TWO RIGIDITY RESULTS FOR STABLE MINIMAL HYPERSURFACES

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Abstract. The aim of this paper is to prove two results concerning the rigidity of complete, immersed, orientable, stable minimal hypersurfaces: we show that they are hyperplane in $\mathbb{R}^4$, while they do not exist in positively curved closed Riemannian $(n+1)$-manifold when $n \leq 5$; in particular, there are no stable minimal hypersurfaces in $\mathbb{S}^{n+1}$ when $n \leq 5$. The first result was recently proved also by Chodosh and Li, and the second is a consequence of a more general result concerning minimal surfaces with finite index. Both theorems rely on a conformal method, inspired by a classical work of Fischer-Colbrie.

Key Words: stable minimal hypersurface, rigidity

AMS subject classification: 53C42, 53C21

1. Introduction

It is well-known that a minimal surface $M^2 \subset \mathbb{R}^3$ is a critical point of the area functional $\mathcal{A}_t$ for all compactly supported variations, i. e. $\frac{d}{dt}\Big|_{t=0} \mathcal{A}_t = 0$; equivalently, M^2 is minimal if and only if the mean curvature H, i.e. the (normalized) trace of the second fundamental form, is identically zero, or if and only if M^2 can be expressed, locally, as the graph $\Gamma(u)$ of a solution u of the minimal surfaces equation

$$(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_{xx} = 0.$$

In 1914, S. Bernstein showed that an entire (i.e., defined on the whole plane $\mathbb{R}^2$) minimal graph in $\mathbb{R}^3$ is necessarily a plane; the so-called "Bernstein problem" in higher dimension can be then stated in the following way: if the graph $\Gamma(u)$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is a minimal hypersurface in $\mathbb{R}^{n+1}$, does $\Gamma(u)$ have to be necessarily a hyperplane? Many famous mathematicians worked on this problem in the Sixties, in particular Fleming [14] (who gave a new proof in the case $n = 2$), De Giorgi [10] (case $n = 3$), Almgren [1] (case $n = 4$), Simons [24] (the three remaining cases for $n \leq 7$) and, eventually, Bombieri, De Giorgi and Giusti [2], who showed that, for $n \geq 8$, there are minimal entire graphs that are not hyperplanes. We explicitly remark that a minimal graph is area-minimizing, i.e. it is not only a critical point of the area functional, but also a minimum, while this is not true for minimal hypersurfaces that are "non-graphical", and also that area-minimizing implies *stability*, that is the non-negativity of the second variation for the area functional $\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_t \geq 0$ for all compactly supported variations.

A natural generalization of the classical Bernstein problem, thus, is the *stable* Bernstein problem, that is: if $M^n \hookrightarrow \mathbb{R}^{n+1}$ is a complete, orientable, isometrically immersed, stable minimal hypersurface, does M have to be necessarily a hyperplane? In the case $n = 2$ the (positive) answer was given in three different papers, which appeared between 1979 and 1981 (see do Carmo and Peng [11], Fischer-Colbrie and Schoen [16] and Pogorelov [19]).

In higher dimensions, the aforementioned result of Bombieri, De Giorgi and Giusti implies that there exist non-flat orientable, complete, stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 8$, while for $n \leq 5$ the stable Bernstein theorems is true with some additional assumptions (for instance, if one requires bounds on the volume growth of geodesic balls, see e.g. [21]; see also [12], [3], [4], [18] and references therein for other interesting results in the same spirit). Moreover, by [2] and [17], we also note that there are non-flat area-minimizing (and thus minimal and stable) complete orientable hypersurfaces $M^7 \hookrightarrow \mathbb{R}^8$.

Up until recently, without additional hypothesis, the remaining cases ($3 \leq n \leq 6$) were still open, even if the study of minimal (in particular stable or in general with finite index) hypersurfaces immersed into a Riemannian manifold (not only the Euclidean space, then) is a very active field and has attracted a lot of interest. Then, in 2021, Chodosh and Li [6] (see also [7]) showed that a complete, orientable, isometrically immersed, stable minimal immersion $M^3 \to \mathbb{R}^4$ is a hyperplane. Their proof, clever and highly non-trivial, is based on the non-parabolicity of M: they perform careful estimates for the quantity

$$F(t) = \int_{\Sigma_t} |\nabla u|^2$$

(here u is a positive Green's function for the Laplacian and Σ_t is the t-level set of u), relating it to $\int_{\Sigma_t} |A_M|^2$ (A_M is the second fundamental form of M).

In this paper we provide a completely different proof of Chodosh and Li result, based on a conformal deformation of the metric, a comparison result and integral estimates, and we also prove another rigidity result when the ambient space is a complete Riemannian manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature. To be precise, and to fix the notation, we consider smooth, complete, connected, orientable, isometrically immersed hypersurfaces $M^n \hookrightarrow (X^{n+1}, h)$, $n \geq 2$, where (X^{n+1}, h) is a (complete) Riemannian manifold of dimension $n+1$ endowed with metric h. We denote with g the induced metric on M and with H the mean curvature of M; we have that M is *minimal* if $H \equiv 0$ on M. In this latter case we say that M is *stable* if

$$\int_M \left[|A|^2 + \mathrm{Ric}_h(\nu,\nu)\right] \varphi^2 \, dV_g \leq \int_M |\nabla \varphi|^2 \, dV_g \qquad \forall \varphi \in C_0^\infty(M), \tag{1.1}$$

where $A = A_M$ is the second fundamental form of M^n, ν is a unit normal vector to M in X and dV_g is the volume form of g .

