A GENERAL COMPACTNESS THEOREM IN $G(S)BD$

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Abstract. We give a new, simpler proof of a compactness result in $GSBD^p$, $p > 1$, by the same authors, which is also valid in GBD (the case $p = 1$), and shows that bounded sequences converge a.e., after removal of a suitable sequence of piecewise infinitesimal rigid motions, subject to a fixed partition.

1. Introduction

Generalized (special) functions with bounded deformation ($G(S)BD$) have been introduced by G. Dal Maso [11] in order to properly tackle *free discontinuity problems* [12, 2] in linearized elasticity, and in particular the minimization of the Griffith functional

$$\int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)\,\mathrm{d}x + \gamma\,\mathcal{H}^{d-1}(K), \tag{1.1}$$

introduced in [14] to model and approximate brittle fracture growth in linear elastic materials. In this functional, $\Omega \subset \mathbb{R}^d$ is a bounded d-dimensional domain (in practice $d \in \{2, 3\}$), u a vectorial displacement, expected to be smooth, with symmetrized gradient $e(u) = (Du + Du^T)/2$, except across a $(d-1)$-dimensional fracture set K. The tensor $\mathbb{C}$ contains the physical constants of the problem, and defines a positive definite quadratic form on symmetric tensors, while $\gamma > 0$ is the toughness of the material. Showing existence to minimizers of this functional has been a difficult task, developed over many years. The situation mostly evolved after [11] introduced for the first time a reasonable energy space for a weak form of (1.1), where K is replaced with J_u, the intrinsic jump set of u [13]. Existence results could then be proved [3, 15, 8, 10, 6, 7] for weak, then "strong" minimizers (that is, for the original problem in (u, K)). Most of these works rely upon a rigidity result for displacements with small jumps, established in [5].

In particular, the main result in [8] is a compactness result in $GSBD^p$, the subspace of $GSBD$ (which is defined precisely in Section 2.1) of displacements with p-integrable symmetrized gradient and jump set of finite surface. In this result, a sequence which is bounded in energy (roughly, (1.1), with the Lagrangian replaced with $|e(u)|^p$) will converge up to subsequences either to a $GSBD^p$ limit $u(x)$ or to $+\infty$ (with some appropriate semicontinuity properties). This is not really an issue for the study of (1.1), since replacing u with 0 where it is infinite, we recover that the limit of a minimizing sequence is a minimizer.

However, it was observed in [16, 9] that this compactness result is not sufficient for studying more general, non-homogeneous variational problems, where the Lagrangian is not minimal at 0. In that case, one has to study more finely what happens in the "infinity" set. Following similar (yet far more precise) results in the scalar case [16], the authors could show in [9] a more complete compactness result, and in particular the existence of a Caccioppoli partition where, on each set of the partition, the sequence converges to a finite limit after substraction of a suitable sequence of infinitesimal rigid motions (affine functions with skew-symmetric gradients).

In addition, S. Almi and E. Tasso [1] recently extended [8], with a different proof, to sequences merely bounded in GBD (roughly, the case $p = 1$ in [8]), while the proof in [9], relying on a fine result of [4] valid only for $p > 1$, would not work in GBD.

The purpose of this note is to give an alternative proof of the main compactness result of [9], which does not rely on [4] and is also valid in GBD, thus permitting to deal with non-homogeneous problems also in this framework. Precisely, we show the compactness Theorem 1.1 below (the notation is made precise in Section 2.1). The proof of this result is quite simpler, in a sense, than in [9], yet also more interesting. It only relies on a suitable version of the approximate Poincaré-Korn inequality of [5] proven in Theorem 2.3, which asserts that the energy controls how far a function is to rigid motions (hence to finite-dimensional), combined with a multiscale construction. We hope that this scheme can be useful for other purposes. We observe for instance that, combined with the celebrated extension method of Nitsche [18], a simplified version or this proof allows to easily deduce Rellich-type theorems in BD [21, 19, 20, 17].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $u_k \in GBD(\Omega)$ be such that*

$$\sup_{k\in\mathbb{N}} \widehat{\mu}_{u_k}(\Omega) < +\infty. \tag{1.2}$$

Then there exist a subsequence, not relabelled, a Caccioppoli partition $\mathcal{P} = (P_n)_n$ of Ω, a sequence of piecewise rigid motions $(a_k)_k$ with

$$a_k = \sum_{n\in\mathbb{N}} a_k^n \chi_{P_n}, \tag{1.3a}$$

$$|a_k^n(x) - a_k^{n'}(x)| \to +\infty \quad \textit{for } \mathcal{L}^d\textit{-a.e. } x \in \Omega, \ \textit{for all } n \neq n', \tag{1.3b}$$

and $u \in GBD(\Omega)$ such that

$$u_k - a_k \to u \quad \mathcal{L}^d\textit{-a.e. in } \Omega, \tag{1.4a}$$

$$\mathcal{H}^{d-1}(\partial^*\mathcal{P} \cap \Omega) \leq \lim_{\sigma\to+\infty} \liminf_{k\to\infty} \mathcal{H}^{d-1}(J^\sigma_{u_k}), \tag{1.4b}$$

where $J^\sigma_{u_k} := \{x \in J_{u_k} : |[u_k]|(x) \geq \sigma\}$.

If furthermore $(u_k)_k$ is bounded in $GSBD^p(\Omega)$, $p > 1$ (that is, (3.18) below holds), following [9] one obtains in addition to the last estimate:

$$\mathcal{H}^{d-1}((\partial^*\mathcal{P} \cup J_u) \cap \Omega) \leq \liminf_{k\to\infty} \mathcal{H}^{d-1}(J_{u_k}), \tag{1.4c}$$

see Remark 3.1 in Section 3.3.

The plan of the note is as follows: we first define properly the notions which are useful for this work (Sec. 2.1). Then, in Section 2.2 we show that a partial rigidity result of [5] is also valid in GBD, without further integrability assumption. The following section is devoted to the proof of Theorem 1.1. Thanks to the rigidity result, we build an appropriate Caccioppoli partition which will satisfy the thesis of the Theorem in Section 3.1. We end up proving the compactness (1.4a) (Sec. 3.2) and the lower-semicontinuity (1.4b) (Sec. 3.3).

2. Preliminaries

2.1. **Notation.** Given $\Omega \subset \mathbb{R}^d$ open, we use the notation $L^0(\Omega;\mathbb{R}^m)$ for the space of $\mathcal{L}^d$-measurable functions $v\colon \Omega \to \mathbb{R}^m$, endowed with the topology of convergence in measure. For any locally compact subset $B \subset \mathbb{R}^d$, (i.e. any point in B has a neighborhood contained in a compact subset of B), the space of bounded $\mathbb{R}^m$-valued Radon measures on B [respectively, the space of $\mathbb{R}^m$-valued Radon measures on B] is denoted by $\mathcal{M}_b(B;\mathbb{R}^m)$ [resp., by $\mathcal{M}(B;\mathbb{R}^m)$]. If $m = 1$, we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B;\mathbb{R})$, $\mathcal{M}(B)$ for $\mathcal{M}(B;\mathbb{R})$, and $\mathcal{M}_b^+(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu \in \mathcal{M}_b(B;\mathbb{R}^m)$, its total variation is denoted by $|\mu|(B)$.

