

DIFFERENTIABILITY OF INTRINSIC LIPSCHITZ FUNCTIONS WITHIN HEISENBERG GROUPS

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ABSTRACT. We study the notion of intrinsic Lipschitz graphs within Heisenberg groups, focusing our attention on their Hausdorff dimension and on almost everywhere existence of (geometrically defined) tangent subgroups. In particular, a Rademacher type theorem enables us to prove that previous definitions of rectifiable sets in Heisenberg groups are natural ones.

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1. INTRODUCTION

A Carnot group \mathbb{G} is a connected, simply connected stratified nilpotent Lie group such that the first layer of its Lie algebra \mathfrak{g} generates all the algebra. Among Carnot groups, nontrivial examples, i.e. non commutative ones, are provided by Heisenberg groups \mathbb{H}^n (see next section and also e.g. [6], [15], [22], [23], [35], [36]).

The first layer of the stratification of \mathfrak{g} , that can be identified with a linear subspace of the tangent space to \mathbb{G} at the origin, generates, by left translation, a subbundle of the tangent bundle, called the horizontal bundle $H\mathbb{G}$. Through the exponential map, Carnot groups can be identified with Euclidean

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spaces \mathbb{R}^N ($N = \dim \mathfrak{g}$), endowed both with a (non-commutative) group law $(g, h) \rightarrow g \cdot h$ and with a family of (non isotropic) dilations δ_t , $t > 0$, that are group automorphisms. The Lebesgue measure \mathcal{L}^N is the Haar measure of the group.

Besides the usual left invariant Riemannian metric, \mathbb{G} carries another left invariant metric, the so called Carnot-Carathéodory metric, that is the subriemannian metric associated with the horizontal bundle $H\mathbb{G}$. Through all this paper we will deal only with the subriemannian structure without mentioning it anymore. In particular, when saying distance, we mean the Carnot-Carathéodory distance, or an equivalent homogeneous distance, and, when we speak of dimension, we mean the Hausdorff dimension with respect to this distance.

From now on, *intrinsic* will denote properties defined only in terms of the group structure of \mathbb{G} or, equivalently, of its Lie algebra \mathfrak{g} .

We point out that non commutative Carnot groups (Heisenberg groups in particular) are not Riemannian manifolds. In fact, when \mathbb{G} is non commutative, the Carnot-Carathéodory distance makes it a metric space non Riemannian at any scale ([34]), though furnished with a rich structure of translations and (non-isotropic) dilations.

In the last few years there have been a large amount of work dedicated to the study of Geometric Measure Theory on metric spaces and in particular on Carnot groups. For example rectifiable sets, finite perimeter sets, various notions of convex surfaces have been studied.

The notion of rectifiable set, in particular, is a central one both in calculus of variations and in geometric measure theory. Several papers focused on this notion and on properties of rectifiable sets (see, for different notions of rectifiability introduced in the last few years, e.g. [2], [3], [16], [19], [33], [28] and the references therein). Besides their own geometric interest, rectifiable sets in Lie groups appear in several applications, as theoretical computer science, geometry of Banach spaces, mathematical models in neurosciences (see e.g. [8], [11]).

Rectifiable sets, in Euclidean spaces, are generalizations of C^1 or of Lipschitz submanifolds. Hence, to understand the objects that, inside Carnot groups, naturally take the role of C^1 or of Lipschitz submanifolds seems to be preliminary in order to develop a satisfactory theory of *intrinsically* rectifiable sets.

In Euclidean spaces, submanifolds are locally graphs. On the other side, we stress that Carnot groups in general cannot be viewed as cartesian products of subgroups (unlike Euclidean spaces). Therefore we need a notion of *intrinsic graph* fitting the structure of the group \mathbb{G} .

We can meet this request with the notion of intrinsic graph that is associated with a decomposition of \mathbb{G} as product of complementary homogeneous subgroups. A subgroup is homogeneous if it is invariant under group dilations and we say that \mathbb{G}_1 , \mathbb{G}_2 are *complementary subgroups* in \mathbb{G} if they are homogeneous subgroups and if any element $g \in \mathbb{G}$ can be uniquely written as product of one element of \mathbb{G}_1 and one of \mathbb{G}_2 (Definition 3.1).

The simple idea of intrinsic graph is the following one: let \mathbb{G}_1 , \mathbb{G}_2 be complementary subgroups of a group \mathbb{G} , then the intrinsic (left) graph of

$f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is the set

$$\text{graph}(f) = \{g \cdot f(g) : g \in \mathcal{A}\}.$$

More generally, we say that a subset S of a Carnot group \mathbb{G} , is a (*left*) *intrinsic graph*, in direction of a homogeneous subgroup \mathbb{H} , if S intersects each left coset of \mathbb{H} in at most a single point.

Intrinsic graphs appeared naturally while studying non critical level sets of differentiable functions from \mathbb{H}^n to \mathbb{R}^k . Indeed, implicit function theorems for groups ([16], [19], [17], [9], [10]) state precisely that these level sets are always, locally, intrinsic graphs.

We point that, if S is an intrinsic graph, then it keeps being an intrinsic graph after left translations or homogeneous automorphisms of the group (dilations in particular) and also that neither Euclidean graphs are necessarily intrinsic graphs nor the opposite.

Always in view of a study of rectifiable sets, we can give a notion of *intrinsic Lipschitz continuity* and of *intrinsic differentiability* for intrinsic graphs. Clearly, to respect the geometry of the ambient space \mathbb{G} , both notions have to be invariant under left translations and dilations of \mathbb{G} .

Definition. $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *intrinsic Lipschitz* (Definition 3.10) if, at each point $p \in \text{graph}(f)$, there is an *intrinsic cone* (Definition 3.9), with vertex p , axis \mathbb{G}_2 and fixed opening, intersecting $\text{graph}(f)$ only at p .

Definition. $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *intrinsic differentiable at* $g \in \mathbb{G}_1$ if there is a homogeneous subgroup \mathbb{T}_g of \mathbb{G} such that, in $p = g \cdot f(g) \in \text{graph}(f)$, the left coset $p \cdot \mathbb{T}_g$ is the *tangent plane* to $\text{graph}(f)$ in p (Definition 3.15).

Notice that both notions are really intrinsic, indeed they are invariant under left translations of the graph (Propositions 3.11), i.e. if we left translate an intrinsic Lipschitz graph we obtain an intrinsic Lipschitz graph; an appropriate similar statement holds for intrinsic differentiability.

The core of this paper is the study of these two notions and their relationship when the ambient group \mathbb{G} is the Heisenberg group \mathbb{H}^n (identified through the exponential map with \mathbb{R}^{2n+1}).

Coming to the paper, in Section 4.2 we prove that the graph of an intrinsic Lipschitz function is a controlled deformation (in measure) of the domain of the function. That is the intrinsic Hausdorff dimension of the graph is the same as the intrinsic dimension of the projection of the graph on the first factor \mathbb{G}_1 , even if projections are not Lipschitz maps (Remark 4.2). As a consequence, we prove that 1-codimensional intrinsic Lipschitz graphs are boundaries of sets of locally finite \mathbb{H} -perimeter ([21] and [16]).

In Section 4.3 we prove two geometric characterizations of intrinsic differentiability, the first one in terms of blow ups and the second one in terms of intrinsic cones.

In Section 4.4 we restrict ourselves to 1-codimensional graphs (i.e. to functions valued in 1-dimensional subgroups). We prove an extension theorem for intrinsic Lipschitz functions (Theorem 4.25) as well as our main result, a Rademacher type theorem (Theorem 4.29), that reads as follows:

Theorem. *If \mathbb{W}, \mathbb{V} are complementary subgroups of \mathbb{H}^n with \mathbb{V} 1-dimensional, if $\mathcal{U} \subset \mathbb{W}$ is relatively open and $f : \mathcal{U} \rightarrow \mathbb{V}$ is intrinsic Lipschitz, then f is intrinsic differentiable in \mathcal{U} , almost everywhere with respect to the Haar measure of \mathbb{W} .*

As a straightforward consequence of the previous theorems, in Section 4.5, we show that, at least in codimension 1, one can equivalently define rectifiable sets as countable union of intrinsic Lipschitz graphs or of intrinsic differentiable graphs.

Several comments are now in order.

- The restriction to 1-codimensional graphs, for Rademacher theorem, depends on the structure of our proof, since it relies on the theory of sets of finite \mathbb{H} -perimeter, that by their own nature are 1-codimensional. However, we point out that, in general, passing from 1-codimension to larger codimension is not as immediate as in Euclidean spaces (where it is possible to work "by components"); indeed it is neither true that a function is intrinsic Lipschitz if its components are intrinsic Lipschitz, nor the opposite. This depends on the non commutativity of the group.
- Since \mathbb{G} is a metric space, also \mathbb{G}_1 and \mathbb{G}_2 carry an induced metric. Hence it is possible to say that $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$, is *metric Lipschitz continuous* when there is $L > 0$ such that, for all $g_1, g_2 \in \mathbb{G}_1$

$$d(f(g_1), f(g_2)) \leq Ld(g_1, g_2).$$

This notion is apparently more intuitive than intrinsic Lipschitz continuity. Nevertheless intrinsic Lipschitz functions fit better the structure of the group \mathbb{G} . From one side, the notion of metric Lipschitz is not invariant under left translations of the graph; on the other side, intrinsic C^1 submanifolds in Heisenberg groups, the so called \mathbb{H} -regular surfaces (see [19]), are known to be, locally, intrinsic Lipschitz graphs, but they are not, in general, metric Lipschitz graphs.

- In his seminal paper [32], Pansu proved a Rademacher theorem for (metric) Lipschitz functions between Carnot groups. Our result is not contained in Pansu theorem because the notions of intrinsic Lipschitz continuity and differentiability do not coincide with metric Lipschitz continuity and P-differentiability. Notice that only if \mathbb{G} is isomorphic with the Cartesian product $\mathbb{G}_1 \times \mathbb{G}_2$ we have that metric and Lipschitz continuity coincide as well as P-differentiability and intrinsic differentiability.
- Recently, Cheeger introduced a new approach to Rademacher theorem in metric spaces (see [7], [24] and the references therein). The functions we consider, as we said repeatedly, are not metric Lipschitz. Nevertheless, given any $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ it is possible to build a new distance d_f on \mathbb{G}_1 , non equivalent with the Carnot Carathéodory metric, such that $f : (\mathbb{G}_1, d_f) \rightarrow \mathbb{G}_2$ is (metric) Lipschitz. The d_f distance of $g_1, g_2 \in \mathbb{G}_1$ is the distance of the corresponding points on graph (f) , i.e. $d_f(g_1, g_2) = d(g_1 \cdot f(g_1), g_2 \cdot f(g_2))$. Unfortunately, d_f , at least for $\mathbb{G} = \mathbb{H}^1$, does not support, in general, a Poincaré inequality. Eventually, we have not been able to get any positive result on this line.

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2. DEFINITIONS AND NOTATIONS

2.0.1. *Carnot groups.* A *graded group* of step k is a connected, simply connected Lie group \mathbb{G} whose Lie algebra \mathfrak{g} , of dimension N , is the direct sum of k subspaces \mathfrak{g}_i , $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for $1 \leq i, j \leq k$ and $\mathfrak{g}_i = 0$ for $i > k$.

A *Carnot group* \mathbb{G} of step k is a graded group of step k , where \mathfrak{g}_1 generates all \mathfrak{g} . That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, k$.

The exponential map is a one to one diffeomorphism from \mathfrak{g} to \mathbb{G} . Let X_1, \dots, X_N be a base for \mathfrak{g} such that X_1, \dots, X_{m_1} is a base for \mathfrak{g}_1 and, for $1 < j \leq k$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for \mathfrak{g}_j . Then any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1 X_1 + \cdots + p_N X_N)$ and we can identify p with the N -tuple $(p_1, \dots, p_N) \in \mathbb{R}^N$ and \mathbb{G} with (\mathbb{R}^N, \cdot) . The explicit expression of the group operation \cdot , determined by the Campbell-Hausdorff formula (see [6] or [15]), has the form

$$(1) \quad p \cdot q = p + q + Q(p, q), \quad \text{for all } p, q \in \mathbb{R}^N \equiv \mathbb{G}$$

where $Q(p, q) = (Q_1(p, q), \dots, Q_N(p, q)) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$. Here, $Q_i(p, q) = 0$, for $i = 1, \dots, m_1$ and, for $1 < j \leq k$ and $m_{j-1} + 1 \leq i \leq m_j$, we have, $Q_i(p, q) = Q_i(p_1, \dots, p_{m_{j-1}}, q_1, \dots, q_{m_{j-1}})$.

If $p \in \mathbb{G}$, $p^{-1} = (-p_1, \dots, -p_N)$ is the inverse of p and $e = (0, \dots, 0)$ is the identity of \mathbb{G} .

If \mathbb{G} is a graded group, for all $\lambda > 0$, the (*non isotropic*) *dilations* $\delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$ are automorphisms of \mathbb{G} defined as $\delta_\lambda(x_1, \dots, x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \dots, \lambda^{\alpha_N} x_N)$, where $\alpha_i = j$, if $m_{j-1} < i \leq m_j$.

Given any homogeneous norm $\|\cdot\|$, the distance in \mathbb{G} is defined as

$$(2) \quad d(x, y) = d(y^{-1} \cdot x, 0) = \|y^{-1} \cdot x\|, \quad \text{for all } x, y \in \mathbb{G}.$$

Such a distance d is always comparable with the Carnot-Carathéodory distance of \mathbb{G} and is left translation invariant and 1-homogeneous, that is, for $x, y, z \in \mathbb{G}$ and $\lambda > 0$,

$$(3) \quad d(z \cdot x, z \cdot y) = d(x, y) \quad , \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y).$$

For $r \geq 0$ and $p \in \mathbb{G}$, we denote by $U(p, r)$ and by $B(p, r)$ the open and the closed ball associated with d .

From distance d , (spherical) Hausdorff measures are obtained following Carathéodory's construction as in [14] Section 2.10.2. Precisely, if $m \geq 0$, \mathcal{S}_d^m is the (*intrinsic spherical*) *Hausdorff measure* defined, for $\mathcal{A} \subset \mathbb{G}$, as

$$\mathcal{S}_d^m(\mathcal{A}) := \lim_{\delta \rightarrow 0} \mathcal{S}_{d,\delta}^m(\mathcal{A}),$$

where $\mathcal{S}_{d,\delta}^m(\mathcal{A}) = \inf \{ \sum_i r_i^m : \mathcal{A} \subset \bigcup_i B(p_i, r_i), r_i \leq \delta \}$.

Translation invariance and dilation homogeneity of Hausdorff measures follow from (3) and, for $\mathcal{A} \subseteq \mathbb{G}$, $p \in \mathbb{G}$ and $r \in [0, \infty)$,

$$\mathcal{S}_d^m(p \cdot \mathcal{A}) = \mathcal{S}_d^m(\mathcal{A}) \quad \text{and} \quad \mathcal{S}_d^m(\delta_r \mathcal{A}) = r^m \mathcal{S}_d^m(\mathcal{A}).$$

A *homogeneous subgroup* of a Carnot group \mathbb{G} (see [35] 5.2.4) is a subgroup \mathbb{H} such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda > 0$.

The (*linear*) *dimension* of a (sub)group is the dimension of its Lie algebra.

The *metric dimension* of a subgroup, or of a subset, is its intrinsic Hausdorff dimension.

2.0.2. *Heisenberg groups.* \mathbb{H}^n is the n -dimensional Heisenberg group, identified with \mathbb{R}^{2n+1} through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = (p' + q', p_{2n+1} + q_{2n+1} - \frac{1}{2}\langle Jp', q' \rangle_{\mathbb{R}^{2n}})$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $(2n \times 2n)$ -symplectic matrix.

For any $q \in \mathbb{H}^n$ and for any $r > 0$ left translations $\tau_q : \mathbb{H}^n \rightarrow \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are defined as

$$\tau_q(p) := q \cdot p \quad \text{and as} \quad \delta_r p := (rp', r^2 p_{2n+1}).$$

We denote as \mathfrak{h} the Lie algebra of \mathbb{H}^n . The standard basis of \mathfrak{h} is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i - \frac{1}{2}(Jp')_i \partial_{2n+1}, \quad Y_i := \partial_{i+n} + \frac{1}{2}(Jp')_{i+n} \partial_{2n+1}, \quad T := \partial_{2n+1}.$$

The *horizontal subspace* \mathfrak{h}_1 is the subspace of \mathfrak{h} spanned by X_1, \dots, X_n and Y_1, \dots, Y_n . Denoting by \mathfrak{h}_2 the linear span of T , the 2-step stratification of \mathfrak{h} is expressed by

$$(4) \quad \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The Lie algebra \mathfrak{h} is also endowed with a scalar product $\langle \cdot, \cdot \rangle$ making the vector fields X_1, \dots, X_n and Y_1, \dots, Y_n and T orthonormal. Thus (4) turns out to be an orthonormal decomposition of \mathfrak{h} as a vector space.

The centre of \mathbb{H}^n is the subgroup $\mathbb{T} := \exp(\mathfrak{h}_2) = \{(0, \dots, 0, p_{2n+1})\}$.

The *horizontal bundle* $H\mathbb{H}^n$ is the subbundle of the tangent bundle $T\mathbb{H}^n$ whose fibers $H\mathbb{H}_p^n$ are spanned by the horizontal vectors $X_1(p), \dots, Y_n(p)$. The scalar product $\langle \cdot, \cdot \rangle$ induces naturally on each fiber $H\mathbb{H}_p^n$ a scalar product here denoted as $\langle \cdot, \cdot \rangle_p$.

If $p \in \mathbb{H}^n$, we indicate as $\|p\|$ its Koranyi norm, i.e.

$$\|p\| = \sqrt[4]{\|p'\|_{\mathbb{R}^{2n}}^4 + |p_{2n+1}|^2}$$

and by $d(\cdot, \cdot)$ the associated distance defined as in (2).

Because the topologies induced by d and by the Euclidean distance coincide, the topological dimension of \mathbb{H}^n is $2n + 1$. On the contrary, the metric dimension of \mathbb{H}^n is $2n + 2$ ([31]).

