Compendium for: Tightness of Random Walks in Infinite Dimensional Spaces and Manifolds

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Abstract

This compendium contains results used in the paper *Tightness of Random Walks in Infinite Dimensional Spaces and Manifolds* [7], that are here collected for convenience of the reader.

Contents

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1 Dposets,Nets

Definition 1.1 (dposet,net)**.** *A set $\mathfrak{T}$ with a reflexive and transitive binary relation $\leq$ is called* directed *[6] if*

$$\forall \tau_1, \tau_2 \in \mathfrak{T} \ , \ \exists \tau_3 \in \mathfrak{T} \ , \tau_1 \leq \tau_3, \tau_2 \leq \tau_3 \quad .$$

We will suppose that $\mathfrak{T}$ also enjoys the antisymmetric properties, so that it is a partially ordered set; see Remark 2B3 from [5], and references therein.

Suppose that $(\mathfrak{T}, \leq)$ is a partially ordered set and is a directed set, then the following are equivalent [1]*:*

- *$(\mathfrak{T}, \leq)$ has no maxima;*
- *$(\mathfrak{T}, \leq)$ has no maximals;*
- $$\forall \tau_1, \tau_2 \in \mathfrak{T} \ , \ \exists \tau_3 \in \mathfrak{T} \ , \tau_1 < \tau_3, \tau_2 < \tau_3 \quad .$$

A partially ordered directed set with no maxima will be abbreviated to ***dposet*** *in the following.*

Functions $f : \mathfrak{T} \to S$ whose domain is a dposet will be called ***nets****.*

A subset of $\mathfrak{T}$ that is totally ordered by $\leq$ is called a **chain**. Nets are a generalization of sequences (since $\mathbb{N}$ is a dposet); the concept of *subsequence* is replaced by the concept of **subnet** $f_{|\widetilde{\mathfrak{T}}}$ where $\widetilde{\mathfrak{T}} \subseteq \mathfrak{T}$ is **cofinal**:

$$\forall \tau \in \mathfrak{T}, \ \exists \tilde{\tau} \in \widetilde{\mathfrak{T}} \text{ such that } \tau \leq \tilde{\tau} \quad . \tag{1.1}$$

Most definitions and results that are valid for *sequences* can be reformulated for *nets*. Let (S, τ_S) be a Hausdorff topological space.

- Let $f : \mathfrak{T} \to S$; we define that
$$\lim_{\tau \in \mathfrak{T}} f(\tau) = x \in S$$
if for all $A \in \tau_S$ with $x \in A$ there exists $\hat{\tau}$ such that
$$\forall \tau \geq \hat{\tau} \quad , \quad f(\tau) \in A \quad .$$
- $C \subseteq S$ is closed iff for any net $f : \mathfrak{T} \to C$ converging to
$$\lim_{\tau \in \mathfrak{T}} f(\tau) = x \in S$$
we have $x \in C$.
- For $f : \mathfrak{T} \to \mathbb{R}$ we define
$$\limsup_{\tau \in \mathfrak{T}} f(\tau) \stackrel{\text{def}}{=} \inf_{\hat{\tau} \in \mathfrak{T}} \sup_{\tau \geq \hat{\tau}} f(\tau)$$
and symmetrically for lim inf.

[1] See 06V from [5].

Remark 1.2. *All of the above can be formulated for directed sets that have a maximum $\tilde{\tau}$, but then it is quite trivial:* $\lim_{\tau\in\mathfrak{T}} f(\tau) = f(\tilde{\tau})$ *and so on.*

Remark 1.3. In [6] and other texts a *net* is a function $f : \mathfrak{T} \to S$ whose domain is a directed set. Since we will always assume that the topological space S is Hausdorff, then all results in [6] that we will need are equally valid for this definition of *net*. Indeed the family of neighbourhoods of a point $x_0 \in S$ is a *dposet* when ordered $U \le V \iff U \supseteq V$.

Some results are actually more intuitive with nets. The following theorem[2] is of fundamental importance in topology (and in particular in connection with Prokhorov's Theorem, in the form presented in 4.13 later on).

