

SCUOLA NORMALE SUPERIORE

CLASSE DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Rectifiability in Carnot Groups

Tesi di Perfezionamento in Matematica

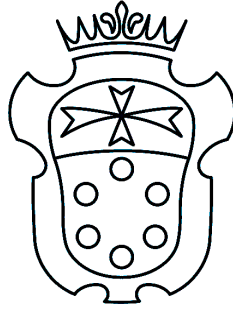
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Prof. Enrico Le Donne

Anno Accademico 2021 – 2022
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ABSTRACT. This thesis is devoted to the study of the theory of rectifiability of sets and measures in the non smooth context of Carnot groups. The focus is on the study of the notion of \mathcal{P} -rectifiability and its relation with other notions of rectifiability in Carnot groups.

A \mathcal{P} -rectifiable measure of integer dimension h in a Carnot group is a Radon measure with positive lower and finite upper h -densities almost everywhere such that the tangent measures are almost everywhere Haar measures of homogeneous subgroups of the Carnot group of homogeneous dimension h .

The results discussed in this thesis have been obtained in the papers [25, 31–33].

In Chapter 1 we shall revise the basic notions of Measure Theory, and we shall introduce Carnot groups with a special focus on the notions of rectifiability, intrinsic regular functions, and submanifolds.

In Chapter 2 we summarize part of the results obtained in collaboration with A. Merlo in [31, 33]. We prove that in arbitrary Carnot groups \mathcal{P} -rectifiable measures of dimension h with a unique complemented tangent almost everywhere have h -density. We also characterize \mathcal{P} -rectifiable measures with complemented tangents by means of a covering property with intrinsically differentiable graphs in Carnot groups. These results complement and extend in several directions the study by Mattila–Serapioni–Serra Cassano in the Heisenberg groups \mathbb{H}^n .

In Chapter 3 we give the proof of a Marstrand–Mattila type rectifiability criterion in Carnot groups for \mathcal{P} -rectifiable measures with tangents that admit at least one normal complementary subgroup. This result extends to the Carnot setting the Marstrand–Mattila rectifiability criterion in Euclidean spaces. We exploit such a criterion to derive as a consequence the Preiss’s Theorem for one-dimensional Radon measures in the first Heisenberg group \mathbb{H}^1 endowed with the Korányi norm. The results in Chapter 3 have been obtained in collaboration with A. Merlo in [32].

In Chapter 4 we present the results obtained with E. Le Donne in [25]. In some Carnot group of homogeneous dimension 13 we construct an analytic hypersurface, which is also a $C_{\mathbb{H}}^1$ -hypersurface, that is purely unrectifiable with respect to Carnot groups of homogeneous dimension 12. This gives an example of a $C_{\mathbb{H}}^1$ -hypersurface that is not Pauls rectifiable. As a consequence Franchi–Serapioni–Serra Cassano’s notion of $C_{\mathbb{H}}^1$ -rectifiability differs from Pauls’s notion of rectifiability in arbitrary Carnot groups. We further present a proof of the fact that in \mathbb{H}^n , with $n \geq 2$, every Euclidean C^∞ -hypersurface can be almost everywhere covered by bi-Lipschitz images of subsets of codimension-one subgroups of \mathbb{H}^n .

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Introduction

This thesis is about Geometric Measure Theory in Carnot groups, with a particular focus on the theory of rectifiability of sets and measures. The results discussed in this manuscript are obtained in collaboration with my PhD advisor Le Donne, and my colleague Merlo [25, 31–33].

In my PhD studies I obtained several other results that will be discussed in the last section of this introduction. Among them I would like to mention a line of research devoted to the study of the isoperimetric problem in spaces with curvature bounded from below that I have been pursuing with several coauthors in [20, 24, 34–36].

The aim of this introduction is to sketch the birth, the development, and the main achievements of the theory of rectifiability: starting from the classical Euclidean setting, passing through the arbitrary metric setting, and finally specializing in Carnot groups. After that, I will discuss the main contributions I gave to the topic of rectifiability in Carnot groups.

The presentation is inspired to the classical references [89, 102, 174] for the Euclidean part. For the development of the theory in metric spaces I shall refer to the introductions of the fundamental papers that I am going to discuss below, while for the discussion related to Carnot groups I shall mainly refer to [145, 208].

Since it is virtually impossible to catch in few pages the huge developments of the theory of rectifiability, I will cut some topics off from the discussion. Nevertheless, for a more informative and general treatment, the reader is referred to the very nice and recent survey by Mattila [177].

Rectifiability in Euclidean spaces

At the beginning of 1900, the fundamental works of Lebesgue, Carathéodory, and Hausdorff introduced the generalizations of the notions of length and area in Euclidean spaces: in fact, the m -dimensional Hausdorff measure \mathcal{H}^m is an outer measure which generalizes the notion of m -dimensional area.

After that, the birth of the theory of rectifiability dates back to three fundamental works of Besicovitch between 1920 and 1940 [55–57].

Besicovitch was mainly concerned with the study of one-dimensional objects in \mathbb{R}^2 : namely, he aimed at studying the geometry of Borel sets $E \subseteq \mathbb{R}^2$ such that $\mathcal{H}^1(E) < +\infty$. His remarkable works revealed that a lot could be said for such sets: e.g., they split in a *regular* part, on which the one-density of the Hausdorff measure \mathcal{H}^1 is \mathcal{H}^1 -almost everywhere equal to one, and an *irregular* part, which we would call - with the modern language - the *purely unrectifiable* part. Moreover, on the regular part, the set is infinitesimally linear \mathcal{H}^1 -almost everywhere: this amounts to saying that it has \mathcal{H}^1 -almost everywhere a well defined one-dimensional tangent space. Finally, the regular part of the set E can be \mathcal{H}^1 -almost everywhere covered by Lipschitz curves.

In [103], Federer generalized many of the ideas and results of Besicovitch for k -dimensional sets in \mathbb{R}^n . Anyway one of the main results Besicovitch proved had been left open: is it true that a Borel set $E \subseteq \mathbb{R}^n$ such that $\mathcal{H}^k(E) < +\infty$ and the density of $\mathcal{H}^k \llcorner E$ is one almost everywhere has a well defined tangent k -plane \mathcal{H}^k -almost everywhere? Further, is it true that, under the same hypotheses, E is covered \mathcal{H}^k -almost everywhere by countably many Lipschitz images of subsets of \mathbb{R}^k in \mathbb{R}^n ?

Let us now introduce some terminology. Given a Radon measure ϕ on \mathbb{R}^n , and given $k \geq 0$, we define the *lower and upper k -densities of ϕ at $x \in \mathbb{R}^n$* , respectively, as

$$(1) \quad \Theta_*^k(\phi, x) := \liminf_{r \rightarrow 0} \frac{\phi(\overline{B}_r(x))}{r^k}, \quad \Theta^{k,*}(\phi, x) := \limsup_{r \rightarrow 0} \frac{\phi(\overline{B}_r(x))}{r^k},$$

where $\overline{B}_r(x)$ is the closed Euclidean ball of radius $r > 0$ and center $x \in \mathbb{R}^n$. We say that ϕ has *k -density at x* if

$$0 < \Theta_*^k(\phi, x) = \Theta^{k,*}(\phi, x) < +\infty,$$

and we denote by $\Theta^k(\phi, x)$ the k -density of ϕ at x .

We shall say that a Borel set $E \subseteq \mathbb{R}^n$ is *(ϕ, k) -rectifiable*, with $k \in \mathbb{N}$, if there exist countably many $f_i : U_i \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that

$$(2) \quad \phi \left(E \setminus \bigcup_{i=1}^{+\infty} f_i(U_i) \right) = 0.$$

The query that had been left open by Federer was settled first in the case $k = 2$ and $n = 3$ by Marstrand [171], and then for every k , and n by Mattila [179]. In particular, as a result of the work by Mattila [179] the following equivalence holds for Borel sets E , provided one chooses the correct multiplicative constant in the definition of the Hausdorff measure:

$$(3) \quad E \subseteq \mathbb{R}^n \text{ is } (\mathcal{H}^k, k)\text{-rectifiable if and only if } \Theta^k(\mathcal{H}^k \llcorner E, x) = 1 \text{ for } \mathcal{H}^k\text{-almost every } x \in E.$$

One fundamental step in order to prove the previous equivalence is the following so-called Marstrand–Mattila rectifiability criterion, that we shall state for arbitrary Radon measures. In the following statement we denote by $\text{Tan}_k(\phi, x)$ the set of k -tangent measures to ϕ at x , for which we refer the reader to Definition 1.52. We recall that a Radon measure ϕ on \mathbb{R}^n is said to be *k -rectifiable*, with $k \in \mathbb{N}$, if $\phi \ll \mathcal{H}^k$ and \mathbb{R}^n is (ϕ, k) -rectifiable. For the following statement, see [174, Theorem 16.7].

Proposition 0.1 (Marstrand–Mattila rectifiability criterion). *Let ϕ be a Radon measure on \mathbb{R}^n , and let k be a natural number such that $0 \leq k \leq n$. Then ϕ is k -rectifiable if and only if for ϕ -almost every $x \in \mathbb{R}^n$ one has*

- (i) $0 < \Theta_*^k(\phi, x) \leq \Theta^{k,*}(\phi, x) < +\infty$,
- (ii) $\text{Tan}_k(\phi, x) \subseteq \{\lambda \mathcal{H}^k \llcorner V : \lambda > 0, \text{ and } V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{R}^n\}$.

The latter proposition can be interpreted in the following way: in Euclidean spaces, the global k -rectifiability property - by means of covering with Lipschitz images of subsets of \mathbb{R}^k - for a measure ϕ arises as a consequence of the infinitesimal (flat) structure of the measure, and vice-versa.

After the foundational contributions described above, there was still one open question in order to complete the picture of the rectifiability in Euclidean spaces. Namely, is the proper analogue of (3) true for arbitrary Radon measures, and not just for sets? The answer is affirmative, and it is due to the breakthrough contribution by Preiss in [202].

Theorem 0.2 (Preiss's Theorem). *Let ϕ be a Radon measure on \mathbb{R}^n , and let $k \geq 0$. Assume that ϕ has k -density positive and finite for ϕ -almost all $x \in \mathbb{R}^n$. Hence k is an integer, and*

ϕ is k -rectifiable. Vice-versa if ϕ is k -rectifiable with $k \in \mathbb{N}$, hence ϕ has k -density positive and finite for ϕ -almost all $x \in \mathbb{R}^n$.

