CHARACTERIZATION OF FRACTIONAL SOBOLEV–POINCARÉ AND (LOCALIZED) HARDY INEQUALITIES

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Abstract. In this paper, we prove capacitary versions of the fractional Sobolev–Poincaré inequalities. We characterize localized variant of the boundary fractional Sobolev–Poincaré inequalities through uniform fatness condition of the domain in $\mathbb{R}^n$. Existence type results on the fractional Hardy inequality in the supercritical case $sp > n$ for $s \in (0,1)$, $p > 1$ are established.

1. Introduction and main results

The central aim of this paper is to study the Sobolev–Poincaré inequality, pointwise Hardy inequality and the Hardy inequality under some assumptions on the domain in the case of the fractional Sobolev spaces. Precise condition on the domain will be clarified later. It is well known that the classical Sobolev–Poincaré inequality states that for a bounded domain $\Omega \subset \mathbb{R}^n$ with C^1 boundary and $1 \le p < n$, there exists a constant $C = C(n,p) > 0$ such that

$$\left(\int_\Omega |u(x) - u_\Omega|^{p^*} dx\right)^{1/p^*} \le C \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p} \text{ for all } u \in W^{1,p}(\Omega), \tag{1.1}$$

where $p^* = \frac{np}{n-p}$ denotes the Sobolev critical exponent and the space $W^{1,p}(\Omega)$ is the usual classical Sobolev space, see for example [16, Chapter 4] in the case of ball. A capacitary variant of the Sobolev–Poincaré inequality eq. (1.1) were considered in [22] and for weighted case, see [30]. The well known classical (boundary) Hardy inequality states that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $1 \le p < \infty$, there exists a constant $C = C(n,p,\Omega) > 0$ such that for any $u \in C_c^\infty(\Omega)$

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^p} dx \le C \int_\Omega |\nabla u(x)|^p dx, \tag{1.2}$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$. The existence of the Hardy inequality eq. (1.2) for every open set $\Omega \subset \mathbb{R}^n$ when $p > n$ has been investigated independently by [27] and [35]. Also observe that both references deal with the case $p \le n$, as well, where the validity of eq. (1.2) has been established through the uniform fatness condition of the complement Ω^c. One can obtain the classical Hardy inequality eq. (1.2) by applying appropriately the Hardy–Littlewood–Wiener maximal function theorem on a pointwise Hardy inequality, see [19, 23, 24] where they have introduced pointwise Hardy inequality

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through a maximal operator. Necessary and sufficient conditions are provided for pointwise Hardy inequalities in [25] and see [26] for weighted case.

Let $\Omega \subseteq \mathbb{R}^n$ be any open set, and let $0 < s < 1$, $1 \leq p < \infty$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty\},$$

endowed with the so-called fractional Sobolev norm, given by

$$||u||_{s,p,\Omega} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p,\Omega}\right)^{\frac{1}{p}},$$

where

$$[u]^p_{s,p,\Omega} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy,$$

is the Gagliardo seminorm. For the study of fractional Sobolev spaces in a systematic way we refer to [8, 10, 12, 14, 18] and references therein. At this stage, we consider two more Banach spaces $W^{s,p}_0(\Omega)$ and $W^{s,p}_\Omega(\mathbb{R}^n)$ defined as the closure of the space $C^\infty_c(\Omega)$ with the norms $||\cdot||_{s,p,\Omega}$ and $||\cdot||_{s,p,\mathbb{R}^n}$ respectively. These two spaces arise naturally in studying weak solutions of the Dirichlet problems involving regional fractional p-Laplacian and fractional p-Laplacian operators respectively, see [7, 9, 11, 17, 31] and references therein. If Ω is a bounded Lipschitz domain and $1 < p < \infty$, then one has

$$W^{s,p}_\Omega(\mathbb{R}^n) = \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\},$$

see [9, Proposition B.1]. Moreover, $W^{s,p}_\Omega(\mathbb{R}^n) = W^{s,p}_0(\Omega)$ provided $sp \neq 1$, see for instance [7, Proposition B.1].

In the spirit of local case, we introduce what we call variational Sobolev capacity in fractional Sobolev spaces.

Definition 1.1. Let $0 < s < 1$, $p \in [1,\infty)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. For a compact set $K \subset \Omega$, variational (s,p)-Sobolev capacity is defined by

$$\text{Cap}_{s,p}(K,\Omega) := \inf\left\{[u]^p_{s,p,\Omega} : u \in C^\infty_c(\Omega),\ u \geq 1 \text{ on } K\right\}. \tag{1.3}$$

For an open set $A \subset \Omega$, variational (s,p)-Sobolev capacity is defined by

$$\text{Cap}_{s,p}(A,\Omega) = \sup\left\{\text{Cap}_{s,p}(K,\Omega) : K \subset A,\ K \text{ is compact }\right\},$$

and for an arbitrary set $E \subset \Omega$, variational (s,p)-Sobolev capacity is defined by

$$\text{Cap}_{s,p}(E,\Omega) = \inf\left\{\text{Cap}_{s,p}(A,\Omega) : E \subset A,\ A \text{ is open }\right\}.$$

Using standard approximation argument, we can replace $C^\infty_c(\Omega)$ by a bigger space $W^{s,p}_0(\Omega) \cap C(\Omega)$ in the definition of capacity eq. (1.3).

Remark 1.2. It is worth mentioning that in the definition of capacity (1.3) one can restricts the function $u \in C^\infty_c(\Omega)$ such that $u = 1$ in a neighbourhood $\mathcal{N}(K) \subset \Omega$ of K and $0 \leq u \leq 1$ in Ω, see for instance [32, Theorem 2.1].

Definition 1.3. Let $0 < s < 1$, $p \in [1, \infty)$. We say that a property holds (s,p)-quasi everywhere (in short (s,p)-q.e.) if it holds except for a set of capacity zero.
We say a function $u : \Omega \to \mathbb{R}$ is (s,p)-quasi continuous (in short (s,p)-q.c.) in Ω if for every $\epsilon > 0$ there exists an open set $E \subset \Omega$ such that $\mathrm{Cap}_{s,p}(E, \Omega) < \epsilon$ and $u|_{\Omega \setminus E}$ is continuous.

Remark 1.4. We observe that, for any $\lambda \in \mathbb{R}$, the set $\{x \in \Omega : u(x) \neq \lambda\} \cup E$ is open in Ω and hence the set $\{x \in \Omega : u(x) = \lambda\} \cap E^c$ is closed in Ω, although $\{x \in \Omega : u(x) \neq \lambda\}$ need not be open. Indeed, by definition of the (s,p)-quasi continuous, $u|_{\Omega \setminus E}$ is continuous. Thus, the set $\{x \in \Omega : u(x) \neq \lambda\} \setminus E$ is open in $\Omega \setminus E$ with respect to the relative topology. Therefore, there exists an open set O in Ω such that $\{x \in \Omega : u(x) \neq \lambda\} \setminus E = O \setminus E$ and this implies $\{x \in \Omega : u(x) \neq \lambda\} \cup E = O \cup E$ is open in Ω. In particular, from this observation we have $Z(u; E^c) = \{x \in \Omega : u(x) = 0\} \cap E^c$ is a closed set in Ω.