Theorem 1.4. *Let S be a Hausdorff topological space, $K \subseteq S$; the following are equivalent.*

- *K is pre-compact[3];*
- *for any dposet $\mathfrak{T}$ and any net $f : \mathfrak{T} \to S$ there is a converging subnet.*

2 Continuous functions

Definition 2.1. *For S a Hausdorff topological space and $I \subseteq \mathbb{R}$ an interval, let $C(I;S)$ be the set of continuous functions $x : I \to S$. In the first part of the paper [7] we have $S = H$, the separable Hilbert Space H, so:*

- *if I is not compact then $C(I;S)$ is a Frechét space where the topology[4] is defined by the seminorms*
$$[f]_{I_k,\infty} \quad where \quad [f]_{I,\infty} = \sup_{t\in I} |f(t)|_H$$
and I_k are compact, $I_k \subset I_{k+1}$, $\bigcup_k I_k = I$;
- *whereas if I is compact then $C(I;S) = C_b(I;S)$ is the usual Banach space with norm*
$$\|f\|_\infty = \sup_{t\in I} |f(t)|_H \quad .$$

Note that in any case $C(I;S)$ is separable. When instead in the second part of the paper [7] we have $S = M$, a closed subset of H, then $C(I;M) \subseteq C(I;H)$ so $C(I;M)$ is nonetheless a complete separable metric space.

Remark 2.2. Consider the restriction map
$$r_T : C(\mathbb{R}^+;H) \to C([0,T];H) \tag{2.1}$$
given by $r_T f = f_{|[0,T]}$; then the topology on $C(\mathbb{R}^+;H)$ is the initial topology with respect to the maps r_n and the Banach spaces $C([0,n];H)$.

Hence the following result can be applied, by setting $W = C(\mathbb{R}^+;H)$, $W_n = C([0,n];H)$.

[2] Derived from Chapter 2 in [6].
[3] This means that the closure of K is compact
[4] The topology does not depend on the choice of the sequence I_n.

Theorem 2.3. *Let W be a set and $f : W \to W_n$ be separating functions where W_n are Hausdorff topological spaces; endow W with the initial topology. A set $K \subset W$ is compact if and only if*

- *for each $n \in \mathbb{N}$, $r_n(K)$ is compact in W_n,*
- *the image $P(K)$ of K under the product map*

$$P : W \to \prod_n W_n \quad , \quad x \mapsto (r_n(x))_n$$

is closed.

3 Measures

In the following S will be a Hausdorff topological space.

Definition 3.1. Let $\mu : \mathcal{F} \to \mathbb{R}$ be a finitely-additive function defined on an algebra $\mathcal{F}$ containing the open sets.

μ is **regular** [5] if for each $E \in \mathcal{F}$ and $\varepsilon > 0$, there exist $F, G \in \mathcal{F}$ with $G \subseteq E \subseteq F$, F closed and G open and such that $|\mu|(F \setminus G) < \varepsilon$ where $|\mu|$ is the total variation [6] of μ.

Definition 3.2. [7] Let $\mathcal{F}$ be the field generated by open sets; we call `rba` the vector space of all $\mu : \mathcal{F} \to \mathbb{R}$ regular bounded finitely-additive functions.

Theorem 3.3. [8] *Suppose that S is normal, then there is a linear isomorphism between $J \in C_b(S)^*$ and $\mu \in$* `rba` *such that*

$$\forall f \in C_b(S) \quad , \quad J(f) = \int_S f \, \mathrm{d}\mu \quad ;$$

and this isomorphism preserves order.

Unfortunately there are different definitions of *Radon measure*. We use the definition from [8, 2, 1]

Definition 3.4. A **Radon measure** in S is a finite non negative measure μ on the Borel sets $\mathcal{B}(S)$ of S such that for each $B \in \mathcal{B}(S)$ there exists a a compact set $B \subseteq A$ such that $\mu(B \setminus K) < \varepsilon$.

We recall that a *Polish space* is a space homeomorphic to a separable complete metric space. If S is a Polish space, then each Borel finite measure is Radon (Theorem 3.1 in [8], or Theorem 7.1.7 in [1]).

4 Probability Theory

In this section, for convenience of the reader, we recall some definitions and results in Probability Theory from the literature.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, S a Hausdorff topological space.

