

STABILITY OF THE BETTI NUMBER UNDER KATO BOUNDS ON THE RICCI CURVATURE

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ABSTRACT. We show two stability results for a closed Riemannian manifold whose Ricci curvature is small in the Kato sense and whose first Betti number is equal to the dimension. The first one is a geometric stability result stating that such a manifold is Gromov-Hausdorff close to a flat torus. The second one states that, under a stronger assumption, such a manifold is diffeomorphic to a torus: this extends a result by Colding and Cheeger–Colding obtained in the context of a lower bound on the Ricci curvature.

1. INTRODUCTION

The celebrated Bochner theorem states that if a closed Riemannian manifold (M^n, g) has non-negative Ricci curvature then its first Betti number satisfies

$$b_1(M) \leq n,$$

with equality if and only if (M^n, g) is isometric to a flat torus. This inequality was improved by Gromov [Gro81] and Gallot [Gal83] who found $\varepsilon(n) > 0$ such that if

$$\text{Ric} \geq -\frac{\varepsilon(n)}{\text{diam}(M, g)^2} g,$$

then $b_1(M) \leq n$. Gromov also made a conjecture about the equality case, which was proven true by Colding [Col97] and Cheeger–Colding [CC97]: there exists $\delta(n) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying

$$b_1(M) = n, \quad \text{Ric} \geq -\frac{\delta(n)}{D^2} g,$$

then M is diffeomorphic to a torus. The proof of this latter result consists in two steps. First, one shows that for any $\varepsilon \in (0, 1)$ there exists $\delta(n, \varepsilon) > 0$ such that if (M^n, g) with diameter D satisfies

$$b_1(M) = n, \quad \text{Ric} \geq -\frac{\delta(n, \varepsilon)}{D^2} g,$$

then there exists a flat torus \mathbb{T}^n and an εD -almost isometry¹ $\Phi: M \rightarrow \mathbb{T}^n$. Secondly, one proves a topological rigidity statement: if ε is sufficiently small, then M is diffeomorphic to \mathbb{T}^n . For more details, we refer to the very instructive texts [Gal98, Che01] presenting the work of Cheeger and Colding.

The Bochner estimate has been generalized in several directions. For a Riemannian manifold (M^n, g) , let $\text{Ric}_- : M \rightarrow \mathbb{R}_+$ be the lowest non-negative function such that for any $x \in M$,

$$\text{Ric}_x \geq -\text{Ric}_-(x)g_x.$$

¹for any $\eta > 0$, a η -almost isometry is a Borel map Φ between two metric spaces (X, d_X) and (Y, d_Y) such that: (i) $|d_X(x_0, x_1) - d_Y(\Phi(x_0), \Phi(x_1))| \leq \eta$ for any $x_0, x_1 \in X$, (ii) for any $y \in Y$ there exists $x \in X$ such that $d_Y(\Phi(x), y) \leq \eta$

Then Gallot obtained in [Gal88] that for every $p > n/2$, there exists $\varepsilon(n, p) > 0$ such that if (M^n, g) with diameter D satisfies

$$D^2 \left(\int_M \text{Ric}^p d\nu_g \right)^{\frac{1}{p}} \leq \varepsilon(n, p),$$

then $b_1(M) \leq n$; here and throughout, ν_g is the Riemannian volume measure induced by g on M , and $\int_A f d\nu_g := \nu_g(A)^{-1} \int_A f d\nu_g$ for any Borel set $A \subset M$ and any measurable function f defined on A . To our knowledge, no topological rigidity result has been obtained so far from this integral condition. It seems to us that the segment inequality proven in [Che21] and the results from [PW97, PW01] may imply such a rigidity result. Another direction is the one of metric measure spaces satisfying a suitable synthetic Ricci curvature lower bound. In this context, a rigidity result à la Bochner and geometric stability results hold, see [GR18, MW19, MMP21].

In this paper, we obtain geometric and topological results under a Kato bound. More precisely, let us introduce the following definition.

Definition 1.1. Let (M^n, g) be a complete Riemannian manifold with heat kernel $H(t, x, y)$ and diameter D . For any $T > 0$, we set

$$k_T(M^n, g) := \sup_{x \in M} \iint_{[0, T] \times M} H(t, x, y) \text{Ric}_-(y) d\nu_g(y) dt.$$

We say that the number $k_{D^2}(M^n, g)$ is the Kato constant of (M^n, g) .

The first occurrence of $k_T(M^n, g)$ in the study of Riemannian manifolds seems to be [GP15]. The geometric and analytic consequences of a bound on $k_T(M^n, g)$ have been extensively studied since then, see e.g. [Ros19, RS20, Car19, RW20, CR21, BG21, CMT21, CMT22]. It is useful to note that if $\text{Ric} \geq -\kappa^2 g$ then $k_T(M^n, g) \leq \kappa^2 T$, hence a smallness assumption on $k_T(M^n, g)$ should be understood as a control on the part of the manifold where $\text{Ric} \gtrsim T^{-1}$.

The Bochner estimate extends to the case of Riemannian manifolds with small Kato constant, as proven in [Ros19] and improved in [Car19]: there exists $\delta(n) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D such that $k_{D^2}(M^n, g) \leq \delta(n)$, then $b_1(M) \leq n$. Our first main result provides an answer to a question raised in [Car19] about the equality case.

Theorem A. *For any $\varepsilon \in (0, 1)$ there exists $\delta(n, \varepsilon) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying*

$$b_1(M) = n \quad \text{and} \quad k_{D^2}(M^n, g) \leq \delta(n, \varepsilon),$$

then M is εD -almost isometric to a flat torus.

Remark 1.2. From [CR21, Theorem 4.3] we can replace the smallness assumption $k_{D^2}(M^n, g) \leq \delta(n, \varepsilon)$ with an integral condition involving the Ricci curvature only, namely

$$\sup_{x \in M} \int_0^D r \int_{B_r(x)} \text{Ric}_-(y) d\nu_g(y) dr \leq \delta(n, \varepsilon).$$

Our second main result provides a topological stability theorem under a so-called strong Kato bound. This assumption appeared naturally in our previous work [CMT21, CMT22] where we obtained, among other results, Reifenberg regularity.

Theorem B. *Let $f: [0, 1] \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying*

$$\int_0^1 \frac{\sqrt{f(t)}}{t} dt < +\infty. \tag{SK}$$

Then there exists $\delta(n, f) > 0$ such that if a closed Riemannian manifold (M^n, g) of diameter D satisfies

$$b_1(M) = n, \quad k_{D^2}(M^n, g) \leq \delta(n, f),$$

and

$$k_{tD^2}(M^n, g) \leq f(t) \quad \text{for all } t \in (0, 1],$$

then M is diffeomorphic to a torus.

Remark 1.3. Our proof actually shows a stronger result: for any $\alpha \in (0, 1)$ there exists $\delta(n, f, \alpha) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying $b_1(M) = n$, $k_{D^2}(M^n, g) \leq \delta(n, f, \alpha)$ and $k_{tD^2}(M^n, g) \leq f(t)$ for any $t \in (0, 1]$, with f satisfying (SK), then there exist a flat torus $(\mathbb{T}^n, \mathbf{d}_{\mathbb{T}^n})$ and a diffeomorphism $\mathcal{A}: M \rightarrow \mathbb{T}^n$ such that for any $x, y \in M$,

$$\alpha \left(\frac{\mathbf{d}_g(x, y)}{D} \right)^{\frac{1}{\alpha}} \leq \frac{\mathbf{d}_{\mathbb{T}^n}(\mathcal{A}(x), \mathcal{A}(y))}{D} \leq \alpha^{-1} \left(\frac{\mathbf{d}_g(x, y)}{D} \right)^{\alpha}.$$

Here and throughout, \mathbf{d}_g is the Riemannian distance induced by g .

According to [RS17], an L^p smallness condition on Ric_- yields the strong Kato bound, so that Theorem B has the following corollary.

Corollary 1.4. For any $p > n/2$, there exists $\varepsilon(n, p) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying

$$b_1(M) = n \quad \text{and} \quad D^2 \left(\int_M \text{Ric}^p \, d\nu_g \right)^{\frac{1}{p}} \leq \varepsilon(n, p),$$

then M is diffeomorphic to a torus.

Similarly, using [CR21, Theorem 4.3], we get the following corollary involving a suitable Morrey norm.

Corollary 1.5. For any $\alpha \in (0, 2]$, there exists $\varepsilon(n, \alpha) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying

$$b_1(M) = n \quad \text{and} \quad \sup_{\substack{x \in M \\ r \in (0, D)}} D^\alpha r^{2-\alpha} \int_{B_r(x)} \text{Ric}_-(y) \, d\nu_g(y) \leq \varepsilon(n, \alpha),$$

then M is diffeomorphic to a torus.

The scheme of our proofs is close to Gallot's adaptation of Colding's original argument [Gal98]. We consider the Albanese map $\mathcal{A}: M \rightarrow \mathbb{T}^n$. We lift it to a harmonic map $\widehat{\mathcal{A}} := (\widehat{\mathcal{A}}_1, \dots, \widehat{\mathcal{A}}_n)$ defined on a suitable abelian cover $\widehat{M} \rightarrow M$ which is equivariant under an action of \mathbb{Z}^n . Estimates from [Car19] imply that if $k_{D^2}(M^n, g) \leq \delta \leq 1/(16n)$, then $\widehat{\mathcal{A}}$ is $(1 + C(n)\sqrt[3]{\delta})$ -Lipschitz and for any $r \in [D, \delta^{-1/6}D]$,

$$\left(r^2 \int_{B_r} |\nabla d\widehat{\mathcal{A}}|^2 \, d\nu_{\widehat{g}} \right)^{\frac{1}{2}} + \int_{B_r} \left| \langle d\widehat{\mathcal{A}}_i, d\widehat{\mathcal{A}}_j \rangle - \delta_{i,j} \right| \leq C(n)\sqrt[3]{\delta}.$$

A new point in our proof of Theorem A is an almost rigidity result (Theorem 3.1) which implies that under such an estimate, the restriction of $\widehat{\mathcal{A}}$ to a ball of radius $64n^2D$ realizes an εD -almost isometry with a Euclidean ball of the same radius. We prove this almost rigidity result by means of the analysis developed in [CMT21]. Then we can follow the lines of Gallot's argument to get Theorem A.

Our proof of Theorem B uses a Reifenberg regularity result that we proved in our previous paper [CMT22, Theorem 5.19], see Proposition 3.4. To put us in a position to apply this result in our context, we prove first that

$$\int_{\widehat{M}} \frac{e^{-\frac{d_g^2(\hat{o}, y)}{4t}}}{(4\pi t)^{\frac{n}{2}}} d\nu_{\hat{g}}(y) \simeq 1 \quad (1)$$

for a suitable t of order D^2 , then we prove an estimate à la Cheeger–Yau [CY81] in order to compare the heat kernel \widehat{H} of \widehat{M} to the integrand of (1).