Let $\pi : \mathbb{H}^n \rightarrow \mathbb{R}^{2n}$ be $\pi(p) = \pi(p_1, \dots, p_{2n}, p_{2n+1}) := (p_1, \dots, p_{2n})$. Notice that any $p \in \mathbb{H}^n$ can be uniquely written as $p = (\pi(p), p_{2n+1}) = (\pi(p), 0) \cdot p_{\mathbb{T}}$, where $p_{\mathbb{T}} = (0, \dots, 0, p_{2n+1}) \in \mathbb{T}$ and $(\pi(p), 0) \in H\mathbb{H}_e^n$.

If k is nonnegative integer, \mathcal{L}^k is the k -dimensional Lebesgue measure. Then, \mathcal{L}^{2n+1} is the bi-invariant Haar measure of \mathbb{H}^n , i.e., if $\mathcal{E} \subset \mathbb{R}^{2n+1}$ is measurable, then $\mathcal{L}^{2n+1}(p \cdot \mathcal{E}) = \mathcal{L}^{2n+1}(\mathcal{E})$, for all $p \in \mathbb{H}^n$. Moreover, if $\lambda > 0$ then $\mathcal{L}^{2n+1}(\delta_\lambda(\mathcal{E})) = \lambda^{2n+2} \mathcal{L}^{2n+1}(\mathcal{E})$. In particular, for all $p \in \mathbb{H}^n$ and $r > 0$,

$$\mathcal{L}^{2n+1}(B(p, r)) = r^{2n+2} \mathcal{L}^{2n+1}(B(p, 1)) = r^{2n+2} \mathcal{L}^{2n+1}(B(e, 1)).$$

2.0.3. *Calculus.* The notion of *P-differentiability* for functions acting between graded groups, was introduced by Pansu in [32].

Definition 2.1. Let \mathbb{G}_1 and \mathbb{G}_2 be graded groups, with homogeneous norms $\|\cdot\|_1, \|\cdot\|_2$ and dilations $\delta_\lambda^1, \delta_\lambda^2$, then $L : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is said to be *H-linear* (see [27]), if L is a *homogeneous homomorphism*, that is if L is a group homomorphism from \mathbb{G}_1 to \mathbb{G}_2 and if,

$$L(\delta_\lambda^1 g) = \delta_\lambda^2 L(g), \quad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0.$$

The norm of L is $\|L\| = \sup\{\|L(g)\|_2 : \|g\|_1 \leq 1\}$.

Definition 2.2. Let \mathbb{G}_1 and \mathbb{G}_2 be graded groups with homogeneous norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *P-differentiable* in $g_0 \in \mathcal{A}$ if there is a *H-linear* function $df_{g_0} : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that

$$\|(df_{g_0}(g_0^{-1} \cdot g))^{-1} \cdot f(g_0)^{-1} \cdot f(g)\|_2 = o(\|g_0^{-1} \cdot g\|_1), \quad \text{as } \|g_0^{-1} \cdot g\|_1 \rightarrow 0.$$

The *H-linear* function df_{g_0} is the *P-differential* of f in g_0 . We say that f is *continuously P-differentiable* in \mathcal{A} , $f \in C_H^1(\mathcal{A}, \mathbb{G}_2)$, if f is P-differentiable at every $g \in \mathcal{A}$ and if df_g depends continuously on g .

3. INTRINSIC GRAPHS

3.1. Complementary subgroups and graphs.

Definition 3.1. Two homogeneous subgroups \mathbb{G}_1 and \mathbb{G}_2 of a Carnot group \mathbb{G} are *complementary subgroups* in \mathbb{G} , and we write

$$\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2,$$

if $\mathbb{G}_1 \cap \mathbb{G}_2 = \{e\}$ and, for each $g \in \mathbb{G}$, there are $g_{\mathbb{G}_1} \in \mathbb{G}_1$ and $g_{\mathbb{G}_2} \in \mathbb{G}_2$ such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}$.

If $\mathbb{G}_1, \mathbb{G}_2$ are complementary subgroups in \mathbb{G} and one of them is a normal subgroup then \mathbb{G} is said to be a *semidirect product* of \mathbb{G}_1 and \mathbb{G}_2 .

We have (see [4])

Proposition 3.2. *If $\mathbb{G}_1, \mathbb{G}_2$ are complementary subgroups in \mathbb{G} , then each $g \in \mathbb{G}$ has unique components $g_{\mathbb{G}_1} \in \mathbb{G}_1, g_{\mathbb{G}_2} \in \mathbb{G}_2$, such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}$. The maps*

$$g \rightarrow g_{\mathbb{G}_1} \text{ and } g \rightarrow g_{\mathbb{G}_2}$$

are continuous and there is a constant $c = c(\mathbb{G}_1, \mathbb{G}_2) > 0$ such that

$$(5) \quad c(\|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|) \leq \|g\| \leq \|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|.$$

Definition 3.3. If $\mathbb{G}_1, \mathbb{G}_2$ are complementary subgroups of \mathbb{G} we say that $S \subset \mathbb{G}$ is a *(left) graph from \mathbb{G}_1 to \mathbb{G}_2* (or over \mathbb{G}_1 along \mathbb{G}_2) if

$$S \cap (\xi \cdot \mathbb{G}_2) \text{ contains at most one point,}$$

for all $\xi \in \mathbb{G}_1$. Equivalently, if there is $f : \mathcal{E} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that

$$S = \{\xi \cdot f(\xi) : \xi \in \mathcal{E}\}$$

and we say that S is the *graph of f* , $S = \text{graph}(f)$.

Remark 3.4. A more general definition of graph inside \mathbb{G} can be considered. Assume that \mathbb{H} is a homogeneous subgroup of \mathbb{G} . Even if no complementary subgroup of \mathbb{H} exists in \mathbb{G} , we can say that a set $S \subset \mathbb{G}$ is a \mathbb{H} -graph (or a graph along \mathbb{H}) if S intersect each coset of \mathbb{H} in at most one point. Such a notion has been used many times in the literature, mainly inside \mathbb{H}^n . Many authors e.g. considered sets as $S = \{(x_1, \dots, y_n, \varphi(x_1, \dots, y_n))\} \subset \mathbb{H}^n$, were φ is a real valued function. These sets are \mathbb{T} -graphs. But we recall that \mathbb{T} has no complementary subgroup in \mathbb{H}^n (see the comments after Proposition 4.1).

If $S = \text{graph}(f)$ with $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ then both $\delta_\lambda S$ and $q \cdot S$ are graphs from \mathbb{G}_1 to \mathbb{G}_2 . The proof of the following proposition is straightforward.

Proposition 3.5. *Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} , $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ and $S = \text{graph}(f)$. Then, for all $\lambda > 0$, the dilated set $\delta_\lambda S$ is a graph, precisely*

$$\delta_\lambda S = \text{graph}(f_\lambda),$$

with $f_\lambda := \delta_\lambda \circ f \circ \delta_{1/\lambda} : \delta_\lambda \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$.

Proposition 3.6. *Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} , $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ and $S = \text{graph}(f)$. Then, for any $q \in \mathbb{G}$, there are $\mathcal{A}_q \subset \mathbb{G}_1$ and $f_q : \mathcal{A}_q \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$, such that*

$$q \cdot S = \text{graph}(f_q) = \{\eta \cdot f_q(\eta) : \eta \in \mathcal{A}_q\}.$$

Proof. As in [4], observe that the map $\tau_q : \mathbb{G}_1 \rightarrow \mathbb{G}_1$ defined as $\tau_q(x) := (q \cdot x)_{\mathbb{G}_1}$ is injective. Indeed, from

$$q \cdot x = (q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2}, \quad q \cdot x' = (q \cdot x')_{\mathbb{G}_1} \cdot (q \cdot x')_{\mathbb{G}_2}, \quad (q \cdot x)_{\mathbb{G}_1} = (q \cdot x')_{\mathbb{G}_1}$$

we get $q \cdot x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = q \cdot x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$. Hence $x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$ and finally $x = x'$, because of the uniqueness of the components. Hence,

$$q \cdot S = \{(q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2} \cdot \varphi(x) : x \in \mathcal{A}\} = \{y \cdot \varphi_q(y) : y \in \mathcal{E}_q\},$$

where, $\mathcal{A}_q = \{(q \cdot x)_{\mathbb{G}_1} : x \in \mathcal{A}\}$ and, for $y = (q \cdot x)_{\mathbb{G}_1} \in \mathcal{A}_q$,

$$(6) \quad \varphi_q(y) = (q \cdot \tau_q(y)^{-1})_{\mathbb{G}_2} \cdot \varphi(\tau_q(y)^{-1}).$$

□

Observe that, if \mathbb{G} is the semidirect product of \mathbb{G}_1 and \mathbb{G}_2 , then the algebraic form of the translated graph can be explicitly given (see the following Proposition 4.5).

Remark 3.7. Let $S = \text{graph}(f) \subset \mathbb{G}$, where (say) $f : \mathcal{E} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$. Then, for $p, q \in \mathbb{G}$,

$$(7) \quad (f_p)_q = f_{q \cdot p}.$$

Indeed, $\text{graph}((f_p)_q) = q \cdot \text{graph}(f_p) = q \cdot (p \cdot \text{graph}(f)) = q \cdot p \cdot \text{graph}(f)$.

Given that $\mathbb{G}_1, \mathbb{G}_2$ are metric spaces, continuous functions from \mathbb{G}_1 to \mathbb{G}_2 are defined as usual.

Proposition 3.8. *If $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is continuous then any translated function $f_q : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is continuous.*

Proof. Given (6), it is enough to prove that the map from \mathbb{G}_1 to \mathbb{G}_1 given by $y \mapsto (\tau_q(y))^{-1}$ is continuous. This fact follows from the – so called – invariance of domain theorem (see e.g. [26]). \square

3.2. Intrinsic Lipschitz graphs. The notion of *intrinsic Lipschitz continuity*, for functions acting between complementary subgroups \mathbb{G}_1 and \mathbb{G}_2 of \mathbb{G} , was originally suggested by Corollary 3.17 of [19] (see the definition given in [20]).

We propose here a geometric definition. We say that $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$, is intrinsic Lipschitz continuous if, at each $p \in \text{graph}(f)$, there is an (intrinsic) closed cone with vertex p , axis \mathbb{G}_2 and fixed opening, intersecting $\text{graph}(f)$ only in p .

Definition 3.9. Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} , $q \in \mathbb{G}$ and $\alpha > 0$. The intrinsic closed cone $C_{\mathbb{G}_1, \mathbb{G}_2}(q, \alpha)$ with base \mathbb{G}_1 , axis \mathbb{G}_2 , vertex q , opening α is

$$C_{\mathbb{G}_1, \mathbb{G}_2}(q, \alpha) := q \cdot C_{\mathbb{G}_1, \mathbb{G}_2}(e, \alpha)$$

where

$$C_{\mathbb{G}_1, \mathbb{G}_2}(e, \alpha) := \{p \in \mathbb{G} : \|p_{\mathbb{G}_1}\| \leq \alpha \|p_{\mathbb{G}_2}\|\}.$$

If $0 < \alpha < \beta$, $C_{\mathbb{G}_1, \mathbb{G}_2}(q, \alpha) \subset C_{\mathbb{G}_1, \mathbb{G}_2}(q, \beta)$ and $C_{\mathbb{G}_1, \mathbb{G}_2}(e, 0) = \mathbb{G}_2$, moreover, for all $t > 0$,

$$\delta_t(C_{\mathbb{G}_1, \mathbb{G}_2}(e, \alpha)) = C_{\mathbb{G}_1, \mathbb{G}_2}(e, \alpha).$$

Definition 3.10. Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} . We say that $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is *intrinsic Lipschitz (continuous)* in \mathcal{A} , if there is $M > 0$ such that, for all $q \in \text{graph}(f)$,

$$(8) \quad C_{\mathbb{G}_1, \mathbb{G}_2}(q, 1/M) \cap \text{graph}(f) = \{q\}.$$

The Lipschitz constant of f in \mathcal{A} is the infimum of the numbers M such that (8) holds. An intrinsic Lipschitz function, with Lipschitz constant not exceeding $L > 0$, is called a *L-Lipschitz function*.

The following Proposition is easy to prove, nevertheless contains one of the crucial reasons for our notion of intrinsic Lipschitz functions.

Proposition 3.11. *For all $q \in \mathbb{G}$, $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is intrinsic L-Lipschitz in \mathcal{A} if and only if $f_q : \mathcal{A}_q \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is intrinsic L-Lipschitz in \mathcal{A}_q .*

Proof. By definition, $\text{graph}(f_q) = q \cdot \text{graph}(f)$. Hence, $p \in \text{graph}(f_q)$ if and only if $p = q \cdot \bar{p}$ for $\bar{p} \in \text{graph}(f)$. Then, if f is intrinsic Lipschitz

$$\{p\} = \{q \cdot \bar{p}\} = q \cdot (C_{\mathbb{G}_1, \mathbb{G}_2}(\bar{p}, \alpha) \cap \text{graph}(f)) = C_{\mathbb{G}_1, \mathbb{G}_2}(p, \alpha) \cap \text{graph}(f_q),$$

hence, f_q is intrinsic Lipschitz. The inverse deduction follows from (7). \square

An equivalent algebraic characterizations of intrinsic Lipschitz functions is given by the following

Proposition 3.12. *Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} . Then $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is intrinsic Lipschitz in \mathcal{A} , if and only if there is $L > 0$ such that, for all $q \in \text{graph}(f)$ and for all $x \in \mathcal{A}_{q^{-1}}$,*

$$(9) \quad \|f_{q^{-1}}(x)\| \leq L \|x\|.$$

Notice that intrinsic Lipschitz functions are continuous. Indeed, if $f(e) = e$ then, by (9), f is continuous in e . To prove the continuity in $x \in \mathcal{E}$, observe that $f_{q^{-1}}$ is continuous in e , where $q = x \cdot f(x)$.

Remark 3.13. Since \mathbb{G}_1 and \mathbb{G}_2 are metric spaces, being subsets of \mathbb{G} , it makes sense to speak also of *metric Lipschitz continuous* functions from \mathbb{G}_1 to \mathbb{G}_2 . As usual, $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is (metric) Lipschitz if there is $L > 0$ such that, for all $g, \bar{g} \in \mathbb{G}_1$,

$$(10) \quad \|f(g)^{-1} \cdot f(\bar{g})\| = d(f(g), f(\bar{g})) \leq L d(g, \bar{g}) = L \|g^{-1} \cdot \bar{g}\|.$$

The notions of intrinsic Lipschitz continuity and of metric Lipschitz continuity are independent properties, as the following examples show. In particular, observe that (10) is not conserved after a left graph translations.

To see these facts, let $\mathbb{G} = \mathbb{H}^1 \cong \mathbb{R}^3$ and let \mathbb{V}, \mathbb{W} be the complementary subgroups defined as $\mathbb{V} = \{v = (v_1, 0, 0)\}$, $\mathbb{W} = \{w = (0, w_2, w_3)\}$

- (1) Let $\varphi : \mathbb{W} \rightarrow \mathbb{V}$, be defined as $\varphi(w) := (1 + |w_3|^{1/2}, 0, 0)$. Then φ satisfies (10) with $L = 1$, hence φ is metric Lipschitz. On the contrary φ is not intrinsic Lipschitz. Indeed, let $p := (1, 0, 0) \in \text{graph}(\varphi)$, from Proposition 4.5 we have $\varphi_{p^{-1}}(w) = (|w_2 + w_3|^{1/2}, 0, 0)$. Hence (9) does not hold for $\varphi_{p^{-1}}$. This shows that (10) is not invariant under graph translations.
- (2) Let $\psi : \mathbb{W} \rightarrow \mathbb{V}$ be defined as $\psi(w) := (1 + |w_3 - w_2|^{1/2}, 0, 0)$. ψ is intrinsic Lipschitz, indeed, with $p = (1, 0, 0)$ and $\varphi(w) := (|w_3|^{1/2}, 0, 0)$ we have $\psi(w) = \varphi_p(w)$, so that ψ is intrinsic Lipschitz because φ is intrinsic Lipschitz. On the contrary ψ is not metric Lipschitz, in the sense of (10), as can be easily observed.

Analogously, it can be checked that

- (1) the constant function $\varphi : \mathbb{V} \rightarrow \mathbb{W}$ defined as $\varphi(v) := (0, 1, 0)$, for all $v \in \mathbb{V}$, is metric Lipschitz, but it is not intrinsic Lipschitz;
- (2) $\psi : \mathbb{V} \rightarrow \mathbb{W}$ defined as $\psi(v) := (0, 1, -v_1)$ for all $v \in \mathbb{V}$, is intrinsic Lipschitz but it is not metric Lipschitz.

3.3. Intrinsic differentiable graphs. We say that a function acting between complementary subgroups is intrinsic differentiable in a point if its graphs admits, in the corresponding point, a tangent subgroup. This is the content of the two following definitions.

Definition 3.14. Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} . We say that $L : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is an *intrinsic linear function* if $\text{graph}(L)$ is an homogeneous subgroup of \mathbb{G} .

Definition 3.15. Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in \mathbb{G} and $f : \mathcal{A} \subset \mathbb{G}_1 \rightarrow \mathbb{G}_2$ with \mathcal{A} relatively open in \mathbb{G}_1 . For $\bar{p} := \bar{g} \cdot f(\bar{g})$ we consider the function $f_{\bar{p}^{-1}}$ defined in the neighborhood $\mathcal{A}_{\bar{p}^{-1}}$ of e in \mathbb{G}_1 , (see Proposition

3.5). We say that f is *intrinsic differentiable* in $\bar{g} \in \mathcal{A}$ if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that, for all $g \in \mathcal{A}_{\bar{g}^{-1}}$,

$$(11) \quad \|df_{\bar{g}}(g)^{-1} \cdot f_{\bar{g}^{-1}}(g)\| = o(\|g\|), \quad \text{as } \|g\| \rightarrow 0,$$

where $o(t)/t \rightarrow 0$ as $t \rightarrow 0^+$. The map $df_{\bar{g}}$ is called the *intrinsic differential* of f .

Intrinsic differentiability is invariant by left translations of the graph. Indeed, let $q_1 = g_1 \cdot f(g_1)$ and $q_2 = g_2 \cdot f(g_2) \in \text{graph}(f)$; then f is intrinsic differentiable in $g_1 \in \mathbb{G}_1$ if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e . Consequently, f is intrinsic differentiable in g_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv (f_{q_1^{-1}})_{q_2}$ is intrinsic differentiable in g_2 .

Remark 3.16. If $f(e) = e$ then f is intrinsic differentiable in e if there is an intrinsic linear map $df_e : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ such that, for all $g \in \mathbb{G}_1$,

$$(12) \quad \|df_e(g)^{-1} \cdot f(g)\| = o(\|g\|), \quad \text{as } \|g\| \rightarrow 0.$$

In this case, intrinsic differentiability looks like P-differentiability, but, in general, P-differentiability and intrinsic differentiability are *different*.

