CHARACTERIZATION OF ABSOLUTELY CONTINUOUS CURVES IN WASSERSTEIN SPACES

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ABSTRACT. Let X be a separable, complete metric space and $\mathscr{P}_p(X)$ be the space of Borel probability measures with finite moment of order $p > 1$, metrized by the Wasserstein distance. In this paper we prove that every absolutely continuous curve with finite p-energy in the space $\mathscr{P}_p(X)$ can be represented by a Borel probability measure on $C([0,T];X)$ concentrated on the set of absolutely continuous curves with finite p-energy in X. Moreover this measure satisfies a suitable property of minimality which entails an important relation on the energy of the curves. We apply this result to the geodesics of $\mathscr{P}_p(X)$ and to the continuity equation in Banach spaces.

1. Introduction

In the recent years, techniques of optimal transportation of measures applied to the study of evolution problems, like linear or nonlinear diffusion equations, have been revealed a useful tool. Fokker-Planck equation, in the linear case, and porous medium equation, in the nonlinear case, are the most important examples. [JKO98], [Ott01] showed that all these partial differential equations can be interpreted as "gradient flow equation" of a suitable energy functional in the space of Borel probability measures, metrized by the so-called 2-Kantorovitch-Rubinstein-Wasserstein distance (for an overview on optimal transportation problems and application to partial differential equations we refer to the books [Vil03] and [AGS05]). In all these models a crucial role is played by the continuity equation

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0, \qquad \text{in } (0,T) \times \mathbb{R}^n; \tag{1}$$

here μ_t, $t \in (0,T)$, is a family of Borel probability measures and $\boldsymbol{v}$ is a Borel velocity vector field $\boldsymbol{v} : (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ (through this paper we always use the notation $\boldsymbol{v}_t(x) := \boldsymbol{v}(t,x)$). The equation has to be intended in the sense of distributions. It is important, in the applications, to consider low regularity vector fields, just satisfying, for $p > 1$, the finite p-energy condition

$$\mathcal{E}_p(\boldsymbol{v}) := \int_0^T \int_{\mathbb{R}^n} \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x) \, dt < +\infty. \tag{2}$$

In order to illustrate the motivations which led us to the results of the paper, we consider a continuous time dependent family μ_t, $t \in [0,T]$, of probability measures with finite p-moment, $p > 1$, which is a solution of (1). When the vector field $\boldsymbol{v}$ is sufficiently regular, in such a way that for every $x \in \mathbb{R}^n$ there exists a unique global solution of the Cauchy problem

$$\dot{X}_t(x) = \boldsymbol{v}_t(X_t(x)), \qquad X_0(x) = x, \qquad t \in [0,T], \tag{3}$$

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then the solution of equation (1) is representable by the formula

$$\mu_t = (X_t)_\# \mu_0. \tag{4}$$

The expression $(X_t)_\#\mu_0$ denotes the push forward of the initial measure μ_0 through the map $X_t : \mathbb{R}^n \to \mathbb{R}^n$, which is defined by $(X_t)_\#\mu_0(B) := \mu_0((X_t)^{-1}(B))$ for every Borel set B of $\mathbb{R}^n$. Taking into account (3) and (2), and using Hölder's inequality, we can estimate for every $s, t \in I$, with $s < t$,

$$\int_{\mathbb{R}^n} \|X_s(x) - X_t(x)\|^p \, d\mu_0(x) \le (t-s)^{p-1} \int_s^t \int_{\mathbb{R}^n} \|\boldsymbol{v}_r(x)\|^p \, d\mu_r(x)\, dr.$$

Recalling that the p-Kantorovitch-Rubinstein-Wasserstein distance between Borel probability measures with finite p-moment could be defined as

$$\begin{aligned} W_p^p(\mu,\nu) \; := \; \inf\Bigg\{ & \int_\Omega \|X(\omega) - Y(\omega)\|^p \, dP(\omega) : (\Omega, P) \text{ probability space}, \\ & X, Y \in L^p((\Omega,P);\mathbb{R}^n),\; X_\# P = \mu,\; Y_\# P = \nu \Bigg\}, \end{aligned} \tag{5}$$

we obtain the inequality

$$W_p^p(\mu_s, \mu_t) \le (t-s)^{p-1} \int_s^t \int_{\mathbb{R}^n} \|\boldsymbol{v}_r(x)\|^p \, d\mu_r(x)\, dr. \tag{6}$$

and consequently

$$|\mu'|^p(t) := \lim_{s\to t} \frac{W_p^p(\mu_s,\mu_t)}{|t-s|^p} \le \int_{\mathbb{R}^n} \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x) \qquad \text{for a.e. } t \in (0,T) \tag{7}$$

(as we can see in the sequel of the paper, the above limit exists for a.e. $t \in (0,T)$).

When the vector field $\boldsymbol{v}$ satisfies only (2), the flow X_t associated to $\boldsymbol{v}$, is, in general, not defined and the representation (4) does not make sense. Nevertheless, another type of representation, strictly linked to the previous one, is possible (see Theorem 8.2.1 in [AGS05] and also the lecture notes of the CIME course [Amb05]): every continuous time dependent Borel probability solution $t \mapsto \mu_t$ of the continuity equation (1) satisfying (2) is representable by means of a Borel probability measure η on the space of continuous functions $C([0,T];\mathbb{R}^n)$. The measure η is concentrated on the set of the curves

$$\begin{aligned} \{t \mapsto X_t(x) : x \in \mathbb{R}^n,\ & X_\cdot(x) \text{ is an integral solution of (3) and} \\ & \dot{X}_\cdot(x) \in L^p(0,T;\mathbb{R}^n)\}. \end{aligned}$$

Now the relation between η and μ_t is given by

$$(e_t)_\# \eta = \mu_t \qquad \forall t \in [0,T], \tag{8}$$

where $e_t : C([0,T];\mathbb{R}^n) \to \mathbb{R}^n$ denotes the evaluation map, defined by $e_t(X_\cdot) := X_t$, and the push forward is defined by $(e_t)_\#\eta(B) := \eta(\{X \in C([0,T];\mathbb{R}^n) : X_t \in B\})$ for every Borel set B of $\mathbb{R}^n$. When (8) holds we say that η represents the curve μ_t.

This interpretation of the solution of the continuity equation, closed to the technique of Young measures, turned out to be very useful in order to study uniqueness and stability properties, in the case of low regularity of the velocity vector field $\boldsymbol{v}_t$ (see [Amb04] and also [ALS05]).

Starting from this representation of the solution μ_t of (1) and taking into account the set on which η is concentrated, it is not difficult to show that the estimate (7) still holds.

For the same curve $t \in I \mapsto \mu_t$ there are many Borel vector fields satisfying (2) such that (1) holds. A natural question is the following: does it exist a Borel vector field $\tilde{\boldsymbol{v}}$ such that the continuity equation (1) holds, and $\tilde{\boldsymbol{v}}$ minimizes the p-energy $\mathcal{E}_p(\boldsymbol{v})$ defined in (2)? The answer is positive and the minimizers are characterized by the fact that the equality holds in (7). Moreover this minimizing vector field is unique and, thanks to the equality in (7), it plays a role of tangent vector to the curve $t \mapsto \mu_t$.

This result holds even for separable Hilbert spaces ([AGS05] Theorem 8.3.1). In this paper we extend it to separable reflexive Banach spaces satisfying the bounded approximation property (see Definition 5.2), and to dual of separable Banach spaces, even considering solutions without finite p-moment; both these results are in fact a direct consequence of a more general property which holds in arbitrary separable and complete metric spaces and is interesting by itself. Indeed, many papers have recently appeared dealing with various aspects of measure metric spaces or, in particular, of Riemannian manifolds, strictly connected to Kantorovitch-Rubinstein-Wasserstein distance (see [LV05a], [LV05b], [Stu05b], [Stu05c], in metric measure spaces, and [WO05], [OV00], [Stu05a], [vRS05], [CEMS01]).

In a separable and complete metric space X, without additional structure, the continuity equation does not make sense, but we can still define a metric notion of velocity and p-energy. Given an absolutely continuous curve $u : [0,T] \to X$ (see the subsection 2.2 for the definition and the property of a.e.-differentiability) the metric derivative of u is defined by

$$|u'|(t) := \lim_{s \to t} \frac{d(u(s), u(t))}{|t-s|}$$

and exists for a.e. $t \in [0,T]$. The p-energy of an absolutely continuous curve $u : [0,T] \to X$ is defined by

$$\mathcal{E}_p(u) := \int_0^T |u'|^p(t)\, dt.$$

We say that u has finite p-energy if $\mathcal{E}_p(u) < +\infty$.

As we have seen in the case $X = \mathbb{R}^n$, the application $t \mapsto \mu_t$ can be thought as a curve in the metric space $\mathscr{P}_p(\mathbb{R}^n)$ of Borel probability measures with finite p-moment, endowed with the distance W_p defined in (5). The property (6) says that the curve $t \mapsto \mu_t$ is absolutely continuous, hence the metric derivative

$$|\mu'|(t) = \lim_{s \to t} \frac{W_p(\mu_s, \mu_t)}{|t-s|}$$

exists for a.e. $t \in (0,T)$, and, for (7) and (2), the p-energy of the curve μ,

$$\mathcal{E}_p(\mu) := \int_0^T |\mu'|^p(t)\, dt$$

is finite.

Let us now try to give a brief account of the metric point of view. First of all, we recall that in a separable and complete metric space X, the Kantorovitch-Rubinstein-Wasserstein distance W_p, defined as in (5) (see also subsection 2.6), makes the space $\mathscr{P}_p(X)$, of the Borel probability measures in X with finite p-moment, a separable and complete metric space too.

In Theorem 3.2 we prove that, given an absolutely continuous curve with finite p-energy $t \mapsto \mu_t$ in the space $\mathscr{P}_p(X)$, $t \in [0,T]$, there exists a Borel probability measure $\tilde{\eta}$ on the space $C([0,T];X)$ of the continuous curves in X, which is concentrated on the set of absolutely continuous curves in X with finite p-energy. This measure $\tilde{\eta}$ represents the curve μ through the relation

$$(e_t)_{\#}\tilde{\eta} = \mu_t \qquad \forall t \in [0,T]. \tag{9}$$

Moreover, and this is the most important point, the following inequality holds

$$\int_{C([0,T];X)} |u'|^p(t)\, d\tilde{\eta}(u) \leq |\mu'|^p(t) \qquad \text{for a.e. } t \in (0,T). \tag{10}$$

On the other hand, Theorem 3.1 states that if η is a Borel probability measure on the space $C([0,T];X)$, concentrated on the set of the absolutely continuous curves with finite p-energy, and

$$\int_{C([0,T];X)} \int_0^T |u'|(t)\, dt\, d\eta(u) < +\infty,$$

then the curve $\mu_t := (e_t)_{\#}\eta$ is absolutely continuous with finite p-energy in the space $\mathscr{P}_p(X)$ and the opposite inequality holds

$$|\mu'|^p(t) \leq \int_{C([0,T];X)} |u'|^p(t)\, d\eta(u) \qquad \text{for a.e. } t \in (0,T).$$

Then the equality holds in (10) and, consequently, the measure $\tilde{\eta}$ satisfies a sort of minimality property.

Thanks to this result, in Banach spaces, it is possible to construct, by disintegrating $\tilde{\eta}$ with respect to e_t and denoting by $\tilde{\eta}_x$ the disintegrated measures $\tilde{\eta} = \tilde{\eta}_x \otimes \mu_t$, a Borel vector field

$$\tilde{\boldsymbol{v}}_t(x) := \int_{\{u \in C([0,T];X): u(t)=x\}} \dot{u}(t)\, d\tilde{\eta}_x(u) \qquad \text{for } \mu_t\text{-a.e. } x \in X,$$

satisfying the continuity equation and

$$\int_X \|\tilde{\boldsymbol{v}}_t(x)\|^p\, d\mu_t(x) = |\mu'|^p(t) \qquad \text{for a.e. } t \in (0,T).$$

This assertion is proved in Section 5 also for measures μ_t without finite p-moment and for dual spaces. For, Theorems 3.1 and 3.2 can be extended to pseudo metric spaces (where the distance can also take the value $+\infty$), so that they cover the case of the space of Borel probability measures $\mathscr{P}(X)$ endowed with the Kantorovitch-Rubinstein-Wasserstein distance, which is a pseudo metric space, if X is not bounded. Another field of application of these results is relative to the theory of Monge-Kantorovitch in Wiener spaces (see [FÜ04], [FÜ02]). The extension of Theorems 3.1 and 3.2 to pseudo metric spaces is possible since all the results are expressed in terms of the metric derivative, and this concept involves only the infinitesimal behaviour of the distance along the curve. It is also interesting to note that if two (topologically equivalent) distances on the same space X induce the same class of absolutely continuous curves and the same metric velocity, then the corresponding Kantorovitch-Rubinstein-Wasserstein distances on $\mathscr{P}(X)$ enjoy the same property (see Corollary 3.4).

Another application of Theorems 3.1 and 3.2 provides a characterization of the geodesics of the metric space $\mathscr{P}_p(X)$ in terms of the geodesics of the metric space X, under the hypothesis that X is a geodesic space (i.e., for every couple of points of X there exists a minimizing geodesic which joins the two points). This application is straightforward since the geodesics are a particular class of absolutely continuous curves. The characterization of the geodesics is stated in Theorem 4.2. A similar result was obtained in [LV05a] in the case of locally compact complete length spaces. In our proof local compactness is not required.
We point out that a characterization of the geodesics of the space $\mathscr{P}_p(X)$ is also useful in order to prove convexity along geodesics of functionals defined on $\mathscr{P}_p(X)$ (on this important subject see e.g. [McC97] where the convex functionals in $\mathscr{P}_2(\mathbb{R}^n)$ was firstly studied and [Stu05a], [LV05a]).
We also show that if X is a geodesic space, then $\mathscr{P}_p(X)$ is a geodesic space too. The problem of existence of curves minimizing an energy functional, more general than the length functional, in the space $\mathscr{P}_p(X)$, is studied in [BBS05].