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2. THE DYNKIN CONDITION AND CONSEQUENCES

In this section, we point out relevant properties of the so-called Dynkin condition. We say that a complete Riemannian manifold (M^n, g) satisfies such a condition if there exists $T > 0$ such that

$$k_T(M^n, g) \leq \frac{1}{16n}. \quad (\text{Dy})$$

2.1. Closed manifolds. Let us first mention properties of closed Riemannian manifolds satisfying a Dynkin condition.

2.1.1. Volume doubling. See [Car19, Proposition 3.8] and [CMT21, Proposition 3.3] for the next result.

Proposition 2.1. Let (M^n, g) be a closed Riemannian manifold satisfying (Dy). Then there exists $C(n) > 0$ such that for any $x \in M$ and $0 < s \leq r \leq \sqrt{T}$,

$$\frac{\nu_g(B_r(x))}{\nu_g(B_s(x))} \leq C(n) \left(\frac{r}{s}\right)^{e^2 n}.$$

2.1.2. Heat kernel bounds. See [CMT22, Proposition 2.6] for the following.

Proposition 2.2. Let (M^n, g) be a closed Riemannian manifold satisfying (Dy). Then there exists $C(n) > 0$ such that for any $x, y \in M$ and $t \in (0, T)$,

$$\begin{aligned} \text{i)} \quad & \frac{C(n)^{-1}}{\nu_g(B_{\sqrt{t}}(x))} e^{-C(n)\frac{d_g^2(x, y)}{t}} \leq H(t, x, y) \leq \frac{C(n)}{\nu_g(B_{\sqrt{t}}(x))} e^{-\frac{d_g^2(x, y)}{5t}}, \\ \text{ii)} \quad & |d_x H(t, x, y)| \leq \frac{C(n)}{\sqrt{t}\nu_g(B_{\sqrt{t}}(x))} e^{-\frac{d_g^2(x, y)}{5t}}. \end{aligned}$$

When $T \geq \text{diam}^2(M^n, g)$, we can get an estimate depending only on the volume of M .

Proposition 2.3. Let (M^n, g) be a closed Riemannian manifold of diameter D satisfying (Dy). If $T \geq D^2$ then for any $x, y \in M$ and $t \in (D^2, T)$,

$$H(t, x, y) \leq \left(1 + C(n)\frac{D}{\sqrt{t}}\right) \frac{1}{\nu_g(M)}.$$

Proof. Let $t \in (D^2, T)$ and $x, y \in M$. By stochastic completeness, we have that $\int_M H(t, z, y) d\nu_g(z) = 1$, hence there is some $z_0 \in M$ such that $H(t, z_0, y) = \frac{1}{\nu_g(M)}$. By the previous proposition, we have that $|d_z H(t, z, y)| \leq \frac{C(n)}{\sqrt{t\nu_g(M)}}$. Then

$$|H(t, x, y) - H(t, z_0, y)| \leq d_g(x, z_0) \frac{C(n)}{\sqrt{t\nu_g(M)}} \leq \frac{C(n)D}{\sqrt{t\nu_g(M)}}.$$

□

2.2. Non-compact manifolds. For complete non-compact manifolds, the results of [CMT21, Subsection 3.1] yield the following.

Proposition 2.4. Let (M^n, g, o) be a pointed complete Riemannian manifold. Assume that there exists a sequence $\{(M_\alpha^n, g_\alpha, o_\alpha)\}_\alpha$ of pointed closed Riemannian manifolds satisfying (Dy) for a same $T > 0$. Assume that for any $R > 0$ there exists α_R such that for any $\alpha \geq \alpha_R$ there exists a diffeomorphism onto its image

$$\Phi_\alpha : B_R(o) \rightarrow M_\alpha$$

such that the following convergence holds:

$$\lim_{\alpha \rightarrow +\infty} \|\Phi_\alpha^* g_\alpha - g\|_{C^0} = 0.$$

Then (M^n, g) satisfies (Dy), the volume doubling estimate from Proposition 2.1 and the heat kernel estimates from Proposition 2.2.

We recall that a group Γ with neutral element 1 is residually finite if and only if it admits a sequence of normal subgroups $\{\Gamma_j\}$ with finite index such that

$$\bigcap_j \Gamma_j = \{1\}.$$

Then Proposition 2.4 has the following useful application.

Proposition 2.5. Let $\pi : \widehat{M} \rightarrow M$ be a normal covering of a closed Riemannian manifold (M^n, g) with residually finite deck transformation group. If (M, g) satisfies (Dy), then (\widehat{M}, π^*g) satisfies (Dy), the volume doubling estimate from Proposition 2.1 and the heat kernel estimates from Proposition 2.2.

Proof. We start by noticing that if $V : M \rightarrow \mathbb{R}$ is a bounded function then

$$e^{-t\Delta_{\pi^*g}}(V \circ \pi) = (e^{-t\Delta_g} V) \circ \pi.$$

Indeed, since (M, g) is closed, (\widehat{M}, π^*g) is stochastically complete and then we have uniqueness in L^∞ of the solution of the heat equation with fixed initial condition

$$\begin{cases} \left(\frac{\partial}{\partial t} + \Delta_{\pi^*g}\right) u = 0, \\ u(0, \cdot) = V \circ \pi(\cdot). \end{cases}$$

But $e^{-t\Delta_{\pi^*g}}(V \circ \pi)$ and $(e^{-t\Delta_g} V) \circ \pi$ are both solutions of this Cauchy problem hence we get the desired equality. Notice that $\text{Ric}(\pi^*g) = \text{Ric}(g) \circ \pi$, hence for any $x \in \widehat{M}$ and $t > 0$

$$\int_{\widehat{M}} H_{\pi^*g}(t, x, y) \text{Ric}(\pi^*g)(y) d\nu_{\pi^*g}(y) = \int_M H_g(t, \pi(x), z) \text{Ric}(g)(z) d\nu_g(z),$$

where we have noted H_{π^*g} (resp. H_g) the heat kernel of (\widehat{M}, π^*g) (resp. of (M, g)). Hence we get that for any $T > 0$:

$$k_T(M, g) = k_T(\widehat{M}, \pi^*g). \quad (2)$$

As Γ is residually finite, there exists a sequence of normal subgroup $\Gamma_j \triangleleft \Gamma$ of finite index such that

$$\bigcap_j \Gamma_j = \{1\}.$$

For any $j \in \mathbb{N}^*$, we set $\widehat{M}_j := \widehat{M}/\Gamma_j$. We get two covering maps

$$\widehat{M} \xrightarrow{p_j} \widehat{M}_j \xrightarrow{\pi_j} M.$$

Note that \widehat{M}_j is a closed manifold and the above argument implies that for any $T > 0$:

$$k_T(\widehat{M}, \pi^*g) = k_T(\widehat{M}_j, \pi_j^*g).$$

Hence each $(\widehat{M}_j, \pi_j^*g)$ satisfies (Dy). Moreover, if we consider $\widehat{o} \in \widehat{M}$, then for any $R > 0$ there is some j_R such that for any $j \geq j_R$, the restriction of the covering map

$$p_j: B_R(\widehat{o}) \rightarrow B_R(p_j(\widehat{o}))$$

is an isometry. Then the conclusion follows from Proposition 2.4. \square

3. ALMOST RIGIDITY RESULTS

In this section, we provide almost rigidity results which are consequences of our previous work [CMT21, CMT22]. For any $\rho > 0$, we let \mathbb{B}_ρ^n be the Euclidean ball in \mathbb{R}^n centered at 0 with radius ρ . If B is a ball in an n -dimensional Riemannian manifold and $h = (h_1, \dots, h_n) : B \rightarrow \mathbb{R}^n$ is a smooth map, we denote by

$$dh^t dh = [\langle dh_i, dh_j \rangle]_{1 \leq i, j \leq n}$$

its Gram matrix map. We write Id_n for the identity matrix of size n and ω_n for the Lebesgue measure of the unit Euclidean ball in \mathbb{R}^n .

3.1. Harmonic almost splitting.

Theorem 3.1. *For any $\varepsilon \in (0, 1)$, there exists $\delta = \delta(n, \varepsilon) > 0$ such that for any closed Riemannian manifold (M^n, g) satisfying $k_{\rho^2}(M^n, g) \leq \delta$ for some $\rho > 0$, if for some $x \in M$ there exists a harmonic map*

$$h: B_{\delta^{-1}\rho}(x) \rightarrow \mathbb{R}^n$$

such that for any $r \in [\rho, \delta^{-1}\rho)$,

$$\left(r^2 \int_{B_r(x)} |\nabla dh|^2 d\nu_g \right)^{\frac{1}{2}} + \int_{B_r(x)} |dh^t dh - \text{Id}_n| d\nu_g \leq \delta,$$

then h is an $\varepsilon\rho$ -almost isometry between $B_\rho(x)$ and \mathbb{B}_ρ^n .

Remarks 3.2. i) From the proof of Proposition 2.5, we easily see that the result also holds if we assume that (M^n, g) is a normal covering of a closed Riemannian manifold whose deck transformation group is residually finite.
ii) A similar statement holds for RCD(K, N) spaces [BPS21, Proposition 3.7].

Proof. By scaling, there is no loss of generality in assuming $\rho = 1$, what we do from now on. We argue by contradiction. Assume that there exists $\varepsilon \in (0, 1)$ and:

- a sequence of positive numbers $\{\delta_\alpha\}$ such that $\delta_\alpha \downarrow 0$,
- a sequence of pointed closed Riemannian manifolds $\{(M_\alpha^n, g_\alpha, x_\alpha)\}$ such that $k_1(M_\alpha, g_\alpha) \leq \delta_\alpha$ for any α ,

- a sequence of maps $\{h_\alpha: B_{\delta_\alpha^{-1}}(x_\alpha) \rightarrow \mathbb{R}^n\}$ such that for any α ,

$$h_\alpha \text{ is not an } \varepsilon\text{-almost isometry between } B_1(x_\alpha) \text{ and } \mathbb{B}_1^n \quad (3)$$

and for any $r \in [1, \delta_\alpha^{-1}]$,

$$\left(r^2 \int_{B_r(x_\alpha)} |\nabla dh_\alpha|^2 d\nu_{g_\alpha} \right)^{\frac{1}{2}} + \int_{B_r(x_\alpha)} |dh_\alpha {}^t dh_\alpha - \text{Id}_n| d\nu_{g_\alpha} \leq \delta_\alpha.$$

We can assume the following.

