TWO RIGIDITY RESULTS FOR STABLE MINIMAL HYPERSURFACES

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ABSTRACT. The aim of this paper is to prove two results concerning the rigidity of complete, immersed, orientable, stable minimal hypersurfaces: we show that they are hyperplane in $\mathbb{R}^4$, while they do not exist in positively curved closed Riemannian $(n+1)$-manifold when $n \leq 5$; in particular, there are no stable minimal hypersurfaces in $\mathbb{S}^{n+1}$ when $n \leq 5$. The first result was recently proved also by Chodosh and Li, and the second is a consequence of a more general result concerning minimal surfaces with finite index. Both theorems rely on a conformal method, inspired by a classical work of Fischer-Colbrie.

Key Words: stable minimal hypersurface, rigidity

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1. INTRODUCTION

In this paper we consider smooth, complete, connected, orientable, isometrically immersed hypersurfaces $M^n \hookrightarrow (X^{n+1}, h)$, $n \geq 2$, where (X^{n+1}, h) is a (complete) Riemannian manifold of dimension $n+1$ endowed with metric h. We denote with g the induced metric on M and with H the mean curvature of M; M is *minimal* if $H \equiv 0$ on M. In this latter case we say that M is *stable* if

$$\int_M \left[|A|^2 + \mathrm{Ric}_h(\nu, \nu)\right] f^2 \, dV_g \leq \int_M |\nabla f|^2 \, dV_g \qquad \forall f \in C_0^\infty(M), \tag{1.1}$$

where $A = A_M$ is the second fundamental form of M^n, ν is a unit normal vector to M in X and dV_g is the volume form of g . It is well known that minimal hypersurfaces arise as critical points of the area functional and stability is related to the non-negativity of the second variation or, equivalently, the non-negativity of the Jacobi operator

$$L_M := \Delta + |A|^2 + \mathrm{Ric}_h(\nu, \nu).$$

Our goal is to prove two results concerning the rigidity of a stable minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$ when (X^{n+1}, h) is either the flat space $\mathbb{R}^{n+1}$ or a closed (compact without boundary) Riemannian manifold with non-negative sectional curvature and positive Ricci curvature. The study of minimal (in particular stable or in general with finite index) hypersurfaces immersed into a Riemannian manifolds is a very active field and has attracted a lot of interest in the last decades.

Our first result is the following:

Theorem 1.1. *A complete, orientable, immersed,* stable *minimal hypersurface* $M^3 \hookrightarrow \mathbb{R}^4$ *is a hyperplane.*

This theorem has been recently proved by Chodosh and Li in [1, Theorem 1] (see also [2]). Here we provide a completely different proof based on a conformal method, a comparison result and integral estimates. We refer to [1, 2] and references therein for previous and related results.

Our second result concerns minimal hypersurfaces with *finite index*. We recall that a minimal immersion $M^n \hookrightarrow (X^{n+1}, h)$ has finite index if the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator L_M on every compact domain in M with Dirichlet boundary conditions is finite. In particular stability implies finite (equal zero) index. In this paper we prove the following:

Theorem 1.2. *If (X^{n+1}, h) is a closed $(n+1)$-dimensional, $n \leq 5$, manifold with non-negative sectional curvature and positive Ricci curvature, then every complete, orientable, immersed, minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$ with* finite index *must be compact.*

As a byproduct we have the following

Corollary 1.3. *If (X^{n+1}, h) is a closed $(n+1)$-dimensional, $n \leq 5$, manifold with non-negative sectional curvature and positive Ricci curvature, then there is no complete, orientable, immersed,* stable *minimal hypersurface $M^n \hookrightarrow (X^{n+1}, h)$.*

In particular, there is no complete, orientable, immersed, stable minimal hypersurface of the round spheres $M^n \hookrightarrow (\mathbb{S}^{n+1}, g_{\mathrm{std}})$, provided $n \leq 5$. In dimension $n = 2$ this follows from a more general result proved in [6], while, in dimension $n = 3$, it was recently proved in [3, Corolary 1.5]. As far as we know, Corollary 1.3 is new in the cases $n = 4, 5$. We do not know if Theorem 1.2 and Corollary 1.3 hold also in dimension greater than five.

