Chapter 1
Stratified Energies: Ground States with Cracks

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Abstract Elastic bodies admitting cracks are analyzed. Separated pairs of displacement fields and cracks are found as minimizers of the energy in large strain setting. The crack patterns are constructed in terms of varifolds. The discontinuity set of the displacement field is contained in the cracks and may or may not coincide with them.

1.1 Introduction

By following Griffith's pioneering suggestions, a variational view on the analysis of cracks in simple bodies has been proposed in [3] (see also [1]). Minimality of the energy at every time among all virtual crack-displacement pairs at that time is required. An energy conservation statement throughout the time evolution is imposed. The difficulty of managing crack geometries in finding minimizers has suggested the convenient simplification of identifying cracks with the jump sets of displacement fields (see results in [3] [2]). However, appropriate function spaces contain fields with discontinuity sets with closure of positive Lebesgue measure. Theorems allowing the selection of fields with physically significant discontinuity sets — that are

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sets that can be appropriate candidates for describing reasonable crack patterns — seem to be not available at least up to now. A new view has been done in [5]. Separated pairs of displacement fields and cracks are found as minimizers of the energy in simple bodies undergoing large strain. A way for managing the geometry of crack patterns is constructed in terms of special measures, namely varifolds. The discontinuity set of the displacement field is contained in the cracks and may or may not coincide with them. The description of closed cracks is also included in this way. Moreover, an essential point is that no crack is prescribed to exist a priori: its possible existence is eventually obtained by the minimization of the energy of the body. A energetic threshold for the formation of a crack arises naturally. Existence theorems are obtained for a non-standard energy functional including a surface energy which depend on the curvature of the possible crack and on the measure localizing it over the body. The present paper anticipates without proof some of the results collected in [5].

1.2 Curvature varifolds with boundary

Some preliminary notions are necessary to the ensuing developments.

Let $\mathscr{B}$ be an open, bounded subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. For a positive integer k, $1 \leq k \leq n$, the Grassmann manifold of k-planes through the origin in $\mathbb{R}^n$ is indicated by $\mathscr{G}_{k,n}$ and is also identified with the set of projectors $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ onto k-planes, characterized by $\Pi^2 = \Pi$, $\Pi^* = \Pi$, $\operatorname{Rank} \Pi = k$, a set which is a compact subset of $\mathbb{R}^n \otimes \mathbb{R}^n$. Consider also the trivial bundle $\mathscr{G}_k(\mathscr{B}) := \mathscr{B} \times \mathscr{G}_{k,n}$ with natural projection $\pi : \mathscr{G}_k(\mathscr{B}) \to \mathscr{B}$. A *$k$-varifold* on $\mathscr{B}$ is a nonnegative Radon measure V over $\mathscr{G}_k(\mathscr{B})$, namely $V \in \mathscr{M}(\mathscr{G}_k(\mathscr{B}))$. The *weight measure* of V is the Radon measure $\mu_V := \pi_\# V$ where $\pi_\#$ is the natural projection of measures associated with the projection π, and the *mass* of V is $\mathbf{M}(V) := V(\mathscr{G}_k(\mathscr{B})) = \mu_V(\mathscr{B})$.

Denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\mathfrak{b}$ is a $\mathscr{H}^k$-measurable, countably k-rectifiable subset of $\mathscr{B}$ and $\theta \in L^1(\mathfrak{b}, \mathscr{H}^k)$, for $\theta\, \mathscr{H}^k \llcorner \mathfrak{b}$ a.e. $x \in \mathscr{B}$ there exists the approximate tangent k-space $T_x\mathfrak{b}$ to $\mathfrak{b}$ at x. Define

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathscr{G}_k(\mathscr{B})} \varphi(x,\Pi)\, dV_{\mathfrak{b},\theta}(x,\Pi) := \int_{\mathfrak{b}} \theta(x)\varphi(x,\Pi(x))\, d\mathscr{H}^k(x) \tag{1.1}$$

for any $\varphi \in C_c^0(\mathscr{G}_k(\mathscr{B}))$, where $\Pi(x)$ is the orthogonal projection of $\mathbb{R}^n$ onto $T_x\mathfrak{b}$.