- (1) Thanks to [CMT21, Corollary 2.5, Remark 4.9], the sequence $\{(M_\alpha^n, g_\alpha, x_\alpha)\}$ converges in the pointed measured Gromov-Hausdorff topology to a space (X, \mathbf{d}, μ, x) which is infinitesimally Hilbertian in the sense of [Gig15]. This limit space is endowed with a carré du champ Γ and a natural Laplace operator L , that is the Friedrichs extension of the quadratic form $\varphi \in W^{1,2} \mapsto \int_X d\Gamma(\varphi, \varphi)$. For the precise definitions of Γ and L , see [CMT21, Section 1.2] and references therein.
- (2) By [CMT21, Proposition E.10], the sequence $\{h_\alpha\}_\alpha$ converges uniformly on compact sets to a harmonic function $h = (h_1, \dots, h_n): X \rightarrow \mathbb{R}^n$ and for any $i, j \in \{1, \dots, n\}$,

$$\frac{d\Gamma}{d\mu}(h_i, h_j) = \delta_{i,j} \quad \mu\text{-a.e. on } X.$$

Moreover, we know that (X, \mathbf{d}, μ) admits a locally Lipschitz heat kernel $H: (0, +\infty) \times X \times X \rightarrow (0, +\infty)$ which satisfies the following Li-Yau inequality [CMT22, Proposition 2.9 and Remark 2.10]: for any $x \in X$, $t > 0$, and μ -a.e. $y \in X$,

$$|d_y H(t, x, y)|^2 - H(t, x, y) \frac{\partial H}{\partial t}(t, x, y) \leq \frac{n}{2t} H^2(t, x, y).$$

Let $U: (0, +\infty) \times X \times X \rightarrow \mathbb{R}$ be such that for any $x, y \in X$ and $t > 0$,

$$H(t, x, y) = \frac{e^{-\frac{U(t, x, y)}{4t}}}{(4\pi t)^{\frac{n}{2}}}.$$

It is easy to check (see [CMT21, Formula (83)]) that U satisfies, for any $(x, t) \in X \times (0, +\infty)$,

$$LU(t, x, \cdot) \geq -2n$$

in a weak sense, that is to say, $v(\cdot) = U(t, x, \cdot) \in W_{loc}^{1,2}$ and for any non-negative $\varphi \in W_c^{1,2}$:

$$\int_X d\Gamma(\varphi, v) \geq -2n \int_X \varphi d\mu.$$

According to Varadhan formula ([CMT21, Proposition 1.6]), for any $x, y \in X$,

$$\lim_{t \rightarrow 0} U(t, x, y) = \mathbf{d}^2(x, y).$$

For any $x, y \in X$, set

$$\rho(x, y) = |h(x) - h(y)|^2$$

and note that

$$L\rho(x, \cdot) = -2n,$$

hence for any $t > 0$ the function $\rho(x, \cdot) - U(t, x, \cdot)$ is sub-harmonic; passing to the limit $t \downarrow 0$ we get that $\rho(x, \cdot) - \mathbf{d}^2(x, \cdot)$ is subharmonic. Moreover, for each $\xi = (\xi_1, \dots, \xi_n)$, the function $h_\xi := \sum_i \xi_i h_i$ satisfies $|dh_\xi|^2 = |\xi|^2$, so that h_ξ is $|\xi|$ -Lipschitz. As a consequence, for any $x, y \in X$,

$$\rho(x, y) = \sup_{\substack{\xi \in \mathbb{R}^n \\ |\xi|=1}} |\langle \xi, h(x) - h(y) \rangle|^2 = \sup_{\substack{\xi \in \mathbb{R}^n \\ |\xi|=1}} |h_\xi(x) - h_\xi(y)|^2 \leq \mathbf{d}^2(x, y).$$

Therefore, the function $\rho(x, \cdot) - d^2(x, \cdot)$ is sub-harmonic, non-positive and it reaches its maximum value, zero, at $y = x$, hence it is constantly equal to zero. Thus we have shown that $h: X \rightarrow \mathbb{R}^n$ is an isometry onto its image. But $h(X)$ is closed, convex and according to [CT22, Claim 4.1, page 130], its convex hull is \mathbb{R}^n . Then h is an isometry between (X, d) and the Euclidean space \mathbb{R}^n . By uniform convergence, we get that for α large enough, $h_\alpha: B_1(x_\alpha) \rightarrow \mathbb{R}^n$ is an ε -almost isometry between $B_1(x_\alpha)$ and an Euclidean ball of radius 1; this contradicts (3). \square

Remark 3.3. Since (X, d, μ) is an RCD(0, n) space [CMT21, Theorem 5.10], we can also prove the previous result with the Functional Splitting Lemma [ABS19, Lemma 1.21]. We prefer providing a proof which avoids the use of the RCD theory.

3.2. Reifenberg regularity result. Let (M^n, g) be a complete Riemannian manifold. For any $(t, x) \in \mathbb{R}_+ \times M$, we set

$$\theta(t, x) := (4\pi t)^{\frac{n}{2}} H(t, x, x).$$

In [CMT21, CMT22], we observed that if (M^n, g) is closed, then its geometry is controlled by θ . In this regard, we have at our disposal the following Reifenberg regularity result which is a consequence of [CMT22, Theorem 5.19] and the proof of Proposition 2.5.

Proposition 3.4. Let $f: (0, 1] \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying (SK). Then there exists $\beta(n, f) > 0$ such that for any complete Riemannian manifold (M^n, g) which is a normal covering of a closed Riemannian manifold with residually finite deck transformation, if there exist $x \in M$, $R > 0$ and $h: B_R(x) \rightarrow \mathbb{R}^n$ harmonic such that:

- $k_{R^2}(M^n, g) \leq \beta(n, f)$ and $k_{tR^2}(M^n, g) \leq f(t)$ for any $t \in (0, 1]$,
- $\theta(R^2, x) \leq 1 + \beta(n, f)$,
- $\|dh\|_{L^\infty} \leq 2$, $h(x) = 0$ and

$$\left(R^2 \int_{B_R(x)} |\nabla dh|^2 d\nu_g \right)^{\frac{1}{2}} + \int_{B_R(x)} |dh^t dh - \text{Id}_n| d\nu_g \leq \beta(n, f),$$

then the restriction of h to $B_{3R/4}(x)$ is a diffeomorphism onto its image.

4. ALBANESE MAPS

In this section, we recall the construction of the Albanese maps and derive some relevant properties.

4.1. Construction of the Albanese maps. Let (M^n, g) be a closed Riemannian manifold. We write $H_1(M, \mathbb{Z})$ for the first integer-valued homology group of M , $H^1(M, \mathbb{R})$ for its first real-valued cohomology group, $H_{dR}^1(M)$ for its first De Rham cohomology space, and

$$\mathcal{H}^1(M^n, g) := \{\alpha \in \mathcal{C}^\infty(T^*M) : d\alpha = d^*\alpha = 0\}$$

for the space of harmonic 1-forms of (M^n, g) . Let b_1 be the first Betti number of M . Then the torsion free part Γ of $H_1(M, \mathbb{Z})$ is isomorphic to \mathbb{Z}^{b_1} (hence it is a residually finite group) and satisfies

$$\Gamma = \pi_1(M)/\Lambda$$

where

$$\Lambda := \left\{ \gamma \in \pi_1(M) : \int_\gamma \alpha = 0 \text{ for all } [\alpha] \in H^1(M, \mathbb{R}) \right\}.$$

Moreover, Γ is the deck transformation group of the normal covering

$$\widetilde{M}/\Lambda =: \widehat{M} \xrightarrow{\pi} M$$

where \widetilde{M} is the universal cover of M .

By the Hodge–de Rham theorem, there exists a normalized L^2 -orthonormal family of harmonic 1-forms $\alpha_1, \dots, \alpha_{b_1}$, i.e.

$$\int_M \langle \alpha_i, \alpha_j \rangle = \delta_{i,j}$$

for any i, j , such that $[\alpha_1], \dots, [\alpha_{b_1}]$ form a basis of $H_{dR}^1(M)$. We choose $o \in M$ and $\widehat{o} \in \widehat{M}$ such that $\pi(\widehat{o}) = o$. Then for any $i \in \{1, \dots, b_1\}$ there exists a unique harmonic function $\widehat{\mathcal{A}}_i : \widehat{M} \rightarrow \mathbb{R}$ such that

$$d\widehat{\mathcal{A}}_i = \pi^* \alpha_i \quad \text{and} \quad \widehat{\mathcal{A}}_i(\widehat{o}) = 0.$$

This yields a harmonic map

$$\widehat{\mathcal{A}} := (\widehat{\mathcal{A}}_1, \dots, \widehat{\mathcal{A}}_{b_1}) : \widehat{M} \rightarrow \mathbb{R}^{b_1}. \quad (4)$$

Let us now consider the linear map

$$\begin{aligned} \rho & : \Gamma \rightarrow \mathbb{R}^{b_1} \\ \gamma & \mapsto \left(\int_\gamma \alpha_1, \dots, \int_\gamma \alpha_{b_1} \right) \end{aligned} \quad (5)$$

and set

$$\mathbf{\Gamma} := \rho(\Gamma).$$

Then $\rho : \Gamma \rightarrow \mathbf{\Gamma}$ is an isomorphism and $\mathbf{\Gamma}$ is a lattice of \mathbb{R}^{b_1} . We endow $\mathbb{R}^{b_1}/\mathbf{\Gamma}$ with the flat quotient Riemannian metric $g_{\mathbb{R}^{b_1}/\mathbf{\Gamma}}$. Note that $\widehat{\mathcal{A}}$ is $\mathbf{\Gamma}$ -equivariant, that is, for any $\gamma \in \Gamma$ and $\widehat{x} \in \widehat{M}$,

$$\widehat{\mathcal{A}}(\gamma \cdot \widehat{x}) = \widehat{\mathcal{A}}(\widehat{x}) + \rho(\gamma). \quad (6)$$

Then $\widehat{\mathcal{A}}$ induces a harmonic quotient map

$$\begin{aligned} \mathcal{A} & : M = \widehat{M}/\Gamma \rightarrow \mathbb{R}^{b_1}/\mathbf{\Gamma} \\ \Gamma \cdot \widehat{x} & \mapsto \widehat{\mathcal{A}}(\widehat{x}) + \mathbf{\Gamma}. \end{aligned} \quad (7)$$

We say that \mathcal{A} is the Albanese map of M and $\widehat{\mathcal{A}}$ is the lifted Albanese map of M . Note that by construction, the following diagram commutes:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\mathcal{A}}} & \mathbb{R}^{b_1} \\ \downarrow \pi & & \downarrow p \\ M & \xrightarrow{\mathcal{A}} & \mathbb{R}^{b_1}/\mathbf{\Gamma}. \end{array}$$

4.2. Some estimates for harmonic 1-forms. In the next proposition, we derive some estimates for the elements of $\mathcal{H}^1(M^n, g)$ under a smallness assumption on the Kato constant.

Proposition 4.1. Let (M^n, g) be a closed Riemannian manifold of diameter D such that $k_{D^2}(M^n, g) \leq \delta$ for some $\delta \in (0, 1/(16n))$. Then there exists $C(n) > 0$ such that for any $\alpha \in \mathcal{H}^1(M^n, g)$,

- i) $\|\alpha\|_{L^\infty}^2 \leq \left(1 + C(n)\sqrt[3]{\delta}\right) \int_M |\alpha|^2$.
- ii) $\int_M \left| |\alpha|^2 - \int_M |\alpha|^2 \right| \leq C(n)\sqrt[3]{\delta} \int_M |\alpha|^2$.
- iii) $D^2 \int_M |\nabla \alpha|^2 \leq C(n)\delta \int_M |\alpha|^2$.

