ON THE VARIATIONAL NATURE OF THE ANZELLOTTI PAIRING

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Abstract. In this paper we prove that the Anzellotti pairing can be regarded as a relaxed functional with respect to the weak* convergence to the space BV of functions of bounded variation. The crucial tool is a preliminary integral representation of this pairing by means of suitable cylindrical averages.

1. Introduction

A classical problem in Calculus of Variations is to find minimal assumptions assuring the lower semicontinuity with respect to a suitable convergence of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^N$ and u belongs to a given space of weakly differentiable functions. With this problem in mind, it is well known that, if the integrand $f(x, s, \xi)$ admits a linear growth with respect to the gradient variable ξ, the natural functional framework is the space BV of functions of bounded variation.

A fundamental result in this direction has been proved by Dal Maso in [23]. More precisely, assuming that the integrand $f(x, s, \xi)$ is coercive, continuous and convex in the last variable, he introduced a proper lower semicontinuous extension of F to BV and proved that it coincides with the integral representation of the relaxed functional of F. If we drop the coercivity assumption on f, the task of studying the lower semicontinuity and of finding the relaxation of F is highly non-trivial and requires some additional regularity assumption on f in the x variable (see for instance [1, 2, 10, 26, 27, 30, 33]).

Aim of this paper is to investigate the possibility of new progress in this area, by confining our study to the model cases

$$F_\varphi(u) = \int_\Omega \varphi\, \boldsymbol{b}(x, u) \cdot \nabla u\, dx\,, \quad G(u) = \int_\Omega |\boldsymbol{b}(x, u) \cdot \nabla u|\, dx\,, \qquad \varphi \in C_0^1(\Omega)\,,$$

in the perspective to extend this study to more general cases. We remark that all the results presented in this paper are new also in the case of a vector field $\boldsymbol{b}$ independent of u.

The lower semicontinuity of these functionals with respect to the L^1 convergence has been established in [30] in the Sobolev space $W^{1,1}$ by requiring a very weak regularity assumption, i.e. that the divergence of the vector field $\boldsymbol{b}(x, s)$ with respect to x is an L^1 function. Our aim is to extend this result to the space BV, by considering a relaxed functional defined by an abstract relaxation procedure. More precisely, for every fixed

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function $\varphi \in C_c(\Omega)$ and every open set $A \subseteq \Omega$, let us consider the functional $F_\varphi(\cdot, A) : BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega) \to [0, +\infty]$ defined by

$$F_\varphi(u, A) := \begin{cases} \displaystyle\int_A \varphi\, \boldsymbol{b}(x, u) \cdot \nabla u\, dx & \text{if } u \in W^{1,1}_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega), \\ +\infty, & \text{if } u \in (BV_{\rm loc}(\Omega) \ \setminus W^{1,1}_{\rm loc}(\Omega)) \cap L^\infty_{\rm loc}(\Omega)\,, \end{cases}$$

and the associated relaxed functional

$$\overline{F}_\varphi(u, A) := \inf \left\{ \liminf_{n\to+\infty} F_\varphi(u_n, A) : u_n \in W^{1,1}(\Omega)\,, u_n \to u \text{ weak}^* \text{ in } BV(\Omega) \right\}. \tag{2}$$

As it is customary, this relaxed functional can be characterized as the greatest lower semicontinuous extension of F to BV, less than or equal to F. Besides this abstract definition, for the applications it is of paramount importance to have an integral representation of $\overline{F}_\varphi$. To this end, the main difficulty is to find a precise representative for the singular part of the relaxed functional (i.e., the representative where the measure Du is singular).

The main contribution of this paper is to find an integral representation for $\overline{F}_\varphi(u, A)$. More precisely, we prove that, for every $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ and for every open set $A \subseteq \Omega$, it holds that

$$\overline{F}_\varphi(u, A) = \int_A \varphi\, (\boldsymbol{b}(x, u), Du)\,, \tag{3}$$

where $(\boldsymbol{b}(x, u), Du)$ denotes the pairing measure defined in the recent paper [21]. This measure extends the concept of pairing measure introduced by Anzellotti in the celebrated paper [8] by establishing a pairing theory between weakly differentiable vector fields $\boldsymbol{b}(x)$ and BV functions. While the original definition of this measure starts from a distributional viewpoint, our contribution shows that it can be regarded also in a variational sense as a relaxed functional. This variational interpretation seems to be useful in order to study the 1–Laplace operator, both in the case of the associated Euler–Lagrange equations (see [38]) and in the study of the related Dirichlet problem with measure data.

Lower semicontinuity results and representation formulas for the relaxed functional in $BV(\Omega)$ have been obtained by many authors. In the already cited paper [24], Dal Maso showed that, in order to prove lower semicontinuity, in his general setting the coercivity assumption cannot be dropped. In the spirit of the alternatives of Serrin (see [43]), in order to drop this assumption, Fonseca and Leoni in [33] assumed a uniform lower semicontinuity condition in x. Moreover, in [1, 2], the authors required weak differentiability in x and BV in x dependence, respectively. In these cases the precise representatives for the singular parts are the approximately continuous representative and the lower semicontinuous capacitary representative, respectively.

Before describing in more details our results, a few words on the pairing measure are in order.

The pairing theory was initially used to extend the validity of the Gauss–Green formula to divergence-measure vector fields and to non-smooth domains (see [8, 11, 12, 15, 16, 18, 20, 37]). Moreover, it can be considered as a useful abstract tool in several contexts, ranging from applications in the theory of hyperbolic systems of conservation and balance laws (see [12–16, 19, 39] and the references therein) to the theory of capillarity and in the study of the Prescribed Mean Curvature problem (see e.g. [36, 37]) and in the context of continuum mechanics (see e.g. [31, 42, 44]). Another field of application is related to the Dirichlet problem for equations involving the 1–Laplacian operator (see [6, 7, 11, 28, 29, 35, 40, 41]).

In a recent paper [21], the authors introduced a nonlinear version of the pairing suitable for applications to semicontinuity problems. This is the pairing appearing in the representation formula (3). The pairing $(\boldsymbol{b}(x,u), Du)$ generalizes the Anzellotti pairing and inherits its properties. In particular, in that paper we characterized the normal traces of the vector field $\boldsymbol{b}(x,u(x))$ and we performed an analysis of the singular part of the pairing measure. Moreover, we established a generalized Gauss-Green formula.

Let us now describe the contents of the present paper. We underline that all our new results have been obtained assuming that the divergence of the vector field with respect to x is an L^1 function (see Section 3 for the detailed list of assumptions on $\boldsymbol{b}$). We believe that this can be considered as a first step in order to study the general case with a divergence–measure vector field.

In Section 4, we prove a coarea formula for the measure $(\boldsymbol{b}(x,u), Du)$ and its variation (see Theorems 4.1 and 4.4).

Then, in Section 5, we show that the pairing $(\boldsymbol{b}(x,u), Du)$ admits a representation of the form

$$(\boldsymbol{b}(\cdot,u), Du) = \boldsymbol{b}(x,u)\cdot\nabla u\,dx + \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}}, \nu_u;\cdot)\,|D^c u| + \left(\fint_{u^-}^{u^+} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_u;\cdot)\,dt\right)|D^j u|, \qquad u\in BV_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)\,, \tag{4}$$

where $\mathrm{Cyl}(\boldsymbol{b}_t, \nu_u;\cdot)$ plays the role of a precise representative, defined by means of some cylindrical averages (see (7) below). The above formula extends the representation formula for the pairing obtained by Anzellotti in the unpublished paper [9] in the case of a vector field $\boldsymbol{b}(x)$ independent of u.

In the same spirit, we prove a similar representation formula

$$(\boldsymbol{b}(\cdot,u), Du) = \boldsymbol{b}(x,u)\cdot\nabla u\,dx + \mathrm{Tr}(\boldsymbol{b}(\cdot,\widetilde{u}), \partial^*\{u>\widetilde{u}(x)\})(x)|D^c u| + \left(\fint_{u^-(x)}^{u^+(x)} \mathrm{Tr}(\boldsymbol{b}(\cdot,t), \partial^*\{u>t\})(x)\,dt\right)|D^j u|, \quad u\in BV_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)\,, \tag{5}$$

based on the use of the weak normal traces as precise representatives (see Section 2.3 for their definition). This formula generalizes the representation obtained in the recent paper [17] for vector fields $\boldsymbol{b}(x)$ independent of u.

Sections 6 and 7 are devoted to the study of semicontinuity and relaxation. The main result is the representation formula (3) for the relaxed functional. Clearly, this representation formula, coupled with (4) or (5), gives a full integral representation of the relaxed functional.

In order to achieve (3), we need to prove those which in relaxation theory are called the liminf and the limsup inequalities. The first one is a consequence of the lower semicontinuity result (see Proposition 6.1 below), while the second one is obtained by using the blow-up method.

2. Preliminaries

Given $x_0\in\mathbb{R}^N$ and $\rho>0$, $B_\rho(x_0)$ denotes the ball in $\mathbb{R}^N$ centered in x_0 with radius ρ, while $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$.

In the following Ω will always denote a nonempty open subset of $\mathbb{R}^N$. We denote by $\mathcal{M}(\Omega)$ the space of signed Radon measures on Ω.