2. Proof for Theorem 1.1

In this section we give an alternative proof of [1, Theorem 1] (see Theorem 1.1). The main idea is to use a weighted volume comparison for a suitable conformal metric $\widetilde{g}$ together with a new weighted integral estimate inspired by [7]. Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected, orientable, isometrically immersed, stable minimal hypersurface.

2.1. **Conformal change.** It is well known (see e.g. [5, Proposition 1]) that, since $M^n \hookrightarrow \mathbb{R}^{n+1}$ is stable, then there exists a positive function $0 < u \in C^\infty(M)$ satisfying

$$-\Delta_g u = |A|_g^2 u \quad \text{on } M\,. \tag{2.1}$$

Following the line in [5] (see also [4]), let $k > 0$ and consider the conformal metric

$$\widetilde{g} = u^{2k} g.$$

where $g = \imath^* h$ is the induced metric on M (and $\imath$ denotes the inclusion). First of all we prove the following lower bound for a modified Bakry-Emery-Ricci curvature of $\widetilde{g}$. In particular, this implies the non-negativity of the 2-Bakry-Emery-Ricci curvature of $\widetilde{g}$ for a suitable k.

Lemma 2.1. *Let $f := k(n-2)\log u$. Then the Ricci tensor of the metric $\widetilde{g} = u^{2k} g$ satisfies*

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1 - k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g$$

in the sense of quadratic forms. In particular, if $n = 3$ and $k = \frac{2}{3}$, then the 2-Bakry-Emery-Ricci tensor $\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df$ satisfies

$$\mathrm{Ric}^{2,f}_{\widetilde{g}} \geq 0.$$

Proof. Since $f = k(n-2)\log u$, we have

$$df = k(n-2)\frac{du}{u}$$

and

$$\nabla^2_g f = k(n-2)\left(\frac{\nabla^2_g u}{u} - \frac{du \otimes du}{u^2}\right),$$

which implies

$$\Delta_g f = k(n-2)\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|^2_g}{u^2}\right).$$

On the other hand, from the standard formulas for a conformal change of the metric $\tilde{g} = e^{2\varphi} g$, $\varphi \in C^\infty(M)$, $\varphi > 0$ we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla^2_g \varphi - d\varphi \otimes d\varphi\right) - \left[\Delta_g \varphi + (n-2)|\nabla_g \varphi|^2_g\right] g$$

and

$$\nabla^2_{\widetilde{g}} f = \nabla^2_g f - (df \otimes d\varphi + d\varphi \otimes df) + g\left(\nabla f, \nabla \varphi\right) g.$$

Note that, in our case, $\varphi = k \log u$; now we exploit the facts that u is a solution of equation (2.1) to write

$$\begin{aligned}\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f &= \mathrm{Ric}_g - k^2(n-2)\frac{du \otimes du}{u^2} + k|A|^2_g g + k\frac{|\nabla_g u|^2_g}{u^2} g \\ &= \mathrm{Ric}_g - \frac{df \otimes df}{n-2} + k|A|^2 g + \frac{|\nabla_g f|^2_g}{k(n-2)^2} g.\end{aligned}$$

From the Cauchy-Schwarz inequality we have

$$|\nabla_g f|^2_g g \geq df \otimes df,$$

thus

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f \geq \mathrm{Ric}_g + \frac{1-k(n-2)}{k(n-2)^2} df \otimes df + k|A|^2_g g;$$

from Gauss equations in the minimal case we get $\mathrm{Ric}_g = -A^2$; since A is traceless we have the inequality

$$A^2 \leq \frac{n-1}{n}|A|^2 g,$$

and substituting in the previous relation we conclude

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1-k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right)|A|^2_g g.$$

□

2.2. **Completeness.** From now on we take

$$n = 3, \qquad k = \frac{2}{3}$$

and $u \in C^\infty(M)$ a positive solution to (2.1). In the following we prove the completeness of the conformal metric $\widetilde{g} = u^{\frac{4}{3}} g$, following the strategy in [5, Theorem 1] and using some computations in [4].