As we recalled before, stability is related to the non-negativity of the second variation or, equivalently, the non-positivity of the Jacobi operator

$$L_M := \Delta + |A|^2 + \mathrm{Ric}_h(\nu,\nu).$$

The first result is thus the following:

Theorem 1.1. *A complete, orientable, immersed,* stable *minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ is a hyperplane.*

The second result concerns minimal hypersurfaces with *finite index*. We recall that a minimal immersion $M^n \hookrightarrow (X^{n+1}, h)$ has finite index if the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator L_M on every compact domain in M with Dirichlet boundary conditions is finite; in particular stability implies finite (equal zero) index. Before presenting our next result, we need to recall the notion of *bi-Ricci curvature tensor* introduced in [23]: given two orthonormal tangent vectors u, v we define

$$\mathrm{BRic}_h(u,v) = \mathrm{Ric}_h(u,u) + \mathrm{Ric}_h(v,v) - \mathrm{Sect}_h(u,v)\,,$$

where $\mathrm{Sect}_h(u,v)$ denotes the sectional curvature of the plane spanned by u and v. Our second result is the following:

Theorem 1.2. *If (X^{n+1}, h) is a complete $(n+1)$-dimensional, $n \leq 5$, manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature, then every complete, orientable, immersed, minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$ with* finite index *must be compact.*

As a byproduct we have the

Corollary 1.3. *If (X^{n+1}, h) is a complete $(n+1)$-dimensional, $n \leq 5$, manifold with non-negative sectional curvature and uniformly positive Ricci curvature, then there is no complete, orientable, immersed,* stable *minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$.*

In particular, there is no complete, orientable, immersed, stable minimal hypersurface of the round spheres $M^n \hookrightarrow (\mathbb{S}^{n+1}, g_{\mathrm{std}})$, provided $n \leq 5$. In dimension $n = 2$ this follows from a more general result proved in [22], while, in dimension $n = 3$, it was recently proved in [8, Corollary 1.5]. We mention that Theorem 1.2 holds also for complete, orientable, immersed, stable minimal hypersurface of the cylinder $M^n \hookrightarrow (\mathbb{R} \times \mathbb{S}^n, g_{\mathrm{std}})$ (observe that in this case $\mathrm{Sect} \geq 0$ and $\mathrm{BRic} \geq 1$), provided $n \leq 5$. As far as we know, Corollary 1.3 is new in the cases $n = 4, 5$ (see [5] where the same technique is used). We do not know if Theorem 1.2 and Corollary 1.3 hold also in dimension greater than five. We note that, in the same spirit, in [23] the authors obtained a compactness result for stable minimal hypersurfaces of dimension $n \leq 4$ immersed in space with uniformly positive bi-Ricci curvature.

2. Proof of Theorem 1.1

In this section we give an alternative proof of [6, Theorem 1] (see Theorem 1.1). The main idea is to use a weighted volume comparison for a suitable conformal metric $\widetilde{g}$ together with a new weighted integral estimate inspired by [21].

Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected, orientable, isometrically immersed, stable minimal hypersurface.

2.1. **Conformal change.** It is well known (see e.g. [15, Proposition 1]) that, since $M^n \hookrightarrow \mathbb{R}^{n+1}$ is stable, then there exists a positive function $0 < u \in C^\infty(M)$ satisfying

$$-\Delta_g u = |A|_g^2 u \quad \text{on } M\,. \tag{2.1}$$

Following the line in [15] (see also [13]), let $k > 0$ and consider the conformal metric

$$\widetilde{g} = u^{2k} g,$$

where $g = \imath^* h$ is the induced metric on M (and $\imath$ denotes the inclusion). First of all we prove the following lower bound for a modified Bakry-Emery-Ricci curvature of $\widetilde{g}$. In particular, this implies the non-negativity of the 2-Bakry-Emery-Ricci curvature of $\widetilde{g}$ for a suitable k.

Lemma 2.1. *Let $f := k(n-2)\log u$. Then the Ricci tensor of the metric $\widetilde{g} = u^{2k} g$ satisfies*

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1 - k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g$$

in the sense of quadratic forms. In particular, if $n = 3$ and $k = \frac{2}{3}$, then the 2-Bakry-Emery-Ricci tensor $\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df$ satisfies

$$\mathrm{Ric}^{2,f}_{\widetilde{g}} \geq 0.$$

Proof. Since $f = k(n-2)\log u$, we have

$$df = k(n-2)\frac{du}{u}$$

and

$$\nabla^2_g f = k(n-2)\left(\frac{\nabla^2_g u}{u} - \frac{du \otimes du}{u^2}\right),$$

which implies

$$\Delta_g f = k(n-2)\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|_g^2}{u^2}\right).$$

On the other hand, from the standard formulas for a conformal change of the metric $\widetilde{g} = e^{2\varphi} g$, $\varphi \in C^\infty(M)$, $\varphi > 0$ we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla_g^2\varphi - d\varphi \otimes d\varphi\right) - \left[\Delta_g\varphi + (n-2)|\nabla_g\varphi|_g^2\right] g$$

and

$$\nabla_{\widetilde{g}}^2 f = \nabla_g^2 f - (df \otimes d\varphi + d\varphi \otimes df) + g\left(\nabla f, \nabla\varphi\right) g.$$