We say that $v \in L^1(\Omega)$ is a *function of bounded variation* on Ω, and we write $v \in BV(\Omega)$, if $\mathrm{D}_i v \in \mathcal{M}_b(\Omega)$ for $i = 1, \dots, n$, where $\mathrm{D}v = (\mathrm{D}_1 v, \dots, \mathrm{D}_n v)$ is its distributional derivative. A vector-valued function $v\colon \Omega \to \mathbb{R}^m$ is in $BV(\Omega;\mathbb{R}^m)$ if $v_j \in BV(\Omega)$ for every $j=1,\dots,m$. The space $BV_{\mathrm{loc}}(\Omega)$ is the space of $v \in L^1_{\mathrm{loc}}(\Omega)$ such that $\mathrm{D}_i v \in \mathcal{M}(\Omega)$ for $i=1,\dots,d$.

We call *infinitesimal rigid motion* any affine function with skew-symmetric gradient and *piecewise rigid motion* any function of the form $\sum_{j\in\mathbb{N}} a_j \chi_{P_j}$, where $(P_j)_j$ is a Caccioppoli partition of Ω (that is, a partition into sets of finite perimeters, with finite total perimeter) and any a_j is an infinitesimal rigid motion.

Fixed $\xi \in \mathbb{S}^{d-1}$, we let

$$\Pi^\xi := \{y \in \mathbb{R}^d \colon y\cdot\xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R}\colon y + t\xi \in B\} \quad \text{for any } y \in \mathbb{R}^d \text{ and } B \subset \mathbb{R}^d, \tag{2.1}$$

and for every function $v\colon B \to \mathbb{R}^d$ and $t \in B^\xi_y$, let

$$v^\xi_y(t) := v(y+t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t)\cdot\xi. \tag{2.2}$$

Moreover, let $\Pi^\xi(x) := x - (x\cdot\xi)\xi \in \xi^\perp = \{y \in \mathbb{R}^d\colon y\cdot\xi = 0\}$ for every $x \in \mathbb{R}^d$.

Definition 2.1 (“*GBD*” [11])**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let $v \in L^0(\Omega;\mathbb{R}^d)$. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that one of the following equivalent conditions holds true for every $\xi \in \mathbb{S}^{d-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac12 \le \tau \le \frac12$ and $0 \le \tau' \le 1$, the partial derivative $\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big) = \mathrm{D}\big(\tau(v\cdot\xi)\big)\cdot\xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$

$$\big|\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big)\big|(B) \le \lambda_v(B);$$

(b) $\widehat{v}^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$

$$\int_{\Pi^\xi} \Big(\big|\mathrm{D}\widehat{v}^\xi_y\big|\big(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\widehat{v}^\xi_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda_v(B),$$

where $J^1_{\widehat{v}^\xi_y} := \Big\{t \in J_{\widehat{v}^\xi_y} : |[\widehat{v}^\xi_y]|(t) \ge 1\Big\}$, for $J_{\widehat{v}^\xi_y}$ the jump set of $\widehat{v}^\xi_y$, i.e. the set of $t \in \Omega^\xi_y$ for which $[\widehat{v}^\xi_y](t) := (\widehat{v}^\xi_y)^+(t) - (\widehat{v}^\xi_y)^-(t) \neq 0$, $(\widehat{v}^\xi_y)^\pm(t)$ being the unilateral limits of $\widehat{v}^\xi_y$ at t.

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

For $v \in GBD(\Omega)$, denoting by

$$(\widehat{\mu}_v)^\xi_y(B) := |\mathrm{D}\widehat{v}^\xi_y|\big(B \setminus J^1_{\widehat{v}^\xi_y}\big) + \mathcal{H}^0\big(B \cap J^1_{\widehat{v}^\xi_y}\big) \quad \text{for every } B \subset \Omega^\xi_y \text{ Borel} \tag{2.3}$$

($(\widehat{\mu}_v)^\xi_y \in \mathcal{M}^+_b(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\xi$) and by

$$(\widehat{\mu}_v)^\xi(B) := \int_{\Pi^\xi} (\widehat{\mu}_v)^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{for every } B \subset \Omega \text{ Borel}, \tag{2.4}$$

it holds that $(\widehat{\mu}_v)^\xi \in \mathcal{M}^+_b(\Omega)$, $(\widehat{\mu}_v)^\xi \le \lambda_v$ for any λ_v satisfying condition (b) of Definition 2.1 and that

$$\widehat{\mu}_v(B) := \sup_k \sup \Big\{ \sum_{i=1}^k (\widehat{\mu}_v)^{\xi_i}(B_i) \colon (\xi_i)_i \subset \mathbb{S}^{d-1},\, B_1,\dots,B_k \subset B, B_i \cap B_j = \emptyset\, \forall i \neq j \Big\} \tag{2.5}$$

is the smallest measure λ_v that satisfies condition (b) of Definition 2.1.

Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega;\mathbb{M}^{d\times d}_{sym})$ such that for every $\xi \in \mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ there holds

$$e(v)(y+t\xi)\xi\cdot\xi = (\widehat{v}^\xi_y)'(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y; \tag{2.6}$$

the *approximate jump set* J_v (*cf.* [11, Definition 2.4]) is still countably $(\mathcal{H}^{d-1}, d-1)$-rectifiable (*cf.* [11, Theorem 6.2] and [13]) and may be reconstructed from its slices through the identity

$$(J_v^\xi)_y^\xi = J_{\widehat{v}_y^\xi} \quad \text{and} \quad v^\pm(y+t\xi)\cdot\xi = (\widehat{v}_y^\xi)^\pm(t) \quad \text{for } t \in (J_v)_y^\xi, \tag{2.7}$$

where $J_v^\xi := \{x \in J_v \colon [v](x)\cdot\xi \neq 0\}$ (it holds that $\mathcal{H}^{d-1}(J_v \setminus J_v^\xi) = 0$ for $\mathcal{H}^{d-1}$-a.e. $\xi \in \mathbb{S}^{d-1}$). For every $\sigma > 0$ we also denote

$$J_v^\sigma := \{x \in J_v \colon |[v]|(x) \geq \sigma\}. \tag{2.8}$$

By (2.7), for every $\sigma > 0$, every $\xi \in \mathbb{S}^{d-1}$, and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi_\xi$

$$J^\sigma_{\widehat{v}_y^\xi} \subset (J_v^\sigma)_y^\xi, \tag{2.9}$$

where $J^\sigma_{\widehat{v}_y^\xi} = \{t \in J_{\widehat{v}_y^\xi} \colon |[\widehat{v}_y^\xi]|(t) \geq \sigma\}$.