Proof. Set $N := \lfloor \frac{1}{816n} \rfloor$. Proceeding like in [Car19, Lemma 2.22], for instance, we get that for any $\ell \in \{1, \dots, N\}$,

$$k_{\ell D^2}(M^n, g) \leq \ell k_{D^2}(M^n, g) \leq \ell \delta \leq \frac{1}{16n}. \quad (8)$$

We first prove i). Let $\alpha \in \mathcal{H}^1(M^n, g)$. By the Bochner formula,

$$|\nabla \alpha|^2 + \frac{1}{2} \Delta |\alpha|^2 + \text{Ric}(\alpha, \alpha) = 0.$$

We fix $x \in M$. We multiply the previous identity evaluated in $y \in M$ by the heat kernel $H(t, x, y)$ and integrate with respect to $(t, y) \in [0, \ell D^2] \times M$, where $\ell \in \{1, \dots, N\}$ is suitably chosen later. This gives

$$\begin{aligned} & - \iint_{[0, \ell D^2] \times M} \text{Ric}(\alpha, \alpha)(y) \Delta_y H(t, x, y) \, d\nu_g(y) \, dt \\ &= \iint_{[0, \ell D^2] \times M} |\nabla \alpha|^2(y) H(t, x, y) \, d\nu_g(y) \, dt \\ &+ \frac{1}{2} \iint_{[0, \ell D^2] \times M} |\alpha|^2(y) \Delta_y H(t, x, y) \, d\nu_g(y) \, dt \end{aligned}$$

Since $\text{Ric} \geq -\text{Ric}_g$ and $\Delta_y H(t, x, y) = \frac{\partial}{\partial t} H(t, x, y)$, we get

$$\begin{aligned} |\alpha|^2(x) &\leq \int_M H(\ell D^2, x, y) |\alpha|^2(y) \, d\nu_g(y) \\ &+ 2 \|\alpha\|_{L^\infty}^2 \iint_{[0, \ell D^2] \times M} H(t, x, y) \text{Ric}_-(y) \, d\nu_g(y) \, dt \\ &\leq \left(1 + C(n) \frac{1}{\sqrt{\ell}}\right) \int_M |\alpha|^2 + 2k_{\ell D^2}(M^n, g) \|\alpha\|_{L^\infty}^2 \quad \text{by Proposition 2.3} \\ &\leq \left(1 + C(n) \frac{1}{\sqrt{\ell}}\right) \int_M |\alpha|^2 + 2\ell \delta \|\alpha\|_{L^\infty}^2 \quad \text{by (8)}. \end{aligned}$$

Thus

$$(1 - 2\ell \delta) \|\alpha\|_{L^\infty}^2 \leq \left(1 + C(n) \frac{1}{\sqrt{\ell}}\right) \int_M |\alpha|^2.$$

Choosing ℓ of the same order as $\delta^{-\frac{2}{3}}$ yields the desired estimate.

Let us now prove ii). Consider $\alpha \in \mathcal{H}^1(M^n, g)$. Up to rescaling, we may assume that

$$\int_M |\alpha|^2 = 1.$$

Then

$$\begin{aligned} \int_M ||\alpha|^2 - 1| &= \frac{2}{\nu_g(M)} \int_{\{|\alpha|^2 \geq 1\}} (|\alpha|^2 - 1) \\ &\leq \frac{2\nu_g(\{|\alpha|^2 \geq 1\})}{\nu_g(M)} C(n) \sqrt[3]{\delta} \\ &\leq 2C(n) \sqrt[3]{\delta}, \end{aligned}$$

where we have used that (i) and then $\nu_g(\{|\alpha|^2 \geq 1\}) \leq \int_M |\alpha|^2$.

Let us prove iii). We integrate the Bochner formula over M , divide by $\nu_g(M)$, and use i) to get

$$\int_M |\nabla \alpha|^2 = - \int_M \text{Ric}(\alpha, \alpha) \leq C(n) \int_M \text{Ric}_- \int_M |\alpha|^2.$$

But

$$\begin{aligned} D^2 \int_M \text{Ric} &= \frac{1}{\nu_g(M)} \int_{[0,D^2] \times M \times M} H(t, x, y) \text{Ric}(y) d\nu_g(y) d\nu_g(x) dt \\ &\leq k_{D^2}(M^n, g) \leq \delta. \end{aligned}$$

□

Remark 4.2. By the Grothendieck Lemma [Rud73, Theorem 5.1] (see also [Li80] and [GM88, Théorème 4]), the previous proposition implies that for any closed Riemannian manifold (M^n, g) of diameter D such that $k_{D^2}(M^n, g) \leq \delta$ for some $\delta \in (0, 1/(16n))$,

$$b_1(M) \leq \left(1 + C(n) \sqrt[3]{\delta}\right) n,$$

so that:

$$\delta < \delta_n := (nC(n))^{-3} \quad \Rightarrow \quad b_1(M) \leq n.$$

In particular, this provides another proof of [Car19, Proposition 3.12].

4.3. Consequences for the Albanese maps. The previous estimate imply the following.

Proposition 4.3. There exists $\delta(n) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying $b_1(M) = n$ and $k_{D^2}(M^n, g) \leq \delta$ for some $\delta \in (0, \delta(n))$, then the Albanese maps \mathcal{A} and $\widehat{\mathcal{A}}$ satisfy the following properties.

- i) They are $(1 + C(n) \sqrt[3]{\delta})$ -Lipschitz.
- ii) They are surjective.
- iii) For any $r \in [D, \delta^{-1/6}D]$,

$$\left(r^2 \int_{B_r(\widehat{o})} |\nabla d\widehat{\mathcal{A}}|^2 d\nu_{\widehat{g}}\right)^{\frac{1}{2}} + \int_{B_r(\widehat{o})} |d\widehat{\mathcal{A}}^t d\widehat{\mathcal{A}} - \text{Id}_n| d\nu_{\widehat{g}} \leq C(n) \sqrt[3]{\delta} \quad (9)$$

and

$$\nu_{\widehat{g}}(B_r(\widehat{o})) \geq \left(1 - C(n) \sqrt[3]{\delta}\right) \omega_n r^n. \quad (10)$$

Proof. Let π be the projection map from \widehat{M} to M , and let $(\alpha_1, \dots, \alpha_n)$ be the orthonormal basis of harmonic 1-forms used to build $\widehat{\mathcal{A}}$. We let $C(n) > 0$ be a generic constant depending only on n whose value may change from line to line.

Let us prove i). For $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, set $\alpha := \sum_i \xi_i \alpha_i \in \mathcal{H}^1(M^n, g)$ and observe that

$$|\xi|^2 = \int_M |\alpha|^2.$$

Then for any $x \in \widehat{M}$ and $v \in T_x \widehat{M}$, since $d\widehat{\mathcal{A}} = (\pi^* \alpha_1, \dots, \pi^* \alpha_n)$,

$$\left| \langle \xi, d_x \widehat{\mathcal{A}}(v) \rangle \right| = |\alpha(\pi(x))(d_x \pi(v))| \leq |\xi| \left(1 + C(n) \sqrt[3]{\delta}\right)^{\frac{1}{2}} \sqrt{\widehat{g}_x(v, v)},$$

where we have applied i) in Proposition 4.1. This yields i).

Let us now prove ii). Proposition 4.1 implies that $\Omega := \alpha_1 \wedge \dots \wedge \alpha_n$ satisfies

$$\int_M \left| |\Omega| - 1 \right| d\nu_g \leq C(n) \sqrt[3]{\delta}.$$

Hence if M is oriented and up to permutation of α_1 and α_2 then

$$\left| \int_M \Omega - \nu_g(M) \right| \leq C(n) \sqrt[3]{\delta} \nu_g(M),$$

but by construction, setting $\mathbf{\Omega} := dx_1 \wedge \dots \wedge dx_n$, we have $\Omega = \mathcal{A}^* \mathbf{\Omega}$, hence

$$\left| \deg \mathcal{A} - \frac{\nu_g(M)}{\text{vol } \mathbb{R}^n / \Gamma} \right| = \frac{1}{\text{vol } \mathbb{R}^n / \Gamma} \left| \int_M \mathcal{A}^* \mathbf{\Omega} - \nu_g(M) \right| \leq C(n) \sqrt[3]{\delta} \frac{\nu_g(M)}{\text{vol } \mathbb{R}^n / \Gamma}.$$

Hence if $C(n)\sqrt[3]{\delta} < 1$, then $\deg \mathcal{A} \neq 0$ and \mathcal{A} is surjective. If M is not oriented, then using the two-fold oriented cover $M_o \xrightarrow{\pi_o} M$, the same argumentation can be applied to $\pi_o^* \Omega$ in order to get that $\mathcal{A} \circ \pi_o: M_o \rightarrow \mathbb{R}^n / \Gamma$ is surjective.

Let us now prove iii). Observe that (10) is a direct consequence of (9) and [CC00, Theorem 1.2] (see also Theorem 7.1). Thus we are left with proving (9). To this aim, we use the following result: for any $f \in L^1(M)$ and $r \in [D, \sqrt{ND}]$,

$$\int_{B_r(\hat{o})} |f \circ \pi| \leq C(n) \int_M |f|. \quad (11)$$

Together with ii) in Proposition 4.1, this implies that for any $r \in [D, \sqrt{ND}]$,

$$\int_{B_r(\hat{o})} |d\hat{\mathcal{A}}^t d\hat{\mathcal{A}} - \text{Id}_n| \leq C(n) \sqrt[3]{\delta},$$

and similarly iii) in Proposition 4.1 yields that if $r \in [D, \delta^{-\frac{1}{6}} D]$, then

$$r^2 \int_{B_r(\hat{o})} |\nabla d\hat{\mathcal{A}}|^2 \leq C(n) \delta \left(\frac{r}{D}\right)^2 \leq C(n) \delta^{\frac{2}{3}}.$$

Thus we are left with proving (11). Let $\mathcal{D} \subset B_D(\hat{o})$ be a fundamental domain for $\Gamma \rightarrow \widehat{M} \xrightarrow{\pi} M$. Set

$$G(r) = \{\gamma \in \Gamma : \gamma \mathcal{D} \cap B_r(\hat{o}) \neq \emptyset\}.$$

Then

$$B_r(\hat{o}) \subset \bigcup_{\gamma \in G(r)} \gamma \mathcal{D} \subset B_{r+D}(\hat{o}),$$

so that

$$\nu_{\hat{g}}(B_r(\hat{o})) \leq \#G(r) \nu_{\hat{g}}(\mathcal{D}) = \#G(r) \nu_g(M) \leq \nu_{\hat{g}}(B_{r+D}(\hat{o})).$$