Definition 1. V is called a *curvature k-varifold with boundary* if

1. $V = V_{\mathfrak{b},\theta}$ is the integer rectifiable k-varifold associated with $(\mathfrak{b}, \theta, \mathscr{H}^k)$,
2. there exist a function $A \in L^1(\mathscr{G}_k(\mathscr{B}), \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*})$, $A = (A_j^{\ell i})$, and a vector Radon measure $\partial V \in \mathscr{M}(\mathscr{G}_k(\mathscr{B}), \mathbb{R}^n)$ such that

$$\int_{\mathscr{G}_k(\mathscr{B})} (\Pi D_x \varphi + A D_\Pi \varphi + \varphi\, {}^t\mathrm{tr}(AI))\, dV(x,\Pi) = -\int_{\mathscr{G}_k(\mathscr{B})} \varphi\, d\partial V(x,\Pi)$$

for every $\varphi \in C_c^\infty(\mathscr{G}_k(\mathscr{B}))$.

Moreover, for $p \geq 1$ the subclass of curvature k-varifolds with boundary such that $A \in L^p(\mathscr{G}_k(\mathscr{B}))$ is indicated by $CV_k^p(\mathscr{B})$.

The function $x \mapsto A(x,\Pi(x)) \in \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*}$ is called the *curvature* of the varifold V. The vector measure ∂V is called the *varifold boundary measure*.

The following results proven in [7] and [8] collect the geometrical properties of curvature k-varifolds with boundary.

Theorem 1. *Let $V = V_{\mathfrak{b},\theta}$ be a k-varifold with boundary ∂V and curvature A, with $A_j^{\ell i} \in L^1(\mathscr{G}_k(\mathscr{B}))$.*

1. *The following symmetry properties hold:*

$$A_j^{\ell i} = A_\ell^{ji}, \quad A_j^{ji} = 0, \quad A_j^{\ell i} = \Pi_h^\ell A_j^{hi} + \Pi_j^h A_h^{\ell i}, \quad V - a.e.$$

2. *$\Pi_h^i A_j^{\ell h} = A_j^{\ell i}$ V-a.e. in such a way that, by setting $H^i(x) := A_j^{ij}(x,\Pi(x))$, one gets $\Pi_i^h H^h = 0$ V-a.e.; in particular, if $\Pi = \Pi(x)$ is the orthogonal projection over $T_x\mathfrak{b}$, then*
$$H(x,\Pi(x)) \perp T_x\mathfrak{b} \qquad \mu_V - a.e.$$
3. *The projection map $x \to \Pi(x)$ is μ_V-a.e approximately differentiable and*
$$(\nabla^{\mathfrak{b}} \Pi_j^\ell(x))^i = A_j^{\ell i}(x,\Pi(x))$$
for μ_V-a.e. x.
4. *The support of $|\partial V|$ is contained in the support of V and $|\partial V| \perp V$.*
5. *∂V is tangential to $\mathfrak{b}$ in the sense that $(\Pi_j^i)_\# \partial^j V = \partial^i V$ as measures on $\mathscr{G}_k(\mathscr{B})$.*
6. *V is a varifold with locally bounded first variation and generalized mean curvature in the sense of Allard with generalized mean curvature vector $H(x) = H(x,\Pi(x))$ and generalized boundary $\pi_\# \partial V$.*

Theorem 2 (Rectifiability of the boundary). *Let V be a curvature k-varifold with boundary ∂V and $k \geq 1$. There exists a $\mathscr{H}^{k-1}$-countably rectifiable set $\mathscr{C}$ and a function $\sigma \in L^1(\mathscr{C}, \mathscr{H}^{k-1})$ such that $\pi_\# |\partial V| = \sigma \mathscr{H}^{k-1} \llcorner \mathscr{C}$. Moreover, one has*

$$\int \varphi(x,\Pi(x))\, d\partial V(x,\Pi) = \int_{\mathscr{C}} \Big(\int_{\mathscr{G}_{k,n}} \varphi(x,\Pi)\, d\tau_x(\Pi) \Big)\, d\mathscr{H}^{k-1}(x)$$

for every $\varphi \in C_c^\infty(\mathscr{G}_k(\mathscr{B}))$, where for $\mathscr{H}^{k-1}$-a.e. $x \in \mathscr{C}$ the vector valued measure τ_x on $\mathscr{G}_{k,n}$ has the structure

$$\tau_x = \sum_{i=1}^{i_x} m_i^x \alpha_i^x \delta_{p_i^x}, \tag{1.2}$$

where $i_x \in \mathbb{N}$, $\delta_{p_i^x}$ is the Dirac delta supported by a k-plane p_i^x of the Grassmanian $\mathscr{G}_{k,n}$; moreover, the α_i^x's are positive integers and the m_i^x's are unit vectors in $\mathbb{R}^n$. In addition p_i^x contains the tangent $(k-1)$-space $T_x\mathscr{C}$ to $\mathscr{C}$ at x and