The assertion of k being an integer in the previous theorem is due to Marstrand [172], while the consequence of ϕ being k -rectifiable is due to Preiss [202]. Actually, in [202] Preiss proved something stronger. He proved that, for every couple of integers $k \leq n$, there exists a constant $c(k, n) \leq 1$ such that whenever ϕ is a Radon measure on \mathbb{R}^n for which

$$\frac{\Theta_*^k(\phi, x)}{\Theta^{k,*}(\phi, x)} \geq c(k, n),$$

for ϕ -almost every $x \in \mathbb{R}^n$, hence ϕ is k -rectifiable. Finally, the last part of Theorem 0.2 is an easy consequence of the locality and the area formula.

The theory of rectifiability gained a lot of fortune because it soon showed to be the correct language to study some classical problems. We refer here just to one famous example. In 1954 and 1955 De Giorgi, motivated by earlier results by Caccioppoli, published a couple of papers [87, 88] in which he studied the structure of finite perimeter sets in \mathbb{R}^n . He showed that given a set of finite perimeter $E \subseteq \mathbb{R}^n$, the perimeter measure $|D\chi_E|$ is the restriction of the Hausdorff measure \mathcal{H}^{n-1} to the so-called reduced boundary, which is a measure-theoretic notion of boundary smaller than the topological boundary. Moreover, De Giorgi proved that the reduced boundary is $(|D\chi_E|, n-1)$ -rectifiable. Such a fundamental structure result for codimension-one objects in \mathbb{R}^n led to the development of Federer–Fleming’s theory of currents that is one of the cornerstones of Geometric Measure Theory [104]. For more on the subject one can read [177, Sections 13-14].

Analysis and rectifiability on metric measure spaces

Let (X, d) be a complete metric space, and let ϕ be a Radon measure on X . The definitions of lower and upper densities (1) make sense in this metric setting, and also the notion of (ϕ, k) -rectifiability (2) if one takes X -valued Lipschitz maps instead of \mathbb{R}^n -valued Lipschitz maps. So it makes sense to study the notion of rectifiability even in the metric setting. This study fits into the broader framework of the study of analysis in metric spaces, for which we refer to [121, 123–125] for some landmark contributions.

One of the first influential papers for the study of the theory of rectifiability in the metric setting was Kirchheim’s paper [138]. In that paper, he shows both a Rademacher-type theorem and an area formula for Lipschitz maps $f : \mathbb{R}^n \rightarrow (X, \|\cdot\|)$, where $(X, \|\cdot\|)$ is a Banach space. A remarkable consequence of his study is that whenever (X, d) is a complete metric space that is (\mathcal{H}^n, n) -rectifiable, then $\Theta^n(\mathcal{H}^n, x) = 1$ for \mathcal{H}^n -almost every $x \in X$, namely one implication of (3) holds in the general metric setting.

A related study in the metric setting is in the paper by Preiss and Tišer [203], in which they improve a previous result by Besicovitch [55]. In fact, Besicovitch shows that if a Borel $E \subseteq \mathbb{R}^2$ is such that $\mathcal{H}^1(E) < +\infty$ and $\Theta_*^1(\mathcal{H}^1 \llcorner E, x) \geq 3/4$ for \mathcal{H}^1 -almost every $x \in E$, then E is $(\mathcal{H}^1, 1)$ -rectifiable. Preiss and Tišer extend Besicovitch’s result to arbitrary metric spaces and they slightly improve the constant $3/4$ by substituting it with $(2 + \sqrt{46})/12$. As a result, remarkably, on the one hand the equivalence in (3) holds in the arbitrary metric setting in the case $k = 1$. On the other hand, it is conjectured that the best constant in Besicovitch’s result is $1/2$ in the arbitrary metric setting, but nowadays the conjecture is still open. It’s worth to point out that it is also not known if the equivalence in (3) holds for $k > 1$ in the general metric setting.

Later, a further impulse to the study of rectifiability in the metric setting was given by the work of Ambrosio and Kirchheim [14], where the authors study (\mathcal{H}^k, k) -rectifiable sets

in arbitrary metric spaces. They prove another variant of Rademacher theorem for Banach-valued Lipschitz maps defined on subsets of \mathbb{R}^n , and they prove area and coarea formulae for Lipschitz maps defined on rectifiable sets with values in arbitrary metric spaces. As the authors explicitly say in the paper, one of the motivations for such a study was the development of the theory of currents in metric spaces [13], which is one of the landmark contributions of modern Geometric Measure Theory.

The paper [14] has been fundamental also for the development of the theory of rectifiability in Carnot groups, as we will specify below. Indeed, in [14] the authors prove that the first Heisenberg group \mathbb{H}^1 with a sub-Riemannian distance is purely k -unrectifiable with $k = 2, 3, 4$. Notice that the topological dimension of \mathbb{H}^1 is 3, and its metric dimension is 4.

In addition to the previously described results, one of the main contributions to the study of Lipschitz functions in metric spaces is Cheeger's paper [70], see also the contribution of Keith [135]. In these papers, the authors propose and study the notion of *Lipschitz differentiability space*: namely, a metric measure space (X, d, μ) with countably many Lipschitz charts with values in Euclidean spaces such that every real-valued Lipschitz function on X is μ -almost everywhere differentiable with respect to every chart. One of the main results is that every metric measure space that is doubling and supports a Poincaré inequality is a Lipschitz differentiable space, and the dimension of the range of the Lipschitz charts is bounded from above by a constant only depending on the doubling and Poincaré constants. For a partial converse to the previous theorem see [100] and references therein.

One of the remarkable contributions to the study of Lipschitz differentiability spaces was lately given by Bate in [44]. In that paper Bate proves that a metric measure space (X, d, μ) is a Lipschitz differentiable space if and only if it can be written as a countable union of Borel sets $X = \cup U_i$ such that each $\mu \llcorner U_i$ possesses a finite collection of (universal) Alberti representations.

The notion of Alberti representation originated in the seminal work [5], where Alberti proved the Rank-One property for the singular part of the derivative of vector-valued BV functions defined on an open subset of a Euclidean space conjectured by De Giorgi and Ambrosio [18]. Alberti proved that given an arbitrary Radon measure μ on a k -dimensional plane V in \mathbb{R}^n that is singular with respect to $\mathcal{H}^k \llcorner V$, one can associate to it a natural bundle $E(\mu, \cdot)$ whose fibers have dimension at most 1. The fiber $E(\mu, x)$ of this bundle consists of the vectors $v \in \mathbb{R}^k$ such that $v\mu$ is tangent in an appropriate sense to the derivative of a BV function on V . Moreover, the restriction of μ to the set where the dimension of $E(\mu, \cdot)$ is 1 can be written as $\int_I \mu_t dt$, where $\mu_t := \mathcal{H}^{k-1} \llcorner S_t$, and S_t is $(\mathcal{H}^{k-1}, k-1)$ -rectifiable in V .

In the language of [4], which collects several other finer results about the theory of rectifiability in \mathbb{R}^n , the previous result means that on the set where the fiber is one-dimensional, μ is $(k-1)$ -representable: namely, it can be written as the integral of measures that are $(k-1)$ -rectifiable. Another interesting contribution that originated from these ideas is the result by Alberti–Marchese in [6]. In that paper the authors associate to every Radon measure μ on \mathbb{R}^n (the unique minimal) bundle $V(\mu, \cdot)$ such that every real-valued Lipschitz function on \mathbb{R}^n is differentiable along $V(\mu, x)$ for μ -almost every $x \in \mathbb{R}^n$. In the language of [4], an Alberti representation in the sense used by Bate in [44] is roughly speaking a 1-representation of the measure.

In [47] Bate–Li exploited the result by Bate in [44] to characterize the (\mathcal{H}^n, n) -rectifiability of a metric space. More precisely, they show that a metric measure space (X, d, μ) is (\mathcal{H}^n, n) -rectifiable if and only if it can be μ -almost everywhere decomposed in the countable union of Borel sets U_i , with good n -density properties with respect to μ , such that, equivalently,

either (U_i, d, μ) is an n -Lipschitz differentiable space, or each $\mu \llcorner U_i$ has n independent Alberti representations with respect to some Lipschitz chart $\varphi_i : X \rightarrow \mathbb{R}^n$.

Moreover, very recently Bate in [46] gives a remarkable generalization of Marstrand–Mattila rectifiability criterion (cf. Proposition 0.1) for sets in arbitrary metric spaces. In the metric setting we have no homogeneous structure on the space so one has to use the pointed measured Gromov–Hausdorff convergence to give a meaning to item (ii) in Proposition 0.1. We stress that in this case the flat models are finite-dimensional Banach spaces. The proof of [46] uses a former result by Bate [45], in which he proves an appropriate metric analogue of Besicovitch–Federer Projection Theorem.

We stress that every generalization of Marstrand–Mattila criterion, or Preiss’s Theorem, is hardly won outside the Euclidean setting, since the classical proofs of these results heavily use the Euclidean structure.

One of the first Preiss-type results outside the Euclidean setting in this direction has been given by Lorent in [159] where he proved that locally 2-uniform measures in ℓ_∞^3 are rectifiable. Moreover, recently Merlo proved the intrinsic codimension-one Preiss’s Theorem in the Heisenberg groups \mathbb{H}^n endowed with the Korányi norm, see [180, 181]. In the setting of the Heisenberg groups \mathbb{H}^n , the notion of rectifiability for which the results in [180, 181] hold is not the one discussed above, but another one tailored for Carnot groups proposed by Franchi–Serapioni–Serra Cassano, and which we will extensively discuss below.

In a parallel direction, Miranda defined the notion of BV function on metric measure spaces, see [183]. Hence one can say that a Borel set E in a metric measure space (X, d, μ) is a set of finite perimeter if the characteristic function χ_E is a BV function in (X, d, μ) . Miranda [183] and Ambrosio [7, 8] started the study of fine properties of sets of finite perimeter in the general metric setting. Among other results, Ambrosio [8] proved that in a doubling metric measure space supporting a Poincaré inequality the perimeter measure $|D\chi_E|$ of a (locally) finite perimeter set E is asymptotically doubling, it is supported on (a precise subset of) the *essential boundary of E* , which is defined as the set of the points that have neither μ -density 0 nor 1 with respect to E , and it is absolutely continuous with respect to the codimension-one measure \mathcal{H}^h , which is constructed like the Hausdorff measure but using the Gauge $\zeta(B_r(x)) := \mu(B_r(x))/\text{diam}(B_r(x))$. As an interesting contribution to the topic, we stress that recently Lahti [142] generalized Federer’s characterization of sets of finite perimeter in the setting of doubling metric measure spaces supporting a Poincaré inequality. Namely, he proves that a set E is of finite perimeter if and only if (a precise subset of) the essential boundary has \mathcal{H}^h -finite measure.