The group $\Gamma \simeq \mathbb{Z}^n$ is residually finite, hence Proposition 2.5 and Proposition 2.1 imply the volume doubling estimate: for any $0 < r \leq R \leq \sqrt{ND}$,

$$\nu_{\hat{g}}(B_R(\hat{o})) \leq C(n) \left(\frac{R}{r}\right)^{e^2 n} \nu_{\hat{g}}(B_r(\hat{o})).$$

If $r \geq D$ and $r + D \leq \sqrt{ND}$, this yields

$$\#G(r) \nu_g(M) \leq C(n) \left(\frac{r+D}{r}\right)^{e^2 n} \nu_{\hat{g}}(B_r(\hat{o})) \leq C(n) \nu_{\hat{g}}(B_r(\hat{o})),$$

so that

$$\begin{aligned} \int_{B_r(\hat{o})} |f \circ \pi| &= \frac{1}{\nu_{\hat{g}}(B_r(\hat{o}))} \sum_{\gamma \in G(r)} \int_{\mathcal{D} \cap B_r(\hat{o})} |f \circ \pi| \leq \frac{\#G(r) \nu_g(M)}{\nu_{\hat{g}}(B_r(\hat{o}))} \int_M |f| \\ &\leq C(n) \int_M |f|. \end{aligned}$$

If $r + D > \sqrt{ND}$, then the volume doubling estimate yields (see e.g. [HSC01, Subsection 2.3])

$$\nu_{\hat{g}}(B_{r+D}(\hat{o})) \leq C(n) \frac{r+D}{r} \nu_{\hat{g}}(B_r(\hat{o})) \leq C(n) \nu_{\hat{g}}(B_r(\hat{o})),$$

and we can conclude in the same way as above. \square

5. PROOF OF THEOREM A

In this section, we prove Theorem A. All the way through we consider $\varepsilon \in (0, 1)$ and

$$\eta := \frac{\varepsilon}{640n^2} \quad \text{and} \quad R := 64n^2D.$$

Let (M^n, g) be a closed Riemannian manifold of diameter D such that

$$b_1(M) = n \quad \text{and} \quad k_{D^2}(M^n, g) \leq \delta$$

for some $\delta \in (0, 1/(16n))$. We consider the covering $\Gamma \rightarrow \widehat{M} \xrightarrow{\pi} M$ built in the previous section, and associated Albanese maps $\widehat{\mathcal{A}}$ and \mathcal{A} . Then the following holds.

Claim 5.1. *There exists $\delta_0(\varepsilon, n) > 0$ such that if $\delta \leq \delta_0(\varepsilon, n)$, then $\widehat{\mathcal{A}}$ satisfies the following.*

- (a) $\widehat{\mathcal{A}}$ is $(1 + \eta)$ -Lipschitz.
- (b) $\widehat{\mathcal{A}}: B_R(\widehat{o}) \rightarrow \mathbb{B}_R^n$ is an ηR -almost isometry.
- (c) $\mathbb{B}_{(1-\eta)R}^n \subset \widehat{\mathcal{A}}(B_R(\widehat{o}))$.

This is a consequence of Proposition 4.3, Theorem 3.1 and i) in Remark 3.2. The last assertion may be proven with degree theory, see [Che01], [Gal98, Proof of 3.2 and 3.3], [CMT21, Proof of theorem 7.2].

From now on, we assume that

$$\delta \leq \delta_0(\varepsilon, n).$$

Step 1. Proceeding like in [Col97, Gal98], we construct a normal subgroup Γ_0 of Γ with finite index, such that $\widehat{\mathcal{A}}$ induces a map

$$\mathcal{A}_0: M_0 := \widehat{M}/\Gamma_0 \rightarrow \mathbb{R}^n/\rho(\Gamma_0).$$

Let (e_1, \dots, e_n) be the canonical basis of \mathbb{R}^n . Since $4\sqrt{n}D \leq (1 - \eta)R$, it follows from (c) that for any $i \in \{1, \dots, n\}$, there exists $x_i \in B_R(\widehat{o})$ such that

$$4\sqrt{n}De_i = \widehat{\mathcal{A}}(x_i).$$

Moreover, for any $i \in \{1, \dots, n\}$, there exists $\gamma_i \in \Gamma$ such that

$$d_{\widehat{g}}(\gamma_i \cdot \widehat{o}, x_i) \leq D.$$

Then we set

$$\Gamma_0 := \langle \gamma_1, \dots, \gamma_n \rangle \subset \Gamma$$

and

$$\mathbf{\Gamma}_0 := \rho(\Gamma_0) = \bigoplus_{i=1}^n \rho(\gamma_i)\mathbb{Z}.$$

Let us show that

$$\rho(\gamma_1), \dots, \rho(\gamma_n) \text{ form a basis of } \mathbb{R}^n. \quad (12)$$

We know by (a) that the map $\widehat{\mathcal{A}}$ is $(1 + \eta)$ -Lipschitz. Then for any $i \in \{1, \dots, n\}$, since the equivariance (6) of $\widehat{\mathcal{A}}$ yields $\rho(\gamma_i) = \widehat{\mathcal{A}}(\gamma_i \cdot \widehat{o})$, we have

$$|4\sqrt{n}De_i - \rho(\gamma_i)| = \left| \widehat{\mathcal{A}}(x_i) - \widehat{\mathcal{A}}(\gamma_i \cdot \widehat{o}) \right| \leq (1 + \eta)d_{\widehat{g}}(\gamma_i \cdot \widehat{o}, x_i) \leq (1 + \eta)D. \quad (13)$$

Then for any $\xi = \sum_{i=1}^n \xi_i e_i \in \mathbb{R}^n$,

$$\begin{aligned} \left| 4\sqrt{n}D\xi - \sum_{i=1}^n \xi_i \rho(\gamma_i) \right| &= \left| \sum_{i=1}^n \xi_i (4\sqrt{n}De_i - \rho(\gamma_i)) \right| \\ &\leq (1 + \eta)D \sum_{i=1}^n |\xi_i| \leq 2D\sqrt{n}|\xi|, \end{aligned}$$

so that

$$|\xi| \leq \frac{1}{2D\sqrt{n}} \left| \sum_{i=1}^n \xi_i \rho(\gamma_i) \right|. \quad (14)$$

Hence we get (12). This implies that the quotient \mathbb{R}^n/Γ_0 is a torus \mathbb{T}^n which we equip with the natural flat quotient Riemannian metric $g_{\mathbb{R}^n/\Gamma_0}$. We also equip M_0 with the quotient Riemannian metric g_0 induced by \widehat{g} .

Step 2. We establish the following diameter bound on (M_0, g_0) :

$$\text{diam}(M_0) \leq 4(n + \sqrt{n})D. \quad (15)$$

To this aim, let us prove an intermediary result: if $\gamma \in \Gamma_0$ is such that

$$|\rho(\gamma)| \leq \frac{R}{2\sqrt{n} + 1}, \quad (16)$$

then

$$|\mathbf{d}_{\widehat{g}}(\widehat{o}, \gamma \cdot \widehat{o}) - |\rho(\gamma)|| \leq \eta R. \quad (17)$$

Write $\gamma \in \Gamma_0$ as

$$\gamma = \gamma_1^{k_1} \dots \gamma_n^{k_n}$$

for some $k_1, \dots, k_n \in \mathbb{Z}$. Consider $i \in \{1, \dots, n\}$. From (b), we know that $|\mathbf{d}_{\widehat{g}}(x_i, \widehat{o}) - |\widehat{\mathcal{A}}(x_i) - \widehat{\mathcal{A}}(\widehat{o})|| \leq \eta R$. Since $\widehat{\mathcal{A}}(\widehat{o}) = 0$ and $\widehat{\mathcal{A}}(x_i) = 4\sqrt{n}De_i$, this implies

$$\mathbf{d}_{\widehat{g}}(x_i, \widehat{o}) \leq \eta R + 4\sqrt{n}D \leq (1 + 4\sqrt{n})D.$$

Then

$$\mathbf{d}_{\widehat{g}}(\widehat{o}, \gamma \cdot \widehat{o}) \leq \sum_i |k_i| \max_i \mathbf{d}_{\widehat{g}}(\widehat{o}, \gamma_i \cdot \widehat{o}) \leq (2 + 4\sqrt{n})D \sum_i |k_i|.$$

Since $\rho(\gamma) = \sum_i k_i \rho(\gamma_i)$, it follows from (14) that

$$\sum_i |k_i| \leq \sqrt{n} \left(\sum_i k_i^2 \right)^{1/2} \leq \frac{|\rho(\gamma)|}{2D}.$$

Then we get $\mathbf{d}_{\widehat{g}}(\widehat{o}, \gamma \cdot \widehat{o}) \leq (2\sqrt{n} + 1)|\rho(\gamma)|$, so that (16) implies

$$\gamma \cdot \widehat{o} \in B_R(\widehat{o}) \quad (18)$$

and the conclusion (17) follows from (b).

We are now in a position to prove the diameter bound (15). Introduce the Dirichlet domain

$$\mathcal{D}_0 := \left\{ x \in \widehat{M} : \mathbf{d}_{\widehat{g}}(x, \widehat{o}) \leq \mathbf{d}_{\widehat{g}}(\gamma \cdot x, \widehat{o}) \text{ for all } \gamma \in \Gamma_0 \setminus \{1\} \right\}.$$

We are going to show that

$$\mathcal{D}_0 \cap \left\{ x \in \widehat{M} : \mathbf{d}_{\widehat{g}}(x, \widehat{o}) = 2(n + \sqrt{n})D \right\} = \emptyset; \quad (19)$$

then the connectedness of \mathcal{D}_0 will imply $\mathcal{D}_0 \subset B_{2(n+\sqrt{n})D}(\widehat{o})$ and (15) will be established.