[5] Definition 11 in Chapter III Section 5 in [4]
[6] Definition 4 in Chapter III Section 1 in [4]
[7] Definition 1 in Chapter IV Section 6 in [4]
[8] Theorem 2 in Chapter IV Section 6 in [4]

A measurable function $\mathfrak{X} : \mathbb{R}^+ \times \Omega \to S$ is called a *process*; for any fixed $\omega \in \Omega$ the function

$$t \mapsto \mathfrak{X}(t,\omega) = \mathfrak{X}(\omega)_t$$

is called a *path* or a *trajectory*; if each path is continuous then $\mathfrak{X} : \Omega \to C(\mathbb{R}^+; S)$ hence the name *random function*. (Measurability issues are explained in [2], see in particular Remark 3.1.3).

We recall the Kolmogoroff test (Theorem 3.3 in [3]).

Theorem 4.1. *Let $I = [0,T]$ $Z = Z_t, t \in I$ be a process taking values in a complete metric space (M,ρ) such that $\exists C > 0, \delta > 0, \varepsilon > 0$*

$$\forall t,s \in I\ ,\ \mathbb{E}[\rho(Z(t),Z(s))^\delta] \le C|t-s|^{1+\varepsilon}$$

then it has a version with paths Hölder continuous with an arbitrary exponent smaller than ε/δ.

4.1 Narrow Convergence

In the following S will be a Hausdorff topological space.

In all of the following $\alpha \in A$, a dposet.

Definition 4.2 (Narrow Convergence)**.** Given a net of Borel measures μ_α, μ on S, we will say that $\lim_{\alpha\in A} \mu_\alpha = \mu$ **narrowly** if

$$\forall f \in C_b(S)\ ,\quad \lim_\alpha \int_S f(x)\,\mathrm{d}\mu_\alpha(x) = \int_S f(x)\,\mathrm{d}\mu(x) \quad .$$

The same definition can be stated when $\mu_\alpha, \mu \in$`rba`.

Definition 4.3. If Z_α, Z are random variables taking values in S we will say that $\lim_{\alpha\in A} Z_\alpha = Z$ *narrowly* when $\lim_{\alpha\in A} \mu_\alpha = \mu$ *narrowly* where $\mu_\alpha = Z_{\alpha\sharp}\mathbb{P}, \mu = Z_\sharp\mathbb{P}$, *i.e.* if

$$\forall f \in C_b(S)\ ,\quad \lim_{\alpha\in A} \mathbb{E}[f(Z_\alpha)] = \mathbb{E}[f(Z)] \quad .$$

Remark 4.4. In some texts ([2], [8]...) this convergence is called *weak convergence*, but this may create confusion when S is a Hilbert space, where *weak convergence* usually means: convergence of a sequence $(x_n)_n \subset H$ to $x \in H$ in the duality with continuous linear functions:

$$\forall v \in H\ ,\ \lim_{n\in\mathbb{N}} \langle x_n, v\rangle_H = \langle x, v\rangle_H \quad .$$

(There is though an important connection, see Corollary 3.8.5 in [2]). In other texts it is called *distributional convergence*, but this may cause confusion with the *Schwartz distributions*. The term *narrow* seems to have originated in Bourbaki's texts.

We recall this fact from Probability Theory.

Theorem 4.5 (Alexandrov Theorem)**.** *Suppose that S is a Polish space. Let μ_α, μ be probability measures on S; then these are equivalent*

- *narrow convergence of μ_α to μ;*
- $$\limsup_{\alpha\in A} \mu_\alpha(F) \le \mu(F)$$

  for all closed sets $F \subseteq S$;

•

$$\liminf_{\alpha\in A} \mu_\alpha(A) \ge \mu(A)$$

for all open sets $F \subseteq S$.

Proof. By Prop. 3.1 in [8], then μ_α, μ are τ-smooth; so we can apply Alexandrov's Theorem in the form in Theorem 3.5 in [8]. □

This can be applied to nets of r.v. $Z_\alpha, Z : \Omega \to S$, as explained in Definition 4.3.

Some implications in the above Theorem hold also in a more general context (as can be seen by reading the proof of Theorem 3.5 in [8]); as in this proposition.

Proposition 4.6. *Suppose that S is normal; suppose that $\mu_\alpha \to \mu$* narrowly, *where μ_α, μ are in $\mathscr{rba}$; then*

•

$$\limsup_{\alpha\in A} \mu_\alpha(F) \le \mu(F)$$

for all closed sets $F \subseteq S$;

•

$$\liminf_{\alpha\in A} \mu_\alpha(A) \ge \mu(A)$$

for all open sets $F \subseteq S$.

