\mathbb{R}^n} |m|^q\,\mathrm{dx}\right)^{\frac{1}{q}} \left(C_1 \|u\|_{D^{1,p_2}(\mathbb{R}^n)}\right)^{\gamma+1} \quad \text{for all } u \in D_{p_1,p_2}(\mathbb{R}^n),$$

where

$$\frac{1}{q} + \frac{\gamma}{p_2^\star} + \frac{1}{p_2^\star} = 1.$$

Therefore,

$$\frac{J(u)}{\Phi(u)} \le \frac{p_2}{\gamma+1} C_1^{\gamma+1} \|m\|_{L^q(\mathbb{R}^n)} \frac{\|u\|^{\gamma+1}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} \quad \text{for all } u \in D_{p_1,p_2}(\mathbb{R}^n).$$

Fix $\varepsilon > 0$. Since

$$\frac{\|u\|^{\gamma+1}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} \le \|u\|^{\gamma+1-p_2}_{D^{1,p_2}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } \|u\|_{p_1,p_2} \longrightarrow 0,$$

we have

$$\limsup_{\|u\|_{p_1,p_2}\to 0} \frac{J(u)}{\Phi(u)} \le \frac{p_2}{\gamma+1} C_1^{\gamma+1} \|m\|_{L^q(\mathbb{R}^n)}\,\varepsilon. \tag{3.3}$$

Next, we estimate the limsup of $\frac{J(u)}{\Phi(u)}$ as $\|u\|_{D_{p_1,p_2}(\mathbb{R}^n)} \longrightarrow +\infty$ and we show that $\alpha = 0$. For this purpose, we recall that, by (1.10), there exists $A > 0$ such that

$$|F(x,t)| \le M\alpha(x)|t|^\tau, \quad \text{for a.e. } x \in \mathbb{R}^n, \text{ for all } |t| > A,$$

where $1 < \tau < p_1 < p_2$ and $M > 0$. Then, for every $u \in D_{p_1,p_2}(\mathbb{R}^n) \setminus \{0\}$,

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int_{\mathbb{R}^n} F(x,u)\,\mathrm{dx}}{\frac{1}{p_1}\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \frac{1}{p_2}\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} \\
&\le \frac{p_2 M \int_{|u|>A} \alpha(x)|u|^\tau\,\mathrm{dx}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} + \frac{p_2}{\gamma+1} \frac{\int_{|u|\le A} m(x)|u|^{\gamma+1}\,\mathrm{dx}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} =: \mathcal{A}_1 + \mathcal{A}_2.
\end{aligned}$$

The numerator of $\mathcal{A}_2$ can be bounded applying the Hölder and Young inequalities and Sobolev embedding as below

$$\mathcal{A}_2 \le p_2 A^{\gamma+1-\tau} \int_{|u|\le A} m(x)|u|^\tau\,\mathrm{dx}$$

$$\leq p_2 A^{\gamma+1-\tau} \left(\int_{|u|\leq A} |m(x)|^{\frac{p_2^\star}{p_2^\star-\tau}} \,\mathrm{dx} \right)^{\frac{p_2^\star-\tau}{p_2^\star}} \left(\int_{|u|\leq A} |u|^{p_2^\star} \,\mathrm{dx} \right)^{\frac{\tau}{p_2^\star}}$$
$$= p_2 A^{\gamma+1-\tau} \|m\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)} \|u\|^{\tau}_{L^{p_2^\star}(\mathbb{R}^n)} \leq p_2 C A^{\gamma+1-\tau} \|m\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)} \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}.$$

Let us notice that, since

$$\frac{q^\star}{q^\star-\tau} \in \left[\frac{q^\star}{q^\star-1}, \frac{q^\star}{q^\star-(\gamma+1)} \right],$$

it holds $m \in L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)$. On the other hand, up to a constant, the numerator of $\mathcal{A}_1$ can be estimated as follows:

$$\int_{|u|>A} \alpha(x)|u|^{\tau} \,\mathrm{dx} \leq \left(\int_{|u|>A} |\alpha(x)|^{\frac{p_2^\star}{p_2^\star-\tau}} \,\mathrm{dx} \right)^{\frac{p_2^\star-\tau}{p_2^\star}} \left(\int_{|u|>A} |u|^{p_2^\star} \,\mathrm{dx} \right)^{\frac{\tau}{p_2^\star}}$$
$$= \|\alpha\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)} \|u\|^{\tau}_{L^{p_2^\star}(\mathbb{R}^n)}$$
$$\leq C \|\alpha\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)} \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}.$$

Now, assuming that $\|u\|_{p_1,p_2}$ is going to infinity, we study the behaviour of the quantity

$$S := \frac{p_2 C \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}}.$$

We distinguish three cases. First, if $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ has a finite limit and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ tends to infinity, then

$$S = \frac{p_2 C \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)} \left(\frac{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)}}{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} + 1 \right)} \leq \frac{C}{\|u\|^{p_2-\tau}_{D^{1,p_2}(\mathbb{R}^n)} \left(\frac{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)}}{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} + 1 \right)} \leq \varepsilon.$$

Otherwise, if $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ goes to infinity and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ has a finite limit, then $\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} \geq C \|u\|^{p_1}_{D^{1,p_2}(\mathbb{R}^n)}$ and we obtain

$$S = \frac{p_2 C \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} \left(\frac{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)}} + 1 \right)}$$
$$\leq \frac{p_2 C \|u\|^{\tau}_{D^{1,p_2}(\mathbb{R}^n)}}{\tilde{C} \|u\|^{p_1}_{D^{1,p_2}(\mathbb{R}^n)} \left(\frac{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}}{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)}} + 1 \right)}$$
$$\leq \frac{C_2}{\|u\|^{p_1-\tau}_{D^{1,p_2}(\mathbb{R}^n)} \left(\frac{\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)}}{\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)}} + 1 \right)} \leq \varepsilon.$$

Finally, if both the terms $\|u\|_{D^{1,p_1}(\mathbb{R}^n)}$ and $\|u\|_{D^{1,p_2}(\mathbb{R}^n)}$ go to the infinity, then

$$\|u\|^{p_1}_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)} \geq C_{p_1}\big(\|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}\big)^{p_2}.$$

