BV ESTIMATES ON THE TRANSPORT DENSITY WITH DIRICHLET REGION ON THE BOUNDARY

SAMER DWEIK

Abstract. In this paper, we prove BV regularity on the transport density in the mass transport problem to the boundary in two dimensions under certain conditions on the domain, the boundary cost and the mass distribution. Moreover, we show by a counter-example that the smoothness of the mass distribution, the boundary and the boundary cost does not imply that the transport density is $W^{1,p}$, for some $p > 1$.

1. Introduction

Let Ω be a compact domain in the Euclidean space $\mathbb{R}^2$. Let $f^{\pm} \in L^1(\Omega)$ be two densities satisfying the mass balance condition

$$\int_\Omega f^+(x)\,\mathrm{d}x = \int_\Omega f^-(y)\,\mathrm{d}y.$$

The Monge-Kantorovich problem [21] consists in finding a measure Λ on $\Omega \times \Omega$ which minimizes the cost functional

$$\mathcal{C}(\Lambda) = \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\Lambda(x,y)$$

among all transport plans satisfying the measure-preserving conditions $(\Pi_x)_\#\Lambda = f^+$ and $(\Pi_y)_\#\Lambda = f^-$, namely

$$\int_{E\times\Omega} \mathrm{d}\Lambda(x,y) = \int_E f^+(x)\,\mathrm{d}x \quad \text{and} \quad \int_{\Omega\times E} \mathrm{d}\Lambda(x,y) = \int_E f^-(y)\,\mathrm{d}y \;\; \text{for all Borel sets } E \subset \Omega.$$

In fact, this problem has been extensively studied (see, for instance, [2, 18, 23, 24]). In particular, one can show that the maximization of the functional

$$\mathcal{J}(u) = \int_\Omega u\,\mathrm{d}(f^+ - f^-)$$

among all 1-Lipschitz functions u on Ω, is the dual of this Monge-Kantorovich problem (in the sequel, a maximizer for this problem will be called a *Kantorovich potential*). This duality implies that optimal Λ and u satisfy $u(x) - u(y) = |x-y|$ on $\mathrm{spt}(\Lambda)$ (a maximal segment $[x,y]$ that satisfies this equality will be called a *transport ray*). In this transport problem, it is classical to associate with any optimal transport plan Λ a nonnegative measure σ_Λ on Ω (called *transport density*) which represents the amount of transport taking place in each region of Ω. This measure σ_Λ is defined by

$$(1.1) \qquad <\sigma_\Lambda, \varphi> = \int_{\Omega\times\Omega}\int_0^1 \varphi((1-t)x+ty)|x-y|\,\mathrm{d}t\,\mathrm{d}\Lambda(x,y), \;\; \text{for all } \varphi \in C(\Omega).$$

From [19], this transport density σ is unique (i.e., it does not depend on the choice of the optimal transport plan Λ) and it belongs to $L^1(\Omega)$. On the other hand, it is well known (see

[23]) that this transport density σ with the Kantorovich potential u solve together a particular PDE system (called *Monge-Kantorovich system*):

$$(1.2)\qquad \begin{cases} -\nabla\cdot[\sigma\nabla u] = f := f^+ - f^- & \text{in } \ \Omega, \\ \sigma\nabla u\cdot n = 0 & \text{on } \ \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \ \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases}$$

The L^p summability of this transport density σ was already studied in different papers (see [10, 11, 12, 23]), where the authors prove that σ is in $L^p(\Omega)$ as soon as $f^\pm \in L^p(\Omega)$, for all $p \in [1,\infty]$. However, the higher order regularity of σ turns out to be a difficult and delicate problem. We note that in general the transport density σ fails to be more regular than the two densities f^+ and f^- (an example can be found in [13], but anyway the construction is classical). In [20], the authors prove that σ is continuous as soon as $f^\pm$ are two positive continuous densities with compact, disjoint and convex supports. In [18], it has been proven that when $f^\pm$ are Lipschitz with disjoint supports (and with some extra technical condition on the supports), then σ is locally Lipschitz "along transport rays". While in [4], the authors prove a more general result for the case of just summable $f^\pm$ without any extra condition on supports; they prove that if $f^\pm \in L^p(\Omega)$, then σ is in $W^{1,p}_{loc}([x,y])$, for every transport ray $[x,y]$. However, the BV or Sobolev regularity of σ on Ω is still an open question; but in [13], there is a family of counter-examples where the author shows that in general we have the following statements:

$$f^\pm \in W^{1,p}(\Omega) \ \not\Rightarrow \ \sigma \in W^{1,p}(\Omega),$$

$$f^\pm \in BV(\Omega) \ \not\Rightarrow \ \sigma \in BV(\Omega),$$

$$f^\pm \in C^\infty(\overline{\Omega}) \ \not\Rightarrow \ \sigma \in W^{1,3}(\Omega).$$

In the framework of both traffic congestion and membrane reinforcement, in [5] the authors use a variant of the Monge-Kantorovich problem, already present in [3, 6], where the Monge-Kantorovich system (1.2) is complemented with Dirichlet boundary condition:

$$(1.3)\qquad \begin{cases} -\nabla\cdot[\sigma\nabla u] = f \ge 0 & \text{in } \ \mathring{\Omega}, \\ u = 0 & \text{on } \ \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \ \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases}$$

In optimal transport terms (see [15]), this corresponds to transport the mass f to the boundary. But, one can also assume that we have an additional cost g on $\partial\Omega$ (see [22, 16, 17]), i.e. we replace the Dirichlet boundary condition $u = 0$ by the nonhomogeneous boundary condition $u = g$. In this case, the system (1.3) becomes (see also [9]):

$$(1.4)\qquad \begin{cases} -\nabla\cdot[\sigma\nabla u] = f & \text{in } \ \mathring{\Omega}, \\ u = g & \text{on } \ \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \ \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases}$$

This system (1.4) describes the growth of a sandpile on a bounded table, with a wall on the boundary of height g, under the action of a vertical source here modeled by f (we see that in order to solve this system, the function g must be 1-Lipschitz). In other words, we consider the following transport problem:

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\Lambda+\int_{\partial\Omega}g\,\mathrm{d}[(\Pi_y)_{\#}\Lambda]\,:\,(\Pi_x)_{\#}\Lambda=f\ \text{ and }\ \mathrm{spt}[(\Pi_y)_{\#}\Lambda]\subset\partial\Omega\right\}. \tag{1.5}$$

Thanks to [22, 16], one can show that Problem (1.5) has a dual formulation which is the following:

$$\sup\left\{\int_{\Omega}u\,\mathrm{d}f\ :\ u\in \mathrm{Lip}_1(\Omega),\ u=g\ \text{ on }\ \partial\Omega\right\}. \tag{1.6}$$

Moreover, it is easy to see that $\Lambda^\star=(Id,T)_{\#}f$, where T is a Borel selector function of the multivalued map

$$\tilde{T}(x):=\mathrm{argmin}\{|x-y|+g(y)\,:\,y\in\partial\Omega\},\quad \text{for all }\ x\in\Omega,$$

is an optimal transport plan for the problem (1.5) while the Kantorovich potential u is given by

$$u(x)=\min\{|x-y|+g(y)\,:\,y\in\partial\Omega\}. \tag{1.7}$$

On the other hand, it is not difficult to prove that $\Lambda^\star$ is the unique optimal transport plan for Problem (1.5) provided that $f\in L^1(\Omega)$ and g is $\beta-$Lip with $\beta<1$ (see [16]). Let σ be the transport density in Problem (1.5), then it is clear that σ is the transport density between f and $T_{\#}f$. But, this means that the target measure is singular (since it is supported on $\partial\Omega$). Consequently, it is not clear whether the transport density σ belongs to $L^p(\Omega)$ or not, even if $f\in L^p(\Omega)$. In [15, 16], the authors have already studied the L^p summability of this transport density σ. More precisely, $\sigma\in L^p(\Omega)$ as soon as $f\in L^p(\Omega)$, Ω satisfies a uniform exterior ball condition and, g is $\beta-$Lip with $\beta<1$ and semi-concave. Moreover, we have the following L^p estimate on σ (for all $p\in[1,\infty]$):

$$||\sigma||_{L^p(\Omega)}\leq C\,||f||_{L^p(\Omega)}. \tag{1.8}$$

However, the higher order regularity (BV, $W^{1,p},...$) of this transport density σ is an open question! In this paper, we will show that under some assumptions on $\partial\Omega$ and g, we have the following statements:

$$f\in BV(\Omega)\cap L^\infty(\Omega)\ \ \Rightarrow\ \ \sigma\in BV(\Omega),$$

$$f\in W^{1,1}(\Omega)\cap L^\infty(\Omega)\ \ \Rightarrow\ \ \sigma\in W^{1,1}(\Omega).$$