As an example, let $\mathbb{G} = \mathbb{H}^n$, $\mathbb{G}_1 = \mathbb{W} = \{w = (0, p_2, \dots, p_{2n+1})\}$ and $\mathbb{G}_2 = \mathbb{V} = \{v = (p_1, 0, \dots, 0)\}$. Then, by (ii) of Proposition 4.13 and keeping in mind that, in this case, df_w is intrinsic linear if and only if it is H -linear, we get that $f : \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic differentiable in $w \in \mathbb{W}$ if, for all $\bar{w} \in \mathbb{W}$,

$$\|df_w(w^{-1} \cdot \bar{w})^{-1} \cdot f(w)^{-1} \cdot f(\bar{w})\| = o(\|f(w)^{-1} \cdot w^{-1} \cdot \bar{w} \cdot f(w)\|);$$

while f is P-differentiable in $w \in \mathbb{W}$ if, for all $\bar{w} \in \mathbb{W}$,

$$\|df_w(w^{-1} \cdot \bar{w})^{-1} \cdot f(w)^{-1} \cdot f(\bar{w})\| = o(\|w^{-1} \cdot \bar{w}\|).$$

To understand better the difference, notice that P-differentiability of a function acting between two (sub)groups \mathbb{G}_1 and \mathbb{G}_2 depends only on the algebraic and metric structure of \mathbb{G}_1 and \mathbb{G}_2 . On the contrary, intrinsic differentiability depends not only on that, but also on the *interaction* of \mathbb{G}_1 and \mathbb{G}_2 as subgroups of the ambient group \mathbb{G} . In particular, it is easy to see that when $\mathbb{G} := \mathbb{G}_1 \times \mathbb{G}_2$, $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is P-differentiable if and only if $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is intrinsic differentiable.

An analogous observation holds about metric Lipschitz continuity and intrinsic Lipschitz continuity. The first one depends only on the metric structure of the two (sub)groups \mathbb{G}_1 , \mathbb{G}_2 , while the second one depends on the interaction of the two inside the ambient group \mathbb{G} .

This way, intrinsic Lipschitz continuity and intrinsic differentiability are generalizations of metric Lipschitz continuity and of P-differentiability.

4. GRAPHS IN HEISENBERG GROUPS

4.1. Complementary subgroups and graphs. In Heisenberg groups, all the notions, described in the preceding sections for general Carnot groups, can be made more explicit because we have more informations on the structure of complementary subgroups of \mathbb{H}^n . The simple proof of the following Proposition can be found in [4], (see also [28]).

Proposition 4.1. *All homogeneous subgroups of \mathbb{H}^n either are horizontal, i.e. contained in $\exp(\mathfrak{h}_1)$, or contain the subgroup \mathbb{T} . A horizontal subgroup has linear dimension and metric dimension k , with $1 \leq k \leq n$ and it is isomorphic and isometric to \mathbb{R}^k . The subgroups containing \mathbb{T} can have any dimension d , with $1 \leq d \leq 2n + 1$, their metric dimension is $d + 1$ and are normal subgroups.*

All couples \mathbb{V}, \mathbb{W} , of complementary subgroups of \mathbb{H}^n are of the type

- (i) \mathbb{V} horizontal of dimension k , $1 \leq k \leq n$,
- (ii) \mathbb{W} normal of dimension $2n + 1 - k$.

Notice that any horizontal subgroup \mathbb{V} has a complementary normal subgroup \mathbb{W} and, conversely, any normal subgroup \mathbb{W} , with linear dimension larger than n , has a horizontal complementary subgroup \mathbb{V} . On the contrary normal subgroups of dimension less or equal than n do not have complementary subgroups. In particular the centre \mathbb{T} does not have a complementary subgroup.

Remark 4.2. It is easy to check that, for all $p, q \in \mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$,

$$\begin{aligned} (q^{-1})_{\mathbb{V}} &= (q_{\mathbb{V}})^{-1}, & (q^{-1})_{\mathbb{W}} &= q_{\mathbb{V}}^{-1} \cdot (q_{\mathbb{W}})^{-1} \cdot q_{\mathbb{V}}; \\ (p \cdot q)_{\mathbb{V}} &= p_{\mathbb{V}} \cdot q_{\mathbb{V}}, & (p \cdot q)_{\mathbb{W}} &= p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1}. \end{aligned}$$

In particular, if we set

$$\Pi_{\mathbb{V}}(q) = q_{\mathbb{V}} \text{ and } \Pi_{\mathbb{W}}(q) = q_{\mathbb{W}}.$$

it follows from that $\Pi_{\mathbb{V}}$ is a group homomorphism from \mathbb{H}^n to \mathbb{V} while, in general, $\Pi_{\mathbb{W}}$ is *not* a group homomorphism from \mathbb{H}^n to \mathbb{W} .

Moreover observe that $\Pi_{\mathbb{V}}$ is a Lipschitz map from \mathbb{H}^n to \mathbb{V} , while, on the contrary, $\Pi_{\mathbb{W}}$ is not a Lipschitz map from \mathbb{H}^n to \mathbb{W} . Indeed, about $\Pi_{\mathbb{V}}$,

$$\|\Pi_{\mathbb{V}}(p)^{-1} \cdot \Pi_{\mathbb{V}}(q)\| = \|\Pi_{\mathbb{V}}(q) - \Pi_{\mathbb{V}}(p)\| \leq \|q' - p'\|_{\mathbb{R}^{2n}} \leq \|p^{-1} \cdot q\|;$$

about $\Pi_{\mathbb{W}}$, let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, as in (4.1), where \mathbb{V}, \mathbb{W} are the complementary subgroups defined as

$$\mathbb{V} = \{v = (v_1, 0, 0)\}, \quad \mathbb{W} = \{w = (0, w_2, w_3)\}.$$

For $\varepsilon > 0$, let $p = (1, 0, 0)$ and $q = (1, \varepsilon, \varepsilon/2)$, then $p_{\mathbb{W}} = (0, 0, 0)$, $q_{\mathbb{V}} = (1, 0, 0)$ and $q_{\mathbb{W}} = (0, \varepsilon, \varepsilon)$. Then

$$\begin{aligned} \|\Pi_{\mathbb{W}}(q)^{-1} \cdot \Pi_{\mathbb{W}}(p)\| &= \|q_{\mathbb{W}}\| \asymp \varepsilon^{1/2}, \\ \|p^{-1} \cdot q\| &= \|(0, \varepsilon, 0)\| \asymp \varepsilon. \end{aligned}$$

Even if $\Pi_{\mathbb{W}}$ is not Lipschitz, nevertheless (see also [25]) we have the following control on the measure of projection.

Lemma 4.3. *Let \mathbb{W}, \mathbb{V} be complementary subgroups of \mathbb{H}^n , as in Proposition 4.1, with \mathbb{V} k -dimensional. Then there is $c = c(\mathbb{V}, \mathbb{W}) > 0$ such that, for all balls $B(p, r) \subset \mathbb{H}^n$,*

$$(13) \quad \mathcal{L}^{2n+1-k}(\Pi_{\mathbb{W}}(B(p, r))) = c(\mathbb{V}, \mathbb{W})r^{2n+2-k}.$$

Proof. Define

$$c(\mathbb{V}, \mathbb{W}) := \mathcal{L}^{2n+1-k}(B(e, 1)),$$

and observe that $c(\mathbb{V}, \mathbb{W}) > 0$, because $\Pi_{\mathbb{W}}$ is an open mapping.

By group dilations $\Pi_{\mathbb{W}}(B(e, r)) = \Pi_{\mathbb{W}}(\delta_r B(e, 1)) = \delta_r \Pi_{\mathbb{W}}(B(e, 1))$ hence

$$\mathcal{L}^{2n+1-k}(\Pi_{\mathbb{W}}(B(e, r))) = c(\mathbb{V}, \mathbb{W})r^{2n+2-k}.$$

Now observe that for any measurable $\mathcal{E} \subset \mathbb{H}^n$

$$(14) \quad \mathcal{L}^{2n+1-k}(\Pi_{\mathbb{W}}(p \cdot \mathcal{E})) = \mathcal{L}^{2n+1-k}(\Pi_{\mathbb{W}}(\mathcal{E})).$$

Indeed, since $\Pi_{\mathbb{W}}(p \cdot \mathcal{E}) = \{p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1} : q \in \mathcal{E}\} = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot \Pi_{\mathbb{W}}(\mathcal{E}) \cdot p_{\mathbb{V}}^{-1}$, then the (linear) map $w \mapsto p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot w \cdot p_{\mathbb{V}}^{-1}$, from $\mathbb{W} \rightarrow \mathbb{W}$, has determinant equal to 1, hence (14) follows and the proof is completed. \square

Lemma 4.4. *Assume that \mathbb{G} is a step 2 group and $p, q \in \mathbb{G}$, then there is $c = c(\mathbb{G})$ such that*

$$(15) \quad \|p^{-1} \cdot q^{-1} \cdot p \cdot q\| \leq c \|p\|^{1/2} \|q\|^{1/2}.$$

From (15) it follows

$$(16) \quad \|q^{-1} \cdot p \cdot q\| \leq \|p\| + c \|p\|^{1/2} \|q\|^{1/2}$$

Proof. We identify \mathbb{G} with \mathbb{R}^n through exponential coordinates, so that $p \cdot q = p + q + Q(p, q)$.

$$\begin{aligned} p^{-1} \cdot q^{-1} \cdot p \cdot q &= (-p - q + Q(-p, -q)) \cdot (p + q + Q(p, q)) \\ &= Q(-p, -q) + Q(p, q) + Q(-p - q + Q(-p, -q), p + q + Q(p, q)) \end{aligned}$$

Notice that, in step 2 groups,

$$Q(-p - q + Q(-p, -q), p + q + Q(p, q)) = Q(-p - q, p + q) = 0.$$

Indeed, the polynomials Q_j depend only on the variables p_1, \dots, p_{m_1} of the first layer and $Q_1 = \dots = Q_{m_1} = 0$; moreover the Q_j s are antisymmetric.

Finally, Q_{m_1+1}, \dots, Q_n are homogeneous polynomials of degree 2, containing only terms as $p_h q_k$ for $1 \leq h, k \leq m_1$, so that they can be estimated with $\|p\|^{1/2} \|q\|^{1/2}$. This proves (15). To conclude notice that

$$\|q^{-1} \cdot p \cdot q\| \leq \|p\| + \|p^{-1} \cdot q^{-1} \cdot p \cdot q\| \leq \|p\| + c \|p\|^{1/2} \|q\|^{1/2},$$

hence (16) follows from (15). \square

Proposition 4.5. *If \mathbb{W}, \mathbb{V} are complementary subgroups of \mathbb{H}^n , as in Proposition 4.1, and if $S \subset \mathbb{H}^n$ is a (left) graph, then, for any $q = q_{\mathbb{W}} \cdot q_{\mathbb{V}}$, the translated set $q \cdot S$ is again a graph. Precisely*

- (i) *If $S = \{w \cdot f(w) : w \in \mathcal{E}\}$, with $f : \mathcal{E} \subset \mathbb{W} \rightarrow \mathbb{V}$, then $q \cdot S = \{\eta \cdot f_q(\eta) : \eta \in \mathcal{E}_q \subset \mathbb{W}\}$, with $f_q : \mathcal{E}_q \rightarrow \mathbb{V}$ and*

$$\begin{aligned} \mathcal{E}_q &= q \cdot \mathcal{E} \cdot (q_{\mathbb{V}})^{-1}, \\ f_q(\eta) &= q_{\mathbb{V}} \cdot f(q_{\mathbb{W}}^{-1} \cdot q_{\mathbb{V}}^{-1} \cdot \eta \cdot q_{\mathbb{V}}) \end{aligned}$$

- (ii) If $S = \{v \cdot f(v) : v \in \mathcal{F}\}$, with $f : \mathcal{F} \subset \mathbb{V} \rightarrow \mathbb{W}$, then $q \cdot S = \{\eta \cdot f_q(\eta) : \eta \in \mathcal{F}_q \subset \mathbb{V}\}$ with $f_q : \mathcal{F}_q \rightarrow \mathbb{W}$ and

$$\mathcal{F}_q = q_{\mathbb{V}} \cdot \mathcal{F},$$

$$f_q(\eta) = \eta^{-1} \cdot q_{\mathbb{W}} \cdot \eta \cdot f(q_{\mathbb{V}}^{-1} \cdot \eta).$$

4.2. Lipschitz graphs and surface measure of Lipschitz graphs. In Heisenberg groups we can make characterization (9) more explicit, using the algebraic forms of the translated functions f_q , as given in Proposition 4.5. The characterizations are different if f is defined on a horizontal subgroup or on a vertical normal subgroup.

Proposition 4.6. *Let \mathbb{W}, \mathbb{V} be complementary subgroups of \mathbb{H}^n , as in Proposition 4.1. Then*

- (i) $f : \mathcal{A} \subset \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic Lipschitz if and only if the map $\Phi_f : \mathcal{A} \rightarrow \mathbb{H}^n$, defined as $\Phi_f(v) = v \cdot f(v)$, is metric Lipschitz in \mathcal{A} , that is if there is $L > 0$ such that, for all $v, \bar{v} \in \mathcal{A}$,

$$\|\Phi_f(\bar{v})^{-1} \cdot \Phi_f(v)\| \leq L \|\bar{v}^{-1} \cdot v\|.$$

- (ii) $f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic Lipschitz in \mathcal{A} , if and only if there is $L > 0$ such that, for all $w, \bar{w} \in \mathcal{A}$,

$$\|f(w)^{-1} \cdot f(\bar{w})\| \leq L \|f(w)^{-1} \cdot w^{-1} \cdot \bar{w} \cdot f(w)\|.$$

Remark 4.7. Unlike the case (i), even if $f : \mathbb{W} \rightarrow \mathbb{V}$ is very regular, the ‘natural’ parametrization of graph(f) given by

$$\Phi_f : \mathbb{W} \rightarrow \text{graph}(f) \subset \mathbb{H}^n, \quad \Phi_f(w) = w \cdot f(w)$$

may be not a metric Lipschitz map. Hence the characterization of intrinsic Lipschitz functions, given in (i) of Proposition 4.6, cannot be extended to functions from \mathbb{W} to \mathbb{V} .

Indeed, let \mathbb{V}, \mathbb{W} be the complementary subgroups of \mathbb{H}^1 as in Example 3.13. Let $f : \mathbb{W} \rightarrow \mathbb{V}$ be the constant map $f(w) = (1, 0, 0) \in \mathbb{V}$. Then graph(f) is a plane in \mathbb{R}^3 , parallel to \mathbb{W} . The parametrization Φ_f acts as

$$\Phi_f(w) = (1, w_2, w_3 + w_2/2).$$

Then $\Phi_f(e) = (1, 0, 0)$ and, if $\bar{w} = (0, \varepsilon, 0)$, $\Phi_f(\bar{w}) = (1, \varepsilon, \varepsilon/2)$. Then, $\|\Phi_f(e)^{-1} \cdot \Phi_f(\bar{w})\|$ is comparable with $\varepsilon^{1/2}$ while $\|\bar{w}\|$ is comparable with ε .

Proposition 4.8. *Let \mathbb{W}, \mathbb{V} be complementary subgroups of \mathbb{H}^n , as in Proposition 4.1. If $f : \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic L -Lipschitz then f is locally $1/2$ -Hölder continuous, with the Hölder norm depending only on $\mathbb{W}, \mathbb{V}, L$ and $\sup \|f(w)\|$.*

Proof. Notice that, for all $p, q \in \mathbb{H}^n$, $p^{-1} \cdot q \cdot p = (q', q_{2n+1} + \langle Jp', q' \rangle)$, so that

$$\|p^{-1} \cdot q \cdot p\| = \sqrt[4]{\|q'\|_{\mathbb{R}^{2n}}^4 + |q_{2n+1} + \langle Jp', q' \rangle|^2} \leq \|q\| + \|p'\|_{\mathbb{R}^{2n}}^{1/2} \|q'\|_{\mathbb{R}^{2n}}^{1/2}.$$

Hence, from (ii) of Proposition 4.6,

$$\begin{aligned} \|f(w) \cdot f(\bar{w})^{-1}\| &\leq L \|f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w})\| \\ &\leq L \left(\|\bar{w}^{-1} \cdot w\| + \|f(\bar{w})\|^{1/2} \|w' - \bar{w}'\|_{\mathbb{R}^{2n}}^{1/2} \right) \\ &\leq L(1 + \|f(\bar{w})\|^{1/2}) \|\bar{w}^{-1} \cdot w\|^{1/2}. \end{aligned}$$

□

As in Euclidean setting, pointwise limits of intrinsic Lipschitz functions are intrinsic Lipschitz. The statement is trivial for intrinsic Lipschitz functions from \mathbb{V} to \mathbb{W} , given (i) of Proposition 4.6.

Proposition 4.9. *If $f_n : \mathcal{E} \subset \mathbb{W} \rightarrow \mathbb{V}$, $n = 1, 2, \dots$, are equibounded and intrinsic L -Lipschitz in \mathcal{E} , then $\{f_n : n = 1, 2, \dots\}$ is precompact with respect to uniform convergence in \mathcal{E} . Moreover, if there is $f : \mathcal{E} \subset \mathbb{W} \rightarrow \mathbb{V}$ such that*

$$f(w) = \lim_{n \rightarrow +\infty} f_n(w),$$

for all $w \in \mathcal{E}$, then f is intrinsic L -Lipschitz in \mathcal{E} .

Proof. Precompactness follows from Arzelà Ascoli Theorem and the equicontinuity proved in Proposition 4.8. Finally, intrinsic Lipschitz continuity of f follows from the following computation

$$\begin{aligned} &\|f(\bar{w})^{-1} \cdot f(w)\| \\ &\leq \|f(\bar{w})^{-1} \cdot f_n(\bar{w})\| + \|f_n(\bar{w})^{-1} \cdot f_n(w)\| + \|f_n(w)^{-1} \cdot f(w)\| \\ &\leq 2\varepsilon + L (\|f_n(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f_n(w)\|) \\ &\leq 2\varepsilon + L(\|f_n(\bar{w})^{-1} \cdot f(\bar{w})\| + \\ &\quad + \|f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w})\| + \|f(\bar{w})^{-1} \cdot f_n(\bar{w})\|) \\ &\leq (2 + 2L)\varepsilon + L (\|f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(w)\|). \end{aligned}$$

□

We show now some relations between Euclidean Lipschitz or C^1 functions and intrinsic Lipschitz functions. As a consequence, we show that there are plenty of non trivial intrinsic Lipschitz functions in \mathbb{H}^n .