Moreover, we have that $\|u\|^{p_2}_{D^{1,p_2}(\mathbb{R}^n)} \geq \|u\|^{p_1}_{D^{1,p_2}(\mathbb{R}^n)} > 1$ and hence

$$S \leq \frac{C_1\big(\|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}\big)^{\tau}}{C_2\big(\|u\|_{D^{1,p_1}(\mathbb{R}^n)} + \|u\|_{D^{1,p_2}(\mathbb{R}^n)}\big)^{p_1}} \leq C_3\, \|u\|^{\tau-p_1}_{p_1,p_2} \leq \varepsilon.$$

Consequently,

$$\limsup_{\|u\|_{p_1,p_2}\to+\infty} \frac{J(u)}{\Phi(u)} \leq p_2 C_1^{\tau}\left(M\,\|\alpha\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)} + \frac{A^{\gamma+1-\tau}}{\gamma+1}\,\|m\|_{L^{\frac{p_2^\star}{p_2^\star-\tau}}(\mathbb{R}^n)}\right)\varepsilon. \tag{3.4}$$

Putting together (3.3) and (3.4), and by the arbitrariness of $\varepsilon > 0$, we achieve

$$\max\left\{\limsup_{\|u\|_{p_1,p_2}\to+\infty} \frac{J(u)}{\Phi(u)},\ \limsup_{\|u\|_{p_1,p_2}\to 0} \frac{J(u)}{\Phi(u)}\right\} \leq 0.$$

Therefore, $\alpha = 0$, and since (1.11) implies $\beta > 0$, we get the thesis. □

Proof of Theorem 1.3. In light of Lemma 2.1, Lemma 3.1 and Proposition 3.2, all the assumptions of Theorem 1.2 are satisfied with $x_0 = 0$. Regarding the function $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, this is measurable in $\mathbb{R}^n$ and C^1 in $\mathbb{R}$. Furthermore, it holds that $G_u = g$ when (1.7) or (1.8) is satisfied, and hence, by standard arguments, the functional Ψ is well defined and continuously Gâteaux differentiable on $W_{D_{p_1,p_2}(\mathbb{R}^n)}$, with compact derivative. Moreover

$$\Psi'(u)v = \int_{\mathbb{R}^n} g(x,u)v\,\mathrm{d}x \quad \text{for every } u, v \in W_{D_{p_1,p_2}(\mathbb{R}^n)}.$$

Then, by Theorem 1.2, the problem (1.1) has at least two nontrivial solutions (note that $u = 0$ is not a solution of (1.1)) in $D_{p_1,p_2}(\mathbb{R}^n)$ which are critical points of the functional $\Phi - \lambda J - \mu\Psi$. The proof of Theorem 1.3 is now complete. □

4. The supercritical case

In order to overcome the presence of the supercritical exponent in (1.12), we introduce the truncation of $h(x)|t|^{r-2}t$ given by

$$g_K(x,t) \coloneqq \begin{cases} h(x)|t|^{r-2}t & \text{if } |t| \leq K, \\ K^{r-p_2}h(x)|t|^{p_2-2}t & \text{if } |t| > K, \end{cases} \tag{4.1}$$

where $K \geq 1$ is a real number whose value will be fixed later. Then, $g_K(x,t)$ satisfies the subcritical growth

$$|g_K(x,t)| \leq K^{r-p_2}|h(x)||t|^{p_2-1} \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } t \in \mathbb{R}. \tag{4.2}$$

Now, let us consider the truncated problem

$$
\begin{cases} -\operatorname{div}(a_1(x)|\nabla u|^{p_1-2}\nabla u) - \operatorname{div}(a_2(x)|\nabla u|^{p_2-2}\nabla u) = \lambda f(x,u) + \mu g_K(x,u) & \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{4.3}
$$

Definition 4.1. *We say that $u \in D_{p_1,p_2}(\mathbb{R}^n)$ is a weak solution of (4.3) if*

$$
\int_{\mathbb{R}^n} a_1(x)|\nabla u|^{p_1-2}\nabla u\nabla v\,\mathrm{dx} + \int_{\mathbb{R}^n} a_2(x)|\nabla u|^{p_2-2}\nabla u\nabla v\,\mathrm{dx} = \lambda\int_{\mathbb{R}^n} f(x,u)v\,\mathrm{dx} + \mu\int_{\mathbb{R}^n} g_K(x,u)v\,\mathrm{dx},
$$

for every $v \in D_{p_1,p_2}(\mathbb{R}^n)$.

For every $u \in D_{p_1,p_2}(\mathbb{R}^n)$, we define the following functional related to g_K:

$$
\Psi_K(u) \coloneqq \int_{\mathbb{R}^n} G_K(x,u)\,\mathrm{dx} = \int_{\mathbb{R}^n}\left(\int_0^u g_K(x,t)\,\mathrm{dt}\right)\mathrm{dx}\,. \tag{4.4}
$$

Lemma 4.2. *Suppose (1.13) holds. Then Ψ_K' is a compact operator from $D_{p_1,p_2}(\mathbb{R}^n)$ to $(D_{p_1,p_2}(\mathbb{R}^n))^\star$ for every $K > 0$.*

Proof. Let $(u_k) \subset D_{p_1,p_2}(\mathbb{R}^n)$ be a sequence such that $u_k \rightharpoonup u$ in $D_{p_1,p_2}(\mathbb{R}^n)$. Thanks to (4.2), the Hölder inequality, (1.13) and the Sobolev embedding, we obtain that

$$
\begin{aligned}
\int_{|x|>R} g_K(x,u)v\,\mathrm{dx} &\le \int_{|x|>R} K^{r-p_2}|h(x)||u|^{p_2-1}|v|\,\mathrm{dx} \\
&\le K^{r-p_2}\Big(\int_{|x|>R}|h(x)|^{\frac{p_2^\star}{p_2^\star-p_2}}\,\mathrm{dx}\Big)^{\frac{p_2^\star-p_2}{p_2^\star}}\Big(\int_{\mathbb{R}^n}|u|^{p_2^\star}\,\mathrm{dx}\Big)^{\frac{p_2-1}{p_2^\star}}\Big(\int_{\mathbb{R}^n}|v|^{p_2^\star}\,\mathrm{dx}\Big)^{\frac{1}{p_2^\star}} \\
&\le K^{r-p_2}\Big(\int_{|x|>R}|h(x)|^{\frac{p_2^\star}{p_2^\star-p_2}}\,\mathrm{dx}\Big)^{\frac{p_2^\star-p_2}{p_2^\star}} C_1^{p_2}\,\|u\|_{p_1,p_2}^{p_2-1}\,\|v\|_{p_1,p_2}\,,
\end{aligned}
$$

for every $K > 0$, $R > 0$ and $u, v \in D_{p_1,p_2}(\mathbb{R}^n)$. Since $h \in L^{\frac{p_2^\star}{p_2^\star-p_2}}(\mathbb{R}^n)$, we see that

$$
\lim_{R\to+\infty}\int_{|x|>R}|h(x)|^{\frac{p_2^\star}{p_2^\star-p_2}}\,\mathrm{dx} = 0.
$$