More precisely, we prove the following:

Theorem 1.1. *Assume that $\partial\Omega$ is of class $C^{2,1}$ and $g\in C^{2,1}(\partial\Omega)$. Then, the transport density σ in (1.4) is in $BV(\Omega)$ provided that $f\in BV(\Omega)\cap L^\infty(\Omega)$. Moreover, we have the following estimate:*

$$||\sigma||_{BV(\Omega)}\leq C(||f||_{BV(\Omega)}+||f||_{L^\infty(\Omega)}).$$

In addition, we will introduce a counter-example to the $W^{1,p}$ regularity of σ, for large p. To be more precise, we will show that in general even if $\partial\Omega$ and g are smooth, we have the following statement:

$$f\in C^\infty(\overline{\Omega})\ \ \not\Rightarrow\ \ \sigma\in W^{1,5}(\Omega).$$

2. BV estimates on the transport density

Let $\Omega \subset \mathbb{R}^2$ be a compact domain, g be a $\beta-$Lip function on $\partial\Omega$ with $\beta < 1$ and the density $f \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Let Σ be the singular set of the Kantorovich potential u in Problem (1.6) (see (1.7)), i.e. Σ is the set of points $x \in \Omega$ at which u is not differentiable or equivalently, it is the set of points x at which $\tilde{T}(x)$ is not a singleton (it is clear that $|\Sigma| = 0$). We note that if $[x, y]$ is a transport ray, then it is clear that $x \in \overline{\Sigma}$ and $y \in \partial\Omega$. On the other hand, if $\partial\Omega$ is C^2 (let us denote by κ the curvature of $\partial\Omega$) and $g \in C^2(\partial\Omega)$, then by [14, Lemma 2.3], we have that $\overline{\Sigma} \subset \mathring{\Omega}$ and, $\overline{\Sigma}$ is given by

$$\overline{\Sigma} = \Sigma \cup \{x \in \Omega\backslash\Sigma \,:\, 1 - \partial_{\mathbf{t}} g(T(x))^2 - |x - T(x)|\, \Gamma(x) = 0\},$$

where

$$\Gamma(x) = \sqrt{1 - \partial_{\mathbf{t}} g(T(x))^2}\, \kappa(T(x)) - \partial^2_{\mathbf{tt}} g(T(x)) - \partial_{\mathbf{n}} g(T(x))\kappa(T(x)),$$

and the vector $\mathbf{n} := \mathbf{n}(T(x))$ denotes the unit interior normal vector to $\partial\Omega$ at $T(x)$ while $\mathbf{t} := \mathbf{t}(T(x))$ is the corresponding tangent vector (the rotation with angle $-\frac{\pi}{2}$ of the normal vector $\mathbf{n}$). Thanks to [14, Lemma 2.1], we note also that

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - |x - T(x)|\, \Gamma(x) \geq 0, \text{ for all } x \in \Omega. \tag{2.1}$$

From now on, we assume that $\partial\Omega \in C^{2,1}$ and $g \in C^{2,1}(\partial\Omega)$. Fix $x_0 \in \Omega\backslash\overline{\Sigma}$. Let $\alpha(s)$, $s \in [-\varepsilon, \varepsilon]$, be a parametrization of $\partial\Omega$ around $T(x_0)$ such that $|\alpha'| = 1$. For every $s \in [-\varepsilon, \varepsilon]$, we set

$$\tau(s) = \min\{\lambda \geq 0 \,:\, \alpha(s) + \lambda\, \nabla u(\alpha(s)) \in \overline{\Sigma}\}.$$

Set

$$\Delta := \{\alpha(s) + \lambda\, \nabla u(\alpha(s)) \,:\, s \in [-\varepsilon, \varepsilon] \text{ and } \lambda \in [0, \tau(s)]\}.$$

In the sequel, we will show that the transport density $\sigma \in W^{1,1}(\Delta)$ and that there is a uniform constant C such that the following estimate holds:

$$||\sigma||_{W^{1,1}(\Delta)} \leq C(||f||_{W^{1,1}(\Delta)} + ||f||_{L^\infty(\Delta)}).$$

From (1.1), we have

$$< \sigma, \varphi >= \int_\Delta \int_0^1 \varphi((1-t)x + tT(x))|x - T(x)|f(x)\,\mathrm{d}t\,\mathrm{d}x, \text{ for all } \varphi \in C(\Delta).$$

Take a change of variable $x = \alpha(s) + \lambda\, \nabla u(\alpha(s))$. Then, for every $\varphi \in C(\Delta)$, we get that

(2.2)

$$< \sigma, \varphi >= \int_0^1 \int_{-\varepsilon}^{\varepsilon} \int_0^{\tau(s)} \varphi(\alpha(s) + (1-t)\,\lambda\, \nabla u(\alpha(s)))\, \lambda\, f(\alpha(s) + \lambda\, \nabla u(\alpha(s)))\, \mathcal{J}(s, \lambda)\,\mathrm{d}\lambda\,\mathrm{d}s\,\mathrm{d}t,$$

where

$$\mathcal{J}(s, \lambda) = |\det[D_{(s,\lambda)}\, x]|$$

$$= \left| \det \begin{bmatrix} \alpha_1'(s) + \lambda[\partial^2_{x_1^2} u(\alpha(s))\alpha_1'(s) + \partial^2_{x_1 x_2} u(\alpha(s))\alpha_2'(s)] & \partial_{x_1} u(\alpha(s)) \\ \alpha_2'(s) + \lambda[\partial^2_{x_1 x_2} u(\alpha(s))\alpha_1'(s) + \partial^2_{x_2^2} u(\alpha(s))\alpha_2'(s)] & \partial_{x_2} u(\alpha(s)) \end{bmatrix} \right|$$

$$= |[I + \lambda\, D^2 u(\alpha(s))]\alpha'(s) \cdot R_{-\frac{\pi}{2}} \nabla u(\alpha(s))|,$$

where $R_{-\frac{\pi}{2}}(\zeta_1,\zeta_2) := (\zeta_2,-\zeta_1)$ denotes the rotation of the vector (ζ_1,ζ_2) with angle $-\frac{\pi}{2}$. From [14, Proposition 2.2], we know that

$$D^2u(\alpha(s)) = \frac{-\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)+\partial^2_{\mathbf{tt}}g(\alpha(s))+\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)}{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,e(s)\otimes e(s),$$

where

$$e(s) = R_{-\frac{\pi}{2}}\nabla u(\alpha(s)).$$

This implies that

$$\mathcal{J}(s,\lambda) = |[I+\lambda\, D^2u(\alpha(s))]\alpha'(s)\cdot e(s)|$$
$$= \left|\alpha'(s)\cdot e(s) - \lambda\frac{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)}{1-\partial_{\mathbf{t}}g(\alpha(s))^2}[e(s)\otimes e(s)]\,\alpha'(s)\cdot e(s)\right|$$
$$= \left|\left[1-\lambda\frac{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)}{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\right]\alpha'(s)\cdot e(s)\right|.$$

