LOCALLY CONFORMALLY FLAT QUASI–EINSTEIN MANIFOLDS

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ABSTRACT. In this paper we prove that any complete locally conformally flat quasi-Einstein manifold of dimension $n \geq 3$ is locally a warped product with $(n-1)$–dimensional fibers of constant curvature. This result includes also the case of locally conformally flat gradient Ricci solitons.

1. INTRODUCTION

Let (M^n, g), for $n \geq 3$, be a complete quasi–Einstein Riemannian manifold, that is, there exist a smooth function $f : M^n \to \mathbb{R}$ and two constants $\mu, \lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} + \nabla^2 f - \mu\, df \otimes df = \lambda g\,. \tag{1.1}$$

When $\mu = 0$, quasi–Einstein manifolds correspond to gradient Ricci solitons and when f is constant (1.1) gives the Einstein equation and we call the quasi–Einstein metric trivial. We also notice that, for $\mu = \frac{1}{2-n}$, the metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$ is Einstein. Indeed, from the expression of the Ricci tensor of a conformal metric, we get

$$\begin{aligned} \mathrm{Ric}_{\widetilde{g}} &= \mathrm{Ric}_g + \nabla^2 f + \tfrac{1}{n-2} df \otimes df + \tfrac{1}{n-2}\big(\Delta f - |\nabla f|^2\big) g \\ &= \tfrac{1}{n-2}\big(\Delta f - |\nabla f|^2 + (n-2)\lambda\big) e^{\frac{2}{n-2}f}\, \widetilde{g}\,. \end{aligned}$$

In particular, if g is also locally conformally flat, then $\widetilde{g}$ has constant curvature.

Quasi–Einstein manifolds have been recently introduced by J. Case, Y.-S. Shu and G. Wei in [5]. In that work the authors focus mainly on the case $\mu \geq 0$. The case $\mu = \frac{1}{m}$ for some $m \in \mathbb{N}$ is particularly relevant due to the link with Einstein warped products. Indeed in [5], following the results in [11], it is proved a characterization of these quasi–Einstein metrics as base metrics of Einstein warped product metrics (see also [15, Theorem 2]). This characterization on the one hand enables to translate results from one setting to the other and on the other hand permits to furnish several examples of quasi–Einstein manifolds (see [1, Chapter 9], [12]). Observe also that, in case $\frac{1}{2-n} \leq \mu < 0$, the definition of quasi–Einstein metric was used by D. Chen in [7] in the context of finding conformally Einstein product metrics on $M^n \times F^m$ for $\mu = \frac{1}{2-m-n}$, $m \in \mathbb{N} \cup \{0\}$.

As a generalization of Einstein manifolds, quasi–Einstein manifolds exhibit a certain rigidity. This is well known for $\mu = 0$, but we have evidence of this also in the case $\mu \geq 0$. This is expressed for example by triviality results and curvature estimates; see [4, 5, 15]. For instance it is known that, according to Qian version of Myers' Theorem, if $\lambda > 0$ and $\mu > 0$ in (1.1) then M^n is compact (see [14]). Moreover in [11], analogously to the case $\mu = 0$, it is proven that if $\lambda \leq 0$, compact quasi–Einstein manifolds are trivial. A generalization to the complete non-compact setting of this result is obtained in [15] by means of an L^p–Liouville result for the weighted Laplacian which relies upon estimates for the infimum of the scalar curvature (extending the previous work in [5]). These latter are achieved by means of tools coming from stochastic analysis such as the weak maximum principle at infinity combined with Qian's estimates on weighted volumes (see [14] and also [13, Section 2]).

The Riemann curvature operator of a Riemannian manifold (M^n, g) is defined as in [9] by

$$\mathrm{Riem}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\,.$$

In a local coordinate system the components of the $(3,1)$–Riemann curvature tensor are given by $\mathrm{R}^d_{abc} \frac{\partial}{\partial x^d} = \mathrm{Riem}\big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\big)\frac{\partial}{\partial x^c}$ and we denote by $\mathrm{R}_{abcd} = g_{de}\mathrm{R}^e_{abc}$ its $(4,0)$–version.