Let $\varepsilon > 0$. Recalling that (u_k) is a bounded sequence in $D_{p_1,p_2}(\mathbb{R}^n)$, we find $R_\varepsilon > 0$ such that

$$
\int_{|x|>R_\varepsilon} g_K(x,u_k)v\,\mathrm{dx} \le \varepsilon\,\|v\|_{p_1,p_2} \quad \text{and} \quad \int_{|x|>R_\varepsilon} g_K(x,u_k)v\,\mathrm{dx} \le \varepsilon\,\|v\|_{p_1,p_2} \tag{4.5}
$$

for all $v \in D_{p_1,p_2}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. On the other hand, applying Young's inequality, we have

$$
|g_K(x,u_k)| \le K^{r-p_2}\left[\frac{(\nu-1)p_2^\star+\gamma+1-\nu p_2}{\nu(p_2^\star-1)}|h(x)|^{\frac{\nu(p_2^\star-1)}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}\right.
$$

$$+\left.\frac{p_2^\star+\nu p_2-\nu-\gamma-1}{\nu(p_2^\star-1)}|u_k|^{\frac{\nu(p_2-1)(p_2^\star-1)}{p_2^\star+\nu p_2-\nu-\gamma-1}}\right].$$

From $\gamma+1<p_2^\star$ we derive

$$\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1}<p_2^\star,$$

and $u_k\to u$ strongly in $L^{\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1}}(B_{R_\varepsilon})$. Since $h\in L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)$ (by (1.13)), we have

$$|h(x)|^{\frac{\nu(p_2^\star-1)}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}\in L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon}).$$

Therefore, the Nemytskii N_h is continuous from $L^{\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1}}(B_{R_\varepsilon})$ to $L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon})$, and we have

$$\|g_K(\cdot,u_k)-g_K(\cdot,u)\|_{L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon})}\longrightarrow 0\quad\text{ as } k\to+\infty.$$

Consequently, for all $v\in D_{p_1,p_2}(\mathbb{R}^n)$,

$$\begin{aligned}&\int_{B_{R_\varepsilon}}|g_K(x,u_k)-g_K(x,u)||v|\,\mathrm{dx}\\&\qquad\le\left(\int_{B_{R_\varepsilon}}|g_K(x,u_k)-g_K(x,u)|^{\frac{p_2^\star}{p_2^\star-1}}\,\mathrm{dx}\right)^{\frac{p_2^\star-1}{p_2^\star}}\left(\int_{B_{R_\varepsilon}}|v|^{p_2^\star}\,\mathrm{dx}\right)^{\frac{1}{p_2^\star}}\\&\qquad\le\left(\int_{B_{R_\varepsilon}}|g_K(x,u_k)-g_K(x,u)|^{\frac{p_2^\star}{p_2^\star-1}}\,\mathrm{dx}\right)^{\frac{p_2^\star-1}{p_2^\star}}C\,\|v\|_{p_1,p_2}\longrightarrow 0\quad\text{ as } k\to+\infty,\end{aligned}$$

which combined with (4.5) yields

$$\int_{\mathbb{R}^n}|g_K(x,u_k)-g_K(x,u)||v|\,\mathrm{dx}\longrightarrow 0\quad\text{ as } k\to+\infty.$$

The proof of lemma is now complete. □

Proof of Theorem 1.4. Thanks to Lemma 2.1, Lemma 3.1 and Lemma 4.2, all the assumptions of Theorem 1.2 (with $x_0=0$) are satisfied. Then there exists $\rho>0$ such that for every $\lambda\in[a,b]\subset]\omega,+\infty[$, there exists $\delta>0$ such that, for $\mu\in[0,\delta]$, problem (4.3) has at least two nontrivial solutions $u_1,u_2\in D_{p_1,p_2}(\mathbb{R}^n)$, whose norms are less than ρ, i.e. $\|u_i\|_{p_1,p_2}\le\rho$, $i=1,2$. Clearly, $u=0$ is a solution of (4.3) (and ρ does not depend on μ). Now, if we prove that each solution $u_i\in D_{p_1,p_2}(\mathbb{R}^n)$, $i=1,2$, of the truncated problem (4.1) satisfies

$$|u_i(x)|\le K\quad\text{ for a.e. } x\in\mathbb{R}^n,\tag{4.6}$$

it follows from the definition of g_K that $g_K(u)=h(x)|u|^{r-2}u$, and hence the solution u_i is also a solution of the original problem (1.12). In what follows, we prove that there exists $\mu^\star>0$ such that each solution $u_i\in D_{p_1,p_2}(\mathbb{R}^n)$ of the truncated problem satisfies (4.6) whenever

$\mu \in [0, \mu^\star]$. We use a Moser iteration argument [15]. To lighten the notation, we will set $u \coloneqq u_i$, $i = 1, 2$. Let $u_+ \coloneqq \max\{u, 0\}$ and $u_- \coloneqq -\min\{u, 0\}$. For each $L > 0$, we define the following functions

$$u_L \coloneqq \begin{cases} u_+ & \text{as } 0 \le u_+ \le L, \\ L & \text{as } u_+ > L, \end{cases}$$

$z_L \coloneqq u_L^{p_2(\beta-1)} u_+$ and $w_L \coloneqq u_L^{\beta-1} u_+$, where $\beta > 1$ will be fixed later. Choosing z_L as test function in Definition 4.1, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} a_1(x)|\nabla u|^{p_1-2}\nabla u \nabla z_L \,\mathrm{dx} + \int_{\mathbb{R}^n} a_2(x)|\nabla u|^{p_2-2}\nabla u \nabla z_L \,\mathrm{dx} \\ &= \lambda \int_{\mathbb{R}^n} f(x,u) z_L \,\mathrm{dx} + \mu \int_{\mathbb{R}^n} g_K(x,u) z_L \,\mathrm{dx}\,. \end{aligned}$$