We recall from [9] the following lemma on piecewise rigid motions.

Lemma 2.2. *Let $(\mathcal{P}_j)_j$ be a Caccioppoli partition and let $(a_h)_h$ be a sequence of piecewise rigid motions such that* (1.3a) *and* (1.3b) *hold. Then for $\mathcal{H}^{d-1}$-a.e. $\xi \in \mathbb{S}^{d-1}$*

$$|(a_h^j - a_h^i)(x)\cdot\xi| \to +\infty \quad \text{as } h \to +\infty \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega, \text{ for all } i \neq j. \tag{2.10}$$

2.2. **Rigidity in** GBD**.** The following result is obtained in the footsteps of Proposition 2.1 in [5]. Let $Q_\delta = (-\delta/2, \delta/2)^d$.

Theorem 2.3. *There exist $c > 0$ such that for any $\delta > 0$, $u \in GSBD(Q_\delta)$, there exists $\omega \subset Q_\delta$ with $|\omega| \leq c\delta\mathcal{H}^{d-1}(J_u^1)$ and an infinitesimal rigid motion a such that*

$$\int_{Q_\delta\setminus\omega} |u-a|\,\mathrm{d}x \leq c\delta\,\widehat{\mu}_u(Q_\delta \setminus J_u^1).$$

Proof. We sketch the proof, highlighting the modifications with respect to [5, Proposition 2.1].

As in [5], we may assume by a rescaling argument $\delta = 1$ (and write Q for Q_1), and that $\mathcal{H}^{d-1}(J_u^1) \leq \frac{1}{32d^3}$, otherwise it is enough to take $\omega = Q$, $a = 0$, $c = 32d^3$. We define the function $T \colon \mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}$ by

$$T(x,\xi,t) := \begin{cases} 1 & \text{if } x \in Q,\ x+t\xi \in Q \text{ and } x+[0,t]\xi \cap J_u^1 \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

By definition of distributional derivative it holds that

$$\xi\cdot\big(u(x+t\xi)-u(x)\big) = \int_{x\cdot\xi}^{x\cdot\xi+t} \mathrm{D}\widehat{u}_y^\xi(s)\,\mathrm{d}s, \quad y := \Pi^\xi(x) \tag{2.12}$$

and $\mathrm{D}\widehat{u}_y^\xi \leq (\widehat{\mu}_u)_y^\xi$ on $[x\cdot\xi, x\cdot\xi+t]$ if $T(x,\xi,t) = 0$ and x, $x+t\xi \in Q$ (recall (2.9)) at least for a.e. $x \in Q$ and $t \in \mathbb{R}$. We remark that (2.11) is the analogue of [5, definition (2.6)] when replacing J_u with J_u^1, and that (2.12) is the analogue of [5, equation (2.5)]. Therefore, exactly as in [5], one obtains that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} T(x,\xi,t)\,\mathrm{d}t\,\mathrm{d}x \leq 4d\mathcal{H}^{d-1}(J_u^1) \quad \text{for any } \xi \in \mathbb{S}^{d-1}$$

and that there exists $t_* \in (1/2, 1)$ (fixed for the rest of the proof) and $q_1 \subset q$ with $\mathcal{L}^d(q_1)/\mathcal{L}^d(q) > 3/4$ such that, defining

$$z_i(z_0) := z_0 + t_* e_i \quad \text{for all } i = 1,\dots,d, \quad z_0(z_0) := z_0, \quad E_{z_0} := \bigcup_{0\leq i<j\leq d} [z_i(z_0), z_j(z_0)] \tag{2.13a}$$

it holds

$$\text{for } z_0 \in q_1 \colon E_{z_0} \cap J_u^1 = \emptyset,\ E_{z_0} \subset Q. \tag{2.13b}$$

For t_* fixed above and any $z_0 \in q$ let us consider

$$F(z_0) := \sum_{0 \le i < j \le d} |\mathrm{D}\widehat{u}^{\xi_{i,j}}_{y_{i,j}(z_0)}|([z_i(z_0) \cdot \xi_{i,j}, z_j(z_0) \cdot \xi_{i,j}]),$$
$$\xi_{i,j} := \frac{z_i(z_0) - z_j(z_0)}{|z_i(z_0) - z_j(z_0)|},\ y_{i,j}(z_0) := \Pi^{\xi_{i,j}}(z_i(z_0)) = z_i(z_0) - (z_i(z_0) \cdot \xi_{i,j})\xi_{i,j}.$$

We notice that $\xi_{i,j} = \frac{e_i - e_j}{\sqrt{2}}$ if $i \neq 0$, while $\xi_{i,j} = -e_j$ if $i = 0$. Fixed $i \neq j$, we integrate for $z_0 \in q$ as $\widetilde{z}_0 = \Pi^{\xi_{i,j}}(z_0)$ ranges in $\xi_{i,j}^\perp$ and $z_0' = z_0 \cdot \xi_{i,j}$ ranges in $q^{\xi_{i,j}}_{\Pi^{\xi_{i,j}}(z_0)}$, using Fubini's Theorem. Moreover, if $\Pi^{\xi_{i,j}}(z_0)$ is fixed to a value $\widetilde{z}_0 \in \Pi^{\xi_{i,j}}$, also $y_{i,j}(z_0)$ is fixed and equal to

$$\widehat{z}_0 := \Pi^{\xi_{i,j}}(z_0) + t_* \Pi^{\xi_{i,j}}(e_i) = \widetilde{z}_0 + t_* \Pi^{\xi_{i,j}}(e_i),$$

so that in such a case $[z_i(z_0), z_j(z_0)] \subset \widehat{z}_0 + \mathbb{R}\xi_{i,j} \cap Q$ and

$$|\mathrm{D}\widehat{u}^{\xi_{i,j}}_{y_{i,j}(z_0)}|([z_i(z_0) \cdot \xi_{i,j}, z_j(z_0) \cdot \xi_{i,j}]) \le (\widehat{\mu}_u)^{\xi_{i,j}}_{\widehat{z}_0}((Q \setminus J^1_u)^{\xi_{i,j}}_{\widehat{z}_0})$$

regardless of the value of $z_0' = z_0 \cdot \xi_{i,j}$ (satisfying $\widetilde{z}_0 + z_0'\xi_{i,j} \in q$ since we integrate over $z_0 \in q$). It follows that (notice that $\mathcal{L}^1(\{s \in \mathbb{R} \colon \widetilde{z}_0 + z_0'\xi_{i,j} \in Q\}) \le \sqrt{2}$)

