VARIATIONAL ANALYSIS IN ONE AND TWO DIMENSIONS OF A FRUSTRATED SPIN SYSTEM: CHIRALITY TRANSITIONS AND MAGNETIC ANISOTROPIC TRANSITIONS

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ABSTRACT. We study the energy of a ferromagnetic/antiferromagnetic frustrated spin system with values on two disjoint circumferences of the 3-dimensional unit sphere in a one-dimensional and two-dimensional domain. It consists on the sum of a term that depends on the nearest and next-to-nearest interactions and a penalizing term that counts the spin's magnetic anisotropy transitions. We analyze the asymptotic behaviour of the energy, that is when the system is close to the helimagnet/ferromagnet transition point as the number of particles diverges. In the one-dimensional setting we compute the Γ-limit of renormalizations of the energy at first and second order. As a result, it is shown how much energy the system spends for any magnetic anistropy transition and chirality transition. In the two-dimensional setting, by computing the Γ-limit of the renormalization of the energy at second order, we we prove the emergence and study the geometric rigidity of chirality transitions.

KEYWORDS: Γ-convergence; Frustrated lattice systems; Chirality transitions

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CONTENTS

Introduction

In this paper we study a frustrated lattice spin system with values on the unit sphere of $\mathbb{R}^3$. Its frustration is a biproduct of competition between ferromagnetic (F) nearest-neighbor (NN) and antiferromagnetic (AF) next-nearest-neighbor (NNN) interactions. The spin of the system u is a vectorial function that takes values in the union of two fixed disjoint circumferences see the figure 1, S_1 and S_2, on the unit sphere, which have the same radius R and are identified by two versors, v_1 and v_2. We set the problem in one and two dimension: in the one-dimensional case the spin field is parametrized over the points of the discrete set $\mathbb{Z}_n = \{i \in \mathbb{Z} : \lambda_n i \in \overline{I}\}$, that is a subset of the thorus $[0,1]$; in the two-dimensional case it is parametrized over the points of the discrete set $\Omega \cap \lambda_n \mathbb{Z}^2$, where $\Omega \subset \mathbb{R}^2$ is an open, bounded, regular domain. In both cases $\{\lambda_n\}_{n\in\mathbb{N}}$ is a vanishing sequence of lattice spaces. In the first setting, the energy of a given spin of the system $u \colon i \in \mathbb{Z}_n(I) \to S_1 \cup S_2$ is

$$\mathcal{E}_n(u) = E_n(u) + P_n(u),$$

with

$$E_n(u) = \sum_{i\in\mathbb{Z}_n(I)} \lambda_n \left[-\alpha u^i \cdot u^{i+1} + u^i \cdot u^{i+2}\right] \quad \text{and} \quad P_n(u) = \lambda_n k_n |D\mathcal{A}(u)|(I),$$

where $\alpha \in (0, +\infty)$ is the frustration parameter of the system that rules the NN and NNN interactions, k_n is a divergent sequence of positive numbers and $|D\mathcal{A}(u)|(I)$ counts the magnetic anistropy transitions that the spin u makes going from one circumference S_i to the other one S_j.

It is easy to see that while the first term of the energy E_n is ferromagnetic and favors the alignment of neighboring spins, the second one, being antiferromagnetic, frustrates it as it favors antipodal next-to-nearest neighboring spins. A more refined analysis, contained in Proposition 3.1 and Remark 3.2, shows that the ground state of the system takes value on one of the two circumferences and for $\alpha \geqslant 4$ is ferromagnetic (the spin is made up of alligned vectors), while for $0 < \alpha \leqslant 4$ it is helimagnetic (the spin consists in rotating vectors with a constant angle $\alpha = \pm \arccos(\alpha/4)$). The property of the latter case is known in literature as chirality simmetry: the two possible choices for the angle correspond to either clockwise or counterclockwise spin rotations, or in other words to a positive or a negative chirality.

In this paper, we address to a system close to the ferromagnet/helimagnet transition point, that is when α is close to 4. Our aim is to provide a careful description of the admissible states and compute their associated energy. In particular, we find the correct scalings to detect the following two fenomena: the spin's magnetic anistropy transitions and its chirality transitions that break the simmetry of minimal configurations.

In [23], the authors studied a one-dimensional ferromagnetic/antiferromagnetic frustrated spin chain with nearest and next-to-nearest interactions close to the helimagnet/ferromagnet transition point as the number of particles diverges. In that case the spin field is valued on the unitary 2d-circumference. The proposed model is very different from that one, where no anisotropy functional P_n was introduced. In that case the presence of periodic boundary conditions allowed to turn E_n into a Modica-Mortola type energy, whose Γ-convergence is well known in literature (see [32] and [33]). Indeed, expanding the functional at the first order, under a suitable rescaling, the spin system makes a chirality transition on a scale of order $\frac{\lambda_n}{\sqrt{\delta_n}}$, when $\frac{\lambda_n}{\sqrt{\delta_n}}$ approaches to a finite nonnegative value, as $n \to +\infty$ (otherwise no chirality transitions emerge).

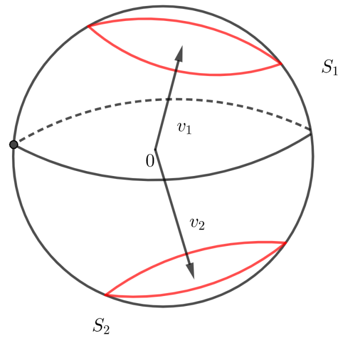

Figure 1. S_1 and S_2 circumferences of anisotropic transitions.

To set up our problem, we let the ferromagnetic interaction parameter α depend on n and be close to 4 from below, that is, we substitute α by $\alpha_n = 4(1-\delta_n)$ for some positive vanishing sequence $\{\delta_n\}_{n\in\mathbb{N}}$. As in [23], the Γ-limit of the energy $\mathcal{E}_n$ (with respect to the weak-star convergence in L^∞)) as $n \to +\infty$ does not provide a detailed description of the fenomena (as a consequence of Theorem 4.3) and suggests that, in order to get more information on the ground states of the system, we need to consider higher order Γ-limits (see [11] and [16]). Thus, we expand

$$\mathcal{E}_n = \min \mathcal{E}_n + \sum_j (R_n)_j + \sqrt{2}\lambda_n \delta_n^{\frac{3}{2}} \mathcal{H}^n,$$

where the functionals $(R_n)_j$ and $\mathcal{H}_n$ are defined in (5.11) and (5.14) respectively. The two fenomena can be detected at different orders and scales. At the first order we are led to normalize the energy of the system and study the asymptotic behavior of a new functional H_n defined as

$$H_n = \mathcal{E}_n - \min \mathcal{E}_n.$$

Rescaling H_n by λ_n, magnetic anisotropic transitions can be made by the spin on a scale of order $\lambda_n k_n$, for n large enough (see Theorem 5.2). In the figure 2 we can see the magnetic anisotropic transitions, i.e. the transitions between the two circumference. In order to continue the analysis at the next order, we can restrict every spin u on some intervals I_j that partition I such that on $\mathbb{Z}_n(I_j)$ it takes values only in one S_j. We need to modify such restrictions $u_{|I_j}$ in a way that they are well-connected on the boundary of the interval I_j, denotating them as $\widetilde{u}_{I_j}$. The functional H_n can be split in two terms:

$$H_n(u) = \sum_j MM_n(\widetilde{u}_{I_j}) + \sum_j (R_n)_j(u).$$

As long as we consider a remainder $R_{n,j}$ for each modification of the spin, the analysis of the global process can be localized in each S_j with the associated energy $MM_n(\widetilde{u}_{I_j})$. The two sums need to be rescaled in different ways, being the first sum a higher order term. Thus, at the second order we deal with the energy

$$\mathcal{H}_n(u) = H_n(u) - \sum_j (R_n)_j(u) = \sum_j MM_n(\widetilde{u}_{I_j}).$$

We transpose the problem valued in the 3d-sphere into a finite number of problems valued in 2d-circumferences with functionals MM_n of Modica-Mortola type, thus generalizing the result contained in [23]. In each S_j at scale $\lambda_n \delta_n^{3/2}$ several regimes are possible (see Theorem

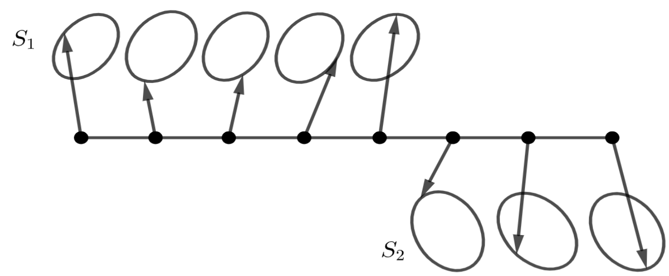

Figure 2. Magnetic anisotropic transitions.

5.4). For n large enough, the spin system makes a chirality transition on a scale of order $\lambda_n/\sqrt{\delta_n}$. As a result, depending on the value of $\lim_n \lambda_n/\sqrt{\delta_n} := l \in [0, +\infty]$ different scenarios may occur. If $l = +\infty$, chirality transitions are forbidden. If $l > 0$, the spin system may have diffuse and regular macroscopic (on an order one scale) chirality transitions in each S_j whose limit energy is finite on $H^1(I_j)$ (provided some boundary conditions are taken into account). When $l = 0$, transitions on a mesoscopic scale are allowed. In this case, the continuum limit energy is finite on $BV(I_j)$ and counts the number of jumps of the chirality of the system.

In the two-dimensional case, the analysis is more involved. The energy of a given spin of the system $u\colon i \in \Omega \cap \lambda_n\mathbb{Z}^2 \to S_1 \cup S_2$ is

$$H_n(u;\Omega) := \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\frac{1}{2}\lambda_n^2 \sum_{(i,j)\in\mathcal{I}^n(\Omega)} \left[\left|u^{i+2,j} - \frac{\alpha_n}{2}u^{i+1,j} + u^{i,j}\right|^2 + \left|u^{i,j+2} - \frac{\alpha_n}{2}u^{i,j+1} + u^{i,j}\right|^2\right],$$

under the assumption that the functional

$$P_n(u;\Omega) := \lambda_n k_n |D\mathcal{A}(u)|(\Omega),$$

is bounded, where the frustation parameter α_n of the system is close to the helimagnet/ferromagnet transition point as the number of particles diverges, k_n is a divergent sequence and $|D\mathcal{A}(u)|(I)$ counts the magnetic anistropy transitions, as in the previous case. Also in the two-dimensional setting we can repeat the construction and restrict every spin u on some open connected sets C_s that partition the set Ω in a way such that u takes values only in one circumference. In order to avoid more complicated notation, we do not impose boundary conditions on $\partial\Omega$ and we will state the result by means of a local convergence.

While in the one-dimensional setting the partition associated with a spin was made by intervals, which guaranty the compactness results stated, in this case the sets C_s could be very wild, as the spacing of the lattice shrinks. Therefore, we require an additional regularity conditions for the components C_s, that is the BVG regularity. Such conditions are satisfied by minimizing sequences, almost minimizing sequences and minimizing sequences under volume preserving hypotheses for the domain with the same anisotropy magnetization.

If the number of magnetic anisotropy transitions is finite, we can apply the Γ-convergence result proved in [21] in each component C_s for the functional

$$\mathcal{H}_n(u;\Omega) := H_n(u,\Omega) - \sum_s R_{nC_s}(u) = \sum_s H_n(u;C_s), \tag{0.1}$$

as it is shown in Theorem 6.3.