Notice that standard calculations show that the left-hand side of the above identity can be estimated as follows:

$$\begin{aligned} &\int_{\mathbb{R}^n} a_1(x)|\nabla u|^{p_1-2}\nabla u \nabla z_L \,\mathrm{dx} + \int_{\mathbb{R}^n} a_2(x)|\nabla u|^{p_2-2}\nabla u \nabla z_L \,\mathrm{dx} \\ &\quad= \int_{\mathbb{R}^n} a_1(x)|\nabla u_+|^{p_1} u_L^{p_2(\beta-1)} \,\mathrm{dx} + p_2(\beta-1)\int_{\mathbb{R}^n} a_1(x)|\nabla u_L|^{p_1} u_L^{p_2(\beta-1)} \,\mathrm{dx} \\ &\qquad+ \int_{\mathbb{R}^n} a_2(x)|\nabla u_+|^{p_2} u_L^{p_2(\beta-1)} \,\mathrm{dx} + p_2(\beta-1)\int_{\mathbb{R}^n} a_2(x)|\nabla u_L|^{p_2} u_L^{p_2(\beta-1)} \,\mathrm{dx} \\ &\quad\ge \int_{\mathbb{R}^n} a_2(x)|\nabla u_+|^{p_2} u_L^{p_2(\beta-1)} \,\mathrm{dx}, \end{aligned}$$

where we employed the following facts

$$\begin{aligned} &\int_{\mathbb{R}^n} a_1(x)|\nabla u_+|^{p_1} u_L^{p_2(\beta-1)} \,\mathrm{dx} \ge 0, \\ &\int_{\mathbb{R}^n} a_i(x)|\nabla u_L|^{p_i} u_L^{p_2(\beta-1)} \,\mathrm{dx} \ge 0 \quad \text{for } i = 1, 2. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} a_2(x)|\nabla u_+|^{p_2} u_L^{p_2(\beta-1)} \,\mathrm{dx} &\le \int_{\mathbb{R}^n} a_1(x)|\nabla u|^{p_1-2}\nabla u \nabla z_L \,\mathrm{dx} + \int_{\mathbb{R}^n} a_2(x)|\nabla u|^{p_2-2}\nabla u \nabla z_L \,\mathrm{dx} \\ &= \lambda \int_{\mathbb{R}^n} f(x,u) z_L \,\mathrm{dx} + \mu \int_{\mathbb{R}^n} g_K(x,u) z_L \,\mathrm{dx} \\ &\le \lambda \int_{\mathbb{R}^n} m(x)|u|^\gamma u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx} + \mu K^{r-p_2} \int_{\mathbb{R}^n} |h(x)||u|^{p_2-1} u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx} \\ &= \lambda \int_{\mathbb{R}^n} m(x) u_L^{p_2(\beta-1)} u_+^{p_2} u_+^{\gamma+1-p_2} \,\mathrm{dx} + \mu K^{r-p_2} \int_{\mathbb{R}^n} |h(x)| u_L^{p_2(\beta-1)} u_+^{p_2} \,\mathrm{dx}\,. \end{aligned} \tag{4.7}$$

Now, invoking the Hölder inequality and (1.6), we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} m(x) u_L^{p_2(\beta-1)} u_+^{p_2} u_+^{\gamma+1-p_2}\,\mathrm{dx} &= \int_{\mathbb{R}^n} m(x) w_L^{p_2} u_+^{\gamma+1-p_2}\,\mathrm{dx} \\
&\le \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \|w_L\|_{L^\alpha(\mathbb{R}^n)}^{\frac{p_2}{\alpha}} \|u_+\|_{L^{p_2^\star}(\mathbb{R}^n)}^{\gamma+1-p_2} \\
&\le (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \|w_L\|_{L^\alpha(\mathbb{R}^n)}^{\frac{p_2}{\alpha}} \\
&\le (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |u_L^{\beta-1} u_+|^\alpha\,\mathrm{dx}\right)^{\frac{p_2}{\alpha}} \\
&\le (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |u_+|^{\beta\alpha}\,\mathrm{dx}\right)^{\frac{p_2}{\alpha}} \\
&\le (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}^{p_2\beta}, \qquad (4.8)
\end{aligned}$$

where we used $\|u_+\|_{L^{p_2^\star}(\mathbb{R}^n)} \le \|u\|_{L^{p_2^\star}(\mathbb{R}^n)} \le C\|u\|_{p_1,p_2} \le C_1\rho$, $|u| = u_+ + u_- \ge u_+ \ge 0$, $\beta := \frac{p_2^\star-\gamma-1}{\nu p_2} + 1 > 1$ and $\alpha\beta = p_2^\star$. On the other hand, the Hölder inequality and $u_L \le u_+$ yield

$$\begin{aligned}
\int_{\mathbb{R}^n} h(x) u_L^{p_2(\beta-1)} u_+^{p_2}\,\mathrm{dx} &= \int_{\mathbb{R}^n} h(x) w_L^{p_2}\,\mathrm{dx} \\
&\le \left(\int_{\mathbb{R}^n} |h(x)|^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}}\,\mathrm{dx}\right)^{\frac{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}{\nu p_2^\star}} \left(\int_{\mathbb{R}^n} w_L^\alpha\,\mathrm{dx}\right)^{\frac{p_2}{\alpha}} \\
&\le \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |u_L^{\beta-1} u_+|^\alpha\,\mathrm{dx}\right)^{\frac{p_2}{\alpha}} \\
&\le \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |u_+|^{\beta\alpha}\,\mathrm{dx}\right)^{\frac{p_2}{\alpha}} \\
&\le \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}}(\mathbb{R}^n)} \|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}^{p_2\beta}, \qquad (4.9)
\end{aligned}$$

where we used $\alpha\beta = p_2^\star$ and $\beta = \frac{p_2^\star-\gamma-1}{\nu p_2} + 1 > 1$. Recalling the Sobolev embedding $D_{p_1,p_2}(\mathbb{R}^n) \hookrightarrow L^{p_2^\star}(\mathbb{R}^n)$, we obtain

(4.10)