One cannot hope for a general rectifiability result for the essential boundary of a finite perimeter set in arbitrary metric measure spaces. Nevertheless, when some additional structure is available on the space, one can hope to prove rectifiability results. In a series of three papers [10, 65, 66] Ambrosio–Bruè–Semola and Bruè–Pasqualetto–Semola succeeded in showing that in an $\text{RCD}(K, N)$ space of essential dimension n - which is roughly speaking a metric-measure generalization of a Riemannian manifold with geometric dimension n , with Ricci curvature bounded from below by K , and analytic dimension bounded above by N - the perimeter measure $|D\chi_E|$ of every set of finite perimeter E is concentrated on the reduced boundary, which is a subset of the essential boundary, and the reduced boundary is $(\mathcal{H}^{n-1}, n-1)$ -rectifiable.

Carnot groups

Due to the multitude of applications, sub-Riemannian geometry has attracted a lot of attention in the mathematical community in the recent years. Simplifying a bit, a sub-Riemannian manifold is a generalization of Riemannian manifold for which the metric is

induced by a smooth scalar product defined only on a sub-bundle of the tangent bundle. For a more general definition one can consult [2]. According to a fundamental result due to Mitchell [185], see also the contributions of Bellaïche [52], Jean [129], and Gromov [120], the infinitesimal model of a sub-Riemannian manifold, namely the class of its Gromov-Hausdorff tangents, is represented by the class of (quotients of) Carnot groups, that we now introduce.

For surveys on Carnot groups we refer the reader to [145, 208]. Carnot groups are connected and simply connected Lie groups \mathbb{G} whose Lie algebra \mathfrak{g} admits a stratification, namely a decomposition into non trivial complementary linear subspaces V_1, \dots, V_κ such that

$$(4) \quad \mathfrak{g} = V_1 \oplus \dots \oplus V_\kappa, \quad [V_j, V_1] = V_{j+1}, \quad \text{for } j = 1, \dots, \kappa - 1, \quad [V_\kappa, V_1] = \{0\},$$

where $[V_j, V_1]$ denotes the subspace of \mathfrak{g} generated by the commutators $[X, Y]$ with $X \in V_j$ and $Y \in V_1$. The number κ is called the step of the stratification. A distinguished subclass of Carnot groups is the one of Heisenberg groups $\{\mathbb{H}^n\}_{n \in \mathbb{N}}$: they are Carnot groups whose Lie algebras \mathfrak{h}^n can be endowed with a 2-step stratification as follows:

$$\mathfrak{h}^n = V_1 \oplus V_2,$$

where $V_1 := \text{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$, $V_2 := \text{span}\{Z\}$, and the only non trivial commutators are $[X_i, Y_i] = Z$ for every $i = 1, \dots, n$. For a survey on Heisenberg groups see [206]. Carnot groups have been studied from very different point of views such as Differential Geometry [69], they naturally emerge in Harmonic analysis and studying subelliptic Differential Equations [61, 106–108, 205], and they have been used for models in Neuroimaging [83].

One can endow a Carnot group with a geodesic left-invariant distance that admits dilations. A natural class of such distances is given by the so-called Carnot–Carathéodory distances. Roughly speaking every such a distance is constructed by taking the length distance associated to the length functional on curves that is the integral of a norm on V_1 (extended left-invariantly) of the derivative of the curve. The only paths admitted are the ones that tangentially follow the left-invariant bundle generated by V_1 . Even the fact that such a distance is finite is non trivial, it is a result due to Chow and Rashevskii, and it heavily relies on the fact that V_1 generates by brackets the Lie algebra. Such a bracket generation condition had already appeared in a very influential work by Hörmander in 1967 [126] as a sufficient condition to prove hypoellipticity of differential operators.

From the geometric viewpoint, Carnot groups represent a different world with respect to the Euclidean one since their Hausdorff dimension is strictly greater than their topological dimension, unless they are Abelian. Anyway, Carnot groups are natural objects to consider not only because they arise as infinitesimal models in sub-Riemannian geometry, but also because they naturally appear as asymptotic cones of connected nilpotent Lie groups, see Pansu’s work [199], they appear as boundaries at infinity of rank-one symmetric spaces [64], and their homogeneous structure allows to study harmonic analysis on them [106].

Carnot groups also play a prominent and natural role in Metric Geometry: Le Donne showed that Carnot groups (equipped with Carnot–Carathéodory distances) can be axiomatically characterized as the only metric spaces that are locally compact, geodesic, homogeneous with respect to at least one dilation, and isometrically homogeneous, see [144]. Moreover, if a doubling geodesic metric measure space has almost everywhere a unique pointed Gromov–Hausdorff tangent, such a tangent is almost everywhere isometric to a Carnot group, see [143].

Several additional subjects have been studied in the framework of sub-Riemannian manifolds and Carnot groups: e.g., the Bernstein problem [209], the isoperimetric problem [69, 157], the study of different notions of curvature [3, 42, 48], the relation between the

heat flow and the entropy [17], the study of complete stable surfaces [127], and the list is far from being complete. For further references we refer to the survey [208].

Carnot groups also appear in several landmark contributions in geometry and analysis in the last three decades. In [200] Pansu proved a Rademacher-type theorem for Lipschitz maps between Carnot groups, and he uses it to show that quasi-isometries of the quaternionic Heisenberg groups are at finite distance from isometries.

In [72, 73] Cheeger and Kleiner show that the first Heisenberg group with an arbitrary left-invariant homogeneous distance cannot be bi-Lipschitz embedded in L^1 . This result has both a theoretical and an applied interest. From the theoretical point of view, in [75] Cheeger and Kleiner, following the study started by Cheeger in [70], had showed that a Rademacher-type theorem for maps between PI spaces (i.e., doubling spaces that admit a Poincaré inequality) and Banach spaces with the Radon-Nikodým Property (RNP) holds. A Banach space has the RNP if every Lipschitz curve in it is almost everywhere differentiable. Such a property holds for L^p spaces with $p > 1$, and it does not for L^1 as the map $t \mapsto \chi_{[0,t]}$ shows. The result of [75] immediately gives non-embeddability theorems in Banach spaces with the RNP, from which L^1 is cut away. Thus, from one hand, [72, 73] complement the study started in [75]. For generalizations of the results in [72, 73] to arbitrary nilpotent Lie groups we refer the reader to the very recent [99]. It is worth noticing that the results in [73] have also motivated an interesting line of research whose aim is to study monotone sets and horizontally affine functions in arbitrary Carnot groups, see the contributions of [148, 189, 190].

From the applied point of view, the results in [72, 73] gave a counterexample to the so-called Goemans-Linial conjecture in computer science. The fact that \mathbb{H}^1 would have given a counterexample to such a conjecture was suggested by Lee and Naor [155]. Another example had previously been given by Khot and Vishnoi [137]. Goemans-Linial conjecture stated that every metric space (X, d) such that (X, \sqrt{d}) is isometric to a subset of a Hilbert space could be bi-Lipschitz embedded in L^1 . The positive solution to such a conjecture would have given the possibility of writing an algorithm that approximates within a constant factor and in polynomial time an NP-hard computable quantity. For recent advances on related topics, one can read Naor–Young remarkable papers [193, 194].

Rectifiability on Carnot groups. The study of Geometric Measure Theory and rectifiability in Carnot groups was pioneered by the works of Ambrosio–Kirchheim [14], and Franchi–Serapioni–Serra Cassano [111].

As anticipated above, in [14] the authors proved that the first Heisenberg group \mathbb{H}^1 is purely k -unrectifiable for $k = 2, 3, 4$. This means that, for every $k = 2, 3, 4$, every Lipschitz map $f : U \subseteq \mathbb{R}^k \rightarrow \mathbb{H}^1$ is such that $\mathcal{H}^k(f(U)) = 0$. This result was generalized by Magnani in [162] where he proved that a Carnot group is purely k -unrectifiable if and only if there do not exist sub-algebras of dimension k in V_1 . These negative results showed that the classical notion of rectifiability, where the models are Lipschitz images of subsets of \mathbb{R}^k , is not feasible in Carnot groups. In fact, for example, \mathbb{H}^1 has metric dimension 4 but it is purely 4-unrectifiable according to the latter notion of rectifiability, and one would avoid this for any reasonable notion of rectifiability tailored for Carnot groups.

In [111], in the setting of the Heisenberg groups \mathbb{H}^n , the authors propose a notion of codimension-one rectifiability where the models with which you cover a set are hypersurfaces that are locally defined as zero-level sets of functions that are continuously differentiable only along horizontal directions, and with non vanishing horizontal gradients. Such intrinsic hypersurfaces are called $C_{\mathbb{H}}^1$ -hypersurfaces. By using the latter notion of rectifiability, the authors proved the rectifiability of the reduced boundary of sets with locally finite perimeter in the Heisenberg groups \mathbb{H}^n . The positive result in [111] led a lot of authors in the last two

decades to study different notions of rectifiability in Carnot groups, modelled on different classes of sets.

In [113] the authors study the notion of $C_{\mathbb{H}}^1$ -hypersurface in arbitrary Carnot groups, and they prove an implicit function theorem for such hypersurfaces, see also [81] for a generalization in Carnot–Carathéodory spaces. Then, in [112] the authors generalize the result of [111] to all the Carnot groups of step 2. The proof closely follows De Giorgi’s scheme and it is essentially divided in two parts: first, one proves that all the blow-ups at almost every point of the boundary are monotone in one horizontal direction ν of the algebra while being invariant along ν^\perp ; second, one proves that such sets, called *constant horizontal normal sets*, are halfspaces in exponential coordinates. Then the conclusion is classical. The first step is true in every Carnot group, while the second is false. This does not mean that the rectifiability of the reduced boundary of sets of finite perimeter is false in some Carnot group. Indeed, as of today, the $C_{\mathbb{H}}^1$ -rectifiability problem for sets of finite perimeter in arbitrary Carnot groups is still open.