As usual, $\mathcal{L}^N$ stands for the Lebesgue measure on $\mathbb{R}^N$ and $\mathcal{H}^k$ for the k-dimensional Hausdorff measure on $\mathbb{R}^N$. The Lebesgue measure of the unit ball in $\mathbb{R}^N$ is denoted by ω_N, hence $\mathcal{L}^N(B_\rho(x_0)) = \omega_N \rho^N$.

For every $x \in \mathbb{R}^N$, $I^{x_0,r}(y) := (y-x)/r$ denotes the homothety with scaling factor r mapping x in 0, and the pushforward $I^{x,r_i}_{\#}\mu$ of a Radon measure μ in $\mathbb{R}^N$ is the measure acting on a test function ϕ as

$$\int_{\mathbb{R}^N} \phi\, d(I^{x,r}_{\#}\mu) = \int_{\mathbb{R}^N} \phi \circ I^{x_0,r}\, d\mu.$$

Let $u \in L^1_{\rm loc}(\Omega, \mathbb{R}^m)$. We say that u has an approximate limit at $x_0 \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N\left(B_r(x_0)\right)} \int_{B_r(x_0)} |u(x) - z|\; dx = 0.$$

The set $C_u \subset \Omega$ of points where this property holds is called the *approximate continuity set* of u, whereas the set $S_u := \Omega \setminus C_u$ is called the *approximate discontinuity set* of u. For any $x \in C_u$ the approximate limit z is uniquely determined and is denoted by $z := \widetilde{u}(x)$.

We say that $x \in \Omega$ is an *approximate jump point* of u if there exist $a, b \in \mathbb{R}^m$, $a \neq b$, and a unit vector $\nu \in \mathbb{R}^N$ such that

$$\begin{aligned} &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B^i_r(x))} \int_{B^i_r(x)} |u(y) - a|\, dy = 0, \\ &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B^e_r(x))} \int_{B^e_r(x)} |u(y) - b|\, dy = 0, \end{aligned} \tag{6}$$

where $B^i_r(x) := \{y \in B_r(x) :\ (y-x)\cdot\nu > 0\}$, and $B^e_r(x) := \{y \in B_r(x) :\ (y-x)\cdot\nu < 0\}$. The triplet (a, b, ν), uniquely determined by (6) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. The set of approximate jump points of u will be denoted by J_u.

The space $BV(\Omega)$ is defined as the space of all functions $u\colon \Omega \to \mathbb{R}$ belonging to $L^1(\Omega)$ whose distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure with total variation $|Du|$ bounded in Ω. We indicate by $D^a u$ and $D^s u$ the *absolutely continuous* and the *singular part* of the measure Du with respect to the Lebesgue measure. We recall that $D^a u$ and $D^s u$ are mutually singular, moreover we can write

$$Du = D^a u + D^s u \qquad \text{and} \qquad D^a u = \nabla u\, \mathcal{L}^N,$$

where ∇u is the *Radon-Nikodým derivative* of $D^a u$ with respect to the Lebesgue measure. In addition,

$$D^s u = D^c u + (u^+ - u^-)\nu_u\, \mathcal{H}^{N-1} \llcorner J_u\,,$$

where J_u is a *countably $\mathcal{H}^{N-1}$-rectifiable* Borel set (see [5, Definition 2.57]) contained in S_u, such that $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. The remaining part $D^c u$ is called the *Cantor part* of Du.

A set $E \subset \Omega$ is of finite perimeter if its characteristic function χ_E belongs to $BV(\Omega)$. If $\Omega \subset \mathbb{R}^N$ is the largest open set such that E is locally of finite perimeter in Ω, we call *reduced boundary* $\partial^* E$ of E the set of all points $x \in \Omega$ in the support of $|D\chi_E|$ such that the limit

$$\widetilde{\nu}_E(x) := \lim_{\rho\to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^N$ and satisfies $|\widetilde{\nu}_E(x)| = 1$. The function $\widetilde{\nu}_E\colon \partial^* E \to S^{N-1}$ is called the *measure theoretic unit interior normal* to E.

A fundamental result of De Giorgi (see [5, Theorem 3.59]) states that $\partial^* E$ is countably $(N-1)$-rectifiable, $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$, and $\widetilde{\nu}_E$ coincides (up to the sign) with the normal $\nu_{\partial^* E}$ defined in Section 2.3. Moreover, the measure theoretic interior normal can be choosen as normal vector to $\partial^* E$, in the sense of Section 2.3.

If $u \in BV(\Omega)$, then the level set $E_t := \{u > t\}$ is of finite perimeter for a.e. $t \in \mathbb{R}$, and we can choose the sign of the normal vectors so that $\widetilde{\nu}_{E_t}(x) = \nu_{\Sigma_t}(x) = \nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma_t$, where $\Sigma_t := \partial^*\{u > t\}$.

Moreover, we can choose an orientation on J_u such that $u^+(x) > u^-(x)$ for every $x \in J_u$. We also set $u^-(x) = u^+(x) := \widetilde{u}(x)$ for every $x \in C_u$, and $u^*(x) := [u^+(x) + u^-(x)]/2$ for every $x \in C_u \cup J_u$.

The measure Du can be disintegrated on the level sets of u using the following coarea formula (see [32, Theorem 4.5.9]).

Theorem 2.1 (Coarea formula). *If $u \in BV(\Omega)$, then for $\mathcal{L}^1$–a.e. $t \in \mathbb{R}$ the set $\{u > t\}$ has finite perimeter in Ω and the following coarea formula holds*

$$\int_\Omega g \, d|Du| = \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u>t\}\cap\Omega} g \, d\mathcal{H}^{N-1} = \int_{-\infty}^{+\infty} dt \int_{\{u^- \le t \le u^+\}} g \, d\mathcal{H}^{N-1},$$

for every Borel function $g : \Omega \to [0, +\infty]$. Moreover, for $\mathcal{L}^1$–a.e. $t \in \mathbb{R}$,

(a) $\partial^*\{u > t\} \subset \{u^- \le t \le u^+\}$,
(b) $\mathcal{H}^{N-1}\Big(\{u^- \le t \le u^+\} \setminus \big(\partial^*\{u > t\}\big)\Big) = 0$,

and, in particular,

(a′) $\partial^*\{u > t\} \cap (\Omega \setminus S_u) \subseteq \{x \in \Omega \setminus S_u \colon \widetilde{u}(x) = t\}$,
(b′) $\mathcal{H}^{N-1}\Big(\{x \in \Omega \setminus S_u \colon \widetilde{u}(x) = t\} \setminus \big((\Omega \setminus S_u) \cap \partial^*\{u > t\}\big)\Big) = 0$.

2.1. **Divergence–measure fields.** We will denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distributions is a bounded Radon measure in Ω. Similarly, $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ will denote the space of all vector fields $\boldsymbol{A} \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure in Ω. We set $\mathcal{DM}^\infty = \mathcal{DM}^\infty(\mathbb{R}^N)$. Moreover, we denote by $\mathcal{DL}^1(\Omega)$ (resp. $\mathcal{DL}^1_{\rm loc}(\Omega)$) the subset of $\mathcal{DM}^\infty(\Omega)$ (resp. $\mathcal{DM}^\infty_{\rm loc}(\Omega)$) of vector fields whose divergence is in $L^1(\Omega)$ (resp. $L^1_{\rm loc}(\Omega)$).

We recall that, if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, then $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$ (see [12, Proposition 3.1]). As a consequence, the set

$$\Theta_{\boldsymbol{A}} := \left\{ x \in \Omega : \limsup_{r \to 0+} \frac{|\operatorname{div} \boldsymbol{A}|(B_r(x))}{r^{N-1}} > 0 \right\},$$

is a Borel set, σ-finite with respect to $\mathcal{H}^{N-1}$, and the measure $\operatorname{div} \boldsymbol{A}$ can be decomposed as

$$\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} + \operatorname{div}^c \boldsymbol{A} + \operatorname{div}^j \boldsymbol{A},$$

where $\operatorname{div}^a \boldsymbol{A}$ is absolutely continuous with respect to $\mathcal{L}^N$, $\operatorname{div}^c \boldsymbol{A}(B) = 0$ for every set B with $\mathcal{H}^{N-1}(B) < +\infty$, and

$$\operatorname{div}^j \boldsymbol{A} = h \, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}}$$

for some Borel function h (see [4, Proposition 2.3]).

2.2. **Anzellotti's pairing.** As in Anzellotti [8] (see also [12]), for every $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ we define the linear functional $(\boldsymbol{A}, Du)\colon C^\infty_0(\Omega) \to \mathbb{R}$ by

$$\langle (\boldsymbol{A}, Du)\,,\, \varphi\rangle := -\int_\Omega u^* \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u\, \boldsymbol{A} \cdot \nabla\varphi \, dx.$$

The distribution $(\boldsymbol{A}, Du)$ is a Radon measure in Ω, absolutely continuous with respect to $|Du|$ (see [8, Theorem 1.5] and [12, Theorem 3.2]), hence the equation

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du)$$

holds in the sense of measures in Ω. Furthermore, Chen and Frid in [12] proved that the absolutely continuous part of this measure with respect to the Lebesgue measure is given by $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N$.