Note that, in our case, $\varphi = k\log u$; now we exploit the facts that u is a solution of equation (2.1) to write

$$\begin{aligned}\mathrm{Ric}_{\widetilde{g}} + \nabla_{\widetilde{g}}^2 f &= \mathrm{Ric}_g - k^2(n-2)\frac{du\otimes du}{u^2} + k|A|_g^2 g + k\frac{|\nabla_g u|_g^2}{u^2} g \\ &= \mathrm{Ric}_g - \frac{df\otimes df}{n-2} + k|A|^2 g + \frac{|\nabla_g f|_g^2}{k(n-2)^2} g.\end{aligned}$$

From the Cauchy-Schwarz inequality we have

$$|\nabla_g f|_g^2 g \geq df \otimes df,$$

thus

$$\mathrm{Ric}_{\widetilde{g}} + \nabla_{\widetilde{g}}^2 f \geq \mathrm{Ric}_g + \frac{1-k(n-2)}{k(n-2)^2} df\otimes df + k|A|_g^2 g;$$

from Gauss equations in the minimal case we get $\mathrm{Ric}_g = -A^2$; since A is traceless we have the inequality

$$A^2 \leq \frac{n-1}{n}|A|^2 g,$$

and substituting in the previous relation we conclude

$$\mathrm{Ric}_{\widetilde{g}} + \nabla_{\widetilde{g}}^2 f - \frac{1-k(n-2)}{k(n-2)^2} df\otimes df \geq \left(k - \frac{n-1}{n}\right)|A|_g^2 g.$$

□

2.2. **Completeness.** In this subsection we are going to prove that the conformal metric $\widetilde{g} = u^{2k} g$ is complete, provided $n = 3$ and $k = \frac{2}{3}$. In order to do this we follow the strategy in [15] and we use some computations in [13]. First, we recall that in the proof of [15, Theorem 1], given a reference point $O \in M^n$, the author showed the existence of a $\widetilde{g}$-minimizing geodesic,

$$\gamma(s) : [0,\infty) \to M^n,$$

where s is the g-arclength and $M^n \hookrightarrow \mathbb{R}^{n+1}$ is the usual complete, connected, orientable, isometrically immersed, stable minimal hypersurface. For the sake of completeness, we report the argument here. First of all, for every $R > 0$, we consider the geodesic ball (of g) centered at O of radius R, $B_R(O)$. Then, we first claim that there exists a $\widetilde{g}$-minimizing geodesic, γ_R, joining O to any boundary point of $B_R(O)$. Indeed, consider $u_R := u + \eta$, where η is a smooth function such that $\eta \equiv 0$ in $B_R(O)$ and $\eta \equiv 1$ in $M^n \setminus B_{R+1}(O)$. Since u_R is bounded below, the metric

$$\widetilde{g}_R = u_R^{\frac{2(n-1)}{n}} g$$

is complete, and these geodesics exist. Therefore, for every $R_i > 0$, since $\partial B_{R_i}(O)$ is compact, there exists $x_i \in \partial B_{R_i}(O)$ so that x_i is closest (in $\widetilde{g}_R$) to O. Let γ_i be the $\widetilde{g}_R$-minimizing geodesic joining O to x_i. Note that $\gamma_i \subset B_{R_i}(O)$ or another point would be closer to O. Since $u_{R_i} = u$ in $B_{R_i}(O)$, then γ_i is a $\widetilde{g}$-minimizing geodesic. We parametrize γ_i with respect to g-arclength. In particular, since $|\dot{\gamma}_i(s)|_g = 1$ for every s, up to subsequences, the sequence $\dot{\gamma}_i(0)$ converges to a limit vector as $R_i \to \infty$. Thus, by ODE theory and Ascoli-Arzelà, γ_i converges on compact sets of $[0, \infty)$ to a limiting curve γ which is a $\widetilde{g}$-minimizing geodesic and is parametrized by g-arclength.

Remark 2.2. *(i) We observe that the completeness of the metric $\widetilde{g} = u^{2k}g$ will follow if we can show that the $\widetilde{g}$-length of γ is infinite, i.e.*

$$\int_\gamma d\tilde{s} = \int_\gamma u^k \, ds = +\infty.$$

Indeed, by construction, the $\widetilde{g}$-length of every other divergent geodesic starting from O (i.e. its image does not lie in any ball $B_R(O)$) must be greater or equal than the one of γ.

(ii) Note that γ has unit speed with respect to g and to $\widetilde{g}$, when it is parametrized by the arclength s and $\tilde{s}$, respectively.

From now on

$$n = 3 \quad \text{and} \quad k = \frac{2}{3}.$$

Lemma 2.3. *The metric $\widetilde{g} = u^{\frac{4}{3}}g$ is complete.*

Proof. We do part of the computations for every n. We consider the $\widetilde{g}$-minimizing geodesic γ just constructed and as observed in Remark 2.2 the completeness of $\widetilde{g}$ is equivalent to prove that γ has infinite $\widetilde{g}$ length, i.e.

$$\int_0^{+\infty} u^k(\gamma(s)) \, ds = +\infty.$$

Since γ is minimizing, by the second variation formula, following the computations in the proof of Theorem 1 (with $H = 0$) in [13], we obtain

$$\begin{aligned}(n-1)\int_0^{+\infty} (\varphi_s)^2 u^{-k} \, ds \geq & \int_0^{+\infty} \varphi^2 u^{-k} \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds \\ & - k(n-2)\int_0^{+\infty} \varphi^2 u^{-k} (\log u)_{ss} \, ds + k \int_0^{+\infty} \varphi^2 u^{-k} \frac{|\nabla u|^2}{u^2} \, ds\end{aligned}$$

for every smooth function φ with compact support in $(0, +\infty)$ and for every $k > 0$. Since A is trace-free, we have

$$|A|^2 \geq A_{11}^2 + A_{22}^2 + \ldots + A_{nn}^2 + 2\sum_{j=2}^n A_{1j}^2 \geq \frac{n}{n-1} A_{11}^2 + 2\sum_{j=2}^n A_{1j}^2,$$

thus

$$k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \geq \left(\frac{kn}{n-1} - 1 \right) A_{11}^2 + (2k-1)\sum_{j=2}^n A_{1j}^2.$$