The set $\mathcal{F}_0 := \sum_{i=1}^n \left[-\frac{1}{2}, \frac{1}{2}\right) \rho(\gamma_i)$ is a fundamental domain for the action of Γ_0 on \mathbb{R}^n ; it is included in the Euclidean ball centered at the origin with radius

$$\frac{\sqrt{n}}{2} \max_i |\rho(\gamma_i)|.$$

By (13), for any i ,

$$|\rho(\gamma_i)| \leq |4\sqrt{n}De_i| + (1 + \eta)D \leq (4\sqrt{n} + 2)D,$$

so that

$$\frac{\sqrt{n}}{2} \max_i |\rho(\gamma_i)| \leq (2n + \sqrt{n})D.$$

For any $x \in \widehat{M}$ there exists $\gamma_0 \in \Gamma_0$ such that $\widehat{\mathcal{A}}(\gamma_0.x) \in \mathcal{F}_0$. By the equivariance (6) of $\widehat{\mathcal{A}}$ and the previous inequality, we get that

$$\left| \widehat{\mathcal{A}}(x) + \rho(\gamma_0) \right| = \left| \widehat{\mathcal{A}}(\gamma_0.x) \right| \leq (2n + \sqrt{n})D. \quad (20)$$

Now assume that $\mathbf{d}_{\widehat{g}}(x, \widehat{o}) = 2(n + \sqrt{n})D$. We are going to show that $x \notin \mathcal{D}_0$. Since $2(n + \sqrt{n})D \leq R$, from (b) we get

$$\left| \widehat{\mathcal{A}}(x) \right| = \left| \widehat{\mathcal{A}}(x) - \widehat{\mathcal{A}}(\widehat{o}) \right| \leq 2(n + \sqrt{n})D + \eta R.$$

Consequently,

$$|\rho(\gamma_0)| \leq \left| \widehat{\mathcal{A}}(x) + \rho(\gamma_0) \right| + \left| \widehat{\mathcal{A}}(x) \right| \leq (4n + 3\sqrt{n})D + \eta R.$$

By our choices of η and R we have

$$(4n + 3\sqrt{n})D + \eta R \leq \frac{R}{2\sqrt{n} + 1}.$$

Then we are in a position to apply (17). We get

$$\mathbf{d}_{\widehat{g}}(\gamma_0.\widehat{o}, \widehat{o}) \leq (4n + 3\sqrt{n})D + 2\eta R$$

and then

$$\mathbf{d}_{\widehat{g}}(\gamma_0.x, \widehat{o}) \leq \mathbf{d}_{\widehat{g}}(\gamma_0.x, \gamma_0.\widehat{o}) + \mathbf{d}_{\widehat{g}}(\gamma_0.\widehat{o}, \widehat{o}) \leq 2(n + \sqrt{n})D + (4n + 3\sqrt{n})D + 2\eta R.$$

Since

$$2(n + \sqrt{n})D + (4n + 3\sqrt{n})D + 2\eta R \leq R,$$

we can use (b) and (20) to deduce that

$$\mathbf{d}_{\widehat{g}}(\gamma_0.x, \widehat{o}) \leq \left| \widehat{\mathcal{A}}(\gamma_0.x) \right| + \eta R \leq (2n + \sqrt{n})D + \eta R < 2(\sqrt{n} + 2n)D = \mathbf{d}_{\widehat{g}}(x, \widehat{o}).$$

Thus $x \notin \mathcal{D}_0$ and (19) is proven.

Step 3. We prove that $\mathcal{A}_0 : M_0 \rightarrow \mathbb{R}^n/\Gamma_0$ is a $3\eta R$ -almost isometry. From Proposition 4.3, we know that $\widehat{\mathcal{A}}$ is surjective, hence \mathcal{A}_0 is surjective too. Thus we are left with proving the distance estimate. Let us introduce the following intermediate projection maps π_0 and p_0 :

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\mathcal{A}}} & \mathbb{R}^n \\ \downarrow \pi_0 & & \downarrow p_0 \\ M_0 & \xrightarrow{\mathcal{A}_0} & \mathbb{R}^{b_1}/\Gamma_0. \end{array}$$

Let $x, y \in M_0$. Since $\mathcal{D}_0 \subset B_{2(n+\sqrt{n})D}(\widehat{o})$, we can choose $\widehat{x} \in B_{2(n+\sqrt{n})D}(\widehat{o})$ such that $\pi_0(\widehat{x}) = x$. Let $c : [0, 1] \rightarrow M_0$ be a minimizing geodesic joining x and y . Let $\widehat{c} : [0, 1] \rightarrow \widehat{M}$ be such that $\pi_0 \circ \widehat{c} = c$ and $\widehat{x} = \widehat{c}(0)$. By the diameter bound (15), we know that $\widehat{y} := \widehat{c}(1)$ belongs to $B_{6(n+\sqrt{n})D}(\widehat{o}) \subset B_R(\widehat{o})$. Moreover, $\pi_0(\widehat{y}) = y$. Thus

$$\begin{aligned} \mathbf{d}_{g_{\mathbb{R}^n}/\Gamma_0}(\mathcal{A}_0(x), \mathcal{A}_0(y)) &= \mathbf{d}_{g_{\mathbb{R}^n}/\Gamma_0}(p_0(\widehat{\mathcal{A}}(\widehat{x})), p_0(\widehat{\mathcal{A}}(\widehat{y}))) \leq \left| \widehat{\mathcal{A}}(\widehat{x}) - \widehat{\mathcal{A}}(\widehat{y}) \right| \\ &\leq \mathbf{d}_{g_0}(x, y) + \eta R \end{aligned} \quad (21)$$

thanks to (b).

It remains to prove that

$$\mathbf{d}_{g_0}(x, y) - \mathbf{d}_{g_{\mathbb{R}^n}/\Gamma_0}(\mathcal{A}_0(x), \mathcal{A}_0(y)) \leq 3\eta R.$$

We start by showing that if $\mathcal{A}_0(x) = \mathcal{A}_0(y)$ then $\mathbf{d}_{g_0}(x, y) \leq \eta R$. Since $\mathcal{A}_0 \circ c = \mathcal{A}_0 \circ \pi_0 \circ \widehat{c} = p_0 \circ \widehat{\mathcal{A}} \circ \widehat{c}$, the curve $\widehat{\mathcal{A}} \circ \widehat{c}$ is a lift of the curve $\mathcal{A}_0 \circ c$ joining $\vec{v} := \widehat{\mathcal{A}}(\widehat{x}) \in \mathbb{R}^n$

to $\vec{w} := \widehat{\mathcal{A}}(\widehat{y})$. Moreover, the length of \widehat{c} is less than $4(n + \sqrt{n})D$, hence (b) implies that the length of $\widehat{\mathcal{A}} \circ \widehat{c}$ is less than $4(1 + \eta)(n + \sqrt{n})D$. Since $p_0(\vec{v}) = p_0(\vec{w})$ there exists $\gamma \in \Gamma_0$ such that

$$\widehat{\mathcal{A}}(\widehat{y}) = \vec{w} = \vec{v} + \rho(\gamma) = \widehat{\mathcal{A}}(\widehat{x}) + \rho(\gamma)$$

and

$$|\rho(\gamma)| \leq 4(1 + \eta)(n + \sqrt{n})D \leq 8(n + \sqrt{n})D.$$

As a consequence, notice that $\gamma^{-1}.\widehat{y}$ satisfies

$$\begin{aligned} d_{\widehat{g}}(\gamma^{-1}.\widehat{y}, \widehat{o}) &\leq d_{\widehat{g}}(\widehat{o}, \gamma.\widehat{o}) + d_{\widehat{g}}(\widehat{y}, \widehat{o}) \leq \eta R + |\rho(\gamma)| + 6(n + \sqrt{n})D \\ &\leq 14(\sqrt{n} + n)D + \eta R, \end{aligned}$$

where we used that $\widehat{\mathcal{A}}$ is an ηR -almost isometry in the second inequality. Since $14(\sqrt{n} + n)D + \eta R \leq R$, we get that $\gamma^{-1}y \in B_R(\widehat{o})$. Moreover, $\widehat{\mathcal{A}}(\widehat{y}) = \widehat{\mathcal{A}}(\widehat{x}) + \rho(\gamma)$, thus by the invariance of $\widehat{\mathcal{A}}$ we have $\widehat{\mathcal{A}}(\gamma^{-1}.\widehat{y}) = \widehat{\mathcal{A}}(\widehat{x})$. Then we can apply (b) and obtain

$$d_{g_0}(x, y) \leq d_{\widehat{g}}(\gamma^{-1}.\widehat{y}, \widehat{x}) \leq \eta R.$$

Now assume that $v := \mathcal{A}_0(x)$ and $w := \mathcal{A}_0(y)$ are distinct. We can choose \vec{v} and \vec{w} such that $p_0(\vec{v}) = v$, $p_0(\vec{w}) = w$,

$$d_{g_{\mathbb{R}^n/\Gamma_0}}(v, w) = |\vec{v} - \vec{w}| \quad \text{and} \quad \vec{v}, \vec{w} \in \mathbb{B}_{2(\sqrt{n}+n)D}^n.$$

Since $2(\sqrt{n} + n)D < (1 - \eta)R$, from (c) we know that there exist $\widehat{x}', \widehat{y}' \in B_R(\widehat{o})$ such that $\widehat{\mathcal{A}}(\widehat{x}') = \vec{v}$ and $\widehat{\mathcal{A}}(\widehat{y}') = \vec{w}$. Then $\mathcal{A}_0(\pi_0(\widehat{x}')) = p_0(\widehat{\mathcal{A}}(\widehat{x}')) = v = \mathcal{A}_0(x)$ and $\mathcal{A}_0(\pi_0(\widehat{y}')) = p_0(\widehat{\mathcal{A}}(\widehat{y}')) = w = \mathcal{A}_0(y)$, thus

$$\begin{aligned} d_{g_0}(x, y) &\leq 2\eta R + d_{g_0}(\pi_0(\widehat{x}'), \pi_0(\widehat{y}')) && \text{by the previous paragraph} \\ &\leq 2\eta R + d_{\widehat{g}}(\widehat{x}', \widehat{y}') \\ &\leq 3\eta R + |\vec{v} - \vec{w}| && \text{by (b)} \\ &= 3\eta R + d_{g_{\mathbb{R}^n/\Gamma_0}}(\mathcal{A}_0(x), \mathcal{A}_0(y)). \end{aligned}$$

Thus we have shown that $\mathcal{A}_0: M_0 \rightarrow \mathbb{R}^n/\Gamma_0$ is a $3\eta R$ -almost isometry.

Step 4. We conclude. Repeating the arguments of Step 3 with the commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{\mathcal{A}_0} & \mathbb{R}^n/\Gamma_0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\mathcal{A}} & \mathbb{R}^n/\Gamma. \end{array}$$

we get that $\mathcal{A}: M \rightarrow \mathbb{R}^n/\Gamma$ is a $9\eta R$ -almost isometry. Since $9\eta R = 9\varepsilon D/10$, this concludes the proof of Theorem A.

6. PROOF OF THEOREM B

In this section, we prove Theorem B. Let $f: [0, 1] \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying (SK). Let (M^n, g) be a closed Riemannian manifold of diameter D such that

- $b_1(M) = n$,
- $k_{D^2}(M^n, g) \leq \delta$ for some $\delta \in (0, 1/(16n))$,
- $k_{tD^2}(M^n, g) \leq f(t)$ for any $t \in (0, 1]$.

We consider the covering $\Gamma \rightarrow \widehat{M} \xrightarrow{\pi} M$ built in Section 4, and associated Albanese maps $\widehat{\mathcal{A}}$ and \mathcal{A} . Set $\widehat{g} := \pi^*g$. From (2), we get that

- $k_{(2D)^2}(\widehat{M}^n, \widehat{g}) \leq 4\delta$,

- $k_{t(2D)^2}(\widehat{M}^n, \widehat{g}) \leq 4f(t)$ for any $t \in (0, 1]$.

Let $\beta(n, 4f)$ be given by Proposition 3.4. Set

$$\eta := \beta(n, 4f)/4.$$

Let \widehat{H} be the heat kernel of \widehat{M} , and $\widehat{\theta}(t, x) := (4\pi t)^{n/2} \widehat{H}(t, x, x)$ for any $(t, x) \in \mathbb{R}_+ \times \widehat{M}$. Then the following holds.