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |w_L|^{p_2^\star}\,\mathrm{dx}\right)^{\frac{p_2}{p_2^\star}} \\
&\le C_1^{p_2} \int_{\mathbb{R}^n} a_2(x) |\nabla w_L|^{p_2}\,\mathrm{dx} \\
&= C_1^{p_2} \int_{\mathbb{R}^n} a_2(x) |(\beta-1) u_+ u_L^{\beta-2} \nabla u_L + u_L^{\beta-1} \nabla u_+|^{p_2}\,\mathrm{dx}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{p_2-1}C_1^{p_2}\int_{\mathbb{R}^n} a_2(x)|(\beta-1)u_+u_L^{\beta-2}\nabla u_L|^{p_2}+a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}\\
&=2^{p_2-1}C_1^{p_2}\int_{\{u_+\leq L\}} a_2(x)|(\beta-1)u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}+2^{p_2-1}C_1^{p_2}\int_{\mathbb{R}^n} a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}\\
&\leq 2^{p_2-1}C_1^{p_2}(\beta-1)^{p_2}\int_{\mathbb{R}^n} a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}+2^{p_2-1}C_1^{p_2}\int_{\mathbb{R}^n} a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}\\
&=2^{p_2-1}C_1^{p_2}[(\beta-1)^{p_2}+1]\int_{\mathbb{R}^n} a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}\\
&=2^{p_2-1}C_1^{p_2}\beta^{p_2}\Big[\Big(\frac{\beta-1}{\beta}\Big)^{p_2}+\frac{1}{\beta^{p_2}}\Big]\int_{\mathbb{R}^n} a_2(x)|u_L^{\beta-1}\nabla u_+|^{p_2}\,\mathrm{dx}
\end{aligned}
$$

(4.11)

$$
\leq 2^{p_2}C_1^{p_2}\beta^{p_2}\int_{\mathbb{R}^n} a_2(x)u_L^{p_2(\beta-1)}|\nabla u_+|^{p_2}\,\mathrm{dx},
$$

where the last inequality is a consequence of $\beta>1$, $\frac{1}{\beta^{p_2}}<1$ and $\big(\frac{\beta-1}{\beta}\big)^{p_2}<1$. Let us observe that the Sobolev embedding and $\|u_+\|_{p_1,p_2}\leq\rho$ give

$$
\|u_+\|_{L^{p_2^\star}(\mathbb{R}^n)}\leq C_1\|u_+\|_{p_1,p_2}\leq C_1\rho. \tag{4.12}
$$

Combining (4.7), (4.8), (4.9), (4.10) and (4.12), we obtain

$$
\begin{aligned}
\|w_L\|^{p_2}_{L^{p_2^\star}(\mathbb{R}^n)}=\Big(\int_{\mathbb{R}^n}|w_L|^{p_2^\star}\,\mathrm{dx}\Big)^{\frac{p_2}{p_2^\star}}\leq 2^{p_2}C_1^{p_2}\beta^{p_2}\Big[&\lambda(C_1\rho)^{\gamma+1-p_2}\|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)}\|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}\\
&+\mu K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2^\star}}(\mathbb{R}^n)}\|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}\Big]
\end{aligned}
$$

from which

$$
\Big(\int_{\mathbb{R}^n}|w_L|^{p_2^\star}\,\mathrm{dx}\Big)^{\frac{p_2}{p_2^\star}}\leq\beta^{p_2}C_{\lambda,\mu,K}\|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}, \tag{4.13}
$$

where

$$
C_{\lambda,\mu,K}\coloneqq 2^{p_2}C_1^{p_2}\left[\lambda(C_1\rho)^{\gamma+1-p_2}\|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)}+\mu K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}\right].
$$

Applying Fatou's lemma, sending $L\to+\infty$ in (4.13), we have

$$
\|u_+\|_{L^{\beta p_2^\star}(\mathbb{R}^n)}\leq\beta^{\frac{1}{\beta}}C_{\lambda,\mu,K}^{\frac{1}{p_2\beta}}\|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}, \tag{4.14}
$$

where $u_+^{\beta\alpha}=u_+^{p_2^\star}\in L^1(\mathbb{R}^n)$. Since $\beta=\frac{p_2^\star}{\alpha}>1$ and $u_+\in L^{p_2^\star}(\mathbb{R}^n)$, the inequality (4.14) holds for this choice of β. Thus, since $\beta^2\alpha=\beta p_2^\star$, it follows that (4.14) also holds with β replaced by β^2. Hence,

$$
\|u_+\|_{L^{\beta^2p_2^\star}(\mathbb{R}^n)}\leq(\beta^2)^{\frac{1}{\beta^2}}C_{\lambda,\mu,K}^{\frac{1}{p_2\beta^2}}\|u_+\|_{L^{\beta^2\alpha}(\mathbb{R}^n)}\leq\beta^{\frac{2}{\beta^2}+\frac{1}{\beta}}C_{\lambda,\mu,K}^{\frac{1}{p_2\beta^2}+\frac{1}{p_2\beta}}\|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}.
$$

By iterating this process and recalling that $\beta\alpha = p_2^\star$, we get

$$\|u_+\|_{L^{\beta^m p_2^\star}(\mathbb{R}^n)} \le \beta^{\sum_{i=1}^m i\beta^{-i}} C_{\lambda,\mu,K}^{\frac{1}{p_2}\sum_{i=1}^m \beta^{-i}} \|u_+\|_{L^{p_2^\star}(\mathbb{R}^n)}. \tag{4.15}$$

Taking the limit as $m \to \infty$ in (4.15), we find

$$\|u_+\|_{L^\infty(\mathbb{R}^n)} \le \beta^{\sigma_1} C_{\lambda,\mu,K}^{\sigma_2} C_1\rho,$$

where

$$\sigma_1 := \sum_{i=1}^{+\infty} i\beta^{-i} < +\infty, \quad \sigma_2 := \frac{1}{p_2}\sum_{i=1}^{+\infty} \beta^{-i} = \frac{\nu}{p_2^\star - \gamma - 1} < +\infty,$$

because $\beta = \frac{p_2^\star-\gamma-1}{\nu p_2} + 1 > 1$. Next we seek K and μ verifying the inequality below

$$\beta^{\sigma_1} C_{\lambda,\mu,K}^{\sigma_2} C_1\rho \le K.$$

Pick $K \ge 1$ satisfying

$$\frac{1}{2^{p_2}C_1^{p_2}}\left(\frac{K}{\beta^{\sigma_1}C_1\rho}\right)^{\frac{1}{\sigma_2}} - \lambda(C_1\rho)^{\gamma+1-p_2}\|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} > 0,$$

and fix $\mu^\star$ such that

$$\mu \le \mu^\star := \frac{1}{K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}}\left[\frac{1}{2^{p_2}C_1^{p_2}}\left(\frac{K}{\beta^{\sigma_1}C_1\rho}\right)^{\frac{1}{\sigma_2}} - \lambda(C_1\rho)^{\gamma+1-p_2}\|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)}\right].$$