Remark 1.5. It is important to note that from [33, Theorem 2.2], for a compact set $K \subset \Omega$ we have the following characterization for $\mathrm{Cap}_{s,p}(K, \Omega)$ via (s,p)-q.e. property

$$\mathrm{Cap}_{s,p}(K, \Omega) = \inf \left\{ [u]^p_{s,p,\Omega} : u \in W^{s,p}_0(\Omega),\ u \geq 1 \ (s,p)\text{-q.e. on } K \right\}.$$

In recent years, many researchers have shown their interest in studying variational Sobolev capacities, see [1, 2, 3, 20] for the case of classical Sobolev spaces and [10, 32, 33, 34, 36] for the case of fractional Sobolev spaces.

Before outlining the main results in the present paper in a precise manner, we need to introduce some terminologies and definitions. Let $0 \leq \alpha < 1$ and $R > 0$. For a locally integrable function u, the fractional maximal function is defined by

$$M_{\alpha,R}(u)(x) := \sup_{0<r<R} r^{\alpha} \fint_{B_r(x)} |u(y)| dy.$$

If $R = \infty$, then we shall simply write $M_{\alpha,R}$ by M_α and for $\alpha = 0,\ R = \infty$, we have the usual maximal function. Let $0 < \beta < \infty$, the fractional sharp maximal function of a locally integrable function u is defined by

$$u^{\#}_{\beta,R}(x) := \sup_{0<r<R} r^{-\beta} \fint_{B_r(x)} |u(y) - u_{B_r(x)}| dy.$$

If $R = \infty$, then we shall simply write $u^{\#}_{\beta,R}$ by $u^{\#}_{\beta}$.

Definition 1.6 (Pointwise fractional p-Hardy inequality). Let $0 < s < 1$, $0 \leq \alpha < 1$ and $p \in [1, \infty)$. We say that an open set $\Omega \subsetneq \mathbb{R}^n$ with non-empty boundary admits *pointwise fractional p-Hardy's inequality* if there exist constants $C > 0$ and $\sigma \geq 1$ such that

$$|u(x)| \leq C\delta(x)^{s-\frac{\alpha}{p}} \left(M_{\alpha,\, \sigma\delta(x)}(|D^s_p u|)^p(x) \right)^{1/p} \text{ for all } u \in C^\infty_c(\Omega), \tag{1.4}$$

and for all $x \in \Omega$, where $|D^s_p u|(x) := \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dy \right)^{\frac{1}{p}}$.

Definition 1.7 (Uniformly (s,p)-fat set). Let $0 < s < 1$ and $1 < p < \infty$ with $sp > 1$. We say that a closed set $E \subset \mathbb{R}^n$ is *uniformly (s,p)-fat set* if there exists a constant $\gamma > 0$ such that

$$\mathrm{Cap}_{s,p}(E \cap \overline{B_r(x)}, 2B_r(x)) \geq \gamma\, \mathrm{Cap}_{s,p}(\overline{B_r(x)}, 2B_r(x)), \text{ for all } x \in E \text{ and } r > 0.$$

Let us now describe our results in this paper before formulating these. Theorem 1.8 is a capacitary version of the fractional Sobolev–Poincaré inequality, which is motivated by the result of [22], and whereas, theorem 1.9 gives a characterization of uniformly (s,p)-fat set through a boundary fractional Sobolev–Poincaré type inequality. As an application of theorem 1.9, at the end of section 3, we provide various classes of domains that are uniformly (s,p)-fat set. The existence issue regarding fractional p-Hardy's inequality theorem 1.10 in the supercritical case $sp > n$ for any proper open set is addressed in theorem 1.10. This result can be obtained by proving an appropriate pointwise fractional Hardy type inequality and applying the maximal function theorem.

Our main results are stated below.

Theorem 1.8. *Let $0 < s < 1$, $p \in (1,\infty)$ with $sp > 1$ and suppose $u \in W^{s,p}(B)$ be a (s,p)-quasi continuous function, where $B = B_r(x_0) \subset \mathbb{R}^n$ is an open ball of radius $r > 0$. Let $1 \leq q \leq p_s^*$ for $sp < n$ and $1 \leq q < \infty$ for $sp \geq n$. Then there exists a constant $C = C(n,s,p,q) > 0$ such that*

$$\left(\fint_B |u(x)|^q dx\right)^{\frac{1}{q}} \leq C \left(\frac{1}{Cap_{s,p}\left(Z(u;E^c) \cap \frac{1}{2}\overline{B}, B\right)} \int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}},$$

where the closed set $Z(u;E^c)$ as in remark 1.4.

Theorem 1.9. *Let $0 < s < 1$, $p \in (1,\infty)$ with $sp > 1$ and let Ω be any proper open set in $\mathbb{R}^n$. Let $1 \leq q \leq p_s^*$ for $sp < n$ and $1 \leq q < \infty$ for $sp \geq n$. Then $\mathbb{R}^n \setminus \Omega$ is uniformly (s,p)-fat set with a constant γ if and only if for any $z \in \mathbb{R}^n \setminus \Omega$, $r > 0$*

$$\left(\fint_{B_r(z)} |u(x)|^q dx\right)^{\frac{1}{q}} \leq C\,\gamma^{-\frac{1}{p}}\, r^{s-\frac{n}{p}} \left(\int_{B_r(z)} \int_{B_r(z)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \tag{1.5}$$

for all $u \in C_c^\infty(\Omega)$, and where $C = C(n,s,p,q)$ is a constant.

Theorem 1.10. *Let Ω be any open set in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $0 < s < 1$ and $p > 1$ such that $sp > n$. Then there exists a constant $C = C(n,s,p) > 0$ such that the fractional Hardy inequality holds that is*

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}} dx \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy, \quad \text{for all } u \in W_\Omega^{s,p}(\mathbb{R}^n).$$

Furthermore, the regional fractional Hardy inequality holds that is

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}} dx \leq C \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy, \quad \text{for all } u \in W_0^{s,p}(\Omega).$$

Recently, in [13] the authors studied capacitary versions of fractional Poincaré, pointwise, and localized fractional Hardy inequalities in a metric measure space. However, their results involve the Assouad codimension of the domain, and certain restrictions on functions. The study of Hardy inequalities in fractional Sobolev spaces has emerged as an intriguing research area in recent times. There is numerous literature available on this topic. For details discussion on the sharp constants in fractional Hardy inequalities, we refer to [5, 6, 15, 29] and references therein.

This paper organized in the following way: In section 2 we collect some known results and discussed some necessary preliminaries. Proofs of theorems 1.8 to 1.10 along with some further results are given in section 3.

2. Preliminaries and Known results

Throughout the paper we shall assume the following notations, unless mentioned otherwise explicitly:

- Ω is an open connected set in $\mathbb{R}^n$, $0 < s < 1$, $1 \leq p < \infty$, $n \in \mathbb{N}$.
- $p_s^* = \frac{np}{n-sp}$ is the fractional Sobolev critical exponent for $sp < n$.
- $\overline{\Omega}$ is the closure of Ω.
- $|\Omega|$ is the Lebesgue measure of Ω.
- $u_\Omega = \fint_\Omega u dx = \frac{1}{|\Omega|} \int_\Omega u dx$ is the average of the function u in Ω.
- X^c is the complement of the set X in the appropriate ambient space.
- $B_r(x)$ is an open ball centered at x of radius $r > 0$.
- $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$.
- $c,\ C,\ C(*, *, \cdots, *) > 0$ denote generic constants that will appear in the estimate and need not be the same as in the preceding steps; the value depends on the quantities indicated by $*$'s.