As a final observation, we point out that in the setting of Banach spaces as in Section 5, using the fact that $\mathscr{P}_p(X)$ is a geodesic space and the existence of the minimal vector field $\tilde{\boldsymbol{v}}$, we recover the Benamou-Brenier formula (introduced, for numerical purposes, in [BB00] in the case $X = \mathbb{R}^n$, $p = 2$ and absolutely continuous measures with compact support)

$$W_p^p(\mu, \nu) = \min \left\{ \int_0^1 \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt : (\mu_t, \boldsymbol{v}_t) \in \mathscr{A}(\mu, \nu) \right\},$$

where $\mathscr{A}(\mu, \nu)$ is the set of the $(\mu_t, \boldsymbol{v}_t)$ such that $t \mapsto \mu_t$ is continuous, $\mu_0 = \mu$, $\mu_1 = \nu$, and $\boldsymbol{v}_t$ is a Borel vector field satisfying $\int_0^1 \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt < +\infty$, and $\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0$ (see the beginning of Section 5 for the notion of solution of the continuity equation in Banach spaces). In Corollary 4.3 we give a metric version of this formula in geodesic metric spaces.

The paper is organized as follows:
in Section 2 we recall the main definitions and we state the preliminary results we need in the sequel;
in Section 3 we state and prove our main results;
Section 4 is devoted to geodesics in the space $\mathscr{P}_p(X)$;
finally, in Section 5, we apply our results to the continuity equation in Banach spaces.

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2. Notation and preliminary results

2.1. **Ambient space and continuous curves.** In this paper the space X will be a complete and separable metric space with metric $d : X \times X \to [0, +\infty)$.
$I := [0, T]$, $T > 0$, is a compact interval of $\mathbb{R}$ and $\Gamma := C(I; X)$ is the separable and complete metric space of continuous curves in X, endowed with the metric of the uniform convergence induced by d:

$$d_\infty(u, \tilde{u}) = \sup_{t \in I} d(u(t), \tilde{u}(t)). \tag{11}$$

2.2. **Absolutely continuous curves in metric spaces and metric derivative.** Let (Y,d) be a metric space. We say that a curve $u : I \to Y$ belongs to $AC^p(I;Y)$, $p \geq 1$, if there exists $m \in L^p(I)$ such that

$$d(u(s),u(t)) \leq \int_s^t m(r)\,dr \qquad \forall s,t \in I \quad s \leq t. \tag{12}$$

A curve $u \in AC^1(I;Y)$ is called absolutely continuous in Y, and a curve $u \in AC^p(I;Y)$, for $p > 1$, is called absolutely continuous with finite p-energy.
The elements of $AC^p(I;Y)$ satisfy the nice property of a.e. metric differentiability. Precisely we have the following Theorem (see [AGS05] for the proof).

Theorem 2.1. *If $u \in AC^p(I;Y)$, $p \geq 1$, then for $\mathscr{L}^1$-a.e. $t \in I$ there exists the limit*

$$\lim_{h \to 0} \frac{d(u(t+h),u(t))}{|h|}. \tag{13}$$

We denote the value of this limit by $|u'|(t)$ and we call it metric derivative *of u at the point t. The function $t \mapsto |u'|(t)$ belongs to $L^p(I)$ and*

$$d(u(s),u(t)) \leq \int_s^t |u'|(r)\,dr \qquad \forall s,t \in I \quad s \leq t.$$

Moreover $|u'|(t) \leq m(t)$ for $\mathscr{L}^1$-a.e. $t \in I$, for each m such that (12) holds.

Remark 2.2. When X is a Banach space satisfying the Radon-Nikodým property (see [DU77]) (resp. X is the dual of a separable Banach space) we have that
$u \in AC^p(I;X)$ if and only if

(i) u is differentiable for a.e. $t \in I$, (resp. u is weakly-$*$ differentiable for a.e. $t \in I$)
(ii) $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \in L^p(I;X)$,
(resp. $\dot{u}(t) := w^* - \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \in L^p_{w*}(I;X)$)
(iii) $u(t) - u(s) = \int_s^t \dot{u}(r)\,dr \qquad \forall s,t \in I, \quad s \leq t.$

Moreover we have

$$\|\dot{u}(t)\| = |u'|(t) \qquad \text{for a.e. } t \in I. \tag{14}$$

See for instance the Appendix of [Bré73] or [Amb95] and [AK00] for the proof. The integral in (iii) is the Bochner integral (resp. the weak-* integral). We recall also that a reflexive Banach space satisfies the Radon-Nikodým property.

Remark 2.3. If $d : Y \times Y \to [0,+\infty]$ satisfies all the usual axioms of the distance but can also assume the value $+\infty$, we call it *pseudo distance* [1] and the space (Y,d) *pseudo metric space*. A pseudo distance induces on Y a topology (the topology generated by the open balls) exactly as a distance and, defining

$$\tilde{d}(x,y) := d(x,y) \wedge 1,$$

[1] Often pseudo distance means a $d : Y \times Y \to [0,+\infty]$ which satisfies all the usual axioms of the distance, except the property $d(x,y) = 0 \Leftrightarrow y = x$. In place of this property, it satisfies only $d(x,x) = 0$. We do not consider this variant. In [Bou58] the application d is called écart.

the space $\tilde{Y} := (Y, \tilde{d})$ is a bounded metric space, topologically equivalent to Y.
We observe explicitly that $C(I;Y) = C(I;\tilde{Y})$, the definition of absolutely continuous curves in Y makes sense and Theorem 2.1 holds. Moreover, if $u \in AC^p(I;Y)$ then $d(u(t), u(s)) < +\infty$ for every $s, t \in I$, and the metric derivative of u with respect to $\tilde{d}$ coincides with the metric derivative of u with respect to d. Then it follows that $AC^p(I;Y) = AC^p(I;\tilde{Y})$.

2.3. $L^p(I;X)$ **spaces.** We say that a function $u : I \to X$ belongs to $\mathcal{L}^p(I;X)$, $p \in [1, +\infty)$ if u is Lebesgue measurable and

$$\int_0^T d^p(u(t), \bar{x})\, dt < +\infty$$

for some (and thus for every) $\bar{x} \in X$.
The metric space $L^p(I;X)$ is the space of equivalence class (with respect to the equality a.e.) of functions in $\mathcal{L}^p(I;X)$, endowed with the distance

$$d_p(u, v) := \left(\int_0^T d^p(u(t), v(t))\, dt \right)^{\frac{1}{p}}.$$

Since X is separable and complete, the space $L^p(I;X)$ is separable and complete too.
We recall a compactness criterion in $L^p(I;X)$ (it follows by Theorem 2, Proposition 1.10, Remark 1.11 of [RS03] since the last two can be extended to $L^p(I;X)$).

Theorem 2.4. *A family $\mathscr{A} \subset L^p(I;X)$ is precompact if $\mathscr{A}$ is bounded,*

$$\lim_{h \downarrow 0} \sup_{u \in \mathscr{A}} \int_0^{T-h} d^p(u(t+h), u(t))\, dt = 0,$$

and there exists a function $\psi : X \to [0, +\infty]$ whose sublevels $\lambda_c(\psi) := \{x \in X : \psi(x) \le c\}$ are compact for every $c \ge 0$, such that

$$\sup_{u \in \mathscr{A}} \int_0^T \psi(u(t))\, dt < +\infty. \tag{15}$$

2.4. **The Sobolev spaces** $W^{1,p}(I;X)$**.** In the finite dimensional case $X = \mathbb{R}^n$ it is well known that the Sobolev spaces $W^{1,p}(I;\mathbb{R}^n)$, for $p > 1$ can be characterized by

$$\left\{ u \in L^p(I;\mathbb{R}^n) : \sup_{0<h<T} \int_0^{T-h} (\Delta_h u(t))^p\, dt < +\infty \right\},$$

where $\Delta_h u$, for $h \in (0, T)$, denotes the differential quotient

$$\Delta_h u(t) := \frac{|u(t+h) - u(t)|}{h}, \qquad t \in [0, T-h].$$

When X is a separable, complete metric space and $p > 1$, still denoting by $\Delta_h u$, for $h \in (0, T)$, the differential quotient

$$\Delta_h u(t) := \frac{d(u(t+h), u(t))}{h}, \qquad t \in [0, T-h],$$

we can define

$$W^{1,p}(I;X) := \left\{ u \in L^p(I;X) : \sup_{0<h<T} \int_0^{T-h} (\Delta_h u(t))^p \, dt < +\infty \right\}. \tag{16}$$

The following Lemma shows that the spaces $AC^p(I;X)$ are strictly linked to the Sobolev spaces $W^{1,p}(I;X)$, as in the well known case $X = \mathbb{R}$.

Lemma 2.5. *Let $p > 1$. If $u \in AC^p(I;X)$ then (the equivalence class of) $u \in W^{1,p}(I;X)$. If $u \in W^{1,p}(I;X)$ then there exists a unique continuous representative $\tilde{u} \in C(I;X)$ (in particular $\tilde{u}(t) = u(t)$ for $\mathscr{L}^1$-a.e. $t \in I$). Moreover $\tilde{u} \in AC^p(I;X)$ and the application $T : W^{1,p}(I;X) \to \Gamma$ defined by $Tu = \tilde{u}$ is a Borel map.*

Proof. The proof of first assertion can be carried out exactly as in the case $X = \mathbb{R}$ (see for example [Bre83] Proposition VIII.3), by using

$$(\Delta_h u(t))^p \le \frac{1}{h} \int_t^{t+h} |u'|^p(r) \, dr.$$

Now we assume that $u \in W^{1,p}(I;X)$ and we consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ dense in X. Defining $u_n(t) := d(u(t), y_n)$, the triangular inequality implies

$$|u_n(t+h) - u_n(t)| \le d(u(t+h), u(t)). \tag{17}$$

The fact that $u \in W^{1,p}(I;X)$ and $p > 1$ implies that $u_n \in W^{1,p}(I)$ again for [Bre83] Proposition VIII.3. Hence there exist $\tilde{u}_n$ absolutely continuous such that $\tilde{u}_n = u_n$ a.e. and $\tilde{u}_n$ is a.e. differentiable.
We introduce the negligible set

$$N = \bigcup_{n\in\mathbb{N}} (\{t \in I : \tilde{u}_n(t) \neq u_n(t)\} \cup \{t \in I : \tilde{u}_n'(t) \text{ does not exists}\}),$$

and we define $m(t) := \sup_n |\tilde{u}_n'(t)|$ for all $t \in I \setminus N$. Clearly, by the density of $\{y_n\}$, we have for all $t, s \in I \setminus N$, with $s < t$,

$$d(u(t), u(s)) = \sup_n |\tilde{u}_n(t) - \tilde{u}_n(s)| \le \sup_n \int_s^t |\tilde{u}_n'(r)| \, dr \le \int_s^t m(r) \, dr. \tag{18}$$

We show that $m \in L^p(I)$. Actually, by (17), if $t \in I \setminus N$ then

$$|\tilde{u}_n'(t)| = \lim_{h\to 0} \frac{|\tilde{u}_n(t+h) - \tilde{u}_n(t)|}{|h|} \le \liminf_{h\to 0} |\Delta_h u(t)|, \tag{19}$$

which implies $m(t) \le \liminf_{h\to 0} |\Delta_h u(t)|$. By Fatou's Lemma and $u \in W^{1,p}(I;X)$ we obtain

$$\int_0^T |m(t)|^p \, dt \le C. \tag{20}$$

(18) and Hölder's inequality show that $u : I \setminus N \to X$ is uniformly continuous, thus, by the completeness of X, it admits a unique continuous extension $\tilde{u} : I \to X$ which also satisfies

$$d(\tilde{u}(t), \tilde{u}(s)) \le \int_s^t m(r) \, dr \qquad \forall t, s \in I, \quad s < t, \tag{21}$$

and then, for (20), $\tilde{u} \in AC^p(I;X)$.
We have thus proved that $u \in AC^p(I;X)$ if and only if $u \in \Gamma$ and $\sup_{0<h<1} \int_0^{1-h} (\Delta_h u(t))^p\,dt < +\infty$.
In order to prove that T is a Borel map, we observe that $W^{1,p}(I;X)$ and $AC^p(I;X)$ are Borel subsets of $L^p(I;X)$ and Γ respectively, since the map

$$u \mapsto \sup_{0<h<T} \int_0^{T-h} \frac{d^p(u(t+h),u(t))}{h^p}\,dt$$

is lower semi continuous from $L^p(I;X)$ to $[0,+\infty]$ and from Γ to $[0,+\infty]$.
Moreover T is an isometry from $(W^{1,p}(I;X),d_p)$ to (Γ,d_p) and the thesis follows by observing that the Borel sets of (Γ,d_∞) coincides with the Borel sets of (Γ,d_p). This last assertion is a general fact: if Y is a separable and complete metric space (Polish space) and Y_w is the same space with an Hausdorff topology weaker than the original, then the Borel sets of Y coincide with the Y_w ones (see for instance [Sch73] Corollary 2, pag 101). $\square$

2.5. **Borel probability measures, narrow topology and tightness.** Given a separable metric space Y, we denote with $\mathscr{P}(Y)$ the set of Borel probability measures on Y. We say that a sequence $\mu_n \in \mathscr{P}(Y)$ *narrowly converges* to $\mu \in \mathscr{P}(Y)$ if

$$\lim_{n\to+\infty} \int_Y \varphi(y)\,d\mu_n(y) = \int_Y \varphi(y)\,d\mu(y) \qquad \forall \varphi \in C_b(Y), \tag{22}$$

where $C_b(Y)$ is the space of continuous bounded real functions defined on Y.