$$\begin{aligned}
&\int_q |\mathrm{D}\widehat{u}^{\xi_{i,j}}_{y_{i,j}(z_0)}|([z_i(z_0) \cdot \xi_{i,j}, z_j(z_0) \cdot \xi_{i,j}])\,\mathrm{d}z_0 \\
&\qquad = \int_{\widetilde{z}_0 = \Pi^{\xi_{i,j}}_{z_0} \in \xi_{i,j}^\perp} \mathrm{d}\mathcal{H}^{d-1}(\widetilde{z}_0) \int_{(\widetilde{z}_0 + \mathbb{R}\xi_{i,j}) \cap q} |\mathrm{D}\widehat{u}^{\xi_{i,j}}_{y_{i,j}(z_0)}|([z_i(z_0) \cdot \xi_{i,j}, z_j(z_0) \cdot \xi_{i,j}])\,\mathrm{d}z_0' \\
&\qquad \le \sqrt{2} \int_{\widetilde{z}_0 \in \xi_{i,j}^\perp} (\widehat{\mu}_u)^{\xi_{i,j}}_{\widehat{z}_0}((Q \setminus J^1_u)^{\xi_{i,j}}_{\widehat{z}_0})\,\mathrm{d}\mathcal{H}^{d-1}(\widetilde{z}_0) \\
&\qquad \le \sqrt{2}(\widehat{\mu}_u)^{\xi_{i,j}}(Q \setminus J^1_u).
\end{aligned} \tag{2.14}$$

Summing (2.14) over $0 \le i < j \le d$, we get

$$\int_q F(z_0)\,\mathrm{d}z_0 \le \sqrt{2}(d+1)^2 \widehat{\mu}_u(Q \setminus J^1_u).$$

and there exists $q_2 \subset q$ with $\mathcal{L}^d(q_2)/\mathcal{L}^d(q) > 3/4$ such that for every $z_0 \in q_2$

$$F(z_0) \le 4\sqrt{2}(d+1)^2 \widehat{\mu}_u(Q \setminus J^1_u). \tag{2.15}$$

This is the analogue of [5, condition (2.8)]. At this stage, following [5], it holds that for any z_0 satisfying (2.13b) and (2.15) the affine map $a \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $a(z_i(z_0)) = u(z_i(z_0))$ for all $i = 0, \dots, d$ satisfies

$$|e(a)| \le c\widehat{\mu}_u(Q \setminus J^1_u). \tag{2.16}$$

Arguing exactly as in Step 3 of the proof of Proposition 2.1 in [5] we find a set $q_3 \subset q$ with $\mathcal{L}^d(q_3)/\mathcal{L}^d(q) > 3/4$ such that if $z_0 \in q_3$ then

$$\begin{aligned}
&\text{(2.12) holds for any } [z_i(z_0), y] \text{ for } y \in Q \setminus \omega, \text{ where} \\
&\omega = \bigcup_{i=0}^d \omega_{(i)}, \qquad \omega_{(i)} := \{y \in Q \colon y = z_i + t\xi \text{ with } T(z_i, \xi, t) = 1\}.
\end{aligned} \tag{2.17}$$

With (2.12) and the fact that for any y there are d points in $\{z_0, \dots, z_d\}$ such that the simplex generated by those and y has volume at least $t_*/(d+1)!$, (2.17) implies that

$$\begin{aligned} |w(y)| \leq & c \sum_{i=0}^{d} |\mathrm{D}\widehat{w}^{\xi_{i,y}}_{x_{i,y}(z_0)}|([z_i(z_0) \cdot \xi_{i,y}, y \cdot \xi_{i,y}]) \quad \text{for all } y \in Q \setminus \omega, \text{ where} \\ & w := u - a, \quad \xi_{i,y} := \frac{y - z_i(z_0)}{|y - z_i(z_0)|}, \quad x_{i,y}(z_0) := \Pi^{\xi_{i,y}}(z_i(z_0)) \end{aligned} \tag{2.18}$$

Let us consider the quantity

$$H(z_0) := \sum_{i=0}^{d} H_i(z_0), \quad H_i(z_0) := \int_{Q \setminus \omega} |\mathrm{D}\widehat{w}^{\xi_{i,y}}_{x_{i,y}(z_0)}|([z_i(z_0) \cdot \xi_{i,y}, y \cdot \xi_{i,y}]) \, \mathrm{d}y. \tag{2.19}$$

In the following we prove that there exists $q_4 \subset q_3$ with $\mathcal{L}^d(q_4)/\mathcal{L}^d(q_3) > 3/4$ such that $H(z_0)$ is controlled by $\widehat{\mu}_u(Q \setminus J^1_u)$ times a constant c depending only on d, for every $z_0 \in q_4$. Together with (2.16) and (2.18) this will conclude the proof.

By Fubini's Theorem, we have that for every $i = 0, \dots, d$

$$\int_{q_3} H_i(z_0) \, \mathrm{d}z_0 = \int_{Q \setminus \omega} \int_{q_3} |\mathrm{D}\widehat{w}^{\xi_{i,y}}_{x_{i,y}(z_0)}|([z_i(z_0) \cdot \xi_{i,y}, y \cdot \xi_{i,y}]) \, \mathrm{d}z_0 \, \mathrm{d}y.$$

For fixed $y \in Q \setminus \omega$ we argue similarly to what done to prove (2.14), namely we integrate for $z_0 \in q_3$ as $\widetilde{z}_0 = \Pi^{\xi_{i,y}}(z_0)$ ranges in $\xi^\perp_{i,y}$ and $z'_0 = z_0 \cdot \xi_{i,y}$ ranges in $(q_3)^{\xi_{i,y}}_{\widetilde{z}_0}$, using Fubini's Theorem. Then, given $\widetilde{z}_0$, we have that $x_{i,y}(z_0) = \widetilde{z}_0 + t_* \Pi^{\xi_{i,y}}(e_i) =: \widehat{z}_0$ and that

$$|\mathrm{D}\widehat{w}^{\xi_{i,y}}_{x_{i,y}(z_0)}|([z_i(z_0) \cdot \xi_{i,y}, y \cdot \xi_{i,y}]) \leq (\widehat{\mu}_w)^{\xi_{i,y}}_{\widehat{z}_0}((Q \setminus J^1_u)^{\xi_{i,y}}_{\widehat{z}_0}),$$

regardless of the value of z'_0. Therefore

$$\begin{aligned} \int_{q_3} |\mathrm{D}\widehat{w}^{\xi_{i,y}}_{x_{i,y}(z_0)}|([z_i(z_0) \cdot \xi_{i,y}, y \cdot \xi_{i,y}]) \, \mathrm{d}z_0 &\leq \sqrt{2} \int_{\xi^\perp_{i,y}} (\widehat{\mu}_w)^{\xi_{i,y}}_{\widehat{z}_0}((Q \setminus J^1_u)^{\xi_{i,y}}_{\widehat{z}_0}) \, \mathrm{d}\mathcal{H}^{d-1}(\widetilde{z}_0) \\ &\leq \sqrt{2} (\widehat{\mu}_w)^{\xi_{i,y}}(Q \setminus J^1_u) = \sqrt{2} (\widehat{\mu}_u)^{\xi_{i,y}}(Q \setminus J^1_u) \end{aligned} \tag{2.20}$$