We begin with the following Theorem that is just a special case of Theorem 4.14, (see also the Implicit Function Theorem in [19]).

Theorem 4.10. *\mathbb{H} -regular surfaces, of any dimension, are locally graphs of intrinsic Lipschitz functions.*

Since low codimensional Euclidean C^1 submanifolds, without characteristic points, are \mathbb{H} -regular surfaces, then they are, locally, *intrinsic graphs* of intrinsic Lipschitz functions.

We show also that (low codimensional) intrinsic graphs of Euclidean Lipschitz functions are intrinsic Lipschitz graphs.

Proposition 4.11. *Let \mathbb{W}, \mathbb{V} be complementary subgroups as in Proposition 4.1, with $\dim \mathbb{V} = k$, $1 \leq k \leq n$. Let $\varphi : \mathbb{R}^{2n+1-k} \rightarrow \mathbb{R}^k$ be an Euclidean*

Lipschitz function. We choose a basis of $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$ adapted to the decomposition, so that we can identify \mathfrak{w} , \mathfrak{v} respectively with \mathbb{R}^{2n+1-k} and \mathbb{R}^k . Then the map

$$f : \mathbb{W} \rightarrow \mathbb{V}$$

defined as $f := \exp \circ \varphi \circ \log$ is intrinsic Lipschitz in the neighborhood of each point of \mathbb{W} .

Proof. We have to prove that

$$\|f(w)^{-1} \cdot f(\eta)\| \leq C_L \|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\|,$$

for all w, η in a small neighborhood of the origin of \mathbb{W} .

Let $\mathbb{V} = \exp(\text{span}\{V_1, \dots, V_k\})$ and $\mathbb{W} = \exp(\text{span}\{W_{k+1}, \dots, W_{2n}, T\})$ where $V_1, \dots, V_k, W_{k+1}, \dots, W_{2n} \in \mathfrak{h}_1$. Let M be the $2n \times 2n$ matrix that gives the vector fields V_1, \dots, W_{2n} in terms of X_1, \dots, Y_n . We write

$$p \simeq (\tilde{p}_1, \dots, \tilde{p}_{2n}, p_{2n+1}) = (\tilde{p}', p_{2n+1}).$$

when $p = \exp(\sum_{i=1}^k \tilde{p}_i V_i + \sum_{i=k+1}^{2n} \tilde{p}_i W_i + p_{2n+1} T)$. This way $p' = M^T \tilde{p}'$.

Notice that for $p \simeq (\tilde{p}_1, \dots, \tilde{p}_{2n}, p_{2n+1})$, $q \simeq (\tilde{q}_1, \dots, \tilde{q}_{2n}, q_{2n+1})$ we have

$$p \cdot q \simeq (\tilde{p}' + \tilde{q}', p_{2n+1} + q_{2n+1} - \frac{1}{2} \langle JM^T \tilde{p}', M^T \tilde{q}' \rangle_{\mathbb{R}^{2n}})$$

and

$$\|p\| = (\|M^T \tilde{p}'\|_{\mathbb{R}^{2n}}^4 + p_{2n+1}^2)^{1/4}.$$

Observe that

$$\|f(w)^{-1} \cdot f(\eta)\| = \|\varphi(w) - \varphi(\eta)\|_{\mathbb{R}^k}$$

and

$$\begin{aligned} \|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\| &\approx \|w' - \eta'\|_{\mathbb{R}^{2n}} \\ &+ |\eta_{2n+1} - w_{2n+1} + \langle JM^T \tilde{f}(w)', M^T(\tilde{\eta}' - \tilde{w}') \rangle_{\mathbb{R}^{2n}}|^{1/2}. \end{aligned}$$

We know that

$$\|\varphi(w) - \varphi(\eta)\|_{\mathbb{R}^k} \leq L \|w - \eta\|_{\mathbb{R}^{2n+1-k}} \approx \|w' - \eta'\|_{\mathbb{R}^{2n}} + |\eta_{2n+1} - w_{2n+1}|.$$

Thus, we are left to show that

$$\begin{aligned} |\eta_{2n+1} - w_{2n+1}| &\leq \text{const.} (\|w' - \eta'\|_{\mathbb{R}^{2n}} \\ &+ |\eta_{2n+1} - w_{2n+1} + \langle JM^T \tilde{f}(w)', M^T(\tilde{\eta}' - \tilde{w}') \rangle_{\mathbb{R}^{2n}}|^{1/2}). \end{aligned}$$

The assertion is trivial if

$$\frac{1}{2} |\eta_{2n+1} - w_{2n+1}| \geq |\langle JM^T \tilde{f}(w)', M^T(\tilde{\eta}' - \tilde{w}') \rangle_{\mathbb{R}^{2n}}|,$$

since

$$\|w' - \eta'\|_{\mathbb{R}^{2n}} + |\eta_{2n+1} - w_{2n+1}| \leq \|w' - \eta'\|_{\mathbb{R}^{2n}} + |\eta_{2n+1} - w_{2n+1}|^{1/2},$$

provided w, η are close to e . On the other side, if the reverse inequality holds, then

$$|\eta_{2n+1} - w_{2n+1}| \leq 2\|M\|^2 \sup \|f\| \|w' - \eta'\|_{\mathbb{R}^{2n}},$$

achieving the proof of the proposition. \square

The graph of an intrinsic Lipschitz function f has the same metric dimension as the domain of f , i.e. if s is this metric dimension,

$$\mathcal{S}_d^s(\text{graph}(f) \cap \mathcal{U}) < \infty,$$

for any bounded $\mathcal{U} \subset \mathbb{H}^n$.

A non trivial corollary of the previous estimate is that 1-codimensional intrinsic Lipschitz graphs are boundaries of sets of locally finite \mathbb{H}^n -perimeter.

Upper and lower bounds on the Hausdorff measure of a Lipschitz graph are trivially true in Euclidean spaces. Indeed if $f : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ is intrinsic Lipschitz then the map $\Phi_f : \mathbb{R}^k \rightarrow \mathbb{R}^n$ defined as $\Phi_f(x) := (x, f(x))$ is a Lipschitz parametrization of the Euclidean graph of f and this gives the upper bound; on the other side the projection $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$ is Lipschitz, with Lipschitz constant 1, yielding the lower bound.

Such a proof cannot work here. From one side the projection $\Pi_{\mathbb{W}} : \mathbb{H}^n = \mathbb{W} \cdot \mathbb{V} \rightarrow \mathbb{W}$ is not Lipschitz continuous, on the other side, as observed in Example 4.7, the ‘natural’ parametrization of $\text{graph}(f)$, given by $\Phi_f : \mathbb{W} \rightarrow \mathbb{H}^n$, $\Phi_f(w) := w \cdot f(w)$, is almost never a Lipschitz continuous map between the two metric spaces \mathbb{W} and \mathbb{H}^n .

Notice that it is an open problem to understand if a different Lipschitz continuous parameterization exists. About this, in [12] it has been proved that, if the surface S is somehow more regular than just Lipschitz, then such a parametrization exists. On the contrary, D.Vittone has provided us an example (see [5]) showing that in general biLipschitz parametrizations may not exist.

Theorem 4.12. *Assume that \mathbb{W}, \mathbb{V} are complementary subgroups of \mathbb{H}^n , as in Proposition 4.1, and let $k, 1 \leq k \leq n$, be the dimension of \mathbb{V} . If $f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic L -Lipschitz in \mathcal{A} , and if \mathcal{A} is relatively open in \mathbb{W} , then $\text{graph}(f)$ has metric dimension $2n+2-k$ and there is a geometric constant $c = c(\mathbb{V}, \mathbb{W}) > 0$ such that, for all $p \in \mathbb{H}^n$ and $R > 0$,*

$$(17) \quad \mathcal{S}_d^{2n+2-k}(\text{graph}(f) \cap B(p, R)) \leq c(1+L)^{2n+2-k} R^{2n+2-k}.$$

Symmetrically, if $f : \mathcal{A} \subset \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic L -Lipschitz in \mathcal{A} , if \mathcal{A} is relatively open in \mathbb{V} , then $\text{graph}(f)$ has metric dimension k and, for all $p \in \mathbb{H}^n$ and $R > 0$,

$$\mathcal{S}_d^k(\text{graph}(f) \cap B(p, R)) \leq c(1+L)^k R^k.$$

Proof. When dealing with functions $f : \mathbb{V} \rightarrow \mathbb{W}$, the proof follows the same pattern as the Euclidean one. Indeed, (see (i) of Proposition 4.6) the parametrization map $\Phi_f : \mathbb{V} \rightarrow \mathbb{H}^n$ is a Lipschitz map, and this yields the upper bound on $\mathcal{S}_d^k(\text{graph}(f))$; on the other side, also the projection $\Pi_{\mathbb{V}} : \mathbb{H}^n \rightarrow \mathbb{V}$ is a Lipschitz map (see Remark 4.2), and this provides the lower bound on $\mathcal{S}_d^k(\text{graph}(f))$.

Let us come now to the more interesting case of functions from \mathbb{W} to \mathbb{V} . The lower bound for $\mathcal{S}_d^{2n+2-k}(\text{graph}(f))$ is a consequence of Lemma 4.3. Indeed, assume $\mathcal{S}_d^{2n+2-k}(\text{graph}(f)) < \infty$. Fix $\varepsilon > 0$, choose $r = r(\varepsilon) > 0$ and a covering of $\text{graph}(f)$ with closed balls $B_i = B(p_i, r_i)$ such that $r_i \leq r$

and

$$\sum_i r_i^{2n+2-k} \leq \mathcal{S}_d^{2n+2-k}(\text{graph}(f)) + \varepsilon.$$

Then, by (13),

$$\begin{aligned} \mathcal{L}^{2n+1-k}(\mathcal{A}) &\leq \sum_i \mathcal{L}^{2n+1-k}(\Pi_{\mathbb{W}}(B_i)) = c \sum_i r_i^{2n+2-k} \\ &\leq c \mathcal{S}_d^{2n+2-k}(\text{graph}(f)) + \varepsilon. \end{aligned}$$

To get the upper bound (17), it is enough to prove that for any $p \in \text{graph}(f)$, $R > 0$ and $\varepsilon > 0$, it is possible to cover $\text{graph}(f) \cap B(p, R)$ with less than $N := c \left(\frac{1}{\varepsilon}\right)^{2n+2-k}$ metric balls of radius less than ε . Here c will depend on $R, \mathbb{W}, \mathbb{V}$ and L .

Without loss of generality, we assume that $p = e$. Then we fix ε , $0 < \varepsilon < 1$. Using a Vitali covering argument we choose a covering of $\text{graph}(f) \cap B(e, R)$ with metric balls $B(p_i, 5\varepsilon)$, $p_i = \bar{w}_i \cdot f(\bar{w}_i) \in \text{graph}(f)$, such that the concentric smaller balls $B_i := B(p_i, \varepsilon)$ are pairwise disjoint. We estimate the number N of balls B_i in this Vitali covering.

Define $\mathcal{E} := \Pi_{\mathbb{W}}(B(e, R)) = \{w \in \mathbb{W} : w \cdot f(w) \in B(e, R)\}$ and $\mathcal{E}_i := \Pi_{\mathbb{W}}(B_i)$. From (5), it follows $\mathcal{E} \subset \{w : \|w\| \leq R/c\}$ and $\mathcal{E}_i \subset B(e, (R+1)/c)$.

Since the sets \mathcal{E}_i are pairwise disjoint, to estimate N it is enough to evaluate from below $\mathcal{L}^{2n+1-k}(\mathcal{E}_i)$.

For each E_i consider the group translation $\tau_{p_i^{-1}}$ that moves the point p_i to the origin e . Let $\tilde{E}_i := \{w : w \cdot f_{p_i^{-1}}(w) \in B(e, \varepsilon)\}$. Remember that $f_{p_i^{-1}}$ is intrinsic Lipschitz with the same constant L of f , that $f_{p_i^{-1}}(e) = e$ hence $\|f_{p_i^{-1}}(w)\| \leq L \|w\|$ and $\|w \cdot f_{p_i^{-1}}(w)\| \leq (1+L) \|w\|$. Hence

$$\mathbb{W} \cap B(e, \varepsilon/(1+L)) \subset \tilde{E}_i,$$

hence

$$\mathcal{L}^{2n+1-k}(\tilde{E}_i) \geq \mathcal{L}^{2n+1-k} \left(\mathbb{W} \cap B(e, \frac{\varepsilon}{1+L}) \right) = c(\mathbb{W}) \left(\frac{\varepsilon}{1+L} \right)^{2n+2-k}.$$

Recalling (i) of Proposition 4.5, we have that

$$\tilde{E}_i = \{f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot w \cdot f(\bar{w}_i) : w \in E_i\}.$$

It is easy to check by a straightforward computation (recall also Lemma 4.3), that any map $\chi : \mathbb{W} \rightarrow \mathbb{W}$, (here $\mathbb{W} \equiv \mathbb{R}^{2n+1-k}$) given by

$$w \mapsto \chi(w) := \bar{v}^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \bar{v},$$

for fixed $\bar{v} \in \mathbb{V}$ and $\bar{w} \in \mathbb{W}$, has Jacobian determinant equal to 1. Hence

$$\mathcal{L}^{2n+1-k}(E_i) = \mathcal{L}^{2n+1-k}(\tilde{E}_i) \geq c(\mathbb{W}) \left(\frac{\varepsilon}{1+L} \right)^{2n+2-k}.$$

This way we have

$$N \leq \left((1+L) \frac{R}{c} \right)^{2n+2-k} \left(\frac{1}{\varepsilon} \right)^{2n+2-k}.$$

□

4.3. Intrinsic differentiable functions. Intrinsic differentiable functions, within \mathbb{H}^n , can be characterized as follows (see [4]).

Proposition 4.13. *Let \mathbb{W}, \mathbb{V} be complementary subgroups of \mathbb{H}^n , as in Proposition 4.1. Then,*

- (i) $f : \mathcal{A} \subset \mathbb{V} \rightarrow \mathbb{W}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if $\Phi_f : \mathcal{A} \rightarrow \mathbb{H}^n$, $\Phi_f(g) := g \cdot f(g)$, is P -differentiable in \bar{g} ;
- (ii) $f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{W} \rightarrow \mathbb{V}$, such that

$$\|df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g)\| = o(\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\|)$$

for $g \in \mathcal{A}$ and $\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\| \rightarrow 0$.

To motivate the centrality of intrinsic differentiable functions we recall the following theorem that is the main result in [4], (see also [19]).

Theorem 4.14. *The following statements are equivalent*

- (1) $S \subset \mathbb{H}^n$ is a \mathbb{H} -regular submanifold.
- (2) S is locally the graph of a uniformly intrinsic differentiable function acting between complementary subgroups of \mathbb{H}^n .

Finally, the core of this section is the following theorem where we prove two geometric characterizations of intrinsic differentiability, one in terms of blowups and the other one in terms of intrinsic cones.

Theorem 4.15. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 4.1 and $f : \mathcal{A} \subset \mathbb{W} \rightarrow \mathbb{V}$ with \mathcal{A} relatively open in \mathbb{W} . Let $w_0 \in \mathcal{A}$ and $g_0 = w_0 \cdot f(w_0)$.*

Then the following statements are equivalent.

- (i) f is intrinsic differentiable in $w_0 \in \mathcal{A}$.
- (ii) There is a vertical subgroup \mathbb{T}_{f,g_0} , complementary to \mathbb{V} , that we will call the tangent subgroup to $\text{graph}(f)$ in g_0 , characterized by

$$(18) \quad \lim_{\lambda \rightarrow +\infty} \delta_\lambda(g_0^{-1} \cdot \text{graph}(f)) = \mathbb{T}_{f,g_0},$$

in the sense of Hausdorff convergence in compact subsets of \mathbb{H}^n .

- (iii) There is a vertical subgroup \mathbb{T}'_{f,g_0} , complementary to \mathbb{V} , and, for any $\alpha > 0$, there is $r_0 = r_0(f, w_0, \alpha) > 0$, such that

$$(19) \quad C_{\mathbb{T}'_{f,g_0}, \mathbb{V}}(g_0, \alpha) \cap B(g_0, r_0) \cap \text{graph}(f) = \{g_0\}.$$

Notice that $\mathbb{T}_{f,g_0} = \mathbb{T}'_{f,g_0} = \text{graph}(df_{w_0})$.

Proof. We can assume, without loss of generality, that

$$w_0 = e \text{ and } f(w_0) = g_0 = e.$$

Indeed, each condition (i),(ii),(iii), for f on g_0 , is equivalent to the corresponding condition for $f_{g_0^{-1}}$ on e . In particular, f is intrinsic differentiable in w_0 if and only if $f_{g_0^{-1}}$ is intrinsic differentiable in e ; about (ii) observe that $\text{graph}(f_{g_0^{-1}}) = g_0^{-1} \cdot \text{graph}(f)$ and that Hausdorff convergence is invariant by left translations because the distance is left invariant; finally, about (iii), it is enough to left translate (19).

(i) \implies (iii).

Define $\mathbb{T}'_{f,e} := \text{graph}(df_e)$. If $p \in \mathbb{H}^n$ then $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}} = p_{\mathbb{W}} \cdot df_e(p_{\mathbb{W}}) \cdot df_e(p_{\mathbb{W}})^{-1} \cdot p_{\mathbb{V}}$ and $p_{\mathbb{W}} \cdot df_e(p_{\mathbb{W}}) \in \mathbb{T}'_{f,e}$ while $df_e(p_{\mathbb{W}})^{-1} \cdot p_{\mathbb{V}} \in \mathbb{V}$. Moreover if, for some $x \in \mathbb{W}$, $x \cdot df_e(x) \in \mathbb{V}$ then $x \in \mathbb{V}$ so that $x = e$. This way we have proved that $\mathbb{T}'_{f,e}$ and \mathbb{V} are complementary subgroups.