It is well known that the narrow convergence is induced by a distance on $\mathscr{P}(Y)$ (see [AGS05]) and we call *narrow topology* the topology induced by this distance. In particular the compact subsets of $\mathscr{P}(Y)$ coincides with sequentially compact subsets of $\mathscr{P}(Y)$.
We also recall that if $\mu_n \in \mathscr{P}(Y)$ narrowly converges to $\mu \in \mathscr{P}(Y)$ and $\varphi : Y \to (-\infty,+\infty]$ is a lower semi continuous function bounded from below, then

$$\liminf_{n\to+\infty} \int_Y \varphi(y)\,d\mu_n(y) \geq \int_Y \varphi(y)\,d\mu(y). \tag{23}$$

A subset $\mathscr{T} \subset \mathscr{P}(Y)$ is said to be tight if

$$\forall \varepsilon > 0 \quad \exists K_\varepsilon \subset Y \text{ compact} : \mu(Y \setminus K_\varepsilon) < \varepsilon \quad \forall \mu \in \mathscr{T}, \tag{24}$$

or, equivalently, if there exists a function $\varphi : Y \to [0,+\infty]$ with compact sublevels $\lambda_c(\varphi) := \{y \in Y : \varphi(y) \leq c\}$, such that

$$\sup_{\mu \in \mathscr{T}} \int_Y \varphi(y)\,d\mu(y) < +\infty. \tag{25}$$

The importance of tight sets is due to the following Theorem:

Theorem 2.6 (Prokhorov)**.** *Let Y be a separable and complete metric space. $\mathscr{T} \subset \mathscr{P}(Y)$ is tight if and only if it is relatively compact in $\mathscr{P}(Y)$.*

2.5.1. *Push forward of measures.* If Y, Z are separable metric spaces, $\mu \in \mathscr{P}(Y)$ and $F : Y \to Z$ is a Borel map, the *push forward of* μ *through* F, denoted by $F_\# \mu \in \mathscr{P}(Z)$, is defined as follows:

$$F_\# \mu(B) := \mu(F^{-1}(B)) \qquad \forall B \in \mathscr{B}(Z), \tag{26}$$

where $\mathscr{B}(Z)$ is the family of Borel subsets of Z. It is not difficult to check that this definition is equivalent to

$$\int_Z \varphi(z)\, d(F_\# \mu)(z) = \int_Y \varphi(F(y))\, d\mu(y) \tag{27}$$

for every bounded Borel function $\varphi : Z \to \mathbb{R}$. More generally (27) holds for every $F_\#\mu$-integrable function $\varphi : Z \to \mathbb{R}$. We will often use this fact.
We recall the following composition rule: for every $\mu \in \mathscr{P}(Y)$, for every Borel maps $F : Y \to Z, \quad G : Z \to W$,

$$(G \circ F)_\# \mu = G_\#(F_\# \mu)$$

and the continuity property:

$$F : Y \to Z \quad \text{continuous} \quad \Longrightarrow \quad F_\# : \mathscr{P}(Y) \to \mathscr{P}(Z) \quad \text{narrowly continuous.}$$

2.6. **Kantorovitch-Rubinstein-Wasserstein distance.** We fix $p \geq 1$ and denote by $\mathscr{P}_p(X)$ the space of Borel probability measures having finite p-moment, i.e.

$$\mathscr{P}_p(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X d^p(x, x_0)\, d\mu(x) < +\infty \right\}, \tag{28}$$

where $x_0 \in X$ is an arbitrary point of X (clearly this definition does not depend on the choice of x_0). Notice also that this condition is always satisfied if the diameter of X is finite; in this case $\mathscr{P}_p(X) = \mathscr{P}(X)$.
Given $\mu, \nu \in \mathscr{P}(X)$ we define the set of admissible plans $\Gamma(\mu, \nu)$ as follows:

$$\Gamma(\mu, \nu) := \{ \gamma \in \mathscr{P}(X \times X) : \pi^1_\# \gamma = \mu,\ \pi^2_\# \gamma = \nu \},$$

where $\pi^1(x, y) := x$ and $\pi^2(x, y) := y$ are the projections on the first and the second component respectively.
The p-Kantorovitch-Rubinstein-Wasserstein distance between $\mu, \nu \in \mathscr{P}_p(X)$ is defined by

$$W_p(\mu, \nu) := \left(\min \left\{ \int_{X \times X} d^p(x, y)\, d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\} \right)^{\frac{1}{p}}. \tag{29}$$

Since $\Gamma(\mu, \nu)$ is tight and $\mu \otimes \nu \in \Gamma(\mu, \nu)$ satisfies $\int_{X \times X} d^p(x, y)\, d\mu \otimes \nu(x, y) < +\infty$, the existence of the minimum, in the above definition, is a consequence of standard Direct Methods in Calculus of Variations.
We denote by

$$\Gamma_o(\mu, \nu) := \left\{ \gamma \in \Gamma(\mu, \nu) : \int_{X \times X} d^p(x, y)\, d\gamma(x, y) = W_p^p(\mu, \nu) \right\}$$

the set of optimal plans.
Being X separable and complete, $\mathscr{P}_p(X)$, endowed with the distance W_p, is a separable and complete metric space too (see for instance [Vil03] and [AGS05]).

Remark 2.7. If we take $\mu, \nu \in \mathscr{P}(X)$, without hypothesis on the moments, when X is unbounded, $W_p(\mu,\nu)$ can be equal to $+\infty$ and W_p is a pseudo distance on $\mathscr{P}(X)$, according to Remark 2.3. More generally, taking a pseudo distance d on X, the space $\mathscr{P}(X)$, endowed with the pseudo distance W_p, is a pseudo metric space.

Remark 2.8 (Non separable spaces). It is possible to work with complete metric spaces, not necessarily separable, considering the set of tight measures. Let X be a complete metric space, we define the set of tight probability measures

$$\mathscr{P}^\tau(X) := \{\mu \in \mathscr{P}(X) : \mu \text{ is tight}\}.$$

We have that $\mu \in \mathscr{P}^\tau(X)$ if and only if $\operatorname{supp}\mu$ is separable. Indeed if μ is tight there exists a sequence of compacts K_n such that $\mu(X \setminus K_n) < \frac{1}{n}$ from which it follows that $\operatorname{supp}\mu \subset \overline{\cup_{n\in\mathbb{N}} K_n}$ and then is separable. The other implication follows by Prokorov's Theorem.
On the other hand if $\mu_n \in \mathscr{P}^\tau(X)$ narrowly converges in $\mathscr{P}(X)$ to μ then $\mu \in \mathscr{P}^\tau(X)$ because $\operatorname{supp}\mu \subset \overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n}$ (see e.g. Proposition 5.1.8 of [AGS05]).
In the space $\mathscr{P}^\tau(X)$, W_p is a complete pseudo distance. The completeness can be proved as in [AGS05] by observing that a Cauchy sequence μ_n with respect to W_p in $\mathscr{P}^\tau(X)$ can be considered in $\mathscr{P}(\overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n})$ and $\overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n}$ is a complete separable space.

3. Main theorems

In this section we state and prove our main result which is a characterization of absolutely continuous curves with finite p-energy in the Wasserstein space $\mathscr{P}_p(X)$ as illustrated in the introduction.

Before to state the results, we define, for every $t \in I$, the *evaluation map* $e_t : \Gamma \to X$ in this way

$$e_t(u) = u(t) \tag{30}$$

and notice that e_t is continuous.

Theorem 3.1. *If $\eta \in \mathscr{P}(\Gamma)$ is concentrated on $AC^p(I;X)$, i.e. $\eta(\Gamma \setminus AC^p(I;X)) = 0$, with $p \in [1,+\infty)$, such that*

$$\mu_0 := (e_0)_\# \eta \in \mathscr{P}_p(X) \tag{31}$$

and

$$\int_\Gamma \int_0^T |u'|^p(t)\, dt\, d\eta(u) < +\infty, \tag{32}$$

(i.e. $\int_\Gamma \mathcal{E}_p(u)\, d\eta(u) < +\infty$) then the curve $t \mapsto \mu_t := (e_t)_\# \eta$ belongs to $AC^p(I;\mathscr{P}_p(X))$. Moreover for a.e. $t \in I$, $|u'|(t)$ exists for $\eta-$a.e. $u \in \Gamma$ and

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\, d\eta(u) \qquad \textit{for a.e. } t \in I. \tag{33}$$

Proof. First of all we check that for a.e. $t \in I$, $|u'|(t)$ exists for $\eta-$a.e. $u \in \Gamma$.
We set $\Lambda := \{(t,u) \in I \times \Gamma : |u'|(t) \text{ does not exist}\}$ and we observe that Λ is a Borel subset of $I \times \Gamma$ since the maps $(t,u) \mapsto \frac{d(u(t+h),u(t))}{|h|}$ are continuous from $I\times\Gamma$ to $\mathbb{R}$ for every $h \neq 0$. Since η is concentrated on $AC^p(I;X)$, we have that for η-a.e. $u \in \Gamma$, $\mathscr{L}^1(\{t \in I : (t,u) \in \Lambda\}) = 0$

and then Fubini's Theorem implies that for a.e. $t \in I$, $\eta(\{u \in \Gamma : (t,u) \in \Lambda\}) = 0$.
Now we prove that $\mu_t = (e_t)_\# \eta$ has finite p-moment for every $t \in I$. Given a point $\bar{x} \in X$, we have

$$\begin{aligned}
\int_X d^p(x,\bar{x})\,d\mu_t(x) &= \int_\Gamma d^p(u(t),\bar{x})\,d\eta(u) \\
&\le 2^{p-1}\int_\Gamma (d^p(u(0),\bar{x}) + d^p(u(0),u(t)))\,d\eta(u) \\
&\le 2^{p-1}\int_\Gamma \left(d^p(u(0),\bar{x}) + \left(\int_0^t |u'|(r)\,dr\right)^p\right)\,d\eta(u) \\
&\le 2^{p-1}\int_\Gamma \left(d^p(u(0),\bar{x}) + \int_0^T |u'|^p(r)\,dr\right)\,d\eta(u)
\end{aligned}$$

and this is finite by (31) and (32).
Now we take $s,t \in I$ with $s<t$ and $\gamma_{s,t} := (e_s,e_t)_\#\eta$. Since $\gamma_{s,t} \in \Gamma(\mu_s,\mu_t)$, by the definition of W_p, the fact that η is concentrated on $AC^p(I;X)$, and Hölder's inequality, we have

$$\begin{aligned}
W_p^p(\mu_s,\mu_t) &\le \int_{X\times X} d^p(x,y)\,d\gamma_{s,t}(x,y) = \int_\Gamma d^p(e_s(u),e_t(u))\,d\eta(u) \\
&\le \int_\Gamma \left(\int_s^t |u'|(r)\,dr\right)^p d\eta(u) \le \int_\Gamma |s-t|^{p-1}\int_s^t |u'|^p(r)\,dr\,d\eta(u) \\
&= |s-t|^{p-1}\int_s^t\int_\Gamma |u'|^p(r)\,d\eta(u)\,dr, \qquad (34)
\end{aligned}$$

where the last equality follows by (32) and Fubini-Tonelli Theorem.
The estimates (34) and (32) imply that μ_t is absolutely continuous. The thesis is now a simple consequence of (34) and Lebesgue differentiation Theorem. □

Theorem 3.2. *If μ_t is an absolutely continuous curve in $\mathscr{P}_p(X)$ with finite p-energy, $p>1$, i.e. $\mu_t \in AC^p(I;\mathscr{P}_p(X))$, then there exists $\tilde{\eta} \in \mathscr{P}(\Gamma)$ such that*

(i) *$\tilde{\eta}$ is concentrated on $AC^p(I;X)$,*
(ii) *$(e_t)_\#\tilde{\eta} = \mu_t \qquad \forall t \in I$,*
(iii)

$$|\mu'|^p(t) = \int_\Gamma |u'|^p(t)\,d\tilde{\eta}(u) \qquad \textit{for a.e. } t \in I.$$

Before to prove Theorem 3.2 we state and prove the extension of this result to *pseudo metric spaces* and to $\mathscr{P}(X)$ according to Remarks 2.3 and 2.7.