We start with some known results and some technical lemmas that will be required to prove our results.

Lemma 2.1. *Let $s \in (0, 1)$, $1 < p < \infty$ with $sp > 1$. Then the following properties of capacity hold:*

a) ***(Ball estimate:)*** $\mathit{Cap}_{s,p}\left(\overline{B_r(x)}, 2B_r(x)\right) = C(n, s, p)\, r^{n-sp}$, *for a constant $C(n, s, p) > 0$.*

b) ***(Monotonicity:)*** *If $K_1 \subseteq K_2 \subset \Omega$, where K_i's are compact sets, one has*

$$\mathit{Cap}_{s,p}(K_1, \Omega) \leq \mathit{Cap}_{s,p}(K_2, \Omega).$$

Proof. $a)$ It follows from [32, Theorem 2.2] by choosing the radius of the ball appropriately.
$b)$ It is an immediate consequence of the definition of $\mathrm{Cap}_{s,p}(\cdot, \Omega)$. □

The proof of the following lemma can be found in [28], however we include the proof of it for the sake of completeness.

Lemma 2.2. *Let $\Omega \subsetneq \mathbb{R}^n$ be an open set, $s \in (0, 1)$ and $p \in (0, \infty)$ such that $sp > 1$. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \leq C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \quad \text{for all } u \in C_c^\infty(\Omega),$$

where $C = C(n, s, p)$ is a positive constant and does not depend on the domain Ω.

Proof. Let $u \in C_c^\infty(\Omega)$. Then we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy$$

$$= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy + 2 \int_\Omega |u(x)|^p \left(\int_{\Omega^c} \frac{dy}{|x - y|^{n+sp}} \right) dx.$$

Note that, for $x \in \Omega$, we have

$$\int_{\Omega^c} \frac{dy}{|x-y|^{n+sp}} \le \int_{\mathbb{S}^{n-1}} dw \int_{d_{w,\Omega}(x)}^{\infty} \frac{dr}{r^{sp+1}} = \frac{1}{sp} \int_{\mathbb{S}^{n-1}} \frac{dw}{d_{w,\Omega}(x)^{sp}} = \frac{C(n,s,p)}{m_{sp}(x)^{sp}},$$

where $d_{w,\Omega}(x) = \inf\{|t| : x + tw \in \Omega^c\}$ (see, [29]) and

$$m_{sp}(x)^{sp} = \frac{2\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+sp}{2}\right)}{\Gamma\left(\frac{n+sp}{2}\right)} \Bigg/ \int_{\mathbb{S}^{n-1}} \frac{dw}{d_{w,\Omega}(x)^{sp}}\ .$$

Thus, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy &\le \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy + 2C(n,s,p)\int_\Omega \frac{|u(x)|^p}{m_{sp}(x)^{sp}} dx \\ &\le C(n,s,p)\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy,\end{aligned}$$

where in the last step we have used the fractional Hardy inequality of Loss–Sloane [29, Theorem 1.2]. Hence the lemma follows. ☐

Lemma 2.3. *Suppose $u \in L^1_{loc}(\mathbb{R}^n)$ and let $0 < \beta < \infty$. Then there is a constant $C = C(n,\beta)$ and a set $A \subset \mathbb{R}^n$ with $|A| = 0$ such that*

$$|u(x)-u(y)| \le C|x-y|^\beta \left(u^{\#}_{\beta,4|x-y|}(x) + u^{\#}_{\beta,4|x-y|}(y)\right), \quad \textit{for all } x,y \in \mathbb{R}^n \setminus A.$$

Proof. Let S_u be the set of all Lebesgue points of the function u and set $A = S_u^c$. Then, by Lebesgue differentiation theorem we have $|A| = 0$. Now fix $x \in S_u$, $0 < r < \infty$ and we denote $B_k = B_{\frac{r}{2^k}}(x)$, $k = 0, 1, 2, \cdots$

$$\begin{aligned}|u(x) - u_{B_r(x)}| &= \left|\lim_{m\to\infty}\sum_{k=0}^{m}\left(u_{B_{k+1}} - u_{B_k}\right)\right| \le \sum_{k=0}^{\infty}|u_{B_{k+1}} - u_{B_k}| \\ &= \sum_{k=0}^{\infty}\left|\frac{1}{|B_{k+1}|}\int_{B_{k+1}} u(z)dz - u_{B_k}\right| \le \sum_{k=0}^{\infty}\frac{1}{|B_{k+1}|}\int_{B_{k+1}}|u(z) - u_{B_k}|\,dz \\ &\le \sum_{k=0}^{\infty}\frac{|B_k|}{|B_{k+1}|}\fint_{B_k}|u(z)-u_{B_k}|\,dz = C(n)\sum_{k=0}^{\infty}\left(\frac{r}{2^k}\right)^\beta\left(\frac{r}{2^k}\right)^{-\beta}\fint_{B_k}|u(z)-u_{B_k}|\,dz \\ &\le C(n,\beta) r^\beta u^{\#}_{\beta,r}(x).\end{aligned}$$

Let $y \in B_r(x) \setminus A$ and we have $B_r(x) \subset B_{2r}(y)$. Now

$$\begin{aligned}|u(y) - u_{B_r(x)}| &\le |u(y) - u_{B_{2r}(y)}| + |u_{B_{2r}(y)} - u_{B_r(x)}| \\ &\le C_1\, r^\beta u^{\#}_{\beta,2r}(y) + \fint_{B_r(x)}|u(z) - u_{B_{2r}(y)}|dz \\ &\le C_1 r^\beta u^{\#}_{\beta,2r}(y) + C_2(n)\fint_{B_{2r}(y)}|u(z) - u_{B_{2r}(y)}|dz \le C(n,\beta) r^\beta u^{\#}_{\beta,2r}(y).\end{aligned}$$

Let $x, y \in \mathbb{R}^n \setminus A$ with $x \neq y$ and let $r = 2|x-y|$. Then $x, y \in B_r(x)$ and hence we have

$$|u(x) - u(y)| \leq |u(x) - u_{B_r(x)}| + |u(y) - u_{B_r(x)}| \leq C(n,\beta) r^\beta \left(u^{\#}_{\beta,2r}(x) + u^{\#}_{\beta,2r}(y) \right).$$

This completes the proof of the lemma. □

Lemma 2.4. *Let $0 \leq \alpha < s < 1$, $R > 0$. Suppose $u \in W^{s,p}(\mathbb{R}^n)$. Then for any $x \in \mathbb{R}^n$ there is a constant $C = C(n,s,p)$ such that*

$$u^{\#}_{s-\alpha,R}(x) \leq C M_{\alpha,R}\left(|D^s_p u|\right)(x),$$

where $|D^s_p u|(x) = \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dy \right)^{1/p}$.