$p_i^x = \mathrm{Span}\left\{T_x\mathscr{C}, m_i^x\right\}$. *In the special case of one-dimensional curvature varifold* V *with boundary, the formula* (1.2) *reduces to*

$$\tau_x := \sum_{j=1}^{j_x} \alpha_j t_j \delta_{P_j}$$

where δ_{P_j} *is the Dirac delta function supported by a straight line* P_j *in* $\mathscr{G}_{1,n}$, t_j *is a unit vector that orients* P_j *and* α_j *a positive integer. As a consequence, for the boundary of a curvature* 1*-varifold one gets*

$$\partial V(x,P) = \sum_{i=1}^{\infty} \delta_{x_i}(x) \times \tau_{x_i}(P).$$

Theorem 3 (Compactness [8]). *For* $1 < p < \infty$, *let* $\{V^{(r)}\} \subset CV_k^p(\mathscr{B})$ *be a sequence of curvature k-varifolds* $V^{(r)} = V_{\mathfrak{b}_r,\theta_r}$ *with boundary. The corresponding curvatures and boundaries are indicated by* $A^{(r)} = \{A^{(r)}{}_j^{\ell i}\}$ *and* $\partial V^{(r)}$, *respectively. Assume that for every open set* $\Omega \subset\subset \mathscr{B}$ *there exists a constant* $c = c(\Omega) > 0$ *such that for every* r

$$\mu_{V^{(r)}}(\Omega) + |\partial V^{(r)}|(\mathscr{G}_k(\Omega)) + \int_{\mathscr{G}_k(\Omega)} |A^{(r)}|^p \, dV^{(r)} \leq c(\Omega).$$

There exists a subsequence $\{V^{(r_s)}\}$ *of* $\{V^{(r)}\}$ *and a curvature k-varifold* $V = V_{\mathfrak{b},\theta} \in CV_k^p(\mathscr{B})$, *with curvature* A *and boundary* ∂V, *such that*

$$V^{(r_s)} \rightharpoonup V, \quad A^{(r_s)} \, dV^{(r_s)} \rightharpoonup A \, dV, \qquad \partial V^{(r_s)} \rightharpoonup \partial V,$$

in the sense of measures. Moreover, for any convex and l.s.c. function $f : \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \to [0,+\infty]$, *one gets*

$$\int_{\mathscr{G}_k(\mathscr{B})} f(A) \, dV \leq \liminf_{s\to\infty} \int_{\mathscr{G}_k(\mathscr{B})} f(A^{(r_s)}) \, dV^{(r_s)}.$$

1.3 Transplacement fields and bulk energy

Only Cauchy bodies are called upon in the analyses presented here. They are bodies for which the morphology of each material element is described only by the place in space occupied by its centre of mass. In other words, a body is identified with a region $\mathscr{B}$ of the Euclidean ambient space $\mathbb{R}^3$ that it occupies in a macroscopic reference configuration, taken as reference place. $\mathscr{B}$ is considered here as an open set with Lipshitz boundary. Other configurations are reached by means of transplacements that are usually taken as orientation preserving differentiable bijections $u : \mathscr{B} \longrightarrow \mathbb{R}^3$ mapping $\mathscr{B}$ in the current configuration $u(\mathscr{B})$, a set that is presumed to be always open and endowed with Lipshitz boundary.

The body occupying $\mathscr{B}$ is said to be an *hyperelastic simple body* when it is endowed by a bulk energy which is absolutely continuous with respect to the volume measure and depends on the deformation gradient only. Ground states of such a type of body are described by minimizers of the overall energy. Such minimizers can be determined in terms of Cartesian currents [6] that are described briefly below.