A lot of research has been done in order to push the previous strategy to give more information on the $C_{\mathbb{H}}^1$ -rectifiability problem for finite perimeter sets in arbitrary Carnot groups. In [15] the authors prove that in arbitrary Carnot groups at almost every point of the reduced boundary of a sets of finite perimeter one has *at least* one vertical halfspace as a blow-up. In [170] Marchi defines a special class of Carnot groups, the Carnot groups of type \star , which strictly generalizes the class of step-2 Carnot groups, in which constant horizontal normal sets are vertical halfspaces, and thus the $C_{\mathbb{H}}^1$ -rectifiability of the boundary of sets of finite perimeter holds. Recently, Le Donne–Moisala [147] strictly extends the class of Carnot groups of type \star , introducing the Carnot groups of type \diamond . In [147] they prove that constant horizontal normal sets in Carnot groups of type \diamond are vertical halfspaces and moreover they characterize all the step-3 Carnot groups for which constant horizontal normal sets are vertical halfspaces. It is worth mentioning that recently Don–Le Donne–Moisala–Vittone obtained a weak rectifiability result for the boundary of sets of finite perimeter in arbitrary Carnot groups [95]. Moreover, a fine study of constant horizontal normal sets in the Engel group and in the free Carnot group of rank 2 and step 3 is in [53, 54].

Going beyond the codimension one, the notion of $C_{\mathbb{H}}^1$ -submanifold has been studied in [115] in the setting of Heisenberg groups, then generalized by Magnani [168, 169] in arbitrary Carnot groups, and recently also studied by Julia–Nicolussi Golo–Vittone [132], where area and co-area formulae are proved within the class of rectifiable sets defined by using $C_{\mathbb{H}}^1$ -submanifolds. For an area formula for $C_{\mathbb{H}}^1$ -submanifolds in \mathbb{H}^n , we mention also the recent contribution by Corni–Magnani [85]. Moreover, area and coarea formulae for Euclidean regular submanifolds in Carnot groups have been studied, e.g., in [165, 167].

For the comparison between Euclidean regularity and intrinsic regularity of hypersurfaces in Carnot groups, an important contribution was given by Balogh [39]. He proved that the set of characteristic points of a C^1 Euclidean hypersurface in the Heisenberg groups \mathbb{H}^n is negligible with respect to the intrinsic codimension-one measure, and he constructed $C^{1,\alpha}$ Euclidean examples for which the characteristic set has positive Euclidean codimension-one measure. Such results were generalized by Magnani [168] in arbitrary codimensions and Carnot groups. We mention also the example by Kirchheim–Serra Cassano [139]: they prove the existence of a fractal $C_{\mathbb{H}}^1$ -hypersurface in \mathbb{H}^1 that has Euclidean Hausdorff dimension 2.5. For further comparisons between the Euclidean and the sub-Riemannian dimension of submanifolds in Carnot groups we refer the reader to [41].

Another notion of rectifiability in the setting of Carnot groups is the intrinsically Lipschitz rectifiability, modelled on the notion of intrinsically Lipschitz graph that has been first

proposed in [114] by Franchi–Serapioni–Serra Cassano, and then studied in [110] by Franchi–Serapioni. An intrinsically Lipschitz graph in a Carnot group \mathbb{G} is the graph of a function $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, where $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$ is a splitting of the group with homogeneous complementary subgroups, such that at every point the graph satisfies an intrinsic cone property. Some very deep recent advances on the study of the latter notion have been obtained by Naor–Young in [194], where the authors introduce the notion of foliated corona decomposition of an intrinsically Lipschitz graph in the first Heisenberg group \mathbb{H}^1 .

The relation between the notion of intrinsically Lipschitz rectifiability and the notion of $C_{\mathbb{H}}^1$ -rectifiability, i.e., the one modelled on $C_{\mathbb{H}}^1$ -submanifolds, has been initially investigated in [109, 116]. Such a study is tightly linked to a Rademacher-type theorem for intrinsically Lipschitz functions. Roughly speaking a function between homogeneous complementary subgroups in a Carnot group is said to be intrinsically differentiable at a point if the graph of such a function has a unique homogeneous complemented subgroup as a blow-up at that point, in the local Hausdorff topology. In [116] the authors prove a Rademacher-type theorem for intrinsically Lipschitz functions with one-dimensional target in the Heisenberg groups, i.e., they prove that such functions are intrinsically differentiable almost everywhere. In [109] the authors generalize this result to Carnot groups of type \star . What they prove in general is the following: in every Carnot group in which De Giorgi’s $C_{\mathbb{H}}^1$ -rectifiability theorem holds for boundaries of sets of finite perimeter, a Rademacher theorem for intrinsically Lipschitz functions with one-dimensional target holds. As a consequence, the intrinsically Lipschitz rectifiability and the $C_{\mathbb{H}}^1$ -rectifiability are the same notion, in codimension one, in groups of type \star and also in the more general framework of groups with semigenerated Lie algebras, among which one can find the groups of type \diamond , according to the recent study in [147]. It is nowadays an open problem to understand whether in codimension one $C_{\mathbb{H}}^1$ -rectifiability and intrinsically Lipschitz rectifiability are the same notion in arbitrary Carnot groups.

In arbitrary Carnot groups, the Rademacher theorem for intrinsically Lipschitz functions in higher codimensions is a challenging problem. When the target of an intrinsically Lipschitz function is normal, Rademacher theorem for intrinsically Lipschitz functions holds as an immediate consequence of Pansu’s theorem, see [29]. In higher codimensions in the Heisenberg groups, Rademacher theorem holds as proved in the recent remarkable work by Vittone [213], in which he also exploits the theory of currents. In general, Rademacher theorem is false, as the counterexample of [134] shows. It is worth to point out also the recent Rademacher theorem for Lipschitz functions defined on $C_{\mathbb{H}}^1$ -submanifolds with low codimension in the Heisenberg groups, see [133].

A study related to the one discussed above is that of the notion of uniformly intrinsically differentiable functions, which, roughly speaking, are the functions that parametrize $C_{\mathbb{H}}^1$ -submanifolds. This study was initiated by Ambrosio–Serra Cassano–Vittone in the Heisenberg groups in codimension one [16], and later pursued by other authors, see, e.g., [37, 93]. For more details we refer the reader to Section 0.3.5, and to Section 4 of Chapter 1.

The problem of linking the $C_{\mathbb{H}}^1$ -rectifiability and the intrinsically Lipschitz rectifiability with the *infinitesimal* notion of having flat blow-ups was raised for the first time in the work by Mattila–Serapioni–Serra Cassano [178] in the setting of Heisenberg groups \mathbb{H}^n . From the results in [178] one deduces that in \mathbb{H}^n the natural *infinitesimal* notion of rectifiable measure - namely the one given in terms of the existence of flat tangent measures almost everywhere - agrees with the one given in terms of intrinsically Lipschitz graphs in low dimensions, and with the one given in terms of $C_{\mathbb{H}}^1$ -submanifolds in low codimensions. Eventually, it took about ten years to conclude that a Rademacher theorem for intrinsically Lipschitz functions in low codimensions holds in \mathbb{H}^n , see the above discussed work by Vittone [213]. As a consequence, at least in \mathbb{H}^n , the natural infinitesimal definition of rectifiability always agrees with the one

given in terms of coverings with intrinsically Lipschitz graphs, or equivalently with intrinsic almost everywhere differentiable intrinsic graphs. An analysis similar to the one of [178] has been pursued by Idu–Magnani–Maiale in [128] in the setting of homogeneous groups and for measures with horizontal tangents.

Other notions of rectifiability modelled on Lipschitz images of (homogeneous subgroups of) Carnot groups have been proposed by Pauls and Cole–Pauls in [84, 201]. We refer the reader to Chapter 4 for more details about these definitions. An interesting open question asks whether in \mathbb{H}^1 the notion of rectifiability by means of $C_{\mathbb{H}}^1$ -hypersurfaces is equivalent to the one of Cole–Pauls given in [84]. In [60] Bigolin–Vittone prove that there is a $C_{\mathbb{H}}^1$ -hypersurface in \mathbb{H}^1 and a neighborhood of it that cannot be bi-Lipschitz parametrized with an open set of the vertical plane in \mathbb{H}^1 . In [92] Di Donato–Fässler–Orponen show that $C_{\mathbb{H}}^1$ -hypersurfaces with an Hölder regular intrinsic normal in \mathbb{H}^n can be almost everywhere covered by bi-Lipschitz images of subsets of the vertical plane. In \mathbb{H}^1 the same result can be strengthened to the class of intrinsically Lipschitz graphs with an extra Hölder regularity on the vertical coordinate.

The relationship between the above discussed notions of rectifiability and density properties have been recently investigated in the remarkable works by Merlo [180, 181]. In [181], Merlo proved that in \mathbb{H}^n endowed with the Korányi norm, if a Radon measure has positive and finite $(2n + 1)$ -density almost everywhere, then all the tangent measures are flat, i.e., they are Haar measures restricted to codimension-one homogeneous subgroups in \mathbb{H}^n . Hence, in [180] Merlo proved a Marstrand–Mattila rectifiability criterion in codimension one for arbitrary Carnot groups that coupled with the result in [181] gives Preiss’s $C_{\mathbb{H}}^1$ -rectifiability theorem in codimension one in all the Heisenberg groups endowed with the Korányi norm. This is a rather remarkable result in high dimensions because Preiss’s proof [202] very deeply relies on the Euclidean structure of \mathbb{R}^n .

Main contributions

The relationship between the notion of $C_{\mathbb{H}}^1$ -rectifiability, intrinsically Lipschitz rectifiability, and Pauls’s rectifiability has been poorly understood. Moreover, a study of the rectifiability of sets and measures privileging the infinitesimal point of view was only performed by Mattila–Serapioni–Serra Cassano in [178] in the setting of Heisenberg groups, and by Idu–Magnani–Maiale [128] for homogeneous groups (but only for horizontal rectifiable sets). An analogous study is missing in arbitrary Carnot groups. The aim of this thesis is to give contributions in better understanding such topics.