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1. Notation and preliminaries

Given $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the integer part of x. For a set K we denote with $co(K)$ the convex hull of K. We denote by $v \cdot w$ the euclidean scalar product of the vectors v, $w \in \mathbb{R}^3$ and by the S^2 the unit sphere of $\mathbb{R}^3$. For all $v \in \mathbb{R}^3$ we define the euclidean orthogonal projection on v as π_v and the projection on the orthogonal complement of v as $\pi_{v^\perp}$. Fixing $v_1, v_2 \in S^2$ and $R \in (0,1)$, we define the set

$$S_i := \left\{ w \in S^2 \,:\, |\pi_{v_i^\perp}(w)| = R,\, \langle w, v_i\rangle > 0 \right\}, \quad \text{for } i \in \{1,2\}. \tag{1.1}$$

It is easy to observe that the set S_i is a circumference centered in $v_i\sqrt{1-R^2}$ and It can be easily verified that for $R < R_{Max} := \sqrt{\frac{1-v_1\cdot v_2}{2}}$ the sets S_1 and S_2 are disjoint. Throughout the paper we will assume that $R \in (0, R_{Max})$. We call a collection $\mathcal{C}$ of subsets of a set S an open partition of an open S if $\mathcal{C}$ does not contain empty sets and

$$\overline{S} = \bigcup_{C \in \mathcal{C}} \overline{C}, \quad C_1 \cap C_2 = \varnothing, \quad \forall C_1, C_2 \in \mathcal{C}.$$

Let $w = (w^1, w^2)$, $\overline{w} = (\overline{w}^1, \overline{w}^2)$ two vector of $\mathbb{R}^2$, we define the function

$$\chi[w, \overline{w}] := \operatorname{sign}(w^1\overline{w}^2 - w^2\overline{w}^1).$$

For all $w \in BV(I, \mathbb{R}^3)$ we denote with $Dw \in \mathcal{M}_b(I, \mathbb{R}^3)$ the distributional differential of w, and with $|Dw| \in \mathcal{M}_b(I)$ the total variation measure of Dw. We say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset BV(I, \mathbb{R}^3)$ converge, as $n \to +\infty$, to a function $u \in BV(I, \mathbb{R}^3)$ in the weak star topology of BV if and only if

$$u_n \to u \quad \text{in } \mathrm{L}^1(I, \mathbb{R}^3) \qquad \text{and} \qquad Du_n \overset{*}{\rightharpoonup} Du \text{ in } \mathcal{M}_b(I, \mathbb{R}^3)$$

sometimes we denote it with $u_n \xrightarrow{BV} u$.

2. The variational problem in dimension one

2.1. **The setting.** We denote $I = (0,1)$ and we consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ that vanishes as $n \to +\infty$. It represents a sequence of spacings of the lattice $\mathbb{Z}_n(I) := \{i \in \mathbb{Z} \,:\, \lambda_n i \in \overline{I}\}$.
We introduce the class of functions valued in $S_1 \cup S_2$ which are piecewise constant on the edges of the lattice $\mathbb{Z}_n(I)$ that satisfy a joint boundary condition:

$$\mathcal{PC}_{\lambda_n} := \left\{ v \colon I \to S_1 \cup S_2 \,:\, v(t) = v(\lambda_n i) \text{ for } t \in \lambda_n[i + [0,1)],\, v^0 \cdot v^1 = v^{\left\lfloor \frac{1}{\lambda_n} \right\rfloor - 1} \cdot v^{\left\lfloor \frac{1}{\lambda_n} \right\rfloor} \right\}.$$

We will identify a piecewise function $v\colon I \to S_1 \cup S_2$ with the function defined on the points of the lattice given by $i \in \mathbb{Z}_n(I) \mapsto v^i := v(\lambda_n i)$. Conversely, given values $v^i \in S_1 \cup S_2$ for $i \in \mathbb{Z}_n(I)$, we define $v\colon I \to S_1 \cup S_2$ by $v(t) := v^i$ for $t \in \lambda_n[i + [0,1)]$.
There exists a natural projection map $\mathcal{A}\colon \mathcal{PC}_{\lambda_n} \to \{v_1, v_2\}$ defined as follows

$$\mathcal{A}(u(t)) = \begin{cases} v_1 & \text{if } u(t) \in S_1, \\ v_2 & \text{otherwise.} \end{cases} \tag{2.1}$$

Furthermore, we observe that if $u \in \mathcal{PC}_{\lambda_n}$, the interval I can be partitioned in regions where the function u takes value only in one of the two circumferences. In other words, there exist $M(u) \in \mathbb{N}$ and a collection of open intervals, $\{I_j^d\}_{j\in\{1,\dots,M(u)\}}$, such that

$$\{I_j^d\}_{j\in\{1,\dots,M(u)\}} \text{ is an open partition of } I, \tag{2.2}$$

$$u(t) \in S_d, \quad \forall t \in I_j^d,\ \forall j \in \{1,\dots,M(u)\}. \tag{2.3}$$

These two properties imply that this partition is unique. We observe that

$$M(u) = \frac{|D\mathcal{A}(u)|(I)}{|v_1 - v_2|} + 1.$$

The following definition will be useful throughout the section.

Definition 2.1. Let $u \in \mathcal{PC}_{\lambda_n}$. We will say that $\mathcal{C}_n(u) = \{I_j^d \,|\, j \in \{1,\dots,M(u)\}\}$ is the open partition associated with u if $M(u) = \frac{|D\mathcal{A}(u)|(I)}{|v_1-v_2|} + 1$ and the collection of open intervals $\{I_j^d\}_{j\in\{1,\dots,M(u)\}}$ satisfies (2.2) and (2.3).

2.2. **A useful abstract result.** Now we cite an abstract Γ-convergence result proved in [3] that will be applied in Section 4. For this purpose, we introduce the following notation. Let $K \subset \mathbb{R}^N$ be a compact set and for all $\xi \in \mathbb{Z}$ let $f^\xi\colon \mathbb{R}^{2N} \to \mathbb{R}$ be a function such that

- (**H1**) $f^\xi(x,y) = f^{-\xi}(y,x)$,
- (**H2**) for all $\xi \in \mathbb{Z}$, $f^\xi(x,y) = +\infty$ if $(x,y) \notin K^2$,
- (**H3**) for all $\xi \in \mathbb{Z}$ there exists $C^\xi \geqslant 0$ such that

$$\sup_{(x,y)\in K^2} |f^\xi(x,y)| \leqslant C^\xi \quad \text{and} \quad \sum_{\xi\in\mathbb{Z}} C^\xi < +\infty.$$

For any $n \in \mathbb{N}$ we define the functional space

$$D_n(I,\mathbb{R}^N)\colon = \big\{u\colon \mathbb{R} \to \mathbb{R}^N : u \text{ is constant in } \lambda_n(i + [0,1)) \text{ for all } i \in \mathbb{Z}_n(I)\big\}$$

and the sequence of functions $F_n\colon \mathrm{L}^\infty(I,\mathbb{R}^N) \to (-\infty,+\infty]$ as follows:

$$F_n(u) := \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}} \sum_{i\in R_n^\xi(I)} \lambda_n f^\xi(u^i, u^{i+\xi}) & \text{for } u \in D_n(I,\mathbb{R}^N)\,, \\ +\infty & \text{for } u \in \mathrm{L}^\infty(I,\mathbb{R}^N)\backslash D_n(I,\mathbb{R}^N)\,, \end{cases} \tag{2.4}$$

where $R_n^\xi(I) := \{i \in \mathbb{Z}_n(I) : i + \xi \in \mathbb{Z}_n(I)\}$. For any open and bounded set $A \subset \mathbb{R}$ and for every $v\colon \mathbb{Z} \to \mathbb{R}^N$, we define the discrete average of v in A as

$$(v)_{1,A} := \frac{1}{\#(\mathbb{Z}\cap A)} \sum_{i\in\mathbb{Z}\cap A} v^i.$$

Theorem 2.2 (See [3])**.** *Let $\{f^\xi\}_{\xi\in\mathbb{Z}}$ a family of functions that satisfies **H1**, **H2**, **H3**. Then the sequence F_n converges, as $n \to +\infty$ with respect to the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^N)$, in the sense of the Γ-convergence to*

$$F(u) := \begin{cases} \int_I f_{hom}(u(t))\, dt & \text{for } u \in \mathrm{L}^\infty(I, co(K)), \\ +\infty & \text{for } u \in \mathrm{L}^\infty(I,\mathbb{R}^N)\backslash \mathrm{L}^\infty(I, co(K)), \end{cases}$$

where $f_{hom}\colon \mathbb{R}^N \to \mathbb{R}$ is given by the following homogenization formula

$$f_{hom}(z) = \lim_{\rho\to 0}\lim_{k\to+\infty}\frac{1}{k}\inf\left\{\sum_{\xi\in\mathbb{Z}}\sum_{\beta\in R_1^\xi((0,k))} f^\xi(v(\beta), v(\beta+\xi)) \,:\, (v)_{1,(0,k)} \in \overline{B(z,\rho)}\right\}.$$

3. The energy model

Let $\alpha > 0$ be a fixed parameter and let $\{k_n\}_{n\in\mathbb{N}} \in \mathbb{R}^+$ be a divergent sequence of positive numbers. Denoting

$$\mathcal{I}^n(I) := \mathbb{Z}^n(I)\backslash\left\{\left\lfloor\frac{1}{\lambda_n}\right\rfloor - 1, \left\lfloor\frac{1}{\lambda_n}\right\rfloor\right\},$$

we define the energy of the system as the sum of two addends. The first addend is a bulk scaled energy of a frustrated F-AF spin chain, $E_n\colon \mathcal{PC}_{\lambda_n} \to (-\infty,+\infty)$, having the following form:

$$E_n(u) := \lambda_n \sum_{i\in\mathcal{I}^n(I)} \left[-\alpha u^i \cdot u^{i+1} + u^i \cdot u^{i+2}\right]. \tag{3.1}$$

The second addend of the energy, $P_n\colon \mathcal{PC}_{\lambda_n} \to [0,+\infty)$, is a term of confinement in $S_1 \cup S_2$ and is defined as follows:

$$P_n(u) := \lambda_n k_n \left|D\mathcal{A}(u)\right|(I),$$

where $\mathcal{A}$ is the function defined in formula (2.1). We consider the family of energies $\mathcal{E}_n\colon \mathcal{PC}_{\lambda_n} \to (-\infty,+\infty)$ defined as

$$\mathcal{E}_n(u) = E_n(u) + P_n(u).$$

Furthermore, we define the functional $H_n\colon \mathcal{PC}_{\lambda_n} \to [0,+\infty)$ as

$$H_n(u) := \frac{1}{2}\lambda_n \sum_{i\in\mathcal{I}^n(I)} \left|u^{i+2} - \frac{\alpha}{2}u^{i+1} + u^i\right|^2.$$

With a slight abuse of notation, we extend the energies E_n, P_n, $\mathcal{E}_n$ and H_n to the space $\mathrm{L}^\infty(I,\mathbb{R}^3)$, setting their value as $+\infty$ in the space $\mathrm{L}^\infty(I,\mathbb{R}^3)\backslash\mathcal{PC}_{\lambda_n}$. If $u \in \mathcal{PC}_{\lambda_n}$, since $|u^i| = 1$ for all $i \in \mathbb{Z}_n(I)$, thanks to the boundary condition contained in the definition of $\mathcal{PC}_{\lambda_n}$, we can rewrite the energy $\mathcal{E}_n$ in terms of H_n as

$$\mathcal{E}_n = H_n + P_n - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I). \tag{3.2}$$

Thanks to this decomposition, we characterize the ground states of E_n.

Proposition 3.1. *Let $0 < \alpha \leqslant 4$. Then*

$$\min_{u\in\mathrm{L}^\infty(I,\mathbb{R}^3)} \mathcal{E}_n(u) = -\lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I).$$

Furthermore, a minimizer u_n of E_n over $\mathrm{L}^\infty(I,\mathbb{R}^3)$ takes values only in one of the two sets S_1 or S_2 and satisfies

$$u_n^i \cdot u_n^{i+1} = \frac{\alpha}{4} \quad \textit{and} \quad u_n^i \cdot u_n^{i+2} = \frac{\alpha^2}{8} - 1, \quad \forall i \in \left\{0, \dots, \left\lfloor \frac{1}{\lambda_n} \right\rfloor - 2\right\}. \tag{3.3}$$

Proof. By formula (3.2), since H_n and P_n are nonnegative, we infer

$$\mathcal{E}_n = H_n + P_n - \lambda_n \left(1 + \frac{\alpha^2}{8}\right) \#\mathcal{I}^n(I) \geqslant -\lambda_n \left(1 + \frac{\alpha^2}{8}\right) \#\mathcal{I}^n(I).$$

Therefore if there exists some function that nullifies H_n and P_n, then a function $u \in \mathrm{L}^\infty(I,\mathbb{R}^3)$ verifies $H_n(u) = P_n(u) = 0$ if and only if u is a minimizer of $\mathcal{E}_n$. By [23, Proposition 3.2] every minimizer $u \in \mathcal{PC}_{\lambda_n}$ of H_n satisfies $H_n(u) = P_n(u) = 0$, verifies (3.3) and takes values only in one of the two sets S_1 and S_2. Hence the thesis is obtained. □

Remark 3.2. The case $\alpha > 4$ is trivial. Indeed, the ground states are all ferromagnetic, i.e. $u_n^i = \bar{u} \in S_1 \cup S_2$ for all $i \in \mathbb{Z}_n(I)$. Denoting with $\mathcal{E}_n^{(\alpha=4)}$ the energy of formula (3.2) for $\alpha = 4$, we have that for all $u \in \mathcal{PC}_{\lambda_n}$

$$\mathcal{E}_n(u) = \mathcal{E}_n^{(\alpha=4)}(u) - \lambda_n(\alpha - 4) \sum_{i \in \mathcal{I}^n(I)} u^i \cdot u^{i+1}.$$

By the above proposition, the energy $\mathcal{E}_n^{(\alpha=4)}$ is minimized on ferromagnetic states, which trivially also holds true for the second term in the above sum. The minimal value of $\mathcal{E}$ is

$$\min_{u \in \mathrm{L}^\infty(I,\mathbb{R}^3)} \mathcal{E}_n(u) = -(\alpha - 1)\#\mathcal{I}^n(I).$$

4. Zero order Γ-convergence of the energy $\mathcal{E}_n$

In this section we study the Γ-convergence of the energy $\mathcal{E}_n$ defined in (3.2) at the zero order. In view of this aim we will, consider the space

$$\mathfrak{D} := \left\{u \in \mathrm{L}^\infty(I, co(S_1) \cup co(S_2)) \, : \, \exists \mathcal{C}_n(u) \text{ finite open partition associated with } u\right\}, \tag{4.1}$$

see 2.1 for the definition of $\mathcal{C}_n$. We observe that $\mathcal{A}(\mathfrak{D}) = BV(I, \{v_1, v_2\})$, where $\mathcal{A}$ is the function defined in 2.1. The following convergence law will be used.