Proof. Using Hölder inequality we have

$$\begin{aligned} |u((y) - u_{B_r(x)}| &= \left| u(y) - \frac{1}{|B_r(x)|} \int_{B_r(x)} u(z) dz \right| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - u(z)| dz \\ &\leq \frac{1}{|B_r(x)|^{1/p}} \left(\int_{B_r(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} |y-z|^{n+sp} dz \right)^{1/p}. \end{aligned}$$

Integrating over the ball $B_r(x)$ we obtain

$$\begin{aligned} \fint_{B_r(x)} |u((y) - u_{B_r(x)}| dy &\leq \frac{C r^{\frac{n+sp}{p}}}{|B_r(x)|^{1/p}} \fint_{B_r(x)} \left(\int_{B_r(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dz \right)^{1/p} dy \\ &\leq \frac{C r^{\frac{n+sp}{p}}}{|B_r(x)|^{1/p}} \fint_{B_r(x)} |D^s_p u|(y) dy \leq C r^{s-\alpha} r^\alpha \fint_{B_r(x)} |D^s_p u|(y) dy, \end{aligned}$$

where $C = C(n,s,p)$. Thus we get

$$r^{\alpha-s} \fint_{B_r(x)} |u((y) - u_{B_r(x)}| dy \leq C r^\alpha \fint_{B_r(x)} |D^s_p u|(y) dy,$$

and consequently we have

$$u^{\#}_{s-\alpha,R}(x) \leq C M_{\alpha,R}\left(|D^s_p u|\right)(x), \text{ for any } x \in \mathbb{R}^n \text{ and } R > 0.$$

□

By the above two lemmas we immediately get the following result.

Corollary 2.5. *Let $u \in W^{s,p}(\mathbb{R}^n)$ and $0 \leq \alpha < s < 1$. Then, there is a set $A \subset \mathbb{R}^n$ with $|A| = 0$ such that for all $x, y \in \mathbb{R}^n \setminus A$ we have*

$$|u(x) - u(y)| \leq C|x-y|^{s-\alpha} \left(M_{\alpha,4|x-y|}(|D^s_p u|)(x) + M_{\alpha,4|x-y|}(|D^s_p u|)(y) \right), \tag{2.1}$$

where $C = C(n,s,\alpha,p)$ is a positive constant.

Remark 2.6. In view of the above corollary, we say that eq. (2.1) holds for almost every x and y.

Proposition 2.7. *Let Ω be an open set in $\mathbb{R}^n$ and $0 < s < 1$, $1 \le p < \infty$. Assume that $u \in W^{s,p}(\Omega)$. Let $1 \le q \le p_s^*$ for $sp < n$ and $1 \le q < \infty$ for $sp \ge n$. Then there exists a constant $C = C(n,s,p,q)$ such that*

$$\left(\fint_B |u(x) - u_B|^q dx \right)^{\frac{1}{q}} \le C \left(r^{sp-n} \int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

for each ball $B = B_r(x_0) \subset \Omega$.

Proof. Let $sp < n$. Then, applying improved fractional Sobolev–Poincaré inequality (see, [21, 33]) for $B_r(x_0)$ we have

$$\left(\fint_B |u(x) - u_B|^{p_s^*} dx \right)^{\frac{1}{p_s^*}} \le C \left(r^{sp-n} \int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}. \tag{2.2}$$

Now by Hölder inequality together with eq. (2.2) we obtain the desired inequality in this case.

Let $sp \ge n$. We choose $0 < s' < s$ such that $s'p < n$. Therefore, we have $u \in W^{s',p}(\Omega)$ by [12, Proposition 2.1]. If $q > p$ satisfying $q = \frac{np}{n-s'p}$. Then by eq. (2.2) we get

$$\left(\fint_B |u(x) - u_B|^q dx \right)^{\frac{1}{q}} \le C r^{s'-\frac{n}{p}} \left(\int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dxdy \right)^{\frac{1}{p}}. \tag{2.3}$$

Now, for $x, y \in B$ we have $|x-y| < 2r$. Since $s' < s$, therefore we get

$$\left(\frac{|x-y|}{2r} \right)^{n+sp} < \left(\frac{|x-y|}{2r} \right)^{n+s'p}.$$

Using this to eq. (2.3), we obtain

$$\left(\fint_B |u(x) - u_B|^q dx \right)^{\frac{1}{q}} \le C(n,s,p,q) r^{s-\frac{n}{p}} \left(\int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.$$

Finally, for $q \le p$, the result follows by using Hölder inequality and the above case with some $\ell > p \ge q$ on the left-hand side. □

Proposition 2.8 (Fractional (q,p)-Poincaré inequality)**.** *Let $0 < s < 1$, $1 < p < \infty$ with $sp > 1$. Let $1 \le q \le p_s^*$ for $sp < n$ and $1 \le q < \infty$ for $sp \ge n$. Then there exists a constant $C = C(n,s,p,q)$ such that*

$$\left(\fint_B |u(x)|^q dx \right)^{\frac{1}{q}} \le C \left(r^{sp-n} \int_B \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

for each ball $B = B_r(x_0) \subset \mathbb{R}^n$ and each $u \in W_0^{s,p}(B)$.

Proof. Since the claim follows from Hölder inequality for $q = 1$, we thus assume that $q > 1$. Let $u \in C_c^\infty(B)$. Then $u = 0$ in $2B \setminus B$. Again, by Hölder inequality we have

$$|u_{2B}| \le \fint_{2B} |u(x)| dx = \fint_{2B} |u(x)| \chi_B(x) dx \le 2^{\frac{n}{q}-n} \left(\fint_{2B} |u(x)|^q dx \right)^{1/q}. \tag{2.4}$$

Now, using Minkowski inequality and the proposition 2.7 for $2B$ with the estimate eq. (2.4), we obtain

$$\left(\fint_{2B}|u(x)|^q dx\right)^{1/q} \le \left(\fint_{2B}|u(x)-u_{2B}|^q dx\right)^{1/q} + |u_{2B}|$$
$$\le C(n,s,p,q) r^{s-\frac{n}{p}}[u]_{s,p,2B} + 2^{\frac{n}{q}-n}\left(\fint_{2B}|u(x)|^q dx\right)^{1/q}.$$

Since $2^{\frac{n}{q}-n} < 1$ and using (Proposition 3.1, [4]) to estimate the seminorm in the above, we obtain

$$\left(\fint_{2B}|u(x)|^q dx\right)^{1/q} \le C r^{s-\frac{n}{p}}\left(\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\right)^{\frac{1}{p}}$$

Finally, the average integral in the left hand side of the above can be taken on B since $u=0$ in $2B\setminus B$. Hence the result follows by density. □

3. On Sobolev-Poincaré and localized Hardy inequalities

This section is devoted to proofs of theorems 1.8 to 1.10. We also address some further results in this context.

Proof of theorem 1.8. If $u_B = \fint_B u = 0$. Then, by proposition 2.7 we have

$$\left(\fint_B |u(x)|^q dx\right)^{\frac{1}{q}} \le C\left(r^{sp-n}\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\right)^{\frac{1}{p}}. \tag{3.1}$$

Using the capacity estimate lemma 2.1 we get

$$\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right) \le \mathrm{Cap}_{s,p}\left(\frac{1}{2}\overline{B}, B\right) = C(n,p,s) r^{n-sp}. \tag{3.2}$$

By exploiting this estimate in eq. (3.1) to get the desired result in this case.