Yet, by [14, Lemma 2.1], one has

$$e(s) = \sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\mathbf{t}(s) - \partial_{\mathbf{t}}g(\alpha(s))\,\mathbf{n}(s).$$

Hence, by (2.1), we get

$$\mathcal{J}(s,\lambda) = \frac{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\lambda\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}}. \tag{2.3}$$

Recalling (2.2), for every $\varphi\in C(\Delta)$, we have

$$<\sigma,\varphi>$$
$$= \int_0^1\int_{-\varepsilon}^{\varepsilon}\int_0^{(1-t)\tau(s)}\varphi(\alpha(s)+\lambda\,\nabla u(\alpha(s)))\frac{\lambda}{(1-t)^2}\,f\left(\alpha(s)+\frac{\lambda}{1-t}\nabla u(\alpha(s))\right)\mathcal{J}\left(s,\frac{\lambda}{1-t}\right)\mathrm{d}\lambda\,\mathrm{d}s\,\mathrm{d}t$$
$$= \int_{-\varepsilon}^{\varepsilon}\int_0^{\tau(s)}\int_0^{1-\frac{\lambda}{\tau(s)}}\varphi(\alpha(s)+\lambda\nabla u(\alpha(s)))\frac{\lambda}{(1-t)^2}\,f\left(\alpha(s)+\frac{\lambda}{1-t}\nabla u(\alpha(s))\right)\mathcal{J}\left(s,\frac{\lambda}{1-t}\right)\mathrm{d}t\,\mathrm{d}\lambda\,\mathrm{d}s.$$

This implies that

$$\sigma(s,\lambda) = \int_0^{1-\frac{\lambda}{\tau(s)}}\frac{\lambda}{(1-t)^2}\,f\left(\alpha(s)+\frac{\lambda}{1-t}\,\nabla u(\alpha(s))\right)\frac{\mathcal{J}(s,\frac{\lambda}{1-t})}{\mathcal{J}(s,\lambda)}\,\mathrm{d}t \tag{2.4}$$
$$= \int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\,\mathrm{d}t.$$

Then, we have

$$\partial_\lambda\sigma = -f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)} \tag{2.5}$$
$$+\int_0^{\tau(s)-\lambda}\left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot\nabla u(\alpha(s))\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t$$

$$+\int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_\lambda\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_\lambda\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t$$

and

$$\partial_s\sigma=\tau'(s)\,f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)} \tag{2.6}$$

$$+\int_0^{\tau(s)-\lambda}\left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot[I+(t+\lambda)D^2u(\alpha(s))]\alpha'(s)\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t$$

$$+\int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_s\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t.$$

Yet,

$$\partial_{x_1}\sigma=\partial_\lambda\sigma\,\partial_{x_1}\lambda+\partial_s\sigma\,\partial_{x_1}s \quad \text{and} \quad \partial_{x_2}\sigma=\partial_\lambda\sigma\,\partial_{x_2}\lambda+\partial_s\sigma\,\partial_{x_2}s.$$

Moreover, one has

$$D_x(s,\lambda) \tag{2.7}$$

$$=\begin{bmatrix} \frac{\partial_{x_2}u(\alpha(s))}{\mathcal{J}(s,\lambda)} & -\frac{\partial_{x_1}u(\alpha(s))}{\mathcal{J}(s,\lambda)} \\ -\frac{\alpha_2'(s)+\lambda[\partial^2_{x_1x_2}u(\alpha(s))\alpha_1'(s)+\partial^2_{x_2^2}u(\alpha(s))\alpha_2'(s)]}{\mathcal{J}(s,\lambda)} & \frac{\alpha_1'(s)+\lambda[\partial^2_{x_1^2}u(\alpha(s))\alpha_1'(s)+\partial^2_{x_1x_2}u(\alpha(s))\alpha_2'(s)]}{\mathcal{J}(s,\lambda)} \end{bmatrix}.$$

In (2.5), we have

$$\left|-f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}\right|$$

$$+\left|\int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_\lambda\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_\lambda\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t\right|$$

$$\le ||f||_\infty\left[\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}+\int_0^{\tau(s)-\lambda}\frac{|\mathcal{J}(s,\lambda)\,\partial_\lambda\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_\lambda\mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t\right]. \tag{2.8}$$

Yet,

$$\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}=\frac{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\tau(s)\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\lambda\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}$$

$$=1-[\tau(s)-\lambda]\frac{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)}{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\lambda\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}$$

$$\le 1+\mathrm{diam}(\Omega)\frac{\max\left\{0,-\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]\right\}}{1-\beta^2}$$

$$\le 1+\mathrm{diam}(\Omega)\frac{(1+\beta)||\kappa||_\infty+||D^2g||_\infty}{1-\beta^2}. \tag{2.9}$$

On the other hand, thanks to the fact that $\partial_\lambda \mathcal{J}(s,t+\lambda) = \partial_\lambda \mathcal{J}(s,\lambda)$ and $\mathcal{J}(s,t+\lambda) - \mathcal{J}(s,\lambda) = t\,\partial_\lambda \mathcal{J}(s,\lambda)$, we have

$$\frac{|\mathcal{J}(s,\lambda)\,\partial_\lambda \mathcal{J}(s,t+\lambda) - \mathcal{J}(s,t+\lambda)\,\partial_\lambda \mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2} = \frac{|\mathcal{J}(s,\lambda) - \mathcal{J}(s,t+\lambda)|\,|\partial_\lambda \mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}$$
$$= t\,\frac{\partial_\lambda \mathcal{J}(s,\lambda)^2}{\mathcal{J}(s,\lambda)^2}.$$

Hence,

$$\int_0^{\tau(s)-\lambda} \frac{|\mathcal{J}(s,\lambda)\,\partial_\lambda \mathcal{J}(s,t+\lambda) - \mathcal{J}(s,t+\lambda)\,\partial_\lambda \mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t \le [\tau(s)-\lambda]^2\,\frac{\partial_\lambda \mathcal{J}(s,\lambda)^2}{\mathcal{J}(s,\lambda)^2}.$$

Yet, by (2.9), we know that

$$-1 \le [\tau(s)-\lambda]\frac{\partial_\lambda \mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)} \le \mathrm{diam}(\Omega)\frac{(1+\beta)||\kappa||_\infty + ||D^2 g||_\infty}{1-\beta^2}. \tag{2.10}$$

Recalling (2.8), we get

$$\begin{aligned}
&\left| -f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}\right| \\
&+\left|\int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_\lambda \mathcal{J}(s,t+\lambda) - \mathcal{J}(s,t+\lambda)\,\partial_\lambda \mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t\right| \\
&\le C||f||_\infty,
\end{aligned} \tag{2.11}$$

where the constant C depends only on $\mathrm{diam}(\Omega)$, β, $||\kappa||_\infty$ and $||D^2 g||_\infty$. On the other hand, in (2.6), we have

$$\begin{aligned}
&\left|\tau'(s)\,f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}\right| \\
&+\left|\int_0^{\tau(s)-\lambda} f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_s \mathcal{J}(s,t+\lambda) - \mathcal{J}(s,t+\lambda)\,\partial_s \mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t\right| \\
&\le ||f||_\infty\left[C|\tau'(s)| + \int_0^{\tau(s)-\lambda}\frac{|\mathcal{J}(s,\lambda)\,\partial_s \mathcal{J}(s,t+\lambda) - \mathcal{J}(s,t+\lambda)\,\partial_s \mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}\,\mathrm{d}t\right].
\end{aligned} \tag{2.12}$$