Claim 6.1. *There exists $\delta_0(n, f) \in (0, \eta]$ such that if $\delta \leq \delta_0(n, f)$, then*

$$\widehat{\theta}((2D)^2, \widehat{\delta}) \leq 1 + \eta \quad (22)$$

and

$$\left((2D)^2 \int_{B_{2D}(\widehat{\delta})} |\nabla d\widehat{A}|^2 d\nu_{\widehat{g}} \right)^{\frac{1}{2}} + \int_{B_{2D}(\widehat{\delta})} |d\widehat{A}^t d\widehat{A} - \text{Id}_n| d\nu_{\widehat{g}} \leq \eta. \quad (23)$$

This claim puts us in a position to apply Proposition 3.4: we get that the map $\widehat{A}: B_{3D/2}(\widehat{\delta}) \rightarrow \mathbb{R}^n$ is a diffeomorphism onto its image. Therefore, the Albanese map $\mathcal{A}: M \rightarrow \mathbb{R}^n/\Gamma$ is a local diffeomorphism, hence it is a finite cover. Since a torus is finitely covered by tori only, we get that M is diffeomorphic to a torus. Then the Albanese map is necessarily a diffeomorphism, by construction. As a consequence, the conclusion of Theorem B holds.

Let us now prove Claim 6.1. From Proposition 4.3, we know that if we choose $\delta_0(n, f) \leq \min(\delta(n), 2^{-6}, (C(n)^{-1}\eta)^3)$, then (23) holds. Let us prove that we can also choose $\delta_0(n, f)$ such that (22) holds. To this aim, we introduce the following almost Euclidean heat kernel on \widehat{M} : for any $\varepsilon \in (0, 1)$, $x, y \in \widehat{M}$ and $t > 0$, we set

$$\mathbb{H}_\varepsilon(t, x, y) := \frac{1}{(1 + \varepsilon)(4\pi t)^{n/2}} e^{-(1+\varepsilon)\frac{d_g^2(x, y)}{4t}}. \quad (24)$$

Step 1. We prove the following Cheeger–Yau type estimate: for any $\varepsilon \in (0, 1)$ and any integer $\ell \geq 4$, there exists $\delta_1(n, f, \varepsilon, \ell) > 0$ such that if

$$k_{D^2}(M^n, g) \leq \delta \leq \delta_1(n, f, \varepsilon, \ell),$$

then for any $x, y \in \widehat{M}$ and $t \in (0, \ell D^2]$,

$$\mathbb{H}_\varepsilon(t, x, y) \leq \widehat{H}(t, x, y). \quad (25)$$

Let us set $\Gamma(\tau) := e^{8\sqrt{nk_\tau(M^n, g)}}$ for any $\tau > 0$. Since \widehat{M} satisfies the Dynkin condition (Dy), it satisfies the Li–Yau inequality from [Car19, Proposition 3.3]. Then we can proceed as in the proof of [CMT21, Proposition 2.12] to get that for any $s \in (0, t)$, any positive solution u of the heat equation on $\widehat{M} \times [0, \ell D^2]$ satisfies

$$\log \left(\frac{u(s, x)}{u(t, y)} \right) \leq \left(\frac{t}{s} \right)^{n/2} e^{\Gamma(t)\frac{d_g^2(x, y)}{4(t-s)}} e^{\frac{n}{2} \int_s^t \frac{\Gamma(\tau)-1}{\tau} d\tau}.$$

Apply this inequality with $u(\cdot, \cdot) = \widehat{H}(\cdot, x, \cdot)$ to get

$$\begin{aligned} \frac{(4\pi s)^{n/2} \widehat{H}(s, x, x)}{(4\pi t)^{n/2} \widehat{H}(t, x, y)} &\leq e^{\Gamma(t)\frac{d_g^2(x, y)}{4(t-s)}} e^{\frac{n}{2} \int_s^t \frac{\Gamma(\tau)-1}{\tau} d\tau} \\ &\leq e^{\Gamma(\ell D^2)\frac{d_g^2(x, y)}{4(t-s)}} e^{C(n) \int_0^{\ell D^2} \sqrt{k_\tau(M^n, g)} \frac{d\tau}{\tau}}. \end{aligned}$$

Letting $s \downarrow 0$ yields

$$\frac{e^{-\Gamma(\ell D^2)\frac{d_g^2(x, y)}{4t}}}{(4\pi t)^{n/2}} e^{-C(n) \int_0^{\ell D^2} \sqrt{k_\tau(M^n, g)} \frac{d\tau}{\tau}} \leq \widehat{H}(t, x, y).$$

Since $k_{\ell D^2}(M^n, g) \leq \ell k_{D^2}(M^n, g) \leq \ell \delta$, we know that there exists $\delta_2(n, f, \varepsilon, \ell) > 0$ such that if $\delta \leq \delta_2(n, f, \varepsilon, \ell)$, then

$$\Gamma(\ell D^2) \leq 1 + \varepsilon.$$

To conclude, let us show that there exists $\delta_1(n, f, \varepsilon, \ell) \leq \delta_2(n, f, \varepsilon, \ell)$ such that if $\delta \leq \delta_1(n, f, \varepsilon, \ell)$, then

$$e^{C(n) \int_0^{\ell D^2} \sqrt{nk_\tau(M^n, g)} \frac{d\tau}{\tau}} \leq 1 + \varepsilon. \quad (26)$$

Since $k_\tau(M^n, g) \leq f(\tau/D^2)$ for any $\tau \in (0, D^2)$, for any $\sigma \in (0, 1)$ we have

$$\begin{aligned} \int_0^{\ell D^2} \sqrt{k_\tau(M^n, g)} \frac{d\tau}{\tau} &\leq \int_0^{\sigma D^2} \sqrt{k_\tau(M^n, g)} \frac{d\tau}{\tau} + \log(\ell/\sigma) \sqrt{k_{\ell D^2}(M^n, g)} \\ &\leq \int_0^\sigma \sqrt{f(\tau)} \frac{d\tau}{\tau} + \log(\ell/\sigma) \sqrt{\ell k_{D^2}(M^n, g)}. \end{aligned}$$

Therefore, to get (26), first we choose $\sigma(f) \in (0, 1)$ such that

$$e^{C(n) \int_0^{\sigma(f)} \frac{\sqrt{f(s)}}{s} ds} \leq \sqrt{1 + \varepsilon},$$

then we choose $\delta_1(n, f, \varepsilon, \ell) \leq \delta_2(n, f, \varepsilon, \ell)$ such that

$$e^{C(n) \log(\ell/\sigma(f)) \sqrt{\ell \delta_1(n, f, \varepsilon, \ell)}} \leq \sqrt{1 + \varepsilon}.$$

Step 2. We prove that for any $\varepsilon \in (0, 1)$ and $t > D^2$,

$$\int_{\widehat{M}} \mathbb{H}_\varepsilon(t, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) \geq \frac{1 - C(n) \sqrt[3]{\delta}}{(1 + \varepsilon)^{\frac{n}{2} + 1}} \left(1 - C(n) \left(\frac{D}{\sqrt{t}} \right)^{n+2} - C(n) e^{-\frac{D^2}{5\sqrt[3]{\delta t}}} \right). \quad (27)$$

Set $s := t/(1 + \varepsilon)$. Then

$$\int_{\widehat{M}} \mathbb{H}_\varepsilon(t, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) = \frac{1}{(1 + \varepsilon)^{1 + \frac{n}{2}}} \int_{\widehat{M}} \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\frac{d_{\widehat{g}}^2(\widehat{\partial}, y)}{4s}} d\nu_{\widehat{g}}(y).$$

By Cavalieri's principle,

$$\begin{aligned} \int_{\widehat{M}} \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\frac{d_{\widehat{g}}^2(\widehat{\partial}, y)}{4s}} d\nu_{\widehat{g}}(y) &= \int_0^{+\infty} \frac{e^{-\frac{r^2}{4s}}}{(4\pi s)^{\frac{n}{2}}} \frac{r}{2s} \nu_{\widehat{g}}(B_r(\widehat{\partial})) dr \\ &\geq \int_D^{D\delta^{-\frac{1}{6}}} \frac{e^{-\frac{r^2}{4s}}}{(4\pi s)^{\frac{n}{2}}} \frac{r}{2s} \nu_{\widehat{g}}(B_r(\widehat{\partial})) dr \\ &\geq \left(1 - C(n) \sqrt[3]{\delta} \right) \int_D^{D\delta^{-\frac{1}{6}}} \frac{e^{-\frac{r^2}{4s}}}{(4\pi s)^{\frac{n}{2}}} \frac{r}{2s} \omega_n r^n dr, \end{aligned}$$

where we use the volume lower bound (10) to get the last line. Set

$$\phi_{n,s}(r) := \frac{e^{-\frac{r^2}{4s}}}{(4\pi s)^{\frac{n}{2}}} \frac{r}{2s} \omega_n r^n$$

and note that

$$\int_0^{+\infty} \phi_{n,s}(r) dr = 1.$$

Then

$$\int_{\widehat{M}} \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\frac{d_{\widehat{g}}^2(\widehat{\partial}, y)}{4s}} d\nu_{\widehat{g}}(y) \geq \left(1 - C(n) \sqrt[3]{\delta} \right) \left(1 - \int_0^D \phi_{n,s}(r) dr - \int_{D\delta^{-\frac{1}{6}}}^{+\infty} \phi_{n,s}(r) dr \right). \quad (28)$$

A direct computation shows that

$$\int_0^D \phi_{n,s}(r) dr \leq C(n) \left(\frac{D}{\sqrt{s}} \right)^{n+2} \quad \text{and} \quad \int_{D\delta^{-\frac{1}{6}}}^{+\infty} \phi_{n,s}(r) dr \leq C(n) e^{-\frac{D^2}{5\sqrt[3]{\delta s}}}. \quad (29)$$

Hence we get (27).