where the equality above follows from the fact that a is an infinitesimal rigid motion. Summing (2.20) over i (and arguing as done for (2.15)) we get that

$$\int_{q_3} H(z_0) \, \mathrm{d}z_0 \leq c \widehat{\mu}_u(Q \setminus J^1_u),$$

thus we find $q_4 \subset q_3$ with $\mathcal{L}^d(q_4)/\mathcal{L}^d(q_3) > 3/4$ and $H(z_0) \leq c\widehat{\mu}_u(Q \setminus J^1_u)$ for every $z_0 \in q_4$; we conclude the proof by picking z_0 in $q_1 \cap q_2 \cap q_4$ (which has positive measure) and integrating (2.18) over $y \in Q \setminus \omega$. □

3. Proof of the compactness theorem

In this section we prove Theorem 1.1. In Subsection 3.1 we construct a suitable partition $\mathcal{P} = (P_n)_n$ of Ω and a sequence piecewise rigid functions $(a_k)_k$ satisfying (1.3); in the next two subsections we prove the existence of $u \in GBD(\Omega)$ satisfying (1.4a) and the lower semicontinuity condition (1.4b) on the surface measure of $\partial^* \mathcal{P}$, respectively.

3.1. **Construction of a partition.** For $\delta > 0$, let $\mathcal{Q}^\delta = \{z + (-\delta/2, \delta/2]^d : z \in \delta\mathbb{Z}^d,\ z + (-\delta/2, \delta/2)^d \subset \Omega\}$. Let $\eta > 0$, small, and define

$$\mathcal{B}^\delta(u_k) = \{Q \in \mathcal{Q}^\delta : \mathcal{H}^{d-1}(J^1_{u_k} \cap Q) > \eta\delta^{d-1}\}, \quad \mathcal{G}^\delta(u_k) = \mathcal{Q}^\delta \setminus \mathcal{B}^\delta(u_k).$$

By construction, one has

$$\Big|\bigcup\nolimits_{Q\in\mathcal{B}^\delta(u_k)} Q\Big| \le \frac{\delta}{\eta}\mathcal{H}^{d-1}(J^1_{u_k}). \tag{3.1}$$

We fix a first value $\delta_0 > 0$, small enough so that $\mathcal{G}^{\delta_0}(u_k) \neq \emptyset$ for all $k \ge 1$, and for $j \ge 0$, denote $\delta_j = \delta_0 2^{-j}$. Upon extracting a subsequence, we may assume that $\mathcal{B}^{\delta_0}(u_k)$ is not depending on k. By a diagonal argument, we may (and will) assume even that for any $j \ge 0$, $\mathcal{B}^{\delta_j}(u_k)$ does not depend on k if k is large enough (depending on j). We denote then $\mathcal{B}^{\delta_j}$ (and $\mathcal{G}^{\delta_j}$) the corresponding limiting sets, dropping the dependence in u_k.

Thanks to Theorem 2.3, for any $j \ge 0$ and each $Q \in \mathcal{Q}^{\delta_j}$, there is $\omega^Q_k \subset Q$ and a^Q_k, an infinitesimal rigid motion, such that

$$\int_{Q\setminus\omega^Q_k} |u_k - a^Q_k|dx \le c\delta_j\widehat{\mu}_{u_k}(Q \setminus J^1_{u_k}) \tag{3.2}$$

and $|\omega^Q_k| \le c\delta_j\mathcal{H}^{d-1}(J^1_{u_k} \cap Q)$. In particular:

$$Q \in \mathcal{G}^{\delta_j}(u_k) \quad\Rightarrow\quad |\omega^Q_k| \le c\eta|Q|. \tag{3.3}$$

Considering the finite family of sequences of infinitesimal rigid motions $(a^Q_k - a^{Q'}_k)_k$, $\{Q, Q'\} \subset \mathcal{G}^{\delta_0}$ (which does not depend on k), we may assume, upon extracting a subsequence, that either $|a^Q_k(x) - a^{Q'}_k(x)| \to \infty$ a.e., or $\sup_k \sup_{|x|\le 1} |a^Q_k(x) - a^{Q'}_k(x)| < +\infty$. By a diagonal argument, similarly, for $j \ge 1$, considering $(a^Q_k - a^{Q'}_k)_k$, $\{Q, Q'\} \in \mathcal{G}^{\delta_j}(u_k)$, which for k large enough is $\mathcal{G}^{\delta_j}$, not depending on k, we may assume the same.

We let for $j \ge 0$

$$B_j = \Big(\Omega \setminus \bigcup_{Q\in\mathcal{Q}^{\delta_j}} Q\Big) \cup \bigcup_{l\ge j}\bigcup_{Q\in\mathcal{B}^{\delta_l}} Q.$$

Thanks to (3.1), it holds that

$$|B_j| \le \Big|\Omega \setminus \bigcup\nolimits_{Q\in\mathcal{Q}^{\delta_j}} Q\Big| + \frac{2\delta_j}{\eta}\sup_k \widehat{\mu}_{u_k}(\Omega), \tag{3.4}$$

so that $\lim_j |B_j| = 0$; moreover $(B_j)_j$ is decreasing, that is $B_{j+1} \subseteq B_j$ for all $j \ge 0$.

We define a partition $(P^j_n)_{n=1}^{N_j}$ of $\Omega \setminus B_j$ as follows: the sequences (a^Q_k), $Q \in \mathcal{G}^{\delta_j}$, for k large enough so that $\mathcal{G}^{\delta_j}(u_k) = \mathcal{G}^{\delta_j}$, can be grouped in equivalent classes for the relationship $a^Q_k \sim a^{Q'}_k$ when $\sup_k \sup_{|x|\le 1} |a^Q_k(x) - a^{Q'}_k(x)| < +\infty$. Then, we say that $Q \sim Q'$ whenever $a^Q_k \sim a^{Q'}_k$. We define, for each equivalence class $\mathcal{C}_n$ in $\mathcal{G}^{\delta_j}$, $n = 1, \dots, N_j$, the set $P^j_n = \bigcup_{Q\in\mathcal{C}_n} Q \setminus B_j$.