As discussed above while depicting the Euclidean theory of rectifiability, in the Euclidean setting the notion of rectifiable set, and more in general that of rectifiable measure, can be given in two equivalent ways. Either one could prescribe the *infinitesimal* behaviour of the measure by saying that it has *flat* tangent measures almost everywhere, i.e., Hausdorff measures on vector subspaces of dimension $k \in \mathbb{N}$; or, following a *global* approach, one could say that the measure is absolutely continuous with respect to the Hausdorff k -dimensional measure, and that it is supported on a countable union of k -dimensional Lipschitz graphs, compare with Proposition 0.1.

One of the big efforts in the study of the theory of rectifiability in Carnot groups, as discussed above, is trying to understand what is the correct class of *building blocks* to consider in order to give a satisfactory *global* definition of rectifiable set, or measure. As said above, several building blocks have been considered in the literature: $C_{\mathbb{H}}^1$ -submanifolds, intrinsically Lipschitz graphs, and (bi)-Lipschitz images of subsets of homogeneous groups.

Privileging the *infinitesimal* viewpoint, a notion that makes sense in arbitrary Carnot groups has been proposed in [180] by Merlo, namely the notion of \mathcal{P} -rectifiable measure,

which we soon introduce. We recall that a subgroup \mathbb{V} of \mathbb{G} is said to be *homogeneous* if it is closed under the action of the natural family of dilations $\{\delta_\lambda\}_{\lambda>0}$ on \mathbb{G} , see Section 2 of Chapter 1 for details. In the discussion below, every Carnot group \mathbb{G} shall be endowed with a left-invariant homogeneous (with respect to δ_λ) distance d .

We recall that, given the stratification in (4), the homogeneous dimension of \mathbb{G} is the number $\sum_{i=1}^k i \dim V_i$. The homogeneous dimension of a Carnot group, or of any homogeneous subgroup of it, is also the Hausdorff dimension with respect to every left-invariant homogeneous distance.

Definition 0.3 (*\mathcal{P} -rectifiable measures*). Let \mathbb{G} be a Carnot group of homogeneous dimension Q . Fix a natural number $1 \leq h \leq Q$. A Radon measure ϕ on \mathbb{G} is said to be *\mathcal{P}_h -rectifiable* (or *\mathcal{P} -rectifiable of dimension h*) if for ϕ -almost every $x \in \mathbb{G}$ we have

- (i) $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < +\infty$,
- (ii) $\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{H}^h \llcorner \mathbb{V}(x) : \lambda \geq 0\}$, where $\mathbb{V}(x)$ is a homogeneous subgroup of \mathbb{G} of homogeneous dimension h ,

where $\Theta_*^h(\phi, x)$ and $\Theta^{h,*}(\phi, x)$ are, respectively, the lower and the upper h -density of ϕ at x , i.e., the obvious analogues of (1.4), $\text{Tan}_h(\phi, x)$ is the set of h -tangent measures to ϕ at x , see Definition 1.52, and \mathcal{H}^h is the Hausdorff measure of dimension h .

As we shall notice, not only one could consider the study of \mathcal{P} -rectifiability reversed with respect to previous studies in the literature but it also has a twofold advantage. On the one hand the definition of \mathcal{P} -rectifiable measure is natural and intrinsic with respect to the (homogeneous) structure of Carnot groups and it is equivalent to the usual one in the Euclidean setting; on the other hand we do not have to handle the problem of distinguishing, in the definition, between the low-dimensional and the low-codimensional rectifiability.

In Chapter 2 we study structure results for \mathcal{P} -rectifiable measures in arbitrary Carnot groups. In particular we prove the following results in arbitrary Carnot groups:

- The support of every \mathcal{P}_h -rectifiable measure can be covered by sets with the cone property with arbitrarily small opening. Notice that such sets can be also taken such that they have Hausdorff tangents everywhere, compare with Proposition 2.26. The cones of the covering have an axis that is a homogeneous subgroup of homogeneous dimension h , see Theorem 2.1;
- The support of every \mathcal{P}_h -rectifiable measure with tangents that are complemented almost everywhere can be covered by sets that are simultaneously intrinsically Lipschitz graphs with arbitrarily small Lipschitz constant, and intrinsically differentiable graphs almost everywhere, see Theorem 2.25;
- Every \mathcal{P}_h -rectifiable measure with tangents that are complemented almost everywhere has h -density almost everywhere, see Theorem 2.12;
- The measures $\mathcal{H}^h \llcorner \Gamma$, for some $\Gamma \subseteq \mathbb{G}$ with $0 < \mathcal{H}^h(\Gamma) < +\infty$, are \mathcal{P}_h -rectifiable with tangents that are complemented almost everywhere if and only if either Preiss's tangent cone is supported on the same complemented homogeneous subgroup of homogeneous dimension h (which might depend on the point) almost everywhere; or Γ is \mathcal{H}^h -almost everywhere covered by graphs that are h -dimensional and intrinsically differentiable almost everywhere with complemented Hausdorff tangents, see Theorem 2.30.

Moreover, we prove that whenever $\mathcal{H}^h \llcorner \Gamma$, for some $\Gamma \subseteq \mathbb{G}$ with $0 < \mathcal{H}^h(\Gamma) < +\infty$, is \mathcal{P}_h -rectifiable with tangents that are complemented almost everywhere, then the density of the centered Hausdorff measure $\mathcal{C}^h \llcorner \Gamma$ is 1 for $\mathcal{H}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, see Theorem 2.30.

For a detailed discussion of the previous statements we refer the reader to the introductions of the sections of Chapter 2. All in all, from the previous results, we could conclude that, in Carnot groups, the correct *building blocks* to consider in order to give a *global* definition of rectifiability that agrees with the infinitesimal one seem to be intrinsically differentiable graphs. In a work in collaboration with Merlo [33], we also provide an area formula for such building blocks. We are not including such a contribution in this thesis, but for a survey we refer the reader to Section 0.3.4. We stress that, due to the existence of intrinsically Lipschitz graphs that are nowhere intrinsically differentiable, see [134], one cannot give a geometric area formula for arbitrary intrinsically Lipschitz graphs, and in general the result in Theorem 2.30 is false if one substitutes intrinsically differentiable with intrinsically Lipschitz.

In Chapter 3 we prove a Marstrand–Mattila rectifiability criterion for \mathcal{P} -rectifiable measures whose tangents admit at least one normal complementary subgroup, see Theorem 3.1. This result provides a generalization of the classical Marstrand–Mattila rectifiability criterion in Euclidean spaces, see Proposition 0.1, in the setting of Carnot groups. One nice consequence of such a result is the proof of the one-dimensional Preiss’s Theorem for measures in the first Heisenberg group \mathbb{H}^1 endowed with the Korányi norm, see Theorem 3.2. Moreover, joining Rademacher theorem for intrinsically Lipschitz functions with normal targets, the result in Theorem 2.30, and Marstrand–Mattila rectifiability criterion in Theorem 3.1, we derive a rather complete characterization of rectifiability for measures $\mathcal{H}^h \llcorner \Gamma$, for some $\Gamma \subseteq \mathbb{G}$ with $0 < \mathcal{H}^h(\Gamma) < +\infty$, in the co-normal case in arbitrary Carnot groups, see Corollary 3.3.

Finally, in Chapter 4 we provide an example of an analytic and non-characteristic hypersurface S in a Carnot group of homogeneous dimension 13 such that the image of every Lipschitz function from a subset of a Carnot group of homogeneous dimension 12 into S is \mathcal{H}^{12} -negligible, see Theorem 4.1. Such an example implies that the notion of Pauls’s rectifiability might not be equivalent to the notion of $C_{\mathbb{H}}^1$ -rectifiability in arbitrary Carnot groups. In the setting of the Heisenberg groups \mathbb{H}^n , with $n \geq 2$, we show that we cannot have such an example. In particular, we show that every C^∞ -hypersurface in \mathbb{H}^n , with $n \geq 2$, is almost everywhere covered by bi-Lipschitz images of subsets of codimension-one subgroups of \mathbb{H}^n that, being isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$, are Carnot groups, see Theorem 4.2.

Let us finish this section by taking a look for one moment at the big picture. On the one hand, the problem of understanding the link between all the different notions of rectifiability presented above in the utmost level of generality - e.g., when the tangents are possibly not complemented - in arbitrary Carnot groups seems out of reach as of today. On the other hand, the various definitions of \mathcal{P} -rectifiability are rather natural in the setting of Carnot groups since they rely on the idea that a good definition of rectifiability selects sets or measures with flat models. In this direction, the results in Chapter 2 and Chapter 3 are a systematic study of the notion of \mathcal{P} -rectifiability, trying to understand how to connect, in the most general scenario, the *infinitesimal approach* to rectifiability with the *global* one - i.e., the one given by covering with the correct class of *building blocks*. The result in Chapter 4 aims to highlight how, in arbitrary Carnot groups, the *global* definition of rectifiability is very sensitive to the choice of different *building blocks*, and requires special attention and studies.

Based on these results, one could speculate that the notion of \mathcal{P} -rectifiability gives the possibility to establish a robust theory of rectifiability of sets and measures in arbitrary Carnot groups; finally, in a different direction, and due to possible pathological behaviours, further studies of fine properties of intrinsic Lipschitz graphs - also in specific Carnot groups - and of the notion of Carnot rectifiability à la Pauls are still needed.

Other contributions

In this section I will list the other contributions I gave during my PhD studies.

0.1. Polynomial functions on Lie groups. In this subsection I shall discuss my contribution to the study of polynomial maps on Lie groups. The reference for the discussion below is my work with Le Donne [26].

Let \mathbb{G} be a Lie group with Lie algebra \mathfrak{g} , seen as left-invariant vector fields on \mathbb{G} . We fix a left-Haar measure μ on \mathbb{G} . Let $S \subseteq \mathfrak{g}$ be a subset that is Lie bracket generating, i.e., the only sub-algebra of \mathfrak{g} that contains S is \mathfrak{g} .

We say that a distribution f on \mathbb{G} is *S-polynomial* if for all $X \in S$ there exists $k \in \mathbb{N}$ such that the iterated derivative $X^k f$ is zero in the sense of distributions on \mathbb{G} . We say that a distribution f is *S-polynomial with degree at most k* if for every $X \in S$ we have that $X^k f$ is zero in the sense of distributions on \mathbb{G} . For basic definitions and properties of distributions on Lie groups we refer the reader to the account given in [26, Section 2] and references therein.