In [9] it is proved that, for every $\boldsymbol{A} \in \mathcal{DL}^1_{\rm loc}(\Omega)$,

$$(\boldsymbol{A}, Du) = \operatorname{Cyl}(\boldsymbol{A}, \nu_u; \cdot)\, |Du|\,, \qquad |Du|\text{–a.e. in } \Omega,$$

where

$$\operatorname{Cyl}(\boldsymbol{A}, \nu_u; x) := \lim_{\rho\downarrow 0} \lim_{r\downarrow 0} \frac{1}{\mathcal{L}^N(C_{r,\rho}(x, \nu_u(x)))} \int_{C_{r,\rho}(x,\nu_u(x))} \boldsymbol{A}(y) \cdot \nu_u(x)\, dy \tag{7}$$

and, for every $\zeta \in \mathbb{S}^{N-1}$,

$$C_{r,\rho}(x, \zeta) := \left\{y \in \mathbb{R}^N :\ |(y - x) \cdot \zeta| < r,\ |(y - x) - [(y - x) \cdot \zeta]\zeta| < \rho\right\}.$$

(The existence of the limit in (7) for $|Du|$–a.e. $x \in \Omega$ is part of the statement.)

As a consequence, it holds that

$$\lim_{r\downarrow 0} \frac{(\boldsymbol{A}, Du)(B_r(x))}{|Du|(B_r(x))} = \operatorname{Cyl}(\boldsymbol{A}, \nu_u; x) \qquad \text{for } |Du|\text{–a.e. } x \in \Omega.$$

2.3. **Weak normal traces on oriented countably $\mathcal{H}^{N-1}$-rectifiable sets.** We recall the notion of the traces of the normal component of a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ on an oriented countably $\mathcal{H}^{N-1}$–rectifiable set $\Sigma \subset \Omega$, introduced in [3, Propositions 3.2, 3.4 and Definition 3.3]. In that paper the authors proved that there exist the normal traces $\operatorname{Tr}^+(\boldsymbol{A}, \Sigma), \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)$ belonging to $L^\infty(\Sigma, \mathcal{H}^{N-1} \llcorner \Sigma)$ and satisfying

$$\operatorname{div} \boldsymbol{A} \llcorner \Sigma = \left[\operatorname{Tr}^+(\boldsymbol{A}, \Sigma) - \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)\right]\, \mathcal{H}^{N-1} \llcorner \Sigma. \tag{8}$$

In what follows we use the notation

$$\operatorname{Tr}^*(\boldsymbol{A}, \Sigma) := \frac{\operatorname{Tr}^+(\boldsymbol{A}, \Sigma) + \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)}{2}\,.$$

If $\boldsymbol{A} \in \mathcal{DL}^1_{\rm loc}(\Omega)$, then $\operatorname{Tr}^+(\boldsymbol{A}, \Sigma) = \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)$ and $\operatorname{div} \boldsymbol{A} \llcorner \Sigma = 0$.

2.4. **Representation formulas for the pairing measure.** In the following theorem, the pairing is characterized in terms of normal traces of the field $\boldsymbol{A}$ on level sets of u.

Theorem 2.2 (see [17], Thm. 3.9)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\mathbb{R}^N)$ and $u \in BV_{\rm loc}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$. Then, the following equality holds in the sense of measures*

$$(\boldsymbol{A}, Du) = \fint_{u^-(x)}^{u^+(x)} \operatorname{Tr}^*(\boldsymbol{A}, \partial^*\{u > t\})(x)\, dt |Du|\,, \tag{9}$$

where we use the convention $\fint_a^a f(t)\, dt := f(a)$. Moreover,

(i) *absolutely continuous part: $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N$;*
(ii) *jump part: $(\boldsymbol{A}, Du)^j = \operatorname{Tr}^*(\boldsymbol{A}, J_u)(x)|D^j u|$;*

(iii) *Cantor part:* $(\boldsymbol{A}, Du)^c = \mathrm{Tr}^*(\boldsymbol{A}, \partial^*\{u > \widetilde{u}(x)\})(x)|D^c u|$.

Corollary 2.3. *Let* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$*, and let* $E \subseteq \Omega$ *be a set of finite perimeter, with* $\overline{E} \subset \Omega$*. Then*

$$(\boldsymbol{A}, D\chi_E) = \mathrm{Tr}^*(\boldsymbol{A}, \partial^* E)(x)\mathcal{H}^{N-1} \llcorner \partial^* E.$$

3. Assumptions on the vector field $\boldsymbol{b}$

Let $\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a function satisfying the following assumptions:

(i) $\boldsymbol{b}$ is a locally bounded Borel function;

(ii) the function $\boldsymbol{b}(x, \cdot)$ is Lipschitz continuous in $\mathbb{R}$, uniformly with respect to x, i.e. there exists a constant $L > 0$ such that

$$|\boldsymbol{b}(x,t) - \boldsymbol{b}(x,s)| \leq L\,|t-s|, \qquad \forall t, s \in \mathbb{R}, \text{ for } \mathcal{L}^N\text{–a.e. } x \in \Omega\,;$$

(iii) for every $t \in \mathbb{R}$, $\boldsymbol{b}_t := \boldsymbol{b}(\cdot, t) \in \mathcal{DL}^1_{\mathrm{loc}}(\Omega)$;

(iv) the least upper bound

$$\sigma := \sup_{t\in\mathbb{R}} |\operatorname{div}_x \boldsymbol{b}_t|$$

belongs to $L^1_{\mathrm{loc}}(\Omega)$.

We remark that, at the price of some additional technicality, assumption (iv) could be replaced by the weaker assumption

(iv′) for every $m > 0$, the least upper bound

$$\sigma_m := \sup_{|t|\leq m} |\operatorname{div}_x \boldsymbol{b}_t|$$

belongs to $L^1_{\mathrm{loc}}(\Omega)$.

The results of Section 4 will be mainly proved replacing (ii) with the following weaker assumption:

(ii′) for $\mathcal{L}^N$–a.e. $x \in \Omega$, the function $\boldsymbol{b}(x, \cdot)$ is continuous in $\mathbb{R}$.

Let us extend $\boldsymbol{b}$ to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$, so that the vector field

$$\boldsymbol{B}(x,t) := \int_0^t \boldsymbol{b}(x,s)\,ds, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{10}$$

is defined for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Moreover $\boldsymbol{B}(x,0) = 0$ for every $x \in \mathbb{R}^N$ and, from (ii′), for every $x \in \mathbb{R}^n$ one has $\boldsymbol{b}(x,t) = \partial_t \boldsymbol{B}(x,t)$ for every $t \in \mathbb{R}$.

Theorem 3.1 (See [21]). *Let* $\boldsymbol{b}$ *satisfy assumptions (i)-(ii′)-(iii)-(iv), let* $\boldsymbol{B}$ *be defined by* (10)*, and let* $u \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$*. Then the distribution* $(\boldsymbol{b}(\cdot,u), Du)$*, defined by*

$$\begin{aligned}\langle(\boldsymbol{b}(\cdot,u), Du), \varphi\rangle := &-\int_\Omega \varphi(x)\,(\operatorname{div}_x \boldsymbol{B})(x, u(x))\,dx \\ &-\int_\Omega \boldsymbol{B}(x,u(x)) \cdot \nabla\varphi(x)\,dx, \qquad \forall \varphi \in C^\infty_c(\Omega),\end{aligned} \tag{11}$$

is a Radon measure in Ω*, and satisfies*

$$|(\boldsymbol{b}(\cdot,u), Du)|(E) \leq \|\boldsymbol{b}\|_{L^\infty(K,\mathbb{R}^N)}|Du|(E), \qquad \textit{for every Borel set } E \Subset \Omega\,, \tag{12}$$

where $K := \overline{E} \times \left[-\|u\|_{L^\infty(\overline{E})}, \|u\|_{L^\infty(\overline{E})}\right]$.

In other words, the composite function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, *defined by* $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$, *belongs to* $L^\infty_{\rm loc}(\Omega, \mathbb{R}^N)$, *and the following equality holds in the sense of measures:*

$$\operatorname{div} \mathbf{v} = (\operatorname{div}_x \boldsymbol{B})(x, u(x))\, \mathcal{L}^N + (\boldsymbol{b}(\cdot, u), Du). \tag{13}$$

From (12) it follows that $(\boldsymbol{b}(\cdot, u), Du) \ll |Du|$, hence there exists a function $\Theta(\boldsymbol{b}, u; \cdot) \in L^1(\Omega, |Du|)$ such that

$$(\boldsymbol{b}(\cdot, u), Du) = \Theta(\boldsymbol{b}, u; \cdot)\, |Du|\,, \qquad |Du|\text{–a.e. in } \Omega. \tag{14}$$

Remark 3.2. By the definition (11) of the pairing and the definition (10) of $\boldsymbol{B}$, it follows that, for every $\varphi \in C^\infty_c(\Omega)$,

$$\begin{aligned} \langle (\boldsymbol{b}(\cdot, u), Du), \varphi \rangle = & -\int_\Omega \varphi(x) \int_0^{u(x)} \operatorname{div}_x \boldsymbol{b}_t(x)\, dt\, dx \\ & - \int_\Omega \int_0^{u(x)} \boldsymbol{b}_t(x) \cdot \nabla\varphi(x)\, dt\, dx. \end{aligned} \tag{15}$$

4. Coarea formula for the pairing measure

In this section we establish a coarea formula for the pairing measure $(\boldsymbol{b}(\cdot, u), Du)$, and we draw some consequences that will be used in order to prove its integral representation (see Theorem 5.1 below).