Observe that for any $j \ge 1$, if $Q \in \mathcal{G}^{\delta_j}$, $Q' \in \mathcal{G}^{\delta_{j+1}}$ with $Q' \subset Q$ (and k is large enough), then one has:

$$\int_{Q'\setminus(\omega^Q_k\cup\omega^{Q'}_k)} |a^Q_k - a^{Q'}_k|dx \le c\delta\widehat{\mu}_{u_k}(Q \setminus J^1_{u_k}),$$

so that, provided $\eta > 0$ was chosen small enough (to ensure for instance that $|\omega^Q_k \cup \omega^{Q'}_k| \le |Q'|/2$, which is guaranteed if $\eta < 2^{-d}/(4c)$, *cf.* (3.3)), $\sup_k \sup_{|x|\le 1} |a^Q_k(x) - a^{Q'}_k(x)| < +\infty$.

Hence given $Q \in Q^{\delta_l}$, $Q' \in Q^{\delta_{l'}}$, for $l' \ge l \ge j$, and such that $Q' \subset Q$ and $|Q' \setminus B_j| > 0$, one obtains by induction, $\sup_k \sup_{|x|\le 1} |a^Q_k(x) - a^{Q'}_k(x)| < +\infty$ (as all the intermediate cubes are all "good" at their respective scale).

It follows that: for any smaller scales $l, l' \geq j$, and any $Q \in \mathcal{G}^{\delta_l}$, $Q' \in \mathcal{G}^{\delta_{l'}}$ with both $Q \setminus B_j$ and $Q' \setminus B_j$ of positive measure and contained in the same component P^j_n, one finds that $\sup_k \sup_{|x|\leq 1} |a^Q_k(x) - a^{Q'}_k(x)| < +\infty$. Indeed, there are $\tilde{Q}, \tilde{Q}' \in \mathcal{G}^{\delta_j}$ with $Q \subset \tilde{Q}$, $Q' \subset \tilde{Q}'$ and $\tilde{Q} \sim \tilde{Q}'$.

In particular this shows that for $j' \geq j$ and $n \in \{1, \dots, N_j\}$, there is $n' \in \{1, \dots, N_{j'}\}$ such that $P^j_n \subset P^{j'}_{n'}$. We may always number the sets $(P^j_n)_n$, $j \geq 1$, according to the numbering of $(P^{j-1}_n)_n$, so that in fact $P^j_n \subset P^{j'}_n$ for any $j' \geq j$ and any $n \in \{1, \dots, N_j\}$. As a consequence, we may define, for $1 \leq n < 1 + \sup_j N_j \in \mathbb{N} \cup \{+\infty\}$, the set $P_n = \bigcup_j P^j_n$ (where the union starts at the first j such that $n \leq N_j$). These sets partition $\Omega \setminus \bigcap_j B_j$, hence, up to a Lebesgue-negligible set, Ω.

For each n, we choose an arbitrary $Q \in \mathcal{G}^{\delta_j}$, at some arbitrary scale $j \geq 0$, with $|Q \setminus B_j| > 0$ and $Q \setminus B_j \subset P_n$, and we associate to P_n the subsequence $(a^Q_k)_k$, hence denoted $(a^n_k)_k$. It follows that for any other such cube Q at any other scale j, one has $\sup_k \sup_{|x|\leq 1} |a^n_k(x) - a^Q_k(x)| < +\infty$, while $\lim_k |a^{n'}_k(x) - a^Q_k(x)| = +\infty$ a.e. if $n' \neq n$.

3.2. **Compactness.** We introduce the smooth, one-Lipschitz truncations $t_\sigma(x) := \sigma \tanh(x/\sigma)$, for $\sigma > 0$. We also let $v_k = \sum_n a^n_k \chi_{P_n}$. Note that at the scale $j \geq 0$, one has that

$$v_k|_{\Omega \setminus B_j} = \sum_n a^n_k \chi_{P_n \setminus B_j} = \sum_{n=1}^{N_j} a^n_k \chi_{P^j_n},$$

showing that $v_k|_{\Omega \setminus B_j}$ is built up of N_j infinitesimal rigid motions.

For each scale $j \geq 0$ let

$$w^j_k = \left(\sum_{Q \in \mathcal{G}^{\delta_j}(u_k)} a^Q_k \chi_Q - v_k \right) (1 - \chi_{B_j}).$$

By construction, $\sup_k \sup_{x \in \Omega \setminus B_j} |w^j_k(x)| < +\infty$ and since $v_k|_{\Omega \setminus B_j}$ is built up of a bounded number of affine maps, the sequence of functions $(w^j_k)_k$ is finite-dimensional, and we may extract a subsequence such that it converges to some limit w^j. By a diagonal argument, we may assume this is true for all $j \geq 0$.

For $e \in \mathbb{R}^d$, $|e| = 1$, $\sigma > 0$, we consider the sequences of functions $u^{e,\sigma}_k := t_\sigma(e \cdot (u_k - v_k))$. We let $\omega^j_k := \bigcup_{Q \in \mathcal{G}^{\delta_j}(u_k)} \omega^Q_k$, then $|\omega^j_k| \leq c\delta_j \mathcal{H}^{d-1}(J^1_{u_k})$. Thus:

$$\begin{aligned} \int_\Omega |u^{e,\sigma}_k - t_\sigma(e \cdot w^j_k)| dx &\leq \sigma \left| B_j \cup \omega^j_k \right| + \int_{\Omega \setminus (B_j \cup \omega^j_k)} |e \cdot (u_k - v_k - w^j_k)| dx \\ &\leq \sigma \eta_j + \sum_{Q \in \mathcal{G}^{\delta_j}(u_k)} \int_{Q \setminus \omega^Q_k} |u_k - a^Q_k| dx \\ &\leq \sigma \eta_j + C\delta_j \end{aligned} \tag{3.5}$$

where we have let $\eta_j = |B_j| + c\delta_j \sup_k \mathcal{H}^{d-1}(J^1_{u_k})$, and $C = c \sup_k \widehat{\mu}_{u_k}(\Omega \setminus J^1_{u_k}) < +\infty$, and used (3.2). Using that $w^j_k - w^j_l \to 0$ as $k, l \to \infty$ in $L^1(\Omega)$, we find that:

$$\limsup_{k,l \to \infty} \int_\Omega |u^{e,\sigma}_k - u^{e,\sigma}_l| dx \leq 2(\sigma \eta_j + C\delta_j).$$

Sending $j \to +\infty$ we find that $\limsup_{k,l\to\infty} \int_\Omega |u^{e,\sigma}_k - u^{e,\sigma}_l| dx = 0$, that is, $(u^{e,\sigma}_k)_k$ are Cauchy sequences and converge to some limit $u^{e,\sigma}$ in $L^1(\Omega)$.

We show now that the limit $u^{e,\sigma}$ is $t_\sigma(e\cdot u)$ for some well-defined measurable function u. Let us consider a subsequence such that $u_k^{e,1}\to u^{e,1}$ a.e. In that case:

- either $|u^{e,1}(x)|=1$, which happens if and only if $\lim_{k\to\infty}|e\cdot(u_k(x)-v_k(x))|=+\infty$, and in particular for any $\sigma>0$, $|u^{e,\sigma}(x)|=\sigma$;
- or, by continuity, $e\cdot(u_k(x)-v_k(x))\to\tanh^{-1}(u^{e,1}(x))$, and we also have that $u^{e,\sigma}(x)=t_\sigma(\tanh^{-1}(u^{e,1}(x)))$ for any $\sigma>0$.