1.3.1 Sobolev maps and related Cartesian currents.

Let $I(k,n)$ be the space of multi-indices in $(1,\dots,n)$ of length k. Denote also by 0 the empty multi-index of length 0. For any α, the *complementary* multi-index to α in $(1,...,n)$ is indicated by $\bar{\alpha}$, $\bar{\alpha}\in I(n-k,n)$, and $\sigma(\alpha,\bar{\alpha})$ is the sign of the permutation from $(1,\dots,n)$ into $(\alpha_1,\dots,\alpha_k,\bar{\alpha}_1,...,\bar{\alpha}_{n-k})$. For $(e_1,e_2,\dots,e_n)$ and $(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, $\Lambda_r(\mathbb{R}^n\times\mathbb{R}^N)$ is the vector space of skew-symmetric tensors over $\mathbb{R}^n\times\mathbb{R}^N$ of the form

$$\xi=\sum_{|\alpha|+|\beta|=r}\xi^{\alpha\beta}e_\alpha\wedge\varepsilon_\beta=\sum_{\max(0,r-n)}^{\min(r,N)}\xi_{(k)},\qquad \xi_{(k)}=\sum_{\substack{|\alpha|+|\beta|=r\\|\beta|=k}}\xi^{\alpha\beta}e_\alpha\wedge\varepsilon_\beta.$$

For any linear map $G:\mathbb{R}^n\to\mathbb{R}^N$, the notation $M(G)$ is used for the simple n-vector in $\Lambda_n(\mathbb{R}^n\times\mathbb{R}^N)$ tangent to the graph of G and defined by

$$M(G):=\Lambda_n(\mathrm{Id}\times G)(e_1\wedge\cdots\wedge e_n)=(e_1,G(e_1))\wedge\cdots\wedge(e_n,Ge_n)).$$

For $u:\mathscr{B}\to\mathbb{R}^N$ an a.e. approximately differentiable map, denote by Du its approximate gradient. u has a Lusin representative on the subset $\widetilde{\mathscr{B}}$ of Lebesgue points of both u and Du, and $|\mathscr{B}\setminus\widetilde{\mathscr{B}}|=0$. Let $\widetilde{u}(x)$ and $D\widetilde{u}(x)$ be the Lebesgue values of u and Du at $x\in\widetilde{\mathscr{B}}$, respectively. Assume that $|M(Du)|\in L^1(\mathscr{B})$. By following [6], the *graph* of u, defined by

$$\mathscr{G}_u:=\left\{(x,y)\in\mathscr{B}\times\mathbb{R}^N\,\middle|\,x\in\widetilde{\mathscr{B}},\ y=\widetilde{u}(x)\right\},$$

is a n-rectifiable subset of $\mathscr{B}\times\mathbb{R}^N$ with approximate tangent vector n-space at $(x,\widetilde{u}(x))$ generated by the vectors $(e_1,D\widetilde{u}(x)e_1),\dots,(e_n,D\widetilde{u}(x)e_n))$ in $\mathbb{R}^n\times\mathbb{R}^N$. The *n-current integration over the graph* of u is defined by the linear functional on smooth n-forms $\omega=\omega(x,y)$ with compact support in $\mathscr{B}\times\mathbb{R}^N$ given by

$$G_u(\omega)=\int<\omega,\xi>\,d\mathscr{H}^n\llcorner\mathscr{G}_u, \tag{1.3}$$

where $\xi(x):=\frac{M(D\widetilde{u}(x))}{|M(D\widetilde{u}(x))|}$, for $x\in\widetilde{\mathscr{B}}$, is the unit n-vector that orients the approximate tangent n-space to $\mathscr{G}_u$ at $(x,\widetilde{u}(x))$; moreover, G_u has finite mass $\mathbf{M}(G_u):=\sup_{||\omega||_\infty\le1}G_u(\omega)<\infty$, since

$$\mathbf{M}(G_u) = \int_{\mathscr{B}} |M(Du(x))|\, dx = \mathscr{H}^n(\mathscr{G}_u).$$

In particular, G_u is a vector valued measure on $\mathscr{B} \times \mathbb{R}^N$, actually an *integer rectifiable n*-current with multiplicity 1 on $\mathscr{B} \times \mathbb{R}^N$. The *boundary* of the current G_u can be defined by duality as the $(n-1)$-current acting on compactly supported smooth $(n-1)$-forms ω in $\mathscr{B} \times \mathbb{R}^N$, namely $\partial G_u(\omega) := G_u(d\omega)$, $\omega \in \mathscr{D}^{n-1}(\mathscr{B} \times \mathbb{R}^N)$, where $d\omega$ is the differential of ω.