Theorem 4.1 (Coarea formula for the pairing measure)**.** *Let* $\boldsymbol{b}$ *satisfy assumptions (i)-(ii')-(iii)-(iv), and let* $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. *Then*

$$\langle (\boldsymbol{b}(\cdot, u), Du), \varphi \rangle = \int_{\mathbb{R}} \langle (\boldsymbol{b}_t, D\chi_{\{u>t\}}), \varphi \rangle\, dt, \qquad \forall \varphi \in C^\infty_c(\Omega)\,, \tag{16}$$

$$(\boldsymbol{b}(\cdot, u), Du)(B) = \int_{\mathbb{R}} (\boldsymbol{b}_t, D\chi_{\{u>t\}})(B)\, dt, \qquad \forall \text{ Borel set } B \subset \Omega\,. \tag{17}$$

Proof. Assume, for simplicity, that $u \geq 0$ and let $C > \|u\|_\infty$. Using the representation (15), we have that

$$\begin{aligned} \langle (\boldsymbol{b}(\cdot, u), Du), \varphi \rangle & = -\int_0^C \int_\Omega \chi_{\{u>t\}} \varphi \operatorname{div}_x \boldsymbol{b}_t\, dx\, dt - \int_0^C \int_\Omega \chi_{\{u>t\}} \boldsymbol{b}_t \cdot \nabla\varphi\, dx\, dt \\ & = \int_0^C \langle (\boldsymbol{b}_t, D\chi_{\{u>t\}}), \varphi \rangle\, dt, \end{aligned}$$

where, in the last equality, we have used the fact that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$,

$$\operatorname{div}(\chi_{\{u>t\}} \boldsymbol{b}_t) = \chi^*_{\{u>t\}} \operatorname{div} \boldsymbol{b}_t + (\boldsymbol{b}_t, D\chi_{\{u>t\}}).$$

The general case follows with minor modifications.

Finally, since both sides of (16) are real measures in Ω, they coincide not only as distributions, but also as measures, hence (17) follows. □

The following approximation result is in the spirit of [22, Proposition 4.11], [20, Proposition 4.15], [12, Theorem 1.2], [8, Lemma 2.2].

Theorem 4.2 (Approximation by C^∞ fields)**.** *Let* $\boldsymbol{b}$ *satisfy assumptions (i)-(ii')-(iii)-(iv). Then there exists a sequence of vector fields* $\boldsymbol{b}^k\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ *satisfying the same assumptions, such that* $\boldsymbol{b}^k_t \in C^\infty(\Omega, \mathbb{R}^N)$ *for every* $t \in \mathbb{R}$ *and*

$$(\boldsymbol{b}^k(\cdot, u), Du) \stackrel{*}{\rightharpoonup} (\boldsymbol{b}(\cdot, u), Du), \qquad \forall u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega),$$

locally in the weak sense of measures in* Ω. *If, in addition,* $\boldsymbol{b}$ *satisfies (ii), then also the vector fields* $\boldsymbol{b}^k$ *satisfy (ii).*

Proof. Using the same construction described in the proof of [22, Proposition 4.11], we obtain locally uniformly bounded vector fields $\boldsymbol{b}_t^k \in C^\infty(\Omega, \mathbb{R}^N)$ satisfying (i)-(ii′)-(iii)-(iv), and, for every $t \in \mathbb{R}$, $\boldsymbol{b}_t^k \to \boldsymbol{b}_t$ in $L^1_{loc}(\Omega)$. If, in addition, $\boldsymbol{b}$ satisfies (ii), then it is verified that also the vector fields $\boldsymbol{b}^k$ satisfy (ii).

Moreover, for every $t \in \mathbb{R}$ and $v \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$,

$$\lim_{k\to+\infty} \int_\Omega v\,\varphi\,\operatorname{div} \boldsymbol{b}_t^k\,dx = \int_\Omega v^*\,\varphi\,d\operatorname{div}\boldsymbol{b}_t, \qquad \forall \varphi \in C_c(\Omega)$$

(see [22], formula (4.8)). We underline that, since by assumption $\operatorname{div}\boldsymbol{b}_t \in L^1_{loc}(\Omega)$, then the above relation can be written as

$$\lim_{k\to+\infty} \int_\Omega v\,\varphi\,\operatorname{div} \boldsymbol{b}_t^k\,dx = \int_\Omega v\,\varphi\,\operatorname{div}\boldsymbol{b}_t\,dx, \qquad \forall \varphi \in C_c(\Omega)\,. \tag{18}$$

Let us fix $u \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and $\varphi \in C_c(\Omega)$. To simplify the notation, we assume without loss of generality that $u \geq 0$. By the representation formula (15) and Fubini's Theorem, we have that

$$\begin{aligned}
\langle(\boldsymbol{b}^k(\cdot,u), Du), \varphi\rangle &= -\int_\Omega \varphi(x) \int_0^{u(x)} \operatorname{div}_x \boldsymbol{b}_t^k(x)\,dt\,dx \\
&\quad - \int_\Omega \int_0^{u(x)} \boldsymbol{b}_t^k(x)\cdot\nabla\varphi(x)\,dt\,dx \\
&= -\int_0^\infty \int_\Omega \chi_{\{u>t\}}(x)\,\varphi(x)\,\operatorname{div}_x \boldsymbol{b}_t^k(x)\,dx\,dt \\
&\quad - \int_0^\infty \int_\Omega \chi_{\{u>t\}}(x)\,\boldsymbol{b}_t^k(x)\cdot\nabla\varphi(x)\,dx\,dt \\
&=: -I_1^k - I_2^k.
\end{aligned}$$

For every $t \in \mathbb{R}$, by (18) with $v = \chi_{\{u>t\}}$ we deduce that, as $k \to \infty$,

$$\zeta^k(t) := \int_\Omega \chi_{\{u>t\}}(x)\,\varphi(x)\,\operatorname{div}_x \boldsymbol{b}_t^k(x)\,dx \to \int_\Omega \chi_{\{u>t\}}(x)\,\varphi(x)\,\operatorname{div}_x \boldsymbol{b}_t(x)\,dx\,.$$

Let $K \Subset \Omega$ denote the support of φ and let $a := \|u\|_{L^\infty(K)}$. Since

$$|\zeta^k(t)| \leq \chi_{[0,a]}(t)\,\|\varphi\|_\infty \int_K \sigma\,,$$

by the Dominated Convergence Theorem we deduce that

$$\lim_{k\to\infty} I_1^k = \int_0^\infty \int_\Omega \chi_{\{u>t\}}(x)\,\varphi(x)\,\operatorname{div}_x \boldsymbol{b}_t^k(x)\,dx\,dt = \int_\Omega \varphi(x) \int_0^{u(x)} \operatorname{div}_x \boldsymbol{b}_t^k(x)\,dt\,dx\,. \tag{19}$$

Let us compute the limit of I_2^k. Since, for every $t \in \mathbb{R}$, $\boldsymbol{b}_t^k \to \boldsymbol{b}_t$ in $L^1_{loc}(\Omega)$, it holds that

$$\psi^k(t) := \int_\Omega \chi_{\{u>t\}}(x)\,\boldsymbol{b}_t^k(x)\cdot\nabla\varphi(x)\,dx \to \int_\Omega \chi_{\{u>t\}}(x)\,\boldsymbol{b}_t(x)\cdot\nabla\varphi(x)\,dx\,.$$

Moreover, there exists a constant $M > 0$ such that $\|\boldsymbol{b}^k\|_{L^\infty(K\times[-a,a])} \leq M$ for every $k \in \mathbb{N}$, so that

$$\psi^k(t)| \leq M\,\|\nabla\varphi\|_1\,,$$

and hence, by the Dominated Convergence Theorem,

$$\lim_{k\to+\infty} I_2^k = \int_0^\infty \int_\Omega \chi_{\{u>t\}}(x)\, \boldsymbol{b}_s(x)\cdot\nabla\varphi(x)\,dx\,dt = \int_\Omega \int_0^{u(x)} \boldsymbol{b}_t^k(x)\cdot\nabla\varphi(x)\,dt\,dx\,. \tag{20}$$

The conclusion now follows from (19) and (20). □

Proposition 4.3. *Let $\boldsymbol{b}$ satisfy assumptions (i)-(ii′)-(iii)-(iv), and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then*

$$\text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R}: \quad \Theta(\boldsymbol{b},u;x) = \Theta(\boldsymbol{b}_t,\chi_{\{u>t\}};x) \quad \text{for } |D\chi_{\{u>t\}}|\text{-a.e. } x\in\Omega\,.$$

Proof. The proof is essentially the same of Proposition 5.2 in [22], and it is based on the use of the coarea formula (Theorem 4.1) and the approximation result by smooth fields (Theorem 4.2). □