Date: February 3, 2011.

In all the paper the Einstein convention of summing over the repeated indices will be adopted.

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v,w,v,w) = \mathrm{R}_{abcd}v^a w^b v^c w^d > 0$. The Ricci tensor is obtained by the contraction $\mathrm{R}_{ac} = g^{bd}\mathrm{R}_{abcd}$ and $\mathrm{R} = g^{ac}\mathrm{R}_{ac}$ will denote the scalar curvature. The so called Weyl tensor is then defined by the following decomposition formula (see [9, Chapter 3, Section K]) in dimension $n \geq 3$,

$$\mathrm{W}_{abcd} = \mathrm{R}_{abcd} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \frac{1}{n-2}(\mathrm{R}_{ac}g_{bd} - \mathrm{R}_{ad}g_{bc} + \mathrm{R}_{bd}g_{ac} - \mathrm{R}_{bc}g_{ad}). \tag{1.2}$$

The Weyl tensor satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero, as it can be easily seen by the above formula.
In dimension three W is identically zero for every Riemannian manifold, it becomes relevant instead when $n \geq 4$ since its vanishing is a condition equivalent for (M^n, g) to be *locally conformally flat*, that is, around every point $p \in M^n$ there is a conformal deformation $\widetilde{g}_{ab} = e^f g_{ab}$ of the original metric g, such that the new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p). In dimension $n = 3$, on the other hand, locally conformally flatness is equivalent to the vanishing of the Cotton tensor

$$\mathrm{C}_{abc} = \nabla_c \mathrm{R}_{ab} - \nabla_b \mathrm{R}_{ac} - \tfrac{1}{2(n-1)}\big(\nabla_c \mathrm{R}\, g_{ab} - \nabla_b \mathrm{R}\, g_{ac}\big)\,.$$

When $n \geq 4$ note that one can compute, (see [1]), that

$$\nabla^d \mathrm{W}_{abcd} = -\frac{n-3}{n-2}\mathrm{C}_{abc}.$$

Hence if we assume that the manifold is locally conformally flat, the Cotton tensor is identically zero also in this case.

In this paper we will consider a generic $\mu \in \mathbb{R}$ and we will prove the following

Theorem 1.1. *Let (M^n, g), $n \geq 3$, be a complete locally conformally flat quasi–Einstein manifold. Then*
(i) *if $\mu = \frac{1}{2-n}$, then (M^n, g) is globally conformally equivalent to a spaceform.*
(ii) *if $\mu \neq \frac{1}{2-n}$, then, around any regular point of f, the manifold (M^n, g) is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature.*

Remark 1.2. This result was already known in the case where $\mu = 0$, i.e. for gradient Ricci solitons (see [3] and [6]). Nevertheless, the strategy of the proof is completely new and can be used as the main step to classify locally conformally flat shrinking and steady gradient Ricci solitons.

Remark 1.3. Very recently, similar results have been obtained by C. He, P. Petersen and W. Wylie [10] in the case when $0 < \mu < 1$, assuming a slightly weaker condition than locally conformally flatness.

2. Proof of Theorem 1.1

As observed in the introduction, if $\mu = \frac{1}{2-n}$, (M^n, g) is globally conformally equivalent to a space form.

From now on, we will consider the case $\mu \neq \frac{1}{2-n}$.

Lemma 2.1. *Let (M^n, g) be a quasi–Einstein manifold. Then the following identities hold*

$$\mathrm{R} + \Delta f - \mu|\nabla f|^2 = n\lambda \tag{2.1}$$

$$\nabla_b \mathrm{R} = 2\mathrm{R}_{ab}\nabla^a f + 2\mu R\nabla_b f - 2\mu^2|\nabla f|^2\nabla_b f - 2n\mu\lambda\nabla_b f + \mu\nabla_b|\nabla f|^2 \tag{2.2}$$

$$\nabla_c \mathrm{R}_{ab} - \nabla_b \mathrm{R}_{ac} = -\mathrm{R}_{cabd}\nabla^d f + \mu\big(\mathrm{R}_{ab}\nabla_c f - \mathrm{R}_{ac}\nabla_b f\big) - \lambda\mu\big(g_{ab}\nabla_c f - g_{ac}\nabla_b f\big) \tag{2.3}$$

Proof. Equation (2.1): we simply contract equation (1.1).