Corollary 3.3. *Let (X,d) be a pseudo metric space, separable and d-complete.*
If $\eta \in \mathscr{P}(\Gamma)$ is concentrated on $AC^p(I;X)$ with $p \in [1,+\infty)$, such that

$$\int_\Gamma\int_0^T |u'|^p(t)\,dt\,d\eta(u) < +\infty,$$

then the curve $t \mapsto \mu_t := (e_t)_\# \eta$ *belongs to* $AC^p(I; (\mathscr{P}(X), W_p))$*. Moreover for a.e.* $t \in I$, $|u'|(t)$ *exists for* $\eta-$*a.e.* u *and*

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\, d\eta(u) \qquad \textit{for a.e. } t \in I.$$

If the curve $t \mapsto \mu_t$ *belongs to* $AC^p(I; (\mathscr{P}(X), W_p))$*, with* $p > 1$*, then there exists* $\tilde\eta \in \mathscr{P}(\Gamma)$ *such that (i), (ii), (iii) of Theorem 3.2 hold.*

Proof of Corollary 3.3. The proof of the first part is exactly the proof of Theorem 3.1 since the hypothesis (31) was only needed in order to ensure that the p-moment of μ_t is finite.
In order to prove the second part we take into account the Remark 2.3. We define

$$\tilde d(x, y) := d(x, y) \wedge 1,$$

and we denote by $\tilde X$ the bounded metric space $(X, \tilde d)$, topologically equivalent to X. Then $\mathscr{P}(X) = \mathscr{P}(\tilde X) = \mathscr{P}_p(\tilde X)$ and $AC^p(I; X) = AC^p(I; \tilde X)$, since the metric derivative with respect to d coincides with the metric derivative with respect to $\tilde d$. Denoting by $\tilde W_p$ the Wasserstein distance with respect to $\tilde d$, we have that $\tilde W_p(\mu, \nu) \le W_p(\mu, \nu)$.
If $\mu_t \in AC^p(I; \mathscr{P}(X))$, then $\mu_t \in AC^p(I; \mathscr{P}_p(\tilde X))$ and, applying Theorem 3.2, we can find $\tilde\eta \in \mathscr{P}(\Gamma)$ concentrated on $AC^p(I; \tilde X) = AC^p(I; X)$ such that

$$\int_\Gamma |u'|^p(t)\, d\tilde\eta(u) = |\mu'|^p_{\tilde W_p}(t) \quad \text{for a.e. } t \in I$$

where $|\mu'|_{\tilde W_p}(t)$ denotes the metric derivative with respect to the distance $\tilde W_p$.
Applying the first part of this Corollary to $\tilde\eta$ we obtain

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\, d\tilde\eta(u) = |\mu'|^p_{\tilde W_p}(t) \le |\mu'|^p(t) \tag{35}$$

for a.e. $t \in I$, which completes the proof. ∎

Reasoning as in the proof of the above corollary we can prove the following corollary, interesting by itself.

Corollary 3.4. *If* $X_1 := (X, d_1)$, $X_2 := (X, d_2)$ *are two topologically equivalent, separable and complete pseudo metric spaces, such that* $AC^p(I; X_1) = AC^p(I; X_2)$ *and for every* $u \in AC^p(I; X_i)$, $i = 1, 2$ *it holds*

$$|u'|_{d_1}(t) = |u'|_{d_2}(t) \qquad \textit{for a.e. } t \in I,$$

then $AC^p(I; (\mathscr{P}(X), W_{p,d_1})) = AC^p(I; (\mathscr{P}(X), W_{p,d_2}))$ *and for every* $\mu \in AC^p(I; (\mathscr{P}(X), W_{p,d_i}))$, $i = 1, 2$ *it holds*

$$|\mu'|_{W_{p,d_1}}(t) = |\mu'|_{W_{p,d_2}}(t) \qquad \textit{for a.e. } t \in I.$$

Remark 3.5. By using Corollary 2, (pag. 101) of [Sch73], the Corollary 3.4 can be proved even when the topology induced by one of the distances is weaker than the other one.

In the case of a complete but non separable metric space, Theorem 3.2 can be stated considering the set of tight measures as in Remark 2.8.

Corollary 3.6. *Let (X, d) be a complete metric space (in general non separable).*
If the curve $t \mapsto \mu_t$ belongs to $AC^p(I; (\mathscr{P}^\tau(X), W_p))$, with $p > 1$, then there exist a closed separable subspace $X_0 \subset X$ and a measure $\tilde{\eta} \in \mathscr{P}(C(I; X_0))$ such that $\operatorname{supp}\mu_t \subset X_0$ for every $t \in I$ and (i), (ii), (iii) of Theorem 3.2 hold.

Proof. It is sufficient to apply the second part of Corollary 3.3 in the separable complete metric space

$$X_0 := \overline{\cup_{s\in\mathbb{Q}\cap I} \operatorname{supp}\mu_s},$$

by observing that $\operatorname{supp}\mu_t \subset X_0$ for every $t \in I$. Indeed for $t \in I$ we take a sequence $t_n \in \mathbb{Q}\cap I$ convergent to t, by the narrow continuity of the curve μ, μ_{t_n} narrowly converges to μ_t and then, recalling Remark 2.8, we have $\operatorname{supp}\mu_t \subset X_0$. □

Proof of Theorem 3.2. We prove the theorem in the particular case $T = 1$, i.e. $I = [0, 1]$. Obviously it is not restrictive.

For any integer $N \geq 1$, we divide the unitary interval I in 2^N equal parts, and we denote the nodal points by

$$t^i := \frac{i}{2^N} \qquad i = 0, 1, \ldots, 2^N.$$

We also denote by X_i, with $i = 0, 1, \ldots, 2^N$, $2^N + 1$ copies of the same space X and define the product space

$$\boldsymbol{X}_N := X_0 \times X_1 \times \ldots \times X_{2^N}.$$

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$$\gamma_N^i \in \Gamma_o(\mu_{t^i}, \mu_{t^{i+1}}) \qquad i = 0, 1, \ldots, 2^N - 1,$$

there exists (see, for example, Lemma 5.3.2 and Remark 5.3.3 of [AGS05]) $\gamma_N \in \mathscr{P}(\boldsymbol{X}_N)$ such that

$$\pi^i_\#\gamma_N = \mu_{t^i} \qquad \text{and} \qquad \pi^{i,i+1}_\#\gamma_N = \gamma^i_N,$$

where we denoted by $\pi^i : \boldsymbol{X}_N \to X_i$ the projection on the i-th component and by $\pi^{i,j} : \boldsymbol{X}_N \to X_i \times X_j$ the projection on the (i, j)-th component.
We define $\sigma : \boldsymbol{x} = (x_0, \ldots, x_{2^N}) \in \boldsymbol{X}_N \to \sigma_{\boldsymbol{x}} \in L^p(I; X)$ by

$$\sigma_{\boldsymbol{x}}(t) := x_i \qquad \text{if} \quad t \in [t^i, t^{i+1}),$$

and we also set

$$\eta_N := \sigma_\#\gamma_N \in \mathscr{P}(L^p(I; X)).$$

Step 1. (Tightness of $\{\eta_N\}_{N\in\mathbb{N}}$) In order to obtain the existence of a narrow limit point η of the sequence $\{\eta_N\}_{N\in\mathbb{N}}$, by Prokhorov's theorem, it is sufficient to prove its tightness by exhibiting a function $\Phi : L^p(I; X) \to [0, +\infty]$ whose sublevels $\lambda_c(\Phi) := \{u \in L^p(I; X) : \Phi(u) \leq c\}$ are compact in $L^p(I; X)$ for any $c \in \mathbb{R}_+$, and

$$\sup_{N\in\mathbb{N}} \int_{L^p(I;X)} \Phi(u)\, d\eta_N(u) < +\infty. \tag{36}$$

First of all we observe that $\mathscr{A} := \{\mu_t : t \in I\}$ is compact (because it is a continuous image of a compact) in $\mathscr{P}_p(X)$ and consequently in $\mathscr{P}(X)$. In particular $\mathscr{A}$ is bounded in $\mathscr{P}_p(X)$

and then, given a point $\bar{x} \in X$, there exists C_1 such that

$$\int_X d^p(x,\bar{x})\, d\mu_t(x) = W_p^p(\mu_t, \delta_{\bar{x}}) \leq C_1 \qquad \forall t \in I. \tag{37}$$

Since, by Prokhorov's Theorem, $\mathscr{A}$ is tight there exists $\psi : X \to [0,+\infty]$ whose sublevels $\lambda_c(\psi) := \{x \in X : \psi(x) \leq c\}$ are compact in X for any $c \in \mathbb{R}_+$, such that

$$C_2 := \sup_{t\in I} \int_X \psi(x)\, d\mu_t(x) < +\infty. \tag{38}$$

We define $\Phi : L^p(I;X) \to [0,+\infty]$ as follows

$$\Phi(u) := \int_0^1 d^p(u(t),\bar{x})\, dt + \int_0^1 \psi(u(t))\, dt + \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h),u(t))}{h}\, dt,$$

where $\bar{x}$ is a given point of X.
The compactness of $\lambda_c(\Phi)$ in $L^p(I;X)$ is immediate since the hypotheses of Theorem 2.4 are satisfied, and Φ is lower semi continuous by a simple application of Fatou's Lemma.

The proof of (36) requires some computations.
As a first step we show that

$$\sup_{N\in\mathbb{N}} \int_{L^p(I;X)} \int_0^1 (d^p(u(t),\bar{x}) + \psi(u(t)))\, dt\, d\eta_N(u) < +\infty.$$

By (37) and (38)

$$\begin{aligned}
\int_{L^p(I;X)} \int_0^1 & d^p(u(t),\bar{x}) + \psi(u(t))\, dt\, d\eta_N(u) = \\
&= \int_0^1 \int_{\boldsymbol{X}_N} d^p(\sigma_{\boldsymbol{x}}(t),\bar{x}) + \psi(\sigma_{\boldsymbol{x}}(t))\, d\gamma_N(\boldsymbol{x})\, dt \\
&= \sum_{i=0}^{2^N-1} \int_{t^i}^{t^{i+1}} \int_X d^p(x,\bar{x}) + \psi(x)\, d\mu_{t^i}(x)\, dt \\
&= \frac{1}{2^N} \sum_{i=0}^{2^N-1} \int_X d^p(x,\bar{x}) + \psi(x)\, d\mu_{t^i}(x) \\
&\leq \frac{1}{2^N} \sum_{i=0}^{2^N-1} (C_1 + C_2) = C_1 + C_2.
\end{aligned}$$

As a second step we show that

$$\sup_{N\in\mathbb{N}} \int_{L^p(I;X)} \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h),u(t))}{h}\, dt\, d\eta_N(u) < +\infty. \tag{39}$$

In order to prove (39), we show that if $\boldsymbol{x} \in \boldsymbol{X}_N$ then

$$\sup_{0<h<1} \int_0^{1-h} \frac{d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))}{h}\, dt \leq \left(2^p + \frac{(2^N)^p}{2^N-1}\right) \sum_{i=0}^{2^N-1} d^p(x_i,x_{i+1}). \tag{40}$$

If $h < 1/2^N$ then

$$\begin{aligned}\int_0^{1-h} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt &= \sum_{i=0}^{2^N-2}\int_{t^i}^{t^{i+1}} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt \\ &= h\sum_{i=0}^{2^N-2} d^p(x_i,x_{i+1})\end{aligned}$$

since $\sigma_{\boldsymbol{x}}(t+h)=\sigma_{\boldsymbol{x}}(t)$ if $t\in[t^i,t^{i+1}-h)$.
If $1/2^N \le h < 1$ we take the integer $k \ge 1$ such that

$$\frac{k}{2^N} \le h < \frac{k+1}{2^N}, \tag{41}$$

so that the triangular inequality yields

$$d(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t)) \le \sum_{i=0}^{k} d(\sigma_{\boldsymbol{x}}(t+t^{i+1}),\sigma_{\boldsymbol{x}}(t+t^i)),$$

and, using Holder's discrete inequality,

$$d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t)) \le (k+1)^{p-1}\sum_{i=0}^{k} d^p(\sigma_{\boldsymbol{x}}(t+t^{i+1}),\sigma_{\boldsymbol{x}}(t+t^i)).$$

Then

$$\begin{aligned}\int_0^{1-h} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt &\le \int_0^{1-t^k} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt \\ &\le \int_0^{1-t^k}(k+1)^{p-1}\sum_{i=0}^{k} d^p(\sigma_{\boldsymbol{x}}(t+t^{i+1}),\sigma_{\boldsymbol{x}}(t+t^i))\,dt \\ &= (k+1)^{p-1}\sum_{i=0}^{k}\frac{1}{2^N}\sum_{j=0}^{2^N-k-1} d^p(x_{i+j+1},x_{i+j}) \qquad (42)\end{aligned}$$

and, observing that, in (42), $d^p(x_{j+1},x_j)$ is counted at most $k+1$ times, we have

$$\int_0^{1-h} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt \le \frac{(k+1)^p}{2^N}\sum_{j=0}^{2^N-1} d^p(x_{j+1},x_j). \tag{43}$$

Using (41) we obtain

$$\int_0^{1-h} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt \le h\frac{(k+1)^p}{k}\sum_{j=0}^{2^N-1} d^p(x_{j+1},x_j).$$

Since the function $\lambda \mapsto \frac{(\lambda+1)^p}{\lambda}$ is increasing in the interval $(1/(p-1),+\infty)$, decreasing in $(0,1/(p-1)$ and $1 \le k \le 2^N-1$ we have

$$\frac{(k+1)^p}{k} \le \begin{cases} 2^p & \text{if } k \le \frac{1}{p-1} \\ \frac{(2^N)^p}{2^N-1} & \text{if } k > \frac{1}{p-1}\end{cases}$$

and (40) is proved.
Since

$$
\begin{aligned}
\int_{\boldsymbol{X}_N} \sum_{i=0}^{2^N-1} d^p(x_i, x_{i+1})\, d\gamma_N(\boldsymbol{x}) &= \sum_{i=0}^{2^N-1} W_p^p(\mu_{t^i}, \mu_{t^{i+1}}) \\
&\le \sum_{i=0}^{2^N-1} \left(\int_{t^i}^{t^{i+1}} |\mu'|(t)\, dt \right)^p \\
&\le \frac{1}{(2^N)^{p-1}} \sum_{i=0}^{2^N-1} \int_{t^i}^{t^{i+1}} |\mu'|^p(t)\, dt \\
&= \frac{1}{(2^N)^{p-1}} \int_0^1 |\mu'|^p(t)\, dt \qquad (44)
\end{aligned}
$$

we have, taking into account (40),

$$\int_{L^p(I;X)} \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h), u(t))}{h}\, dt\, d\eta_N(u) \le (2^p + 2) \int_0^1 |\mu'|^p(t)\, dt$$

which is finite because $|\mu'| \in L^p(I)$.