If $u_B \ne 0$, without loss of generality we can take $u_B = 1$. Choose $\phi \in C_c^\infty(B)$ such that $\phi = 1$ in $\frac{1}{2}\overline{B}$, $|\nabla\phi| \le c/r$ and $0 \le \phi \le 1$. We define $\psi = \phi(u_B - u)$. Clearly, $\psi \in W_0^{s,p}(B)$ be a (s,p)-quasi continuous function and $\psi = 1$ in $Z(u;E^c)\cap\frac{1}{2}\overline{B}$. Therefore, by remark 1.5 we obtain

$$\begin{aligned}
\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right) &\le \int_B\int_B \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{n+sp}}dxdy \\
&\le 2^{p-1}\left(\int_B\int_B \frac{|\phi(x)|^p|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy + \int_B\int_B \frac{|u_B-u(y)|^p|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}dxdy\right) \\
&\le 2^{p-1}\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy + 2^{p-1}\int_B |u(y)-u_B|^p \int_{B\cap\{x:|x-y|\le\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}dxdy \\
&\quad + 2^{p-1}\int_B |u(y)-u_B|^p \int_{B\cap\{x:|x-y|>\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}dxdy \\
&=: 2^{p-1}\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy + 2^{p-1}(I_1+I_2),
\end{aligned} \tag{3.3}$$

where $\eta > 0$ will be chosen later,

$$I_1 := \int_B |u(y) - u_B|^p \int_{B\cap\{x:|x-y|\leq\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}} dxdy,$$

and

$$I_2 := \int_B |u(y) - u_B|^p \int_{B\cap\{x:|x-y|>\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}} dxdy.$$

We estimate the above integrals $I_1,\ I_2$ in the following:

Estimate for I_1 *:* Using the properties of the function ϕ, we have

$$\begin{aligned} I_1 = \int_B |u(y) - u_B|^p &\int_{B\cap\{x:|x-y|\leq\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}} dxdy \\ &\leq \frac{c^p}{r^p}\int_B |u(y) - u_B|^p \int_{B\cap\{x:|x-y|\leq\eta\}} \frac{dx}{|x-y|^{n+sp-p}} dy \\ &\leq c^p r^{-p}\int_B |u(y) - u_B|^p \int_{|x-y|\leq\eta} \frac{dx}{|x-y|^{n+sp-p}} dy \leq \frac{C\eta^{p-sp}}{r^p}\int_B |u(y)-u_B|^p dy. \end{aligned}$$

Estimate for I_2 *:*

$$\begin{aligned} I_2 = \int_B |u(y) - u_B|^p &\int_{B\cap\{x:|x-y|>\eta\}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}} dxdy \\ &\leq 2^p \int_B |u(y) - u_B|^p \int_{B\cap\{x:|x-y|>\eta\}} \frac{dx}{|x-y|^{n+sp}} dy \\ &\leq 2^p \int_B |u(y) - u_B|^p \int_{|x-y|>\eta} \frac{dx}{|x-y|^{n+sp}} dy \leq C\eta^{-sp}\int_B |u(y)-u_B|^p dy. \end{aligned}$$

Plugging the above two estimates into eq. (3.3), we obtain

$$\begin{aligned} &\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right) \\ &\qquad \leq 2^{p-1}\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy + 2^{p-1}C\left(\frac{\eta^{p-sp}}{r^p} + \eta^{-sp}\right)\int_B |u(y)-u_B|^p dy. \end{aligned}$$

Now, choose $\eta = r$ the radius of the ball B and by proposition 2.7 for $q = p$, we have

$$\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right) \leq C\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy.$$

Therefore, by above estimate we get

$$u_B = 1 \leq C\left(\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right)^{-1}\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}}. \tag{3.4}$$

Now using Minkowski inequality, the estimate eq. (3.4) together with proposition 2.7, we obtain

$$\begin{aligned}\left(\fint_B |u(x)|^q dx\right)^{\frac{1}{q}} &\le \left(\fint_B |u(x)-u_B|^q dx\right)^{\frac{1}{q}} + u_B \\ &\le C\, r^{s-\frac{n}{p}} \left(\int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \\ &+ C\left(\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right)^{-1} \int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \\ &\le C\left(\frac{1}{\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap\frac{1}{2}\overline{B}, B\right)} \int_B\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}}.\end{aligned}$$

In the last estimate we used eq. (3.2). This completes the proof of the theorem. □

Remark 3.1. In view of theorem 1.8, if we assume that $u \in W^{s,p}(B)$ be a continuous function then we may replace $\mathrm{Cap}_{s,p}\left(Z(u;E^c)\cap \frac{1}{2}\overline{B}, B\right)$ by $\mathrm{Cap}_{s,p}\left(Z(u)\cap \frac{1}{2}\overline{B}, B\right)$, $Z(u)$ is the zero set of u.

Remark 3.2. Theorem 1.8 is essentially the best possible for the case $q = p_s^*$ in the following way: Let $1 < sp < n$ and $F \subset \mathbb{R}^n$ closed set. Take $x_0 \in F$ be any and consider the ball $B_r(x_0)$ with $r > 0$ small enough. Suppose there exists $C_F > 0$ such that

$$\left(\int_{B_r(x_0)} |u(x)|^{p_s^*} dx\right)^{1/p_s^*} \le C_F \left(\int_{B_r(x_0)}\int_{B_r(x_0)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{1/p},$$

where $u \in W^{s,p}(B_r(x_0))$ is a (s,p)-quasi continuous function with $u = 0$ in $F \cap \frac{1}{2}\overline{B_r(x_0)}$. Let $v \in W_0^{s,p}(\frac{3}{4}B_r(x_0))$ be a (s,p)-quasi continuous function with $v = 1$ in $F \cap \frac{1}{2}\overline{B_r(x_0)}$. Define $w := 1 - v \in W^{s,p}(B_r(x_0))$. Then w is a (s,p)-quasi continuous function and $w = 0$ in $F \cap \frac{1}{2}\overline{B_r(x_0)}$. Hence,

$$\begin{aligned}C_F &\left(\int_{B_r(x_0)}\int_{B_r(x_0)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \\ &= C_F\left(\int_{B_r(x_0)}\int_{B_r(x_0)} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \\ &\ge \left(\int_{B_r(x_0)} |w|^{p_s^*} dx\right)^{\frac{1}{p_s^*}} \ge \left|B_r(x_0)\setminus \frac{3}{4}B_r(x_0)\right|^{\frac{1}{p_s^*}} \ge c|B_r(x_0)|^{\frac{1}{p_s^*}}.\end{aligned}$$

Therefore, taking infimum over all such v we infer that

$$C_F\, \mathrm{Cap}_{s,p}\left(F \cap \frac{1}{2}\overline{B_r(x_0)}, B_r(x_0)\right)^{1/p} \ge c|B_r(x_0)|^{\frac{1}{p_s^*}} = cr^{\frac{n-sp}{p}}.$$

The proof of the following theorem is adapted from the proof of [23, Theorem 6.23].