Lemma 2.2. *The metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete.*

Proof. As shown in [5, Theorem 1] we can construct a minimizing geodesic in the metric $\widetilde{g} = u^{2k} g$, $\gamma = \gamma(s)$, where s is the arclength in the metric g. As observed in [5, Theorem 1] the completeness of $\widetilde{g}$ is equivalent to prove that γ has infinite $\widetilde{g}$ length, i.e.

$$\int_0^{+\infty} u^k(\gamma(s))\, ds = +\infty.$$

Since γ is minimizing, by the second variation formula, following the computation in the proof of Theorem 1 (with $H = 0$) in [4], we obtain

$$\begin{aligned}(n-1)\int_0^{+\infty} (\varphi_s)^2 u^{-k}\, ds \geq \int_0^{+\infty} \varphi^2 u^{-k} \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds \\ - k(n-2)\int_0^{+\infty} \varphi^2 u^{-k} (\log u)_{ss}\, ds + k \int_0^{+\infty} \varphi^2 u^{-k} \frac{|\nabla u|^2}{u^2}\, ds\end{aligned}$$

for every smooth function φ with compact support in $(0, +\infty)$ and for every $k > 0$. Since A is trace-free, we have

$$|A|^2 \geq A_{11}^2 + A_{22}^2 + \ldots + A_{nn}^2 + 2\sum_{j=2}^n A_{1j}^2 \geq \frac{n}{n-1} A_{11}^2 + 2\sum_{j=2}^n A_{1j}^2,$$

thus

$$k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \geq \left(\frac{kn}{n-1} - 1 \right) A_{11}^2 + (2k-1)\sum_{j=2}^n A_{1j}^2.$$

In particular, if

$$k \geq \frac{n-1}{n} \tag{2.2}$$

we have

$$\int_0^{+\infty} \varphi^2 u^{-k} \left(k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) ds \geq 0.$$

Using this estimate, the fact that $|\nabla u|^2 \geq (u_s)^2$ and integrating by parts, we obtain

$$\begin{aligned}(n-1)\int_0^{+\infty} (\varphi_s)^2 u^{-k}\, ds \geq 2k(n-2)\int_0^{+\infty} \varphi \varphi_s u^{-k-1} u_s\, ds \\ + k\left[1 - k(n-2)\right] \int_0^{+\infty} \varphi^2 u^{-k-2} (u_s)^2\, ds.\end{aligned}$$

Let now $\varphi = u^k \psi$, with ψ smooth with compact support in $(0, +\infty)$. We have

$$\begin{aligned}\varphi^2 u^{-k} &= u^k \psi^2, \\ \varphi_s &= k\psi u^{k-1} u_s + u^k \psi_s, \\ (\varphi_s)^2 u^{-k} &= k^2 \psi^2 u^{k-2} (u_s)^2 + u^k (\psi_s)^2 + 2k\psi\psi_s u^{k-1} u_s,\end{aligned}$$

and substituting in the previous relation we get

$$(n-1)\int_0^{+\infty} u^k (\psi_s)^2\, ds \geq -2k\int_0^{+\infty} \psi\psi_s u^{k-1} u_s\, ds + k(1-k)\int_0^{+\infty} \psi^2 u^{k-2} (u_s)^2\, ds. \tag{2.3}$$

Let

$$I := \int_0^{+\infty} \psi\psi_s u^{k-1} u_s\, ds;$$

thus we have

$$I = \frac{1}{k}\int_0^{+\infty} \psi\psi_s (u^k)_s \, ds = -\frac{1}{k}\int_0^{+\infty} u^k(\psi_s)^2 \, ds - \frac{1}{k}\int_0^{+\infty} \psi\psi_{ss}u^k \, ds.$$