Equation (2.2): we take the divergence of the equation (1.1)

$$\begin{aligned} \operatorname{div}\mathrm{Ric}_b &= -\nabla^a\nabla_a\nabla_b f + \mu\nabla^a\nabla_a f\nabla_b f + \mu\nabla^a\nabla_b f\nabla_a f \\ &= -\nabla_b\Delta f - \mathrm{R}_{ab}\nabla^a f + \mu\Delta f\nabla_b f + \tfrac{1}{2}\mu\nabla_b|\nabla f|^2\,, \end{aligned}$$

where we interchanged the covariant derivatives. Now, using equation (2.1), we get

$$\begin{aligned}\operatorname{div}\mathrm{Ric}_b &= \nabla_b \mathrm{R} - \mu\nabla_b|\nabla f|^2 - \mathrm{R}_{ab}\nabla^a f - \mu \mathrm{R}\nabla_b f + \mu^2|\nabla f|^2\nabla_b f + n\mu\lambda\nabla_b f + \tfrac{1}{2}\mu\nabla_b|\nabla f|^2 \\ &= \nabla_b \mathrm{R} - \mathrm{R}_{ab}\nabla^a f - \mu R\nabla_b f + \mu^2|\nabla f|^2\nabla_b f + n\mu\lambda\nabla_b f - \tfrac{1}{2}\mu\nabla_b|\nabla f|^2\end{aligned}$$

Finally, using Schur's Lemma $\nabla \mathrm{R} = 2\operatorname{div}\mathrm{Ric}$, we obtain equation (2.2).

Equation (2.3): taking the covariant derivative of (1.1) we obtain

$$\begin{aligned}\nabla_c \mathrm{R}_{ab} - \nabla_b \mathrm{R}_{ac} &= -\big(\nabla_c\nabla_b\nabla_a f - \nabla_b\nabla_c\nabla_a f\big) \\ &\quad + \mu\big(\nabla_c\nabla_a f\nabla_b f + \nabla_c\nabla_b f\nabla_a f - \nabla_b\nabla_a f\nabla_c f - \nabla_b\nabla_c f\nabla_a f\big) \\ &= -\mathrm{R}_{cabd}\nabla^d f + \mu\big(\mathrm{R}_{ab}\nabla_c f - \mathrm{R}_{ac}\nabla_b f\big) - \mu\lambda\big(g_{ab}\nabla_c f - g_{ac}\nabla_b f\big)\,,\end{aligned}$$

where we interchanged the covariant derivatives and we have used again equation (1.1). □

In any neighborhood, where $|\nabla f| \neq 0$, of a level set $\Sigma_\rho = \{p \in M^n \mid f(p) = \rho\}$ of a regular value ρ of f, we can express the metric g as

$$g = \tfrac{1}{|\nabla f|^2}\, df \otimes df + g_{ij}(f,\theta)\, d\theta^i \otimes d\theta^j\,, \tag{2.4}$$

where $\theta = (\theta^1, \dots, \theta^{n-1})$ denotes intrinsic coordinates for Σ_ρ. In the following computations we will agree that $\partial_0 = \frac{\partial}{\partial f}$, $\partial_j = \frac{\partial}{\partial\theta^j}$, $\nabla_0 = \nabla_{\partial_0}$, $\mathrm{R}_{00} = \mathrm{Ric}(\partial_0,\partial_0)$, $\mathrm{R}_{0j} = \mathrm{Ric}(\partial_0,\partial_j)$ and so on. According to (2.4), we compute easily that

$$\nabla_j f = 0, \qquad \nabla_0 f = |\nabla f|^2, \qquad g^{00} = |\nabla f|^2\,. \tag{2.5}$$