Step 3. We conclude. Consider $\varepsilon \in (0, 1)$. From the proof of [CMT22, Lemma 5.7], we know that for any integer $\ell \geq 4$,

$$\widehat{\theta}((2D)^2, \widehat{\partial}) \leq \widehat{\theta}(\ell D^2, \widehat{\partial}) e^{C(n)\sqrt{\delta\ell}}.$$

We are going to show that we can choose δ small enough and ℓ large enough to ensure that

$$\widehat{\theta}(\ell D^2, \widehat{\partial}) \leq \sqrt{1 + \eta} \quad (30)$$

and

$$e^{C(n)\sqrt{\delta\ell}} \leq \sqrt{1 + \eta}. \quad (31)$$

For the sake of brevity, let us set $\tau := \ell D^2$. Assume that $\delta \leq \delta_1(n, f, \varepsilon, \ell)$ where $\delta_1(n, f, \varepsilon, \ell)$ is given by Step 1. The semi-group law yields

$$\begin{aligned} \widehat{\theta}(\tau, \widehat{\partial}) &= (4\pi\tau)^{\frac{n}{2}} \int_{\widehat{M}} \widehat{H}^2(\tau/2, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) \\ &= (4\pi\tau)^{\frac{n}{2}} \int_{\widehat{M}} \left(\widehat{H}^2(\tau/2, \widehat{\partial}, y) - \mathbb{H}_\varepsilon^2(\tau/2, \widehat{\partial}, y) \right) d\nu_{\widehat{g}}(y) \\ &\quad + \frac{1}{1 + \varepsilon} \int_{\widehat{M}} \mathbb{H}_\varepsilon(\tau/4, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) \\ &=: I + II. \end{aligned} \quad (32)$$

By Step 1 and stochastic completeness,

$$II \leq \frac{1}{1 + \varepsilon} \leq 1. \quad (33)$$

By Step 1 we also know that $0 \leq \widehat{H}^2 - \mathbb{H}_\varepsilon^2 = (\widehat{H} - \mathbb{H}_\varepsilon)(\widehat{H} + \mathbb{H}_\varepsilon) \leq (\widehat{H} - \mathbb{H}_\varepsilon)2\widehat{H}$. Moreover, the heat kernel upper bound from Proposition 2.2 and the volume lower bound (10) imply that if

$$\ell\delta^{\frac{1}{3}} \leq 2, \quad (34)$$

then

$$\widehat{H}(\tau/2, \widehat{\partial}, y) \leq \frac{C(n)}{\tau^{\frac{n}{2}}},$$

so that

$$\begin{aligned} I &\leq (4\pi\tau)^{\frac{n}{2}} \int_{\widehat{M}} \left(\widehat{H}(\tau/2, \widehat{\partial}, y) - \mathbb{H}_\varepsilon(\tau/2, \widehat{\partial}, y) \right) 2\widehat{H}(\tau/2, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) \\ &\leq C(n) \int_{\widehat{M}} \left(\widehat{H}(\tau/2, \widehat{\partial}, y) - \mathbb{H}_\varepsilon(\tau/2, \widehat{\partial}, y) \right) d\nu_{\widehat{g}}(y) \\ &= C(n) \left(1 - \int_{\widehat{M}} \mathbb{H}_\varepsilon(\tau/2, \widehat{\partial}, y) d\nu_{\widehat{g}}(y) \right), \end{aligned} \quad (35)$$

by stochastic completeness. By combining (32), (33), (35), and thanks to Step 2, we eventually get

$$\begin{aligned} \widehat{\theta}(\tau, \widehat{\partial}) &\leq 1 + C(n) \left(1 - \frac{1 - C(n)\sqrt[3]{\delta}}{(1 + \varepsilon)^{\frac{n}{2} + 1}} \left(1 - C(n) \left(\frac{1}{\sqrt{\ell}} \right)^{n+2} - C(n)e^{-\frac{1}{5\sqrt[3]{\delta\ell}}} \right) \right) \\ &\leq 1 + C(n) \left(\varepsilon + \sqrt[3]{\delta} + \left(\frac{1}{\sqrt{\ell}} \right)^{n+2} + e^{-\frac{1}{5\sqrt[3]{\delta\ell}}} \right). \end{aligned}$$

Then we choose successively:

- (1) $\varepsilon \in (0, 1)$ such that $C(n)\varepsilon \leq \frac{\eta}{12}$,
- (2) $\ell \geq 4$ such that $C(n) \left(\frac{1}{\sqrt{\ell}} \right)^{n+2} \leq \frac{\eta}{12}$,

(3) $\delta \leq \delta_1(n, f, \varepsilon, \ell)$ such that (31) and (34) hold together with

$$C(n)\sqrt[3]{\delta} \leq \frac{\eta}{12} \quad \text{and} \quad C(n)e^{-\frac{1}{5\sqrt[3]{\delta\ell}}} \leq \frac{\eta}{12}.$$

This implies $\widehat{\theta}(\ell D^2, \widehat{o}) \leq 1 + \eta/3 \leq \sqrt{1 + \eta}$ and concludes the proof.

7. APPENDIX : ALMOST SURJECTIVITY

In this appendix, we follow the lines of the proof of [CC00, Theorem 1.2] to establish the following.

Theorem 7.1. *Let (M^n, g) be a complete Riemannian manifold and $k \in \{1, \dots, n\}$. There exist $\eta(n, k) \in (0, 1)$ and $C(n, k) > 0$ such that for any $o \in M$ and $r > 0$, if there exists $\Phi: B_r(o) \rightarrow \mathbb{R}^k$ smooth and $\varepsilon \in (0, \eta(n, k))$ such that*

- i) $\Phi(o) = 0$,
- ii) $\|d\Phi\|_{L^\infty(B)} \leq 1 + \varepsilon$,
- iii) $\int_{B_r(o)} |d\Phi^t d\Phi - \text{Id}_k| d\nu_g \leq \varepsilon$,
- iv) $r \int_{B_r(o)} |\nabla d\Phi| d\nu_g \leq \varepsilon$,

then

$$\mathcal{H}^k(\mathbb{B}_r^k \setminus \Phi(B_r(o))) \leq C(n, k)r^k \varepsilon.$$

Proof. By scaling, there is no loss of generality in assuming $r = 1$, what we do from now on. Set $B := B_1(o)$ and

$$w := \sqrt{\det d\Phi^t d\Phi} = |d\Phi_1 \wedge \dots \wedge d\Phi_k|,$$

and recall that the coarea formula gives that for any $f \in L^1(B)$,

$$\int_B f d\nu_g = \int_{\mathbb{R}^k} \left(\int_{\Phi^{-1}(z)} \frac{f}{w} d\mathcal{H}^{n-k} \right) dz. \quad (36)$$

Acting as in [CC00], we introduce a function $J: \mathbb{R}^k \rightarrow \mathbb{R}_+$ which provides a weighted measure of the fibers $\Phi^{-1}(z)$. Let $\chi: \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth function such that $\chi = 0$ on $[0, 1/4]$ and $\chi = 1$ on $[1/2, +\infty)$. For any $z \in \mathbb{R}^k$, set

$$J(z) := \int_{\Phi^{-1}(z)} \chi \circ w^2 d\mathcal{H}^{n-k}.$$

Note that if $z \notin \Phi(B)$, then $J(z) = 0$. Moreover, the presence of χ in the integrand ensures that the integral may be taken over

$$\Sigma_z := \Phi^{-1}(z) \cap \{w > 0\}$$

which is a smooth $(n - k)$ -dimensional submanifold of B . By ii), we know that $\Phi(B) \subset \mathbb{B}_{1+\varepsilon}^k$. Therefore, by the Poincaré inequality,

$$\mathcal{H}^k(\mathbb{B}_{1+\varepsilon}^k \setminus \Phi(B)) \bar{J} \leq \int_{\mathbb{B}_{1+\varepsilon}^k} |J(z) - \bar{J}| dz \leq C(n, k) \int_{\mathbb{B}_{1+\varepsilon}^k} |\nabla J(z)| dz \quad (37)$$

where

$$\bar{J} := \int_{\mathbb{B}_{1+\varepsilon}^k} J.$$

Let us estimate $|\nabla J(z)|$. For any $v \in \mathbb{R}^k$ and $x \in \{w > 0\}$, we let $X_v(x)$ be the unique element in $T_x \Sigma_{\Phi(x)}^\perp$ such that $d_x \Phi(X_v(x)) = v$. Then there exists $\xi: B \rightarrow \mathbb{R}^k$ such that $\sum_{\alpha=1}^k \xi_\alpha \nabla \Phi_\alpha = X_v$. We easily get that

$$|X_v| \leq \frac{2^{k-1}}{w} |v|.$$

Moreover

$$\nabla_v J(z) = \int_{\Sigma_z} (\chi' \circ w^2) \nabla_{X_v} w^2 d\mathcal{H}^{n-k} + \int_{\Sigma_z} (\chi \circ w^2) \langle \vec{H}, X_v \rangle d\mathcal{H}^{n-k} \quad (38)$$

where \vec{H} is the mean curvature vector of Σ_z . We easily compute that

$$\langle \vec{H}, X_v \rangle = - \sum_{\alpha=1}^k \sum_{i=1}^{n-k} \xi_\alpha \nabla d\Phi_\alpha(e_i, e_i)$$

where (e_1, \dots, e_{n-k}) is an orthonormal basis of $T_x \Sigma_z$. Hence

$$\left| \langle \vec{H}, X_v \rangle \right| \leq C(n, k) \frac{|\nabla d\Phi|}{w} |v|.$$

We also have

$$|\nabla_{X_v} w^2| \leq C(n) |v| |\nabla d\Phi|.$$

Using the fact that integration in (38) is done only on the set $\{w \geq 1/2\}$, we get that

$$|\nabla J(z)| \leq C(n, k) \int_{\Phi^{-1}(z)} \frac{|\nabla d\Phi|}{w} d\mathcal{H}^{n-k}.$$

Then the coarea formula (36) applied to $|\nabla d\phi|$ together with iv yields that

$$\int_{\mathbb{R}^k} |\nabla J(z)| dz \leq C(n, k) \varepsilon \nu_g(B). \quad (39)$$

Now we bound \bar{J} from below. The coarea formula gives that

$$\bar{J} = \frac{1}{\omega_k} \int_B (\chi \circ w^2) w d\nu_g \geq \frac{1}{\sqrt{2}\omega_k} \nu_g(\{w^2 > 1/2\}).$$

But there is a constant $C(n) > 0$ such that $|w^2 - 1| \leq C(n) |d\Phi^t d\Phi - \text{Id}_k|$, so that

$$\begin{aligned} \nu_g(\{w^2 < 1/2\}) &\leq \nu_g(\{|d\Phi^t d\Phi - \text{Id}_k| > 1/(2C(n))\}) \\ &\leq 2C(n) \int_B |d\Phi^t d\Phi - \text{Id}_k| d\nu_g \leq 2C(n) \nu_g(B) \varepsilon. \end{aligned}$$

As a consequence, if $4C(n)\varepsilon < 1$ then $\nu_g(\{w^2 \geq 1/2\}) \geq \frac{1}{2} \nu_g(B)$, so that

$$\bar{J} \geq \frac{1}{2\sqrt{2}\omega_k} \nu_g(B). \quad (40)$$

Combining (37), (39) and (40), we get

$$\mathcal{H}^k(\mathbb{B}_{1+\varepsilon}^k \setminus \Phi(B)) \leq C(n, k) \varepsilon.$$

Finally,

$$\mathcal{H}^k(\mathbb{B}_1^k \setminus \Phi(B)) \leq \mathcal{H}^k(\mathbb{B}_{1+\varepsilon}^k \setminus \Phi(B)) + \mathcal{H}^k(\mathbb{B}_{1+\varepsilon}^k \setminus \mathbb{B}_1^k) \leq C(n, k) \varepsilon + C(k) \varepsilon. \quad \square$$

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