Moreover, for every $t > 1$ and using Young's inequality for every $\varepsilon > 0$, we have

$$\begin{aligned} 2kI &= 2ktI + 2k(1-t)I \\ &= -2t\int_0^{+\infty} u^k(\psi_s)^2 \, ds - 2t\int_0^{+\infty} \psi\psi_{ss}u^k \, ds + 2k(1-t)\int_0^{+\infty} \psi\psi_s u^{k-1}u_s \, ds \\ &\le -2t\int_0^{+\infty} u^k(\psi_s)^2 \, ds - 2t\int_0^{+\infty} \psi\psi_{ss}u^k \, ds \\ &\quad + k(t-1)\varepsilon\int_0^{+\infty} \psi^2 u^{k-2}(u_s)^2 \, ds + \frac{k(t-1)}{\varepsilon}\int_0^{+\infty} u^k(\psi_s)^2 \, ds. \end{aligned}$$

Assuming

$$k < 1 \tag{2.4}$$

and choosing

$$\varepsilon := \frac{1-k}{t-1}$$

we obtain

$$2kI \le -2t\int_0^{+\infty} \psi\psi_{ss}u^k \, ds + k(1-k)\int_0^{+\infty} \psi^2 u^{k-2}(u_s)^2 \, ds + \left[\frac{k(t-1)^2}{1-k} - 2t\right]\int_0^{+\infty} u^k(\psi_s)^2 \, ds.$$

From (2.3) we get

$$0 \le \left[\frac{k(t-1)^2}{1-k} - 2t + (n-1)\right]\int_0^{+\infty} u^k(\psi_s)^2 \, ds - 2t\int_0^{+\infty} \psi\psi_{ss}u^k \, ds$$

for every $t > 1$ and every k satisfying (2.2) and (2.4). Let

$$P(t) := \frac{k(t-1)^2}{1-k} - 2t + (n-1)$$

and choose $k = \frac{n-1}{n}$. It is easy to see that $P(t)$ is negative for some $t > 1$ if $n = 3$: indeed

$$P(t) = (n-1)t^2 - 2nt + 2(n-1) = 2t^2 - 6t + 4 = -2(t-1)(2-t).$$

Therefore, if $n = 3$, $k = \frac{2}{3}$ and $t = \frac{3}{2}$, we deduce

$$0 \le -\int_0^{+\infty} u^{\frac{2}{3}}(\psi_s)^2 \, ds - 6\int_0^{+\infty} u^{\frac{2}{3}}\psi\psi_{ss} \, ds$$

for every ψ smooth with compact support in $(0, +\infty)$. Now we choose $\psi = s\eta$ with η smooth with compact support in $(0, +\infty)$: thus

$$\psi_s = \eta + s\eta_s, \quad \psi_{ss} = 2\eta_s + s\eta_{ss},$$

and we get

$$\int_0^{+\infty} u^{\frac{2}{3}}\eta^2 \, ds \le \int_0^{+\infty} u^{\frac{2}{3}}\left(-14s\eta\eta_s - 6s^2\eta\eta_{ss} - s^2(\eta_s)^2\right) \, ds.$$

Choose η so that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2 respectively for $R \le s \le 2R$. Then

$$\int_0^R u^{\frac{2}{3}} \, ds \le \int_0^{+\infty} u^{\frac{2}{3}}\eta^2 \, ds \le C\int_R^{+\infty} u^{\frac{2}{3}} \, ds$$

for some $C > 0$ independent of R. We conclude that

$$\int_0^{+\infty} u^{\frac{2}{3}}\, ds = +\infty,$$

i.e. $\widetilde{g} = u^{\frac{4}{3}} g$ is complete. □

2.3. **Weighted integral estimates.** From lemma 2.1 and lemma 2.2 we have that the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete and it has non-negative 2-Bakry-Emery-Ricci curvature. Using well known comparison results (see [8, Theorem A.1]) we immediately obtain the following weighted Bishop-Gromov volume estimate for a geodesic ball $B_R^{\widetilde{g}}(x_0)$ centered at $x_0 \in M$, of radius R, with respect to the metric $\widetilde{g}$.