Proof. Since S is normal then Urysohn's Lemma holds in S. Given $C \subseteq A \subseteq S$ where C is closed and A open, there exists a continuous function $f : S \to [0,1]$ such that

$$\mathbb{1}_C \le f \le \mathbb{1}_A \tag{4.1}$$

so

$$\limsup_\alpha \mu_\alpha(C) \le \int f \, \mathrm{d}\mu \le \limsup_\alpha \mu_\alpha(A) \tag{4.2}$$

and then in particular

$$\limsup_\alpha \mu_\alpha(C) \le \mu(A)$$
$$\mu(C) \le \limsup_\alpha \mu_\alpha(A)$$

using the fact that μ is regular then we conclude.

□

(Note that this is a fundamental step in the proof the Riesz–Markov representation theorem [9]).

4.1.1 Properties

Lemma 4.7. *If $\mu_\alpha \to \mu$ narrowly, if $f : S \to \mathbb{R}$ is continuous and μ-integrable and*

$$\lim_{R\to\infty} \sup_\alpha \int_{|f|\ge R} |f| \, \mathrm{d}\mu_\alpha = 0 \tag{4.3}$$

then

$$\lim_\alpha \int_S f \, \mathrm{d}\mu_\alpha(x) = \int_S f \, \mathrm{d}\mu(x) \quad .$$

(The proof is the same as Lemma 3.8.7 from [2], where though it is stated for sequences and not nets).

We assume that S is a Polish space; so as consequence of Alexandrov's theorem [9] we state.

Lemma 4.8. *If $\mu_\alpha \to \mu$ narrowly, if $f : S \to \mathbb{R}^+$ is lower semi continuous and non negative, and*

$$\int_S f \, \mathrm{d}\mu = L \in [0, \infty]$$

then

$$\liminf_\alpha \int_S f \, \mathrm{d}\mu_n \geq L \quad .$$

Corollary 4.9. *If $\mu_\alpha \to \mu$ narrowly, if $f : S \to \mathbb{R}$ is continuous and if there is $\varepsilon > 0$ such that*

$$s = \sup_\alpha \int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu_\alpha < \infty$$

then

$$\int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu \leq s$$

and

$$\lim_\alpha \int_S f(x) \, \mathrm{d}\mu_\alpha(x) = \int_S f(x) \, \mathrm{d}\mu(x) \quad .$$

Proof. We check that (4.3) is satisfied. Setting

$$\nu_\alpha(A) = \int_A |f| \, \mathrm{d}\mu_\alpha$$

then

$$\int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu_\alpha = \int_{\mathbb{R}} |t|^{1+\varepsilon} \, \mathrm{d}\nu_\alpha$$

and

$$\int_{|f| \geq R} |f| \, \mathrm{d}\mu_\alpha = \nu_\alpha\{|t| \geq R\}$$

so by Markov inequality

$$\int_{|f| \geq R} |f| \, \mathrm{d}\mu_\alpha \leq \frac{\int_{\mathbb{R}} |t|^{1+\varepsilon} \, \mathrm{d}\nu_\alpha}{R} \leq \frac{s}{R} \quad .$$

We have that $f_\sharp \mu_\alpha \to f_\sharp \mu$ narrowly, so by the hypothesis and the previous Lemma

$$s \geq \liminf_{n \to \infty} \int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu_\alpha = \liminf_{n \to \infty} \int_{\mathbb{R}} |t|^{1+\varepsilon} \, \mathrm{d}f_\sharp \mu_\alpha \geq \int_{\mathbb{R}} |t|^{1+\varepsilon} \, \mathrm{d}f_\sharp \mu = \int_S |f|^{1+\varepsilon} \, \mathrm{d}\mu$$

in particular f is μ-integrable. So we can apply the previous Lemma 4.7 □

[9] See note at Theorem 3.5 in [8].