In this coordinate system, we have the following formulae:

Lemma 2.2. *If (M^n,g) is a quasi–Einstein manifold. Then, for $j = 1,\dots,n-1$, we have*

$$\nabla_j|\nabla f|^2 = -2|\nabla f|^4\mathrm{R}_{0j} \tag{2.6}$$

$$\nabla_0|\nabla f|^2 = -2|\nabla f|^4\mathrm{R}_{00} + 2\mu|\nabla f|^4 + 2\lambda|\nabla f|^2 \tag{2.7}$$

$$\nabla_j \mathrm{R} = 2(1-\mu)\,|\nabla f|^4\mathrm{R}_{0j} \tag{2.8}$$

$$\nabla_0 \mathrm{R} = 2(1-\mu)\,|\nabla f|^4\mathrm{R}_{00} - 2(n-1)\mu\lambda|\nabla f|^2 + 2\mu\mathrm{R}|\nabla f|^2 \tag{2.9}$$

$$\nabla_0 \mathrm{R}_{j0} - \nabla_j \mathrm{R}_{00} = \mu|\nabla f|^2\,\mathrm{R}_{0j}\,. \tag{2.10}$$

Moreover, if (M^n,g) has $\mathrm{W}=0$, for $i,j = 1,\dots,n-1$, we have

$$\begin{aligned}\nabla_0 \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{i0} &= \tfrac{\mu(n-2)+1}{n-2}\mathrm{R}_{ij}|\nabla f|^2 + \tfrac{1}{n-2}\mathrm{R}_{00}|\nabla f|^4 g_{ij} + \\ &\quad - \tfrac{1}{(n-1)(n-2)}\mathrm{R}|\nabla f|^2 g_{ij} - \lambda\mu|\nabla f|^2 g_{ij}\,.\end{aligned} \tag{2.11}$$

Proof. Equation (2.6): we compute

$$\nabla_j|\nabla f|^2 = 2\,g^{00}\nabla_j\nabla_0 f\nabla_0 f + 2g^{kl}\nabla_j\nabla_k f\nabla_l f.$$

Using (1.1) and (2.5) we thus obtain

$$\begin{aligned}\nabla_j|\nabla f|^2 &= 2g^{00}\left(-\mathrm{R}_{0j} + \mu df\,(\partial_j)\,df\,(\partial_0) + \lambda g_{0j}\right)\nabla_0 f \\ &= -2|\nabla f|^4\mathrm{R}_{0j} + 2\mu|\nabla f|^2\nabla_j f\nabla_0 f\nabla_0 f + 2\lambda|\nabla f|^2 g_{0j}\nabla_0 f \\ &= -2|\nabla f|^4\mathrm{R}_{0j}\,.\end{aligned}$$

Equation (2.7): we have as before,

$$\begin{aligned}\nabla_0|\nabla f|^2 &= 2|\nabla f|^4\nabla^2_{00} f \\ &= 2|\nabla f|^4\left(-\mathrm{R}_{00} + \mu df(\partial_0)df(\partial_0) + \lambda g_{00}\right) \\ &= -2|\nabla f|^4\mathrm{R}_{00} + 2\mu|\nabla f|^4 + 2\lambda|\nabla f|^2.\end{aligned}$$

Equation (2.8): using equation (2.2) and equation (2.6) one has

$$\nabla_j \mathrm{R} = 2|\nabla f|^4\mathrm{R}_{0j} - 2\mu|\nabla f|^4\mathrm{R}_{0j} = 2(1-\mu)|\nabla f|^2\mathrm{R}_{0j}\,.$$

Equation (2.9): it follows as before from equations (2.2) and (2.7).