Corollary 2.3. *Let $x_0 \in M^3$. Then, for every $R > 0$, there exists $C > 0$ such that the f-volume*

$$\mathrm{Vol}_f\, B_R^{\widetilde{g}}(x_0) := \int_{B_R^{\widetilde{g}}(x_0)} e^{-f}\, dV_{\widetilde{g}} \leq C R^5,$$

where $f = \frac{2}{3}\log u$. Equivalently, in terms of u and the volume form of g,

$$\int_{B_R^{\widetilde{g}}(x_0)} u^{\frac{4}{3}}\, dV_g \leq C R^5.$$

The last ingredient that we need in the proof of Theorem 1.1 is the following weighted integral inequality in the spirit of [7, Theorem 1].

Lemma 2.4. *For every $0 < \delta < \frac{1}{100}$, there exists $C > 0$ such that*

$$\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}} \psi^{5+\delta}\, dV_g \leq C \int_M u^{-2-\frac{2\delta}{3}} |\nabla\psi|^{5+\delta}\, dV_g \qquad \forall \psi \in C_0^\infty(M).$$

Proof. We do part of the computations for every n. From [7, Formulas (2.1)-(2.3)] we get

$$\int_M |A|^p f^2 \leq C \int_M |A|^{p-2} |\nabla f|^2 \qquad \forall f \in C_0^\infty(M), \tag{2.5}$$

for every $p \in [4, 4 + \sqrt{8/n}]$ and for some $C > 0$. Take $f = u^\alpha \psi$, with ψ smooth with compact support, u the solution of (2.1) and $\alpha < 0$. Since, from Cauchy-Schwarz and Young's inequalities,

$$|\nabla(u^\alpha \psi)|^2 \leq 2\psi^2 |\nabla(u^\alpha)|^2 + 2u^{2\alpha}|\nabla\psi|^2\,, \tag{2.6}$$

then (2.5) becomes

$$\int_M |A|^p u^{2\alpha}\psi^2 \leq 2C\left[\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\right] \qquad \forall \psi \in C_0^\infty(M). \tag{2.7}$$

Now we tackle the first integral on the right-hand side of (2.7) firstly integrating by parts

$$\begin{aligned}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 = &-\int_M |A|^{p-2}\psi^2 u^\alpha \Delta u^\alpha - \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2}\rangle \\ &-2\int_M |A|^{p-2} u^\alpha \psi \langle \nabla u^\alpha, \nabla\psi\rangle\,,\end{aligned}$$

secondly we use the fact that

$$\Delta u^\alpha = \alpha u^{\alpha-1}\Delta u + \alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 \quad \text{and} \quad |\nabla u^\alpha|^2 = \alpha^2 u^{2\alpha-2}|\nabla u|^2\,,$$

together with Cauchy-Schwarz and Young's inequalities to get

$$\begin{aligned}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \leq &-\alpha\int_M |A|^{p-2}u^{2\alpha-1}\psi^2\Delta u - \frac{\alpha-1}{\alpha}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \\ &-\int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \varepsilon\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,\end{aligned}$$

for all $\varepsilon > 0$. From (2.1) we find

$$\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2 - \frac{\alpha-1}{\alpha}\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 - \int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \varepsilon\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

i.e.