From (2.3), one has

$$\begin{aligned}
\partial_s \mathcal{J}(s,\lambda) &= \frac{-2\,\partial_{\mathbf{t}} g(\alpha(s))\,[D^2 g(\alpha(s))\,\alpha'(s)\cdot\mathbf{t}(s) + \kappa(s)\,\nabla g(\alpha(s))\cdot\mathbf{n}(s)]}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}} \\
&-\lambda\,\frac{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}\,\kappa'(s) - \frac{\partial_{\mathbf{t}} g(\alpha(s))[D^2 g(\alpha(s))\,\alpha'(s)\cdot\mathbf{t}(s)+\kappa(s)\,\nabla g(\alpha(s))\cdot\mathbf{n}(s)]}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}}\,\kappa(s)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}} \\
&+\lambda\,\frac{\partial_s[D^2 g(\alpha(s))\,\mathbf{t}(s)]\cdot\mathbf{t}(s) + D^2 g(\alpha(s))\,\mathbf{t}(s)\cdot\kappa(s)\,\mathbf{n}(s)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}} \\
&+\lambda\,\frac{[D^2 g(\alpha(s))\,\alpha'(s)\cdot\mathbf{n}(s) - \nabla g(\alpha(s))\cdot\kappa(s)\,\mathbf{t}(s)]\,\kappa(s) + \partial_{\mathbf{n}} g(\alpha(s))\,\kappa'(s)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s))^2}} \\
&+\mathcal{J}(s,\lambda)\frac{\partial_{\mathbf{t}} g(\alpha(s))[D^2 g(\alpha(s))\,\alpha'(s)\cdot\mathbf{t}(s) + \kappa(s)\,\nabla g(\alpha(s))\cdot\mathbf{n}(s)]}{1-\partial_{\mathbf{t}} g(\alpha(s))^2}.
\end{aligned}$$

Moreover,

$$\frac{\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_s\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}$$
$$=\frac{\mathcal{J}(s,\lambda)[\partial_s\mathcal{J}(s,t+\lambda)-\partial_s\mathcal{J}(s,\lambda)]-t\,\partial_\lambda\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}.$$

Yet,

$$\partial_s\mathcal{J}(s,t+\lambda)-\partial_s\mathcal{J}(s,\lambda)$$
$$=-t\,\frac{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa'(s)-\frac{\partial_{\mathbf{t}}g(\alpha(s))[D^2g(\alpha(s))\,\alpha'(s)\cdot\mathbf{t}(s)+\kappa(s)\,\nabla g(\alpha(s))\cdot\mathbf{n}(s)]}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}}\,\kappa(s)}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}}$$
$$+\,t\,\frac{\partial_s[D^2g(\alpha(s))\,\mathbf{t}(s)]\cdot\mathbf{t}(s)+D^2g(\alpha(s))\,\mathbf{t}(s)\cdot\kappa(s)\,\mathbf{n}(s)}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}}$$
$$+\,t\,\frac{[D^2g(\alpha(s))\,\alpha'(s)\cdot\mathbf{n}(s)-\nabla g(\alpha(s))\cdot\kappa(s)\,\mathbf{t}(s)]\,\kappa(s)+\partial_{\mathbf{n}}g(\alpha(s))\,\kappa'(s)}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}}$$
$$+\,t\,\partial_\lambda\mathcal{J}(s,\lambda)\frac{\partial_{\mathbf{t}}g(\alpha(s))[D^2g(\alpha(s))\,\alpha'(s)\cdot\mathbf{t}(s)+\kappa(s)\,\nabla g(\alpha(s))\cdot\mathbf{n}(s)]}{1-\partial_{\mathbf{t}}g(\alpha(s))^2}.$$

Then, we get that

$$\int_0^{\tau(s)-\lambda}\frac{|\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_s\mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}\,\mathrm{d}t\le C\int_0^{\tau(s)-\lambda}\frac{t}{\mathcal{J}(s,\lambda)^2}\,\mathrm{d}t,$$

where $C:=C(\mathrm{diam}(\Omega),\,\beta,\,||\kappa||_\infty,\,||\kappa'||_\infty,\,||D^2g||_\infty,\,||D^3g||_\infty)$. Consequently, we infer that we have

$$\int_0^{\tau(s)-\lambda}\frac{|\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_s\mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)^2}\,\mathrm{d}t\le C\,\frac{[\tau(s)-\lambda]^2}{\mathcal{J}(s,\lambda)^2}.$$

Recalling (2.10), we have

$$[\tau(s)-\lambda]\frac{|\partial_\lambda\mathcal{J}(s,\lambda)|}{\mathcal{J}(s,\lambda)}\le C. \tag{2.13}$$

Moreover,

$$\mathcal{J}(s,\lambda)=\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}+\lambda\,\partial_\lambda\mathcal{J}(s,\lambda).$$

If $\partial_\lambda\mathcal{J}(s,\lambda)\ge-\frac{\sqrt{1-\beta^2}}{2\,\mathrm{diam}(\Omega)}$, then we have $\mathcal{J}(s,\lambda)\ge\frac{\sqrt{1-\beta^2}}{2}$. If $\partial_\lambda\mathcal{J}(s,\lambda)\le-\frac{\sqrt{1-\beta^2}}{2\,\mathrm{diam}(\Omega)}$, then by (2.13), we get that

$$\frac{\tau(s)-\lambda}{\mathcal{J}(s,\lambda)}\le C. \tag{2.14}$$

From (2.12), this implies that

$$\left|\tau'(s)\,f\Big(\alpha(s)+\tau(s)\,\nabla u(\alpha(s))\Big)\frac{\mathcal{J}(s,\tau(s))}{\mathcal{J}(s,\lambda)}\right| \tag{2.15}$$
$$+\left|\int_0^{\tau(s)-\lambda}f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\,\frac{\mathcal{J}(s,\lambda)\,\partial_s\mathcal{J}(s,t+\lambda)-\mathcal{J}(s,t+\lambda)\,\partial_s\mathcal{J}(s,\lambda)}{\mathcal{J}(s,\lambda)^2}\mathrm{d}t\right|$$
$$\le C||f||_\infty[|\tau'(s)|+1].$$

On the other hand, it is clear that we have

$$\int_0^{\tau(s)-\lambda} \partial_{x_1} s \left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot[I+(t+\lambda)D^2u(\alpha(s))]\alpha'(s)\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t$$
$$+\int_0^{\tau(s)-\lambda} \partial_{x_1}\lambda \left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot\nabla u(\alpha(s))\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t$$
$$= \int_0^{\tau(s)-\lambda} \partial_{x_1}\left[f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t. \tag{2.16}$$

Yet,

$$\partial_{x_1}\left[f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\right]=\partial_{x_1}[f(x+t\nabla u(x))]=[I+tD^2u(x)]\nabla f(x+t\nabla u(x))\cdot <1,0>.$$

Thanks to [14, Proposition 2.2], we have

$$I+t\,D^2u(x)$$
$$=I-t\,\frac{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)}{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\lambda\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}\,e(s)\otimes e(s)$$
$$=\frac{1-\partial_{\mathbf{t}}g(\alpha(s))^2-(t+\lambda)\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}{1-\partial_{\mathbf{t}}g(\alpha(s))^2-\lambda\left[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s)-\partial^2_{\mathbf{tt}}g(\alpha(s))-\partial_{\mathbf{n}}g(\alpha(s))\kappa(s)\right]}\,e(s)\otimes e(s)$$
$$+\nabla u(\alpha(s))\otimes\nabla u(\alpha(s)).$$

Recalling (2.9), one can see that

$$I+t\,D^2u(x)\le CI.$$

From (2.16), we infer that

$$\Bigg|\int_0^{\tau(s)-\lambda} \partial_{x_1} s \left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot[I+(t+\lambda)D^2u(\alpha(s))]\alpha'(s)\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t$$
$$+\int_0^{\tau(s)-\lambda} \partial_{x_1}\lambda \left[\nabla f\Big(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s))\Big)\cdot\nabla u(\alpha(s))\right]\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t\Bigg|$$
$$\le C\int_0^{\tau(s)-\lambda}|\nabla f|(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s)))\,\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t. \tag{2.17}$$