Theorem 4.4 (Coarea formula for the variation)**.** *Let $\boldsymbol{b}$ satisfy assumptions (i)-(ii′)-(iii)-(iv), and let $u \in BV_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$. Then*

$$\langle |(\boldsymbol{b}(\cdot,u),Du)|\;,\;\varphi\rangle = \int_{\mathbb{R}} \langle \left|(\boldsymbol{b}_t, D\chi_{\{u>t\}})\right|\;,\;\varphi\rangle\,dt, \qquad \forall\varphi\in C_c^\infty(\Omega).$$

Proof. To simplify the notation let $\mu := (\boldsymbol{b}(\cdot,u),Du)$ and $\mu_t := (\boldsymbol{b}_t, D\chi_{\{u>t\}})$, $t\in\mathbb{R}$. By (14), we have that $\mu = \Theta(\boldsymbol{b},u)\,|Du|$, so that

$$|\mu| = |\Theta(\boldsymbol{b},u)|\,|Du|\,, \qquad |\mu_t| = |\Theta(\boldsymbol{b}_t,\chi_{\{u>t\}})|\,|D\chi_{\{u>t\}}|$$

(see [5, Proposition 1.23]). Let $B\subset\Omega$ be a Borel set. By the coarea formula in BV (see [5, Theorem 3.40]) and Proposition 4.3 it holds that

$$\begin{aligned}|\mu|(B) &= \int_B |\Theta(\boldsymbol{b},u)|\,d|Du| = \int_{\mathbb{R}} dt \int_B |\Theta(\boldsymbol{b},u)|\,d|D\chi_{\{u>t\}}| \\ &= \int_{\mathbb{R}} dt \int_B |\Theta(\boldsymbol{b}_t,\chi_{\{u>t\}})|\,d|D\chi_{\{u>t\}}| = \int_{\mathbb{R}} |\mu_t|(B)\,dt\,,\end{aligned}$$

concluding the proof. □

Lemma 4.5. *Let $\boldsymbol{b}$ satisfy (i)–(iv), and let $u\in BV_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$. Then, for every $\tau\in\mathbb{R}$ and every $\varphi\in C_c^\infty(\Omega)$, it holds that*

$$\begin{aligned}&|\langle(\boldsymbol{b}(\cdot,u),Du),\varphi\rangle - \langle(\boldsymbol{b}_\tau,Du),\varphi\rangle| \\ &\leq L\,\|\varphi\|_\infty \left[\int_{\operatorname{spt}\varphi} |\widetilde{u}-\tau|\,d|D^d u| + \int_{J_u\cap\operatorname{spt}\varphi} \left(\int_{u^-}^{u^+} |t-\tau|\,dt\right) d\mathcal{H}^{N-1}\right],\end{aligned}$$

where $\operatorname{spt}\varphi\Subset\Omega$ denotes the support of φ.

Proof. Using the coarea formula (16) and (ii) we obtain that

$$\begin{aligned}I_\tau &:= |\langle(\boldsymbol{b}(\cdot,u),Du),\varphi\rangle - \langle(\boldsymbol{b}_\tau,Du),\varphi\rangle| = \left|\int_{\mathbb{R}} \langle(\boldsymbol{b}_t-\boldsymbol{b}_\tau, D\chi_{\{u>t\}}),\varphi\rangle\,dt\right| \\ &\leq \|\varphi\|_\infty \int_{\mathbb{R}}\int_{\operatorname{spt}\varphi} \|\boldsymbol{b}_t-\boldsymbol{b}_\tau\|_\infty\,d|D\chi_{\{u>t\}}|\,dt \\ &\leq L\,\|\varphi\|_\infty \int_{\mathbb{R}}\int_{\operatorname{spt}\varphi} |t-\tau|\,d|D\chi_{\{u>t\}}|\,dt\,.\end{aligned}$$

We now consider $\operatorname{spt}\varphi$ as the disjoint union of $\operatorname{spt}\varphi \setminus J_u$ and $J_u \cap \operatorname{spt}\varphi$, and we use the coarea formula in BV, obtaining

$$
\begin{aligned}
I_\tau &\leq L\,\|\varphi\|_\infty \left[\int_{\mathbb{R}} \int_{\operatorname{spt}\varphi\setminus J_u} |t-\tau|\, d|D\chi_{\{u>t\}}|\, dt + \int_{\mathbb{R}} \int_{J_u\cap \operatorname{spt}\varphi} |t-\tau|\, d|D\chi_{\{u>t\}}|\, dt\right] \\
&= L\,\|\varphi\|_\infty \left[\int_{\operatorname{spt}\varphi} |\widetilde{u}-\tau|\, d|D^d u| + \int_{J_u\cap \operatorname{spt}\varphi} \left(\int_{u^-}^{u^+} |t-\tau|\, dt\right) d\mathcal{H}^{N-1}\right]. \qquad \square
\end{aligned}
$$

5. Integral representation of the pairing

In this section we are interested in finding an integral representation of the pairing measure and of its total variation. We prove that the pairing measure can be represented by an integral functional defined on the space $BV(\Omega)$, provided that in the support of the singular part of the measure we choose a suitable precise representative of the vector field $\boldsymbol{b}$.

We recall the general form of an integral functional defined in $BV(\Omega)$. Given the integrand $f(x,t,\xi) = \boldsymbol{b}(x,t)\cdot\xi$, for every open set $A \subset \Omega$, let us define the functional $\mathcal{F}(\cdot, A)\colon BV(\Omega) \to]-\infty, +\infty]$ by setting

$$
\begin{aligned}
\mathcal{F}(u,A) = &\int_A \boldsymbol{b}(x,u)\cdot \nabla u\, dx \\
&+ \int_A \overline{f}\Big(x, \widetilde{u}, \frac{D^c u}{|D^c u|}\Big)\, d|D^c u| + \int_{J_u\cap A} d\mathcal{H}^{N-1} \int_{u^-}^{u^+} \overline{f}(x,t,\nu_u)\, dt,
\end{aligned}
$$

where $\overline{f}(\cdot, s, \xi)$ is a proper precise representative of $f(\cdot, t, \xi) = \boldsymbol{b}_t \cdot \xi$.

We show that, in our case, this representative is the limit of cylindrical averages introduced in [9] for vector fields $\boldsymbol{b}(x)$ whose divergence belongs to L^1.

Theorem 5.1 (Integral representation of the pairing measure)**.** *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv), and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then it holds that*

$$
(\boldsymbol{b}(\cdot,u), Du) = \operatorname{Cyl}(\boldsymbol{b}_{\widetilde{u}}, \nu_u; \cdot)\, |D^d u| + \left(\fint_{u^-}^{u^+} \operatorname{Cyl}(\boldsymbol{b}_t, \nu_u; \cdot)\, dt\right) |D^j u|.
$$

In other words, the density Θ defined at (14) *is given by*

$$
\Theta(\boldsymbol{b}, u; x) = \begin{cases} \operatorname{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x)\,, & |D^d u|\text{–a.e. } x \in \Omega, \\ \displaystyle\fint_{u^-(x)}^{u^+(x)} \operatorname{Cyl}(\boldsymbol{b}_t, \nu_u; x)\, dt\,, & \mathcal{H}^{N-1}\text{–a.e. } x \in J_u. \end{cases} \tag{21}
$$

Moreover, $\operatorname{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x)|\nabla u(x)| = \boldsymbol{b}(x, u(x)) \cdot \nabla u(x)$ *for $\mathcal{L}^N$-a.e. $x \in \Omega$.*

Proof. By assumption (iii), for every $t \in \mathbb{R}$ we have that

$$
\begin{aligned}
\frac{d(\boldsymbol{b}_t, Du)}{d|Du|} &= \operatorname{Cyl}(\boldsymbol{b}_t, \nu_u; x), \qquad |D^d u|\text{–a.e. in } \Omega, \\
\operatorname{Tr}^+(\boldsymbol{B}_t, J_u) = \operatorname{Tr}^-(\boldsymbol{B}_t, J_u) &= \operatorname{Cyl}(\boldsymbol{B}_t, \nu_u; \cdot), \qquad \mathcal{H}^{N-1}\text{–a.e. in } J_u
\end{aligned}
$$

(see [9, Theorems 2.6 and 3.6]).

The representation of the jump part (i.e. of Θ on J_u) follows directly from [21, Theorem 5.6] and the simple computation

$$\mathrm{Cyl}(\boldsymbol{B}_{u^+(x)}, \nu_u; x) - \mathrm{Cyl}(\boldsymbol{B}_{u^-(x)}, \nu_u; x) = \int_{u^-(x)}^{u^+(x)} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_u; x)\, dt\,.$$

It remains to prove that

$$\Theta(\boldsymbol{b}, Du; \cdot) = \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}}, \nu_u; \cdot) \qquad |D^d u|\text{–a.e. in } \Omega\,.$$

First, we remark that there exists a Borel set $N \subset \Omega$, with $|D^d u|(N) = 0$, such that the limit of cylindrical averages $\mathrm{Cyl}(\boldsymbol{b}_t, \nu_u; \cdot)$ exists for every $x \in \Omega \setminus N$ and every $t \in \mathbb{R}$ (see e.g. the proof of [21, Lemma 4.2]). As a consequence, the map $x \mapsto \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x)$ belongs to $L^\infty_{\mathrm{loc}}(\Omega, |D^d u|)$.