Then there exist $\eta \in \mathscr{P}(L^p(I;X))$ and a subsequence N_k such that $\eta_{N_k} \to \eta$ narrowly in $\mathscr{P}(L^p(I;X))$ if $k \to +\infty$.

Step 2. **(η is concentrated on $W^{1,p}(I;X)$)** Let us define the sequence of lower semi continuous functions $f_N : L^p(I;X) \to [0, +\infty]$ as follows

$$f_N(u) := \sup_{1/2^N \le h < 1} \int_0^{1-h} \frac{d^p(u(t+h), u(t))}{h^p}\, dt;$$

clearly they satisfy the monotonicity property

$$f_N(u) \le f_{N+1}(u) \qquad \forall\, u \in L^p(I;X). \tag{45}$$

We prove that

$$\int_{L^p(I;X)} f_N(u)\, d\eta_N(u) \le C. \tag{46}$$

We fix $1/2^N \le h < 1$ and take the integer $k \ge 1$ as in (41). From (43) and

$$\frac{1}{2^N} \le (2^N)^{p-1} \frac{h^p}{k^p} \tag{47}$$

which is the first inequality of (41) rewritten, it follows that

$$\int_0^{1-h} d^p(\sigma_{\boldsymbol{x}}(t+h), \sigma_{\boldsymbol{x}}(t))\, dt \le h^p \frac{(k+1)^p}{k^p} (2^N)^{p-1} \sum_{j=0}^{2^N-1} d^p(x_{j+1}, x_j)$$

which implies

$$\sup_{1/2^N \le h < 1} \int_0^{1-h} \frac{d^p(\sigma_{\boldsymbol{x}}(t+h), \sigma_{\boldsymbol{x}}(t))}{h^p}\, dt \le 2^p (2^N)^{p-1} \sum_{j=0}^{2^N-1} d^p(x_{j+1}, x_j).$$

Integrating and using (44) we obtain (46) with $C := 2^p \int_0^1 |\mu'|^p(t)\,dt$.
By (45) and (46) we have that

$$\int_{L^p(I;X)} f_N(u)\,d\eta_{N_k}(u) \le C \tag{48}$$

for every k such that $N_k \ge N$.
The lower semi continuity of f_N, (23) and the bound (48) yield

$$\int_{L^p(I;X)} f_N(u)\,d\eta(u) \le C \qquad \forall\, N \in \mathbb{N},$$

and consequently, by monotone convergence Theorem, we have that

$$\int_{L^p(I;X)} \sup_{N\in\mathbb{N}} f_N(u)\,d\eta(u) \le C,$$

and

$$\sup_{N\in\mathbb{N}} f_N(u) < +\infty \quad \text{ for } \eta - \text{a.e. } u \in L^p(I;X). \tag{49}$$

Since

$$\sup_{N\in\mathbb{N}} f_N(u) = \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h),u(t))}{h^p}\,dt,$$

(49) shows that η is concentrated on $W^{1,p}(I;X)$ by it's very definition (16).

Recalling that $W^{1,p}(I;X)$ is a Borel subset of $L^p(I;X)$, the measure η can be considered as a Borel measure on $W^{1,p}(I;X)$, i.e. $\eta \in \mathscr{P}(W^{1,p}(I;X))$.
Thanks to Lemma 2.5 we can define

$$\tilde\eta := T_\#\eta \in \mathscr{P}(\Gamma)$$

which is concentrated, by definition, on $AC^p(I;X)$.

Step 3. (Proof of (ii) and (iii)) In order to prove (iii), we show preliminarily that for all $s_1, s_2 \in I$, $s_1 < s_2$, we have

$$\int_{L^p(I;X)} \int_{s_1}^{s_2} \frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta(u) \le \int_{s_1}^{s_2+h} |\mu'|^p(t)\,dt \tag{50}$$

for every $h \in (0, 1-s_2)$.
We fix $h \in (0,1-s_2)$ and for every $N \in \mathbb{N}$ such that $\frac{1}{2^N} \le h$ we take $k \ge 1$ such that (41) holds. Setting

$$s_1^N := \frac{\bar{j}}{2^N} := \max\left\{ \frac{j}{2^N} : \frac{j}{2^N} \le s_1 \right\}, \qquad s_2^N := \frac{\bar{i}}{2^N} := \min\left\{ \frac{j}{2^N} : \frac{j}{2^N} \le s_1 \right\},$$

and reasoning as in the proof of (43) we obtain

$$\int_{s_1}^{s_2} d^p(\sigma_{\boldsymbol{x}}(t+h), \sigma_{\boldsymbol{x}}(t))\,dt \le \frac{(k+1)^p}{2^N} \sum_{j=\bar{j}}^{\bar{i}+k} d^p(x_{j+1}, x_j)$$

and by (47)

$$\int_{s_1}^{s_2} d^p(\sigma_{\boldsymbol{x}}(t+h),\sigma_{\boldsymbol{x}}(t))\,dt \le h^p \frac{(k+1)^p}{k^p}(2^N)^{p-1}\sum_{j=\bar{j}}^{\bar{i}+k} d^p(x_{j+1},x_j).$$

Integrating we obtain

$$\int_{L^p(I;X)}\int_{s_1}^{s_2}\frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta_N(u) \le \left(\frac{k+1}{k}\right)^p \int_{s_1^N}^{s_2^N+h}|\mu'|^p(t)\,dt$$

from which (50) follows passing to the limit for $N\to+\infty$.

Clearly we have

$$\begin{aligned}\int_{L^p(I;X)}\int_{s_1}^{s_2}\frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta(u) &=\\ &=\int_{W^{1,p}(I;X)}\int_{s_1}^{s_2}\frac{d^p(Tu(t+h),Tu(t))}{h^p}\,dt\,d\eta(u)\\ &=\int_{\Gamma}\int_{s_1}^{s_2}\frac{d^p(\tilde{u}(t+h),\tilde{u}(t))}{h^p}\,dt\,d\tilde{\eta}(\tilde{u}).\end{aligned}$$

By this last relation and (50), by Fatou's Lemma and the fact that $\tilde{\eta}$ is concentrated on $AC^p(I;X)$, letting h going to 0, we obtain

$$\int_{\Gamma}\int_{s_1}^{s_2}|u'|^p(t)\,dt\,d\tilde{\eta}(u) \le \int_{s_1}^{s_2}|\mu'|^p(t)\,dt \tag{51}$$

for every $s_1,s_2\in I$ such that $s_1<s_2$. By Fubini and Lebesgue differentiation Theorems, (51) yields

$$\int_{\Gamma}|u'|(t)^p\,d\tilde{\eta}(u)\le|\mu'|^p(t)\qquad\text{for a.e. } t\in I. \tag{52}$$

In order to show (ii) we prove that for every $t\in I$,

$$\int_{\Gamma}\varphi(u(t))\,d\tilde{\eta}(u)=\int_X\varphi(x)\,d\mu_t(x)\qquad\forall\varphi\in C_b(X). \tag{53}$$

We fix $\varphi\in C_b(X)$ and we observe that the function

$$g(t):=\int_X\varphi(x)\,d\mu_t(x)$$

is uniformly continuous in I, consequently the sequence of piecewise constant functions

$$g_N(t):=g(t^i)=\int_X\varphi(x)\,d\mu_{t^i}(x)\qquad\text{if } t\in[t^i,t^{i+1}),$$

converges uniformly to g in I when $N\to+\infty$. Then, for every test function $\zeta\in C_b(I)$, we have that

$$\lim_{N\to+\infty}\int_0^1\zeta(t)g_N(t)\,dt=\int_0^1\zeta(t)g(t)\,dt. \tag{54}$$

On the other hand

$$\begin{aligned}\int_0^1 \zeta(t) g_N(t)\,dt &= \int_0^1 \zeta(t) \int_{L^p(I;X)} \varphi(u(t))\,d\eta_N(u)\,dt \\ &= \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta_N(u).\end{aligned}$$

Since the map

$$u \mapsto \int_0^1 \zeta(t)\varphi(u(t))\,dt$$

is continuous and bounded from $L^p(I;X)$ to $\mathbb{R}$, then by the narrow convergence of η_{N_k} we have

$$\lim_{k\to+\infty} \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta_{N_k}(u) = \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta(u).$$

By Fubini's Theorem and the definition of $\tilde{\eta}$,

$$\begin{aligned}\int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta(u) &= \int_\Gamma \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\tilde{\eta}(u) \\ &= \int_0^1 \zeta(t) \int_\Gamma \varphi(u(t))\,d\tilde{\eta}(u)\,dt.\end{aligned}$$

By the uniqueness of the limit then

$$\int_0^1 \zeta(t) \int_\Gamma \varphi(u(t))\,d\tilde{\eta}(u)\,dt = \int_0^1 \zeta(t) \int_X \varphi(x)\,d\mu_t(x)\,dt \qquad \forall \zeta \in C_b(I),$$

from which

$$\int_\Gamma \varphi(u(t))\,d\tilde{\eta}(u) = \int_X \varphi(x)\,d\mu_t(x) \qquad \text{for a.e. } t \in I. \tag{55}$$

The applications $t \mapsto \int_X \varphi(x)\,d\mu_t(x)$ and $t \mapsto \int_\Gamma \varphi(u(t))\,d\tilde{\eta}(u)$ are continuous because the applications $t \in I \mapsto \mu_t \in \mathscr{P}(X)$ and $t \in I \mapsto (e_t)_\#\tilde{\eta} \in \mathscr{P}(X)$ are narrowly continuous (recall that $e_t : \Gamma \to X$ is continuous). Then (55) is true for every $t \in I$ and (53) is proved.

Finally, by (52) and (33) of Theorem 3.1 applied to $\tilde{\eta}$ we obtain (iii). □

4. Application: Wasserstein geodesics

In this section we apply Theorem 3.2 in order to give a characterization of the geodesics of the metric space $(\mathscr{P}_p(X), W_p)$ in terms of the geodesics of the space (X,d), under the further assumption that (X,d) is a length space.

In this section I denotes the unitary interval $[0,1]$.

Recalling that the length of $u \in AC(I;X)$ is defined by $L(u) := \int_0^1 |u'|(t)\,dt$, we say that X is a *length space* if for every $x, y \in X$,

$$d(x,y) = \inf\{L(u) : u \in AC(I;X),\ u(0) = x,\ u(1) = y\} \tag{56}$$

and X is a *geodesic space* if for every $x, y \in X$, the inf in (56) is a minimum, i.e. there exists $u \in AC(I;X)$ such that

$$u(0) = x, \quad u(1) = y, \quad d(x,y) = L(u). \tag{57}$$

We call u satisfying (57) *minimizing geodesic* of X. A curve $u : I \to X$ satisfying

$$d(u(t), u(s)) = |t - s| d(u(0), u(1)) \qquad \forall s, t \in I \tag{58}$$

is called *constant speed minimizing geodesic*.
If u is a constant speed minimizing geodesic then it is a minimizing geodesic, since $|u'|(t) = d(u(0), u(1))$ for every $t \in I$. Conversely every minimizing geodesic can be reparametrized in such a way (58) holds.
We define the set

$$G := \{u : I \to X : u \text{ is a constant speed minimizing geodesics of } X\}$$

and we observe that it is immediate to check that G is a closed subset of Γ.
The following property will be useful: $u \in G$, for $p > 1$, if and only if

$$\int_0^1 |u'|^p(t)\, dt \le d^p(u(0), u(1)). \tag{59}$$

Indeed if $u \in G$ then the equality holds in (59). Conversely if $u \notin G$ then $|u'|$ is not constant and the strict convexity of $\alpha \mapsto |\alpha|^p$ yields $\int_0^1 |u'|^p(t)\, dt > d^p(u(0), u(1))$. In other words, the elements of G are the unique minimizers of the p-energy.

Proposition 4.1. *If X is a length (resp. geodesic) space then $\mathscr{P}_p(X)$ is a length (resp. geodesic) space too.*

Proof. First of all we define the Lipschitz constant $\mathrm{Lip} : \Gamma \to [0, +\infty]$ as

$$\mathrm{Lip}(u) = \sup_{s,t \in I, s \neq t} \frac{d(u(s)), d(u(t))}{|s - t|}$$

and we observe that Lip is a lower semi continuous function. Moreover for every $u \in AC(I; X)$ we have that $|u'|(t) \le \mathrm{Lip}(u)$ for a.e. $t \in I$. We say that u is a Lipschitz curve if $\mathrm{Lip}(u) < +\infty$.

We recall that (see for instance [AGS05]) every $u \in AC(I; X)$ of length $L(u)$ can be reparametrized in such a way that $\mathrm{Lip}(u) = L(u) = |u'|(t)$ for a.e. $t \in I$.