In particular, if

$$k \geq \frac{n-1}{n} \tag{2.2}$$

we have

$$\int_0^{+\infty} \varphi^2 u^{-k} \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds \geq 0.$$

Using this estimate, the fact that $|\nabla u|^2 \geq (u_s)^2$ and integrating by parts, we obtain

$$(n-1)\int_0^{+\infty}(\varphi_s)^2u^{-k}\,ds \geq 2k(n-2)\int_0^{+\infty}\varphi\varphi_s u^{-k-1}u_s\,ds + k\left[1-k(n-2)\right]\int_0^{+\infty}\varphi^2u^{-k-2}(u_s)^2\,ds.$$

Let now $\varphi = u^k\psi$, with ψ smooth with compact support in $(0,+\infty)$. We have

$$\begin{aligned}\varphi^2u^{-k} &= u^k\psi^2,\\ \varphi_s &= k\psi u^{k-1}u_s + u^k\psi_s,\\ (\varphi_s)^2u^{-k} &= k^2\psi^2u^{k-2}(u_s)^2 + u^k(\psi_s)^2 + 2k\psi\psi_s u^{k-1}u_s,\end{aligned}$$

and substituting in the previous relation we get

$$(n-1)\int_0^{+\infty}u^k(\psi_s)^2\,ds \geq -2k\int_0^{+\infty}\psi\psi_s u^{k-1}u_s\,ds + k(1-k)\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds. \tag{2.3}$$

Let

$$I := \int_0^{+\infty}\psi\psi_s u^{k-1}u_s\,ds;$$

thus we have

$$I = \frac{1}{k}\int_0^{+\infty}\psi\psi_s(u^k)_s\,ds = -\frac{1}{k}\int_0^{+\infty}u^k(\psi_s)^2\,ds - \frac{1}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds.$$

Moreover, for every $t>1$ and using Young's inequality for every $\varepsilon>0$, we have

$$\begin{aligned}2kI &= 2ktI + 2k(1-t)I\\ &= -2t\int_0^{+\infty}u^k(\psi_s)^2\,ds - 2t\int_0^{+\infty}\psi\psi_{ss}u^k\,ds + 2k(1-t)\int_0^{+\infty}\psi\psi_s u^{k-1}u_s\,ds\\ &\leq -2t\int_0^{+\infty}u^k(\psi_s)^2\,ds - 2t\int_0^{+\infty}\psi\psi_{ss}u^k\,ds\\ &\quad + k(t-1)\varepsilon\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds + \frac{k(t-1)}{\varepsilon}\int_0^{+\infty}u^k(\psi_s)^2\,ds.\end{aligned}$$

Assuming

$$k<1 \tag{2.4}$$

and choosing

$$\varepsilon := \frac{1-k}{t-1}$$

we obtain

$$\begin{aligned}2kI \leq &-2t\int_0^{+\infty}\psi\psi_{ss}u^k\,ds + k(1-k)\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds\\ &+\left[\frac{k(t-1)^2}{1-k}-2t\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds.\end{aligned}$$

From (2.3) we get

$$0 \leq \left[\frac{k(t-1)^2}{1-k}-2t+(n-1)\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds - 2t\int_0^{+\infty}\psi\psi_{ss}u^k\,ds$$

for every $t>1$ and every k satisfying (2.2) and (2.4). Let

$$P(t) := \frac{k(t-1)^2}{1-k} - 2t + (n-1)$$

and choose $k = \frac{n-1}{n}$. It is easy to see that $P(t)$ is negative for some $t > 1$ if $n = 3$: indeed

$$P(t) = (n-1)t^2 - 2nt + 2(n-1) = 2t^2 - 6t + 4 = -2(t-1)(2-t).$$

Therefore, if $n = 3$, $k = \frac{2}{3}$ and $t = \frac{3}{2}$, we deduce

$$0 \leq -\int_0^{+\infty} u^{\frac{2}{3}} (\psi_s)^2 \, ds - 6 \int_0^{+\infty} u^{\frac{2}{3}} \psi \psi_{ss} \, ds$$

for every ψ smooth with compact support in $(0, +\infty)$. Now we choose $\psi = s\eta$ with η smooth with compact support in $(0, +\infty)$: thus

$$\psi_s = \eta + s\eta_s, \quad \psi_{ss} = 2\eta_s + s\eta_{ss},$$

and we get

$$\int_0^{+\infty} u^{\frac{2}{3}} \eta^2 \, ds \leq \int_0^{+\infty} u^{\frac{2}{3}} \left(-14 s\eta\eta_s - 6s^2 \eta\eta_{ss} - s^2 (\eta_s)^2 \right) \, ds.$$

Choose η such that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2, respectively, for $R \leq s \leq 2R$ (C is a positive constant). Then

$$\int_0^{R} u^{\frac{2}{3}} \, ds \leq \int_0^{+\infty} u^{\frac{2}{3}} \eta^2 \, ds \leq C \int_R^{+\infty} u^{\frac{2}{3}} \, ds$$

for some $C > 0$ independent of R. We conclude that

$$\int_0^{+\infty} u^{\frac{2}{3}} \, ds = +\infty,$$

i.e. $\widetilde{g} = u^{\frac{4}{3}} g$ is complete. □

2.3. **Weighted integral estimates.** From lemma 2.1 and lemma 2.3 we have that the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete and it has non-negative 2-Bakry-Emery-Ricci curvature. Using well known comparison results (see [20]) we immediately obtain the following weighted Bishop-Gromov volume estimate for a geodesic ball $B_R^{\widetilde{g}}(x_0)$ centered at $x_0 \in M$, of radius R, with respect to the metric $\widetilde{g}$.