To simplify the notation, let us denote by $\mu := (\boldsymbol{b}(\cdot, u), Du)$ the pairing measure. We have to prove that

$$\frac{d\mu^d}{d|Du|}(x) = \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x)\,, \quad \text{for } |D^d u|\text{–a.e. } x \in \Omega.$$

Let us choose $x \in \Omega$ such that

(a) x belongs to the support of $D^d u$, that is $|D^d u|(B_r(x)) > 0$ for every $r > 0$;

(b) there exists the limit $\lim_{r\downarrow 0} \dfrac{\mu^d(B_r(x))}{|Du|(B_r(x))} = \dfrac{d\mu^d}{d|Du|}(x)$;

(c) $\lim_{r\downarrow 0} \dfrac{|D^j u|(B_r(x))}{|Du|(B_r(x))} = 0$;

(d) $\lim_{r\downarrow 0} \dfrac{(\boldsymbol{b}_{\widetilde{u}(x)}, Du)(B_r(x))}{|Du|(B_r(x))} = \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x)$;

(e) $\lim_{r\downarrow 0} \dfrac{1}{|Du|(B_r(x))} \displaystyle\int_{B_r(x)} |\widetilde{u}(y) - \widetilde{u}(x)|\, d|D^d u|(y) = 0$.

We remark that these conditions are satisfied for $|D^d u|$-a.e. $x \in \Omega$. In particular, (e) holds since $|D^d u|$-a.e. $x \in \Omega$ is a Lebesgue point of $\widetilde{u}$ with respect to $|Du|$.

Since

$$\begin{aligned}
&\left| \frac{(\boldsymbol{b}(\cdot, u), Du)(B_r(x))}{|Du|(B_r(x))} - \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x) \right| \\
&\leq \left| \frac{(\boldsymbol{b}(\cdot, u), Du)(B_r(x))}{|Du|(B_r(x))} - \frac{(\boldsymbol{b}_{\widetilde{u}(x)}, Du)(B_r(x))}{|Du|(B_r(x))} \right| + \left| \frac{(\boldsymbol{b}_{\widetilde{u}(x)}, Du)(B_r(x))}{|Du|(B_r(x))} - \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}(x)}, \nu_u; x) \right|,
\end{aligned}$$

by (d) it is enough to prove that

$$I_r := \left| \frac{(\boldsymbol{b}(\cdot, u), Du)(B_r(x))}{|Du|(B_r(x))} - \frac{(\boldsymbol{b}_{\widetilde{u}(x)}, Du)(B_r(x))}{|Du|(B_r(x))} \right| \longrightarrow 0\,, \qquad \text{as } r \searrow 0, \tag{22}$$

i.e.

$$\frac{d(\boldsymbol{b}(\cdot, u), Du)}{d|Du|}(x) = \frac{d(\boldsymbol{b}_{\widetilde{u}(x)}, Du)}{d|Du|}(x)\,.$$

By Lemma 4.5, choosing $\tau = \widetilde{u}(x)$ and taking a sequence $\phi_j \in C_c^\infty(B_r(x))$, $\phi_j(y) \to 1$ in $B_r(x)$, with $0 \le \phi_j \le 1$, we get

$$\begin{aligned} I_r &\le \frac{L}{|Du|(B_r(x))}\Big[\int_{B_r(x)} |\widetilde{u}(y) - \widetilde{u}(x)|\, d|D^d u|(y) \\ &\quad + \int_{B_r(x)\cap J_u} \left(\int_{u^-(y)}^{u^+(y)} |t - \widetilde{u}(x)|\, dt\right) d\mathcal{H}^{N-1}(y)\Big] \\ &\le \frac{L}{|Du|(B_r(x))}\int_{B_r(x)} |\widetilde{u}(y) - \widetilde{u}(x)|\, d|D^d u|(y) + 2L\|u\|_\infty \frac{|D^j u|(B_r(x))}{|Du|(B_r(x))}\,. \end{aligned}$$

Finally, by (c) and (e) we conclude that (22) holds. □

Corollary 5.2 (Integral representation of the pairing functional)**.** *Let* $\boldsymbol{b}$ *satisfy assumptions (i)–(iv), and let* $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$*. Then it holds that*

$$\int_\Omega \phi(\boldsymbol{b}(x,u), Du) = \int_\Omega \phi \fint_{u^-}^{u^+} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_u; x)\, dt|Du|, \qquad \phi \in C_c(\Omega)\,, \tag{23}$$

where we use the compact notation

$$\begin{aligned} \fint_{u^-}^{u^+} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_u; x)\, dt|Du| &:= \boldsymbol{b}(x,u)\cdot\nabla u\, dx + \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}}, \nu_u; x)\, |D^c u| \\ &\qquad + \fint_{u^-}^{u^+} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_u; x)\, dt|D^j u|. \end{aligned} \tag{24}$$

Remark 5.3. As a direct consequence of the above corollary, it holds that

$$\frac{(\boldsymbol{b}(\cdot,u), Du)}{|D^c u|}(x) = \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}}, \nu_u; x)$$

for $|D^c u|$-a.e. $\in \Omega$.

Theorem 5.4. *Let* $\boldsymbol{b}$ *satisfy assumptions (i)–(iv), and* $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$*. Then, for* $|Du|$*-a.e.* $x \in \Omega$*,*

$$\Theta(\boldsymbol{b}, u; x) = \fint_{u^-(x)}^{u^+(x)} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u > t\})(x)\, dt\,, \tag{25}$$

where we use the convention $\fint_a^a f(t)\, dt := f(a)$*. In particular,*

$$\Theta(\boldsymbol{b}, u; x) = \mathrm{Tr}(\boldsymbol{b}(\cdot,u), \partial^*\{u > \widetilde{u}(x)\})(x), \qquad \textit{for } |D^d u|\textit{-a.e. } x \in \Omega. \tag{26}$$

Proof. It suffices to prove that, for every Borel set $B \subset \Omega$, it holds that

$$\int_B \boldsymbol{b}(\cdot,u), Du) = \int_B \fint_{u^-(x)}^{u^+(x)} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u > t\})(x)\, dt\, |Du|. \tag{27}$$

For every $t \in \mathbb{R}$ such that $\partial^*\{u > t\}$ is locally of finite perimeter (and hence for a.e. $t \in \mathbb{R}$), by Corollary 2.3 we deduce that

$$\Theta(\boldsymbol{b}_t, D\chi_{\{u>t\}}, x) = \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u > t\})(x) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^*\{u > t\}. \tag{28}$$

Then, by using the coarea formula (16) for the pairing measure, formula (28), Fubini's theorem and Theorem 2.1 we get

$$
\begin{aligned}
\int_B \boldsymbol{b}(\cdot,u), Du) &= \int_{\mathbb{R}} \int_B (\boldsymbol{b}_t, D\chi_{\{u>t\}})\,dt = \int_{\mathbb{R}} \int_{B\cap\partial^*\{u>t\}} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u>t\}) d\mathcal{H}^{N-1}\,dt \\
&= \int_{B\setminus J_u} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u>\widetilde{u}(x)\})\,d\mathcal{H}^{N-1} + \int_{B\cap J_u} \int_{u^-(x)}^{u^+(x)} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u>t\})\,dt\,d\mathcal{H}^{N-1} \\
&= \int_B \fint_{u^-(x)}^{u^+(x)} \mathrm{Tr}(\boldsymbol{b}_t, \partial^*\{u>t\})\,dt\,|Du|\,,
\end{aligned}
$$

so that (27) is proved. □

6. Lower semicontinuity of the pairing

In this section, by using the nonautonomous chain rule formula (13) for the divergence, we study the lower semicontinuity with respect to the L^1 convergence of the functionals $F, G^+ \colon BV(\Omega)\cap L^\infty(\Omega) \to \mathbb{R}$ defined by

$$
F(u) := \int_\Omega |(\boldsymbol{b}(x,u), Du)|\,, \quad G(u) := \int_\Omega (\boldsymbol{b}(x,u), Du)\,, \quad G^+(u) := \int_\Omega (\boldsymbol{b}(x,u), Du)^+\,.
$$

We start by proving the following continuity result (see [30] for the analogous result in $W^{1,1}$).