We take $\mu, \nu \in \mathscr{P}_p(X)$ and an optimal plan $\gamma \in \Gamma_o(\mu, \nu)$.
We fix $\varepsilon > 0$ and we define the multi-valued application $\Sigma_\varepsilon : X \times X \to 2^{C(I;X)}$ as follows:

$$\Sigma_\varepsilon(x, y) := \{u \in AC(I; X) : u(0) = x,\ u(1) = y,\ d(x, y) + \varepsilon \ge \mathrm{Lip}(u)\}.$$

Since X is a length space $\Sigma_\varepsilon(x, y)$ is not empty. Indeed taking $u_\varepsilon \in AC(I; X)$ such that $u_\varepsilon(0) = x$, $u_\varepsilon(1) = y$ and $d(x, y) + \varepsilon \ge L(u_\varepsilon)$, then the reparametrization of u_ε satisfying $\mathrm{Lip}(u) = L(u)$ belongs to $\Sigma_\varepsilon(x, y)$. In order to obtain a γ-measurable selection of the multi-function Σ_ε we can apply Aumann's selection Theorem (see Theorem III.22 of [CV77]). For this application it is sufficient to show that the graph of Σ_ε, defined by

$$\mathcal{G}(\Sigma_\varepsilon) := \{(x, y, u) \in X \times X \times C(I; X) : u \in \Sigma_\varepsilon(x, y)\}$$

is Borel measurable. This is true since $\mathcal{G}(\Sigma_\varepsilon)$ is closed. Indeed, taking $(x_n, y_n, u_n) \in \mathcal{G}(\Sigma_\varepsilon)$ convergent to (x, y, u), $d(x_n, x) \to 0$, $d(y_n, y) \to 0$ and $d_\infty(u_n, u) \to 0$ by the uniqueness of the limit we obtain that $u(0) = x$ and $u(1) = y$. Moreover, using the lower semi continuity of the Lipschitz constant, we can pass to the limit in $d(x_n, y_n) + \varepsilon \ge \mathrm{Lip}(u_n)$ obtaining that $d(x, y) + \varepsilon \ge \mathrm{Lip}(u)$, which yields that u is a Lipschitz curve and then $u \in AC(I; X)$.

Denoting by $S_\varepsilon : X \times X \to \Gamma$ this measurable selection, we observe that it is well defined the measure

$$\eta := (S_\varepsilon)_\# \gamma \in \mathscr{P}(\Gamma).$$

The curve

$$\mu_t := (e_t)_\# \eta, \qquad t \in I,$$

satisfies $\mu_0 = \mu$, $\mu_1 = \nu$ and

$$W_p(\mu, \nu) + \varepsilon \geq \int_0^1 |\mu'|(t)\, dt.$$

Indeed, applying Theorem 3.1 to η we obtain

$$\begin{aligned} \int_0^1 |\mu'|(t)\, dt &\leq \int_0^1 \int_\Gamma |u'|(t)\, d\eta(u)\, dt \leq \int_0^1 \int_\Gamma \mathrm{Lip}(u)\, d\eta(u)\, dt \\ &= \int_\Gamma \mathrm{Lip}(u)\, d\eta(u) \leq \int_{X\times X} (d(x,y)+\varepsilon)\, d\gamma(x,y) \\ &\leq \left(\int_{X\times X} d^p(x,y)\, d\gamma(x,y) \right)^{\frac{1}{p}} + \varepsilon \end{aligned}$$

which concludes the proof in the case of a length space X. When X is a geodesic space the proof works with $\varepsilon = 0$. □

A consequence of Theorem 3.2 is the following characterization of geodesics.

Theorem 4.2. *Let X be a separable and complete length space.*
A curve μ_t is a constant speed minimizing geodesic of $\mathscr{P}_p(X)$ if and only if there exists $\eta \in \mathscr{P}(\Gamma)$ such that

(i) *η is concentrated on G,*
(ii) *$(e_t)_\# \eta = \mu_t \qquad \forall t \in I$,*
(iii) *$W_p^p(\mu_0, \mu_1) = \int_\Gamma d^p(u(0), u(1))\, d\eta(u)$.*

We observe explicitly that if (i), (ii), (iii) hold then

$$W_p^p(\mu_s, \mu_t) = \int_\Gamma d^p(u(s), u(t))\, d\eta(u), \quad \forall\, s, t \in I,$$

that is $\gamma_{t,s} := (e_t, e_s)_\# \eta \in \Gamma_o(\mu_t, \mu_s)$.

Proof. Let μ_t be a constant speed minimizing geodesic of $\mathscr{P}_p(X)$ and let η be given by Theorem 3.2 applied to μ_t. We have only to check (i) and (iii).
By (iii) of Theorem 3.2 and the fact that μ_t is a constant speed geodesic, we have

$$\begin{aligned} \int_\Gamma d^p(u(0), u(1))\, d\eta(u) &\geq W_p^p(\mu_0, \mu_1) = \int_0^1 |\mu'|^p(t)\, dt = \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt \\ &= \int_\Gamma \int_0^1 |u'|^p(t)\, dt\, d\eta(u) \geq \int_\Gamma d^p(u(0), u(1))\, d\eta(u) \end{aligned}$$

which shows that η is concentrated on G, by (59), and (iii) holds.

Conversely, we assume that (i), (ii), (iii) hold. Setting $\gamma_{t,s} := (e_t, e_s)_\# \eta \in \Gamma(\mu_t, \mu_s)$, for $t, s \in I$, and using the fact that η is concentrated on G, we obtain

$$
\begin{aligned}
W_p^p(\mu_t, \mu_s) &\leq \int_{X \times X} d^p(x,y)\, d\gamma_{t,s}(x,y) = \int_\Gamma d^p(u(t), u(s))\, d\eta(u) \\
&= |t-s|^p \int_\Gamma d^p(u(0), u(1))\, d\eta(u) = |t-s|^p \int_{X \times X} d^p(x,y)\, d\gamma(x,y) \\
&= |t-s|^p W_p^p(\mu_0, \mu_1).
\end{aligned}
$$

The triangular inequality shows that the above inequality cannot be strict. □

The following Corollary is a metric generalization of the Benamou-Brenier formula.

Corollary 4.3. *Let X be a separable and complete geodesic space. Then for every μ, $\nu \in \mathscr{P}_p(X)$ we have*

$$
W_p^p(\mu, \nu) = \min \left\{ \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt : \eta \in \mathscr{A}(\mu, \nu) \right\},
$$

where the set of admissible measures is $\mathscr{A}(\mu,\nu) := \{\eta \in \mathscr{P}(\Gamma) : \eta(\Gamma \setminus AC^p(I;X)) = 0,\ (e_0)_\#\eta = \mu,\ (e_1)_\#\eta = \nu,\ \int_\Gamma \mathcal{E}_p(u)\, d\eta(u) < +\infty\}$.

Proof. Corollary 3.3 shows that

$$
W_p^p(\mu, \nu) \leq \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt
$$

for every $\eta \in \mathscr{A}(\mu, \nu)$. Since, by Proposition 4.1, $\mathscr{P}_p(X)$ is a geodesic space, taking μ_t a constant speed minimizing geodesic such that $\mu_0 = \mu$ and $\mu_1 = \nu$, the measure η given by Theorem 4.2 is admissible and realizes the equality. □

Remark 4.4. A sufficient condition on X, ensuring that a length space X is a geodesic space, is that all the closed balls of X are compact and any two points of X can be connected by an absolutely continuous curve (see e.g. [AT04]).
A more general condition, which replaces the compactness of the closed balls, is that there exists another topology τ on X, weaker than the topology induced by the metric, such that d is lower semi continuous with respect to τ and all d-bounded subsets of X are precompact in the topology τ. The sufficiency of this condition can be proved using the same arguments of the proof of Proposition 3.3.1 of [AGS05].

5. Application: the continuity equation in Banach spaces

We consider a separable Banach space $(X, \|\cdot\|)$ satisfying the Radon-Nicodým property (resp. the dual $X = E^*$ of a separable Banach space E). Through this section we treat the separable case pointing out, inside brackets, the necessary changes for the dual case. The norm $\|\cdot\|$ denotes the norm in X in both cases and the norm $\|\cdot\|_*$ denotes the norm of X^* in the separable case and the norm of E in the dual case. According to Remark 2.8 we work in the space $\mathscr{P}^\tau(X)$, observing that in the separable case $\mathscr{P}^\tau(X) = \mathscr{P}(X)$.

In this section we denote by I the closed interval $[0,T]$ and we fix the exponent $p>1$. Given a narrowly continuous curve $\mu : I \to \mathscr{P}^\tau(X)$, $t \mapsto \mu_t$, we can associate to it the probability measure $\bar\mu \in \mathscr{P}^\tau(I\times X)$ defined by

$$\int_{I\times X} \varphi(t,x)\, d\bar\mu(t,x) := \int_0^T \int_X \varphi(t,x)\, d\mu_t(x)\, dt$$

for every bounded Borel function $\varphi : I\times X \to \mathbb{R}$. We say that a time dependent vector field $\boldsymbol{v} : I\times X \to X$ belongs to $L^p(\bar\mu;X)$ (resp. $L^p_{w^*}(\bar\mu;X)$) if $\boldsymbol{v}$ is $\bar\mu$ Bochner integrable (resp. if $\boldsymbol{v}$ is weakly-* $\bar\mu$ measurable, i.e. for every $f\in E$ the maps $\varphi_f(t,x) := \langle \boldsymbol{v}_t(x), f\rangle$ are $\bar\mu$ measurable) and $\int_{I\times X} \|\boldsymbol{v}_t(x)\|^p\, d\bar\mu(t,x)$ is finite (see e.g. [DU77] for the definition and the properties of the Bochner integral and of the weak-* integral (or Gelfand integral), here we recall only that the weak-* integral of $\boldsymbol{v}$ is defined by

$$\left\langle \int_{I\times X} \boldsymbol{v}_t(x)\, d\bar\mu(t,x), f \right\rangle = \int_{I\times X} \langle \boldsymbol{v}_t(x), f\rangle\, d\bar\mu(t,x) \qquad \forall f\in E).$$

We say that $(\mu,\boldsymbol{v})$, where μ is narrowly continuous and $\boldsymbol{v}\in L^p(\bar\mu;X)$ satisfies the *continuity equation*

$$\partial_t\mu_t + \operatorname{div}(\boldsymbol{v}_t\mu_t) = 0 \tag{60}$$

if the relation

$$\frac{d}{dt}\int_X \phi\, d\mu_t = \int_X \langle D\phi, \boldsymbol{v}_t\rangle\, d\mu_t \qquad \forall \phi\in C^1_b(X) \quad (\text{resp. } \forall\phi\in C^1_{b*}(X)) \tag{61}$$

holds in the sense of distributions in $(0,T)$; here $C^1_b(X)$ denotes the space of functions $\phi : X\to\mathbb{R}$ such that ϕ is bounded, Fréchet differentiable, and the application $D\phi : X \to X^*$ is continuous and bounded (in all this paragraph D denotes the Fréchet differential), whereas $C^1_{b*}(X)$ denotes the space of functions $\phi : X\to\mathbb{R}$ such that ϕ is bounded, Fréchet differentiable and the application $D\phi : X \to X^*$ is continuous, bounded and, for every $x\in X$, $D\phi(x)\in E\subset E^{**}$. Here E is identified with the image of the canonical injection of E in E^{**}.
We define the following set,

$$\begin{aligned} EC_p(X) := \big\{(\mu,\boldsymbol{v}) : \mu : I\to\mathscr{P}^\tau(X) &\text{ is narrowly continuous, } \boldsymbol{v}\in L^p(\bar\mu;X),\\ (\text{resp. } \boldsymbol{v}\in L^p_{w^*}(\bar\mu;X)),\ &(\mu,\boldsymbol{v}) \text{ satisfies the continuity equation}\big\}. \end{aligned}$$

In the space $\mathscr{P}^\tau(X)$ we always use the pseudo distance W_p.

Theorem 5.1. *If $\mu\in AC^p(I;\mathscr{P}^\tau(X))$ then there exists a vector field $\boldsymbol{w} : I\times X\to X$ such that $(\mu,\boldsymbol{w})\in EC_p(X)$ and*

$$\|\boldsymbol{w}_t\|_{L^p(\mu_t;X)} \le |\mu'|(t) \qquad \textit{for a.e. } t\in I. \tag{62}$$

Proof. In order to carry out the proof for both cases we put $X_0 = X$ and $\Gamma_0 = \Gamma$ in the separable case, whereas in the dual case we take X_0 a separable Banach space $X_0\subset X$ containing the support of every μ_t, and $\Gamma_0 := C(I;X_0)$. Let $\eta := \tilde\eta \in \mathscr{P}(\Gamma_0)$ be given by Corollary 3.3 (resp. $\eta := \tilde\eta\in\mathscr{P}(\Gamma_0)$ given by Corollary 3.6). We denote by $\bar\eta\in\mathscr{P}(I\times\Gamma_0)$ the measure $\bar\eta := \frac{1}{T}\mathscr{L}^1_{|I}\otimes\eta$. Defining the evaluation map $e : I\times\Gamma_0 \to I\times X_0$ by $e(t,u) = (t, e_t(u))$, it is immediate to check that $e_\#\bar\eta = \bar\mu$.

The disintegration of $\bar{\eta}$ with respect to e yields a Borel family of probability measures $\bar{\eta}_{t,x}$ on Γ_0 concentrated on $\{u : e_t(u) = x\}$ such that for every $\varphi : I \times \Gamma_0 \to \mathbb{R}$, $\varphi \in L^1(\bar{\eta})$, we have

$$u \mapsto \varphi(t, u) \in L^1(\bar{\eta}_{t,x}) \text{ for } \bar{\mu}\text{-a.e. } (t, x) \in I \times X_0, \tag{63}$$

$$(t, x) \mapsto \int_{\Gamma_0} \varphi(t, u)\, d\bar{\eta}_{t,x}(u) \in L^1(\bar{\mu}), \tag{64}$$

$$\int_{I\times\Gamma_0} \varphi(t, u)\, d\bar{\eta}(t, u) = \int_{I\times X_0} \int_{\Gamma_0} \varphi(t, u)\, d\bar{\eta}_{t,x}(u)\, d\bar{\mu}(t, x)$$

and the measures $\bar{\eta}_{t,x}$ are uniquely determined for $\bar{\mu}$-a.e. $(t, x) \in I \times X_0$.