Let $A=\{x:|u^{e,1}(x)|=1\}=\{x:|u^{e,\sigma}(x)|=\sigma\}$: then, for $j\geq 0$,

$$\int_{A\setminus B_j}|u_k^{e,\sigma}-t_\sigma(e\cdot w_k^j)|dx\geq\int_{A\setminus B_j}|u_k^{e,\sigma}|dx-\int_{A\setminus B_j}|w_k^j|dx\overset{k\to\infty}{\longrightarrow}\sigma|A\setminus B_j|-\int_{A\setminus B_j}|w^j|dx.$$

On the other hand thanks to (3.5):

$$\int_{A\setminus B_j}|u_k^{e,\sigma}-t_\sigma(e\cdot w_k^j)|dx\leq\sigma\eta_j+C\delta_j,$$

hence:

$$\sigma|A\setminus B_j|\leq\int_{A\setminus B_j}|w^j|dx+\sigma\eta_j+C\delta_j.$$

Dividing by σ and letting $\sigma\to\infty$, we deduce:

$$|A\setminus B_j|\leq\eta_j,$$

so that $|A|=0$. It follows that $\tanh^{-1}u^{e,1}$ is finite a.e.

To sum up, we have shown that for any e with $|e|=1$, there is a measurable function u^e such that $t_\sigma(e\cdot(u_k-v_k))\to t_\sigma(u^e)$ in $L^1(\Omega)$ for any $\sigma>0$. It is then obvious to check that $u^e=e\cdot u$ for some measurable vector-valued function u, and, up to a subsequence, to deduce that $u_k-v_k\to u$ a.e. in Ω.

3.3. **Lower semicontinuity.** We argue similarly to what done in [9, Step 2 in Section 3] to prove the $GSBD^p$ analogue of (1.4b), that is [9, equation (1.5d)].

Let us fix $\sigma>1$, $\xi\in\mathbb{S}^{d-1}$ in a set of full $\mathcal{H}^{d-1}$-measure of $\mathbb{S}^{d-1}$ for which (2.10) holds (*cf.* Lemma 2.2), and define

$$\mathrm{I}_y^{\sigma,\xi}(u_k):=|\mathrm{D}(\widehat{u}_k)_y^\xi|\big(\Omega_y^\xi\setminus J^\sigma_{(\widehat{u}_k)_y^\xi}\big)\,. \tag{3.6}$$

Since

$$\mathrm{I}_y^{\sigma,\xi}(u_k)\leq(\widehat{\mu}_{u_k})_y^\xi(\Omega_y^\xi)+(\sigma-1)\mathcal{H}^0\big(\Omega_y^\xi\cap(J^1_{(\widehat{u}_k)_y^\xi}\setminus J^\sigma_{(\widehat{u}_k)_y^\xi})\big)\leq\sigma(\widehat{\mu}_{u_k})_y^\xi(\Omega_y^\xi),$$

it holds that

$$\int_{\Pi^\xi}\mathrm{I}_y^{\sigma,\xi}(u_k)\,\mathrm{d}\mathcal{H}^{d-1}(y)\leq\sigma\widehat{\mu}^\xi_{u_k}(\Omega)\leq\sigma\sup_{k\in\mathbb{N}}\widehat{\mu}_{u_k}(\Omega). \tag{3.7}$$

Following exactly [9, Step 2 in Section 3] for $\sigma>1$ fixed (using (3.7) in place of [9, estimate (3.11)]), we get that all the [9, (3.12)-(3.18)] hold for J_v replaced by J_v^σ and $\widehat{I}_y^\xi$ replaced by $\mathrm{I}_y^{\sigma,\xi}$. In particular, for $\mathcal{H}^{d-1}$-a.e. $\xi\in\mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y\in\Pi_\xi$, along a suitable subsequence $(\cdot)_j$ depending on σ, ξ, $\varepsilon\in(0,\sigma^{-2})$ fixed as in [9, (3.13)], and y, it holds that

$$(\widehat{u}_j-\widehat{a}_j)_y^\xi\to\widehat{u}_y^\xi\quad\mathcal{L}^1\text{-a.e. in }\Omega_y^\xi, \tag{3.8}$$

$$|(\widehat{a}_j^{i_1}-\widehat{a}_j^{i_2})_y^\xi(t)|=|(\widehat{a}_j^{i_1}-\widehat{a}_j^{i_2})_y^\xi(0)|\to+\infty\quad\text{for }t\in\Omega_y^\xi\text{ and }i_1\neq i_2, \tag{3.9}$$

$$\lim_{j\to\infty}\Big(\mathcal{H}^0\big(J^\sigma_{(\widehat{u}_j)_y^\xi}\big)+\varepsilon\mathrm{I}_y^{\sigma,\xi}(u_j)\Big)=\liminf_{m\to\infty}\Big(\mathcal{H}^0\big(J^\sigma_{(\widehat{u}_m)_y^\xi}\big)+\varepsilon\mathrm{I}_y^{\sigma,\xi}(u_m)\Big)=M(y)\in\mathbb{R} \tag{3.10}$$

for $(\cdot)_m$ a subsequence of $(\cdot)_k$ independent of y such that

$$H^\sigma_{\varepsilon,\xi}:=\lim_{m\to\infty}\int_{\Pi_\xi}\Big(\mathcal{H}^0\big(J^\sigma_{(\widehat{u}_m)_y^\xi}\big)+\varepsilon\mathrm{I}_y^{\sigma,\xi}(u_m)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y)\in\mathbb{R} \tag{3.11}$$

and

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{S}^{d-1}} H^\sigma_{\varepsilon,\xi}\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\le \liminf_{k\to\infty}\mathcal{H}^{d-1}(J^\sigma_{u_k}). \tag{3.12}$$