Observe that (19) is the same as

$$(20) \quad C_{\mathbb{T}'_{f,e}, \mathbb{V}}(e, \alpha) \cap B(e, r_0) \cap \text{graph}(f) = \{e\}.$$

To get (20) we have to prove that for all $\alpha > 0$ there is $r_0 = r_0(f, e, \alpha) > 0$ such that

$$0 < \|x \cdot f(x)\| < r_0 \implies x \cdot f(x) \notin C_{\mathbb{T}'_{f,e}, \mathbb{V}}(e, \alpha),$$

that is

$$(21) \quad 0 < \|x \cdot f(x)\| < r_0 \implies \left\| \Pi_{\mathbb{T}'_{f,e}}(x \cdot f(x)) \right\| > \alpha \left\| \Pi_{\mathbb{V}}(x \cdot f(x)) \right\|.$$

Remember, once more, that $x \cdot f(x) = x \cdot df_e(x) \cdot df_e(x)^{-1} \cdot f(x)$, hence

$$\Pi_{\mathbb{T}'_{f,e}}(x \cdot f(x)) = x \cdot df_e(x), \quad \Pi_{\mathbb{V}}(x \cdot f(x)) = df_e(x)^{-1} \cdot f(x).$$

Before completing our argument we observe that, by a standard compactness argument, we know that the geometric constant $c = c(\mathbb{W}, \mathbb{V})$, defined as

$$(22) \quad c(\mathbb{W}, \mathbb{V}) := \inf_{\substack{v \in \mathbb{V} \\ x \in \mathbb{W}, \|x\|=1}} \|x \cdot v\|$$

is positive. By homogeneity, from (22), we have

$$\|x\| < \frac{1}{c} \|x \cdot v\|,$$

for all $x \in \mathbb{W}$ and for all $v \in \mathbb{V}$.

Because f is intrinsically differentiable in e , then f is intrinsically differentiable in e hence, by (11), choosing $\varepsilon = \frac{c(\mathbb{W}, \mathbb{V})}{\alpha}$ there is $\delta > 0$ such that

$$\begin{aligned} \left\| \Pi_{\mathbb{V}}(x \cdot f(x)) \right\| &\equiv \left\| df_e(x)^{-1} \cdot f(x) \right\| \\ &< \frac{c(\mathbb{W}, \mathbb{V})}{\alpha} \|x\| \leq \frac{1}{\alpha} \|x \cdot df_e(x)\| \equiv \frac{1}{\alpha} \left\| \Pi_{\mathbb{T}'_{f,e}}(x \cdot f(x)) \right\|, \end{aligned}$$

for $\|x\| < \delta$. Hence choosing $r_0 := c\delta$ we have proved (21) and, eventually, (19).

(iii) \implies (ii).

Fix $B(e, R)$ and choose $\mathbb{T}_{f,e} = \mathbb{T}'_{f,e}$. Observe preliminary that, by a compactness argument, for all $\varepsilon > 0$ there is $\alpha = \alpha(\mathbb{T}_{f,e}, \mathbb{V}, R, \varepsilon) > 0$ such that

$$(23) \quad (\mathbb{T}_{f,e})_{\varepsilon} \cup C_{\mathbb{T}_{f,e}, \mathbb{V}}(e, \alpha) \supset B(e, R),$$

where $(\mathbb{T}_{f,e})_{\varepsilon}$ is a ε -neighborhood of $\mathbb{T}_{f,e}$. Clearly, (23) implies that the complementary set, in $B(e, R)$, of $C_{\mathbb{T}_{f,e}, \mathbb{V}}(e, \alpha)$ is contained in $(\mathbb{T}_{f,e})_{\varepsilon}$.

Now, from (iii), we know that, for all $\alpha > 0$, there is $r_0(\alpha) > 0$ such that for all $0 < r < r_0$,

$$C_{\mathbb{T}_{f,e}, \mathbb{V}}(e, \alpha) \cap B(e, r) \cap \text{graph}(f) = \{e\}.$$

Hence, for all $\lambda > R/r_0$, we have

$$(24) \quad C_{\mathbb{T}_{f,e}, \mathbb{V}}(e, \alpha) \cap B(e, R) \cap \delta_\lambda(\text{graph}(f)) = \{e\}.$$

It follows, from the preliminary observation, that, for all $\varepsilon > 0$ there is $\lambda_0 = \lambda_0(e, r_0, \varepsilon) > 0$ such that, for all $\lambda > \lambda_0$ is contained in $(\mathbb{T}_{f,e})_\varepsilon$.

Conversely, we have to prove that, for all $\varepsilon > 0$ and $R > 0$, $\mathbb{T}_{f,e} \cap B(e, R)$ is contained in a ε -neighborhood of $\delta_\lambda(\text{graph}(f))$, for $\lambda > \lambda_0(R, \varepsilon)$.

We observe preliminary that $\mathbb{T}_{f,e}$ is a graph from \mathbb{W} to \mathbb{V} . We prove this, in a more general way, in the following Lemma.

Lemma 4.16. *If \mathbb{G}_1 and \mathbb{G}_2 are complementary subgroups of \mathbb{H}^n , and if \mathbb{M} is another homogeneous subgroup such that $\mathbb{M} \cap \mathbb{G}_2 = \{e\}$, then \mathbb{M} is a graph over \mathbb{G}_1 . More precisely, with the notations of Proposition 3.2 there is a continuous $\psi : \Pi_{\mathbb{G}_1}(\mathbb{M}) \rightarrow \mathbb{G}_2$ such that $\mathbb{M} = \text{graph}(\psi)$.*

In particular, if, with the notations of (4.1), \mathbb{W}, \mathbb{V} and \mathbb{W}', \mathbb{V} are two couples of complementary subgroups in \mathbb{H}^n , then there is an intrinsic linear function $\psi : \mathbb{W} \rightarrow \mathbb{V}$ such that

$$\mathbb{W}' = \text{graph}(\psi).$$

Proof. It is enough to show that the projection $\Pi_{\mathbb{G}_1} : \mathbb{M} \rightarrow \mathbb{G}_1$ is injective. Indeed, if $p, \bar{p} \in \mathbb{M}$ have the same projection on \mathbb{G}_1 , that is if $p = g_1 \cdot g_2$, $\bar{p} = g_1 \cdot \bar{g}_2$, then, because \mathbb{M} is a group, $g_2^{-1} \cdot g_1^{-1} \cdot g_1 \cdot \bar{g}_2 = g_2^{-1} \cdot \bar{g}_2 \in \mathbb{M} \cap \mathbb{G}_2$. By our assumption $g_2^{-1} \cdot \bar{g}_2 = e$ and, eventually, $p = \bar{p}$.

Finally, any $p \in \mathbb{M}$ can be written as $p = \Pi_{\mathbb{G}_1}(p) \cdot \Pi_{\mathbb{G}_2}(p)$. Hence, defining $g_1 := \Pi_{\mathbb{G}_1}(p) \in \Pi_{\mathbb{G}_1}(\mathbb{M})$ we have $p = g_1 \cdot \psi(g_1)$ with $\psi = \Pi_{\mathbb{G}_2} \circ (\Pi_{\mathbb{G}_1})^{-1}$.

Because projections are continuous (see Proposition 3.2), the continuity of ψ follows from the continuity of $(\Pi_{\mathbb{G}_1})^{-1}$. To see this last point, observe that $\Pi_{\mathbb{G}_1}(\mathbb{M})$ is invariant by dilations and that $(\Pi_{\mathbb{G}_1})^{-1} : \Pi_{\mathbb{G}_1}(\mathbb{M}) \rightarrow \mathbb{M}$ is homogeneous, that is $(\Pi_{\mathbb{G}_1})^{-1} \circ \delta_\lambda = \delta_\lambda \circ (\Pi_{\mathbb{G}_1})^{-1}$, for all $\lambda > 0$.

Let $(q_n)_n \subset \Pi_{\mathbb{G}_1}(\mathbb{M})$, with $q_n \rightarrow \bar{q} \in \Pi_{\mathbb{G}_1}(\mathbb{M})$ as $n \rightarrow +\infty$. Assume for the moment that, possibly taking a subsequence, $(\Pi_{\mathbb{G}_1})^{-1}(q_n) \rightarrow \bar{m}$; then $\bar{m} \in \mathbb{M}$, because \mathbb{M} is closed, and, by continuity of $\Pi_{\mathbb{G}_1}$,

$$q_n = \Pi_{\mathbb{G}_1} \circ (\Pi_{\mathbb{G}_1})^{-1}(q_n) \rightarrow \Pi_{\mathbb{G}_1}(\bar{m});$$

hence $\bar{q} = \Pi_{\mathbb{G}_1} \circ (\Pi_{\mathbb{G}_1})^{-1}(\bar{q}) = \Pi_{\mathbb{G}_1}(\bar{m})$ and finally $\bar{m} = (\Pi_{\mathbb{G}_1})^{-1}(\bar{q})$, that gives the continuity of $(\Pi_{\mathbb{G}_1})^{-1}$.

Finally, we are left with the proof of the existence of a convergent subsequence of $(\Pi_{\mathbb{G}_1})^{-1}(q_n)$. This follows from the homogeneity of $(\Pi_{\mathbb{G}_1})^{-1}$.

Indeed, let $c_n := \left\| (\Pi_{\mathbb{G}_1})^{-1}(q_n) \right\|$ and assume, by contradiction, that c_n is unbounded, then $p_n := \delta_{c_n^{-1}} q_n \rightarrow e$ and $\left\| (\Pi_{\mathbb{G}_1})^{-1}(p_n) \right\| = 1$. Hence, with the same argument as before, we get that, possibly taking a subsequence, $(\Pi_{\mathbb{G}_1})^{-1}(p_n) \rightarrow e$ as $n \rightarrow +\infty$. Absurd, because they have unit norm. \square

Now, coming back to the proof of Theorem 4.15, from Lemma 4.16, there is $\psi : \Pi_{\mathbb{W}}(\mathbb{T}_{f,e}) \subset \mathbb{W} \rightarrow \mathbb{V}$ such that

$$\mathbb{T}_{f,e} = \text{graph}(\psi).$$

For any $\bar{p} = \bar{w} \cdot \psi(\bar{w}) \in \mathbb{T}_{f,e} \cap B(e, R)$ let $w := \delta_{1/\lambda} \bar{w}$. Notice that, because \mathcal{A} is open in \mathbb{W} , there is $\bar{\lambda} > 0$, depending on \mathcal{A} and R , but not on \bar{p} , such that, $w \in B(e, r_0) \cap \mathbb{W} \subset \mathcal{A}$ for all $\lambda > \bar{\lambda}$. With this choice of w we have $\delta_\lambda(w) = \bar{w}$ hence

$$(25) \quad \delta_\lambda(w \cdot f(w)) = \bar{w} \cdot \delta_\lambda(f(w)).$$

We can estimate now $d(\bar{p}, \delta_\lambda(\text{graph}(f)))$.

$$(26) \quad \begin{aligned} d(\bar{p}, \delta_\lambda(\text{graph}(f))) &\leq d(\bar{p}, \delta_\lambda(w \cdot f(w))) \\ &\leq \|\bar{p}^{-1} \cdot \delta_\lambda(w \cdot f(w))\| \leq \|\psi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot \bar{w} \cdot \delta_\lambda f(w)\| \\ &= \|\psi(\bar{w})^{-1} \cdot \delta_\lambda f(w)\|. \end{aligned}$$

On the other side, from (iii), and (24) we know that for all $\alpha > 0$, $r_0 > 0$, there is $\lambda_0 = \lambda_0(r_0, \alpha)$ such that, for all $\lambda > \lambda_0$,

$$(27) \quad C_{\mathbb{T}_{f,e}, \mathbb{V}}(e, \alpha) \cap B(e, r_0) \cap \delta_\lambda(\text{graph}(f)) = \{e\}$$

that in turn is equivalent to

$$(28) \quad \alpha \|\Pi_{\mathbb{V}}(\delta_\lambda(w \cdot f(w)))\| < \|\Pi_{\mathbb{T}_{f,e}}(\delta_\lambda(w \cdot f(w)))\|.$$

for all $\lambda > \lambda_0$ and for $w \cdot f(w) \in B(e, r_0)$. Computing the two projections, we have

$$\delta_\lambda(w \cdot f(w)) = \bar{w} \cdot \delta_\lambda f(w) = \underbrace{\bar{w} \cdot \psi(\bar{w})}_{\in \mathbb{T}_{f,e}} \cdot \underbrace{\psi(\bar{w})^{-1} \cdot \delta_\lambda f(w)}_{\in \mathbb{V}},$$

hence (28) becomes

$$\|\psi(\bar{w})^{-1} \cdot \delta_\lambda f(w)\| < \|\bar{w} \cdot \psi(\bar{w})\| / \alpha \leq R/\alpha.$$

(ii) \implies (i).

We have to show that there is an intrinsic linear function $df_e : \mathbb{W} \rightarrow \mathbb{V}$, such that

$$(29) \quad d(df_e(w), f(w)) = \|df_e(w)^{-1} \cdot f(w)\| = o(\|w\|),$$

as $w \rightarrow e$. We know, by Lemma 4.16, that there is an intrinsic linear function $\psi : \mathbb{W} \rightarrow \mathbb{V}$ such that $\mathbb{T}_{f,e} = \text{graph}(\psi)$. Then we choose

$$df_e(w) := \psi(w),$$

for all $w \in \mathbb{W}$. Let $(w_n)_n \subset \mathbb{W}$, with $w_n \neq e$ and $w_n \rightarrow e$ as $n \rightarrow +\infty$. Define $\lambda_n := 1/\|w_n\|$, then we have

$$(30) \quad \begin{aligned} \lambda_n d(df_e(w_n), f(w_n)) &= d(\delta_{\lambda_n} w_n \cdot df_e(\delta_{\lambda_n} w_n), \delta_{\lambda_n}(w_n \cdot f(w_n))) \\ &\leq d(\delta_{\lambda_n} w_n \cdot df_e(\delta_{\lambda_n} w_n), p_n) + d(p_n, \delta_{\lambda_n}(w_n \cdot f(w_n))) \end{aligned}$$

where we have chosen $p_n = \eta_n \cdot df_e(\eta_n) \in \mathbb{T}_{f,e} \cap B(e, 2)$ such that

$$d(\delta_\lambda(w_n \cdot f(w_n)), \mathbb{T}_{f,e}) = d(\delta_\lambda(w_n \cdot f(w_n)), p_n).$$

Then, from assumption (18),

$$(31) \quad d(p_n, \delta_{\lambda_n}(w_n \cdot f(w_n))) = d(\eta_n \cdot df_e(\eta_n), \delta_{\lambda_n}(w_n \cdot f(w_n))) \rightarrow 0$$

as $n \rightarrow +\infty$. Hence, by Proposition 3.2,

$$(32) \quad d(\delta_\lambda w_n, \eta_n) \rightarrow 0,$$

as $n \rightarrow +\infty$. It follows that

$$(33) \quad d(df_e(\delta_\lambda w_n), df_e(\eta_n)) \rightarrow 0,$$

as $n \rightarrow +\infty$, because df_e is continuous. Hence, once more from Proposition 18 and from (32) and (33),

$$(34) \quad d(\delta_{\lambda_n} w_n \cdot df_e(\delta_{\lambda_n} w_n), p_n) = d(\delta_{\lambda_n} w_n \cdot df_e(\delta_{\lambda_n} w_n), \eta_n \cdot df_e(\eta_n)) \rightarrow 0$$

as $n \rightarrow +\infty$. Finally, (30), (31) and (34) imply (29) and the proof is completed. \square

4.4. 1-codimensional graphs. Through all this last section we assume that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in (4.1), but with the further requirement that \mathbb{V} is 1-dimensional, i.e. $\mathbb{V} = \{\exp(tV), t \in \mathbb{R}\}$, for a fixed $V \in \mathfrak{h}_1$ (recall the observations after Definition 3.1). Without loss of generality, we assume $\|V\| = 1$.

4.4.1. Intrinsic Lipschitz graphs and finite perimeter sets. The aim of this subsection is to show that subgraphs of 1-dimensional valued, intrinsic Lipschitz functions are sets of locally finite \mathbb{H} -perimeter. We begin with some basic definitions.

Let $f : \mathcal{U} \subset \mathbb{W} \rightarrow \mathbb{V}$, with $f(w) = \exp(\varphi(w)V)$ for any $w \in \mathcal{U}$ and $\varphi : \mathcal{U} \rightarrow \mathbb{R}$. We define the *supergraph* E_f^+ and the *subgraph* E_f^- of f as

$$(35) \quad \begin{aligned} E_f^- &:= \{w \cdot \exp(tV) : w \in \mathcal{U}, t < \varphi(w)\} \\ E_f^+ &:= \{w \cdot \exp(tV) : w \in \mathcal{U}, t > \varphi(w)\}. \end{aligned}$$

We say, as in [21], that $\mathcal{E} \subset \mathbb{H}^n$ has *locally finite \mathbb{H} -perimeter* (or, following De Giorgi, \mathcal{E} is \mathbb{H} -Caccioppoli set) if for any bounded open set $\mathcal{U} \subseteq \mathbb{H}^n$

$$|\partial\mathcal{E}|_{\mathbb{H}}(\mathcal{U}) := \sup \left\{ \int_{\mathcal{E}} \sum_{j=1}^n (X_j \phi + Y_j \phi) d\mathcal{L}^{2n+1} \right\} < \infty,$$

where the supremum is taken over all $\phi \in C_0^1(\mathcal{U}, H\mathbb{H}^n)$, such that $\|\phi(p)\| \leq 1$. In such a way, $|\partial\mathcal{E}|_{\mathbb{H}}$ is a Radon measure in \mathbb{H}^n .

Theorem 4.17. *Let \mathbb{W} and \mathbb{V} be complementary subgroups in \mathbb{H}^n with \mathbb{V} 1-dimensional, i.e. $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$ for a fixed $V \in \mathfrak{h}_1$, with $\|V\| = 1$. If $f : \mathbb{W} \rightarrow \mathbb{V}$ is intrinsic Lipschitz then the subgraph E_f^- is a set with locally finite \mathbb{H} -perimeter.*

Proof. The proof is a consequence of Theorems 4.12 and 4.18. \square

The following Theorem is the counterpart, in Heisenberg setting, of Proposition 3.6.2 of [1].

Theorem 4.18. *Let \mathcal{O} be an open sets in \mathbb{H}^n .*

If $\mathcal{S}_d^{2n+1}(\partial\mathcal{O})$ is locally finite in \mathbb{H}^n then \mathcal{O} has locally finite \mathbb{H} -perimeter.