We denote by $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$ the pointwise derivative of the curve u (resp. $\dot{u}(t) := w^* - \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$ the weak-* derivative of the curve u, and we observe explicitly that, in general, X_0 is not weakly-* closed and then $\dot{u}(t) \in X$), which is defined $\mathscr{L}^1$- almost everywhere when u is absolutely continuous (see Remark 2.2).
We observe that the set

$$A := \{(t, u) \in I \times \Gamma_0 : \dot{u}(t) \text{ exists}\}$$

$$(\text{resp. } A := \{(t, u) \in I \times \Gamma_0 : \dot{u}(t) \text{ exists}, |u'|(t) \text{ exists}, \ \|\dot{u}(t)\| = |u'|(t)\})$$

is a Borel set and $\bar{\eta}(A^c) = 0$. Indeed, defining, for $h \neq 0$, the continuous functions $g_h : I \times \Gamma \to X_0$, by $g_h(t, u) = \frac{u(t+h)-u(t)}{h}$, (in the definition of g_h we extend the functions u outside of I by $u(s) = u(0)$ for $s < 0$ and $u(s) = u(T)$ for $s > T$), the completeness of X_0 yields that

$$A^c = \{(t, u) : \limsup_{(h,k)\to(0,0)} \|g_h(t, u) - g_k(t, u)\| > 0\},$$

and then A is a Borel set because of the continuity of the functions $(t, u) \mapsto \|g_h(t, u) - g_k(t, u)\|$ for every $h \neq 0$ and $k \neq 0$. (in the dual case A is a Borel set since, for a dense subset $\{f_n\}_{n\in\mathbb{N}} \subset E$ we have

$$A := \Big\{(t, u) \in I \times \Gamma_0 : \lim_{h\to 0} \left\langle \frac{u(t+h) - u(t)}{h}, f_n \right\rangle \text{ exists}, |u'|(t) \text{ exists},$$
$$|u'|(t) = \sup_{n\in\mathbb{N}} \lim_{h\to 0} \left\langle \frac{u(t+h) - u(t)}{h}, \frac{f_n}{\|f_n\|_E} \right\rangle \Big\}).$$

Since $\bar{\eta}$ is concentrated on $I \times AC^p(I, X_0)$, Fubini's Theorem implies that $\bar{\eta}(A^c) = 0$.
Then the map $\psi : I \times \Gamma_0 \to X$ defined by

$$\psi(t, u) = \dot{u}(t)$$

is well defined for $\bar{\eta}$-a.e. $(t, u) \in I \times \Gamma_0$.
We prove that ψ is $\bar{\eta}$ Bochner (resp. $\bar{\eta}$ weak-*) integrable and $\|\psi\|^p \in L^1(\bar{\eta})$. Taking $f \in X^*$ (resp. $f \in E$), we define for $(t, u) \in A$, $\psi_f(t, u) = \langle f, \dot{u}(t)\rangle$. Since ψ_f is limit of continuous functions is a Borel function in A and then $\bar{\eta}$ measurable. Since X_0 is separable, by Pettis

Theorem ψ is a $\bar\eta$ measurable function (resp. By definition ψ is a $\bar\eta$ weak-* measurable function) and, recalling (iii) of Theorem 3.2,

$$\int_{I\times\Gamma_0}\|\dot u(t)\|^p\,d\bar\eta(t,u)=\int_{I\times\Gamma_0}|u'|^p(t)\,d\bar\eta(u)=\frac{1}{T}\int_0^T|\mu'|^p(t)\,dt<+\infty. \tag{65}$$

Since for $\bar\mu$-a.e. $(t,x)\in I\times X_0$, we have $\bar\eta_{t,x}(\{u:(t,u)\in A^c\})=0$, the map $\psi(t,\cdot)$ is well defined for $\bar\eta_{t,x}$-a.e. $u\in\Gamma_0$ and $\psi(t,\cdot)$ is $\bar\eta_{t,x}$ Bochner integrable (resp. $\psi(t,\cdot)$ is $\bar\eta_{t,x}$ weak-* integrable). Indeed for every $f\in X^*$ (resp. $f\in E$),

$$\int_{I\times\Gamma_0}|\psi_f(t,u)|\,d\bar\eta(t,u)\quad\le\quad\|f\|_*\int_{I\times\Gamma_0}\|\dot u(t)\|\,d\bar\eta(t,u)$$

i.e. $\psi_f\in L^1(\bar\eta)$, and by (63) $\psi_f(t,\cdot)\in L^1(\bar\eta_{t,x})$, then Pettis Theorem yields that $\psi(t,\cdot)$ is $\bar\eta_{t,x}$ measurable (resp. then by definition $\psi(t,\cdot)$ is $\bar\eta_{t,x}$ is weak-* measurable) and $\|\psi(t,\cdot)\|\in L^1(\bar\eta_{t,x})$ by (63) and (65).
Consequently it is well defined the vector field

$$\boldsymbol{w}_t(x):=\int_{\Gamma_0}\dot u(t)\,d\bar\eta_{t,x}(u)\qquad\text{for }\bar\mu\text{-a.e. }(t,x)\in I\times X_0, \tag{66}$$

and $\boldsymbol{w}\in L^p(\bar\mu,X)$ (resp. $\boldsymbol{w}\in L^p_{w*}(\bar\mu,X)$). Indeed for every $f\in X^*$ (resp. $f\in E$),

$$\langle f,\boldsymbol{w}_t(x)\rangle=\int_{\Gamma_0}\psi_f(t,u)\,d\bar\eta_{t,x}(u)$$

and by (64) the maps $(t,x)\mapsto\langle f,\boldsymbol{w}_t(x)\rangle\in L^1(\bar\mu)$. Pettis Theorem implies that $\boldsymbol{w}$ is $\bar\mu$ measurable (resp. By definition $\boldsymbol{w}$ is $\bar\mu$ weak-* measurable) and moreover

$$\begin{aligned}\int_{I\times X_0}\|\boldsymbol{w}_t(x)\|^p\,d\bar\mu(t,x)\quad&\le\quad\int_{I\times X_0}\int_{\Gamma_0}\|\dot u(t)\|^p\,d\bar\eta_{t,x}(u)\,d\bar\mu(t,x)\\&=\quad\int_{I\times\Gamma_0}\|\dot u(t)\|^p\,d\bar\eta(t,u)<+\infty.\end{aligned}$$

The inequality (62) follows from the definition of $\boldsymbol{w}$, Jensen's inequality and (iii) of Theorem 3.2. Indeed for every $[a,b]\subset I$,

$$\begin{aligned}\int_a^b\int_{X_0}\|\boldsymbol{w}_t(x)\|^p\,d\mu_t(x)dt\quad&=\quad\int_{I\times X_0}T\chi_{[a,b]}(t)\,\|\boldsymbol{w}_t(x)\|^p\,d\bar\mu(t,x)\\&\le\quad\int_{I\times X_0}T\chi_{[a,b]}(t)\int_{\Gamma_0}\|\dot u(t)\|^p\,d\bar\eta_{t,x}(u)\,d\bar\mu(t,x)\\&=\quad\int_{I\times\Gamma_0}T\chi_{[a,b]}(t)\,\|\dot u(t)\|^p\,d\bar\eta(t,u)=\int_a^b|\mu'|(t)\,dt.\end{aligned}$$

Now we prove that (61) holds. Taking $\phi\in C^1_b(X)$ (resp $\phi\in C^1_{b*}(X)$), as a consequence of the absolute continuity of μ_t in $(\mathscr{P}(X_0),W_p)$, the application $t\mapsto\int_{X_0}\phi\,d\mu_t$ is absolutely continuous. Indeed for every $s,t\in I$, taking $\gamma_{s,t}\in\Gamma_o(\mu_s,\mu_t)$, we have

$$\begin{aligned}\left|\int_{X_0}\phi\,d\mu_t-\int_{X_0}\phi\,d\mu_s\right| &\leq \int_{X_0\times X_0}|\phi(y)-\phi(x)|\,d\gamma_{s,t}(x,y)\\ &\leq \sup_{x\in X_0}\|D\phi(x)\|_*\int_{X_0\times X_0}\|x-y\|\,d\gamma_{s,t}(x,y)\\ &\leq \sup_{x\in X_0}\|D\phi(x)\|_*\,W_p(\mu_s,\mu_t).\end{aligned}$$

Moreover, by (iii) of Remark 2.2 and the differentiability of test functions,

$$\begin{aligned}\int_{X_0}\phi\,d\mu_t-\int_{X_0}\phi\,d\mu_s &= \int_{\Gamma_0}\phi(u(t))-\phi(u(s))\,d\eta(u)\\ &= \int_{\Gamma_0}\langle D\phi(u(s)),u(t)-u(s)\rangle\,d\eta(u)\\ &\quad+\int_{\Gamma_0}\|u(t)-u(s)\|\,\omega_{u(s)}(u(t))\,d\eta(u)\\ &= \int_{\Gamma_0}\langle D\phi(u(s)),\int_s^t\dot u(r)\,dr\rangle\,d\eta(u)\\ &\quad+\int_{\Gamma_0}\|u(t)-u(s)\|\,\omega_{u(s)}(u(t))\,d\eta(u)\end{aligned}$$

where

$$\omega_x(y)=\frac{\phi(y)-\phi(x)-\langle D\phi(x),y-x\rangle}{\|y-x\|}.$$

Dividing by $t-s$ and passing to the limit for $t\to s$, for a.e. $s\in I$ we have

$$\frac{d}{ds}\int_{X_0}\phi\,d\mu_s=\int_{\Gamma_0}\langle D\phi(u(s)),\dot u(s)\rangle\,d\eta(u).$$

Indeed, for a.e. $s\in I$, a simple application of Lebesgue Theorem yields

$$\lim_{t\to s}\frac{1}{t-s}\int_{\Gamma_0}\langle D\phi(u(s)),\int_s^t\dot u(r)\,dr\rangle\,d\eta(u)=\int_{\Gamma_0}\langle D\phi(u(s)),\dot u(s)\rangle\,d\eta(u)$$

(we observe that, in the dual case, $D\phi(x)\in E$) and we must only prove that

$$\lim_{t\to s}\frac{1}{t-s}\int_{\Gamma_0}\|u(t)-u(s)\|\,\omega_{u(s)}(u(t))\,d\eta(u)=0. \tag{67}$$

We take a Lebesgue point s of the application $t\mapsto|\mu'|^p(t)$ such that $\dot u(s)$ exists for η-a.e. $u\in\Gamma_0$. We define $f_t:\Gamma_0\to\mathbb{R}$ by $f_t(u):=\left\|\frac{u(t)-u(s)}{t-s}\right\|\omega_{u(s)}(u(t))$. Since $\sup_{x\in X_0}\|D\phi(x)\|_*=:C<+\infty$, in particular ϕ is Lipschitz and

$$|\omega_x(y)|\leq\frac{|\phi(y)-\phi(x)|}{\|y-x\|}+\frac{|\langle D\phi(x),y-x\rangle|}{\|y-x\|}\leq 2C.$$

Clearly $f_t(u) \to 0$ when $t \to s$ for η-a.e. $u \in \Gamma_0$ and,

$$|f_t(u)| \le g_t(u) := 2C \left\| \frac{u(t)-u(s)}{t-s} \right\| \qquad \text{for } \eta\text{-a.e. } u \in \Gamma_0.$$

The family $\{g_t : 0 < |t-s| < \delta\}$, for δ sufficiently small, is η-equiintegrable. Indeed

$$\begin{aligned} \int_{\Gamma_0} g_t(u)^p \, d\eta(u) &\le (2C)^p \int_{\Gamma_0} \frac{1}{|t-s|} \left| \int_s^t |u'|^p(r)\, dr \right| d\eta(u) \\ &= \frac{(2C)^p}{|t-s|} \left| \int_s^t |\mu'|^p(r)\, dr \right| \to (2C)^p |\mu'|^p(s) < +\infty, \end{aligned}$$

for $t \to s$, which implies that there exists $\delta > 0$ such that $\int_{\Gamma_0} g_t(u)^p \, d\eta(u) \le C_0$ for every t satisfying $0 < |t-s| < \delta$. By Vitali's convergence Theorem we obtain (67).
Finally, taking into account the definition of $\boldsymbol{w}_t$ we easily obtain

$$\frac{d}{dt} \int_{X_0} \phi \, d\mu_t = \int_{X_0} \langle D\phi, \boldsymbol{w}_t \rangle \, d\mu_t, \qquad \text{for a.e. } t \in I.$$

Since this pointwise derivative is also a distributional derivative, we can conclude. □

Before stating the next Theorem we give two definitions.