Equation (2.10): using equation (2.3), we have

$$\nabla_0 \mathrm{R}_{j0} - \nabla_j \mathrm{R}_{00} = -g^{00}\mathrm{R}_{0j00}\nabla_0 f + \mu\big(\mathrm{R}_{0j}\nabla_0 f - \mathrm{R}_{00}\nabla_j f\big) - \lambda\big(g_{0j}\nabla_0 f - g_{00}\nabla_j f\big) = \mu|\nabla f|^2 \mathrm{R}_{0j}\,.$$

Equation (2.11): using again equation (2.3), we have

$$\begin{aligned}
\nabla_0 \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{i0} &= -g^{00}\mathrm{R}_{0ji0}\nabla_0 f + \mu\big(\mathrm{R}_{ij}\nabla_0 f - \mathrm{R}_{i0}\nabla_j f\big) - \lambda\mu\big(g_{ij}\nabla_0 f - g_{i0}\nabla_j f\big)\\
&= -|\nabla f|^4\Big[\tfrac{1}{n-2}\big(\mathrm{R}_{i0}g_{j0} + \mathrm{R}_{j0}g_{i0} - \mathrm{R}_{00}g_{ij} - \mathrm{R}_{ij}g_{00}\big) - \tfrac{\mathrm{R}}{(n-1)(n-2)}\big(g_{j0}g_{i0} - g_{ij}g_{00}\big)\Big]\\
&\quad +\mu|\nabla f|^2\mathrm{R}_{ij} - \lambda\mu|\nabla f|^2 g_{ij}\\
&= \tfrac{\mu(n-2)+1}{n-2}\mathrm{R}_{ij}|\nabla f|^2 + \tfrac{1}{n-2}\mathrm{R}_{00}|\nabla f|^4 g_{ij} - \tfrac{1}{(n-1)(n-2)}\mathrm{R}|\nabla f|^2 g_{ij} - \lambda\mu|\nabla f|^2 g_{ij}\,.
\end{aligned}$$

where in the second equality we have used the decomposition formula for the Riemann tensor (1.2) and the fact that the Weyl curvature part vanishes. □

Now, if we assume that the manifold is locally conformally flat, the Cotton tensor is identically zero. Locally around every point where $|\nabla f| \neq 0$, from Lemma 2.2, we obtain

$$\begin{aligned}
\mathrm{C}_{0j0} &= \nabla_0 \mathrm{R}_{j0} - \nabla_j \mathrm{R}_{00} - \tfrac{1}{2(n-1)}\big(\nabla_0 \mathrm{R}\, g_{j0} - \nabla_j \mathrm{R}\, g_{00}\big)\\
&= \mu|\nabla f|^2\mathrm{R}_{0j} + \tfrac{1-\mu}{n-1}|\nabla f|^2\,\mathrm{R}_{0j}\\
&= \tfrac{\mu(n-2)+1}{n-1}\,|\nabla f|^2\,\mathrm{R}_{0j}\,.
\end{aligned}$$

Hence if (M^n, g) is locally conformally flat, we have that $\mathrm{R}_{0j} = 0$ for every $j = 1, \dots, n-1$, hence also

$$\nabla_j \mathrm{R} = \nabla_j |\nabla f|^2 = 0\,, \tag{2.12}$$

where we have used again the previous lemma. Hence, one has

$$\begin{aligned}
\Gamma^0_{0j} &= \frac{1}{2}g^{l0}(\partial_0 g_{jl} + \partial_j g_{0l} - \partial_l g_{0j})\\
&= \frac{1}{2}g^{00}(\partial_0 g_{j0} + \partial_j g_{00}) = 0\,,\\
\Gamma^j_{00} &= \frac{1}{2}g^{lj}(\partial_0 g_{0l} + \partial_0 g_{0l} - \partial_l g_{00})\\
&= \frac{1}{2}g^{ij}(-\partial_i g_{00}) = 0\,,
\end{aligned}$$

since $\partial_j g_{00} = \partial_j(|\nabla f|^{-2}) = 0$. An easy computation shows that $\partial_j \mathrm{R}_{00} = 0$. Indeed,