1.3.2 The bulk energy.

By taking apart for a while the description of the possible cracks, it is assumed that the external body forces have conservative nature so that the bulk energy of the body has the usual form

$$\mathscr{E}_{\mathscr{B}}(u) := \int_{\mathscr{B}} e(x,u,Du)\, dx$$

where $e(\cdot)$ is the sum of the elastic energy and the potential of external forces. It is assumed that $e = e(x,u,F)$ satisfies common assumptions listed below:

- (H1) $e : \mathscr{B} \times \hat{\mathbb{R}}^n \times \mathbb{M}^+_{n\times n} \to [0,+\infty]$ is continuous, where $\mathbb{M}^+_{n\times n}$ is the class of real $(n\times n)$-matrices F such that $\det F > 0$.
- (H2) The map $F \mapsto e(x,u,F)$ is *polyconvex*, i.e. there exists a function

  $$Pe(x,u,\xi) : \mathscr{B} \times \hat{\mathbb{R}}^n \times \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n) \to [0,+\infty]$$

  continuous in (x,u) for every ξ, convex and lower semicontinuous in ξ for every (x,u), such that

  $$e(x,u,F) = Pe(x,u,M(F)) \qquad \forall F \in \mathbb{M}^+_{n\times n}, \quad \forall (x,u) \in \mathscr{B} \times \hat{\mathbb{R}}^n.$$

- (H3) $e = e(x,u,F)$ satisfies the growth conditions

  $$e(x,u,F) \geq c_4\, |M(F)|^q \qquad \forall F \in \mathbb{M}^+_{n\times n}, \quad \forall (x,u) \in \mathscr{B} \times \hat{\mathbb{R}}^n,$$

  for some $c_4 > 0$ and $q > 1$.
- (H4) For every $x \in \mathscr{B}$ and $F \in \mathbb{M}^+_{n\times n}$ if for some $u \in \hat{\mathbb{R}}^n$ the inequality $e(x,u,F) < +\infty$ is satisfied, then $\det F > 0$.

The assumptions (H1) and (H4) are essentially suggested by physical plausibility. The hypothesis (H2) is an essence an assumption of material stability while the growth condition (H3) has more technical nature.

1.4 A skeletal model admitting formation of cracks

The aim now is to describe the possible presence of cracks. The setting is selected two-dimensional for the sake of simplicity. The generalization to 3D and extensions are in [5]. Cracks are here represented here by 1-dimensional curvature varifolds with curvature in L^p, $p > 1$, which are quite regular. Basically $V \in CV_1^p(\mathscr{B})$ can be essentially described as (the integration over) a locally finite union of $C^{1,1-1/p}$ curves counted with integer multiplicities. Their boundaries are just Dirac measures concentrated at the endpoints with their tangential directions.

Definition 2. A macroscopic configuration of a body $\mathscr{B} \subset \mathbb{R}^2$ with a crack is a pair composed by the bounded connected open set $\mathscr{B}$ with Lipschitz boundary and a curvature 1-varifold with boundary, namely $V = V_{\mathfrak{b},\theta} \in CV_1^p(\mathscr{B})$ for some $p > 1$.

The gross place occupied by the body and the crack are treated as distinct objects. The crack is not part of the initial boundary: it is selected by a measure over $\mathscr{B}$, namely a curvature varifold, and may or may not be an empty set in the reference place. Since the material bonds across the crack margins are broken, along the deformation, the cracks faces may loose contact. The obvious implication is that the graph of the deformation may have nonzero boundary. An appropriate class of admissible deformations has to be defined.

Weak diffeomorphisms have been found to be natural descriptors of deformations of standard elastic bodies [6]. They are orientation-preserving, allow frictionless contact of parts of the boundary while still prevent self-penetration of the matter. However, they satisfy a condition of zero boundary in the sense of currents, a condition avoiding the formation of ‘holes’ of various nature. To allow fractures, an extended version of them has to be formulated.