Proof of theorem 1.9. Suppose $\mathbb{R}^n \setminus \Omega$ is uniformly (s,p)-fat set with a constant γ and let $z \in \mathbb{R}^n \setminus \Omega$, $r > 0$. Let $u \in C_c^\infty(\Omega)$. Then $\mathbb{R}^n \setminus \Omega \subset Z(u) := \{x \in \mathbb{R}^n : u(x) = 0\}$. Now, using lemma 2.1 and by definition of the (s,p) fat set, we obtain

$$\operatorname{Cap}_{s,p}\left(Z(u) \cap \frac{1}{2}\overline{B_r(z)}, B_r(z)\right) \geq \operatorname{Cap}_{s,p}\left((\mathbb{R}^n \setminus \Omega) \cap \frac{1}{2}\overline{B_r(z)}, B_r(z)\right) \geq \gamma\, r^{n-sp}. \tag{3.5}$$

Again, by theorem 1.8 with the remark 3.1 we have

$$\begin{aligned}\left(\fint_{B_r(z)} |u(x)|^q dx\right)^{\frac{1}{q}} &\leq C\left(\frac{1}{\operatorname{Cap}_{s,p}\left(Z(u) \cap \frac{1}{2}\overline{B_r(z)}, B_r(z)\right)} \int_{B_r(z)} \int_{B_r(z)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} \\ &\leq C\, \gamma^{-1/p}\, r^{s-\frac{n}{p}} \left(\int_{B_r(z)} \int_{B_r(z)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore, the result follows in this case.
To prove other implication. Let $u \in C_c^\infty(\Omega)$, $x \in \Omega$ and $R = \operatorname{dist}(x, \partial\Omega)$. Choose $x_0 \in \partial\Omega$ such that $R = |x - x_0|$. Then by triangle inequality, we have

$$|u_{B_R(x)}| \leq |u_{B_R(x)} - u_{B_R(x_0)}| + |u_{B_R(x_0)}|.$$

Note that, $B_R(x) \cup B_R(x_0) \subset B_{2R}(x)$ and by proposition 2.7 with $q = 1$, we obtain

$$\begin{aligned}|u_{B_R(x)} - u_{B_R(x_0)}| &\leq |u_{B_R(x)} - u_{B_{2R}(x)}| + |u_{B_{2R}(x)} - u_{B_R(x_0)}| \\ &\leq C(n) \fint_{B_{2R}(x)} |u(y) - u_{B_{2R}(x)}| dy \leq C(n,s,p) R^{s-n/p}\, [u]_{s,p,B_{2R}(x)}.\end{aligned} \tag{3.6}$$

On the other hand, by Hölder inequality and hypothesis eq. (1.5) with a constant C_1, we have

$$\begin{aligned}|u_{B_R(x_0)}| \leq \fint_{B_R(x_0)} |u(y)| dy \leq \left(\fint_{B_R(x_0)} |u(y)|^q dy\right)^{1/q} &\leq C_1 R^{s-n/p} [u]_{s,p,B_R(x_0)} \\ &\leq C_1 R^{s-n/p}\, [u]_{s,p,B_{2R}(x)}.\end{aligned} \tag{3.7}$$

Combining eq. (3.6) and eq. (3.7), we have

$$|u_{B_R(x)}| \leq C(n,s,p,C_1) R^{s-n/p}\, [u]_{s,p,B_{2R}(x)},$$

and this implies

$$|u_{B_R(x)}| \leq C(n,s,p,C_1) R^s \left(M_{2R}\left(|D^s_{p,\,B_{2R}(x)} u|\right)^p (x)\right)^{1/p},$$

where

$$|D^s_{p,\,B_{2R}(x)} u|(y) := \left(\int_{B_{2R}(x)} \frac{|u(y) - u(z)|^p}{|y-z|^{n+sp}} dz\right)^{1/p}.$$

By proceeding as in lemma 2.3, we obtain

$$|u(x) - u_{B_R(x)}| \leq C(n,s,p) R^s \left(M_{2R}\left(|D^s_{p,\,B_{2R}(x)} u|\right)^p (x)\right)^{1/p}.$$

Therefore, by above estimates we have for any $x \in \Omega$

$$
\begin{aligned}
|u(x)| &\leq |u(x) - u_{B_R(x)}| + |u_{B_R(x)}| \\
&\leq C(n,s,p,C_1) R^s \left(M_{2R} \left(|D^s_{p,\,B_{2R}(x)} u| \right)^p (x) \right)^{1/p} \\
&= C(n,s,p,C_1) \operatorname{dist}(x,\partial\Omega)^s \left(M_{2R} \left(|D^s_{p,\,B_{2\operatorname{dist}(x,\partial\Omega)}(x)} u| \right)^p (x) \right)^{1/p}.
\end{aligned}
\tag{3.8}
$$

Let $z \in \mathbb{R}^n \setminus \Omega,\ r > 0$. To conclude the proof of the theorem, it is enough to find a positive constant C such that

$$
\int_{B_r(z)} \int_{B_r(z)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} dxdy \geq C\, r^{n-sp} \tag{3.9}
$$

whenever $v \in C_c^\infty(B_r(z))$ and $v(x) = 1$ for $x \in (\mathbb{R}^n \setminus \Omega) \cap \frac{1}{2}\overline{B_r(z)}$. Moreover, we may assume that $0 \leq v \leq 1$, compare to remark 1.2. Let $\sigma = 1/6$.

Step 1: If

$$
\fint_{B_{r/2}(z)} v(y)dy > \frac{\sigma^n}{4},
$$

then by proposition 2.8 with $q = 1$ we have

$$
\begin{aligned}
1 < 4\sigma^{-n} \fint_{B_{r/2}(z)} v(y)dy &\leq C(n) \fint_{B_r(z)} |v(y)|dy \\
&\leq C(n,s,p)\, r^{s-n/p} \left(\int_{B_r(z)} \int_{B_r(z)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p}
\end{aligned}
$$

and eq. (3.9) follows.

Step 2: If

$$
\fint_{B_{r/2}(z)} v(y)dy \leq \frac{\sigma^n}{4}.
$$

Let $F = \left\{ y \in B_{\frac{\sigma r}{2}}(z) : v(y) < \frac{1}{2} \right\}$. Since $v = 1$ in $(\mathbb{R}^n \setminus \Omega) \cap \frac{1}{2}\overline{B_r(z)}$, we have $F \subset \Omega$. By definition, $v \geq \frac{1}{2}$ in $B_{\frac{\sigma r}{2}}(z) \setminus F$ and thus

$$
|B_{\frac{\sigma r}{2}}(z) \setminus F| \leq 2 \int_{B_{\frac{\sigma r}{2}}(z)\setminus F} v(y)dy \leq 2 \int_{B_{\frac{r}{2}}(z)} v(y)dy \leq \frac{\sigma^n}{2} |B_{r/2}(z)|.
$$

This gives that

$$
|F| = |B_{\frac{\sigma r}{2}}(z)| - |B_{\frac{\sigma r}{2}}(z) \setminus F| \geq \frac{\sigma^n}{2} |B_{r/2}(z)|. \tag{3.10}
$$

Let $\phi \in C_c^\infty(B_{r/2}(z))$ such that $\phi = 1$ in $\frac{1}{4}\overline{B_r(z)}$. Define $u(x) = \phi(x)(1 - v(x))$, $x \in \mathbb{R}^n$. Then $u \in C_c^\infty(\Omega)$ and $u = 1 - v$ in $\frac{1}{4}\overline{B_r(z)}$. Moreover, $u = 0$ in $\Omega^c \cap \frac{1}{2}\overline{B_r(z)}$ and for $x \in B_{r/4}(z)$,