In [26, Theorem 1.1] we show that every *S-polynomial* distribution on \mathbb{G} , with a Lie generating S , is represented by an analytic function. Moreover, the vector space of *S-polynomial* distributions with degree at most $k \in \mathbb{N}$ on each connected component of \mathbb{G} is finite-dimensional. It is then natural to ask if an *S-polynomial* distribution on \mathbb{G} is a polynomial in some sense. A notion of *polynomial map* between arbitrary groups that showed to be versatile has been studied, with a special attention toward the nilpotent case, in [156]. In the case we deal with, i.e., the case of maps $f : \mathbb{G} \rightarrow \mathbb{R}$, Leibman's definition in [156] can be generalized for distributions. Let us define the operator D_g acting on distributions f on \mathbb{G} as follows

$$D_g f := f \circ R_g - f,$$

where R_g stands for the right translation by $g \in \mathbb{G}$ and $f \circ R_g$ is defined in the obvious way via duality. We say that a distribution f on \mathbb{G} is *polynomial à la Leibman with degree at most d* if

$$(5) \quad g_1, \dots, g_{d+1} \in \mathbb{G} \Rightarrow D_{g_1} \cdots D_{g_{d+1}} f \equiv 0, \quad \text{in the sense of distributions on } \mathbb{G}.$$

In [26, Item (1) of Theorem 1.2] we prove that the latter notion of being polynomial, which is “discrete” in spirit, is equivalent to the following “differential” definition. We say that a distribution f on \mathbb{G} is *polynomial (in the differential sense) with degree at most d* if

$$(6) \quad X_1, \dots, X_{d+1} \in \mathfrak{g} \Rightarrow X_1 \cdots X_{d+1} f \equiv 0, \quad \text{in the sense of distributions on } \mathbb{G}.$$

Moreover, in [26, Item (4) of Theorem 1.2] we show that every polynomial function on a connected Lie group passes to the maximal nilpotent Lie quotient, see the introduction of [26] for the terminology.

Hence polynomial maps always factor via a nilpotent group. This motivates us to focus the attention on polynomial maps on nilpotent Lie groups. When \mathbb{G} is a connected nilpotent Lie group $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is an analytic and surjective map, and thus one could also give another definition of “polynomial”, namely a map $f : \mathbb{G} \rightarrow \mathbb{R}$ is *polynomial in exponential chart* if $f \circ \exp : \mathfrak{g} \rightarrow \mathbb{R}$ is a polynomial. In case \mathbb{G} is a connected and nilpotent Lie group, we show that the property of being *S-polynomial* propagates to the entire Lie algebra. Namely, we prove that an *S-polynomial* distribution on \mathbb{G} is represented by a polynomial in exponential chart, and thus in particular it is \mathfrak{g} -polynomial of some degree $k \in \mathbb{N}$, see [26, Remark 4.10]. Thus, one of our results in the setting of connected nilpotent Lie groups is the following.

Theorem 0.4. *Let \mathbb{G} be a connected nilpotent Lie group, let f be a distribution on \mathbb{G} , and let S be a Lie generating subset of \mathfrak{g} . If f is S -polynomial, then it is represented by a function that is polynomial in exponential chart.*

We stress that if \mathbb{G} is not nilpotent, being S -polynomial for a Lie generating S may not imply being \mathfrak{g} -polynomial, or even being polynomial in exponential chart, see [26, Appendix A] for such a counterexample in the group of affine orientation-preserving maps $\text{Aff}(\mathbb{R})^+$. For non-nilpotent groups we do not know either if being \mathfrak{g} -polynomial implies being polynomial, or even if being \mathfrak{g} -polynomial passes to the maximal nilpotent Lie quotient.

With the previous result we can prove that on a connected nilpotent Lie group \mathbb{G} the different notions of being polynomial that we discussed above are equivalent and in particular they are equivalent to being S -polynomial for any Lie generating S .

Theorem 0.5. *Let \mathbb{G} be a connected nilpotent Lie group, and let f be a distribution on \mathbb{G} . Then the following are equivalent*

- (1) *f is an S -polynomial distribution for some Lie generating $S \subseteq \mathfrak{g}$,*
- (2) *f is an S -polynomial distribution for all $S \subseteq \mathfrak{g}$,*
- (3) *f is represented by a function that is polynomial in exponential chart,*
- (4) *f is a polynomial distribution (in the differential sense), see (6).*
- (5) *f is a polynomial distribution à la Leibman, see (5).*

0.2. Isoperimetric problem on spaces with curvature bounded from below.

In this subsection I shall discuss my contributions to the study of the isoperimetric problem on spaces with curvature bounded from below. The references for the discussion below are my works [20, 24, 34–36]. These results are the outcome of several collaborations of myself together with Bruè, Fogagnolo, Nardulli, Pasqualetto, Pozzetta, and Semola.

The *isoperimetric problem* can be formulated on every ambient space possessing notions of *volume measure* \mathfrak{m} and *perimeter* Per on (some subclass of) its subsets. Among sets having assigned positive volume, the problem deals with finding those having least perimeter. Among the most basic questions in the context of the isoperimetric problem, one would naturally ask whether there exist minimizers, called *isoperimetric regions* (or *isoperimetric sets*), but also what goes wrong in the minimization process in case such minimizers do not exist. The value of the infimum of the perimeter among sets of a given volume V is called *isoperimetric profile* at V , and denoted by $I(V)$.

A natural class of spaces where to set the isoperimetric problem is given by metric measure spaces $(X, \mathfrak{d}, \mathfrak{m})$. Indeed, the nonnegative Radon measure \mathfrak{m} plays the role of a volume functional, and, together with a distance \mathfrak{d} , it is possible to give a definition of perimeter Per , see [183]. The smooth and more classical counterpart of these spaces is given by Riemannian manifolds. If (M, g) is a Riemannian manifold of dimension N , the natural Riemannian distance and the N -dimensional Hausdorff measure \mathcal{H}^N yield the structure of metric measure space, and the corresponding definition of perimeter recovers the classical well-known notion à la Caccioppoli–De Giorgi. In fact, the theory of BV functions and of the perimeter functional on metric measure spaces has been blossoming in the last decades [7, 8, 11, 183, 184].

The most natural way to approach the existence problem is to argue *by direct method*, that is, by studying the behaviour of a minimizing sequence of sets E_i of fixed volume V whose perimeter is converging to the isoperimetric profile at volume V . It is therefore understood that, by usual precompactness and lower semicontinuity, the problem of existence is non trivial only in case the ambient is *noncompact* (actually, with infinite measure). Already in the smooth ambient, the development of an effective theory of a direct method for the isoperimetric problem is a difficult task. Studying the problem in Euclidean solid cones, in

[204] the authors identified a general mass splitting phenomenon of a minimizing sequence for the problem, where the sequence decomposes into two components, one converging in the space and the other diverging at infinity. Combining this approach with a concentration-compactness argument, in [195] the author performed a better description of the possible mass lost at infinity for the problem on Riemannian manifolds satisfying some asymptotic hypotheses on their ends. The theory has then been successfully applied to get existence theorems in [187], further generalized in [192].

As the examples in [24] point out, already in smooth Riemannian manifolds, the isoperimetric problem becomes trivial, i.e., the isoperimetric profile vanishes, unless it is assumed a lower bound on the Ricci curvature and a positive lower bound on the volume of unit balls. In fact, as one of the two hypotheses is not satisfied, one can find examples where a description of the behaviour of minimizing sequences is actually compromised, see [24]. Therefore, it becomes natural to consider the isoperimetric problem on $\text{RCD}(K, N)$ metric measure spaces, which are spaces encoding synthetic notions of Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in (0, +\infty]$. We are not going to give an account on the huge development of the RCD theory in the last years, and we refer the reader to the survey by Ambrosio in [9] and [207].

Moreover, we shall address only the case of $\text{RCD}(K, N)$ spaces of the form (X, d, \mathcal{H}^N) , i.e., endowed with the N -dimensional Hausdorff measure. We will call such spaces N -dimensional $\text{RCD}(K, N)$ spaces. The case of arbitrary volume measures \mathfrak{m} appears to be more involved and related to a better understanding of the properties of the density of \mathfrak{m} with respect to the Hausdorff measure of the essential dimension of the space. We stress that the class of N -dimensional $\text{RCD}(K, N)$ spaces, that has been recently introduced and studied in the works [21, 67, 90, 140], is the non-smooth generalization of the class of non collapsed Ricci limit spaces [71].

It is remarkable to notice that the development of a theory on such nonsmooth spaces already is a *necessary consequence* also of the approach by direct method of the isoperimetric problem on perfectly smooth Riemannian manifolds, see the introduction of my work with Fogagnolo and Pozzetta [24]. Indeed, nonsmooth $\text{RCD}(K, N)$ spaces (X, d, \mathcal{H}^N) arise as limits in the pointed Gromov–Hausdorff sense of smooth manifolds M with Ricci and volume of unit balls bounded below along sequences of points diverging on M .

Capitalizing on the methods developed in [24, 35, 195], we are able to give a description of the behaviour of perimeter minimizing sequences for the isoperimetric problem on RCD spaces as follows. The following result is in my work with Nardulli and Pozzetta in [34]. It generalizes a previous result of myself with Fogagnolo and Pozzetta [24], by using as a tool the results I obtained with Pasqualetto and Pozzetta [35]. For the notion of pmGH convergence and L^1 -strong convergence, as well as for the notation, we refer the reader to [34, Section 2] and references therein.