Definition 4.1 (Convergence law)**.** Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}$ and $f \in \mathfrak{D}$. We say that f_n L-converges to f (we write $f_n \xrightarrow{L} f \in \mathfrak{D}$) as $n \to +\infty$ if and only if $f_n \overset{*}{\rightharpoonup} f$ in the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^3)$ and $\mathcal{A}(f_n) \to \mathcal{A}(f)$ in the weak-star topology of $BV(I; \{v_1, v_2\})$, as $n \to +\infty$.

Remark 4.2. We observe that the convergence law introduced in the definition above is induced by the topology on $\mathfrak{D}$ defined as the smaller topology containing the set

$$\{A : A \text{ is open set of weak-star topology of } \mathrm{L}^\infty \text{ or of the } BV(I, \{v_1, v_2\}) \text{ topology}\}.$$

Firstly, we study the Γ-convergence of the energy E_n. The following theorem relies on a straightforward application of Theorem 2.2.

Theorem 4.3. *The sequence E_n converges in the sense of the Γ-convergence to the functional*

$$E(u) := \begin{cases} \int_I f_{hom}(u(t))\,dt & \text{if } u \in \mathrm{L}^\infty(I, co(S_1 \cup S_2)), \\ +\infty & \text{otherwise}, \end{cases}$$

with respect to the weak-star topology of $\mathrm{L}^\infty(I;\mathbb{R}^3)$, where $f_{hom}\colon co(S_1 \cup S_2) \to \mathbb{R}$ is defined as

$$f_{hom}(z) = \lim_{\rho\to 0}\lim_{k\to+\infty}\frac{1}{k}\inf\left\{\sum_{i=1}^{k-2}\left[-\alpha u^i\cdot u^{i+1} + u^i\cdot u^{i+2}\right] \,:\, (u)_{1,(0,k)} \in \overline{B(z,\rho)}\right\}. \tag{4.2}$$

Proof. The result immediately follows applying Theorem 2.2 to

$$f^\xi(u,v) = \begin{cases} -\frac{\alpha}{2}u\cdot v & \text{if } |\xi| = 1, \\ \frac{1}{2}u\cdot v & \text{if } |\xi| = 2, \\ 0 & \text{otherwise}, \end{cases}$$

where $u, v \in K := S_1 \cup S_2$, extended to $+\infty$ outside K. □

Remark 4.4. The function f_{hom} defined in (4.2) does not depend on the parameter λ_n. Therefore, in the theorem above the Γ-limit does not depend on the choice of λ_n.

Remark 4.5. An analogous statement of Theorem 4.3 above can be obtained if the functional E_n is defined only in $\mathrm{L}^\infty(I, S_i)$ for some $i \in \{1,2\}$ (see [23, Theorem 3.4]). The Γ-limit has the same form and it is finite on $\mathrm{L}^\infty(I, co(S_i))$.

The following theorem is the main result of this section.

Theorem 4.6. *Assume that there exists $\lim_{n\to+\infty}\lambda_n k_n =: \eta \in (0,+\infty]$. Then the following Γ-convergence and compactness results hold true.*

(i) *If $\eta \in (0,+\infty)$ then $\mathcal{E}_n$ converges in the sense of Γ-convergence to the functional*

$$\mathcal{E}(u) = \begin{cases} \int_I f_{hom}(u(t))\,dt + \eta|D\mathcal{A}(u)|(I) & \text{if } u \in \mathfrak{D}, \\ +\infty & \text{otherwise}, \end{cases}$$

with respect to the L-convergence of Definition 4.1, where f_{hom} is defined in (4.2) *and $\mathfrak{D}$ is the set defined in* (4.1). *Moreover if $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I,\mathbb{R}^3)$ satisfies*

$$\sup_{n\in\mathbb{N}}\mathcal{E}_n(u_n) < +\infty,$$

then, up to a subsequence, $u_n \xrightarrow{L} u \in \mathfrak{D}$.

(ii) *If $\eta = +\infty$ then E_n converges in the sense of Γ-convergence to the functional*

$$\mathcal{E}(u) := \begin{cases} \int_I f_{hom}(u(t))\,dt & \text{if } u \in \mathrm{L}^\infty(I, co(S_1)) \text{ or } u \in \mathrm{L}^\infty(I, co(S_2)), \\ +\infty & \text{otherwise}, \end{cases}$$

with respect to the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^3)$, where f_{hom} is defined in (4.2). *Moreover for all $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I,\mathbb{R}^3)$ such that*

$$\sup_{n\in\mathbb{N}}\mathcal{E}_n(u_n) < +\infty$$

then, up to a subsequence, $u_n \overset{*}{\rightharpoonup} u$ for some $u \in \mathrm{L}^\infty(I, co(S_1))$ or $u \in \mathrm{L}^\infty(I, co(S_2))$.

Proof. (i) We start to prove the compactness result. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I,\mathbb{R}^3)$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < H, \tag{4.3}$$

for some $H > 0$. Thus we have that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$. Moreover, by the definition of the space $\mathcal{PC}_{\lambda_n}$, we have that for all u_n there exists a finite open partition $\mathcal{C}_n(u_n) = \{(I^d_j)_n \,|\, j \in \{1,\dots,M(u_n)\}\}$ associated with u_n, where $M(u_n) - 1 = \frac{|D\mathcal{A}(u_n)|(I)}{|v_1 - v_2|} \in \mathbb{N}$. By formula (3.2) and by the definition of the function $\mathcal{A}$, we compute

$$\begin{aligned}
\mathcal{E}_n(u_n) &= H_n(u_n) + P_n(u_n) - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \geqslant P_n(u_n) - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\
&= k_n\lambda_n\,|D\mathcal{A}(u_n)|\,(I) - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\
&= k_n\lambda_n(M(u_n) - 1)|v_1 - v_2| - \lambda_n\left(1 + \frac{\alpha^2}{8}\right)\#\mathcal{I}^n(I) \\
&\geqslant -C(\alpha) + k_n\lambda_n(M(u_n) - 1)|v_1 - v_2|,
\end{aligned} \tag{4.4}$$

for some constant $C = C(\alpha) > 0$, where the last inequality is obtained by observing that $\lambda_n\#\mathcal{I}^n(I) = \lambda_n\left\lfloor\frac{1}{\lambda_n}\right\rfloor - \lambda_n \to 1$ as $n \to +\infty$ and thus it is bounded. Therefore by formulas (4.3) and (4.4) we obtain that

$$\sup_{n\in\mathbb{N}} M(u_n) < C(\eta, H, \alpha, |v_1 - v_2|).$$

Hence the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the hypotheses of the Corollary A.3, and so we deduce the existence of $u \in \mathfrak{D}$ such that, up to a subsequence, $u_n \xrightarrow{L} u$.

Now we prove the liminf inequality. Let $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \xrightarrow{L} u \in \mathfrak{D}$ as $n \to +\infty$, i.e. $u_n \overset{*}{\rightharpoonup} u$ in L^∞ and $\mathcal{A}(u_n) \to \mathcal{A}(u)$ in $BV(I,\{v_1,v_2\})$. By the liminf inequality of the Theorem 4.3 we have

$$\liminf_{n\to+\infty} E_n(u_n) \geqslant \int_I f_{hom}(u(t))\,dt. \tag{4.5}$$

On the other hand, by the lower semi continuity of the total variation respect the convergence in $BV(I,\{v_1,\,v_2\})$ we have

$$\liminf_{n\to+\infty} P_n(u_n) = \liminf_{n\to+\infty} k_n\lambda_n|D\mathcal{A}(u_n)|(I) \geqslant \eta|D\mathcal{A}(u)|(I). \tag{4.6}$$

Hence by formulas (4.5) and (4.6) we obtain

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) \geqslant \liminf_{n\to+\infty} E_n(u_n) + \liminf_{n\to+\infty} P_n(u_n) \geqslant \int_I f^0_{hom}(t)\,dt + \eta|D\mathcal{A}(u)|.$$

We finally prove the limsup inequality. Let $u \in \mathrm{L}^\infty(I, co(S_1) \cup co(S_2))$. We can assume that $u \in \mathfrak{D}$ and furthermore, by a standard density argument and the locality of the construction, we can assume that

$$u(t) = \begin{cases} a_1 & \text{if } t \in [0,\frac{1}{2}], \\ a_2 & \text{if } t \in (\frac{1}{2},1], \end{cases}$$

where $a_1 \in co(S_1)$ and $a_2 \in co(S_2)$. Let $\{v^j_n\}_{n\in\mathbb{N}} \in \mathrm{L}^\infty(I,S_j)$ be the recovery sequence for the constant function a_j obtained by the Γ-convergence result in Remark 4.5 with $2\lambda_n$ as the spacing of the lattice (see Remark 4.4), i.e.

$$f_{hom}(a_j) = \lim_{n\to+\infty} E_n(v_n^j) = \lim_{n\to+\infty} 2\lambda_n \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor -2} \left[-\alpha(v_n^j)^i \cdot (v_n^j)^{i+1} + (v_n^j)^i \cdot (v_n^j)^{i+2}\right]. \tag{4.7}$$

We define

$$u_n(t) = \begin{cases} v_n^1(2t) & \text{if } t \in [0, \frac{1}{2}], \\ v_n^2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \tag{4.8}$$

and compute

$$\begin{aligned} E_n(u_n) =& \frac{1}{2} \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor -2} 2\lambda_n \left[-\alpha(v_n^1)^i \cdot (v_n^1)^{i+1} + (v_n^1)^i \cdot (v_n^1)^{i+2}\right] \\ &+ \frac{1}{2} \sum_{i=1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor -2} 2\lambda_n \left[-\alpha(v_n^2)^i \cdot (v_n^2)^{i+1} + (v_n^2)^i \cdot (v_n^2)^{i+2}\right] \\ &+ \sum_{i=\left\lfloor \frac{1}{2\lambda_n}\right\rfloor -1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor} \lambda_n \left[-\alpha u_n^i \cdot u_n^{i+1} + u_n^i \cdot u_n^{i+2}\right]. \end{aligned} \tag{4.9}$$

We observe that

$$\left| \sum_{i=\left\lfloor \frac{1}{2\lambda_n}\right\rfloor -1}^{\left\lfloor \frac{1}{2\lambda_n}\right\rfloor} \lambda_n \left[-\alpha u_n^i \cdot u_n^{i+1} + u_n^i \cdot u_n^{i+2}\right] \right| \leqslant C(\alpha)\lambda_n \to 0, \text{ as } n \to +\infty. \tag{4.10}$$

By formulas (4.7), (4.9), (4.10), we obtain that

$$E_n(u_n) \to \frac{f_{hom}(a_1) + f_{hom}(a_2)}{2} = \int_I f_{hom}(u(t))\, dt, \quad \text{as } n \to +\infty. \tag{4.11}$$

Observing that for all $n \in \mathbb{N}$

$$\mathcal{A}(u_n)(t) = \mathcal{A}(u)(t) = \begin{cases} v_1 & \text{if } t \in [0, \frac{1}{2}], \\ v_2 & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

then $|D\mathcal{A}(u_n)|(I) = |D\mathcal{A}(u)|(I) = |v_1 - v_2|$ and

$$\lim_{n\to+\infty} P_n(u_n) = \lim_{n\to+\infty} \lambda_n k_n |v_1 - v_2| = \eta |v_1 - v_2|. \tag{4.12}$$