If $\mathcal{S}_d^{2n+1}(\partial\mathcal{O}) < \infty$ then there is a geometric constant $c > 0$ such that

$$|\partial\mathcal{O}|_{\mathbb{H}} \leq c \mathcal{S}_d^{Q-1} \llcorner \partial\mathcal{O}.$$

In addition, if U is an open ball such that $\partial\mathcal{O} \cap U$ is an intrinsic Lipschitz graph, then

$$|\partial\mathcal{O}|_{\mathbb{H}}(U) = c \mathcal{S}_d^{2n+1}(\partial\mathcal{O} \cap U).$$

Proof. Suppose first that \mathcal{O} is bounded and $\mathcal{S}_d^{2n+1}(\partial\mathcal{O}) < \infty$; then we show that $|\partial\mathcal{O}|_{\mathbb{H}}(\mathbb{H}^n) < \infty$. Indeed, if $\varepsilon > 0$, we can cover $\partial\mathcal{O}$ with open metric balls $\{U_{\varepsilon,j}, j \in \mathbb{N}\}$, with radius $r_{\varepsilon,j} < \varepsilon$, such that

$$\sum_j r_{\varepsilon,j}^{2n+1} < c \mathcal{S}_d^{2n+1}(\partial\mathcal{O}) = C < \infty.$$

Denote

$$S_\varepsilon := \bigcup_j U_{\varepsilon,j} \quad \text{and} \quad \mathcal{O}_\varepsilon := \mathcal{O} \cup S_\varepsilon.$$

We have

$$(36) \quad \mathcal{O}_\varepsilon \rightarrow \mathcal{O} \quad \text{in } L^1, \text{ as } \varepsilon \rightarrow 0.$$

Indeed, $\mathcal{L}^{2n+1}(\mathcal{O}_\varepsilon \Delta \mathcal{O}) = \mathcal{L}^{2n+1}(\mathcal{O}_\varepsilon \setminus \mathcal{O}) \leq \mathcal{L}^{2n+1}(S_\varepsilon) \leq c \sum_j r_{\varepsilon,j}^{2n+2} \leq C \varepsilon$. Suppose now, for a while, that we have proven

$$(37) \quad |\partial\mathcal{O}_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) \leq c |\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n).$$

The proof of our Theorem follows from (36), (37) and the L^1 -lower semicontinuity of the perimeter, since $|\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) \leq \sum_j |\partial U_{\varepsilon,j}|_{\mathbb{H}}(\mathbb{H}^n) \leq c \sum_j r_{\varepsilon,j}^{2n+1} \leq c < \infty$. Thus, we are left with the proof of (37). To this end, observe that

$$(38) \quad \partial\mathcal{O}_\varepsilon \subset \bar{\mathcal{O}}^c, \quad \text{dist}(\partial\mathcal{O}_\varepsilon, \bar{\mathcal{O}}^c) > 0$$

and that

$$(39) \quad \mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c = S_\varepsilon \cap \bar{\mathcal{O}}^c.$$

Using (38), (39) and general properties of the perimeter we get (37) as follows

$$\begin{aligned} |\partial\mathcal{O}_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n) &= |\partial\mathcal{O}_\varepsilon|_{\mathbb{H}}(\bar{\mathcal{O}}^c) = |\partial(\mathcal{O}_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{H}}(\bar{\mathcal{O}}^c) \\ &= |\partial(S_\varepsilon \cap \bar{\mathcal{O}}^c)|_{\mathbb{H}}(\bar{\mathcal{O}}^c) = |\partial S_\varepsilon|_{\mathbb{H}}(\bar{\mathcal{O}}^c) \leq |\partial S_\varepsilon|_{\mathbb{H}}(\mathbb{H}^n). \end{aligned}$$

Finally we drop the assumption of the boundedness of \mathcal{O} . Let U be any fixed open ball such that $U \cap \partial\mathcal{O} \neq \emptyset$ and $\mathcal{S}_d^{2n+1}(\partial\mathcal{O} \cap U) < \infty$. Notice that $\partial(U \cap \mathcal{O}) \subset \partial U \cup (\partial\mathcal{O} \cap U)$, hence, because $\mathcal{S}_d^{2n+1}(\partial U) < \infty$,

$$\mathcal{S}_d^{2n+1}(\partial(U \cap \mathcal{O})) < \infty.$$

Thus, from the first part of the proof, it follows that $|\partial(U \cap \mathcal{O})|_{\mathbb{H}}(\mathbb{H}^n) < \infty$, and, once more by the locality of the \mathbb{H} -perimeter,

$$|\partial\mathcal{O}|_{\mathbb{H}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{H}}(U) = |\partial(U \cap \mathcal{O})|_{\mathbb{H}}(U) < \infty.$$

This achieves the proof of the first part of the theorem.

Finally, if $\partial\mathcal{O} \cap U$ is an intrinsic Lipschitz graph, then its measure theoretic boundary in U ¹ coincides with $\partial\mathcal{O} \cap U$ and then the assertion follows from [16], Theorem 7.1. \square

¹ If $\mathcal{E} \subset \mathbb{H}^n$, as it is usual in the literature, we define the *measure theoretic boundary* $\partial_{*,\mathbb{H}}\mathcal{E}$ (called also *essential boundary*) as

4.4.2. *Extension of intrinsic Lipschitz functions.* Our proof of the extension theorem follows the ideas of the classical Mc Shane-Whitney extension theorem (see [23]). We need, given $L > 0$, an elementary intrinsic L -Lipschitz function ϕ_L , that is a function vanishing at the origin and with ‘maximal slope’ L in all directions. In general metric spaces, such a real valued Lipschitz function is simply a multiple of the distance function from the origin. Remember that here we have to check intrinsic Lipschitz continuity and not metric Lipschitz continuity.

Lemma 4.20. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$, for a fixed $V \in \mathfrak{h}_1$. For $L > 0$, define $\phi_L : \mathbb{W} \rightarrow \mathbb{V}$ as*

$$\phi_L(w) := \exp(L \|w\| V).$$

Then there is $L_1 = L_1(L, \mathbb{W}, \mathbb{V}) \geq L$ such that the function ϕ_L is intrinsic L_1 -Lipschitz. Here $L_1 = 2 \max\{2, \|M\| (1 + 2L)\}$, where M is a matrix depending on \mathbb{W}, \mathbb{V} only, that will be made precise during the proof.

We notice also that

$$(40) \quad \text{graph}(\phi_L) = \partial C_{\mathbb{W}, \mathbb{V}}^+(e, 1/L),$$

where

$$C_{\mathbb{W}, \mathbb{V}}^+(e, 1/L) := S_{\mathbb{H}}^+(V) \cap C_{\mathbb{W}, \mathbb{V}}(e, 1/L).$$

Proof. We have to prove that

$$(41) \quad \|\phi_L(w)^{-1} \cdot \phi_L(\eta)\| \leq L_1 \|\phi_L(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \phi_L(w)\|$$

for all $\eta, w \in \mathbb{W}$. Notice that

$$\|\phi_L(w)^{-1} \cdot \phi_L(\eta)\| = L \|\|\eta\| - \|w\|\|.$$

Let $\mathbb{V} = \exp(tV)$ and $\mathbb{W} = \exp(\text{span}\{W_2, \dots, W_{2n}, T\})$ where $V, W_2, \dots, W_{2n} \in \mathfrak{h}_1$. Let M be the $2n \times 2n$ matrix that gives the vector fields V, W_2, \dots, W_{2n} in terms of X_1, \dots, Y_n . We write

$$p \simeq (\tilde{p}_1, \dots, \tilde{p}_{2n}, p_{2n+1}) = (\tilde{p}', p_{2n+1}).$$

when $p = \exp(\tilde{p}_1 V + \sum_{i=2}^{2n} \tilde{p}_i W_i + p_{2n+1} T)$. It turns out that $p' = M^T \tilde{p}'$. Notice that for $p \simeq (\tilde{p}_1, \dots, \tilde{p}_{2n}, p_{2n+1})$, $q \simeq (\tilde{q}_1, \dots, \tilde{q}_{2n}, q_{2n+1})$ we have

$$p \cdot q \simeq (\tilde{p}' + \tilde{q}', p_{2n+1} + q_{2n+1} - \frac{1}{2} \langle JM^T \tilde{p}', M^T \tilde{q}' \rangle_{\mathbb{R}^{2n}})$$

and

$$\|p\| = (\|M^T \tilde{p}'\|_{\mathbb{R}^{2n}}^4 + p_{2n+1}^2)^{1/4}.$$

With this choice of coordinates we have

$$\phi_L(w) \simeq L \|w\| (1, 0, \dots, 0) := L \|w\| e_1$$

Definition 4.19. (Measure theoretic boundary) Let $\mathcal{E} \subset \mathbb{H}^n$ be a measurable set, we say that $x \in \partial_{*, \mathbb{H}} \mathcal{E}$ if

$$\limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^{2n+2}(\mathcal{E} \cap U(x, r))}{\mathcal{L}^{2n+2}(U(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^{2n+2}(\mathcal{E}^c \cap U(x, r))}{\mathcal{L}^{2n+2}(U(x, r))} > 0.$$

and $\tilde{p}_1 = 0$ for all $p \in \mathbb{W}$. Therefore, for all w and $\eta \in \mathbb{W}$,

$$\begin{aligned} & \phi_L(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \phi_L(w) \\ & \simeq (0, \tilde{\eta}_2 - \tilde{w}_2, \dots, \tilde{\eta}_{2n} - \tilde{w}_{2n}, \eta_{2n+1} - w_{2n+1} \\ & \quad + \frac{1}{2} \langle JM^T \tilde{w}', M^T \tilde{\eta}' \rangle_{\mathbb{R}^{2n}} - L \|w\| \langle JM^T (\tilde{\eta}' - \tilde{w}'), M^T e_1 \rangle_{\mathbb{R}^{2n}}). \end{aligned}$$

We define $\mathcal{H} = \mathcal{H}(\eta, w)$, $\mathcal{V} = \mathcal{V}(\eta, w)$ and $\mathcal{I} = \mathcal{I}(\eta, w)$ as

$$\mathcal{H}^2 := \|M^T(\tilde{\eta}' - \tilde{w}')\|^2$$

$$\begin{aligned} \mathcal{V} & := \eta_{2n+1} - w_{2n+1} \\ & \quad + \frac{1}{2} \langle JM^T \tilde{w}', M^T \tilde{\eta}' \rangle_{\mathbb{R}^{2n}} - L \|w\| \langle JM^T (\tilde{\eta}' - \tilde{w}'), M^T e_1 \rangle_{\mathbb{R}^{2n}} \end{aligned}$$

and $\mathcal{I} := \|\phi_L(w)^{-1} \cdot w^{-1} \cdot \eta \cdot \phi_L(w)\| = \sqrt[4]{\mathcal{H}^4 + \mathcal{V}^2}$.

Then inequality (41) can be written as

$$(42) \quad L \left| \|\eta\| - \|w\| \right| \leq L_1 \mathcal{I}, \quad \text{for all } \eta, w \in \mathbb{W}.$$

Assume w.l.g. that $\|\eta\| \leq \|w\|$, $\|w\| > 0$ and observe that (41) is invariant by Heisenberg dilations. Hence with

$$\eta \rightarrow \delta_{1/\|w\|} \eta, \quad w \rightarrow \delta_{1/\|w\|} w$$

proving (42) is equivalent to proving that

$$(43) \quad L(1 - \|\eta\|) \leq L_1 \mathcal{I}, \quad \text{for } \|\eta\| \leq 1 \text{ and } \|w\| = 1.$$

Notice that, for $-1 \leq \sigma \leq 0$, $1 - \sqrt[4]{1 + \sigma} \leq -\sigma$ hence

$$(44) \quad \begin{aligned} 0 & \leq 1 - \|\eta\| \leq 1 - \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^4 - \eta_{2n+1}^2 \\ & = \|M^T \tilde{w}'\|_{\mathbb{R}^{2n}}^4 - \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^4 + w_{2n+1}^2 - \eta_{2n+1}^2 \end{aligned}$$

Observe that, for $\|\eta\| \leq 1$ and $\|w\| = 1$,

$$(45) \quad \begin{aligned} & \|M^T \tilde{w}'\|_{\mathbb{R}^{2n}}^4 - \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^4 \\ & = \left(\|M^T \tilde{w}'\|_{\mathbb{R}^{2n}}^2 - \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^2 \right) \left(\|M^T \tilde{w}'\|_{\mathbb{R}^{2n}}^2 + \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^2 \right) \\ & \leq 2 \left| \|M^T \tilde{w}'\|_{\mathbb{R}^{2n}}^2 - \|M^T \tilde{\eta}'\|_{\mathbb{R}^{2n}}^2 \right| \\ & = 2 \left| \langle M^T (\tilde{w}' - \tilde{\eta}'), M^T (\tilde{w}' + \tilde{\eta}') \rangle_{\mathbb{R}^{2n}} \right| \\ & \leq 4 \|M^T (\tilde{w}' - \tilde{\eta}')\|_{\mathbb{R}^{2n}} \leq 4\mathcal{H} \\ & \leq 4\mathcal{I}. \end{aligned}$$

Finally, by (45), to conclude the proof, we have to estimate $|w_{2n+1}^2 - \eta_{2n+1}^2|$ in terms of \mathcal{I} . We consider two cases.

Case 1: Assume

$$|\mathcal{V}| < \frac{1}{2} |\eta_{2n+1} - w_{2n+1}|.$$

Then, with $M^T \tilde{w}' = (\mu_1, \dots, \mu_{2n})$ and $M^T \tilde{\eta}' = (\nu_1, \dots, \nu_{2n})$, we have

$$\begin{aligned}
& \frac{1}{2} |\eta_{2n+1} - w_{2n+1}| \\
& < \frac{1}{2} |\langle JM^T \tilde{w}', M^T \tilde{\eta}' \rangle_{\mathbb{R}^{2n}}| + L \|w\| |\langle JM^T (\tilde{\eta}' - \tilde{w}'), M^T e_1 \rangle_{\mathbb{R}^{2n}}| \\
& = \frac{1}{2} |\langle JM^T (\tilde{w}' - \tilde{\eta}'), M^T \tilde{\eta}' \rangle_{\mathbb{R}^{2n}}| + L \|w\| |\langle JM^T (\tilde{\eta}' - \tilde{w}'), M^T e_1 \rangle_{\mathbb{R}^{2n}}| \\
& \leq \frac{\|M^T\|}{2} (1 + 2L) \|M^T (\tilde{w}' - \tilde{\eta}')\|.
\end{aligned}$$

Hence $|\eta_{2n+1} - w_{2n+1}| \leq \|M^T\| (1 + 2L) \mathcal{I}$ and then,

$$(46) \quad |w_{2n+1}^2 - \eta_{2n+1}^2| \leq 2|w_{2n+1} - \eta_{2n+1}| \leq 2\|M^T\| (1 + 2L) \mathcal{I}.$$

Case 2: Assume

$$|\mathcal{V}| \geq \frac{1}{2} |\eta_{2n+1} - w_{2n+1}|.$$

Then

$$(47) \quad \begin{aligned} |w_{2n+1}^2 - \eta_{2n+1}^2| & \leq 2|w_{2n+1} - \eta_{2n+1}| \leq \\ & \leq 2\sqrt{2}|w_{2n+1} - \eta_{2n+1}|^{1/2} \leq 4\sqrt[4]{\mathcal{V}^2} \leq 4\mathcal{I}. \end{aligned}$$

From (44), (46) and (47) we get (42) with

$$L_1 = 2 \max\{2, \|M^T\| (1 + 2L)\}.$$

□

Lemma 4.21. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$. Then, for each $\alpha > 0$ there is $\alpha_1 = \alpha_1(\alpha, \mathbb{W}, \mathbb{V}) \leq \alpha$, such that, for all $v = \exp(tV) \in \mathbb{V}$, with $t > 0$,*

$$C_{\mathbb{W}, \mathbb{V}}^+(v, \alpha_1) := v \cdot C_{\mathbb{W}, \mathbb{V}}^+(e, \alpha_1) \subset C_{\mathbb{W}, \mathbb{V}}^+(e, \alpha).$$

Proof. Let $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \in C_{\mathbb{W}, \mathbb{V}}^+(e, \alpha)$. Then $(v \cdot p)_{\mathbb{W}} = v \cdot p_{\mathbb{W}} \cdot v^{-1}$ and $(v \cdot p)_{\mathbb{V}} = v \cdot p$. First we notice that, because $\dim \mathbb{V} = 1$ and \mathbb{V} is horizontal,

$$\|(v \cdot p)_{\mathbb{V}}\| = \|p\| + \|v\|.$$

Indeed, if $v = \exp \theta V$ and $\bar{v} = \exp \bar{\theta} V$, then $v \cdot \bar{v} = \exp(\theta + \bar{\theta})V$ and $\|v \cdot \bar{v}\| = |\theta + \bar{\theta}| = |\theta| + |\bar{\theta}|$ when θ and $\bar{\theta}$ are positive.

$$\|(v \cdot p)_{\mathbb{W}}\| = \|v \cdot p_{\mathbb{W}} \cdot v^{-1}\| \leq \|p_{\mathbb{W}}\| + c \|v\|^{1/2} \|p_{\mathbb{W}}\|^{1/2}$$

by Lemma 4.4. Now, using that $p \in C_{\mathbb{W}, \mathbb{V}}^+(e, \alpha_1)$, we have

$$\begin{aligned}
\|(v \cdot p)_{\mathbb{W}}\| & \leq (1 + c/2) \|p_{\mathbb{W}}\| + c/2 \|v\| \\
& \leq \alpha_1 (1 + c/2) \|p_{\mathbb{V}}\| + c/2 \|v\| \\
& \leq \max\{\alpha_1 (1 + c/2), c/2\} (\|p_{\mathbb{V}}\| + \|v\|).
\end{aligned}$$

Hence, if we choose $\alpha_1 > 0$ such that $\alpha = \max\{\alpha_1 (1 + c/2), c/2\}$, it follows that $v \cdot p \in C_{\mathbb{W}, \mathbb{V}}^+(e, \alpha)$. □

Since $\mathbb{V} = \{\exp tV\}$, it can be identified with \mathbb{R} so that it carries an order and we can define the supremum and the infimum of families of \mathbb{V} -valued functions. If $f_\alpha : \mathbb{W} \rightarrow \mathbb{V}$ for $\alpha \in A$ with $f_\alpha(w) = \exp(f_{\alpha,1}(w)V)$ and $f_{\alpha,1} : \mathbb{W} \rightarrow \mathbb{R}$, we define $\inf_{\alpha \in A} f_\alpha : \mathbb{W} \rightarrow \mathbb{V}$ as

$$\inf_{\alpha \in A} f_\alpha(w) := \exp\left(\inf_{\alpha \in A} f_{\alpha,1}(w)V\right),$$

for all $w \in \mathbb{W}$ such that $\inf_{\alpha \in A} f_{\alpha,1}(w)$ is finite; analogously we define $\sup_{\alpha \in A} f_\alpha$, $\max\{f_{\alpha_1}, f_{\alpha_2}\}$, $\min\{f_{\alpha_1}, f_{\alpha_2}\}$, etc.