Notice that $\mu^\star < +\infty$ because

$$\begin{aligned}
\mu^\star &= \frac{1}{2^{p_2}C_1^{p_2}K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}}\left(\frac{K}{\beta^{\sigma_1}C_1\rho}\right)^{\frac{1}{\sigma_2}} - \frac{\lambda(C_1\rho)^{\gamma+1-p_2}\|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)}}{K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}} \\
&< \frac{K^1\sigma_1}{2^{p_2}C_1^{p+\frac{1}{\sigma_2}}\beta^{\frac{\sigma_1}{\sigma_2}}\rho^{\frac{1}{\sigma_2}}K^{r-p_2}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}} \\
&= \frac{1}{2^{p_2}C_1^{p+\frac{1}{\sigma_2}}\beta^{\frac{\sigma_1}{\sigma_2}}\rho^{\frac{1}{\sigma_2}}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}}\frac{1}{K^{r-p_2-\frac{1}{\sigma_2}}} \\
&= \frac{1}{2^{p_2}C_1^{p+\frac{1}{\sigma_2}}\beta^{\frac{\sigma_1}{\sigma_2}}\rho^{\frac{1}{\sigma_2}}\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}} < +\infty,
\end{aligned}$$

where in the last inequality we used

$$\frac{1}{K^{r-p_2-\sigma_1}} = \frac{1}{K^{r-p_2-\frac{p_2^\star-\gamma-1}{\nu}}} \le 1,$$

since $K \geq 1$ and

$$r - p_2 - \frac{p_2^\star - \gamma - 1}{\nu} > r - p_2 - (p_2^\star - \gamma - 1) = r - p_2^\star + \gamma + 1 - p_2 > 0.$$

Then we have

$$\|u_+\|_{L^\infty(\mathbb{R}^n)} \leq \beta^{\sigma_1} C_{\lambda,\mu,K}^{\sigma_2} C_1 \rho \leq K \quad \text{for every } \mu \in [0, \mu^\star].$$

Set $\tilde{\delta} := \min\{\delta, \mu^\star\}$, where $\delta > 0$ is determined as in Theorem 1.2. From $|u| = u_+ + u_-$, we deduce that $\|u\|_{L^\infty(\mathbb{R}^n)} \leq K$ for all $\mu \in [0, \tilde{\delta}]$. Recalling that $u = u_i, i = 1, 2$, we get $\|u_i\|_{L^\infty(\mathbb{R}^n)} \leq K$ for all $\mu \in [0, \tilde{\delta}]$ and $i = 1, 2$. Consequently, (4.6) holds and the proof of Theorem 1.4 is now complete. $\square$

5. The exponential case

As in the proof of Theorem 1.4, we consider a suitable truncated function to overcome the presence of the exponential growth. More precisely, we define

$$g_K(x,t) := \begin{cases} h(x)e^{\mu} & \text{if } |t| \leq K, \\ \frac{e^K}{K^{p_2-1}} h(x)|t|^{p_2-2}t & \text{if } |t| \geq K, \end{cases}$$

where the value of $K \geq 1$ will be fixed later. Notice that g_K has subcritical growth because

$$\begin{aligned} |g_K(x,t)| &\leq |h(x)|e^K + \frac{e^K}{K^{p_2-1}} h(x)|t|^{p_2-1} \\ &\leq |h(x)|e^K \Big(1 + \frac{|t|^{p_2-1}}{K^{p_2-1}}\Big) \\ &\leq |h(x)|e^K (1 + |t|^{p_2-1}) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } t \in \mathbb{R}. \end{aligned} \tag{5.1}$$

Let us introduce Ψ_K as in (4.4) but considering g_K defined as above.

Lemma 5.1. *Suppose (1.16) holds. Then the functional Ψ_K' is a compact operator from $D_{p_1,p_2}(\mathbb{R}^n)$ to $(D_{p_1,p_2}(\mathbb{R}^n))^\star$, for every $K > 0$.*

Proof. Let $(u_k) \subset D_{p_1,p_2}(\mathbb{R}^n)$ be such that $u_k \rightharpoonup u$ in $D_{p_1,p_2}(\mathbb{R}^n)$. By interpolation in $L^p(\mathbb{R}^n)$ spaces and (1.16), we know that $h \in L^{\frac{p_2^\star}{p_2^\star - p_2}}(\mathbb{R}^n)$ with

$$\frac{\nu p_2^\star}{(\nu - 1) + \gamma + 1 - \nu p_2} > \frac{p_2^\star}{p_2^\star - p_2} > \frac{p_2^\star}{p_2^\star - 1}.$$

Using the Hölder inequality and (5.1), we can see that

$$\int_{|x|>R} g_K(x, u_k) v \, \mathrm{dx} \leq \int_{|x|>R} |g_K(x, u_k) v| \, \mathrm{dx}$$

$$
\begin{aligned}
&\le \int_{|x|>R} e^K |h(x)|(1+|u_k|^{p_2-1})|v|\,\mathrm{dx} \\
&= e^K \int_{|x|>R} |h(x)||v|\,\mathrm{dx} + e^K \int_{|x|>R} |h(x)||u_k|^{p_2-1}|v|\,\mathrm{dx} \\
&\le e^K \Big(\int_{|x|>R} |h(x)|^{\frac{p_2^\star}{p_2^\star-1}}\,\mathrm{dx} \Big)^{\frac{p_2^\star-1}{p_2^\star}} \Big(\int_{|x|>R} |v|^{p_2^\star}\,\mathrm{dx} \Big)^{\frac{1}{p_2^\star}} \\
&+ e^K \Big(\int_{|x|>R} |h(x)|^{\frac{p_2^\star}{p_2^\star-p_2}}\,\mathrm{dx} \Big)^{\frac{p_2^\star-p_2}{p_2^\star}} \Big(\int_{|x|>R} |u_k|^{p_2^\star}\,\mathrm{dx} \Big)^{\frac{p_2-1}{p_2^\star}} \Big(\int_{|x|>R} |v|^{p_2^\star}\,\mathrm{dx} \Big)^{\frac{1}{p_2^\star}}.
\end{aligned}
$$