Theorem 0.6 (Asymptotic mass decomposition). *Let $K \leq 0$ and $N \geq 2$. Let (X, d, \mathcal{H}^N) be a noncompact $\text{RCD}(K, N)$ space. Assume there exists $v_0 > 0$ such that $\mathcal{H}^N(B_1(x)) \geq v_0$ for every $x \in X$. Let $V > 0$. For every minimizing (for the perimeter) sequence of bounded sets $\Omega_i \subseteq X$ of volume V , up to passing to a subsequence, there exist a nondecreasing bounded sequence $\{N_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$, disjoint finite perimeter sets $\Omega_i^c, \Omega_{i,j}^d \subseteq \Omega_i$, and points $p_{i,j}$, with $1 \leq j \leq N_i$ for any i , such that the following claims hold*

- $\lim_i d(p_{i,j}, p_{i,\ell}) = \lim_i d(p_{i,j}, o) = +\infty$, for any $j \neq \ell \leq \overline{N}$ and any $o \in X$, where $\overline{N} := \lim_i N_i < +\infty$;
- Ω_i^c converges to $\Omega \subseteq X$ in the sense of finite perimeter sets, $\mathcal{H}^N(\Omega_i^c) \rightarrow_i \mathcal{H}^N(\Omega)$, and $\text{Per}(\Omega_i^c) \rightarrow_i \text{Per}(\Omega)$. Moreover Ω is an isoperimetric region in X ;

- for every $0 < j \leq \bar{N}$, $(X, \mathbf{d}, \mathcal{H}^N, p_{i,j})$ converges in the pmGH sense to a pointed RCD(K, N) space $(X_j, \mathbf{d}_j, \mathcal{H}^N, p_j)$. Moreover there are isoperimetric regions $Z_j \subseteq X_j$ such that $\Omega_{i,j}^d \rightarrow_i Z_j$ in L^1 -strong and $\text{Per}(\Omega_{i,j}^d) \rightarrow_i \text{Per}(Z_j)$;
- it holds that

$$(7) \quad I_X(V) = \text{Per}(\Omega) + \sum_{j=1}^{\bar{N}} \text{Per}(Z_j), \quad V = \mathcal{H}^N(\Omega) + \sum_{j=1}^{\bar{N}} \mathcal{H}^N(Z_j).$$

The previous Theorem 0.6 states a general behaviour for minimizing sequences of the isoperimetric problem. Roughly speaking, the mass of a sequence splits into at most finitely many pieces and it is totally recovered by finitely many isoperimetric regions sitting in spaces possibly “located at infinity” with respect the original ambient space. Notice that Theorem 0.6 is not an existence theorem, nor it is a nonexistence result, instead it is a general tool for treating the problem by direct method. With such theorem it is then possible to recover the main existence and nonexistence results previously proved in [20, 24, 187], and, actually, to suitably extend those to the nonsmooth RCD setting.

We exploited the previous asymptotic mass decomposition result in Theorem 0.6 to prove genuinely new - even on noncompact smooth Riemannian manifold - existence results for the isoperimetric problem, and new sharp differential inequalities for the isoperimetric profile. We refer the reader to the introductions of [20, 36] for a detailed account on the results - and their consequences - that we are going to discuss below.

We stress that almost all of the results obtained in the paper [36] I wrote together with Pasqualetto, Pozzetta, and Semola, are new even for smooth, non compact manifolds with lower Ricci curvature bounds and for Alexandrov spaces with lower sectional curvature bounds. They answer several open questions in [38, 50, 154, 182, 196].

On simply connected model spaces with constant sectional curvature $K/(N-1) \in \mathbb{R}$ and dimension $N \geq 2$ the isoperimetric profile $I_{K,N}$ solves the following second order differential equation on its domain

$$(8) \quad -I''_{K,N} I_{K,N} = K + \frac{(I'_{K,N})^2}{N-1}.$$

Equivalently, setting $\psi_{K,N} := I_{K,N}^{\frac{N}{N-1}}$, we have

$$(9) \quad -\psi''_{K,N} = \frac{KN}{N-1} \psi_{K,N}^{\frac{2-N}{N}}.$$

Combining the existence of isoperimetric regions for any volume, the regularity theory in Geometric Measure Theory, and the second variation formula

$$(10) \quad \frac{d^2}{dt^2} \Big|_{t=0} \text{Per}(E_t) = \int_{\partial E} \left(H^2 - \|\text{II}\|^2 - \text{Ric}(\nu, \nu) \right) d\text{Per},$$

where $t \mapsto E_t$ denotes the parallel deformation of E via equidistant sets, ν and II denote a choice of the unit normal to ∂E and its second fundamental form, respectively, and $\text{Ric}(\nu, \nu)$ indicates the Ricci curvature of M in the direction of ν ; in [49–51, 191, 197] it was proved that the isoperimetric profile of a smooth, compact, N -dimensional Riemannian manifold with $\text{Ric} \geq K$ verifies the inequality \geq in (8) and (9) in a weak sense.

In [36] we obtain the following far reaching extension to the setting of RCD(K, N) metric measure spaces $(X, \mathbf{d}, \mathcal{H}^N)$ with a uniform lower bound on the volume of unit balls, without any assumption on the existence of isoperimetric regions. We stress again that the classical argument to show Theorem 0.7 in the compact setting uses in a crucial way the existence of isoperimetric regions for every volume, that we do not have at disposal in the present setting.

Indeed, for the proof of Theorem 0.7, one key tool will be the previously discussed asymptotic mass decomposition theorem in Theorem 0.6.

Theorem 0.7. *Let $K \in \mathbb{R}$, and $N \geq 2$. Let $(X, \mathbf{d}, \mathcal{H}^N)$ be an $\text{RCD}(K, N)$ space. Assume that there exists $v_0 > 0$ such that $\mathcal{H}^N(B_1(x)) \geq v_0$ for every $x \in X$.*

Let $I : (0, \mathcal{H}^N(X)) \rightarrow (0, \infty)$ be the isoperimetric profile of X . Then the following holds.

(1) *the inequality*

$$-I''I \geq K + \frac{(I')^2}{N-1} \quad \text{holds in the viscosity sense on } (0, \mathcal{H}^N(X)),$$

(2) *if $\psi := I^{\frac{N}{N-1}}$ then*

$$-\psi'' \geq \frac{KN}{N-1} \psi^{\frac{2-N}{N}} \quad \text{holds in the viscosity sense on } (0, \mathcal{H}^N(X)).$$

In particular, the above holds for non compact smooth Riemannian manifolds with Ricci curvature bounded from below and volume of unit balls uniformly bounded away from zero, without further restrictions on their geometry at infinity. The proof combines the generalized existence of isoperimetric regions (cf. Theorem 0.6), the interpretation of the differential inequalities in the viscosity sense and the forthcoming Laplacian comparison Theorem 0.8 to estimate first and second variation of the area via equidistant sets in the non smooth setting, in an original way.

The other key tool that we develop to prove Theorem 0.7 is a sharp bound on the Laplacian of the signed distance function from isoperimetric regions inside $\text{RCD}(K, N)$ metric measure spaces $(X, \mathbf{d}, \mathcal{H}^N)$. It is the isoperimetric analogue of the result in [188] for perimeter minimizers.

For every $k, \lambda \in \mathbb{R}$, let us introduce the comparison functions

$$(11) \quad s_{k,\lambda}(r) := \cos_k(r) - \lambda \sin_k(r),$$

where

$$(12) \quad \cos_k'' + k \cos_k = 0, \quad \cos_k(0) = 1, \quad \cos_k'(0) = 0,$$

and

$$(13) \quad \sin_k'' + k \sin_k = 0, \quad \sin_k(0) = 0, \quad \sin_k'(0) = 1.$$

Theorem 0.8. *Let $(X, \mathbf{d}, \mathcal{H}^N)$ be an $\text{RCD}(K, N)$ metric measure space for some $K \in \mathbb{R}$ and $N \geq 2$, and let $E \subseteq X$ be an isoperimetric region. Then, denoting by f the signed distance function from \overline{E} , there exists $c \in \mathbb{R}$ such that*

$$(14) \quad \Delta f \geq -(N-1) \frac{s'_{\frac{K}{N-1}, \frac{c}{N-1}} \circ (-f)}{s_{\frac{K}{N-1}, \frac{c}{N-1}} \circ (-f)} \quad \text{on } E, \quad \text{and} \quad \Delta f \leq (N-1) \frac{s'_{\frac{K}{N-1}, -\frac{c}{N-1}} \circ f}{s_{\frac{K}{N-1}, -\frac{c}{N-1}} \circ f} \quad \text{on } X \setminus \overline{E}.$$

The bounds in (14) are understood in the sense of distributions, and we always consider open representatives for isoperimetric regions, which is possible due to one of the main results of my work with Pasqualetto and Pozzetta [35]. The bounds above are sharp, since equalities are attained in the model spaces with constant sectional curvature.

Notice that the distance function might not be globally smooth even when (X, \mathbf{d}) is isometric to a smooth Riemannian manifold and $E \subseteq X$ has smooth boundary, in which case (14) is equivalent to the requirement that ∂E has constant mean curvature equal to c . Hence we decided to call any $c \in \mathbb{R}$ such that (14) holds a *mean curvature barrier* for E .

In [36] we derive several consequences of Theorem 0.7 and Theorem 0.8 among which:

- a sharp and rigid isoperimetric inequality for $\text{RCD}(0, N)$ spaces (X, d, \mathcal{H}^N) with Euclidean volume growth, see [36, Theorem 1.3]. The rigidity part improves the previous results obtained in [1, 38, 63, 105];
- uniform lower bounds, semi-concavity and Lipschitz properties of the isoperimetric profile in a fixed range of volumes, only depending on the lower Ricci curvature bound, the dimension and a lower bound on the volume of unit balls;
- the strict subadditivity of the isoperimetric profile for small volumes (only depending on K , N and the uniform lower bound on the volume of unit balls). This implies in turn that isoperimetric regions with small volume are connected. Moreover, in the asymptotic mass decomposition, minimizing sequences for small volumes do not split: either they converge to an isoperimetric region, or they drift off to exactly one isoperimetric region in a pointed limit at infinity. All the previous conclusions hold for every volume when $K = 0$;
- uniform, scale invariant diameter estimates for isoperimetric regions of small volume, without further assumptions, and for any volume when $K = 0$ and (X, d, \mathcal{H}^N) has Euclidean volume growth. This answers a question in [196];
- uniform density estimates and uniform almost minimality properties for isoperimetric sets. They allow to bootstrap L^1 -convergence to Gromov–Hausdorff convergence and convergence of the perimeters for sequences of isoperimetric sets, and to prove the stability of mean curvature barriers, compare with [136];
- the exact asymptotic behaviour of the isoperimetric profile for small volumes and, when $K = 0$, for large volumes, see [36, Theorem 1.4 and Theorem 1.5]. This extends some results in [158] that hold for unbounded convex bodies to the much more general setting of $\text{RCD}(K, N)$ spaces (X, d, \mathcal{H}^N) .

We also stress that the application of Theorem 0.6 gives raise to new existence results in spaces with curvature bounded from below. For the following statement in the smooth case, we refer the reader to [20, Theorem 1.3] while the exact result stated below is in [36, Item (1) of Theorem 1.4]. For some comments on this statement we refer the reader to the introduction of my work with Bruè, Fogagnolo, and Pozzetta [20], where we developed all the machineries to prove Theorem 0.9.