Now, recalling (2.5), (2.6) & (2.7) and combining (2.11), (2.15) & (2.17), we infer that

$$|\partial_{x_1}\sigma|\le C\left[\frac{||f||_\infty(|\tau'(s)|+1)}{\mathcal{J}(s,\lambda)}+\int_0^{\tau(s)-\lambda}|\nabla f|(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s)))\,\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t\right]. \tag{2.18}$$

Hence, we get that

$$||\partial_{x_1}\sigma||_{L^1(\Delta)}$$

$$\leq C\Big[||f||_\infty \int_\Delta \frac{|\tau'(s)|+1}{\mathcal{J}(s,\lambda)}\,\mathrm{d}x + \int_\Delta \int_0^{\tau(s)-\lambda} |\nabla f|(\alpha(s)+(t+\lambda)\,\nabla u(\alpha(s)))\,\frac{\mathcal{J}(s,t+\lambda)}{\mathcal{J}(s,\lambda)}\mathrm{d}t\,\mathrm{d}x\Big]$$

$$\leq C\Big[||f||_\infty\Big(\int_{-\varepsilon}^{\varepsilon}|\tau'(s)|\,\mathrm{d}s+1\Big)+||\nabla f||_{L^1(\Delta)}\Big].$$

In the same way, we show that

$$||\partial_{x_2}\sigma||_{L^1(\Delta)} \leq C\Big[||f||_\infty\Big(\int_{-\varepsilon}^{\varepsilon}|\tau'(s)|\,\mathrm{d}s+1\Big)+||\nabla f||_{L^1(\Delta)}\Big].$$

Hence, we get

$$||\nabla\sigma||_{L^1(\Delta)} \leq C\Big[||f||_\infty\Big(\int_{-\varepsilon}^{\varepsilon}|\tau'(s)|\,\mathrm{d}s+1\Big)+||\nabla f||_{L^1(\Delta)}\Big]. \tag{2.19}$$

Now, we claim that the map $s \mapsto \tau(s)$ is Lipschitz (we note that the Lipschitz regularity of this map τ was already proved in [7, Theorem 2.12] but in the particular case $g = 0$). More precisely, we show that

$$\tau'(s) \leq C(\mathrm{diam}(\Omega),\,\beta,\,||\kappa||_\infty,\,||\kappa'||_\infty,\,||D^2 g||_\infty,\,||D^3 g||_\infty),\ \text{ for all } s \in (-\varepsilon,\varepsilon). \tag{2.20}$$

For this aim, we show that there is a uniform constant C such that for every $s_0 \in (-\varepsilon,\varepsilon)$, there is a $\delta > 0$ such that

$$\tau(s) \leq \tau(s_0) + C|s-s_0|,\ \text{ for all } s \in (s_0-\delta, s_0+\delta). \tag{2.21}$$

Fix $s_0 \in (-\varepsilon,\varepsilon)$. First, we assume that

$$1-\partial_{\mathbf{t}}g(\alpha(s_0))^2 - \tau(s_0)\Big[\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\,\kappa(s_0) - \partial^2_{\mathbf{tt}}g(\alpha(s_0)) - \partial_{\mathbf{n}}g(\alpha(s_0))\kappa(s_0)\Big] = 0. \tag{2.22}$$

Let $x_0 \in \overline{\Sigma}$ be a point such that $\alpha(s_0) \in \tilde{T}(x_0)$. Let us denote by $P(x_0)$ a projection point of x_0 onto $\partial\Omega$, i.e.

$$P(x_0) := \mathrm{argmin}\{|x_0-y| : y \in \partial\Omega\}.$$

Then, we have

$$|x_0-\alpha(s_0)| + g(\alpha(s_0)) \leq |x_0 - P(x_0)| + g(P(x_0)).$$

Since $|x_0-P(x_0)| \leq \frac{\mathrm{diam}(\Omega)}{2}$ and thanks to the fact that g is $\beta-$Lip with $\beta<1$, this implies that

$$\tau(s_0) \leq \frac{(1+\beta)\,\mathrm{diam}(\Omega)}{2(1-\beta)}.$$

From (2.22), we get that

$$\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\,\kappa(s_0) - \partial^2_{\mathbf{tt}}g(\alpha(s_0)) - \partial_{\mathbf{n}}g(\alpha(s_0))\kappa(s_0) \geq \frac{2(1-\beta)^2}{\mathrm{diam}(\Omega)}.$$

Now, let $\delta > 0$ be small enough. Then it is clear that, for all $s \in (s_0-\delta, s_0+\delta)$, one can assume that

$$\Gamma(s) = \sqrt{1-\partial_{\mathbf{t}}g(\alpha(s))^2}\,\kappa(s) - \partial^2_{\mathbf{tt}}g(\alpha(s)) - \partial_{\mathbf{n}}g(\alpha(s))\kappa(s) \geq \frac{(1-\beta)^2}{\mathrm{diam}(\Omega)}.$$

Hence,

$$\tau(s) \leq \frac{1-\partial_{\mathbf{t}}g(\alpha(s))^2}{\Gamma(s)} = \frac{1-\partial_{\mathbf{t}}g(\alpha(s))^2}{\Gamma(s_0)} + \frac{[1-\partial_{\mathbf{t}}g(\alpha(s))^2](\Gamma(s_0)-\Gamma(s))}{\Gamma(s)\,\Gamma(s_0)}$$

$$= \tau(s_0) + \frac{\partial_{\mathbf{t}} g(\alpha(s_0))^2 - \partial_{\mathbf{t}} g(\alpha(s))^2}{\Gamma(s_0)} + \frac{[1-\partial_{\mathbf{t}} g(\alpha(s))^2](\Gamma(s_0)-\Gamma(s))}{\Gamma(s_0)\,\Gamma(s)} \le \tau(s_0) + C|s-s_0|,$$

where the constant C depends only on $\mathrm{diam}(\Omega)$, β, $||D^2 g||_\infty$, $||D^3 g||_\infty$, $||\kappa||_\infty$ and $||\kappa'||_\infty$. Now, we assume that

$$1 - \partial_{\mathbf{t}} g(\alpha(s_0))^2 - \tau(s_0)\,\Gamma(s_0) > 0. \tag{2.23}$$

Thanks to [1, Lemma 4.5] or [14, Proposition 2.6], one can show that there is a Lipschitz arc $\gamma : [-\delta', \delta'] \mapsto \overline{\Sigma}$ (for some $\delta' > 0$) such that $|\gamma'| = 1$, $\gamma(0) = x_0$ and $\gamma'(0) = n$, where $n \cdot e_1 > 0$, $e_1 = R_{-\frac{\pi}{2}}\, e_2$, $e_2 = \nabla u(\alpha(s_0))$ and n is a normal vector to $[p, e_2]$, for some vector p in the set of limiting gradients $D^\star u(x_0)$. For every $s \in [-\delta, \delta]$, let $t(s) \in [-\delta', \delta']$ be such that $\alpha(s) \in \tilde{T}(\gamma(t(s)))$. Then, we have

$$[\gamma(t(s)) - \alpha(s)] \cdot R_{\frac{\pi}{2}} \nabla u(\alpha(s)) = 0. \tag{2.24}$$

Yet,

$$\gamma(t(s)) - \alpha(s) = \tau(s_0)\, e_2 + t(s)\,\gamma'(0) - (s-s_0)\,\mathbf{t}(s_0) + o(s-s_0) + o(t(s)). \tag{2.25}$$

Moreover,

$$\nabla u(\alpha(s)) = e_2 + D^2 u(\alpha(s_0))(\alpha(s) - \alpha(s_0)) + o(s-s_0).$$