Taking into account that $h \in L^{\frac{p_2^\star}{p_2^\star-p_2}}(\mathbb{R}^n) \cap L^{\frac{p_2^\star}{p_2^\star-1}}(\mathbb{R}^n)$, we have

$$
\lim_{R\to+\infty} \int_{|x|>R} |h(x)|^{\frac{p_2^\star}{p_2^\star-p_2}}\,\mathrm{dx} = 0 \quad \text{and} \quad \lim_{R\to+\infty} \int_{|x|>R} |h(x)|^{\frac{p_2^\star}{p_2^\star-1}}\,\mathrm{dx} = 0,
$$

which implies (4.5). Fix $\varepsilon > 0$. Applying the Young inequality, we obtain

$$
\begin{aligned}
|g_K(x,u)| &\le |h(x)|e^K(1+|u|^{p_2-1}) \le |h(x)|e^K + |h(x)|e^K|u|^{p_2-1} \\
&\le |h(x)|e^K + \frac{(\nu-1)p_2^\star+\gamma+1-\nu p_2}{\nu(p_2^\star-1)} e^K |h(x)|^{\frac{\nu(p_2^\star-1)}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}} \\
&+ \frac{p_2^\star+\nu p_2-\nu-\gamma-1}{\nu(p_2^\star-1)} e^K |u|^{\frac{\nu(p_2-1)(p_2^\star-1)}{p_2^\star+\nu p_2-\nu-\gamma-1}}.
\end{aligned}
$$

Since $\gamma+1 < p_2^\star$, we have

$$
\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1} < p_2^\star \tag{5.2}
$$

and $u_k \to u$ strongly in $L^{\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1}}(B_{R_\varepsilon})$. In view of (1.16), we get

$$
h \in L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(B_{R_\varepsilon}) \cap L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon}),
$$

which leads to

$$
e^K|h(\cdot)| + \frac{(\nu-1)p_2^\star+\gamma+1-\nu p_2}{\nu(p_2^\star-1)} e^K |h(\cdot)|^{\frac{\nu(p_2^\star-1)}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}} \in L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon}).
$$

Therefore, the Nemytskii operator N_h is continuous from $L^{\frac{\nu(p_2-1)p_2^\star}{p_2^\star+\nu p_2-\nu-\gamma-1}}(B_{R_\varepsilon})$ to $L^{\frac{p_2^\star}{p_2^\star-1}}(B_{R_\varepsilon})$. Arguing as in the proof of Lemma 4.2, we conclude that, for all $v \in D_{p_1,p_2}(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} |g_K(x,u_k) - g_K(x,u)||v|\,\mathrm{dx} \longrightarrow 0 \quad \text{as } k \to +\infty.
$$

This proves that Ψ_K' is a compact operator. □

Proof of Theorem 1.5. In view of Lemma 2.1, Lemma 3.1, (3.3), (3.4), Lemma 5.1, all the assumptions of Theorem 1.2 are satisfied. Therefore, problem (1.15) has at least three non-trivial solutions $u_i \in D_{p_1,p_2}(\mathbb{R}^n)$, $i = 1, 2, 3$ whose norms are less than ρ (note that $u = 0$ is not a solution of (1.15)). As in the proof of Theorem 1.4, we will prove that

$$|u_i(x)| \leq K \quad \text{for a.e. } x \in \mathbb{R}^n,$$

for all $i = 1, 2, 3$. However, some appropriate modifications will be done to implement the Moser iteration scheme. For simplicity, we set $u \coloneqq u_i$, $i = 1, 2, 3$. Then we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} a_2(x) u_L^{p_2(\beta-1)} |\nabla u_+|^{p_2} \,\mathrm{dx} &\leq \lambda \int_{\mathbb{R}^n} f(x,u) z_L \,\mathrm{dx} + \mu \int_{\mathbb{R}^n} g_K(x,u) z_L \,\mathrm{dx} \\
&\leq \lambda \int_{\mathbb{R}^n} m(x) |u|^{\gamma} u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx} \\
&+ \mu e^K \int_{\mathbb{R}^n} |h(x)| (|u|^{p_2-1} + 1) u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx} \\
&\leq \lambda (C_1 \rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}^{p_2\beta} \\
&+ \mu e^K \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)} \|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}^{p_2\beta} + \mu e^K \int_{\mathbb{R}^n} |h(x)| u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx},
\end{aligned}$$

where in the last inequality we used (4.8) and (4.9). By interpolation in $L^p(\mathbb{R}^n)$ spaces and the fact that $\beta = \frac{p_2^\star-\gamma-1}{\nu p_2} + 1$, we have $h \in L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)$, with

$$\frac{\nu p_2^\star}{(\nu-1)p_2^\star + \gamma + 1 - \nu p_2} > \frac{\nu p_2^\star}{(\nu-1)p_2^\star + \gamma + 1 - \nu} > \frac{p_2^\star}{p_2^\star - 1}.$$

Thanks to the definition of u_L and (1.16), we can estimate as follows

$$\begin{aligned}
\int_{\mathbb{R}^n} |h(x)| u_L^{p_2(\beta-1)} u_+ \,\mathrm{dx} &\leq \int_{\mathbb{R}^n} |h(x)| u_+^{p_2\beta-p_2+1} \,\mathrm{dx} \\
&\leq \left(\int_{\mathbb{R}^n} |h(x)|^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}} \,\mathrm{dx} \right)^{\frac{(\nu-1)p_2^\star+\gamma+1-\nu}{\nu p_2^\star}} \left(\int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \right)^{\frac{p_2\beta-p_2+1}{p_2^\star}}.
\end{aligned} \tag{5.3}$$

Since $\frac{p_2\beta-p_2+1}{p_2^\star} < \frac{p_2\beta}{p_2^\star}$ and

$$\int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \leq 1 \quad \text{or} \quad \int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \geq 1,$$

we have

$$\left(\int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \right)^{\frac{p_2\beta-p_2+1}{p_2^\star}} \leq 1 \quad \text{or} \quad \left(\int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \right)^{\frac{p_2\beta-p_2+1}{p_2^\star}} \leq \left(\int_{\mathbb{R}^n} u_+^{p_2^\star} \,\mathrm{dx} \right)^{\frac{p_2\beta}{p_2^\star}}.$$

Hence,

$$\left(\int_{\mathbb{R}^n} u_+^{p_2^\star}\,\mathrm{dx}\right)^{\frac{p_2\beta-p_2+1}{p_2^\star}} \le \max\left\{1, \left(\int_{\mathbb{R}^n} u_+^{p_2^\star}\,\mathrm{dx}\right)^{\frac{p_2\beta}{p_2^\star}}\right\}.$$

By (5.3), we can infer that

$$\int_{\mathbb{R}^n} |h(x)| u_L^{p_2(\beta-1)} u_+\,\mathrm{dx} \le \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} \max\left\{1, \left(\int_{\mathbb{R}^n} u_+^{p_2^\star}\,\mathrm{dx}\right)^{\frac{p_2\beta}{p_2^\star}}\right\}.$$