Definition 3. Let $\mathscr{B} \subset \mathbb{R}^2$ be a body with crack $V \in CV_1^p(\mathscr{B})$. A *weak diffeomorphism* on $\mathscr{B}$ *admitting cracks* described by V is an a.e. approximately differentiable map $u : \mathscr{B} \to \hat{\mathbb{R}}^2$ such that

1. $|Du|, \det Du \in L^1(\mathscr{B})$;
2. $\pi_\#|\partial G_u| \le \mu_V$, where $\mu_V := \pi_\# V$;
3. $\det Du(x) > 0$ for a.e. $x \in \mathscr{B}$;
4. for every compactly supported smooth function $f : \mathscr{B} \times \hat{\mathbb{R}}^2 \to [0, +\infty)$

$$\int_{\mathscr{B}} f(x, u(x)) \det Du(x)\, dx \le \int_{\hat{\mathbb{R}}^2} \sup_{x \in \mathscr{B}} f(x, y)\, dy.$$

In this case, one writes $u \in \mathrm{dif}^{1,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2)$. Moreover, for $q > 1$ the class $\mathrm{dif}^{q,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2)$ is defined by

$$\mathrm{dif}^{q,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2) := \left\{ u \in \mathrm{dif}^{1,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2) \,\middle|\, |M(Du)| \in L^q(\mathscr{B}) \right\}.$$

Condition (ii) implies that the Green formulas hold true in $\mathscr{B}$ outside the crack and prescribes that the boundary current has finite mass, namely $\mathbf{M}(\partial G_u) < \infty$.

Theorem 4. *Let $\{V^{(r)}\} \subset CV_1^p(\mathscr{B})$, with $p > 1$, be a sequence of curvature varifolds describing cracks in the body $\mathscr{B}$, with equibounded total variations, i.e.* $\sup_r \mu_{V^{(r)}}(\mathscr{B}) < \infty$. *Moreover, assume $u_r \in \mathrm{dif}^{1,1}(\mathscr{B}, V^{(r)}, \widetilde{\mathbb{R}}^2)$. Suppose also that there exist $u \in L^1(\mathscr{B}, \hat{\mathbb{R}}^2)$, $v \in L^1(\mathscr{B}, \Lambda_n(\mathbb{R}^2 \times \hat{\mathbb{R}}^2))$, and $V \in CV_1^p(\mathscr{B})$ such that $u_r \rightharpoonup u$, $M(Du_r) \rightharpoonup v$ weakly in L^1, and $V_r \rightharpoonup V$ as measures. Then $v = M(Du)$ and, moreover, if* $\det Du > 0$ *a.e.,* $u \in \mathrm{dif}^{1,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2)$.

1.4.1 The energy functional

Bulk and crack contributions to the energy are involved as usual. The crack in this skeletal model is one dimensional. The part of the energy associated with the crack is then splitted in two contributions: (i) the energy along the margins, which depend on the curvature of the margins themselves and is represented by the curvature of a varifold and (ii) the energy at the tips, the corners and the junctions of the fracture, that are represented by the boundary of the same varifold.

The energy $\mathscr{E}(u,V)$ reads

$$\begin{aligned}\mathscr{E}(u,V) &:= \mathscr{E}(u,V,\mathscr{B}) \\ &= \int_{\mathscr{B}} e(x,u,Du)\,dx + c_1 \int_{\mathscr{G}_1(\mathscr{B})} |A|^p\,dV + c_2\mathbf{M}(V) + c_3\mathbf{M}(\partial V)\end{aligned} \tag{1.4}$$

where the c_i's are positive constants and the hypotheses (H1) (H2), (H3) and (H4) of Section 1.3 on the bulk energy density $e = e(x,u,F)$ are satisfied.

As regards to crack energy term, the p-norm $|A|^p$ of the curvature can be replaced by $\phi(|A|)$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function satisfying $\phi(t) \geq c_5 t^p$.

The term $c_2\mathbf{M}(V)$ is the Griffith-like part of the surface energy of the crack.

1.4.2 Ground states: existence theorems

It may be convenient to prescribe a *comparison varifold* $\widetilde{V} \in CV_1^p(\mathscr{B})$ such that all competing varifolds V satisfy the bound $\mu_{\widetilde{V}} \leq \mu_V$. The comparison varifold $\widetilde{V}$ can be of course zero when an initial crack is absent. In the opposite case, $\widetilde{V}$ describes a crack from which the competing cracks may extend without excluding that they may have portions unrelated with $\widetilde{V}$.