$A \subset B_{r/4}(z)$ we have $|D^s_{p,\,A} u|(x) = |D^s_{p,\,A} v|(x)$. Since, by definition $|u(y)|^p$ is finite for every $y \in F$, there exists a radius $0 < R_y \le 2\,\mathrm{dist}(y, \partial\Omega) =: 2\,\delta(y)$ such that

$$M_{2\delta(y)}\left(|D^s_{p,B_{2\delta(y)}(y)} u|\right)^p (y) \le 2 \fint_{B_{R_y}(y)} |D^s_{p,B_{2\delta(y)}(y)} u|^p(x)dx. \tag{3.11}$$

Observe that the balls $B_{R_y}(y)$ form a cover of F and they have a uniformly bounded radii. By $5R$-covering lemma (see for example [23, Lemma 1.13]) there exist pairwise disjoint balls $B_{R_j}(y_j)$ where $y_j \in F$ and $R_j = R_{y_j}$ are as above, such that $F \subset \bigcup_{j=1}^{\infty} B_{5R_j}(y_j)$. It follows from eq. (3.10)

$$|B_{r/2}(z)| \le \frac{2}{\sigma^n}|F| \le C(n)\sum_{j=1}^{\infty} |B_{R_j}(y_j)|. \tag{3.12}$$

Let $j \in \mathbb{N}$. Since $y_j \in F \cap B_{\frac{\sigma r}{2}}(z) \subset \Omega \cap B_{\frac{\sigma r}{2}}(z)$ and $z \notin \Omega$, we have $\delta(y_j) := \mathrm{dist}(y_j, \partial\Omega) < \frac{\sigma r}{2}$. If $x \in B_{R_j}(y_j)$, then

$$|x - z| \le |x - y_j| + |y_j - z| \le R_j + \frac{\sigma r}{2} \le 2\,\mathrm{dist}(y_j, \partial\Omega) + \frac{\sigma r}{2} < \frac{\sigma r}{2}(2+1) = \frac{r}{4},$$

and this gives that $B_{R_j}(y_j) \subset B_{r/4}(z)$. Since $y_j \in F$, we have $u(y_j) = 1 - v(y_j) > \frac{1}{2}$. By eq. (3.8) and the choice of the radius R_j in eq. (3.11), we obtain

$$\begin{aligned} \frac{1}{2^p} \le |u(y_j)|^p &\le C(\delta(y_j))^{sp} M_{2\delta(y_j)}\left(|D^s_{p,B_{2\delta(y_j)}(y_j)} u|\right)^p (y_j) \\ &\le C\sigma^{sp} r^{sp} \fint_{B_{R_j}(y_j)} |D^s_{p,B_{2\delta(y_j)}(y_j)} u|^p(x)dx, \end{aligned}$$

where $C = C(n, s, p, C_1) > 0$ and consequently

$$|B_{R_j}(y_j)| \le C\, r^{sp} \int_{B_{R_j}(y_j)} |D^s_{p,B_{2\delta(y_j)}(y_j)} v|^p(x)dx \quad \text{for all } j \in \mathbb{N},$$

where we used the definition of v. Using this into eq. (3.12), we get

$$|B_{r/2}(z)| \le C\, r^{sp} \sum_{j=1}^{\infty} \int_{B_{R_j}(y_j)} |D^s_{p,B_{2\delta(y_j)}(y_j)} v|^p(x)dx \le C\, r^{sp} \int_{B_r(z)} |D^s_{p,B_r(z)} v|^p(x)dx,$$

where we also used the fact that the balls $B_{R_j}(y_j) \subset B_r(z)$ are pairwise disjoint. This shows that eq. (3.9) holds and the proof is complete. □

Proof of theorem 1.10. Let us choose $q \ge 1$ such that $n < sq < sp$. Now fix $x \in \Omega$ and $x_0 \in \partial\Omega$ such that $|x - x_0| = \mathrm{dist}(x, \partial\Omega) = \delta(x) = R$. We denote $\chi = \chi_{B_{4R}(x_0)}$ the characteristic function of $B_{4R}(x_0)$. Let $u \in C^\infty_c(\Omega)$ and consider the natural zero extension of u to $\mathbb{R}^n \setminus \Omega$, then by corollary 2.5 we have for almost every x,

$$|u(x)| = |u(x) - u(x_0)| \le C|x - x_0|^{s - \frac{n}{q}} \left(M_{\frac{n}{q}}(|D^s_p u|\chi)(x) + M_{\frac{n}{q}}(|D^s_p u|\chi)(x_0)\right). \tag{3.13}$$

Using Hölder inequality we have

$$\frac{r^{\frac{n}{q}}}{|B_r(x)|}\int_{B_r(x)}\left(|D_p^s u|\chi\right)(y)dy \leq \frac{r^{\frac{n}{q}}}{|B_r(x)|^{1/q}}\left(\int_{B_r(x)}\left(\left(|D_p^s u|\chi\right)(y)\right)^q dy\right)^{1/q}$$
$$\leq C(n)\left(\int_{\mathbb{R}^n}\left(\left(|D_p^s u|\chi\right)(y)\right)^q dy\right)^{1/q},$$

and consequently we get

$$M_{\frac{n}{q}}(|D_p^s u|\chi)(x) \leq C(n)\ ||(|D_p^s u|\chi)||_{L^q(\mathbb{R}^n)}.$$

Similar computation yields that

$$M_{\frac{n}{q}}(|D_p^s u|\chi)(x_0) \leq C(n)\ ||(|D_p^s u|\chi)||_{L^q(\mathbb{R}^n)}.$$

Combining the above two estimates in eq. (3.13) we obtain

$$|u(x)| \leq C|x-x_0|^{s-\frac{n}{q}}\left(\int_{B_{4R}(x_0)}(|D_p^s u|(z))^q dz\right)^{1/q}$$
$$\leq CR^{s-\frac{\alpha}{q}}\left(R^{\alpha-n}\int_{B_{5R}(x)}(|D_p^s u|(z))^q dz\right)^{1/q}$$
$$\leq C\left(\operatorname{dist}(x,\partial\Omega)\right)^{s-\frac{\alpha}{q}}\left(M_{\alpha,5R}(|D_p^s u|)^q(x)\right)^{1/q}.$$

The above inequality holds for almost every $x \in \mathbb{R}^n$. Integrating with respect to the variable x over Ω with $\alpha = 0$, we infer that

$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp}}dx \leq C\int_\Omega \left(M(|D_p^s u|^q(x))\right)^{\frac{p}{q}}dx \leq C\int_{\mathbb{R}^n}|D_p^s u|^p(x)dx$$
$$= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy.$$

In the above estimate we used the Hardy-Littlewood-Wiener maximal function theorem. By density, we conclude the first part of theorem 1.10. The second part follows from the first part together with lemma 2.2. This completes the proof of the theorem. □

In the following theorem, we prove the validity of pointwise fractional p-Hardy inequality eq. (1.4).