Corollary 2.4. *Let $x_0 \in M^3$. Then, for every $R > 0$, there exists $C > 0$ such that the f-volume*

$$\mathrm{Vol}_f \, B_R^{\widetilde{g}}(x_0) := \int_{B_R^{\widetilde{g}}(x_0)} e^{-f} \, dV_{\widetilde{g}} \leq CR^5,$$

where $f = \frac{2}{3} \log u$. Equivalently, in terms of u and the volume form of g,

$$\int_{B_R^{\widetilde{g}}(x_0)} u^{\frac{4}{3}} \, dV_g \leq CR^5.$$

The last ingredient that we need in the proof of Theorem 1.1 is the following weighted integral inequality in the spirit of [21, Theorem 1].

Lemma 2.5. *For every $0 < \delta < \frac{1}{100}$, there exists $C > 0$ such that*

$$\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}} \psi^{5+\delta} \, dV_g \leq C \int_M u^{-2-\frac{2\delta}{3}} |\nabla \psi|^{5+\delta} \, dV_g \qquad \forall \psi \in C_0^\infty(M).$$

Proof. Again, we do part of the computations for every n. From [21] we get

$$\int_M |A|^p \varphi^2 \leq C \int_M |A|^{p-2} |\nabla \varphi|^2 \qquad \forall \varphi \in C_0^\infty(M), \tag{2.5}$$

for every $p \in [4, 4+\sqrt{8/n}]$ and for some $C = C(n,p) > 0$. For the sake of completeness we report here the proof of (2.5). We take $\varphi = |A|^{1+q}\psi$, $q \geq 0$, with $\psi \in C_0^\infty(M)$, in the stability inequality (1.1) obtaining

$$\int_M |A|^{4+2q}\psi^2 \leq [(1+q)^2+\varepsilon] \int_M |A|^{2q}|\nabla|A||^2\psi^2 + \frac{1+q}{\varepsilon}\int_M |A|^{2+2q}|\nabla\psi|^2,$$

for every $\varepsilon > 0$, where we used Young's inequality. On the other hand, multiplying Simons' inequality (see [9, Lemma 2.1] for a proof)

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2. \tag{2.6}$$

by $|A|^{2q}\psi^2$ and integrating by parts, we get

$$\left(\frac{2}{n}+1+2q-\varepsilon\right)\int_M |A|^{2q}|\nabla|A||^2\psi^2 \leq \int_M |A|^{4+2q}\psi^2 + \frac{1}{\varepsilon}\int_M |A|^{2+2q}|\nabla\psi|^2$$

for every $\varepsilon > 0$, where we used again Young's inequality. Since $q \geq 0$, for $\varepsilon > 0$ sufficiently small, we obtain

$$\left\{1 - [(1+q)^2+\varepsilon]\left(\frac{2}{n}+1+2q-\varepsilon\right)^{-1}\right\}\int_M |A|^{4+2q}\psi^2 \leq C\int_M |A|^{2+2q}|\nabla\psi|^2.$$

Let $q := \frac{p-4}{2}$. For $\varepsilon > 0$ small enough, we have

$$1 - [(1+q)^2+\varepsilon]\left(\frac{2}{n}+1+2q-\varepsilon\right)^{-1} > 0$$

if $p \in [4, 4+\sqrt{8/n}]$ and we finally obtain

$$\int_M |A|^p\psi^2 \leq C\int_M |A|^{p-2}|\nabla\psi|^2 \qquad \forall\psi \in C_0^\infty(M).$$

Taking $\psi = \varphi^{p/2}$, by Holder's inequality we get (2.5).

Take $\varphi = u^\alpha\psi$, with ψ smooth with compact support, u the solution of (2.1) and $\alpha < 0$. Since, from Cauchy-Schwarz and Young's inequalities,

$$|\nabla(u^\alpha\psi)|^2 \leq 2\psi^2|\nabla(u^\alpha)|^2 + 2u^{2\alpha}|\nabla\psi|^2\,, \tag{2.7}$$

then (2.5) becomes

$$\int_M |A|^p u^{2\alpha}\psi^2 \leq 2C\left[\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\right] \quad \forall\psi \in C_0^\infty(M). \tag{2.8}$$

Now we tackle the first integral on the right-hand side of (2.8) firstly integrating by parts

$$\begin{aligned}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 = -\int_M |A|^{p-2}\psi^2 u^\alpha\Delta u^\alpha - \int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle \\ -2\int_M |A|^{p-2}u^\alpha\psi\langle\nabla u^\alpha, \nabla\psi\rangle\,,\end{aligned}$$

secondly we use the fact that

$$\Delta u^\alpha = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 \quad \text{and} \quad |\nabla u^\alpha|^2 = \alpha^2 u^{2\alpha-2}|\nabla u|^2\,,$$

together with Cauchy-Schwarz and Young's inequalities to get

$$\begin{aligned}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \leq -\alpha\int_M |A|^{p-2}u^{2\alpha-1}\psi^2\Delta u - \frac{\alpha-1}{\alpha}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \\ -\int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \varepsilon\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,\end{aligned}$$

for all $\varepsilon > 0$. From (2.1) we find

$$\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2 - \frac{\alpha-1}{\alpha}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2$$
$$-\int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \varepsilon\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

i.e.