Proposition 6.1. *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv), let $\varphi \in C^1_c(\Omega)$ be a fixed test function, and let $G_\varphi \colon BV(\Omega)\cap L^\infty(\Omega)\to\mathbb{R}$ be the functional defined by*

$$
G_\varphi(u) := \langle (\boldsymbol{b}(x,u), Du)\,,\, \varphi\rangle = \int_\Omega \varphi\, d(\boldsymbol{b}(x,u), Du)\,, \qquad u\in BV(\Omega)\cap L^\infty(\Omega).
$$

Then, for every sequence $(u_j)\subset BV(\Omega)\cap L^\infty(\Omega)$ converging to $u\in BV(\Omega)\cap L^\infty(\Omega)$ in the L^1 convergence, and satisfying

$$
L := \sup_j \|u_j\|_{L^\infty(\Omega)} < +\infty,
$$

it holds that

$$
\lim_{j\to+\infty} G_\varphi(u_j) = G_\varphi(u). \tag{29}
$$

Moreover, by assuming, instead of (iv), the stronger condition $\sigma\in L^N_{\mathrm{loc}}(\Omega)$ (respectively $\sigma\in L^\infty_{\mathrm{loc}}(\Omega)$), the continuity (29) *holds if (u_j) converges to u weakly* in $BV(\Omega)$ (respectively strongly in $L^1_{\mathrm{loc}}(\Omega)$).*

Proof. Using (15) we have that

$$
G_\varphi(u_j) - G_\varphi(u) = -\int_\Omega \varphi(x) \int_{u(x)}^{u_j(x)} \mathrm{div}_x\, \boldsymbol{b}_t(x)\,dt\,dx - \int_\Omega \int_{u(x)}^{u_j(x)} \boldsymbol{b}_t(x)\cdot\nabla\varphi(x)\,dt\,dx.
$$

Since $\boldsymbol{b}$ is a locally bounded vector field, the second integral converges to 0 by the Lebesgue Dominated Convergence theorem. The first integral can be written as

$$
\iint_{K\times[-L,L]} \mathrm{sign}(u(x)-u_j(x))\chi_{D_j}(x,s)\ \mathrm{div}_x\, \boldsymbol{b}_s(x)\,dx\,ds, \tag{30}
$$

where $K \subset \Omega$ is the support of φ and $D_j \subset \Omega \times [-L, L]$ is the set of pairs (x, s) such that s belongs to the segment of endpoints $u(x)$ and $u_j(x)$. Since

$$\left|\chi_{D_j}(x,s)\varphi(x)\,\operatorname{div}_x \boldsymbol{b}_s(x)\right| \le \|\varphi\|_{L^\infty(\Omega)}\,|\operatorname{div}_x \boldsymbol{b}_s(x)| \le \|\varphi\|_{L^\infty(\Omega)}\,\sigma(x) \in L^1(K\times[-L,L]),$$

the integral (30) converges to 0 by the Lebesgue Dominated Convergence theorem.

Let us prove the second part of the theorem. If the sequence $(u_j) \subset BV(\Omega) \cap L^\infty(\Omega)$ weak*-converges to $u \in BV(\Omega)\cap L^\infty(\Omega)$, by the Poincaré inequality (see [5, Remark 3.50]) we have that (u_j) weakly converges to u in $L^{\frac{N}{N-1}}(\Omega)$. Since

$$\left|\int_\Omega \varphi(x)\int_{u(x)}^{u_j(x)} \operatorname{div}_x \boldsymbol{b}_t(x)\,dt\,dx\right| \le \|\varphi\|_\infty \int_K |u_j(x)-u(x)|\sigma(x)\,dx,$$

then, if $\sigma \in L^N_{\mathrm{loc}}(\Omega)$, the integral on the right-hand side converges to 0. The same conclusion holds if (u_j) converges to u strongly in $L^1_{\mathrm{loc}}(\Omega)$ and $\sigma \in L^\infty_{\mathrm{loc}}(\Omega)$. □

Remark 6.2. In [24] Dal Maso proved the lower semicontinuity of integral functionals with coercive integrands and he showed, by exploiting Aronszajn's example, that this result is sharp, in the sense that, in general, the coercivity assumption cannot be dropped. Indeed, Dal Maso constructed a continuous function $\omega\colon \Omega \to \mathbb{R}$, where $\Omega = (0,1)\times(0,1)$ and $x = (x_1, x_2)$, and a sequence of functions $\{u_n\}$ converging to $u(x) = x_2$ in $L^\infty(\Omega)$, such that

$$\int_\Omega |(\sin\omega(x), \cos\omega(x))\cdot\nabla u(x))|\,dx > \liminf_{n\to\infty}\int_\Omega |(\sin\omega(x),\cos\omega(x))\cdot\nabla u_n(x))|\,dx.$$

Let us remark that the integrand $|\boldsymbol{b}(x)\cdot\xi|$ of Dal Maso's example does not satisfy our condition $\operatorname{div}\boldsymbol{b} \in L^1$.

Theorem 6.3. *[Lower semicontinuity] Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv). Then the functionals F, G^+ are lower semicontinuous on $BV(\Omega)\cap L^\infty(\Omega)$ with respect to the L^1 convergence.*

Proof. Let us define the auxiliary functionals $H, H^+\colon BV(\Omega)\cap L^\infty(\Omega) \to \mathbb{R}$ by

$$H(u) := -\int_\Omega (\boldsymbol{b}(x,u), Du)\,, \qquad H^+(u) := \int_\Omega (-(\boldsymbol{b}(x,u), Du))^+\,.$$

Since $F(u) = [G^+(u) + H^+(u)]/2$, it suffices to prove that $G^+(u)$ and $H^+(u)$ are lower semicontinuous on $BV(\Omega)\cap L^\infty(\Omega)$ with respect to the L^1 convergence. We shall prove the claim only for G^+, being the proof for H^+ similar.

Let us prove that $\liminf_n G^+(u_n) \ge G^+(u)$ for every sequence $(u_n) \subset BV(\Omega)\cap L^\infty(\Omega)$ converging to $u \in BV(\Omega)\cap L^\infty(\Omega)$ in $L^1(\Omega)$.

For every $k \in \mathbb{N}$ let us consider the Lipschitz function $\sigma_k\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\sigma_k(t) := \begin{cases} 1 & |t| \le k-1, \\ -|t| + k & k-1 < |t| \le k, \\ 0 & |t| > k. \end{cases}$$

The vector field

$$\boldsymbol{b}^k\,(x,t) := \sigma_k\,(t)\;\boldsymbol{b}(x,t)$$

satisfies

$$\boldsymbol{b}^k(x,t) = 0 \text{ for } |t| \ge k, \qquad \boldsymbol{b}^k(x,t) = \boldsymbol{b}(x,t) \text{ for } |t| \le k-1.$$

We claim that, for every $v \in BV(\Omega) \cap L^\infty(\Omega)$,

$$\Theta(\boldsymbol{b}^k, v; x) = \sigma_k(\widetilde{v}(x))\,\Theta(\boldsymbol{b}, v; x)\,, \qquad \text{for } |D^d v|\text{-a.e. } x \in \Omega, \tag{31}$$

$$\Theta(\boldsymbol{b}^k, v; x) = \fint_{v^-(x)}^{v^+(x)} \sigma_k(t)\,\mathrm{Cyl}(\boldsymbol{b}_t, \nu_v; x)\,dt\,, \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega \cap J_v. \tag{32}$$

Specifically, both relations are a consequence of the representation formula (21) and of the equality

$$\begin{aligned}\mathrm{Cyl}(\boldsymbol{b}_t^k, \nu_v; x) &= \lim_{\rho\downarrow 0}\lim_{r\downarrow 0} \frac{1}{\mathcal{L}^N(C_{r,\rho}(x, \nu_v(x)))} \int_{C_{r,\rho}(x,\nu_v(x))} \sigma_k(t)\,\boldsymbol{b}_t(y)\cdot\nu_v(x)\,dy \\ &= \sigma_k(t)\ \mathrm{Cyl}(\boldsymbol{b}_t, \nu_v; x)\,.\end{aligned}$$

We now proceed as in [25]. Given $\varphi \in C_c^1(\Omega)$, let us consider the following functionals, defined in $BV(\Omega) \cap L^\infty(\Omega)$:

$$\begin{aligned}
F_\varphi^1(v) &:= \int_\Omega \varphi(x)\,\mathrm{Cyl}(\boldsymbol{b}_{\widetilde{v}(x)}, \nu_v; x)|D^d v| + \int_{\Omega\cap J_v} \varphi(x) \fint_{v^-(x)}^{v^+(x)} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_v; x)\,dt\,d\mathcal{H}^{N-1}\,, \\
F_\varphi^2(v) &:= \int_\Omega \varphi(x)\Big[\mathrm{Cyl}(\boldsymbol{b}_{\widetilde{v}(x)}, \nu_v; x)\Big]^+ |D^d v| \\
&\quad + \int_{\Omega\cap J_v} \varphi(x) \fint_{v^-(x)}^{v^+(x)} \Big[\mathrm{Cyl}(\boldsymbol{b}_t, \nu_v; x)\,dt\Big]^+ d\mathcal{H}^{N-1}\,, \\
F_\varphi^3(v) &:= \int_\Omega \varphi(x)\Big[\mathrm{Cyl}(\boldsymbol{b}_{\widetilde{v}(x)}, \nu_v; x)\Big]^+ |D^d v| \\
&\quad + \int_{\Omega\cap J_v} \varphi(x)\Big[\fint_{v^-(x)}^{v^+(x)} \mathrm{Cyl}(\boldsymbol{b}_t, \nu_v; x)\,dt\Big]^+ d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Let Φ denote the set of all functions $\varphi \in C_c^1(\Omega)$ such that $0 \le \varphi \le 1$. Then it holds that

$$G^+(u_n) = \sup_{\varphi\in\Phi} \int_\Omega \varphi(x)(\boldsymbol{b}(x, u_n), Du_n) = \sup_{\varphi\in\Phi} \int_\Omega \varphi(x)(\boldsymbol{b}(x, u_n), Du_n)^+. \tag{33}$$

Using the representation formulas (31) and (32), from (33) we deduce that

$$G^+(u_n) = \sup_{\varphi\in\Phi} \int_\Omega F_\varphi^1(u_n) = \sup_{\varphi\in\Phi} \int_\Omega F_\varphi^3(u_n)\,. \tag{34}$$