$$\partial_j \mathrm{R}_{00} = \nabla_j \mathrm{R}_{00} + 2\Gamma^0_{j0}\mathrm{R}_{00} = \nabla_j \mathrm{R}_{00} = \nabla_0 \mathrm{R}_{j0} = \partial_0 \mathrm{R}_{0j} - \Gamma^i_{00}\mathrm{R}_{ij} - \Gamma^0_{0j}\mathrm{R}_{00} = 0\,,$$

where we used equations (2.10) and (2.12). Now we want to show that the mean curvature of the level set Σ_ρ is constant on the level set. We recall that, since $\nabla f/|\nabla f|$ is the unit normal vector to Σ_ρ, the second fundamental form h verifies

$$h_{ij} = -\frac{\nabla^2_{ij} f}{|\nabla f|} = \frac{\mathrm{R}_{ij} - \lambda g_{ij}}{|\nabla f|}\,, \tag{2.13}$$

for $i, j = 1, \dots, n-1$. Thus, the mean curvature H of Σ_ρ satisfies

$$\mathrm{H} = g^{ij}\,h_{ij} = \frac{\mathrm{R} - \mathrm{R}_{00}|\nabla f|^2 - (n-1)\lambda}{|\nabla f|}\,, \tag{2.14}$$

which clearly implies that the mean curvature is constant on Σ_ρ, since all the quantities on the right hand side do. Now we want to compute the components C_{ij0} of the Cotton tensor. Using equations (2.9), (2.11) and (2.12), we have

$$
\begin{aligned}
\mathrm{C}_{ij0} &= \nabla_0 \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{i0} - \tfrac{1}{2(n-1)}\big(\nabla_0 \mathrm{R}\, g_{ij} - \nabla_j \mathrm{R}\, g_{i0}\big)\\
&= \nabla_0 \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{i0} - \tfrac{1}{2(n-1)}\nabla_0 \mathrm{R}\, g_{ij}\\
&= \tfrac{\mu(n-2)+1}{n-2}\mathrm{R}_{ij}|\nabla f|^2 + \tfrac{1}{n-2}\mathrm{R}_{00}|\nabla f|^4 g_{ij} - \tfrac{1}{(n-1)(n-2)}\mathrm{R}|\nabla f|^2 g_{ij} - \lambda\mu|\nabla f|^2 g_{ij} +\\
&\quad -\tfrac{1}{2(n-1)}\Big[2(1-\mu)\,|\nabla f|^4 \mathrm{R}_{00} - 2(n-1)\mu\lambda|\nabla f|^2 + 2\mu \mathrm{R}|\nabla f|^2\Big] g_{ij}\\
&= \tfrac{\mu(n-2)+1}{n-2}\Big(\mathrm{R}_{ij} + \tfrac{1}{n-1}\mathrm{R}_{00}|\nabla f|^2 g_{ij} - \tfrac{1}{n-1}\mathrm{R}\, g_{ij}\Big)|\nabla f|^2\,.
\end{aligned}
$$

Finally, using the expression (2.13) and (2.14), we obtain

$$
\mathrm{C}_{ij0} = \tfrac{\mu(n-2)+1}{n-2}\big(h_{ij} - \tfrac{1}{n-1}\mathrm{H}\, g_{ij}\big)|\nabla f|^3\,.
$$

Again, since all the components of the Cotton tensor vanish and we are assuming that $\mu \neq \frac{1}{2-n}$, we obtain that

$$
h_{ij} = \tfrac{1}{n-1}\mathrm{H}\, g_{ij}\,. \tag{2.15}
$$

For any given $p \in \Sigma_\rho$, we suppose now to take orthonormal coordinates centered at p, still denoted by $\theta^1, \dots, \theta^{n-1}$. From the Gauss equation (see also [3, Lemma 3.2] for a similar argument), one can see that the sectional curvatures of (Σ_ρ, g_{ij}) at p with the induced metric g_{ij}, are given by