But, we recall that

$$D^2 u(\alpha(s_0)) = \frac{-\Gamma(s_0)}{1 - \partial_{\mathbf{t}} g(\alpha(s_0))^2}\, e_1 \otimes e_1.$$

Hence, we get

$$\nabla u(\alpha(s))$$
$$= e_2 - \frac{\Gamma(s_0)}{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}[e_1 \cdot (s-s_0)\mathbf{t}(s_0)]e_1 + o(s-s_0) = e_2 - \frac{\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}(s-s_0)\, e_1 + o(s-s_0).$$

Then,

$$R_{\frac{\pi}{2}} \nabla u(\alpha(s)) = -e_1 - \frac{\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}(s-s_0)\, e_2 + o(s-s_0). \tag{2.26}$$

By (2.25) & (2.26), we get

$$[\gamma(t(s)) - \alpha(s)] \cdot R_{\frac{\pi}{2}} \nabla u(\alpha(s))$$
$$= [\tau(s_0)\, e_2 + t(s)\gamma'(0) - (s-s_0)\,\mathbf{t}(s_0) + o(s-s_0) + o(t(s))] \cdot \left[-e_1 - \frac{(s-s_0)\,\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}} e_2 + o(s-s_0) \right]$$
$$= \frac{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2 - \tau(s_0)\,\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}(s-s_0) - [\gamma'(0)\cdot e_1]\, t(s) - \frac{\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}[\gamma'(0)\cdot e_2](s-s_0)\, t(s)$$
$$+ o(s-s_0).$$

Thanks to (2.24), we infer that

$$\frac{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2 - \tau(s_0)\,\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}(s-s_0) - [\gamma'(0)\cdot e_1] t(s) - \frac{\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}} g(\alpha(s_0))^2}}[\gamma'(0)\cdot e_2](s-s_0)\, t(s)$$
$$+ o(s-s_0) = 0.$$

This implies that

$$t(s)=\frac{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2-\tau(s_0)\,\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\,[\gamma'(0)\cdot e_1]}(s-s_0)+o(s-s_0).$$

Hence, we have

$$\tau(s)=|\gamma(t(s))-\alpha(s)|$$

$$=[\tau(s_0)\,e_2+t(s)\,\gamma'(0)-(s-s_0)\,\mathbf{t}(s_0)+o(s-s_0)]\cdot\left[e_2-\frac{(s-s_0)\,\Gamma(s_0)}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}}e_1+o(s-s_0)\right]$$

$$=\tau(s_0)+[\gamma'(0)\cdot e_2]\,t(s)-\partial_{\mathbf{t}}g(\alpha(s_0))\,(s-s_0)+o(s-s_0)$$

$$=\tau(s_0)+\left[\frac{[1-\partial_{\mathbf{t}}g(\alpha(s_0))^2-\tau(s_0)\,\Gamma(s_0)][\gamma'(0)\cdot e_2]}{\sqrt{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\,[\gamma'(0)\cdot e_1]}-\partial_{\mathbf{t}}g(\alpha(s_0))\right](s-s_0)+o(s-s_0)$$

$$\leq\tau(s_0)+\left[\frac{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2-\tau(s_0)\,\Gamma(s_0)}{\sqrt{1-\beta^2}\,[\gamma'(0)\cdot e_1]}+\beta\right]|s-s_0|+o(s-s_0). \tag{2.27}$$

Yet,

$$\gamma'(0)\cdot e_1=n\cdot e_1=\frac{[e_2-p]}{|e_2-p|}\cdot e_2=\frac{1-p\cdot e_2}{|p-e_2|}=\frac{|p-e_2|}{2}.$$

We recall that $p\in D^{\star}u(x_0)$. So, let $x\in\tilde{T}(x_0)$ be such that $\nabla u(x)=p$. Then, we have the following:

$$\alpha(s_0)-x=-[u(x_0)-g(\alpha(s_0))]e_2+[u(x_0)-g(x)]p=[u(x_0)-g(\alpha(s_0))][p-e_2]+[g(\alpha(s_0))-g(x)]p$$

$$=\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p. \tag{2.28}$$

Thanks to [14, Proposition 2.2 & Lemma 2.3], we have

$$|\nabla u(x)-\nabla u(\alpha(s_0))-D^2u(\alpha(s_0))[x-\alpha(s_0)]|\leq C|x-\alpha(s_0)|^2.$$

Then,

$$\left|p-e_2-\frac{\Gamma(s_0)}{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\,e_1\otimes e_1(\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p)\right|\leq C|\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p|^2. \tag{2.29}$$

But, we have

$$e_1\otimes e_1(\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p)=\tau(s_0)[e_1\cdot p]e_1+[g(\alpha(s_0))-g(x)][e_1\cdot p]e_1$$

$$=[\tau(s_0)+g(\alpha(s_0))-g(x)][e_1\cdot p]e_1.$$

Moreover, one has

$$p-e_2=[e_1\cdot p]e_1+([e_2\cdot p]-1)e_2=[e_1\cdot p]e_1-\frac{|p-e_2|^2}{2}e_2.$$

Then,

$$e_1\otimes e_1(\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p)=[\tau(s_0)+g(\alpha(s_0))-g(x)]\left[p-e_2+\frac{|p-e_2|^2}{2}e_2\right].$$

By (2.29), we get

$$\left|p-e_2-\frac{\Gamma(s_0)}{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}\left[\tau(s_0)+g(\alpha(s_0))-g(x)\right]\left[p-e_2+\frac{|p-e_2|^2}{2}e_2\right]\right| \leq C|\tau(s_0)[p-e_2]+[g(\alpha(s_0))-g(x)]p|^2.$$

Thanks to (2.28) and the fact that g is $\beta-$Lip with $\beta<1$, we have

$$|g(\alpha(s_0))-g(x)| \leq \beta|\alpha(s_0)-x| \leq \frac{\beta}{1-\beta}\tau(s_0)|p-e_2|.$$

Hence,

$$\frac{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2-\tau(s_0)\,\Gamma(s_0)}{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2}|p-e_2| \leq C|p-e_2|^2.$$

This implies that

$$\frac{1-\partial_{\mathbf{t}}g(\alpha(s_0))^2-\tau(s_0)\,\Gamma(s_0)}{\gamma'(0)\cdot e_1} \leq C,$$

where C is a uniform constant depending only on diam(Ω), β, $||D^2g||_\infty$, $||D^3g||_\infty$, $||\kappa||_\infty$ and $||\kappa'||_\infty$. Recalling (2.27), we get (2.21) which yields (see, for instance, [8, Theorem 7.3]) that τ is Lipschitz and, this also concludes the proof of the claim (2.20).

Consequently, we get the following Sobolev estimates on the transport density σ:

Theorem 2.1. *Assume that $\partial\Omega$ is of class $C^{2,1}$, g is $\beta-$Lipschitz with $\beta<1$ and, $g\in C^{2,1}(\partial\Omega)$. Then, the transport density σ is in $W^{1,1}(\Omega)$ as soon as $f\in W^{1,1}(\Omega)\cap L^\infty(\Omega)$. Moreover, there is a constant C that depends only on diam(Ω), Per(Ω), β, $||D^2g||_\infty$, $||D^3g||_\infty$, $||\kappa||_\infty$ and $||\kappa'||_\infty$ (where κ is the curvature of $\partial\Omega$) such that the following estimate holds*

$$||\sigma||_{W^{1,1}(\Omega)} \leq C(||f||_{W^{1,1}(\Omega)}+||f||_{L^\infty(\Omega)}).$$

Proof. This follows immediately from (2.19), (2.20) & (1.8). □

Finally, we are ready to prove Theorem 1.1.