Since $F_\varphi^1 \le F_\varphi^2 \le F_\varphi^3$, from (34) it follows that

$$G^+(u_n) = \sup_{\varphi\in\Phi} \int_\Omega F_\varphi^2(u_n)\,.$$

As a consequence, recalling that $0 \leq \sigma_k \leq 1$, for every $\varphi \in \Phi$ we deduce that

$$\begin{aligned}
G^+(u_n) &\geq F^2_\varphi(u_n) \\
&\geq \int_\Omega \varphi(x)\sigma_k(\widetilde{u}_n(x)) \left[\mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}_n(x)}, \nu_{u_n}; x)\right]^+ |D^d u_n| \\
&\quad + \int_{\Omega\cap J_{u_n}} \varphi(x) \fint_{u_n^-(x)}^{u_n^+(x)} \sigma_k(t) \left[\mathrm{Cyl}(\boldsymbol{b}_t, \nu_{u_n}; x)\, dt\right]^+ d\mathcal{H}^{N-1} \\
&\geq \int_\Omega \varphi(x)\sigma_k(\widetilde{u}_n(x))\ \mathrm{Cyl}(\boldsymbol{b}_{\widetilde{u}_n(x)}, \nu_{u_n}; x)\, |D^d u_n| \\
&\quad + \int_{\Omega\cap J_{u_n}} \varphi(x) \fint_{u_n^-(x)}^{u_n^+(x)} \sigma_k(t)\ \mathrm{Cyl}(\boldsymbol{b}_t, \nu_{u_n}; x)\, dt\, d\mathcal{H}^{N-1} \\
&= \int_\Omega \varphi(x)\, \mathrm{Cyl}(\boldsymbol{b}^k_{\widetilde{u}_n(x)}, \nu_{u_n}; x)\, |D^d u_n| \\
&\quad + \int_{\Omega\cap J_{u_n}} \varphi(x) \fint_{u_n^-(x)}^{u_n^+(x)} \mathrm{Cyl}(\boldsymbol{b}^k_t, \nu_{u_n}; x)\, dt\, d\mathcal{H}^{N-1} \\
&= \int_\Omega \varphi(x)\, (\boldsymbol{b}^k(x, u_n), Du_n)\,.
\end{aligned}$$

Let us choose $k > \|u\|_\infty + 1$, let $T_k z := \min\{\max\{z, -k\}, k\}$, $z \in \mathbb{R}$, and let us define the functional

$$G^k_\varphi(v) := \int_\Omega \varphi(x)\, (\boldsymbol{b}^k(x, v), Dv), \qquad v \in BV(\Omega) \cap L^\infty(\Omega),$$

so that the previous inequality reads

$$G^+(u_n) \geq G^k_\varphi(u_n)\,. \tag{35}$$

From the representation formula (15) it holds that

$$G^k_\varphi(v) = G^k_\varphi(T_k v), \qquad \forall v \in BV(\Omega) \cap L^\infty(\Omega),$$

and, by our choice of k, $G^k_\varphi(u) = G_\varphi(u)$.

For every $\varphi \in \Phi$, from (35) and Proposition 6.1 applied to the functional G^k_φ and the uniformly bounded sequence $(T_k u_n)_n$, we deduce that

$$\liminf_{n\to+\infty} G^+(u_n) \geq \liminf_{n\to+\infty} G^k_\varphi(u_n) = \liminf_{n\to+\infty} G^k_\varphi(T_k u_n) \geq G^k_\varphi(u) = G_\varphi(u).$$

Taking the supremum for $\varphi \in \Phi$ we finally conclude that $\liminf\limits_{n\to+\infty} G^+(u_n) \geq G^+(u)$. $\square$

7. Pairing as relaxed functional

For every function $\varphi \in C^1_c(\Omega)$ and for every open set $A \subset \Omega$ let us consider the functional $F^\varphi(\cdot, A) : BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) \to]-\infty, +\infty]$ defined by

$$F^\varphi(u, A) := \begin{cases} \displaystyle\int_A \varphi\, \boldsymbol{b}(x, u) \cdot \nabla u\, dx & u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega), \\ +\infty, & u \in (BV_{\mathrm{loc}}(\Omega)\ \setminus W^{1,1}_{\mathrm{loc}}(\Omega)) \cap L^\infty_{\mathrm{loc}}(\Omega). \end{cases}$$

Moreover, for every $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ we define the relaxations with respect to the weak* convergence in BV and the L^1 convergence in BV as

$$\overline{F}^\varphi(u,A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} F^\varphi(u_n,A) \colon u_n \in W^{1,1}_{\rm loc}(A),\ u_n \stackrel{*}{\rightharpoonup} u \text{ in } BV_{\rm loc}(A) \right\},$$

$$\overline{F}_1^\varphi(u,A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} F^\varphi(u_n,A) \colon u_n \in W^{1,1}_{\rm loc}(A),\ u_n \to u \text{ strongly in } L^1(A) \right\}.$$

Theorem 7.1. *[Integral representation of the relaxed functionals of F^φ] Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv). Then for every $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ and for every open set $A \subset \Omega$, if we assume $\sigma \in L^N_{\rm loc}(\Omega)$, then it holds that*

$$\overline{F}^\varphi(u,A) = \int_A \varphi \, d(\boldsymbol{b}(x,u), Du)$$

and, if we assume $\sigma \in L^\infty_{\rm loc}(\Omega)$, then it holds that

$$\overline{F}_1^\varphi(u,A) = \int_A \varphi \, d(\boldsymbol{b}(x,u), Du).$$

Proof. Thanks to the continuity results proved in Proposition 6.1 and the argument in [34, Theorem 1.3, part (i)], it is enough to prove the following two inequalities:

(J) for $\mathcal{H}^{N-1}$-a.e. $x_0 \in J_u$, it holds that

$$\frac{d\overline{F}^\varphi(u,\cdot)}{d\mathcal{H}^{N-1} \llcorner J_u}(x_0) \leq \varphi(x_0) \int_{u^-(x_0)}^{u^+(x_0)} \operatorname{Cyl}(\boldsymbol{b}(\cdot,t), \nu_u(x_0); x_0)\, dt\,;$$

(C) for $|D^c u|$-a.e. $x_0 \in \Omega$, it holds that

$$\frac{d\overline{F}^\varphi(u,\cdot)}{d|D^c u|}(x_0) \leq \varphi(x_0) \operatorname{Cyl}(\boldsymbol{b}(\cdot,\widetilde{u}(x_0)), \nu_u(x_0); x_0)\,.$$

Since both results are of local nature, it is not restrictive to assume that $\Omega = \mathbb{R}^N$ and that $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, to simplify the notation we denote $\mu := (\boldsymbol{b}(\cdot,u), Du)$.

Proof of (J). By the definition of relaxed functional we have that

$$\begin{aligned}
\frac{d\overline{F}^\varphi(u,\cdot)}{d\mathcal{H}^{N-1} \llcorner J_u}(x_0) &= \lim_{r\searrow 0} \frac{\overline{F}^\varphi(u, B_r(x_0))}{\omega_{N-1} r^{N-1}} \\
&\leq \lim_{r\searrow 0} \liminf_{\varepsilon\searrow 0} \frac{1}{\omega_{N-1} r^{N-1}} \int_{B_r(x_0)} \varphi(y)\, \boldsymbol{b}(y, \rho_\varepsilon * u(y)) \cdot \nabla(\rho_\varepsilon * u)(y)\, dy\,.
\end{aligned}$$

As $\varepsilon \to 0^+$, the integral above converges to $\int_{B_r(x_0)} \varphi\, d\mu$ (see the proof of Theorem 4.3 in [21], where this convergence is stated in formula (38)). Hence,

$$\begin{aligned}
\frac{d\overline{F}^\varphi(u,\cdot)}{d\mathcal{H}^{N-1} \llcorner J_u}(x_0) &\leq \lim_{r\searrow 0} \frac{1}{\omega_{N-1} r^{N-1}} \int_{B_r(x_0)} \varphi\, d\mu \\
&= \varphi(x_0) \lim_{r\searrow 0} \frac{|Du|(B_r(x_0))}{\omega_{N-1} r^{N-1}} \cdot \frac{\mu(B_r(x_0))}{|Du|(B_r(x_0))} \\
&= \varphi(x_0)\, [u^+(x_0) - u^-(x_0)]\, \Theta(\boldsymbol{b}, u, x_0)\,,
\end{aligned}$$

so that (J) follows from Theorem 5.1.

Proof of (C). Reasoning as in the proof of (J) above, we have that

$$\begin{aligned}\frac{d\overline{F}^{\varphi}(u,\cdot)}{d|D^c u|}(x_0) &= \lim_{r\searrow 0}\frac{\overline{F}^{\varphi}(u,B_r(x_0))}{\omega_{N-1}r^{N-1}}\\ &\leq \varphi(x_0)\,\lim_{r\searrow 0}\frac{\mu(B_r(x_0))}{|Du|(B_r(x_0))} = \varphi(x_0)\,\Theta(\boldsymbol{b},u,x_0)\,,\end{aligned}$$

and again the conclusion follows from Theorem 5.1. □

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