The space

$$\begin{aligned}\mathscr{A}_{q,p,K,\widetilde{V}}(\mathscr{B}) := \Big\{ (u,V) \,\Big|\, & V \in CV_1^p(\mathscr{B}),\ u \in \mathrm{dif}^{q,1}(\mathscr{B}, V, \hat{\mathbb{R}}^2), \\ & ||u||_{L^\infty(\mathscr{B})} \leq K,\ \mu_{\widetilde{V}} \leq \mu_V \Big\},\end{aligned}$$

with $K > 0$, is then the natural functional environment for investigating the existence of minimizers (u,V) for the energy $\mathscr{E}$.

Theorem 5. *Consider* $\mathscr{B} \subset \mathbb{R}^2$, $q, p > 1$, $K > 0$, $\widetilde{V} \in CV_1^p(\mathscr{B})$. *Assume that there exists an element* $(u,V) \in \mathscr{A}_{q,p,K,\widetilde{V}}(\mathscr{B})$ *that satisfies the prescribed Dirichlet boundary conditions. Then the energy functional* (1.4) *attains its minimum in the subclass of* $\mathscr{A}_{q,p,K,\widetilde{V}}(\mathscr{B})$ *of couples* (u,V) *where u satisfies the prescribed boundary conditions.*

The constant K is selected at will for purposes of physical plausibility: it is only necessary for establishing the boundedness of the L^∞ norm of u. In contrast, the constants p and q and the comparison varifold $\widetilde{V}$ have constitutive nature. The a-priori L^∞ bound on the transplacement field has been relaxed in [2] [4] in a different setting, not dealing with the path followed here.

The simpler description of the boundary measure of the one dimensional curvature varifolds allows one to state another existence theorem with a different growth condition for the bulk energy.

Consider the energy functional (1.4) where the bulk energy density $e(x,u,F)$ satisfies (H1), (H2), (H4) of Section 1.3 and impose a different growth condition indicated here by

(H3-1)
$$e(x,u,F) \geq c_4|F|^2 \qquad \forall F \in \mathbb{M}^+_{2\times 2}, \quad \forall (x,u) \in \mathscr{B} \times \hat{\mathbb{R}}^2,$$

for some $c_4 > 0$.

For $K > 0$ and $\widetilde{V} \in CV_1^p(\mathscr{B})$ the class

$$\mathscr{A}_{p,\widetilde{V},K} := \Big\{ (u,V) \Big| V \in CV_1^p(\mathscr{B}),\ p > 1,\ u \in \mathrm{dif}^{1,1}(\mathscr{B},V,\hat{\mathbb{R}}^2),\ Du \in L^2(\mathscr{B}), \\ ||u||_{L^\infty(\mathscr{B})} \leq K,\ \mu_{\widetilde{V}} \leq \mu_V \Big\}, \tag{1.5}$$

is then the natural functional setting for another existence result.

Theorem 6. *Assume that the bulk energy density of* (1.4) *satisfies* (H1), (H2) (H4) *of Section 1.3 and* (H3-1). *Suppose that that there is at least one element* (u_0,V_0) *in the class* (1.5) *with* u_0 *satisfying a given Dirichlet data. Then the functional* (1.4) *has a minimizer in the subclass of* (1.5) *of couples* (u,V) *with u satisfying the prescribed Dirichlet boundary conditions.*

In the previous scheme, a sequence of varifolds accumulating at the boundary of $\mathscr{B}$ vanishes at the limit. It is possible to consider a different situation where the propagation of cracks at the boundary of the body $\mathscr{B}$ is taken into account, and a term involving the crack at the boundary may contribute to the limit energy of minimizing sequences. Such a situation has a clear meaning in terms of transplacements for the Dirichlet problem, where the limit crack may be seen as a rupture of the boundary condition.

A related existence theorem again follows.

Acknowledgements This work has been developed within the programs of the research group in 'Theoretical Mechanics' of the 'Centro di Ricerca Matematica Ennio De Giorgi' of the Scuola Normale Superiore at Pisa.

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