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2 - \int_M u^\alpha\psi^2\langle\nabla u^\alpha, \nabla|A|^{p-2}\rangle + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,,$$

Now, since

$$\nabla|A|^{p-2} = (p-2)|A|^{p-3}\nabla|A| = (p-2)|A|^{\frac{p-2}{2}}|A|^{\frac{p-4}{2}}\nabla|A|\,,$$

then, from Cauchy-Schwarz and Young's inequalities we obtain

$$\left(1-\varepsilon+\frac{\alpha-1}{\alpha}-\frac{p-2}{2t_1}\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 \le \alpha\int_M |A|^p u^{2\alpha}\psi^2 + \frac{(p-2)t_1}{2}\int_M |A|^{p-4}\psi^2u^{2\alpha}|\nabla|A||^2 + \frac{1}{\varepsilon}\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2\,, \tag{2.8}$$

for every $t_1 > 0$. Now we use Simon's identity and the improved Kato's inequality as in [7, Equation (1.34)] to obtain

$$|A|\Delta|A| + |A|^4 \ge \frac{2}{n}|\nabla|A||^2.$$

Multiplying by $|A|^{p-4}f^2$, integrating by parts and using Young's inequality we obtain

$$\int_M |A|^p f^2 \ge \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2f^2 - \frac{1}{t_2}\int_M |A|^{p-2}|\nabla f|^2$$

for every $t_2 > 0$. Choosing $f = u^\alpha\psi$ we obtain

$$\int_M |A|^p u^{2\alpha}\psi^2 \ge \left(\frac{2}{n}+p-3-t_2\right)\int_M |A|^{p-4}|\nabla|A||^2u^{2\alpha}\psi^2 - \left(\frac{1}{t_2}+\varepsilon\right)\int_M |A|^{p-2}\psi^2|\nabla u^\alpha|^2 - \frac{1}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\int_M |A|^{p-2}u^{2\alpha}|\nabla\psi|^2 \tag{2.9}$$

for every $\varepsilon > 0$, since

$$|\nabla(u^\alpha\psi)|^2 \le (1+t_2\varepsilon)\psi^2|\nabla(u^\alpha)|^2 + \left(1+\frac{1}{t_2\varepsilon}\right)u^{2\alpha}|\nabla\psi|^2\,.$$

Now let $\delta > 0$. Using (2.9) in (2.8) with

$$\alpha = -1-\frac{\delta}{3} \le -1$$

we obtain

$$\left(1+\frac{2+\frac{\delta}{3}}{1+\frac{\delta}{3}}-2\varepsilon-\frac{p-2}{2t_1}-\frac{1}{t_2}\right)\int_M |A|^{p-2}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2 \le \left[\frac{1}{\varepsilon}+\frac{1}{t_2}\left(1+\frac{1}{t_2\varepsilon}\right)\right]\int_M |A|^{p-2}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2 + \left[\frac{(p-2)t_1}{2}-\frac{2}{n}-p+3+t_2\right]\int_M |A|^{p-4}\psi^2u^{-2-\frac{2\delta}{3}}|\nabla|A||^2\,,$$

for all $\varepsilon, t_1, t_2 > 0$. Let

$$n = 3,\quad p = 5+\delta,\quad t_1 = \frac{2(5+3\delta)}{3(3+\delta)},\quad t_2 = 1$$

we obtain

$$\frac{(p-2)t_1}{2}-\frac{2}{n}-p+3+t_2=0$$

and

$$1+\frac{2+\frac{\delta}{3}}{1+\frac{\delta}{3}}-\frac{p-2}{2t_1}-\frac{1}{t_2}=\frac{39+11\delta-15\delta^2-3\delta^3}{60+56\delta+12\delta^2}.$$

Thus

$$\left(\frac{39+11\delta-15\delta^2-3\delta^3}{60+56\delta+12\delta^2}-2\varepsilon\right)\int_M |A|^{3+\delta}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2\leq\left(1+\frac{2}{\varepsilon}\right)\int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2,$$

for all $\varepsilon>0$. Choosing $0<\delta<1/100$ and ε small enough we obtain

$$\int_M |A|^{3+\delta}\psi^2|\nabla u^{-1-\frac{\delta}{3}}|^2\leq C\int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2\,,$$

for some $C>0$. From (2.7) and Young's inequality we get

$$\begin{aligned}\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2 &\leq C\int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2\\ &\leq\varepsilon'\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2+\frac{C}{\varepsilon'}\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)}\end{aligned}$$

for all $\varepsilon'>0$ and $\psi\in C^\infty_0(M)$. Therefore

$$\begin{aligned}\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2 &\leq C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)}\\ &=C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi^{\frac{2}{5+\delta}}|^{5+\delta}.\end{aligned}$$