Proof. Let ρ_ε be a sequence of mollifiers and, set f_ε to be the mollification of f (i.e. $f_\varepsilon := f\star\rho_\varepsilon$). Let σ_ε be the transport density between f_ε and $T_\# f_\varepsilon$. Then, we have

$$||\sigma_\varepsilon||_{W^{1,1}(\Omega)} \leq C(||f_\varepsilon||_{W^{1,1}(\Omega)}+||f_\varepsilon||_{L^\infty(\Omega)}) \leq C(||f||_{BV(\Omega)}+||f||_{L^\infty(\Omega)}).$$

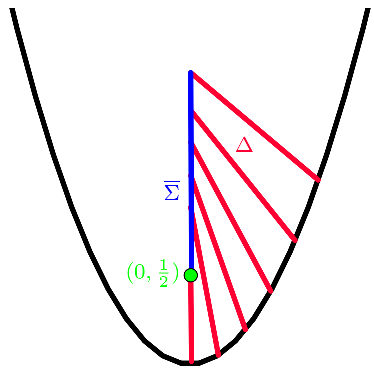
Hence, up to a subsequence, σ_ε converges weakly* in $BV(\Omega)$. Yet, it is not difficult to see that $\sigma_\varepsilon \rightharpoonup \sigma$ in the sense of measures, where σ is the transport density between f and $T_\# f$. This implies that $\sigma\in BV(\Omega)$ and that we have the following BV estimate on σ:

$$||\sigma||_{BV(\Omega)} \leq C(||f||_{BV(\Omega)}+||f||_{L^\infty(\Omega)}). \quad \square$$

3. Counter-example to the $W^{1,5}$ regularity of the transport density

Let Ω be a compact domain such that $\Omega\cap\{x_2\leq M\}=\{x_1^2\leq x_2\}$ (where $M>0$ is large enough), $f=1$ on Ω and, $g=0$ on $\partial\Omega$. Fix $\varepsilon>0$ small enough. For every $s\in[0,\varepsilon]$, the unit interior normal vector to $\partial\Omega$ at (s,s^2) is given by $\mathbf{n}(s)=\frac{(-2s,1)}{\sqrt{1+4s^2}}$ and the normal line is then given by $x_2=-\frac{1}{2s}x_1+s^2+\frac{1}{2}$. Set $\Delta:=\{(x_1,x_2):x_2=-\frac{1}{2s}x_1+s^2+\frac{1}{2},\,0\leq s\leq\varepsilon\}$. It is easy to see that $\Delta \cap \overline{\Sigma} = \{(0, s^2 + \frac{1}{2}) : 0 \le s \le \varepsilon\}$. For every $x \in \Delta$, it is clear that there is a unique $(s, \lambda) \in [0, \varepsilon] \times [0, \tau(s)]$, where $\tau(s) = |(s, s^2) - (0, s^2 + \frac{1}{2})| = \sqrt{s^2 + \frac{1}{4}}$, such that

$$x = \left(s - \lambda \frac{2s}{\sqrt{1+4s^2}}, s^2 + \frac{\lambda}{\sqrt{1+4s^2}} \right).$$



Recalling (2.3) & (2.4), we have

$$\sigma(s, \lambda) = \int_0^{\tau(s)-\lambda} \frac{\mathcal{J}(s, t+\lambda)}{\mathcal{J}(s,\lambda)} \mathrm{d}t, \tag{3.1}$$

where

$$\mathcal{J}(s, \lambda) = 1 - \lambda\, \kappa(s).$$

Hence,

$$\sigma(s, \lambda) = \int_0^{\tau(s)-\lambda} \left[1 - t \frac{\kappa(s)}{1 - \lambda\, \kappa(s)} \right] \mathrm{d}t = \tau(s) - \lambda - \frac{(\tau(s) - \lambda)^2}{2} \frac{\kappa(s)}{1 - \lambda\, \kappa(s)} \tag{3.2}$$

and

$$\kappa(s) = \frac{2}{(1+4s^2)^{\frac{3}{2}}}.$$

On the other hand, we have

$$D_{(s,\lambda)}x = \begin{bmatrix} 1 - \frac{2\lambda}{(1+4s^2)^{\frac{3}{2}}} & -\frac{2s}{\sqrt{1+4s^2}} \\ 2s - \frac{4s\lambda}{(1+4s^2)^{\frac{3}{2}}} & \frac{1}{\sqrt{1+4s^2}} \end{bmatrix}.$$

In particular, one has

$$\tilde{\mathcal{J}}(s, \lambda) = |D_{(s,\lambda)}x| = \sqrt{1+4s^2} \left[1 - \lambda \frac{2}{(1+4s^2)^{\frac{3}{2}}} \right]$$

and

$$D_x(s, \lambda) = \frac{1}{\sqrt{1+4s^2} \left[1 - \lambda \frac{2}{(1+4s^2)^{\frac{3}{2}}} \right]} \begin{bmatrix} \frac{1}{\sqrt{1+4s^2}} & \frac{2s}{\sqrt{1+4s^2}} \\ -2s + \frac{4s\lambda}{(1+4s^2)^{\frac{3}{2}}} & 1 - \frac{2\lambda}{(1+4s^2)^{\frac{3}{2}}} \end{bmatrix}.$$

Yet, we have

$$\partial_{x_1}\sigma = \partial_s \sigma\, \partial_{x_1} s + \partial_\lambda \sigma\, \partial_{x_1} \lambda.$$

But,

$$\partial_\lambda \sigma = -1 + (\tau(s)-\lambda)\frac{\kappa(s)}{1-\lambda\,\kappa(s)} - \frac{(\tau(s)-\lambda)^2}{2}\frac{\kappa(s)^2}{(1-\lambda\,\kappa(s))^2}$$

and

$$\partial_{x_1}\lambda = \frac{-2s}{\sqrt{1+4s^2}}.$$

And, by (2.14), one has

$$\frac{\tau(s)-\lambda}{1-\lambda\kappa(s)} \le C, \text{ for all } s\in[0,\varepsilon] \text{ and } \lambda\in[0,\tau(s)].$$

This yields that $\partial_\lambda\sigma\,\partial_{x_1}\lambda \in L^\infty(\Delta)$. Then, it remains to study the summability of $\partial_s\sigma\,\partial_{x_1}s$ to obtain the summability of $\partial_{x_1}\sigma$. First of all, we see that

$$\partial_{x_1}s = \frac{1}{(1+4s^2)[1-\lambda\kappa(s)]}.$$

Moreover, one has

$$\partial_s\sigma = \left[1-(\tau(s)-\lambda)\frac{\kappa(s)}{1-\lambda\,\kappa(s)}\right]\tau'(s) - \frac{(\tau(s)-\lambda)^2}{2}\frac{\kappa'(s)}{(1-\lambda\,\kappa(s))^2}.$$

Yet, $\kappa'(s)<0$, $\kappa(s)\le 2$ and $\tau'(s)=\frac{s}{\sqrt{s^2+\frac{1}{4}}}\ge 0$, for all $s\in[0,\varepsilon]$. Hence, we infer that we have

$$\partial_s\sigma \ge \left[1-2\frac{\tau(s)-\lambda}{1-\lambda\,\kappa(s)}\right]\tau'(s).$$