Combining (4.11) and (4.12), we deduce the limsup inequality.
(ii) Let us prove first the compactness result. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < H,$$

for some constant $H > 0$. With the same compactness argument used in the previous case, we deduce the existence of $u \in \mathfrak{D}$ such that $u_n \xrightarrow{L} u$ as $n \to +\infty$. By the lower semicontinuity of the map

$$u \to |D\mathcal{A}(u)|(I)$$

with respect to the L-convergence (see Definition 4.1), we get

$$0 = \liminf_{n\to+\infty} \frac{H}{\lambda_n k_n} > \liminf_{n\to+\infty} \frac{1}{\lambda_n k_n} \left[E_n(u_n) + \lambda_n k_n |D\mathcal{A}(u_n)|(I)\right]$$
$$\geqslant \liminf_{n\to+\infty} \left(\frac{C(\alpha)}{\lambda_n k_n} + |D\mathcal{A}(u_n)|(I)\right) \geqslant |D\mathcal{A}(u)|(I),$$

hence $u \in \mathrm{L}^\infty(I, co(S_1)) \cup \mathrm{L}^\infty(I, co(S_2))$.
Let us prove the liminf inequality. Let $u_n \overset{*}{\rightharpoonup} u$ and suppose that

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) < +\infty.$$

Up to the extraction of a subsequence, we can assume that the previous liminf is actually limit. By compactness we infer that $u_n \xrightarrow{L} u \in \mathrm{L}^\infty(I, co(S_1)) \cup \mathrm{L}^\infty(I, co(S_2))$. Hence, by Theorem 4.3, we obtain

$$\liminf_{n\to+\infty} \mathcal{E}_n(u_n) \geqslant \liminf_{n\to+\infty} E_n(u_n) \geqslant \int_I f_{hom}(u(t))\, dt.$$

We finally prove the limsup inequality. Let $u \in \mathrm{L}^\infty(I, co(S_1))$, being the case $u \in \mathrm{L}^\infty(I, co(S_2))$ fully analogous. The recovery sequence obtained from Remark 4.5, $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I, S_1)$, satisfies the limsup inequality. □

5. First and second order Γ-convergence of the energy $\mathcal{E}_n$

In this section we study the system when it is close to the helimagnet/ferromagnet transition point as the number of particles diverges. In what follows we will denote $\alpha_n := 4(1 - \delta_n)$, where $\delta_n > 0$ and $\delta_n \to 0$ as $n \to +\infty$.
Regarding the notation, in this section we denote with d a number that can be 1 or 2. This number stands for the index of the two circumferences S^1, S^2.

5.1. First order Γ-convergence of the energy $\mathcal{E}_n$.

We define the renormalized energy and introduce a new functional whose asymptotic behavior will better describe the ground states of the system. More precisely we define

$$\mathcal{E}_n^{hf} : \mathrm{L}^\infty(I, \mathbb{R}^3) \to (-\infty, +\infty] \quad \text{and} \quad H_n^{hf} : \mathrm{L}^\infty(I, \mathbb{R}^3) \to [0, +\infty]$$

as:

$$E_n^{hf}(u) := \begin{cases} \lambda_n \displaystyle\sum_{i\in\mathcal{I}^n(I)} \left[-\alpha_n u^i \cdot u^{i+1} + u^i \cdot u^{i+2}\right] & \text{if } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1}$$

$$H_n^{hf}(u) := \begin{cases} \dfrac{1}{2}\lambda_n \displaystyle\sum_{i\in\mathcal{I}^n(I)} \left|u^{i+2} - \frac{\alpha_n}{2} u^{i+1} + u^i\right|^2 & \text{if } u \in \mathcal{PC}_{\lambda_n}, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

By Proposition 3.1 we observe that

$$H_n^{hf}(u) = E_n^{hf}(u) - \min E_n^{hf}(u) = E_n^{hf}(u) + \lambda_n \left(1 + \frac{\alpha_n^2}{8}\right) \#\mathcal{I}^n(I).$$

At this point we need to introduce a modified spin chain in order to understand better the asymptotic behaviour of the renormalized energy H_n^{hf}. Let $u \in \mathcal{PC}_{\lambda_n}$ and $\mathcal{C}_n(u) = \{I_j^d \mid j \in \{1, \dots, M(u)\}\}$ the open partition associated with u, with $\overline{I}_j^d = [(t_j^d)_1, (t_j^d)_2]$. Henceforth omits to write superscript in the formulas. We define the auxiliary function $\widetilde{u}_{I_j} : \overline{I}_j = [(t_j)_1, (t_j)_2] \to S_d$ as

$$\widetilde{u}_{I_s}(t) = \begin{cases} u_{|[(t_j)_1,(t_j)_2)}(t) & \text{if } t \in [(t_j)_1, (t_j)_2), \\ w_j & \text{if } t = (t_s)_2, \end{cases} \tag{5.3}$$

where $w_j \in S_d$ is a vector such that the following joint boundary condition is satisfied:

$$u^{\frac{(t_j)_2}{\lambda_n}-1} \cdot w_j = u^{\frac{(t_j)_1}{\lambda_n}} \cdot u^{\frac{(t_j)_1}{\lambda_n}+1}.$$

We split the energy H_n^{hf} as follows:

$$H_n^{hf}(u) = E_n^{hf}(u) - \min E_n^{hf}(u) = \sum_{j=1}^{M(u)} MM_n(\widetilde{u}_{I_j}) + \sum_{j=1}^{M(u)-1} (R_n)_j(u), \quad \forall u \in \mathcal{PC}_{\lambda_n}, \tag{5.4}$$

where

$$MM_n(\widetilde{u}_{I_j}) = \lambda_n \sum_{i \in \mathcal{I}^n(I_j)} \left[-\alpha_n \widetilde{u}^i_{I_j} \cdot \widetilde{u}^{i+1}_{I_j} + \widetilde{u}^i_{I_j} \cdot \widetilde{u}^{i+2}_{I_j}\right] + \lambda_n \left(1 + \frac{\alpha_n^2}{8}\right) \#\mathcal{I}^n(I)|I_j| \tag{5.5}$$

is the local energy associated with the modified spin chain $\widetilde{u}_{I_j}$,

$$\begin{aligned}(R_n)_j(u) := -\lambda_n \Bigg[&- \alpha_n u^{\frac{(t_j)_2}{\lambda_n}-1} \cdot u^{\frac{(t_j)_2}{\lambda_n}} + u^{\frac{(t_j)_2}{\lambda_n}-2} \cdot u^{\frac{(t_j)_2}{\lambda_n}} + u^{\frac{(t_j)_2}{\lambda_n}-1} \cdot u^{\frac{(t_j)_2}{\lambda_n}+1} \\ &+ u^{\frac{(t_j)_2}{\lambda_n}-2} \cdot w_j \Bigg] + \frac{\lambda_n}{M(u)-1} \left[u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}-2} \cdot u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}} - u^{\frac{\left(t_{M(u)}\right)_2}{\lambda_n}-2} \cdot w_{M(u)} \right],\end{aligned} \tag{5.6}$$

is the remainder for each modification. Note that R_n consists of two addends: the first sum is related to the interactions between spins with values in two neighboring intervals I_j and I_{j+1}, for $j \in \{1, \dots, M(u)-1\}$, while the second sum refers to the last interval $I_{M(u)}$.

Remark 5.1. For all $u \in \mathcal{PC}_{\lambda_n}$ we have that $MM_n(\widetilde{u}_{I_j}) \geqslant 0$ for all $j = 1, \dots, M(u)$, see [23, Theorem 4.2].

Theorem 5.2. *Assume that there exists* $\lim_{n\to+\infty} \lambda_n k_n =: \eta \in (0, +\infty)$ *and let*

$$R := \inf\left\{ \liminf_{n\to+\infty} \frac{(R_n)_j(u_n)}{\lambda_n} \,:\, \mathcal{A}(u_n) \xrightarrow{BV} v_1\chi_{(0,\frac{1}{2})} + v_2\chi_{[\frac{1}{2},1)} \right\}.$$

Then the following compactness and Γ*-convergence results hold true.*

(i) *(Compactness) If for* $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I, \mathbb{R}^3)$ *there exists* $C > 0$ *independent of* n *such that*

$$H_n^{hf}(u_n) \leqslant \lambda_n P_n(u_n) \leqslant \lambda_n C \tag{5.7}$$

then, up to subsequence, $\mathcal{A}(u_n) \to v \in BV(I; \{v_1, v_2\})$ *as* $n \to +\infty$ *in the weak-star topology of* $BV(I; \{v_1, v_2\})$.

(ii) *(liminf inequality) For all* $v \in BV(I; \{v_1, v_2\})$ *and for all* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ *such that*

$$\mathcal{A}(u_n) \to v, \quad \text{as } n \to +\infty \text{ in the weak-star topology of } BV(I; \{v_1, v_2\}),$$

and

$$H_n^{hf}(u_n) \leqslant \lambda_n P_n(u_n) \leqslant \lambda_n C, \quad \text{for some } C > 0,$$

then

$$\liminf_{n\to+\infty}\frac{H_n^{hf}(u_n)}{\lambda_n}\geqslant R\frac{|Dv|(I)}{|v_1-v_2|}.$$

(iii) *(limsup inequality) For all $v\in BV(I;\{v_1,v_2\})$ there exists $\{u_n\}_{n\in N}\subset\mathcal{PC}_{\lambda_n}$ such that*

$$\mathcal{A}(u_n)\to v,\quad \text{as } n\to+\infty \text{ in the weak-star topology of } BV(I;\{v_1,v_2\}),$$

and

$$H_n^{hf}(u_n)\leqslant\lambda_n P_n(u_n)\leqslant\lambda_n C,\quad \text{for some } C>0,$$

satisfying

$$\lim_{n\to+\infty}\frac{H_n^{hf}(u_n)}{\lambda_n}=R\frac{|Dv|(I)}{|v_1-v_2|}.$$

Proof. We start proving (i). Since $\eta\in(0,+\infty)$, by the second inequality of formula (5.7) we deduce that the sequence $\{|D\mathcal{A}(u_n)|(I)\}_{n\in\mathbb{N}}$ is bounded and so the sequence $\{\mathcal{A}(u_n)\}_{n\in\mathbb{N}}$ is bounded in the space $BV(I;\{v_1,v_2\})$ (see Proposition A.3). Thus, up to subsequence, it converges to a function $v\in BV(I;\{v_1,v_2\})$ in the weak-star topology of $BV(I;\{v_1,v_2\})$.
Let us prove (ii). By assumption, $\{D\mathcal{A}(u_n)(I)\}_{n\in\mathbb{N}}$ is bounded. Let $\mathcal{C}_n(u_n)=\{(I_j^d)_n\,|\,j\in\{1,\dots,M(u_n)\}\}$ the open partition associated with u_n. By Remark 5.1 and by the definition of R we have

$$\begin{aligned}\liminf_{n\to+\infty}\frac{H_n^{hf}(u_n)}{\lambda_n}&\geqslant\liminf_{n\to+\infty}\sum_{j=1}^{M(u_n)}\frac{MM_n(\widetilde{u}_{n(I_j^d)_n})}{\lambda_n}+\liminf_{n\to+\infty}\sum_{j=1}^{M(u_n)-1}\frac{R_n(u_n)}{\lambda_n}\\&\geqslant\liminf_{n\to+\infty}\sum_{j=1}^{M(u_n)-1}\frac{R_n(u_n)}{\lambda_n}\geqslant\liminf_{n\to+\infty}R\frac{|D\mathcal{A}(u_n)|(I)}{|v_1-v_2|}\geqslant R\frac{|D\mathcal{A}(u)|(I)}{|v_1-v_2|},\end{aligned}\tag{5.8}$$

where in the last step we have used that $M(u_n)-1=\frac{|D\mathcal{A}(u_n)|(I)}{|v_1-v_2|}$ and the lower semicontinuity of total variation.
We finally prove (ii). By a standard density argument we can choose u such that $\mathcal{A}(u)=v_1\chi_{(0,\frac12)}+v_2\chi_{[\frac12,1)}$. By the definition of R and by [23, Theorem 4.2] we gain the existence of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\frac{(R_n)_j(u_n)}{\lambda_n}\to R\ \text{ as } n\to+\infty,\qquad\qquad \mathcal{A}(u_n)\xrightarrow{BV}\mathcal{A}(u)\ \text{ as } n\to+\infty,$$

$$u_n\chi_{(0,\frac12)}\in S_1,\ \ u_n\chi_{[\frac12,1)}\in S_2,\qquad\qquad \frac{MM_n(u_n\chi_{(0,\frac12)})}{\lambda_n\delta_n^{\frac32}},\ \frac{MM_n(u_n\chi_{[\frac12,1)})}{\lambda_n\delta_n^{\frac32}}<C.$$