Notice that, if $f : \mathbb{W} \rightarrow \mathbb{V}$ is continuous and $f(w) = \exp(\varphi(w)V)$ for $\varphi : \mathbb{W} \rightarrow \mathbb{R}$, then

$$(48) \quad \begin{aligned} \overline{E_f^-} &= \{w \cdot \exp(tV) : w \in \mathbb{W}, t \leq \varphi(w)\} \\ \overline{E_f^+} &= \{w \cdot \exp(tV) : w \in \mathbb{W}, t \geq \varphi(w)\}. \end{aligned}$$

Clearly,

$$\begin{aligned} E^-(\inf f_\alpha) &= \bigcap_{\alpha} E^-(f_\alpha) \\ E^+(\inf f_\alpha) &= \bigcup_{\alpha} E^+(f_\alpha). \end{aligned}$$

When dealing with 1-codimensional graphs we can give a different characterization of intrinsic Lipschitz functions as follows.

Lemma 4.22. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$. Then $f : \mathbb{W} \rightarrow \mathbb{V}$, $f(w) := \exp(\varphi(w)V)$, is intrinsic L -Lipschitz if and only if, for all $w \in \mathbb{W}$,*

$$C_{\mathbb{W}, \mathbb{V}}^+(w \cdot f(w), 1/L) \subset \overline{E_f^+} \quad \text{and} \quad C_{\mathbb{W}, \mathbb{V}}^-(w \cdot f(w), 1/L) \subset \overline{E_f^-}.$$

Proof. Assume that f is intrinsic L -Lipschitz, then by definition, for all $w \in \mathbb{W}$ and for all $\alpha < 1/L$,

$$(49) \quad C_{\mathbb{W}, \mathbb{V}}(w \cdot f(w), \alpha) \cap \text{graph}(f) = \{w \cdot f(w)\}.$$

Assume by contradiction that there is $\bar{w} \in \mathbb{W}$ and $\bar{t} > \varphi(\bar{w}) \in \mathbb{R}$ such that

$$\bar{w} \cdot \exp(\bar{t}V) \in C_{\mathbb{W}, \mathbb{V}}^+(\bar{w} \cdot f(\bar{w}), \alpha) \cap E_f^-;$$

but then

$$\bar{w} \cdot \exp(tV) \in C_{\mathbb{W}, \mathbb{V}}^+(\bar{w} \cdot f(\bar{w}), \alpha)$$

for all $t \geq \bar{t}$, in particular $\bar{w} \cdot \exp(\varphi(\bar{w})V) \in C_{\mathbb{W}, \mathbb{V}}^+(\bar{w} \cdot f(\bar{w}), \alpha)$ and this contradicts (49). \square

Lemma 4.23. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$. Then there is $L_2 = L_2(L, \mathbb{W}, \mathbb{V}) \geq L$ such that, if $f_\alpha, f_\beta : \mathbb{W} \rightarrow \mathbb{V}$ are intrinsic L -Lipschitz, then $g := \max\{f_\alpha, f_\beta\}$ is intrinsic L_2 -Lipschitz.*

Proof. Let $w \in \mathbb{W}$, then we apply Lemma 4.21, with $\alpha_1 = 1/L$ and $L_2 := 1/\alpha_2$, to have

$$C_{\mathbb{W}, \mathbb{V}}^+(w \cdot g(w), 1/L_2) \subset \overline{E^+(f_\alpha)} \quad \text{and} \quad C_{\mathbb{W}, \mathbb{V}}^+(w \cdot g(w), 1/L_2) \subset \overline{E^+(f_\beta)}.$$

Hence

$$C_{\mathbb{W}, \mathbb{V}}^+(w \cdot g(w), 1/L_2) \subset \overline{E^+(f_\alpha)} \cap \overline{E^+(f_\beta)} = \overline{E^+(g)}.$$

Notice that, by Lemma 4.21, L_2 is independent of w . Hence, by Lemma 4.22, we have proved that g is intrinsic L_2 -Lipschitz. \square

Proposition 4.24. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$. Then, for all $L > 0$ there is $L_2 = L_2(L, \mathbb{W}, \mathbb{V}) \geq L$ such that, if*

$$f := \inf_{\alpha \in A} f_\alpha,$$

with $f_\alpha : \mathbb{W} \rightarrow \mathbb{V}$ intrinsic L -Lipschitz, then either $f \equiv -\infty$ or f is defined on all of \mathbb{W} and it is intrinsic L_2 -Lipschitz. Notice that L_2 is the same constant as in Lemma 4.23.

Proof. We will prove the assertion showing that, for all $w, \eta \in \mathbb{W}$,

$$\|f(w)^{-1} \cdot f(\eta)\| \leq L_2 \|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\|.$$

Fix w and η in \mathbb{W} , and given $\varepsilon > 0$, choose $\alpha = \alpha(\varepsilon), \beta = \beta(\varepsilon) \in A$, such that

$$\|f_\alpha(w)^{-1} \cdot f(w)\| < \varepsilon \text{ and } \|f_\beta(\eta)^{-1} \cdot f(\eta)\| < \varepsilon.$$

Then define $g := \max\{f_\alpha, f_\beta\}$. Observe that, from Lemma 4.23,

- (i) g is intrinsic L_2 -Lipschitz.
- (ii) $\|g(w)^{-1} \cdot f(w)\| < \varepsilon$ and $\|g(\eta)^{-1} \cdot f(\eta)\| < \varepsilon$.

Hence, from (i) and (ii),

$$\begin{aligned} \|f(w)^{-1} \cdot f(\eta)\| &= \|f(w)^{-1} \cdot g(w) \cdot g(w)^{-1} \cdot g(\eta) \cdot g(\eta)^{-1} \cdot f(\eta)\| \\ &\leq \|f(w)^{-1} \cdot g(w)\| + \|g(\eta)^{-1} \cdot f(\eta)\| + \|g(w)^{-1} \cdot g(\eta)\| \\ &\leq 2\varepsilon + L_2 \|g(w)^{-1} \cdot w^{-1} \cdot \eta \cdot g(w)\| \\ &\leq 2\varepsilon + 2\varepsilon L_2 + L_2 \|f(w)^{-1} \cdot w^{-1} \cdot \eta \cdot f(w)\|. \end{aligned}$$

\square

Theorem 4.25. *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V} = \{\exp tV : t \in \mathbb{R}\}$ for a fixed $V \in \mathfrak{h}_1$, with $\|V\| = 1$. Let \mathcal{C} be a Borel subset of \mathbb{W} and $f : \mathcal{C} \rightarrow \mathbb{V}$, be an intrinsic L -Lipschitz function. Then there are $\tilde{f} : \mathbb{W} \rightarrow \mathbb{V}$ and $L_3 = L_3(L, \mathbb{W}, \mathbb{V}) \geq L$ such that*

- (i) \tilde{f} is intrinsic L_3 -Lipschitz,
- (ii) $\tilde{f}(w) = f(w)$, for all $w \in \mathcal{C}$.

Proof. For each $\bar{w} \in \mathcal{C}$ we define $\phi_{L, \bar{w}} : \mathbb{W} \rightarrow \mathbb{V}$ as the translated function (see Proposition 4.5)

$$\phi_{L, \bar{w}}(w) := f(\bar{w}) \cdot \phi_L(f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w})),$$

where ϕ_L is the function defined in Lemma 4.20. We notice, as in (40), that

$$(50) \quad \text{graph}(\phi_{L, \bar{w}}) = \partial C_{\mathbb{W}, \mathbb{V}}^+(\bar{w} \cdot f(\bar{w}), 1/L).$$

We write

$$\phi_{L, \bar{w}}(w) = \exp(\theta_{\bar{w}}(w)V) \text{ and } f(w) = \exp(\varphi(w)V),$$

where $\theta_{\bar{w}} : \mathbb{W} \rightarrow \mathbb{R}$ and $\varphi : \mathcal{C} \rightarrow \mathbb{R}$. Clearly, since $\phi_{L, \bar{w}}(\bar{w}) = f(\bar{w})$, then

$$(51) \quad \theta_{\bar{w}}(\bar{w}) = \varphi(\bar{w}), \quad \text{for all } \bar{w} \in \mathcal{C}.$$

We define $\tilde{f} : \mathbb{W} \rightarrow \mathbb{V}$ as

$$\tilde{f}(w) := \inf_{\bar{w} \in \mathcal{C}} \phi_{L, \bar{w}}(w)$$

for all $w \in \mathbb{W}$. From Lemma 4.20 we know that all the functions $\phi_{L, \bar{w}}$ are intrinsic L_1 -Lipschitz, where L_1 is the constant defined in Lemma 4.20.

We want to show now that $\tilde{f}(w) = f(w)$ for all $w \in \mathcal{C}$. Given (51), it is enough to show

$$(52) \quad \theta_{\bar{w}}(w) \geq \varphi(w), \quad \text{for all } w, \bar{w} \in \mathcal{C}.$$

Now, f is intrinsic L -Lipschitz, then, keeping in mind (51),

$$(53) \quad \begin{aligned} |\varphi(w) - \theta_{\bar{w}}(\bar{w})| &= |\varphi(w) - \varphi(\bar{w})| = \|f(\bar{w})^{-1} \cdot f(w)\| \\ &\leq L \|f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w})\|. \end{aligned}$$

On the other hand, from (50),

$$(54) \quad \begin{aligned} |\theta_{\bar{w}}(w) - \theta_{\bar{w}}(\bar{w})| &= \|\phi_{L, \bar{w}}(\bar{w})^{-1} \cdot \phi_{L, \bar{w}}(w)\| \\ &= L \|\phi_{L, \bar{w}}(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \phi_{L, \bar{w}}(\bar{w})\| \\ &= L \|f(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot f(\bar{w})\|. \end{aligned}$$

Combining (53) and (54) we get (52).

Finally we apply Proposition 4.24 with $L_3 := L_2(L_1, \mathbb{W}, \mathbb{V})$, to get that \tilde{f} is intrinsic L_3 -Lipschitz. □

4.4.3. Rademacher Theorem. Let $f : \mathcal{U} \subset \mathbb{W} \rightarrow \mathbb{V}$ be an intrinsic Lipschitz map, where \mathcal{U} is a (relatively) open subset; we want to prove here a Rademacher's type result, that is, *if f is intrinsic Lipschitz in \mathcal{U} then f is intrinsic differentiable almost everywhere in \mathcal{U} .*

By Theorem 4.25 there is an intrinsic Lipschitz function $\tilde{f} : \mathbb{W} \rightarrow \mathbb{V}$ extending f ; moreover the notion of intrinsic differentiability is a local one, hence, without loss of generality, from now on we assume that f is intrinsic Lipschitz and defined on all of \mathbb{W} .

One of the key points in our proof of the Rademacher theorem is the fact, proved in Theorem 4.17, that the subgraph E_f^- of an intrinsic Lipschitz function $f : \mathbb{W} \rightarrow \mathbb{V}$ is a set with locally finite \mathbb{H} -perimeter. From this and the results in [16], it follows that at almost every point of the graph of f there is an approximate tangent plane (see (56)) that in turn, together with the intrinsic Lipschitz assumption, yields the intrinsic differentiability of f .

We begin with some notations. For each $\nu \in \mathfrak{h}_1$, we define the *vertical hemispaces* $S_{\mathbb{H}}^+(\nu)$, $S_{\mathbb{H}}^-(\nu)$ and their common boundary, the vertical subgroup $\mathbb{N}(\nu)$, as

$$(55) \quad \begin{aligned} S_{\mathbb{H}}^+(\nu) &:= \exp(\{Z : \langle Z, \nu \rangle \geq 0\}), & S_{\mathbb{H}}^-(\nu) &:= \exp(\{Z : \langle Z, \nu \rangle \leq 0\}), \\ \mathbb{N}(\nu) &:= \exp(\{Z : \langle Z, \nu \rangle = 0\}). \end{aligned}$$

Notice that $\mathbb{N}(\nu)$ is a 1-codimensional normal subgroup, because $\{Z : \langle Z, \nu \rangle = 0\}$ is a one codimensional ideal in \mathfrak{h} . We point out that this definition yields the objects denoted respectively by the symbols $S_{\mathbb{H}}^+(\nu)$, $S_{\mathbb{H}}^-(\nu)$ and $T_{\mathbb{H}}^g(\nu)$ in [16] and [18]. Moreover, notice that the vertical subgroup $\mathbb{N}(\nu)$ and the

1-dimensional subgroup $\mathbb{L}_\nu := \exp(\{t\nu : t \in \mathbb{R}\})$ are complementary subgroups in \mathbb{H}^n .

Now, coming back to Rademacher's theorem, observe that, when dealing with 1-codimensional intrinsic Lipschitz graphs, conditions (ii) and (iii) of Theorem 4.15 can be relaxed. Notice that (ii) and (iii) could be stated, informally, as *uniform convergence* of dilated subgraphs of f to a hemispace; now, here, in Proposition 4.26, we ask only that the dilated subgraphs converge to a hemispace, in $\mathcal{L}_{\text{loc}}^1(\mathbb{H}^n)$. Notice that assumption (56) is precisely in the spirit of De Giorgi's blow-up results for finite perimeter sets (see [13] and [16] for the Heisenberg case).

Proposition 4.26. *Let $f : \mathbb{W} \rightarrow \mathbb{V}$ be an intrinsic L -Lipschitz function and $p = w \cdot f(w) \in \text{graph}(f)$. If there is an approximate tangent plane to $\text{graph}(f)$ in p , that is, if there is $\nu \in \mathfrak{h}_1$ such that*

$$(56) \quad \lim_{r \rightarrow 0} \mathbf{1}_{(E_f^-)_{r,p}} = \mathbf{1}_{S_{\mathbb{H}}^+(\nu)}$$

in $\mathcal{L}_{\text{loc}}^1(\mathbb{H}^n)$, where

$$(E_f^-)_{r,p} = \{q : p \cdot \delta_r q \in E_f^-\} = \delta_{\frac{1}{r}}(p^{-1} \cdot E_f^-),$$

then f is intrinsic differentiable in w .

Proof. As usual, replacing f by $f_{p^{-1}}$, we can assume that $w = f(w) = e$. Now the proof relies on the following two lemmata.

Lemma 4.27. *Keeping the same notations of Proposition 4.26, set*

$$f_{1/r} := \delta_{\frac{1}{r}} \circ f \circ \delta_r,$$

for $r > 0$. Then there is an intrinsic linear function $f_\infty : \mathbb{W} \rightarrow \mathbb{V}$ such that

- (i) $f_{1/r} \rightarrow f_\infty$ as $r \rightarrow 0$ uniformly on compact sets;
- (ii) $\text{graph}(f_\infty) = \mathbb{N}(\nu)$;
- (iii) $\mathbb{N}(\nu)$ and \mathbb{V} are complementary subgroups in \mathbb{H}^n .

Proof. We point first that

$$(57) \quad (E_f^-)_{r,e} = \delta_{1/r}(E_f^-) = E_{f_{1/r}}^- \quad \text{and} \quad \text{graph}(f_{1/r}) = \delta_{1/r}(\text{graph}(f)).$$

For $M > 0$ we consider $\{f_{1/r} : B(e, M) \cap \mathbb{W} \rightarrow \mathbb{V} \text{ for } r > 0\}$. Notice that that $\{f_{1/r}, r > 0\}$ are equibounded in $B(e, M) \cap \mathbb{W}$. Indeed

$$\|f_{1/r}(w)\| = \frac{1}{r} \|f(\delta_r w)\| \leq \frac{L}{r} \|\delta_r w\| = L \|w\| \leq LM,$$

for $w \in B(e, M) \cap \mathbb{W}$. Moreover, $\{f_{1/r}, r > 0\}$ are all intrinsic L -Lipschitz and, as proved in Proposition 4.8, $\{f_{1/r}, r > 0\}$ are equicontinuous in $B(e, M) \cap \mathbb{W}$. By Arzelà–Ascoli theorem and a standard diagonal argument, we obtain thus that there exists a subsequence $(r_k)_{k \in \mathbb{N}}$, $r_k \downarrow 0$ as $k \rightarrow \infty$, such that $f_{1/r_k} \rightarrow f_\infty$ as $k \rightarrow \infty$, uniformly on compact sets. Once (ii) is proved, then f_∞ is uniquely determined by the very definition of intrinsic graph and we can get rid of the subsequence as usual, achieving the proof of statement (i).

Thus, we are left with the proof of (ii). We show first that

$$(58) \quad \text{graph}(f_\infty)^c \subset \mathbb{N}(\nu)^c, \quad \text{i.e.} \quad \mathbb{N}(\nu) \subset \text{graph}(f_\infty).$$

To this end, take a point $\bar{p} \notin \text{graph}(f_\infty)$; for instance suppose $\bar{p} \in E_{f_\infty}^-$. We want to show that $\mathbf{1}_{S_{\mathbb{H}}^+(\nu)}(\bar{p}) = 1$. Then, take a ball $B := B(\bar{p}, \rho)$ such that

$$\text{dist}(B, \text{graph}(f_\infty)) > 0.$$

Because of the uniform convergence of the f_{1/r_k} 's on compact sets, we can assume that $B \subset E_{f_{1/r_k}}^-$ for k large enough, so that $\mathbf{1}_{E_{f_{1/r_k}}^-} \equiv 1$ on B for k large. By (4.4.3) and (57), this implies that $\mathbf{1}_{S_{\mathbb{H}}^+(\nu)} = 1$ a.e. in B . But this implies that $\mathbf{1}_{S_{\mathbb{H}}^+(\nu)} \equiv 1$ on B , by our previous remark, and hence that $\mathbf{1}_{S_{\mathbb{H}}^+(\nu)}(\bar{p}) = 1$. The case $\bar{p} \in (E_{f_\infty}^-)^c \setminus \text{graph}(f_\infty)$ can be handled in the same way. This proves (58). To prove the reversed inclusion, notice first that a point $\bar{p} \in \text{graph}(f_\infty)$ is both the limit of a sequence $(p_n)_{n \in \mathbb{N}}$ in $E_{f_\infty}^-$ and of a sequence $(q_n)_{n \in \mathbb{N}}$ in $(E_{f_\infty}^-)^c \setminus \text{graph}(f_\infty)$. Indeed, if $\bar{p} = \bar{w} \cdot f(\bar{w}) = \bar{w} \cdot \exp(\phi(\bar{w}))$, it is enough to choose

$$p_n = \bar{w} \cdot \exp(\phi(\bar{w}) - 1/n) \quad \text{and} \quad q_n = \bar{w} \cdot \exp(\phi(\bar{w}) + 1/n).$$

On the other hand, we have just shown that $E(f_\infty) \subset S_{\mathbb{H}}^+(\nu)$ and $E(f_\infty)^c \setminus \text{graph}(f_\infty) \subset S_{\mathbb{H}}^-(\nu)$, so that $\bar{p} \in \overline{S_{\mathbb{H}}^+(\nu)} \cap \overline{S_{\mathbb{H}}^-(\nu)} = \mathbb{N}(\nu)$. This achieves the proof of statements (i) and (ii).