Denoting

$$\partial\mathcal{P}^\xi_y:=\bigcup_{n\in\mathbb{N}}\partial\left((P_n)^\xi_y\right)\cap\Omega^\xi_y\subset\Omega^\xi_y,$$

by (3.10) we may assume, up to a further subsequence, that $\mathcal{H}^0\big(J^\sigma_{(\widehat{u}_j)^\xi_y}\big)=N_y\in\mathbb{N}$ for every j and so that there are $\widehat{N}_y\le N_y$ cluster points in the limit, denoted by

$$t_1,\dots,t_{\widehat{N}_y}.$$

Therefore we have that $K\cap J^\sigma_{(\widehat{u}_j)^\xi_y}=\emptyset$ for any K compact subset of (t_l,t_{l+1}), so $|\mathrm{D}(\widehat{u}_k)^\xi_y|\big(K\setminus J^\sigma_{(\widehat{u}_j)^\xi_y}\big)=|\mathrm{D}(\widehat{u}_j)^\xi_y|(K)$; with (3.10) and the Fundamental Theorem of Calculus, this gives that, for $\mathcal{L}^1$-almost any choice of $\overline{t}\in(t_l,t_{l+1})$,

$$t\mapsto(\widehat{u}_j)^\xi_y(t)-(\widehat{u}_j)^\xi_y(\overline{t})\text{ are equibounded w.r.t. } j \text{ in } BV_{\mathrm{loc}}(t_l,t_{l+1}), \tag{3.13}$$

so the bound above is also in $L^\infty_{\mathrm{loc}}(t_l,t_{l+1})$. At this stage we prove, as in [9, (3.20)], that

$$\partial\mathcal{P}^\xi_y\subset\{t_1,\dots,t_{\widehat{N}_y}\}: \tag{3.14}$$

in fact, assuming by contradiction that there exists $l\in\{1,\dots,M_y\}$ and $i_1\neq i_2$ such that $\partial(P_{i_1})^\xi_y\cap(t_l,t_{l+1}),\partial(P_{i_2})^\xi_y\cap(t_l,t_{l+1})\neq\emptyset$, by (3.8) there are two corresponding sequences of infinitesimal rigid motions $(a^{i_1}_j)_j$, $(a^{i_2}_j)_j$ for which

$$\begin{aligned}&(\widehat{u}_j-\widehat{a}^{i_1}_j)^\xi_y\to\widehat{u}^\xi_y\quad\mathcal{L}^1\text{-a.e. in }(P_{i_1})^\xi_y\cap(t_l,t_{l+1}),\\&(\widehat{u}_j-\widehat{a}^{i_2}_j)^\xi_y\to\widehat{u}^\xi_y\quad\mathcal{L}^1\text{-a.e. in }(P_{i_2})^\xi_y\cap(t_l,t_{l+1}),\end{aligned} \tag{3.15}$$

with $\mathcal{L}^1\big((P_{i_1})^\xi_y\cap(t_l,t_{l+1})\big)$, $\mathcal{L}^1\big((P_{i_2})^\xi_y\cap(t_l,t_{l+1})\big)>0$; this gives (with (3.13) and since $\widehat{a}^i_j$ are infinitesimal rigid motions and $\widehat{u}^\xi_y\colon\Omega^\xi_y\to\mathbb{R}$) that $(\widehat{a}^{i_1}_j-\widehat{a}^{i_2}_j)^\xi_y$ is constant in Ω^ξ_y and uniformly bounded w.r.t. j, in contradiction with (3.9). Therefore, (3.14) is proven.

Integrating (3.14) over $y\in\Pi^\xi$ and using Fatou's lemma with (3.10), (3.11) we deduce that

$$\int_{\Pi^\xi}\mathcal{H}^0(\partial\mathcal{P}^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y)\le\lim_{m\to\infty}\int_{\Pi^\xi}\Big(\mathcal{H}^0\big(J^\sigma_{(\widehat{u}_m)^\xi_y}\big)+\varepsilon\,\mathrm{I}^{\sigma,\xi}_y(u_m)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \tag{3.16}$$

for every $\sigma>1$ and $\mathcal{H}^{d-1}$-a.e. $\xi\in\mathbb{S}^{d-1}$ (in view of the choice of the subsequences, see [9]). This implies that $\mathcal{P}$ is a Caccioppoli partition.

Moreover, integrating over $\xi\in\mathbb{S}^{d-1}$ we get that

$$\begin{aligned}\mathcal{H}^{d-1}(\partial^*\mathcal{P}\cap\Omega)&\le C\varepsilon\sigma\sup_{k\in\mathbb{N}}\widehat{\mu}_{u_k}(\Omega)+\int_{\mathbb{S}^{d-1}}H^\sigma_{\varepsilon,\xi}\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\\&\le C\sqrt{\varepsilon}\sup_{k\in\mathbb{N}}\widehat{\mu}_{u_k}(\Omega)+\int_{\mathbb{S}^{d-1}}H^\sigma_{\varepsilon,\xi}\,\mathrm{d}\mathcal{H}^{d-1}(\xi)\end{aligned} \tag{3.17}$$

for a universal constant $C>0$ and every $\sigma>1$, $\varepsilon\in(0,\sigma^{-2})$. Letting $\varepsilon\to0$, in view of (3.12) and the arbitrariness of $\sigma>1$ we conclude (1.4b).

Let us now confirm that $u\in GBD(\Omega)$. For any $\xi\in\mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y\in\Pi^\xi$, setting $\tilde{u}_k:=u_k-a_k$ for any $k\in\mathbb{N}$, it holds that

$$(\widehat{\mu}_{\tilde{u}_k})^\xi_y(B)\le(\widehat{\mu}_{u_k})^\xi_y(B)+\mathcal{H}^0(\partial^*\mathcal{P}^\xi_y\cap B)\quad\text{for every }B\subset\Omega^\xi_y\text{ Borel,}$$

since a_k is a piecewise rigid motion constant on every P_j, $j \in \mathbb{N}$, where $\mathcal{P} = (P_j)_j$. Integrating over Π_ξ and recalling (2.5) we have that

$$\widehat{\mu}_{\tilde{u}_k}(B) \leq \widehat{\mu}_{u_k}(B) + \mathcal{H}^{d-1}(\partial^* \mathcal{P} \cap B) \quad \text{for every } B \subset \Omega \text{ Borel.}$$

Since $\tilde{u}_k$ pointwise converges $\mathcal{L}^d$-a.e. to u, there is an increasing function $\psi_0 \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{s \to +\infty} \psi_0(s) = +\infty$ such that $\|\psi_0(\tilde{u}_k)\|_{L^1(\Omega)}$ is uniformly bounded w.r.t. $k \in \mathbb{N}$ (see e.g. [16, Lemma 2.1]). Then we may apply [11, Corollary 11.2] to deduce that $u \in GBD(\Omega)$.

This concludes the proof of Theorem 1.1.

Remark 3.1. Let us consider a sequence $(u_k)_k$ such that

$$\int_\Omega |e(u_k)|^p \,\mathrm{d}x + \mathcal{H}^{d-1}(J_{u_k}) \leq M, \quad p > 1 \tag{3.18}$$

for $M > 0$ independent of $k \in \mathbb{N}$. Applying Theorem 1.1 we obtain the compactness part of [9, Theorem 1.1] (that is [9, (1.5b)] for $u \in GBD(\Omega)$) without using the Korn-type inequality in [4]. Combining this with the last part of the proof ([9, Steps 2-3 in proof of Theorem 1.1]) we obtain [9, Theorem 1.1], and in particular (1.4c). Nevertheless, the result in [4] is crucial for [9, Theorem 1.2].

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