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2$$
$$-\int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

Now, since

$$\nabla|A|^{p-2} = (p-2)|A|^{p-3}\nabla|A| = (p-2)|A|^{\frac{p-2}{2}}|A|^{\frac{p-4}{2}}\nabla|A|\,,$$

then, from Cauchy-Schwarz and Young's inequalities we obtain

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}-\frac{p-2}{2t_1}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2 \tag{2.9}$$
$$+\frac{(p-2)t_1}{2}\int_M |A|^{p-4}\psi^2u^{2\alpha}|\nabla|A||^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

for every $t_1 > 0$. Now, multiplying by $|A|^{p-4}f^2$ the Simons' inequality (2.6), integrating by parts and using Young's inequality we obtain

$$\int_M |A|^p f^2 \ge \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2 f^2 - \frac{1}{t_2}\int_M |A|^{p-2}|\nabla f|^2$$

for every $t_2 > 0$. Choosing $f = u^\alpha\psi$ we get

$$\int_M |A|^p u^{2\alpha}\psi^2 \ge \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2 u^{2\alpha}\psi^2 \tag{2.10}$$
$$-\left(\frac{1}{t_2}+\varepsilon\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 - \frac{1}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2$$

for every $\varepsilon > 0$, since

$$|\nabla(u^\alpha\psi)|^2 \le (1+t_2\varepsilon)\psi^2|\nabla(u^\alpha)|^2 + \left(1+\frac{1}{t_2\varepsilon}\right)u^{2\alpha}|\nabla\psi|^2\,.$$

Now let $\delta > 0$. Using (2.10) in (2.9) with

$$\alpha = -1-\frac{\delta}{3} \le -1$$

we obtain

$$\left(1+\frac{2+\frac{\delta}{3}}{1+\frac{\delta}{3}} - \left(2+\frac{\delta}{3}\right)\varepsilon - \frac{p-2}{2t_1} - \frac{1+\frac{\delta}{3}}{t_2}\right)\int_M |A|^{p-2}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2$$
$$\le \left[\frac{1}{\varepsilon} + \frac{1+\frac{\delta}{3}}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\right]\int_M |A|^{p-2}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2$$
$$+\left[\frac{(p-2)t_1}{2} - \frac{2+\frac{2\delta}{3}}{n} - \left(1+\frac{\delta}{3}\right)p + 3 + \delta + \left(1+\frac{\delta}{3}\right)t_2\right]\int_M |A|^{p-4}\psi^2u^{-2-\frac{2\delta}{3}}|\nabla|A||^2\,,$$

for all $\varepsilon, t_1, t_2 > 0$. Let

$$n = 3, \quad p = 5+\delta, \quad t_1 = \frac{2(5+3\delta)}{9}, \quad t_2 = 1$$

we obtain
$$\frac{(p-2)t_1}{2}-\frac{2+\frac{2\delta}{3}}{n}-\left(1+\frac{\delta}{3}\right)p+3+\delta+\left(1+\frac{\delta}{3}\right)t_2=0$$
and
$$1+\frac{2+\frac{\delta}{3}}{1+\frac{\delta}{3}}-\frac{p-2}{2t_1}-\frac{1+\frac{\delta}{3}}{t_2}=\frac{117+54\delta-47\delta^2-12\delta^3}{180+168\delta+36\delta^2}.$$
Thus
$$\begin{aligned}\left(\frac{117+54\delta-47\delta^2-12\delta^3}{180+168\delta+36\delta^2}-\left(2+\frac{\delta}{3}\right)\varepsilon\right)&\int_M|A|^{3+\delta}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2\\ \leq\left[\frac{1}{\varepsilon}+\left(1+\frac{\delta}{3}\right)\left(1+\frac{1}{\varepsilon}\right)\right]&\int_M|A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2,\end{aligned}$$
for all $\varepsilon>0$. Choosing $0<\delta<1/100$ and ε small enough we obtain
$$\int_M|A|^{3+\delta}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2\leq C\int_M|A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2\,,$$
for some $C>0$. From (2.8) and Young's inequality we get
$$\begin{aligned}\int_M|A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2&\leq C\int_M|A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2\\&\leq\varepsilon'\int_M|A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2+\frac{C}{\varepsilon'}\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)}\end{aligned}$$
for all $\varepsilon'>0$ and $\psi\in C_0^\infty(M)$. Therefore
$$\begin{aligned}\int_M|A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2&\leq C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)}\\&=C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi^{\frac{2}{5+\delta}}|^{5+\delta}.\end{aligned}$$
The conclusion now follows immediately by replacing ψ with $\psi^{\frac{2}{5+\delta}}$. □