Definition 5.2. *A separable Banach space X satisfies the Bounded Approximation Property (BAP) if there exists a sequence of finite rank linear operators $T_n : X \to X$ such that*

$$\lim_{n\to\infty} \|T_n x - x\| = 0 \qquad \forall\, x \in X. \tag{68}$$

We observe explicitly that if (68) holds, then, by Banach-Steinhaus Theorem, there exists $M \ge 1$ such that

$$\|T_n x\| \le M \, \|x\| \qquad \forall\, x \in X. \tag{69}$$

Definition 5.3. *The dual X of a separable Banach space satisfies the Weak* Bounded Approximation Property (wBAP) if there exists a sequence of finite rank linear operators $T_n : X \to X$ such that*

$$w^* - \lim_{n\to\infty} T_n x = x \qquad \forall\, x \in X, \tag{70}$$

there exists $M \ge 1$ such that

$$\|T_n x\| \le M \, \|x\| \qquad \forall\, x \in X, \tag{71}$$

$$\limsup_{n\to+\infty} \|T_n x\| \le \|x\| \qquad \forall\, x \in X, \tag{72}$$

and

$$T_n \text{ are weakly-* continuous.} \tag{73}$$

Remark 5.4. We observe that, being X separable, Definition 5.2 is equivalent to the following one, valid also for non separable Banach spaces: there exists $M \ge 1$ such that for every $\varepsilon > 0$ and for every compact $K \subset X$ there exists a finite rank linear operator $T_{\varepsilon,K} : X \to X$ such that $\|T_{\varepsilon,K}\| \le M$ and $\sup_{x\in K} \|T_{\varepsilon,K} x - x\| < \varepsilon$ (see [LT77]).

Clearly a Banach space with a Schauder basis has the (BAP). Then, for instance, the property is satisfied by the Sobolev spaces $W^{k,p}(\Omega)$ for $p \in [1, +\infty)$ and $k \geq 0$, where Ω is a smooth bounded open subset of $\mathbb{R}^n$. Even if M is a compact, smooth n-dimensional manifold with or without boundary, the spaces of continuous functions $C^k(M)$ and the Sobolev spaces $W^{k,p}(M)$, for $p \in [1, +\infty)$ and $k \geq 0$, satisfy the property (see [FW01]).
We observe that l^∞ satisfies the (wBAP) as we can see taking the operators $T_n x := (x_1, \ldots, x_n, 0, \ldots)$. The space l^∞ is of fundamental importance since every separable metric space can be isometrically embedded in l^∞ (see e.g. [AK00]).

Theorem 5.5. *Assume that X satisfies the bounded approximation property (BAP) (resp. the property (wBAP)).*
If $(\mu, \boldsymbol{v}) \in EC_p(X)$ then $\mu \in AC^p(I; \mathscr{P}^\tau(X))$ and

$$|\mu'|(t) \leq \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \text{for a.e. } t \in I. \tag{74}$$

Proof. For all $n \in \mathbb{N}$ we set $X_n := T_n(X)$ and $m := \dim(T_n(X))$. Let $\{e_i\}_{i=1,\ldots,m}$ be a basis of X_n. There exist $f_1^n, \ldots, f_m^n \in X^*$ (resp. $f_1^n, \ldots, f_m^n \in E$, thanks to (73)) such that $T_n x = \sum_{i=1}^m \langle f_i^n, x \rangle e_i$.
Denoting by $P_n : X_n \to \mathbb{R}^m$ the linear isomorphism given by $P_n(\sum_{i=1}^m a_i e_i) = (a_1, \ldots, a_m)$, we define the projection $\pi_n : X \to \mathbb{R}^m$, as

$$\pi_n = P_n \circ T_n$$

and the relevement $\tilde{\pi}_n : \mathbb{R}^m \to X$, as

$$\tilde{\pi}_n(a_1, \ldots, a_m) = \sum_{i=1}^m a_i e_i.$$

We observe that

$$\pi_n \circ \tilde{\pi}_n = id_{\mathbb{R}^m} \qquad \text{and} \qquad \tilde{\pi}_n \circ \pi_n = T_n. \tag{75}$$

We define

$$\mu_t^n := \pi_{n\#}\mu_t \qquad \text{and} \qquad \boldsymbol{v}_t^n(y) := \int_X \pi_n(\boldsymbol{v}_t)(x))\, d\mu_{ty}(x)$$

where μ_{ty} is the disintegration of μ_t with respect to π_n which is concentrated on $\{x : \pi_n x = y\}$. Using a test function of the form $\phi = \psi \circ \pi_n$ with $\psi \in C_c^\infty(\mathbb{R}^m)$, it is easy to check that $(\mu_t^n, \boldsymbol{v}_t^n)$ satisfies the continuity equation in the distribution sense on $\mathbb{R}^m$.
For our purposes it's useful to consider on $\mathbb{R}^m$ the norm

$$|||y|||_{\mathbb{R}^m} := \|\tilde{\pi}_n y\|, \tag{76}$$

satisfying, by (75),

$$|||\pi_n x|||_{\mathbb{R}^m} = \|T_n x\| \qquad \forall x \in X, \tag{77}$$

and

$$\limsup_{n \to +\infty} \|\boldsymbol{v}_t^n\|_{L^p(\mu_t^n;\mathbb{R}^m)} \leq \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \text{for a.e. } t \in I. \tag{78}$$

In fact, using the property of disintegration, (77) and Jensen's inequality,

$$\begin{aligned}
\|\boldsymbol{v}_t^n\|^p_{L^p(\mu_t^n;\mathbb{R}^m)} &= \int_{\mathbb{R}^m} |||\boldsymbol{v}_t^n(y)|||^p_{\mathbb{R}^m}\, d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left|\left|\left|\pi_n \int_{\{x:\pi_n x=y\}} \boldsymbol{v}_t(x)\, d\mu_{ty}(x)\right|\right|\right|^p_{\mathbb{R}^m} d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left\|T_n \int_{\{x:\pi_n x=y\}} \boldsymbol{v}_t(x)\, d\mu_{ty}(x)\right\|^p d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left\|\int_{\{x:\pi_n x=y\}} T_n\boldsymbol{v}_t(x)\, d\mu_{ty}(x)\right\|^p d\mu_t^n(y) \\
&\le \int_{\mathbb{R}^m} \int_{\{x:\pi_n x=y\}} \|T_n\boldsymbol{v}_t(x)\|^p\, d\mu_{ty}(x)\, d\mu_t^n(y) \\
&= \int_X \|T_n\boldsymbol{v}_t(x)\|^p\, d\mu_t(x).
\end{aligned}$$

In the separable case, by (69) and (68), Lebesgue dominated convergence Theorem implies

$$\lim_{n\to\infty} \int_X \|T_n\boldsymbol{v}_t(x)\|^p\, d\mu_t(x) = \int_X \|\boldsymbol{v}_t(x)\|^p\, d\mu_t(x)$$

from which (78) follows. In the dual case, (78) follows by using (72) instead of Lebesgue Theorem.
Theorem 8.2.1 of [AGS05] states that there exists $\eta^n \in \mathscr{P}(\Gamma_{\mathbb{R}^m})$ such that $(e_t)_\# \eta^n = \mu_t^n$ and η^n is concentrated on the set $\{u \in AC^p(I;\mathbb{R}^m) : u$ is an integral solution of $\dot u(t) = \boldsymbol{v}_t^n(u(t))\}$.
By the first part of Corollary 3.3 and (14) we conclude that $\mu_t^n \in AC^p(I;\mathscr{P}_p(\mathbb{R}^m))$ and

$$|(\mu^n)'|(t) \le \|\boldsymbol{v}_t^n\|_{L^p(\mu_t;\mathbb{R}^m)} \qquad \text{for a.e. } t \in I \tag{79}$$

where the Wasserstein pseudo distance, and consequently the metric derivative, is made with respect to the distance induced by the norm (76).
Defining

$$\tilde\mu_t^n := (T_n)_\#\mu_t = (\tilde\pi_n)_\#\mu_t^n,$$

in the separable case, (68) implies that $\tilde\mu_t^n \to \mu_t$ narrowly and hence we have

$$W_p(\mu_t,\mu_s) \le \liminf_{n\to\infty} W_p(\tilde\mu_t^n,\tilde\mu_s^n) = \liminf_{n\to\infty} W_p(\mu_t^n,\mu_s^n), \tag{80}$$

by the lower semicontinuity of the Wasserstein distance and the isometric embedding (76).
In the dual case, the inequality (80) can be obtained by introducing the distance

$$d_w(x,y) := \sum_{n=1}^{+\infty} \frac{1}{2^n} |\langle x-y, f_n\rangle|, \tag{81}$$

where $\{f_n\}_{n\ge1}$ is a countable dense subset of $\{f \in E : \|f\|_E \le 1\}$. d_w is a distance on X which metrizes the weak-* topology on bounded sets (see e.g. [Bre83]) and, in general, the topology induced by d_w is weaker than the weak-* topology. We can assume that all the supports of $(T_n)_\#\mu_t$ and μ_t, for all $t \in I$, are contained in a separable subspace X_0, since T_n are of finite rank $(T_n)_\#\mu_t \in \mathscr{P}^\tau(X)$, and $t \mapsto \mu_t$ is narrowly continuous. Taking $\varphi \in C_b((X_0,d_w)) \subset C_b((X_0,w*))$ by (70) $\lim_{n\to\infty}\varphi(T_n x) = \varphi(x)$ for every $x \in X_0$ and

Lebesgue dominated convergence Theorem implies that $(T_n)_{\#}\mu_t$ narrowly converges to μ_t in $\mathscr{P}((X_0, d_w))$. Consequently a sequence of optimal plans $\gamma_n \in \Gamma_o((T_n)_{\#}\mu_t, (T_n)_{\#}\mu_s)$ is tight in $\mathscr{P}((X_0, d_w) \times (X_0, d_w))$. Let γ be a limit point of γ_n in $\mathscr{P}((X_0, d_w) \times (X_0, d_w))$; since Borel sets of X_0 coincides with Borel sets of (X_0, d_w) (see [Sch73]), $\gamma \in \mathscr{P}(X_0 \times X_0)$. By the lower semi continuity of the application $(x, y) \to \|x - y\|^p$ with respect to the product topology induced by d_w in $X_0 \times X_0$ we obtain

$$\liminf_{n\to\infty} \int_{X_0\times X_0} \|x - y\|^p \, d\gamma_n(x, y) \geq \int_{X_0\times X_0} \|x - y\|^p \, d\gamma(x, y),$$

and then (80) follows.

Finally, by (79), (78) and (80) we have

$$W_p(\mu_t, \mu_s) \leq \int_s^t \|\boldsymbol{v}_r\|_{L^p(\mu_r;X)} \, dr,$$

which shows that $\mu_t \in AC^p(I; \mathscr{P}^\tau(X))$ and the inequality (74) holds. □

Using the two previous Theorems, and the fact that the strict convexity of the norm implies the strict convexity of the norm of $L^p(\bar{\mu}; X)$, it is immediate to prove the following

Corollary 5.6. *Assume that X satisfies the bounded approximation property (BAP) (resp. the property (wBAP)).*
If $\mu \in AC^p(I; \mathscr{P}^\tau(X))$ then there exists a vector field $\boldsymbol{v} : I \times X \to X$ such that $(\mu, \boldsymbol{v}) \in EC_p(X)$ and

$$|\mu'|(t) = \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \text{for a.e. } t \in I. \tag{82}$$

Moreover $\boldsymbol{v}_t$ is minimal, since for any $\boldsymbol{w} \in L^p(\bar{\mu}; X)$ (resp. $\boldsymbol{w} \in L^p_{w^*}(\bar{\mu}; X)$) such that $(\mu_t, \boldsymbol{w}_t) \in EC_p(X)$ we have

$$\|\boldsymbol{w}_t\|_{L^p(\mu_t;X)} \geq \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \text{for a.e. } t \in I. \tag{83}$$

If the norm of X is strictly convex, then $\boldsymbol{v}$ is uniquely determined in $L^p(\bar{\mu}; X)$.

Remark 5.7. The proof of Theorem 5.1 works also for Riemannian manifolds. We denote by M a smooth n-dimensional complete Riemannian manifold, by TM its tangent bundle and by T_xM the tangent space on M at the point $x \in M$. For $v, w \in T_xM$, we denote by $\langle v, w\rangle_x$ the scalar product on T_xM and by $|v|_x := \sqrt{\langle v, v\rangle_x}$ the norm on T_xM. On M we put the Riemannian distance which we denote by d.
A vector field $\boldsymbol{v} : M \to TM$ is an application such that $\boldsymbol{v}(x) \in T_xM$. Given a Borel probability measure μ on M and a Borel vector field $\boldsymbol{v}$ we say that $\boldsymbol{v} \in L^p(\mu, TM)$ if $\int_M |\boldsymbol{v}(x)|_x^p \, d\mu(x) < +\infty$.
We say that $(\mu_t, \boldsymbol{v}_t)$ satisfies the continuity equation in M if

$$\frac{d}{dt} \int_M \phi(x) \, d\mu_t(x) = \int_M \langle \boldsymbol{v}_t(x), \nabla\phi(x)\rangle_x \, d\mu_t(x) \qquad \forall \phi \in C_c^\infty(M) \tag{84}$$

where the equality is intended in the sense of distribution in $(0, T)$.
Since we can check that a curve $u \in AC^p(I; M)$ is differentiable for a.e. t and $|\dot{u}(t)|_{u(t)} =$

$|u'|(t)$ for a.e. $t \in I$, we can prove that if $\mu_t \in AC^p(I; \mathscr{P}(M))$, then there exists a time dependent Borel vector field $\boldsymbol{v}$ such that

$$\int_0^T \int_M |\boldsymbol{v}_t(x)|_x^p \, d\mu_t(x) \, dt < +\infty,$$

$(\mu_t, \boldsymbol{v}_t)$ satisfies (84) and

$$|\mu'|(t) \geq \|\boldsymbol{v}_t\|_{L^p(\mu_t; TM)} \qquad \text{for a.e. } t \in I. \tag{85}$$

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