Theorem 3.3. *Let $0 < s < 1$, $1 < p < \infty$ such that $1 < sp \leq n$, $0 \leq \alpha < p$, and let Ω be an open set in $\mathbb{R}^n$ such that it's complement that is $\mathbb{R}^n \setminus \Omega$ is uniformly (s,p)- fat. Assume that $u \in C_c^\infty(\Omega)$. Then pointwise fractional p-Hardy inequality eq. (1.4) holds true for Ω that is there exist constants $C = C(n,p,\alpha) > 0$ and $\sigma > 1$ such that for all $x \in \Omega$*

$$|u(x)| \leq C\delta(x)^{s-\frac{\alpha}{p}}\left(M_{\alpha,\,\sigma\delta(x)}|D_p^s u|^p(x)\right)^{\frac{1}{p}}.$$

Proof. Let $x \in \Omega$. Let us choose $x_0 \in \partial\Omega$ for which $|x - x_0| = \delta(x) = R$. Then, by using the standard telescoping argument as in the proof of lemma 2.3 we obtain

$$|u(x) - u_{B_R(x_0)}| \leq CR^{s-\frac{\alpha}{p}}\left(M_{\alpha,\,2R}|D_p^s u|^p(x)\right)^{\frac{1}{p}}.$$

Thus, by above estimate we have

$$|u(x)| \leq |u(x) - u_{B_R(x_0)}| + |u_{B_R(x_0)}| \leq CR^{s-\frac{\alpha}{p}} \left(M_{\alpha,\, 2R}|D^s_p u|^p(x)\right)^{\frac{1}{p}} + |u_{B_R(x_0)}|. \tag{3.14}$$

Now consider the set $Z(u) = \{x \in \mathbb{R}^n : u(x) = 0\}$, which is a closed as the function $u \in C^\infty_c(\Omega)$. Then, by using theorem 1.8 with remark 3.1 and the monotonicity property of the capacity along with hypothesis, we obtain

$$\begin{aligned}
|u_{B_R(x_0)}| &\leq \left(C\ \mathrm{Cap}_{s,p}\left(Z(u) \cap \frac{1}{2}\overline{B}, B\right)^{-1} \int_{B_R(x_0)} |D^s_p u|^p(x)dx\right)^{\frac{1}{p}} \\
&\leq \left(C\ \mathrm{Cap}_{s,p}\left(\Omega^c \cap \frac{1}{2}\overline{B}, B\right)^{-1} \int_{B_R(x_0)} |D^s_p u|^p(x)dx\right)^{\frac{1}{p}} \\
&\leq C\left(R^{sp-n} \int_{B_R(x_0)} |D^s_p u|^p(x)dx\right)^{\frac{1}{p}} \leq CR^{s-\frac{\alpha}{p}} \left(M_{\alpha,R}|D^s_p u|^p(x)\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.15}$$

Plugging the estimate eq. (3.15) into eq. (3.14) we obtain

$$|u(x)| \leq CR^{s-\frac{\alpha}{p}} \left(M_{\alpha,\, \sigma R}|D^s_p u|^p(x)\right)^{\frac{1}{p}}.$$

Since we have chosen $x \in \Omega$ arbitrarily and hence this completes the proof of the theorem. □

As an application of theorem 1.9 and theorem 3.3 we discuss some examples of domains that are uniformly (s,p)-fat set.

Example 3.4. For $s \in (0,1)$, $1 < p < \infty$. Then all nonempty closed sets in $\mathbb{R}^n$ are uniformly (s,p)-fat set provided $sp > n$.

Proof. Suppose $E \subset \mathbb{R}^n$ be an nonempty closed set. Let $x \in E$ and $r > 0$. Consider $u \in C^\infty_c(2B_r(x))$ such that $u \geq 1$ in $E \cap \overline{B_r(x)}$. Choose a cutoff function $\rho \in C^\infty_c(3B_r(x))$ such that $0 \leq \rho \leq 1$, $\rho = 1$ in $\overline{B_r(x)}$ and $|\nabla\rho| \leq c/r$. Then, by fractional Morrey's inequality (see, [8, Corollary 2.7]) we have

$$|\left(\rho u\right)(y) - \left(\rho u\right)(z)| \leq C|y-z|^{s-\frac{n}{p}}[\rho u]_{s,p,\mathbb{R}^n}. \tag{3.16}$$

Now, let $y \in E \cap \overline{B_r(x)}$ and $z \in B_{3r}(x) \setminus B_{2r}(x)$. Then, we have $(\rho u)(y) \geq 1$ and $(\rho u)(z) = 0$. Thus by eq. (3.16), we obtain

$$[\rho u]^p_{s,p,\mathbb{R}^n} \geq C|y-z|^{n-sp}|\left(\rho u\right)(y) - \left(\rho u\right)(z)| \geq Cr^{n-sp}. \tag{3.17}$$

Therefore, using lemma 2.2 and eq. (3.17) we have

$$[\rho u]^p_{s,p,B_{2r}(x)} \geq Cr^{n-sp}. \tag{3.18}$$

Also, note that

$$[\rho u]^p_{s,p,B_{2r}(x)} = \int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|\left(\rho u\right)(y) - \left(\rho u\right)(z)|^p}{|y-z|^{n+sp}} dydz \tag{3.19}$$

$$
\begin{aligned}
&= \int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|\rho(y)\,(u(y)-u(z)) + u(z)\,(\rho(y)-\rho(z))\,|^p}{|y-z|^{n+sp}} dydz \\
&\leq 2^{p-1} \left(\int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz + \frac{C}{r^p} \int_{B_{2r}(x)} \int_{B_{4r}(z)} \frac{|u(z)|^p}{|y-z|^{n+sp-p}} dydz \right) \\
&\leq 2^{p-1} \left(\int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz + \frac{C}{r^{sp}} \int_{B_{2r}(x)} |u(z)|^p dz \right) \\
&\leq C \int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz.
\end{aligned}
$$

In the last estimate we have used the fractional Poincaré inequality (see, [7]) and then lemma 2.2. Now, combining the estimates eq. (3.18), eq. (3.19) we obtain

$$
\int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz \geq C r^{n-sp}.
$$

Since u is arbitrary and thus taking infimum over all such functions to get the desired result. □

Example 3.5. Let $0 < s < 1$ and $1 \leq p < \infty$. Suppose $E \subset \mathbb{R}^n$ be a closed set such that it satisfies the measure density condition

$$
|E \cap B_r(x)| \geq c\,|B_r(x)| \tag{3.20}
$$

for all $x \in E$ and radii $r > 0$, and for some constant $c > 0$. Then E is uniformly (s,p)-fat set.

Proof. Let $x \in E$ and $r > 0$. Let $u \in C_c^\infty(B_{2r}(x))$ such that $u \geq 1$ in $E \cap \overline{B_r(x)}$. Then, by hypothesis eq. (3.20) and using proposition 2.8 for $q = p$ we obtain

$$
\begin{aligned}
c\,|B_r(x)| \leq |E \cap B_r(x)| \leq \int_{E\cap B_r(x)} |u(y)|^p dy &\leq \int_{B_{2r}(x)} |u(y)|^p dy \\
&\leq C(n,s,p)\, r^{sp} \int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz.
\end{aligned}
$$

Thus,

$$
\int_{B_{2r}(x)} \int_{B_{2r}(x)} \frac{|u(y)-u(z)|^p}{|y-z|^{n+sp}} dydz \geq c \times C(n,s,p)\, r^{n-sp}.
$$

This proves that E is uniformly (s,p)-fat set with a constant $\gamma = c \times C(n,s,p)$. □

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