Theorem 4.10. [10] *Let $p_1, p_2 \in [1, \infty), p_1 < p_2$. Suppose that $Z_\alpha, Z : \Omega \to H$ are random variables taking values in a Hilbert separable space H, such that $Z_\alpha \to Z$ narrowly and that*

$$\sup_\alpha \mathbb{E}[\|Z_\alpha\|_H^{p_2}] < \infty \quad ;$$

then

$$\lim_\alpha \mathbb{E}[\|Z_\alpha\|_H^{p_1}] = \mathbb{E}[\|Z\|_H^{p_1}] \quad .$$

Proof. Let $\varepsilon = p_2 - p_1$; set $Y_\alpha = \|Z_\alpha\|_H, Y = \|Z\|_H$, then $\mu_\alpha = Y_{\alpha\sharp}\mathbb{P}, \mu = Y_\sharp \mathbb{P}$ and

$$f : \mathbb{R} \to \mathbb{R}, f(t) = |t|^{p_1}$$

so

$$\mathbb{E}[\|Z_\alpha\|_H^{p_2}] = \int_{\mathbb{R}} f(t)^{1+\varepsilon}\, d\mu_\alpha \quad , \mathbb{E}[\|Z_\alpha\|_H^{p_1}] = \int_{\mathbb{R}} f(t)\, d\mu_\alpha \quad ,$$

and apply the previous results. □

Theorem 4.11. *Let again*

$$r_T : C(\mathbb{R}^+; H) \to C([0, T]; H) \qquad \text{(as in (2.1))}$$

be the restriction map. Let $W = C(\mathbb{R}^+; H), W_n = C([0, n]; H)$ for simplicity. Suppose μ_α, μ are Radon probability measures on W, these are equivalent:

- $\lim_{\alpha \in A} \mu_\alpha = \mu$ *narrowly in W,*
- *for each $n \in \mathbb{N}$,* $\lim_{\alpha \in A} r_{n\sharp}\mu_\alpha = r_{n\sharp}\mu$ *narrowly in W_n .*

Proof. One implication is trivial. We prove that the second clause implies the first. The balls (for $\varepsilon > 0, f \in W_n$)

$$B^{W_n}(f, \varepsilon) = \{g :\in W_n : [f - g]_{[0,n],\infty} < \varepsilon\}$$

are a base for the topology in W^n; since each f, g can be extended constantly, we can say that

$$\{g \in W : [f - g]_{[0,n],\infty} < \varepsilon\} \quad \text{for} \quad \varepsilon > 0, f \in W, n \in \mathbb{N}$$

are a base for the topology in W. Since W is separable, let f_k a countable dense sequence, then

$$B(k, m, n) \stackrel{\text{def}}{=} \{g \in W : [f_k - g]_{[0,n],\infty} < 1/m\} \quad \text{for } n, m, k \in \mathbb{N}$$

is a countable base.

For each $A \subseteq W$ open there are sequences k_j, m_j, n_j

$$A = \bigcup_{j=1}^{\infty} B(k_j, m_j, n_j)$$

let

$$A_l = \bigcup_{j=1}^{l} B(k_j, m_j, n_j)$$

[10] This seems a standard result, but we could not find a reference for it.

then there are $B_l \in W_{N_l}$ open such that

$$A_l = r_{N_l}^{-1}(B_l)$$

with $N_l = \max_{j \le l} n_j$. We eventually write

$$\liminf_\alpha \mu_\alpha(A) \ge \liminf_\alpha \mu_\alpha(A_l) = \liminf_\alpha r_{N_l \sharp}\mu_\alpha(B_l) \ge r_{N_l \sharp}\mu(B_l) = \mu(A_l)$$

and then pass to the limit on RHS. We conclude by Alexandrov's Theorem 4.5. □

4.2 Tightness

Let S be a Hausdorff topological space.

Definition 4.12. Let $\mathcal{M}$ be a family of Radon measures. It is called[11] **tight** if for every $\varepsilon > 0$ there is a compact set $K \subseteq S$ such that

$$\forall \mu \in \mathcal{M} \ , \ \mu(S \setminus K) < \varepsilon \quad .$$

Note that any finite family is *tight*, by 3.4; and if some families $\mathcal{M}_1, \dots \mathcal{M}_v$ are *tight* then $\bigcup_{j=1}^{v} \mathcal{M}_j$ is tight.

We endow the family $\mathcal{R}(S)$ of all Radon probabilities on S with the weak topology induced by the duality with $C_b(S; \mathbb{R})$; for coherence with the above, we call this topology *narrow topology*.