Therefore

$$\lim_{n\to+\infty}\frac{H_n^{hf}(u_n)}{\lambda_n}=R=R\frac{|D\mathcal{A}(u)|(I)}{|v_1-v_2|}.$$

□

5.2. **Second order Γ-convergence of the energy $\mathcal{E}_n$ as $n \to +\infty$.**
At the second order we split the global functional on the 3-dimensional sphere in a finite number of functionals localized in circumferences, where we repeat the analysis lead in [23]. For each S_i we define a convenient order parameter.
Let $u \in \mathcal{PC}_{\lambda_n}$. We associate each pair $\tilde{u}^i_{I^d_j}, \tilde{u}^{i+1}_{I^d_j}$ (see definition (5.3)) with the corresponding oriented angle $\theta^{ij} \in [-\pi, \pi)$ with vertex the center of the circumference S_j given by

$$\theta^i_{I_j} := \chi\left[\tilde{u}^i_{I^d_j} - \mathcal{A}(\tilde{u}^i_{I^d_j}), \tilde{u}^{i+1}_{I^d_j} - \mathcal{A}(\tilde{u}^{i+1}_{I^d_j})\right] \arccos\Big((\tilde{u}^i_{I^d_j} - \mathcal{A}(\tilde{u}^i_{I^d_j}))\cdot(\tilde{u}^{i+1}_{I^d_j} - \mathcal{A}(\tilde{u}^{i+1}_{I^d_j}))\Big).$$

We set

$$w^i_{I_j} := \sqrt{\frac{2}{\delta_n}}\sin\frac{\theta^i_{I_j}}{2} \tag{5.9}$$

and

$$w(t) = w^i_{I_j} \quad \text{if } t \in \lambda_n\{i + [0,1)\},\ \ i \in \left\{\frac{(t^d_j)_1}{\lambda_n}, \dots, \frac{(t^d_j)_2}{\lambda_n} - 1\right\},\ j \in \{1, \dots, M(u) - 1\}$$

Note that we can define a map

$$T_n \colon u \in \mathcal{PC}_{\lambda_n} \mapsto (w, \mathcal{A}(u)).$$

and denote $\widetilde{\mathcal{PC}}_{\lambda_n} := T_n(\mathcal{PC}_{\lambda_n})$. We observe that if $h = T_n(u) = T_n(v)$ then u and v take values in the same S_i and differ by a constant rotation so that $H^{hf}_n(u) = H^{hf}_n(v)$ and the same holds for the functionals $(R_n)^d_j, MM_n, P_n$. Therefore, with a slight abuse of notation, we now regard $H^{hf}_n, (R_n)^d_j, MM_n, P_n$ as functions defined on $h \in \mathrm{L}^1(I; \mathbb{R} \times \{v_1, v_2\})$:

$$H^{hf}_n(h) := \begin{cases} H^{hf}_n(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.10}$$

$$(R_n)_j(h) := \begin{cases} R_{n,j}(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.11}$$

$$MM_n(h_{|I^d_j}) := \begin{cases} MM_n(\tilde{u}_{I^d_j}) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.12}$$

$$P_n(h) := \begin{cases} P_n(u) & \text{if } h \in \widetilde{\mathcal{PC}}_{\lambda_n}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.13}$$

for $j \in \{1, \dots, M(h)\}$, where $h = T(u)$ and $M(h) := M(u)$.

We want to study the convergence of the functional

$$\mathcal{H}_n(h) = H_n(h) - \sum_{j=1}^{M(h)-1} (R_n)_j(h), \tag{5.14}$$

for $h \in \mathrm{L}^1(I; \mathbb{R} \times \{v_1, v_2\})$. In order to establish the related result, we need a notion of convergence.

Definition 5.3. Let $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ and $h \in \mathrm{L}^1(I;\mathbb{R}\times\{v_1,v_2\})$. We say that h_n L_θ-converges to h (we write $h_n \xrightarrow{L_\theta} h$) if and only if the following conditions are satisfied:

- there exist $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{PC}_{\lambda_n}$ and a positive constant C such that if $\mathcal{C}_n(u_n) = \{(I_j^d)_n \,|\, j \in \{1,\ldots,M(u_n)\}\}$ is the open partition associated with u_n, the following two conditions are satisfied:
  - $h_n = T_n(u_n)$ and $P_n(h_n) < C$,
  - $M(u_n) \to M \in \mathbb{N}$ as $n \to +\infty$,
  - $(I_j^d)_n \to I_j^d$ in the Hausdorff sense, as $n \to +\infty$, for any $j \in \{1,\ldots,M\}$.
- $h_n\chi_{(I_j^d)_n} \to h\chi_{I_j^d}$ in $\mathrm{L}^1(I;\mathbb{R}\times\{v_d\})$, a $n \to +\infty$, for all $j \in \{1,\ldots,M\}$.

Theorem 5.4. *Assume that there exist* $\lim_{n\to+\infty} \lambda_n k_n =: \eta \in (0,+\infty)$ *and* $l := \lim_{n\to+\infty} \frac{\lambda_n}{(2\delta_n)^{\frac{1}{2}}} \in [0,+\infty]$. *Then the following statements are true:*

(i) *(Compactness) If for* $\{h_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^1(I;\mathbb{R}\times\{v_1,v_2\})$ *there exists a constant* $C > 0$ *such that*

$$\mathcal{H}_n(h_n) \leqslant \sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}P_n(h_n) \leqslant \lambda_n\delta_n^{\frac{3}{2}}C, \tag{5.15}$$

then, up to a subsequence, $h_n \xrightarrow{L_\theta} h$ *as* $n \to +\infty$, *where*
- *if* $l = 0$, $h \in BV(I;\{-1,1\}\times\{v_1,v_2\})$;
- *if* $l \in (0,+\infty)$, $h_{|I_j^d} \in \mathrm{H}^1_{|per|}(I_j^d;\mathbb{R}\times\{v_1,v_2\})$ *for all* $j \in \{1,\ldots,M(h)\}$;
- *if* $l = +\infty$, h *is piecewise-constant with values in* $\mathbb{R}\times\{v_1,v_2\}$.

The space $\mathrm{H}^1_{|per|}((a,b);\mathbb{R}\times\{v_1,v_2\})$ *is equal to*

$$\left\{h \in H^1((a,b);\mathbb{R}\times\{v_1,v_2\}) : |w(a)| = |w(b)| \text{ where } h = (w,\mathcal{A}(u))\right\}.$$

(ii) *(liminf inequality)*
- *If* $l = 0$, *for all* $h = (w,\mathcal{A}(u)) \in BV(I;\{-1,1\}\times\{v_1,v_2\})$ *and for all* $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ *such that*

$$h_n \xrightarrow{L_\theta} h \in BV(I;\{-1,1\}\times\{v_1,v_2\}), \quad \text{as } n \to +\infty,$$

and

$$\mathcal{H}_n(h_n) \leqslant \sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}P_n(h_n) \leqslant \lambda_n\delta_n^{\frac{3}{2}}C, \tag{5.16}$$

for some $C > 0$ *then*

$$\liminf_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} \geqslant \frac{8}{3}\sum_{j=1}^{M(h)} |Dw|\,(I_j^d).$$

- *If* $l \in (0,+\infty)$, *for all* $h = (w,\mathcal{A}(u)) \in \mathrm{L}^1(I;\mathbb{R}\times\{v_1,v_2\})$ *such that* $h_{|I_j^d} \in \mathrm{H}^1_{|per|}(I_j^d,\mathbb{R}\times\{v_1,\,v_2\})$ *for all* $j \in \{1,\ldots,M(h)\}$, *and for all* $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ *such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

and satisfies formula (5.16), *then*

$$\liminf_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} \geqslant \sum_{j=1}^{M(h)} \left[\frac{1}{l}\int_{I_j}(w^2(x)-1)^2\,dx + l\int_{I_j}(w'(x))^2\,dx\right].$$

– *If $l = +\infty$, for all h piecewise-constant function with values in $\mathbb{R} \times \{v_1, v_2\}$ and for all $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{PC}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

and satisfies formula (5.16), *then*

$$\liminf_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} \geqslant 0.$$

(iii) *(limsup inequality)*

– *If $l = 0$, for all $h = (w, \mathcal{A}(u)) \in BV(I; \{-1, 1\} \times \{v_1, v_2\})$ there exists $\{h_n\}_{n\in N} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (5.16) *and*

$$\lim_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = \frac{8}{3}\sum_{j=1}^{M} |Dw|\,(I_j).$$

– *If $l \in (0, +\infty)$, for all $h = (w, \mathcal{A}(u)) \in \mathrm{L}^1(I; \mathbb{R} \times \{v_1, v_2\})$ such that $h_{|I_j} \in \mathrm{H}^1_{|per|}(I_j; \mathbb{R} \times \{v_1,\, v_2\})$ for all $j \in \{1, \ldots, M(h)\}$, there exists $\{h_n\}_{n\in N} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (5.16) *and*

$$\lim_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = \sum_{j=1}^{M}\left[\frac{1}{l}\int_{I_j}(w^2(x)-1)^2\,dx + l\int_{I_j}(w'(x))^2\,dx\right].$$

– *If $l = +\infty$, for all h piecewise-constant function with values in $\mathbb{R} \times \{v_1, v_2\}$ there exists $\{h_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{PC}}_{\lambda_n}$ such that*

$$h_n \xrightarrow{L_\theta} h, \quad \text{as } n \to +\infty,$$

satisfies formula (5.16) *and*

$$\liminf_{n\to+\infty} \frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = 0.$$

Proof. We will prove the statement only in case $l = 0$, being the other cases fully analogous. We start proving (i). By formulas (5.15), (5.4) we infer

$$MM_n(h_{n|I_j^n}) \leqslant \lambda_n\delta_n^{\frac{3}{2}}C, \quad \text{for all } j \in \{1, \ldots,\, M(h_n)\} \text{ and } n \in \mathbb{N}. \tag{5.17}$$

It's easy to see that, up to subsequences, $M(h_n)$ is independent of $n \in \mathbb{N}$ and that the intervals $(I_j^d)_n \to I_j = (t_{j-1},\, t_j)$, in the Hausdorff sense (it may happen that some limit intervals are empty). In the following computations we drop for simplicity the dependence on n writing I_j^d in place of $(I_j^d)_n$. If $M = 1$, the proof can be lead exactly as in [23, Theorem 4.2]. Let us assume that $M \geqslant 2$. By the very definition of $\widetilde{u}^i_{nI_j^d}$ see formula 5.3, we observing that

$$1 - \widetilde{u}^i_{nI_j^d} \cdot \widetilde{u}^{i+1}_{nI_j^d} = 2R^2\sin^2\left(\frac{\theta^i_{I_j}}{2}\right),$$

$$1-\widetilde{u}^I_{nI^d_j}\cdot\widetilde{u}^{i+2}_{nI^d_j}=R^2[1-\cos(\theta^i_{I_j}+\theta^{i+1}_{I_j})],$$
$$\sum_{j=1}^M \#\mathcal{I}^n(I^d_j)=\#\mathcal{I}^n(I)-M-1,$$

we gain

$$\begin{aligned}\mathcal{H}_n(h_n)&=\sum_{j=1}^M\lambda_n\sum_{i\in\mathcal{I}^n(I^d_j)}\left\{\alpha_n\left[1-\widetilde{u}^I_{nI^d_j}\cdot\widetilde{u}^{i+1}_{nI^d_j}\right]-\left[1-\widetilde{u}^i_{nI^d_j}\cdot\widetilde{u}^{i+2}_{nI^d_j}\right]\right\}\\
&+\lambda_n\left(1+\frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I)+\lambda_n(1-\alpha_n)\sum_{j=1}^M\#\mathcal{I}^n(I^d_j)\\
&=R^2\sum_{j=1}^M\lambda_n\sum_{i\in\mathcal{I}^n(I^d_j)}\left\{2\alpha_n\sin^2\left(\frac{\theta^i_{I_j}}{2}\right)-\left[1-\cos(\theta^i_{I_j}+\theta^{i+1}_{I_j})\right]\right\}\\
&+\lambda_n\left(1+\frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I)+\lambda_n(1-\alpha_n)(\#\mathcal{I}^n(I)-M-1)\\
&\geqslant R^2\sum_{j=1}^M\lambda_n\sum_{i\in\mathcal{I}^n(I^d_j)}\left\{2\alpha_n\sin^2\left(\frac{\theta^i_{I_j}}{2}\right)-\left[1-\cos(\theta^i_{I_j}+\theta^{i+1}_{I_j})\right]\right\}+\lambda_n\left(2-\alpha_n+\frac{\alpha_n^2}{8}\right)\#\mathcal{I}^n(I),\end{aligned}$$

where we used that $M>1$. Repeating the same computations shown in [23, Theorem 4.2], we eventually obtain

(5.18)
$$\sum_{j=1}^M MM_n(\widetilde{u}_{nI_j})\geqslant R^2\sum_{j=1}^M\lambda_n\sum_{i\in\mathcal{I}^n(I^d_j)}\lambda_n\left\{2\delta_n^2\left[(w^i_{nI^d_j})^2-1\right]^2+(1-\gamma_n)\delta_n(w^{i+1}_{nI^d_j}-w^i_{nI^d_j})^2\right\}.$$

If $\varepsilon>0$ is sufficiently small we have that $I^\varepsilon_j:=(t_{j-1}+\varepsilon,\,t_j-\varepsilon)\subset(I^d_j)_n$, for all $n\in\mathbb{N}$, then we obtain that

$$MM_n(w_{n|I^\varepsilon_j})\leqslant\lambda_n\delta_n^{\frac{3}{2}}C$$

and the formula (5.18) holds with I^ε_j in place of I^d_j. Therefore $\{w_n\chi_{I^\varepsilon_j}\}_{n\in\mathbb{N}}$, up to subsequence, converge in L^1 to $w\in BV(I_j)$ then we deduce the existence of $h\in BV(I,\{-1,1\}\times\{v_1,v_2\})$ such that $h_n:=(w_n,\,\mathcal{A}(u_n))\xrightarrow{L_\theta}h$ as $n\to+\infty$.