2.4. **Final estimate.** Combining Lemma 2.5 with Corollary 2.4 we can conclude the proof of Theorem 1.1. More precisely, let $x_0\in M$ and let $\widetilde{r}$ the distance function from x_0 with respect to the metric $\widetilde{g}=u^{\frac{4}{3}}g$. We choose $\psi:=\eta(\widetilde{r})$ with $0\leq\eta\leq 1$, $\eta\equiv 1$ on $[0,R]$, $\eta\equiv 0$ on $[2R,+\infty)$ and $|\eta'|\leq C/R$ on $[R,2R]$, for some $C>0$ and $R>0$. From Lemma 2.5, for some $0<\delta<1/100$, we have
$$\begin{aligned}\int_M|A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\eta^{5+\delta}\,dV_g&\leq C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|_g^{5+\delta}\,dV_g\\&=C\int_M u^{-2-\frac{2\delta}{3}+\frac{2(5+\delta)}{3}}|\tilde{\nabla}\psi|_{\widetilde{g}}^{5+\delta}\,dV_g\\&\leq\frac{C}{R^{5+\delta}}\int_{B^{\widetilde{g}}_{2R}(x_0)}u^{\frac{4}{3}}\,dV_g\\&\leq\frac{C}{R^\delta},\end{aligned}$$
where we used the fact that $|\tilde{\nabla}\widetilde{r}|_{\widetilde{g}}\equiv 1$ and Corollary 2.4. Since $\delta>0$, letting $R\to+\infty$ we get
$$|A|\equiv 0\quad\text{on } M^3$$
and this concludes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let (X^{n+1}, h) be a complete n-dimensional, $n \le 5$, manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature and consider an orientable, immersed, minimal hypersurface $M^n \to (X^{n+1}, h)$ with finite index. Suppose, by contradiction, that M is non-compact. It is well known (see [15, Proposition 1]) that there exist $0 < u \in C^\infty(M)$ and a compact subset $K \subset M$ such that u solves

$$-\Delta u = \left[|A|^2 + \mathrm{Ric}_h(\nu, \nu)\right] u \qquad \text{on } M \setminus K.$$

Let $k > 0$ and consider the conformal metric

$$\widetilde{g} = u^{2k} g.$$

where g is the induced metric on M. Let s be the arc length with respect to the metric g. Following the construction in [15, Theorem 1], we can construct a minimizing geodesic $\widetilde{\gamma}(s) : [0, +\infty) \to M \setminus K$ in the metric $\widetilde{g}$ which has infinite length in the metric g. Now we can argue exactly as in the proof of estimate (6) in [13], using $H \equiv 0$, obtaining

$$\begin{aligned}(n-1)\int_0^a (\varphi_s)^2\,ds \ge{}& k(n-3)\int_0^a \varphi\varphi_s \frac{u_s}{u}\,ds + \frac{k\left[4 - k(n-1)\right]}{4}\int_0^a \varphi^2 \left(\frac{u_s}{u}\right)^2 ds \\ &+ \int_0^a \varphi^2 \left(k\mathrm{Ric}_h(\nu,\nu) + \sum_{j=2}^n R^h_{1j1j}\right) ds \\ &+ \int_0^a \varphi^2 \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2\right) ds,\end{aligned}$$

for every smooth function φ such that $\varphi(0) = \varphi(a) = 0$ and for every $k > 0$. Arguing as in [23, Section 2] we have the following identity

$$\begin{aligned}k\mathrm{Ric}_h(\nu,\nu) + \sum_{j=2}^n R^h_{1j1j} &= k\mathrm{BRic}_h(e_1,\nu) - k\mathrm{Ric}_h(e_1,e_1) + kR^h_{1\nu1\nu} + \sum_{j=2}^n R^h_{1j1j} \\ &= k\mathrm{BRic}_h(e_1,\nu) + (1-k)\sum_{j=2}^n R^h_{1j1j}.\end{aligned}$$

Since (N^{n+1}, h) is with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature, we have $R^h_{1j1j} \ge 0$ for every $j = 2, \dots, n$ and either

$$\mathrm{BRic}_h(e_1, \nu) \ge R_0 \qquad \text{or} \qquad \mathrm{Ric}_h(\nu,\nu) \ge R_0$$

for some $R_0 > 0$. Therefore, if $k \le 1$, we get

$$\begin{aligned}(n-1)\int_0^a (\varphi_s)^2\,ds \ge{}& k(n-3)\int_0^a \varphi\varphi_s \frac{u_s}{u}\,ds + \frac{k\left[4 - k(n-1)\right]}{4}\int_0^a \varphi^2 \left(\frac{u_s}{u}\right)^2 ds \\ &+ \int_0^a \varphi^2 \left(kR_0 + k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2\right) ds.\end{aligned}$$

Since A is trace-free, we have

$$|A|^2 \ge A_{11}^2 + A_{22}^2 + \ldots + A_{nn}^2 + 2\sum_{j=2}^n A_{1j}^2 \ge \frac{n}{n-1}A_{11}^2 + 2\sum_{j=2}^n A_{1j}^2,$$

thus

$$k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \geq \left(\frac{kn}{n-1} - 1\right) A_{11}^2 + (2k-1)\sum_{j=2}^n A_{1j}^2.$$

Choose

$$k = \frac{n-1}{n} \leq 1.$$

We get

$$\int_0^a (\varphi_s)^2\, ds \geq \frac{n-3}{n}\int_0^a \varphi\varphi_s \frac{u_s}{u}\, ds + \frac{6n-n^2-1}{4n^2}\int_0^a \varphi^2\left(\frac{u_s}{u}\right)^2 ds + \frac{R_0}{n}\int_0^a \varphi^2\, ds.$$