The conclusion now follows immediately by replacing ψ with $\psi^{\frac{2}{5+\delta}}$. □

2.4. **Final estimate.** Combining Lemma 2.4 with Corollary 2.3 we can conclude the proof of Theorem 1.1. More precisely, let $x_0\in M$ and let $\widetilde{r}$ the distance function from x_0 with respect to the metric $\widetilde{g}=u^{\frac{4}{3}}g$. We choose $\psi:=\eta(\widetilde{r})$ with $0\leq\eta\leq 1$, $\eta\equiv 1$ on $[0,R]$, $\eta\equiv 0$ on $[2R,+\infty)$ and $|\eta'|\leq C/R$ on $[R,2R]$, for some $C>0$ and $R>0$. From Lemma 2.4, for some $0<\delta<1/100$, we have

$$\begin{aligned}\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\eta^{5+\delta}\,dV_g &\leq C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\eta|_g^{5+\delta}\,dV_g\\ &=C\int_M u^{-2-\frac{2\delta}{3}+\frac{2(5+\delta)}{3}}|\nabla\eta|_{\widetilde{g}}^{5+\delta}\,dV_g\\ &\leq\frac{C}{R^{5+\delta}}\int_{B^{\widetilde{g}}_{2R}(x_0)}u^{\frac{4}{3}}\,dV_g\\ &\leq\frac{C}{R^\delta},\end{aligned}$$

where we used the fact that $|\nabla\widetilde{r}|_{\widetilde{g}}\equiv 1$ and Corollary 2.3. Since $\delta>0$, letting $R\to+\infty$ we get

$$|A|\equiv 0\quad\text{on } M^3$$

and this concludes the proof of Theorem 1.1.

3. Closed ambient space: proof of Theorem 1.2

Proof of Theorem 1.2. Let (X^{n+1},h) be a closed n-dimensional, $n\leq 5$, manifold with non-negative sectional curvature and positive Ricci curvature and consider an orientable, immersed, minimal hypersurface $M^n\to(X^{n+1},h)$ with finite index. Suppose, by contradiction, that M is non-compact. It is well known (see [5, Proposition 1]) that there exist $0<u\in C^\infty(M)$ and a compact subset $K\subset M$ such that u solves

$$-\Delta u=\left[|A|^2+\mathrm{Ric}_h(\nu,\nu)\right]u\qquad\text{on } M\setminus K.$$

Let $k>0$ and consider the conformal metric

$$\widetilde{g}=u^{2k}g.$$

where g is the induced metric on M. Let s be the arc length with respect to the metric g. Following the construction in [5, Theorem 1], we can construct a minimizing geodesic $\widetilde{\gamma}(s)$: $[0,+\infty)\to M\setminus K$ in the metric $\widetilde{g}$ which has infinite length in the metric g. Now we can argue exactly as in the proof of estimate (6) in [4], using $H\equiv 0$, obtaining

$$\begin{aligned}(n-1)\int_0^a(\varphi_s)^2\,ds&\geq k(n-3)\int_0^a\varphi\varphi_s\frac{u_s}{u}\,ds+\frac{k\left[4-k(n-1)\right]}{4}\int_0^a\varphi^2\left(\frac{u_s}{u}\right)^2\,ds\\&\quad+\int_0^a\varphi^2\left(kRic_h(\nu,\nu)+\sum_{j=2}^n R^h_{1j1j}\right)\,ds\\&\quad+\int_0^a\varphi^2\left(k|A|^2-A_{11}^2-\sum_{j=2}^n A_{1j}^2\right)\,ds,\end{aligned}$$