Now we prove (ii). Let $h=(w,\mathcal{A}(u))\in BV(I;\{-1,1\}\times\{v_1,v_2\})$ By definition 5.3 we have that, up to a subsequence, $M(h_n)$ is independent of n and for $\varepsilon>0$ sufficiently small $I^\varepsilon_j:=(t_j+\varepsilon,\,t_{j+1}-\varepsilon)\subset(I^d_j)_n$ for all $j\in\{1,\dots,M(h_n)\}$ and $n\in\mathbb{N}$, where $(t_j,\,t_{j+1})=I_j$. By formula (5.4) we have

$$\liminf_{n\to+\infty}\frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}=\liminf_{n\to+\infty}\frac{\sum_{j=1}^M MM_n(h_{n|I_j})}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\geqslant\frac{8}{3}\sum_{j=1}^M|Dw|\,(I^\varepsilon_j),$$

where in the last step we have used the Γ-liminf inequality of [23, Theorem 4.2]. Letting $\varepsilon\to0$, we obtain the liminf inequality.

We finally prove (iii). Let $h\in BV(I;\{-1,1\}\times\{v_1,\,v_2\})$. We can find $M>0$ and an open partition of I made by the intervals $\mathcal{C} = \{I_j\}_{j\in\{1,\ldots,M\}}$ such that $h_{|I_j} = (z_j, \overline{v}_j)$ for some $\overline{v}_j \in \{v_1, v_2\}$ and $z_j \in BV(I_j; \{-1,1\})$. For all $j \in \{1,\ldots,M\}$ there exists a sequence $\{(z_j)_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^1(I_j;\mathbb{R})$ (see [23, Theorem 4.2]), such that

$$\lim_{n\to+\infty}(z_j)_n = z_j \text{ in } \mathrm{L}^1(I_j;\mathbb{R}) \quad\text{and}\quad \lim_{n\to+\infty}\frac{MM_n(h_{n|I_j})}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = \frac{8}{3}|Dz|(I_j), \tag{5.19}$$

where $h_{n|I_j} = ((z_j)_n, \overline{v}_j)$. By formulas (5.4) and (5.19) we gain

$$\lim_{n\to+\infty}\frac{\mathcal{H}_n(h_n)}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = \lim_{n\to+\infty}\frac{\sum_{j=1}^{M} MM_n(h_{n_{|I_j}})}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}} = \frac{8}{3}\sum_{j=1}^{M}|Dz|\,(I_j),$$

that is the thesis. □

6. The two-dimensional case

We now analyze the problem in the two-dimensional case. Therefore we need to introduce proper notation and new definitions.

6.1. **Discrete functions.** In this section we denote with d a number that can be 1 or 2. This number is the index of the two circumferences S^1, S^2. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be an infinitesimal sequence of positive lattice spacings. Given $i,j \in \mathbb{Z}$, we denote with $Q_{\lambda_n}(i,j) := (\lambda_n i, \lambda_n j) + [0,\lambda_n)^2$ the half-open square with left-bottom corner in $(\lambda_n i, \lambda_n j)$. For a given set S, we introduce the class of functions with values in S which are piecewise constant on the squares of the lattice $\lambda_n\mathbb{Z}^2$:

$$\mathcal{PC}_{\lambda_n}(S) := \{v \colon \mathbb{R}^2 \to S \,:\, v(x) = v(\lambda_n i, \lambda_n j) \text{ for } x \in Q_{\lambda_n}(i,j)\}.$$

We will identify a function $v \in \mathcal{PC}_{\lambda_n}(S)$ with the function defined on the points of the lattice $\lambda_n\mathbb{Z}^2$ given by $(i,j) \mapsto v^{i,j} := v(\lambda_n i, \lambda_n j)$ for $i,j \in \mathbb{Z}$. Conversely, given values $v^{i,j} \in S$ for $i,j \in \mathbb{Z}$, we define $v \in \mathcal{PC}_{\lambda_n}(S)$ by $v(x) := v^{i,j}$ for $x \in Q_{\lambda_n}(i,j)$.
Given a function $v \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$, we define the discrete partial derivatives $\partial_i^d v, \partial_2^d v \in \mathcal{PC}_{\lambda_n}(\mathbb{R})$ by

$$\partial_1^d v^{i,j} := \frac{v^{i+1,j} - v^{i,j}}{\lambda_n} \quad\text{and}\quad \partial_2^d v^{i,j} := \frac{v^{i,j+1} - v^{i,j}}{\lambda_n}, \quad \forall i,j \in \mathbb{Z},$$

and we denote the discrete gradient of v by $\nabla^d v$, defined as usual. Note that the two derivatives commute. Thus we may define the second order discrete partial derivatives $\partial_{11}^d v$, $\partial_{12}^d v = \partial_{21}^d v$, $\partial_{22}^d v$ by iterative application of these operators in arbitrary order. Similarly, we define higher order discrete partial derivatives.
It is easy to observe that for all $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$ the function $\mathcal{A}(u) \in SBV(\Omega, \{v_1, v_2\})$, where $\mathcal{A}(u)$ is defined in formula 2.1.

6.2. **Assumptions on the model.** Our model is an energy on discrete spin fields defined on square lattices inside a given domain $\Omega \subset \mathbb{R}^2$ belonging to the following class:

$$\mathfrak{A}_0 := \{\Omega \subset \mathbb{R}^2 \,:\, \Omega \text{ is an open, bounded, simply connected, BVG domain}\},$$

where the definition of a BVG domain is given in the article [36] (see also [21, Section 3]. To define the energies in our model, we introduce the set of indices

$$\mathcal{I}^n(\Omega) := \{(i,j) \in \mathbb{Z}^2 \,:\, \overline{Q}_{\lambda_n}(i,j), \overline{Q}_{\lambda_n}(i+1,j), \overline{Q}_{\lambda_n}(i,j+1) \subset \Omega\},$$

for $\Omega \in \mathfrak{A}_0$. Let $\alpha_n := 4(1-\delta_n)$, where $\{\delta_n\} \subset \mathbb{R}^+$ is an infinitesimal sequence, and $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ a divergent sequence. In the following we shall assume that $\varepsilon_n := \frac{\lambda_n}{\sqrt{\delta_n}} \to 0$ and $\lambda_n k_n \to \eta \in (0,+\infty)$ as $n \to +\infty$.
We consider the functionals $H_n, P_n\colon \mathrm{L}^\infty(\mathbb{R}^2; S_1 \cup S_2) \times \mathfrak{A}_0 \to [0,+\infty]$ defined by

$$H_n(u;\Omega) := \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\frac{1}{2}\lambda_n^2 \sum_{(i,j)\in\mathcal{I}^n(\Omega)} \left[\left|u^{i+2,j} - \frac{\alpha_n}{2}u^{i+1,j} + u^{i,j}\right|^2 + \left|u^{i,j+2} - \frac{\alpha_n}{2}u^{i,j+1} + u^{i,j}\right|^2\right],$$

$$P_n(u;\Omega) := \lambda_n k_n |D\mathcal{A}(u)|(\Omega),$$

for $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$ and extended to $+\infty$ elsewhere, where $\mathcal{A}(u)$ is the function defined in the formula (2.1).
As in the one-dimensional case we observe that if $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$, the set Ω can be uniquely partitioned in regions where the function u takes value only in one of the two circumferences. We now make the notation clear. There exist $M(u) \in \mathbb{N}$ and a collection of open connected sets, $\{C_s^d\}_{s\in\{1,\dots,M(u)\}}$, such that

$$\{C_s^d\}_{s\in\{1,\dots,M(u)\}} \text{ is an open partition of } \Omega, \tag{6.1}$$

$$u(x) \in S_d, \quad \forall x \in C_s^d,\ \forall s \in \{1,\dots,M(u)\}. \tag{6.2}$$

The following definition will be useful throughout the section.

Definition 6.1. Let $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$. We will say that $\mathcal{C}_n(u) = \{C_s^d \,|\, s \in \{1,\dots,M(u)\}\}$ is the open partition associated with u if $M(u) \in \mathbb{N}$ and the collection of open connected sets, $\{C_s^d\}_{s\in\{1,\dots,M(u)\}}$ satisfies (6.1), (6.2). We call $\mathcal{C}_n(u)$ the open BVG partition associated with u if C_s^d is also BVG, for all $s \in \{1,\dots,M(u)\}$.

As in the first section, we split the functional H_n in two addends:

$$H_n(u;\Omega) = \sum_{s=1}^{M(u)} \Big[H_n(u;C_s^d) + R_{nC_s^d}(u)\Big],$$

where

$$R_{n,C_s^d}(u) := \frac{1}{\sqrt{2}\lambda_n\delta_n^{\frac{3}{2}}}\frac{1}{2}\lambda_n^2 \sum_{(i,j)\in(C_s^d\cap\mathcal{I}^n(\Omega))\setminus\mathcal{I}^n(C_s^d)} \left[\left|u^{i+2,j} - \frac{\alpha_n}{2}u^{i+1,j} + u^{i,j}\right|^2 + \left|u^{i,j+2} - \frac{\alpha_n}{2}u^{i,j+1} + u^{i,j}\right|^2\right]$$

is the remainder associated with the partition of u, which consists of the interactions between the vectors of the spin field in different elements of $\mathcal{C}(u)$.