In this case the following version of Prokhorov's Theorem holds (here expressed in the form of Theorem 3.6 in [8]).

Theorem 4.13 (Prokhorov's Theorem)**.** *Let $\mathcal{M} \subseteq \mathcal{R}(S)$.*

1. *If S is a completely regular Hausdorff topological space and $\mathcal{M}$ is tight then it is pre-compact in the* narrow topology.

2. *If S is a* Polish space *and $\mathcal{M}$ is pre-compact in the* narrow topology *then $\mathcal{M}$ is tight.*

We agree on this (non standard) definition.

Definition 4.14. Let $\mathfrak{T}$ be a dposet.

Let μ_τ be a net of Radon measures on S: it is **tight** if $\forall \varepsilon > 0$ there is a compact set $C \subseteq S$ such that

$$\limsup_{\tau \in \mathfrak{T}} \mu_\tau(S \setminus C) \le \varepsilon \quad .$$

Let $\mathfrak{X}^\tau$ a net of r.v. taking values in S, for $\tau \in \mathfrak{T}$: it is **tight** if the net of laws $\mu_\tau = \mathfrak{X}^\tau_\sharp \mathbb{P}$ is tight, namely $\forall \varepsilon > 0$ there is a compact set $C \subseteq S$ such that

$$\limsup_{\tau \in \mathfrak{T}} \mathbb{P}(\mathfrak{X}^\tau \notin C) \le \varepsilon \quad .$$

Definition 4.15. For $f : \mathfrak{T} \to \mathcal{R}(S)$ we define the **narrow limit points** $L \subseteq S$ by

$$L = \bigcap_{\hat\tau \in \mathfrak{T}} \overline{\{f(\tau) : \hat\tau \le \tau\}} \quad . \tag{4.4}$$

where "closure" is in the narrow topology of $\mathcal{R}(S)$.

[11] In Definition 3.8.3 from [2] it is called *uniformly tight*.

Theorem 4.16. *Let S be a Polish space, let $\mathfrak{T}$ be a dposet, let $\mu_\alpha, \alpha \in \mathfrak{T}$ be a tight net of Radon probabilities on S: then it has a narrowly converging subnet; in particular the set of narrow limit points is not empty.*

Proof. A possible proof can be obtained by adapting the proof of Theorem 3.6 in [8]; we present a slightly different proof.

If μ is a finite signed Borel measure let $J_\mu \in C_b(S)^*$ be given

$$J_\mu : C_b(S) \to \mathbb{R} \quad , \quad J_\mu(f) = \int_S f \, \mathrm{d}\mu_\mu$$

and recall that such functionals are bounded

$$|J_\mu(f)| \le \|\mu\| \|f\|_\infty$$

where $\|\mu\|$ is the total variation norm.

By Banach-Alaoglu Theorem and Theorem 1.4, the net J_{μ_α} admits a subnet $\tilde{\mathfrak{T}}$ weakly converging to $J \in C_b(S)^*$; obviously J is positive.

By Theorem 3.3 there exists $\nu : F \to \mathbb{R}$ in $\mathtt{rba}$, where F is the algebra generated by open sets, such that J can be represented as J_ν; moreover since J is positive then ν is positive; and obviously $J(1) = 1 = \nu(S)$. By the generalization 4.6 of Alexandroff Theorem, for any open set A,

$$\nu(A) \le \liminf_{\beta \in \tilde{\mathfrak{T}}} \nu_\beta(A)$$

and by hypothesis $\forall n \ge 1$ there is a compact set $C_n \subseteq S$ such that

$$\limsup_{\beta \in \tilde{\mathfrak{T}}} \nu_\beta(S \setminus C_n) \le 1/n \quad ,$$

so

$$\nu(S \setminus C_n) \le 1/n$$

hence by Theorem 3.2 in [8] there is an unique extension of ν to a Radon measure; and again by Alexandroff Theorem we have that ν is the narrow limit of the subsequence. (See Corollary 3 of Theorem 3.5 in [8])

□

Corollary 4.17. *If S is a Polish space and if $\mathfrak{X}^\tau$ a tight net of r.v. taking values in S, for $\tau \in \mathfrak{T}$; then the net of laws $\mathfrak{X}^\tau_\sharp \mathbb{P}$ admits narrowly converging subnets.*

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