To prove (iii), let us first prove that $v \notin \text{graph}(f_\infty)$. We remind that, if we fix $\alpha \in (0, 1/L)$, then $C_{\mathbb{W}, \mathbb{V}}(e, \alpha) \cap \text{graph}(f) = \{e\}$, by Proposition 3.9. Hence, by invariance of cones under dilations and by (57), $C_{\mathbb{W}, \mathbb{V}}(e, \alpha) \cap \text{graph}(f_{1/r}) = \{e\}$ for any $r > 0$. This implies that

$$(59) \quad C_{\mathbb{W}, \mathbb{V}}(e, \alpha/2) \cap \text{graph}(f_\infty) = \{e\}.$$

Indeed, take a point $p = w \cdot f_\infty(w) \in \text{graph}(f_\infty)$, $p \neq e$. By (i), $p = \lim_{r \rightarrow 0} w \cdot f_{1/r}(w) := \lim_{r \rightarrow 0} p_r$. Clearly, $p_r \in \text{graph}(f_{1/r})$ and we can assume $p_r \neq e$. But then $\|\Pi_{\mathbb{W}} p_r\| > \alpha \|\Pi_{\mathbb{V}} p_r\|$ for any $r > 0$ and eventually $\|\Pi_{\mathbb{W}} p\| \geq \alpha \|\Pi_{\mathbb{V}} p\|$, by Proposition 3.2. This implies that $p \notin C_{\mathbb{W}, \mathbb{V}}(e, \alpha/2)$ and hence (59). Since $v \in C_{\mathbb{W}, \mathbb{V}}(e, \alpha/2)$, and $v \neq 0$, then $v \notin \text{graph}(f_\infty)$, then, by (ii), the first assertion of (iii) is proved. On the other hand, by Definition 55, this implies that $\{Z \in \mathfrak{h} : \langle Z, \nu \rangle = 0\} \cap \text{span}(V) = \{0\}$ in \mathfrak{h} , and hence that $\mathfrak{h} = \{Z \in \mathfrak{h} : \langle Z, \nu \rangle = 0\} \oplus \text{span}(V)$. Then (iii) follows. \square

The final step consists of the following lemma.

Lemma 4.28. *With the same assumptions as in Proposition 4.26, for any $\alpha > 0$ there exists $\delta = \delta(\alpha) > 0$ such that*

$$(60) \quad \text{graph}(f) \cap B(e, \delta) \cap C_{\mathbb{N}(\nu), \mathbb{V}}(e, \alpha) = \{e\}.$$

Proof. By contradiction, suppose (60) fails, then there exists a sequence of points $p_n := w_n \cdot f(w_n) \in \text{graph}(f) \cap C_{\mathbb{N}(\nu), \mathbb{V}}(e, \alpha)$, $p_n \neq e$ but converging to e as $n \rightarrow \infty$. Set now $\xi_n := \delta_{1/\|w_n\|}(w_n)$ for any $n \in \mathbb{N}$. Without loss of generality, we may assume $\xi_n \rightarrow \xi_0$ as $n \rightarrow \infty$, with $\|\xi_0\| = 1$. Then we have

$$(61) \quad \begin{aligned} \delta_{1/\|w_n\|}(p_n) &= \xi_n \cdot \delta_{1/\|w_n\|}(f(\delta_{\|w_n\|}(\xi_n))) \\ &= \xi_n \cdot f_{1/\|w_n\|}(\xi_n) \in C_{\mathbb{N}(\nu), \mathbb{V}}(e, \alpha) \cap \text{graph}(f_{1/\|w_n\|}). \end{aligned}$$

By Lemma 4.27, (i) and (ii), this yields that

$$(62) \quad \begin{aligned} \xi_0 \cdot f_\infty(\xi_0) &\in C_{\mathbb{N}(\nu), \mathbb{V}}(e, \alpha) \cap \text{graph}(f_\infty) \\ &= C_{\mathbb{N}(\nu), \mathbb{V}}(e, \alpha) \cap \mathbb{N}(\nu) = \{e\}, \end{aligned}$$

by the very definition of intrinsic cone. This implies that $\xi_0 \in \mathbb{W} \cap \mathbb{V} = \{e\}$, contradicting the fact that $\|\xi_0\| = 1$, and the lemma is proved. \square

Now, from (60), differentiability of f in e follows from Theorem 4.15 and the proof of Proposition 4.26 is completed. \square

Theorem 4.29. *Suppose \mathbb{W} and \mathbb{V} be complementary subgroups of \mathbb{H}^n with \mathbb{V} 1-dimensional. Let $\mathcal{U} \subset \mathbb{W}$ be a relatively open subset and $f : \mathcal{U} \rightarrow \mathbb{V}$ be intrinsic Lipschitz. Then f is intrinsic differentiable $(\mathcal{L}^{2n} \llcorner \mathbb{W})$ -a.e. in \mathcal{U} . Notice that $(\mathcal{L}^{2n} \llcorner \mathbb{W})$ is the Haar measure of \mathbb{W} .*

Proof. By Theorem 4.25, we may assume that f is intrinsic Lipschitz and defined on all of \mathbb{W} . Hence, by Theorem 4.17, we know that E_f^- has locally finite \mathbb{H} -perimeter. Then, by [16], Theorem 6 (or [18], Theorem 3.1), we know that there is a subset

$$\partial^* E_f^- \subset \partial E_f^- = \text{graph}(f)$$

such that

$$|\partial E_f^-|_{\mathbb{H}}(\text{graph}(f) \setminus \partial^* E_f^-) \equiv |\partial E_f^-|_{\mathbb{H}}(\partial E_f^- \setminus \partial^* E_f^-) = 0,$$

and for all $p \in \partial^* E_f^-$ there is $\nu = \nu(p) \in \mathfrak{h}_1$ (the inward unit normal to E_f^- at p) with $\|\nu(p)\| = 1$ and such that (56) holds, i.e.

$$\lim_{r \rightarrow 0} \mathbf{1}_{(E_f^-)_{r,p}} = \mathbf{1}_{S_{\mathbb{H}}^+(\nu(p))},$$

in $\mathcal{L}_{\text{loc}}^1(\mathbb{H}^n)$. By Proposition 4.26, f is differentiable at any point $w \in \mathbb{W}$ such that

$$w \cdot f(w) \in \partial^* E_f^-.$$

Thus the proof follows if we prove

$$(\mathcal{L}^{2n} \llcorner \mathbb{W})(\Pi_{\mathbb{W}}(\text{graph}(f) \setminus \partial^* E_f^-)) = (\mathcal{L}^{2n} \llcorner \mathbb{W})(\mathbb{W} \setminus \Pi_{\mathbb{W}} \partial^* E_f^-) = 0,$$

that indeed is the content of the following Lemmas. \square

Lemma 4.30. *With the same assumptions of Theorem 4.29, let $f : \mathbb{W} \rightarrow \mathbb{V}$ be an intrinsic Lipschitz function and let $\Phi_f : \mathbb{W} \rightarrow \mathbb{H}^n$ be the parametric map defined as $\Phi_f(w) = w \cdot f(w)$. Then there exists $c(\mathbb{W}, \mathbb{V}) > 0$ such that*

$$(\Phi_f)_\#(\mathcal{L}^{2n} \llcorner \mathbb{W}) = c(\mathbb{W}, \mathbb{V}) \langle \nu, V \rangle |\partial E_f^-|_{\mathbb{H}}$$

where $(\Phi_f)_\#(\mathcal{L}^{2n} \llcorner \mathbb{W})$ and ν denote respectively the image of the measure $\mathcal{L}^{2n} \llcorner \mathbb{W}$ under the map Φ_f , and the horizontal generalized inward normal to E_f^- .

Proof. Let $v = \exp(V) \in \mathbb{V}$ and $a = (a_1, \dots, a_{2n})$ such that

$$\mathbb{W} = \{p = (p', p_{2n+1}) \in \mathbb{H}^n : \langle a, p' \rangle_{\mathbb{R}^{2n}} = 0\}.$$

Since at least one of the a_i does not vanish, we assume that it is a_1 and that $a_1 = 1$. Since E_f^- has locally finite \mathbb{H} -perimeter then,

$$(63) \quad \int_{E_f^-} (Vg) d\mathcal{L}^{2n+1} = \int_{\mathbb{H}^n} \langle \nu, V \rangle g d|\partial E_f^-|_{\mathbb{H}},$$

for all continuously differentiable, compactly supported $g \in C_c^1(\mathbb{H}^n)$. We define now a change of variables useful for the following computations. Observe that it is the same change of variables that we used in Lemma 4.20 (and also in Proposition 4.11).

Let $\Psi : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n \cong \mathbb{R}^{2n+1}$ be the map defined as

$$\begin{aligned} \Psi(\xi_1, 0, \dots, 0) &= \delta_{\xi_1} v \in \mathbb{V}, \\ \Psi(0, \xi_2, \dots, \xi_{2n+1}) &= \left(- \sum_{i=2}^{2n} a_i \xi_i, \xi_2, \dots, \xi_{2n+1} \right) \in \mathbb{W}, \end{aligned}$$

and such that $\Psi(\xi_1, \dots, \xi_{2n+1}) = \Psi(0, \xi_2, \dots, \xi_{2n+1}) \cdot \Psi(\xi_1, 0, \dots, 0)$. The components of Ψ can be explicitly computed as follows

$$\Psi_1(\xi) = - \sum_{i=2}^{2n} a_i \xi_i + v_1 \xi_1; \quad \Psi_i(\xi) = \xi_i + \xi_1 v_i, \quad \text{for } i = 2, \dots, 2n;$$

$$\Psi_{2n+1}(\xi) = \xi_{2n+1} - \xi_1 \left[- \sum_{i=2}^n \xi_i v_{n+i} + \sum_{i=1}^n \xi_{n+i} v_i + \sum_{i=2}^{2n} a_i \xi_i \right] / 2.$$

Since $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, it is easy to see that Ψ is 1-1. In addition, $\Psi^{-1}(\mathbb{W}) = \{\xi_1 = 0\}$, identified with a subset of \mathbb{R}^{2n} , and $\Psi(0, \cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{W}$ is still 1-1. Let us prove now that

$$(64) \quad c_1(\mathbb{W}, \mathbb{V}) := \left| \det \frac{\partial \Psi}{\partial \xi} \right| = \left| \sum_{i=1}^{2n} a_i v_i \right| \neq 0,$$

since $v \notin \mathbb{W}$. In particular, Ψ is an invertible linear map. Indeed, we have

$$\det \frac{\partial \Psi}{\partial \xi} = \det \begin{pmatrix} v_1 & -a_2 & \cdots & \cdots & -a_{2n} & 0 \\ v_2 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{2n} & 0 & 0 & \cdots & 1 & 0 \\ * & * & * & \cdots & * & 1 \end{pmatrix} = v_1 + \sum_{i=2}^{2n} a_i v_i.$$

Notice that, for all $g \in C_c^1(\mathbb{H}^n)$, $\frac{\partial}{\partial \xi_1}(g \circ \Psi) = (Vg) \circ \Psi$.

Denoting $f(w) = \exp(\varphi(w)V)$ with $\varphi : \mathbb{W} \rightarrow \mathbb{R}$,

$$(65) \quad \hat{E}_f^- := \Psi^{-1}(E_f^-) = \{ \xi \in \mathbb{R}^{2n+1} : \xi_1 < \varphi(\Psi(0, \xi_2, \dots, \xi_{2n+1})) \}.$$

Moreover

$$(66) \quad \Psi(\varphi(\Psi(0, \cdot), \cdot)) = \Phi_f \circ \Psi(0, \cdot).$$

By (65) and (64), we get

$$(67) \quad \int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1} (g \circ \Psi) d\mathcal{L}^{2n+1} = \frac{1}{c_1(\mathbb{W}, \mathbb{V})} \int_{E_f^-} Vg d\mathcal{L}^{2n+1}.$$

On the other hand, by Fubini theorem, (65) and (66), we have

$$(68) \quad \int_{\hat{E}_f^-} \frac{\partial}{\partial \xi_1} (g \circ \Psi) d\mathcal{L}^{2n+1} = \int_{\{\xi_1=0\}} g \circ \Phi_f \circ \Psi d\mathcal{L}^{2n}.$$

Then, by [29], Theorem 1.19, and classical area formula for linear maps,

$$\begin{aligned} \int_{\mathbb{H}^n} g d((\Phi_f \circ \Psi(0, \cdot))_{\#}(\mathcal{L}^{2n} \llcorner \mathbb{W})) &= \int_{\{\xi_1=0\}} g \circ \Phi_f \circ \Psi(0, \cdot) d\mathcal{L}^{2n} \\ &= c_2(\mathbb{W}) \int_{\mathbb{W}} g \circ \Phi_f d(\mathcal{L}^{2n} \llcorner \mathbb{W}), \end{aligned}$$

for all $g \in C_c^1(\mathbb{H}^n)$, for a positive constant $c_2(\mathbb{W})$. By (63), (67) and (68) we get eventually

$$\int_{\mathbb{H}^n} g d((\Phi_f)_{\#}(\mathcal{L}^{2n} \llcorner \mathbb{W})) = c(\mathbb{W}, \mathbb{V}) \int_{\mathbb{H}^n} \langle \nu, V \rangle g d|\partial E_f^-|_{\mathbb{H}},$$

for all $g \in C_c^1(\mathbb{H}^n)$, and the proof is completed. \square

Corollary 4.31. *Under the same assumptions of Lemma 4.30, we have*

$$(\mathcal{L}^{2n} \llcorner \mathbb{W})(\mathbb{W} \setminus \Pi_{\mathbb{W}}(\partial^* E_f^-)) = 0.$$

Proof. By Lemma 4.30,

$$(\mathcal{L}^{2n} \llcorner \mathbb{W})(\mathbb{W} \setminus \Pi_{\mathbb{W}}(\partial^* E_f^-)) = c(\mathbb{W}, \mathbb{V}) \int_{\mathbb{H}^n \setminus \partial^* E_f^-} \langle \nu, V \rangle d|\partial E_f^-|_{\mathbb{H}} = 0,$$

since $|\partial E_f^-|_{\mathbb{H}}(\mathbb{H}^n \setminus \partial^* E_f^-) = 0$. \square

4.5. Rectifiable sets. Some results of the previous sections can be applied to prove the equivalence of two intrinsic notions of $2n$ -dimensional (or 1 -codimensional) rectifiable sets in \mathbb{H}^n . For a related and deeper analysis about equivalence of different notions of intrinsic rectifiable sets in \mathbb{H}^n we refer to [30].

We begin recalling the definition of \mathbb{H} -rectifiable sets that we gave, with a slightly different notation, in [19].

Definition 4.32. For $k = 1, 2, \dots, 2n$, we say that $M \subset \mathbb{H}^n$ is (k, \mathbb{H}) -rectifiable (or k -dimensional \mathbb{H} -rectifiable) if there exists a sequence of k -dimensional \mathbb{H} -regular surfaces $(S_i)_{i \in \mathbb{N}}$ such that, for any bounded $\mathcal{U} \subset \mathbb{H}^n$,

$$\mathcal{S}_d^{k_m}((M \cap \mathcal{U}) \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0,$$

where the metric dimension k_m is $k_m = k$ for $k = 1, 2, \dots, n$ and $k_m = k + 1$ if $k = n + 1, \dots, 2n$.

Another natural definition, using the notion of intrinsic Lipschitz graphs, is the following one

Definition 4.33. For $k = 1, 2, \dots, 2n$, we say that $M \subset \mathbb{H}^n$ is (k, \mathbb{H}_L) -rectifiable if there exist a sequence of k -dimensional intrinsic Lipschitz graphs G_i , $i = 1, 2, \dots$, such that, for any bounded $\mathcal{U} \subset \mathbb{H}^n$,

$$\mathcal{S}_d^{k_m}((M \cap \mathcal{U}) \setminus \bigcup_{i \in \mathbb{N}} G_i) = 0,$$

where the *metric dimension* k_m is $k_m = k$ for $k = 1, 2, \dots, n$ and $k_m = k + 1$ if $k = n + 1, \dots, 2n$.

We stress here that when we speak of k -dimensional \mathbb{H} -regular surfaces or k -dimensional intrinsic Lipschitz graphs, we refer to the *topological dimension* and not to the *metric dimension* of the surface or of the graph.

Since intrinsic regular surfaces are locally graphs of intrinsic Lipschitz functions it follows that the scope of the second definition is larger than the first one. On the other hand we have the following

Proposition 4.34. *If M is $(2n, \mathbb{H}_L)$ -rectifiable then it is $(2n, \mathbb{H})$ -rectifiable.*

Proof. By definition, each G_i is the graph of a 1-dimensional valued, intrinsic Lipschitz function $f_i : \mathcal{C}_i \subset \mathbb{W}_i \rightarrow \mathbb{V}_i$. By Theorem 4.25, we can assume that $\mathcal{C}_i = \mathbb{W}_i$ for all i . Hence, by Theorem 4.17, the subgraph of f_i has locally finite \mathbb{H} -perimeter and, eventually, it is $(2n, \mathbb{H})$ -rectifiable, by the structure theorem for sets of locally finite \mathbb{H} -perimeter proved in [16]. This proves that all of M is $(2n, \mathbb{H})$ -rectifiable. \square

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