If $n \leq 5$, we have

$$\frac{6n-n^2-1}{4n^2} \geq \delta_0 > 0.$$

Moreover, there exists $C > 0$, such that

$$\frac{n-3}{n}\varphi\varphi_s\frac{u_s}{u} \geq -\delta_0\varphi^2\left(\frac{u_s}{u}\right)^2 - C(\varphi_s)^2.$$

Therefore, there exists $C > 0$, such that

$$C\int_0^a (\varphi_s)^2\, ds \geq \frac{R_0}{n}\int_0^a \varphi^2\, ds$$

for every smooth function φ such that $\varphi(0) = \varphi(a) = 0$. Integrating by parts we obtain

$$\int_0^a \left(\varphi\varphi_s s + CR_0\varphi^2\right) ds \leq 0.$$

Choosing $\varphi(s) = \sin(\pi s\, a^{-1})$, $s \in [0, a]$ one has

$$\left(CR_0 - \frac{\pi^2}{a^2}\right)\int_0^a \sin^2(\pi s\, a^{-1}) ds \leq 0$$

i.e.

$$a^2 \leq \frac{\pi^2}{CR_0}.$$

We conclude that the length (in the metric g) of the geodesic $\widetilde{\gamma}(s)$ is finite and this gives a contradiction. Therefore (M^n, g) must be compact and this concludes the proof of Theorem 1.2. □

Proof of Corollary 1.3. If M is stable, by Theorem 1.2 it must be compact. Moreover there exists $u > 0$ satisfying

$$-\Delta u = \left[|A|^2 + \mathrm{Ric}_h(\nu, \nu)\right] u \qquad \text{on } M.$$

Integrating over M we get a contradiction, since $\mathrm{Ric}_h > 0$ on M. Equivalently, one can use $f \equiv 1$ in the stability inequality (1.1) to get a contradiction. □

Acknowledgments. *The first and the second authors are members of the GNSAGA, Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of INdAM. The third author is member of GNAMPA, Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni of INdAM. The authors thank Douglas Stryker for useful comments regarding Theorem 1.2.*

Data availability statement
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] F. J. Jr. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem.*, Ann. of Math (2) 84, 277–292 (1966).

[2] E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem.*, Invent. Math. 7, 243–268 (1969).

[3] H.-D. Cao, Y. Shen, S. Zhu, *The structure of stable minimal hypersurfaces in* $\mathbb{R}^{n+1}$*. (English summary)*, Math. Res. Lett. 4, no. 5, 637–644 (1997).

[4] Q. Chen, *Curvature estimates for stable minimal hypersurfaces in* $\mathbb{R}^4$ *and* $\mathbb{R}^5$*. (English summary)*, Ann. Global Anal. Geom. 19, no. 2, 177–184 (2001).

[5] X. Cheng, *One end theorem and application to stable minimal hypersurfaces*, Arch Math. 90 (2008), 461–470.

[6] O. Chodosh, C. Li, *Stable minimal hypersurfaces in* $\mathbb{R}^4$, to appear on Acta Math.

[7] O. Chodosh, C. Li, *Stable anisotropic minimal hypersurfaces in* $\mathbb{R}^4$, Forum of Math. Pi, 11, e3 (2023).

[8] O. Chodosh, C. Li, D. Stryker *Complete stable minimal hypersurfaces in positively curved 4-manifolds*, https://arxiv.org/abs/2202.07708v1, 2022.

[9] T. H. Colding, W. P. Minicozzi, II, *A course in minimal surfaces, Graduate Studies in Mathematics*, vol. 121, American Mathematical Society, Providence, RI, 2011.

[10] E. De Giorgi, *Una estensione del teorema di Bernstein. (Italian)*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 19, 79–85 (1965).

[11] M. do Carmo, C. K. Peng, *Stable complete minimal surfaces in* $\mathbb{R}^3$ *are planes.*, Bull. Amer. Math. Soc. (N.S.) 1, no. 6, 903–906 (1979).

[12] M. do Carmo, C. K. Peng, *Stable complete minimal hypersurfaces.*, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), 1349–1358, Sci. Press Beijing, Beijing, 1982.

[13] M. F. Elbert, B. Nelli, H. Rosenberg, *Stable constant mean curvature hypersurfaces*, Proc. Am. Math. Soc. 135 n. 10, 3359–3366 (2007).

[14] W. H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) 11, 69–90 (1962).

[15] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. 82, 121–132 (1985).

[16] D. Fischer-Colbrie, R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature.*, Comm. Pure Appl. Math. 33, no. 2, 199–211 (1980).

[17] R. Hardt, L. Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. 362 (1985), 102–129.

[18] B. Nelli, M. Soret, *Stably embedded minimal hypersurfaces. (English summary)*, Math. Z. 255, no. 3, 493–514 (2007).

[19] A. V. Pogorelov, *On the stability of minimal surfaces. (Russian)*, Dokl. Akad. Nauk SSSR 260, no. 2, 293–295 (1981).

[20] Z. Qian, *Estimates for weighted volumes and applications.*, Quart. J. Math. Oxford Ser. (2) 48, no. 190, 235–242 (1997).
[21] R. Schoen, L. Simon, S.T. Yau, *Curvature estimates for minimal hypersurfaces,* Acta Math. 134, 275–288 (1975).
[22] R. Schoen, S.T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 209–228.
[23] Y. Shen, R. Ye, *On stable minimal surfaces in manifolds of positive bi-Ricci curvatures,* Duke Math. J. (1) 85, 109–116 (1996).
[24] J. Simons, *Minimal varieties in Riemannian manifolds.*, Ann. of Math. (2) 88, 62–105 (1968).

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