$$
\begin{aligned}
\mathrm{R}^{\Sigma}_{ijij} &= \mathrm{R}_{ijij} + h_{ii}h_{jj} - h_{ij}^2\\
&= \tfrac{1}{n-2}\big(\mathrm{R}_{ii} + \mathrm{R}_{jj}\big) - \tfrac{1}{(n-1)(n-2)}\mathrm{R} + \tfrac{1}{(n-1)^2}\mathrm{H}^2\\
&= \tfrac{2}{n-2}\mathrm{R}_{ii} - \tfrac{1}{(n-1)(n-2)}\mathrm{R} + \tfrac{1}{(n-1)^2}\mathrm{H}^2\\
&= \tfrac{2}{(n-1)(n-2)}\mathrm{H}|\nabla f| + \tfrac{2}{n-2}\lambda - \tfrac{1}{(n-1)(n-2)}\mathrm{R} + \tfrac{1}{(n-1)^2}\mathrm{H}^2\,,
\end{aligned}
$$

for $i, j = 1, \dots, n-1$, where in the second equality we made use of the decomposition formula for the Riemann tensor (1.2), the locally conformally flatness of g and of (2.15). Since all the terms on the right hand side are constant on Σ_ρ, we obtain that the sectional curvatures of (Σ_ρ, g_{ij}) are constant, which implies that (M^n, g) is locally a warped product metric with fibers of constant curvature. □

Remark 2.3. Consider the manifold $(M^n, \widetilde{g})$ with the conformal metric $\widetilde{g} = e^{-\frac{2}{n-2}f} g$. Since the locally conformally flat property is conformally invariant this is a still locally conformally flat metric, hence its Cotton tensor is zero. Thus, from equations (2.1) and (2.12) (this latter saying that the modulus of the gradient of f is constant along any regular level set of f), it follows that its Ricci tensor has only two eigenvalues of multiplicities one and $(n-1)$, which are constant along the level sets of f. Indeed,

$$
\begin{aligned}
\mathrm{Ric}_{\widetilde{g}} &= \mathrm{Ric}_g + \nabla^2 f + \tfrac{1}{n-2}df \otimes df + \tfrac{1}{n-2}\big(\Delta f - |\nabla f|^2\big)g\\
&= \Big(\tfrac{1}{n-2} + \mu\Big)df \otimes df + \tfrac{1}{n-2}\big(\Delta f - |\nabla f|^2 + (n-2)\lambda\big)e^{\frac{2}{n-2}f}\,\widetilde{g}\,.
\end{aligned}
$$

Then, arguing as in [6] by means of splitting results for manifolds admitting a Codazzi tensor with only two distinct eigenvalues, we can conclude that $(M^n, \widetilde{g})$ is locally a warped product with $(n-1)$–dimensional fibers of constant curvature which are the level sets of f.
By the structure of the conformal deformation this conclusion also holds for the original Riemannian manifold (M^n, g).

It is well known that, if (M^n, g) is a compact locally conformally flat gradient shrinking Ricci soliton, then it has constant curvature (see [8]). As pointed out to us by the anonymous referee such a conclusion cannot be extended to quasi–Einstein metrics. Indeed, C. Böhm in [2] has found Einstein metrics on $\mathbb{S}^{k+1} \times \mathbb{S}^l$ for $k, l \geq 2$ and $k + l \leq 8$ and these induce a quasi–Einstein metric on $\mathbb{S}^{k+1}$ with $\mu = \frac{1}{l}$ and with the metric on $\mathbb{S}^{k+1}$ being conformally flat (see also [10]).

In the complete, noncompact, case one would like to use Theorem 1.1 to have a classification of LCF quasi–Einstein manifolds (see [3] and [16] for steady and shrinking gradient Ricci solitons, respectively). Possibly one has to assume some curvature conditions as the nonnegativity of the curvature operator or of the Ricci tensor.

Acknowledgments . *We wish to thank Andrea Landi and Alessandro Onelli for several interesting comments on earlier versions of the paper.*
The first three authors are partially supported by the Italian project FIRB–IDEAS "Analysis and Beyond".

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