We now introduce the chirality order parameter associated with a spin field. Let $u \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$ and $\mathcal{C}_n(u) = \{C_s^d \,|\, s \in \{1,\dots,M(u)\}\}$ be the partition associated with u. For $(i,j) \in \mathcal{I}^n(C_s^d)$, we define the horizontal and vertical oriented angles between two adjacent spins by

$$\widetilde{\theta}^{i,j}_{C_s^d} := \chi[u^{i,j}, u^{i+1,j}] \arccos(u^{i,j}\cdot u^{i+1,j}) \in [-\pi,\pi),$$

$$\breve{\theta}^{i,j}_{C_s^d} := \chi[u^{i,j}, u^{i,j+1}] \arccos(u^{i,j}\cdot u^{i,j+1}) \in [-\pi,\pi).$$

Denoting with $\mathcal{D}(\{v_1,v_2\})$ the space of functions defined in Ω with values in $\{v_1,v_2\}$, we define the order parameter $(w,z,\mathcal{A}(u)) \in \mathcal{PC}_{\lambda_n}(\mathbb{R}^2)\times\mathcal{D}(\{v_1,v_2\})$ as

$$w^{i,j} := \begin{cases} \sqrt{\frac{2}{\delta_n}}\sin\frac{\widetilde{\theta}^{i,j}_{C^d_s}}{2} & \text{if } (i,j)\in\mathcal{I}^n(C^d_s) \text{ for some } s\in\{1,\dots,M(u)\},\\ 0 & \text{otherwise,}\end{cases}$$

$$z^{i,j} := \begin{cases} \sqrt{\frac{2}{\delta_n}}\sin\frac{\breve{\theta}^{i,j}_{C^d_s}}{2} & \text{if } (i,j)\in\mathcal{I}^n(C^d_s) \text{ for some } s\in\{1,\dots,M(u)\},\\ 0 & \text{otherwise.}\end{cases}$$

It is convenient to introduce the transformation $T_n\colon \mathcal{PC}_{\lambda_n}(S_1\cup S_2)\to\mathcal{PC}_{\lambda_n}(\mathbb{R}^2)\times\mathcal{D}(\{v_1,v_2\})$ given by

$$T_n(u) := \big(w,z,\mathcal{A}(u)\big).$$

With a slight abuse of notation we define the functional $H_n\colon \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})\times\mathfrak{A}_0\to[0,+\infty)$ by

$$H_n(h;\Omega) = \begin{cases} H_n(u;\Omega) & \text{if } T_n(u)=h \text{ for some } u\in\mathcal{PC}_{\lambda_n}(S_1\cup S_2),\\ +\infty & \text{otherwise.}\end{cases}$$

Notice that H_n does not depend on the particular choice of u, since it is rotation-invariant. The same notation can be adopted for P_n and $R_{nC^d_s}$.
We will study the convergence of the functional

$$\mathcal{H}_n(h;\Omega) := H_n(h,\Omega) - \sum_{s=1}^{M(h)} R_{nC^d_s}(h) = \sum_{s=1}^{M(h)} H_n(h;C^d_s), \tag{6.3}$$

where $h=T_n(u)$ and $M(h):=M(u)$, for some $u\in\mathcal{PC}_{\lambda_n}(S_1\cup S_2)$. Hence, we introduce the functional $\mathcal{H}\colon \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})\times\mathfrak{A}_0\to[0,+\infty)$ by setting

$$\mathcal{H}(h;\Omega) := \begin{cases} \frac{4}{3}\sum_{s=1}^{M(h)}(|D_1w|(C^d_s)+|D_2z|(C^d_s)) & \text{if } h=(w,z,\alpha)\in \mathrm{Dom}(\mathcal{H};\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$\mathrm{Dom}(\mathcal{H};\Omega) := \Big\{(w,z,\alpha)\in\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\}) : \exists\{C^d_s\}_{s\in\{1,\dots,M\}} \text{ open partition of } \Omega$$
$$\text{s.t. } (w_{|C^d_s}, z_{|C^d_s}, \alpha_{|C^d_s})\in BV(C^d_s;\{-1,1\}^2\times\{v_d\}),\ \mathrm{curl}(w_{|C^d_s},z_{|C^d_s})=0 \text{ in } \mathcal{D}'(C^d_s)\Big\}$$

where we have denoted by $\mathcal{D}'(C^d_s)$ the space of distributions, and for all $T\in\mathcal{D}'(C^d_s,\mathbb{R}^2)$ we have

$$\langle \mathrm{curl}(T)_{h,k},\,\xi\rangle := -\langle T^k,\,\partial_h\xi\rangle + \langle T^h,\,\partial_k\xi\rangle \quad \text{for every } \xi\in C^\infty_c(C^d_s).$$

Definition 6.2 (Convergence law)**.** Let $\{h_n\}_{n\in\mathbb{N}}\subset\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$. We say that h_n L_θ-converges to $h\in\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$ (we write $h_n\overset{L_\theta}{\to}h$) if the following conditions are satisfied:

- there exist $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{PC}_{\lambda_n}(S_1\cup S_2)$ and a positive constant C and, for any $n\in\mathbb{N}$, an open BVG partition associated with u_n, $\mathcal{C}_n(u_n)=\{(C^d_s)_n \mid s\in\{1,\dots,M(h_n)\}\}$, such that

 - $h_n = T_n(u_n)$ and $P_n(u_n;\Omega) < C$,
 - $M(u_n) \to M \in \mathbb{N}$ as $n \to +\infty$,
 - $(C^d_s)_n \to C^d_s$ in the Hausdorff sense, as $n \to +\infty$, for any $s \in \{1,\dots,M\}$.
- $h_n\chi_{(C^d_s)_n} \to h\chi_{C^d_s}$ in $\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_d\})$, as $n \to +\infty$, for all $s \in \{1,\dots,M\}$.

Now we state the main theorem of this section. The hypotheses assumed in the main theorem are satisfied by minimizing sequences, almost minimizing sequences and minimizing sequences under volume preserving hypotheses. Thus by a standard argument for the Γ-convergence, we obtain that a minimizing sequences converge to a minimizer of the Γ-lim functional and that the minimal values of the functionals $\mathcal{H}_n$ converge to the minimal values of the functional $\mathcal{H}$.

Theorem 6.3. *Assume that $\Omega \in \mathfrak{A}_0$. Then the following statements hold true:*

i) *(Compactness) Let $\{h_n = T_n(u_n)\}_{n\in\mathbb{N}} \subset T_n(\mathcal{PC}_{\lambda_n}\times\mathcal{D}(v_1,v_2))$ be a sequence such that*

$$\mathcal{H}_n(h_n;\Omega) \leqslant P_n(h_n;\Omega) < C, \tag{6.4}$$

for some constant $C>0$. Assume that the BVG partition associated with u_n, $\mathcal{C}_n(u_n) = \{(C^d_s)_n |\, s \in \{1,\dots,M(u_n)\}\}$, is such that

$$M(u_n) \to M \in \mathbb{N} \quad \textit{as } n \to +\infty,$$

$$(C^d_s)_n \to C^d_s \quad \textit{in the Hausdorff sense, as } n \to +\infty,\ \forall s \in \{1,\dots,M\}.$$

Then there exists $h \in \mathrm{Dom}(\mathcal{H},\Omega)$ such that, up to a subsequence, $h_n \overset{L_\theta}{\to} h$, as $n \to +\infty$.

ii) *(Liminf inequality) Let $\{h_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$ and $h \in \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$. Assume that $P_n(h_n;\Omega) < C$ for some constant $C>0$ and $h_n \overset{L_\theta}{\to} h$, as $n \to +\infty$. Then*

$$\mathcal{H}(h;\Omega) \leqslant \liminf_{n\to+\infty} \mathcal{H}_n(h_n;\Omega).$$

iii) *(Limsup inequality) Let $h \in \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$. Then there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$ such that $h_n \overset{L_\theta}{\to} h$ and*

$$\limsup_{n\to+\infty} \mathcal{H}_n(h_n;\Omega) \leqslant \mathcal{H}(h;\Omega).$$

Proof. We start proving i). Let $\{h_n = (w_n, z_n, \mathcal{A}(u_n))\}_{n\in\mathbb{N}} \subset \mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$ be a sequence satisfiying (6.4). Fixing $\varepsilon>0$ sufficiently small, we have that for all $n\in\mathbb{N}$, up to a subsequence, $(C^d_s)_\varepsilon := \{x \in C^d_s : \mathrm{dist}(x,\partial C^d_s) > \varepsilon\} \subset (C^d_s)^n$ and $u_{n_{|(C^d_s)_\varepsilon}}$ takes values only in one circumference. We infer that

$$\sum_{s=1}^{M} H_n(h_n;(C^d_s)_\varepsilon) \leqslant \mathcal{H}_n(h_n;\Omega) < C.$$

which of course implies that $H_n(h_n;(C^d_s)_\varepsilon) < C$, for all $s \in \{1,\dots,M\}$. We are in position to apply [21, Theorem 2.1 i)] to deduce the existence of $(w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon}) \in BV((C^d_s)_\varepsilon;\{-1,1\}^2)$ such that, up to subsequences, $(w_n,z_n) \to (w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon})$ in $\mathrm{L}^1_{loc}((C^d_s)_\varepsilon;\mathbb{R}^2)$ and $\mathrm{curl}(w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon}) = 0$ in $\mathcal{D}'((C^d_s)_\varepsilon)$. The couples $(w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon})$ can be extended to 0 in $C^d_s \backslash (C^d_s)_\varepsilon$. We preliminary observe that

$$(w_{(C^d_s)_{\varepsilon_2}}, z_{(C^d_s)_{\varepsilon_2}}) = (w_{(C^d_s)_{\varepsilon_1}}, z_{(C^d_s)_{\varepsilon_1}}) \quad \text{a.e. on } (C^d_s)_{\varepsilon_2}, \tag{6.5}$$

for any $0 < \varepsilon_1 < \varepsilon_2$. Indeed, since $(C^d_s)_{\varepsilon_2} \subset (C^d_s)_{\varepsilon_1}$, we have that

$$(w_n,z_n) \to (w_{(C^d_s)_{\varepsilon_1}}, z_{(C^d_s)_{\varepsilon_1}}) \quad \text{in } \mathrm{L}^1_{loc}((C^d_s)_{\varepsilon_2};\mathbb{R}^2).$$

The uniqueness of the limit in the L^1_{loc}-topology implies (6.5). We now define the couples $(w_{C^d_s}, z_{C^d_s})\colon C^d_s \to \mathbb{R}^2$ as

$$(w_{C^d_s}, z_{C^d_s}) := \lim_{\varepsilon\to 0^+} (w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon}).$$

The definition is well-posed; indeed, since by (6.5),

$$\lim_{\varepsilon'\to 0^+} (w_{(C^d_s)_{\varepsilon'}}, z_{(C^d_s)_{\varepsilon'}}) = (w_{(C^d_s)_{\frac{1}{n}}}, z_{(C^d_s)_{\frac{1}{n}}}) \quad \text{a.e. in } (C^d_s)_{\frac{1}{n}},$$

for all $n \in \mathbb{N}$, then

$$\begin{aligned}&\left|\left\{x \in C^d_s \,:\, \nexists \lim_{\varepsilon'\to 0^+} (w_{(C^d_s)_{\varepsilon'}}(x), z_{(C^d_s)_{\varepsilon'}}(x))\right\}\right| \\ &= \left|\bigcup_{n=1}^{+\infty}\left\{x \in (C^d_s)_{\frac{1}{n}} \,:\, \nexists \lim_{\varepsilon'\to 0^+} (w_{(C^d_s)_{\varepsilon'}}(x), z_{(C^d_s)_{\varepsilon'}}(x))\right\}\right| = 0.\end{aligned}$$

Furthermore we set $(w, z)\colon \Omega \to \mathbb{R}^2$

$$(w, z)(x) = (w_{C^d_s}, z_{C^d_s})(x), \tag{6.6}$$

for a.e. $x \in \Omega$ with $x \in C^d_s$, for some $s \in \{1, \dots, M\}$. Of course $(w_{|C^d_s}, z_{|C^d_s}) = (w_{C^d_s}, z_{C^d_s}) \in BV(C^d_s; \{-1, 1\}^2)$, as it is the limit of BV functions. In order to show the L^1_{loc}-convergence, we fix $A \subset\subset C^d_s$. Since $\mathrm{dist}(A, \partial C^d_s) > 0$, there exists $\varepsilon > 0$ such that $A \subset\subset (C^d_s)_\varepsilon$. We obtain:

$$\left\|(w_n, z_n) - (w_{C^d_s}, z_{C^d_s})\right\|_{\mathrm{L}^1(A;\mathbb{R}^2)} = \left\|(w_n, z_n) - (w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon})\right\|_{\mathrm{L}^1(A;\mathbb{R}^2)},$$

which vanishes as $n \to +\infty$, up to subsequences. This leads to the convergence

$$(w_n, z_n) \to (w_{C^d_s}, z_{C^d_s}) \quad \text{in } L^1_{loc}(C^d_s; \mathbb{R}^2).$$

Finally, we prove that $\mathrm{curl}(w_{C^d_s}, z_{C^d_s}) = 0$ in $\mathcal{D}'(C^d_s)$. If $\xi \in C^\infty_c(C^d_s)$, then $\mathrm{supp}\xi \subset (C^d_s)_\varepsilon$ for some $\varepsilon > 0$ and so

$$\begin{aligned}\left\langle \mathrm{curl}(w_{C^d_s}, z_{C^d_s}), \xi\right\rangle &= -\int_{(C^d_s)_\varepsilon} w_{(C^d_s)_\varepsilon}\partial_2\xi\, dx + \int_{(C^d_s)_\varepsilon} z_{(C^d_s)_\varepsilon}\partial_1\xi\, dx \\ &= \left\langle \mathrm{curl}(w_{(C^d_s)_\varepsilon}, z_{(C^d_s)_\varepsilon}), \xi\right\rangle = 0.\end{aligned}$$