for every smooth function φ such that $\varphi(0)=\varphi(a)=0$ and for every $k>0$. Since (N^{n+1},h) is closed with non-negative sectional curvature and positive Ricci curvature, we have $R^h_{1j1j}\geq 0$ for every $j=2,\ldots,n$ and

$$\mathrm{Ric}_h(\nu,\nu)\geq R_0>0.$$

Theofore

$$\begin{aligned}(n-1)\int_0^a(\varphi_s)^2\,ds&\geq k(n-3)\int_0^a\varphi\varphi_s\frac{u_s}{u}\,ds+\frac{k\left[4-k(n-1)\right]}{4}\int_0^a\varphi^2\left(\frac{u_s}{u}\right)^2\,ds\\&\quad+\int_0^a\varphi^2\left(kR_0+k|A|^2-A_{11}^2-\sum_{j=2}^n A_{1j}^2\right)\,ds.\end{aligned}$$

Since A is trace-free, we have

$$|A|^2\geq A_{11}^2+A_{22}^2+\ldots+A_{nn}^2+2\sum_{j=2}^n A_{1j}^2\geq\frac{n}{n-1}A_{11}^2+2\sum_{j=2}^n A_{1j}^2,$$

thus

$$k|A|^2-A_{11}^2-\sum_{j=2}^n A_{1j}^2\geq\left(\frac{kn}{n-1}-1\right)A_{11}^2+(2k-1)\sum_{j=2}^n A_{1j}^2.$$

Choose

$$k=\frac{n-1}{n}.$$

We get

$$\int_0^a(\varphi_s)^2\,ds\geq\frac{n-3}{n}\int_0^a\varphi\varphi_s\frac{u_s}{u}\,ds+\frac{6n-n^2-1}{4n^2}\int_0^a\varphi^2\left(\frac{u_s}{u}\right)^2\,ds+\frac{R_0}{n}\int_0^a\varphi^2\,ds.$$

If $n \leq 5$, we have

$$\frac{6n - n^2 - 1}{4n^2} \geq \delta_0 > 0.$$

Moreover, there exists $C > 0$, such that

$$\frac{n-3}{n}\varphi\varphi_s\frac{u_s}{u} \geq -\delta_0\varphi^2\left(\frac{u_s}{u}\right)^2 - C(\varphi_s)^2.$$

Therefore, there exists $C > 0$, such that

$$C\int_0^a (\varphi_s)^2\,ds \geq \frac{R_0}{n}\int_0^a \varphi^2\,ds$$

for every smooth function φ such that $\varphi(0) = \varphi(a) = 0$. Integrating by parts we obtain

$$\int_0^a \left(\varphi\varphi_s s + CR_0\varphi^2\right)ds \leq 0.$$

Choosing $\varphi(s) = \sin(\pi s\, a^{-1})$, $s \in [0, a]$ one has

$$\left(CR_0 - \frac{\pi^2}{a^2}\right)\int_0^a \sin^2(\pi s\, a^{-1})ds \leq 0$$

i.e.

$$a^2 \leq \frac{\pi^2}{CR_0}.$$

We conclude that the length (in the metric g) of the geodesic $\widetilde{\gamma}(s)$ is finite and this gives a contradiction. Therefore (M^n, g) must be compact and this concludes the proof of Theorem 1.2. □

Proof of Corollary 1.3. If M is stable, by Theorem 1.2 it must be compact. Moreover there exists $u > 0$ satisfying

$$-\Delta u = \left[|A|^2 + \mathrm{Ric}_h(\nu,\nu)\right]u \qquad \text{on } M.$$

Integrating over M we get a contradiction, since $\mathrm{Ric}_h > 0$ on M. Equivalently, one can use $f \equiv 1$ in the stability inequality (1.1) to get a contradiction. □

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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