In particular, for $s\in(0,\varepsilon)$ and $\lambda\in\left(\frac{\tau(s)-\frac{1}{4}}{1-\frac{\kappa(s)}{4}},\tau(s)\right)$ (or equivalently, $0\le\frac{\tau(s)-\lambda}{1-\lambda\,\kappa(s)}\le\frac{1}{4}$), we have

$$\partial_s\sigma \ge \frac{\tau'(s)}{2}.$$

Then, we infer that there is a constant $c>0$ such that the following holds for all $s\in(0,\varepsilon)$ and $\lambda\in\left(\frac{\tau(s)-\frac{1}{4}}{1-\frac{\kappa(s)}{4}},\tau(s)\right)$:

$$\partial_s\sigma\,\partial_{x_1}s \ge c\,\frac{s}{1-\lambda\,\kappa(s)}.$$

Consequently, we get that

$$||\partial_s\sigma\,\partial_{x_1}s||^p_{L^p(\Delta)} \ge c\int_0^\varepsilon\int_{\frac{\tau(s)-\frac{1}{4}}{1-\frac{\kappa(s)}{4}}}^{\tau(s)} \frac{s^p}{(1-\lambda\,\kappa(s))^{p-1}}\,\mathrm{d}\lambda\,\mathrm{d}s$$

$$= \frac{c}{p-2}\int_0^\varepsilon \frac{s^p}{\kappa(s)}\left[(1-\tau(s)\,\kappa(s))^{-p+2} - \left(1-\frac{\tau(s)-\frac{1}{4}}{1-\frac{\kappa(s)}{4}}\,\kappa(s)\right)^{-p+2}\right]\mathrm{d}s$$

$$\ge c\int_0^\varepsilon s^p(1-\tau(s)\,\kappa(s))^{-p+2}\left[1-\left(1-\frac{\kappa(s)}{4}\right)^{p-2}\right]\mathrm{d}s$$

$$\ge c\left[1-\frac{1}{2^{p-2}}\right]\int_0^\varepsilon s^p(1-\tau(s)\,\kappa(s))^{-p+2}\,\mathrm{d}s.$$

Yet,

$$1-\tau(s)\kappa(s) = \frac{4s^2}{1+4s^2}.$$

This implies that

$$||\partial_s\sigma\,\partial_{x_1}s||^p_{L^p(\Delta)} \geq c\int_0^{\varepsilon} s^{-p+4}\,\mathrm{d}s.$$

Hence, $\sigma \notin W^{1,5}(\Omega)$. However, one can show that $\sigma \in W^{1,p}(\Omega)$, for all $p < 5$. This follows immediately using (2.14) and the fact that

$$|\partial_s\sigma\,\partial_{x_1}s| = \left|\left[1-(\tau(s)-\lambda)\frac{\kappa(s)}{1-\lambda\,\kappa(s)}\right]\tau'(s) - \frac{(\tau(s)-\lambda)^2}{2}\frac{\kappa'(s)}{(1-\lambda\,\kappa(s))^2}\right| \frac{1}{(1+4s^2)\,[1-\lambda\kappa(s)]}$$
$$\leq C\frac{s}{1-\lambda\kappa(s)} \in L^p(\Delta),\ \text{for all } p<5.$$

Finally, we get the following (negative) result:

Proposition 3.1. *Let Ω be a compact domain with a smooth boundary and g be a smooth $\beta-$Lip function on $\partial\Omega$ with $\beta < 1$. Then, in general, we have the following statement:*

$$f \in C^\infty(\overline{\Omega}) \quad \not\Rightarrow \quad \sigma \in W^{1,5}(\Omega).$$

We finish this paper with two last remarks.

Remark 3.1. *The $W^{1,p}$ regularity of the transport density σ with $p < 5$ is still an open question. But, it seems that one can construct a counter-example by choosing a suitable domain Ω; one possibility will be to consider the case where the boundary contains a parabola of the form $\{(s, |s|^\alpha) : s \in [-M, M]\}$ (where $\alpha \geq 2$), except that it is not easy to compute explicitly σ since it is difficult to describe the singular set Σ when $\alpha > 2$. In general, we conjecture that we have the following statement:*

$$f \in C^\infty(\overline{\Omega}) \quad \not\Rightarrow \quad \sigma \in W^{1,p}(\Omega),\ \textit{for all } p > 1.$$

Remark 3.2. *It is not obvious whether we can extend the BV estimates on the transport density σ to higher dimension $d > 2$ or not, since it seems difficult to study the Lipschitz regularity of the map τ in this case. More precisely, the proof of Lipschitz regularity of the map τ that we have introduced in Section 2 is based on some estimates where we use rotated gradient of u (see (2.24) & (2.26)). So, the BV regularity of σ in higher dimension is still an open question!*

References

[1] P. Albano and P. Cannarsa: Structural properties of singularities of semiconcave functions. *Ann. Scuola Norm. Sup. Pisa* Cl. Sci. 28, 719-740, 1999.

[2] L. Ambrosio. Lecture notes on optimal transport problems in *Mathematical aspects of evolving interfaces*, Lecture Notes in Math., 1812, 1-52, 2003.

[3] G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge-Kantorovich equation *J. Eur. Math. Soc.* 3 (2), 139–168, 2001.

[4] G. Buttazzo, E. Stepanov, On Regularity of Transport Density in the Monge-Kantorovich Problem, *SIAM Journal on Control and Optimization* 42 (3): 1044-1055.

[5] G. Buttazzo, E. Oudet, B. Velichkov, A free boundary problem arising in PDE optimization, *Calc. Var. and Par. Diff. Eq.* 54 (4), 3829-3856, 2015.

[6] G. Buttazzo, E. Oudet and E. Stepanov, Optimal transportation problems with free Dirichlet regions, in *Variational methods for discontinuous structures*, 41-65, vol 51 of PNLDE, Birkhäuser, Basel, 2002.

[7] P. Cannarsa and P. Cardaliaguet, Representation of equilibrium solutions to the table problem for growing sandpiles, *J. Eur. Math. Soc.* 6, 1-30.

[8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York (1998).

[9] G. Crasta and A. Malusa, A nonhomogeneous boundary value problem in mass transfer theory, in *Calculus of Variations and Partial Differential Equations*, 44 (1-2), 2012, 61-80.

[10] L. De Pascale, L. C. Evans and A. Pratelli, Integral estimates for transport densities, *Bull. of the London Math. Soc.*. 36, n. 3, pp. 383-395, 2004.

[11] L. De Pascale and A. Pratelli, Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem, *Calc. Var. Par. Diff. Eq.* 14, n. 3, pp. 249-274, 2002.

[12] L. De Pascale and A. Pratelli, Sharp summability for Monge Transport density via Interpolation, *ESAIM Control Optim. Calc. Var.* 10, n. 4, pp. 549-552, 2004.

[13]] S. Dweik, Lack of regularity of the transport density in the Monge problem, *Journal de mathématiques pures et appliquées*, 118, 219-234, 2018.

[14] S. Dweik, Optimal region for the transport problem to the boundary, *preprint*, 2022.

[15] S. Dweik and F. Santambrogio, Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques, *Control, Optimisation and Calculus of Variations, 2016.*

[16] S. Dweik, Optimal transportation with boundary costs and summability estimates on the transport density, *Journal of Convex Analysis*, 2017.

[17] S. Dweik, Weighted Beckmann problem with boundary costs, *Quarterly of applied mathematics*, 2018.

[18] L. C. Evans and W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem. *Mem. Amer. Math. Soc.*, 137 (1999), no. 653.

[19] M. Feldman and R. McCann, Uniqueness and transport density in Monge's mass transportation problem, *Calc. Var. Par. Diff. Eq.* 15, n. 1, pp. 81-113, 2002.

[20] I. Fragalà, M.S. Gelli, A. Pratelli, Continuity of an optimal transport in Monge problem, J. Math. Pures Appl. 84 (2005), 1261-1294.

[21] L. Kantorovich, On the transfer of masses, *Dokl. Acad. Nauk. USSR*, (37), 7-8, 1942.

[22] J.M.Mazon, J.Rossi and J.Toledo, An optimal transportation problem with a cost given by the euclidean distance plus import/export taxes on the boundary, *Rev. Mat. Iberoam.* 30 (2014), no. 1, 1-33.

[23] F. Santambrogio, *Optimal Transport for Applied Mathematicians,* in *Progress in Nonlinear Differential Equations and Their Applications,* 87, Birkhäuser Basel (2015).

[24] C. Villani, Optimal Transport, *Old and New*, Springer-Verlag, Berlin, 2009.

Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, 2713, Doha, Qatar.

Email address: sdweik@qu.edu.qa