Now we prove ii). Let $\{h_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$ and $h \in \mathrm{L}^1_{loc}(\mathbb{R}^2; \mathbb{R}^2 \times \{v_1, v_2\})$ such that $P_n(h_n; \Omega) < C$ and $h_n \overset{L_\theta}{\to} h$, as $n \to +\infty$. Up to subsequences we can assume that the lower limit in the right hand side of the liminf inequality is actually a limit. Furthermore we can assume that it is finite, the inequality being otherwise trivial. In particular, we have

$$\mathcal{H}_n(h_n; \Omega) < C,$$

with a possibly larger C. By the definition of L_θ-convergence, $h_n = (w_n, z_n, \mathcal{A}(u_n)) = T_n(u_n)$ for some $u_n \in \mathcal{PC}_{\lambda_n}(S_1 \cup S_2)$. We can assume, up to subsequences, that $M(h_n)$ is independent of n and, by the Hausdorff convergence, for $\varepsilon > 0$ sufficiently small, $(C^d_s)_\varepsilon \subset (C^d_s)^n$ and $u_{n_{|(C^d_s)_\varepsilon}}$ takes values only on one circumference, for all $n \in \mathbb{N}$. By the positivity of $\mathcal{H}_n$, we infer that

$$\mathcal{H}_n(h_n; \Omega) \geqslant \sum_{s=1}^{M} H_n(h_n; (C^d_s)_\varepsilon).$$

Since $h_n \to h$ in $\mathrm{L}^1((C_s^d)_\varepsilon;\mathbb{R}^2\times\{v_d\})$, as $n\to+\infty$, we can apply [21, Theorem 2.1 ii)] so that, passing to the lower limit, we get

$$\liminf_{n\to+\infty}\mathcal{H}_n(h_n;\Omega)\geqslant\sum_{s=1}^{M}\liminf_{n\to+\infty}H_n(h_n;(C_s^d)_\varepsilon)\geqslant\sum_{s=1}^{M}\frac{4}{3}[|D_1w|((C_s^d)_\varepsilon)+|D_2z|((C_s^d)_\varepsilon)],$$

where $h=(w,z,\alpha)$. Letting $\varepsilon\to0^+$ we get the thesis.
Let us prove iii). Let $h\in\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$. We can assume that $h\in\mathrm{Dom}(H;\Omega)$. This means that $h=(w,z,\alpha)\in\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2\times\{v_1,v_2\})$ and the existence of an open partition of Ω, $\mathcal{C}=\{C_s^d|\,s\in\{1,\dots,M\}\}$ such that

$$(w_{|C_s^d},z_{|C_s^d},\alpha_{|C_s^d})\in BV(C_s^d;\{-1,1\}^2\times\{v_d\})\quad\text{and}\quad\mathrm{curl}(w_{|C_s^d},z_{|C_s^d})=0\text{ in }\mathcal{D}'(C_s^d).$$

Applying [21, Theorem 2.1 iii)] to any $(w_{|C_s^d},z_{|C_s^d})$, we get the existence of a sequence $(w_{nC_s^d},z_{nC_s^d})\in\mathrm{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ such that $(w_{nC_s^d},z_{nC_s^d})\to(w_{|C_s^d},z_{|C_s^d})$ in $\mathrm{L}^1(C_s^d;\mathbb{R}^2)$ and

$$\limsup_{n\to+\infty}H_n(w_{nC_s^d},z_{nC_s^d},v_d)\leqslant\frac{4}{3}(|D_1w|(C_s^d)+|D_2z|(C_s^d))$$

Defining $(w_n,z_n,\alpha_n)\colon\mathbb{R}^2\to\mathbb{R}^2\times\{v_1,v_2\}$ by

$$(w_n,z_n,\alpha_n)(x)=(w_{nC_s^d}(x),z_{nC_s^d}(x),v_d),$$

if $x\in\Omega$ such that $x\in C_s^d$ for some $s\in\{1,\dots,M\}$, and arbitrarily extended outside Ω, and summing on $s\in\{1,\dots,M\}$ the previous inequality we get the thesis. ☐

Appendix A. Some properties of L^∞ functions with values in a compact set

In this appendix we recall some classical properties of the Lebesgue space $\mathrm{L}^\infty(I,K)$, where $K\subset\mathbb{R}^N$ is a compact set.

Proposition A.1. *Let $K\subset\mathbb{R}^N$ be a compact set and let $\{f_n\}_{n\in\mathbb{N}}\subset\mathrm{L}^\infty(I,K)$. Then, up to subsequences, $f_n\overset{*}{\rightharpoonup}f\in\mathrm{L}^\infty(I,co(K))$ as $n\to+\infty$ in the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^N)$. Moreover for all $u\in\mathrm{L}^\infty(I,co(K))$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}\subset\mathrm{L}^\infty(I,K)$ of piecewise functions such that $u_n\overset{*}{\rightharpoonup}u$ as $n\to+\infty$.*

Proof. Since the set K is bounded then, up to a subsequence, there exists $f\in\mathrm{L}^\infty(I,\mathbb{R}^N)$ such that $f_n\overset{*}{\rightharpoonup}f$ as $n\to+\infty$. We now prove that $f(t)\in co(K)$ for almost every $t\in(0,1)$. For every $\xi\notin co(K)$ there exist an affine function $h_\xi:\mathbb{R}^N\to\mathbb{R}$ and $\alpha<0$ such that

$$h_\xi(\xi)>0>\alpha>h_\xi(x),\quad\forall x\in co(K).$$

By the weak-star convergence of $\{f_n\}_{n\in\mathbb{N}}$ we have that for any measurable set $A\subset(0,1)$

$$\int_A h_\xi(f(t))dt=\lim_{n\to+\infty}\int_A h_\xi(f_n(t))dt\leqslant|A|\alpha<0.$$

Hence by the arbitrariness of A we obtain

$$h_\xi(f(t))<0\quad\text{for a.e. }t\in(0,1).\tag{A.1}$$

Recalling that

$$co(K)=\bigcap_{j\in\mathbb{N}}\left\{y\in\mathbb{R}^N:\ h_{\xi_j}(y)<0,\ \xi_j\in\mathbb{Q}^N\cap co(K)^c\right\},$$

by formula (A.1) we obtain

$$f(t) \in co(K) \quad \text{for a.e. } t \in (0,1).$$

Now we can prove the second statement of the proposition. By a standard density argument, it is enough to prove the claim for $u = a\chi_J$, where J is an open interval and $a \in co(K)$. We define the following function:

$$h(t) := \begin{cases} a_1 & \text{if } t \in (0,\lambda), \\ a_2 & \text{if } t \in [\lambda, 1), \end{cases}$$

where $a = \lambda a_1 + (1-\lambda)a_2$ with $a_1, a_2 \in K$ and for some $\lambda \in [0,1]$. Then the sequence $u_n(t) := h(nt)$ converges to u in the weak-star topology of L^∞ by Riemann-Lebesgue's lemma. □

Corollary A.2. *Let $K \subset \mathbb{R}^N$ be a compact set. The closure of the set $\mathrm{L}^\infty(I,K)$ with respect to the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^N)$ is the set $\mathrm{L}^\infty(I, co(K))$.*

Proof. Since the space $\mathrm{L}^1(I,\mathbb{R}^N)$ is separable, every bounded subset of $\mathrm{L}^\infty(I,\mathbb{R}^N)$ is metrizable with respect to the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^N)$. Hence the set $\mathrm{L}^\infty(I,K)$ is metrizable. Therefore, by the above proposition, we have that the set $\mathrm{L}^\infty(I,co(K))$ is the weak-star closure of the set $\mathrm{L}^\infty(I,K)$. □

Proposition A.3. *Let K_1, $K_2 \subset \mathbb{R}^N$ be two compact sets and let $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(I, K_1 \cup K_2)$ be such that for all $n \in \mathbb{N}$ exist $M(n) \in \mathbb{N}$ and $0 = t_1^{(n)} < \cdots < t_{M(n)}^{(n)} = 1$ for which*

$$f_n(t) \in K_j \text{ for some } j \in \{1,2\} \text{ and a.e. } t \in (t_i^{(n)}, t_{i+1}^{(n)}),$$

for all $i \in \{1,\dots,M(n)-1\}$. Finally we suppose that

$$\sup_{n\in\mathbb{N}} M(n) < +\infty. \tag{A.2}$$

Then, up to subsequences, $f_n \overset{*}{\rightharpoonup} f$ in the weak-star topology of $\mathrm{L}^\infty(I,\mathbb{R}^N)$ and $f \in \mathrm{L}^\infty(I, co(K_1) \cup co(K_2))$. Moreover if $co(K_1) \cap co(K_2) = \varnothing$, there exist $M \in \mathbb{N}$ and $0 = t_1 < \cdots < t_M = 1$ such that

$$f(t) \in co(K_j) \text{ for some } j \in \{1,2\} \text{ and a.e. } t \in (t_i, t_{i+1}),$$

for all $i \in \{1,\dots,M\}$,

Proof. By Proposition A.1 we have, up to a subsequence, that $f_n \overset{*}{\rightharpoonup} f \in \mathrm{L}^\infty(I, co(K_1 \cup K_2))$. Accordingly, by assumption (A.2) we can find $M \in \mathbb{N}$, independent of $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ there exist $0 = t_1^{(n)} < \cdots < t_M^{(n)} = 1$ for which $f_n(t) \in K_j$ for some $j \in \{1,2\}$ and a.e. $t \in (t_i^{(n)}, t_{i+1}^{(n)})$, for all $i \in \{1,\dots,M-1\}$. Up to subsequence, we can calculate

$$\lim_{n\to+\infty} t_i^{(n)} = t_i, \quad \forall i \in \{1,\cdots,M\},$$

so that $0 = t_1 < \cdots < t_M = 1$. Let us fix $i \in \{1,\cdots,M-1\}$. For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that we have

$$(t_i + \varepsilon, t_{i+1} - \varepsilon) \subset (t_i^{(n)}, t_{i+1}^{(n)}) \quad \forall n \geqslant n_0.$$

We define the following two sets:

$$A_1 = \{n \geqslant n_0 \, : \, f_n(t) \in K_1 \text{ for a.e. } t \in (t_i + \varepsilon, t_{i+1} - \varepsilon)\},$$
$$A_2 = \{n \geqslant n_0 \, : \, f_n(t) \in K_2 \text{ for a.e. } t \in (t_i + \varepsilon, t_{i+1} - \varepsilon)\}.$$

One of the following three cases may occur:

$$1.\ \#A_1 = \infty,\ \#A_2 < \infty; \quad 2.\ \#A_1 < \infty,\ \#A_2 = \infty; \quad 3.\ \#A_1 = \infty,\ \#A_2 = \infty.$$

In the first case we have that $f_n \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), K_1)$ for all $n \geqslant n_0$, up to a finite number of indices of the sequence. Thus, by Proposition A.1 $f_n \overset{*}{\rightharpoonup} f \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), co(K_1))$ and hence by the arbitrariness of $\varepsilon > 0$ we obtain $f \in \mathrm{L}^\infty((t_i, t_{i+1}), co(K_1))$. The second case is fully analogous to the first case. In the third case we can find two subsequences $\{n_k^{(1)}\}_{k\in\mathbb{N}}$ and $\{n_k^{(2)}\}_{k\in\mathbb{N}}$ such that $f_{n_k^{(1)}} \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), K_1)$ and $f_{n_k^{(2)}} \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), K_2)$ for all $k \in \mathbb{N}$. By Proposition A.1 there exist $f_1 \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), co(K_1))$ and $f_2 \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), co(K_2))$ such that $f_{n_k^{(1)}} \overset{*}{\rightharpoonup} f_1$ and $f_{n_k^{(2)}} \overset{*}{\rightharpoonup} f_2$. One the other hand we recall that $f_n \overset{*}{\rightharpoonup} f \in \mathrm{L}^\infty(I, co(K_1 \cup K_2))$. Then by the uniqueness of the limit in the weak-star topology we have $f_1(t) = f_2(t) = f(t)$ for almost every $t \in (t_i+\varepsilon, t_{i+1}-\varepsilon)$ and so $f \in \mathrm{L}^\infty((t_i+\varepsilon, t_{i+1}-\varepsilon), co(K_1) \cap co(K_2))$. Hence by the arbitrariness of $\varepsilon > 0$ we have $f \in \mathrm{L}^\infty((t_i, t_{i+1}), co(K_1) \cap co(K_2))$. If we repeat the above argument for all $i \in \{1, \cdots, M\}$ we obtain the thesis. If $co(K_1) \cap co(K_2) = \varnothing$ the case $\#A_1 = \#A_2 = \infty$ cannot occur and therefore we obtain the last claim of the statement. $\square$

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