Similarly to (4.13), we find

$$\begin{aligned}
\|w_L\|^{p_2}_{L^{p_2^\star}(\mathbb{R}^n)} &\le 2^{p_2} C_1^{p_2} \beta^{p_2} \Big[\lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)} \\
&\quad + \mu e^K \Big(\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)} \|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)} + \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} \max\{1, \|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}\}\Big)\Big] \\
&\le 2^{p_2} C_1^{p_2} \beta^{p_2} \Big[\lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} + \mu e^K \Big(\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} \\
&\quad + \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)} \Big)\Big] \max\{1, \|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}\} \\
&= \beta^{p_2} C_{\lambda,\mu,K} \max\{1, \|u_+\|^{p_2\beta}_{L^{\beta\alpha}(\mathbb{R}^n)}\},
\end{aligned}$$

where

$$\begin{aligned}
C_{\lambda,\mu,K} \coloneqq 2^{p_2} C_1^{p_2} \Bigg[&\lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \\
&+ \mu e^K \Big(\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} + \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)} \Big)\Bigg].
\end{aligned}$$

Applying Fatou's Lemma as $L \to +\infty$ in the previous inequality, we get

$$\left(\int_{\mathbb{R}^n} u_+^{p_2^\star\beta}\,\mathrm{dx}\right)^{\frac{p_2}{p_2^\star}} \le \beta^{p_2} C_{\lambda,\mu,K} \max\{1, \|u_+\|^{p_2\beta}_{L^{\beta\alpha}}\}.$$

Setting $\tilde{C} \coloneqq 1 + C_{\lambda,\mu,K}$, we have

$$\|u_+\|_{L^{p_2^\star\beta}(\mathbb{R}^n)} \le \beta^{\frac{1}{\beta}} \tilde{C}^{\frac{1}{p_2\beta}} \max\{1, \|u_+\|_{L^{\beta\alpha}(\mathbb{R}^n)}\}. \tag{5.4}$$

Since $\alpha\beta = p_2^\star$ and $\alpha\beta^2 = \beta p_2^\star$, we see that (5.4) also holds with β replaced by β^2, and thus

$$\max\{1, \|u_+\|_{L^{\beta^2 p_2^\star}(\mathbb{R}^n)}\} \le \beta^{\frac{2}{\beta}+\frac{1}{\beta}} \tilde{C}^{\frac{1}{p_2\beta}+\frac{1}{p_2\beta^2}} \max\{1, \|u_+\|_{L^{p_2^\star}(\mathbb{R}^n)}\}.$$

Setting $a_m \coloneqq \max\{1, \|u_+\|_{L^{\beta^m p_2^\star}(\mathbb{R}^n)}\}$ and iterating this process, we achieve

$$a_m \le \beta^{\frac{1}{p_2}\sum_{i=1}^m i\beta^{-i}} \tilde{C}^{\frac{1}{p_2}\sum_{i=1}^m \beta^{-i}} a_0 \quad \text{for all } m \in \mathbb{N}. \tag{5.5}$$

Taking the limit as $m \to \infty$ in (5.5), we find

$$a_\infty \coloneqq \max\{1, \|u_+\|_{L^\infty(\mathbb{R}^n)}\} \leq \beta^{\sigma_1} \tilde{C}^{\sigma_2} a_0 = \beta^{\sigma_1} \tilde{C}^{\sigma_2} \max\{1, C_1\rho\},$$

where $\sigma_1 \coloneqq \frac{1}{p_2} \sum_{i=1}^{+\infty} i\beta^{-i}$ and $\sigma_2 \coloneqq \frac{1}{p_2} \sum_{i=1}^{+\infty} \beta^{-i} = \frac{\nu}{p_2^\star - \gamma - 1}$. Next, we choose K and μ in such a way that the inequality

$$\beta^{\sigma_1} \tilde{C}^{\sigma_2}_{\lambda,\mu,K} \max\{1, C_1\rho\} \leq K$$

holds, where

$$\begin{aligned}\tilde{C} \coloneqq 1 + 2^{p_2} C_1^{p_2} \Bigg[&\lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} \\ &+ \mu e^K \Big(\|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} + \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)} \Big) \Bigg].\end{aligned}$$

It is enough to select $K \geq 1$ such that

$$\frac{1}{2^{p_2} C_1^{p_2}} \left[\left(\frac{K}{\beta^{\sigma_1}} \max\{1, C_1\rho\} \right)^{\frac{1}{\sigma_2}} - 1 \right] - \lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)} =: A > 0,$$

and

$$\mu \leq \mu^\star \coloneqq \frac{A}{e^K B},$$

where

$$B \coloneqq \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu}}(\mathbb{R}^n)} + \|h\|_{L^{\frac{\nu p_2^\star}{(\nu-1)p_2^\star+\gamma+1-\nu p_2}}(\mathbb{R}^n)}.$$

Notice that

$$\begin{aligned}\mu^\star &= \frac{1}{2^{p_2} C_1^{p_2} e^K B} \left(\frac{K}{\max\{1, C_1\rho\} \beta^{\sigma_1}} \right)^{\frac{1}{\sigma_2}} - \frac{1}{2^{p_2} C_1^{p_2} e^K B} - \frac{\lambda (C_1\rho)^{\gamma+1-p_2} \|m\|_{L^{\frac{\nu p_2^\star}{(\nu-1)(p_2^\star-\gamma-1)}}(\mathbb{R}^n)}}{e^K B} \\ &< \frac{1}{2^{p_2} C_1^{p_2} \beta^{\frac{\sigma_1}{\sigma_2}} (\max\{1, C_1\rho\})^{\frac{1}{\sigma_2}} B} \frac{K^{\frac{p_2^\star-\gamma-1}{\nu}}}{e^K} \\ &\leq \frac{1}{2^{p_2} C_1^{p_2} \beta^{\frac{\sigma_1}{\sigma_2}} (\max\{1, C_1\rho\})^{\frac{1}{\sigma_2}} B} \left(\frac{p_2^\star - \gamma - 1}{\nu e} \right)^{\frac{p_2^\star-\gamma-1}{\nu}} < +\infty,\end{aligned}$$

where in the last inequality we used

$$\max_{x \geq 0} \varphi(x) = \max_{x \geq 0} \frac{x^n}{e^x} = \varphi\left(\frac{p_2^\star - \gamma - 1}{\nu} \right).$$

Arguing as in the proof of Theorem 1.4, we deduce the thesis of Theorem 1.5. $\square$

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