Theorem 0.9. *Let (X, d) be an Alexandrov space of dimension $N \in \mathbb{N}$ with non negative curvature. Suppose that (X, d, \mathcal{H}^N) satisfies*

$$\lim_{r \rightarrow +\infty} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} =: \text{AVR}(X, d, \mathcal{H}^N) > 0,$$

where $x \in X$, and ω_N is the volume of the unit ball in \mathbb{R}^N . Then there exists $V_0 \geq 0$ such that for every $V \geq V_0$ there exists an isoperimetric region of volume V .

0.3. Other results. In this final subsection I will sketch the statements of other results I obtained during my studies.

0.3.1. *Carnot–Carathéodory structures and their limits.* In this subsection I will state the main result obtained in [27], together with Le Donne, and Nicolussi Golo, about the convergence of distances associated to converging Carnot–Carathéodory structures.

In [27] we dealt with the following general problem. Let M be a smooth manifold endowed with a family of vector fields and a continuously varying norm on the tangent spaces. Let us consider the length distance associated to the trajectories that infinitesimally follow such a family. What are the weakest notion of convergence and the most general assumptions on the

family of vector fields and the norm that ensure the uniform convergence of the associated length distances?

Such a question is natural while studying metric geometry. For example, a better understanding of such a question gives an effective way of approximating sub-Finsler distances with Finsler distances, compare with [152, 153]. Moreover, a remarkable situation in which the convergence of the distances associated to converging sub-Finsler structures emerges is while studying asymptotic or tangent cones of sub-Finsler structures, see the celebrated works of Mitchell and Bellaïche [52, 185] (and the account in [129]), and the work of Pansu [199].

Let us fix from now on a finite-dimensional real Banach space \mathbb{E} , and a smooth manifold M . A *Lipschitz-vector-field structure* $\mathbf{f} : M \rightarrow \mathbb{E}^* \otimes TM$ on M modelled by \mathbb{E} is a Lipschitz choice, for every point $p \in M$, of a linear map between \mathbb{E} and $T_p M$. We say that a sequence of Lipschitz-vector-field structures $\{\mathbf{f}_n\}_{n \in \mathbb{N}}$ converges to a Lipschitz-vector-field structure \mathbf{f}_∞ if, on every compact subset of M , $\{\mathbf{f}_n\}_{n \in \mathbb{N}}$ is an equi-Lipschitz family that converges to \mathbf{f}_∞ uniformly.

Let \mathbf{f} be a Lipschitz-vector-field structure on M modelled by \mathbb{E} . We say that $N : M \times \mathbb{E} \rightarrow \mathbb{R}$ is a *continuously varying norm* on $M \times \mathbb{E}$ if N is continuous, and $N(p, \cdot)$ is a norm on \mathbb{E} for every $p \in M$. Attached to a couple (\mathbf{f}, N) there is a natural notion of energy and length associated to every $u \in L^\infty([0, 1]; \mathbb{E})$, which we will sometimes call *control*, see [27]. Taking the infimum of the energy (or equivalently of the length) of all the controls associated to the curves that connect two points, one defines the *Carnot–Carathéodory distance* $d_{(\mathbf{f}, N)}$ associated to (\mathbf{f}, N) , see [27, Definition 1.2] for details.

In [27] we aim at understanding which kind of convergence is expected from the sequence of distances $\{d_{(\mathbf{f}_n, N_n)}\}_{n \in \mathbb{N}}$ when we have that the sequence $\{(\mathbf{f}_n, N_n)\}_{n \in \mathbb{N}}$ converges. The key hypothesis in order to have the local uniform convergence of the distances is a kind of essential non-holonomicity of the limit vector-field structure \mathbf{f}_∞ . We refer to [27, Definition 1.3] for the precise notion of essential non-holonomicity, and we just sketch its definition here. First of all we introduce the notion of essentially open map. We say that a continuous map $f : M \rightarrow N$ between two topological manifolds of the same dimension k is *essentially open at* $p \in M$ *at scale* U if U is a neighborhood of p homeomorphic to the k -dimensional Euclidean ball, with ∂U homeomorphic to the sphere \mathbb{S}^{k-1} , and there exists V a neighborhood of $f(p)$ homeomorphic to the k -dimensional Euclidean ball, such that $f(\partial U) \subseteq V \setminus f(p)$ and the map $f : \partial U \rightarrow V \setminus f(p)$ induces a nonconstant map between the $(k-1)$ -homology groups. Then, a set \mathcal{F} of Lipschitz vector fields on a smooth manifold M of dimension m is *essentially non-holonomic* at a point $p \in M$ whenever there exists a sequence of points $p_n \in M$ that converges to p such that p_n is connected to p with the concatenation, starting at p , of line flows of m vector fields in \mathcal{F} for times (t_1, \dots, t_m) , and moreover such concatenation is essentially open around (t_1, \dots, t_m) . We stress that one can prove that the latter notion is weaker than the bracket-generating condition in the case the vector fields are smooth.

We are now ready to give the main theorem of our paper [27], see [27, Theorem 1.4]. For some applications, i.e., a sub-Finsler Mitchell’s Theorem, and a convergence result for distances on Lie groups, we refer the reader to [27, Theorem 1.5], and to [27, Theorem 1.6], respectively.

Theorem 0.10. *Let M be a smooth manifold, and let \mathbb{E} be a finite-dimensional real Banach space. Let $\hat{\mathbf{f}}$ be an essentially non-holonomic Lipschitz-vector-field structure modelled by \mathbb{E} , and let $\hat{N} : M \times \mathbb{E} \rightarrow [0, +\infty)$ be a continuously varying norm. Then the following hold.*

- (1) *if M is connected, then $d_{(\hat{\mathbf{f}}, \hat{N})}(p, q) < \infty$ for every $p, q \in M$;*
- (2) *$d_{(\hat{\mathbf{f}}, \hat{N})}$ induces the manifold topology on M ;*

- (3) Let $\{\mathbf{f}_n\}_{n \in \mathbb{N}}$ be a sequence of Lipschitz-vector-field structures on M modelled by \mathbb{E} , and let $\{N_n\}_{n \in \mathbb{N}}$ be a sequence of continuously varying norms on $M \times \mathbb{E}$. Let us assume that $\mathbf{f}_n \rightarrow \hat{\mathbf{f}}$ in the sense of Lipschitz-vector-field structures described above, and $N_n \rightarrow \hat{N}$ uniformly on compact subsets of $M \times \mathbb{E}$.

Then $d_{(\mathbf{f}_n, N_n)} \rightarrow d_{(\hat{\mathbf{f}}, \hat{N})}$ locally uniformly on M , i.e., every $o \in M$ has a neighborhood U such that $d_{(\mathbf{f}_n, N_n)} \rightarrow d_{(\hat{\mathbf{f}}, \hat{N})}$ uniformly on $U \times U$ as $n \rightarrow +\infty$.

- (4) If in the hypotheses of item (3) we additionally have that $d_{(\hat{\mathbf{f}}, \hat{N})}$ is a boundedly compact (or equivalently complete) distance, we conclude that

$$\lim_{n \rightarrow +\infty} d_{(\mathbf{f}_n, N_n)} = d_{(\hat{\mathbf{f}}, \hat{N})},$$

uniformly on compact subsets of $M \times M$. Moreover, for every $x \in M$, we have $(M, d_{(\mathbf{f}_n, N_n)}, x) \rightarrow (M, d_{(\hat{\mathbf{f}}, \hat{N})}, x)$ in the pointed Gromov–Hausdorff topology as $n \rightarrow +\infty$.

0.3.2. Unextendable intrinsically Lipschitz graphs. In this subsection I will state the main result obtained in [30] together with Merlo about the existence, in some Carnot groups, of positive-measured intrinsically Lipschitz curves that cannot be extended to entire intrinsically Lipschitz curves.

Questions about Lipschitz Extension Properties, LEP from now on, that are classical in Geometric Measure Theory, can be asked also for intrinsically Lipschitz graphs in Carnot groups. For example, is it true that every intrinsically Lipschitz map $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{V}$, where \mathbb{W}, \mathbb{V} are complementary subgroups of a Carnot group, can be extended to an entire intrinsically Lipschitz map $\tilde{\varphi} : \mathbb{W} \rightarrow \mathbb{V}$? The answer is positive when the subgroup \mathbb{V} is horizontal, i.e., contained in the first layer of the stratification of the Carnot group, cf. [213, Theorem 1.5], and [116, Theorem 4.25].

Hence, we have the validity of the LEP for low-codimensional intrinsically Lipschitz graphs in arbitrary Carnot groups. Moreover, in the recent [94], the authors prove that every $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{V}$, where \mathbb{W}, \mathbb{V} are complementary subgroups of the n -th Heisenberg group \mathbb{H}^n , and \mathbb{W} is horizontal, can be extended to an entire intrinsically Lipschitz map $\tilde{\varphi} : \mathbb{W} \rightarrow \mathbb{V}$, cf. [94, Theorem 1.2].

In [30] we show that the previous example is special. Namely, we provide a negative answer to the validity of the LEP for intrinsically Lipschitz maps defined on subsets of horizontal subgroups of a Carnot group. It is the first example in which the LEP of intrinsically Lipschitz graphs is known to fail on Carnot groups. We recall that with $\mathbb{F}_{2,3}$ we denote the free Carnot group of rank 2 and step 3, and with V_1 we denote its horizontal layer. Up to a choice of an adapted basis $\mathcal{B} := (X_1, X_2, X_3, X_4, X_5)$ of the Lie algebra, we identify $\mathbb{F}_{2,3}$ with \mathbb{R}^5 through the exponential map. We endow the Lie algebra of $\mathbb{F}_{2,3}$ with an auxiliary inner product that makes \mathcal{B} an orthonormal basis, and we fix an arbitrary left-invariant homogeneous distance on $\mathbb{F}_{2,3}$. The Hausdorff measures on $\mathbb{F}_{2,3}$ are computed with respect to such a distance. Finally, for every $e \in V_1$ we denote $\mathfrak{N}(e) := \{\exp(te) : t \in \mathbb{R}\}$, and $\mathbb{V}(e) := \exp(e^\perp)$. Hence the main result of our paper [30] reads as follows.

Theorem 0.11. *Let $\mathbb{F}_{2,3}$ be the free Carnot group of rank 2 and step 3, and let V_1 be the first layer of a stratification of its Lie algebra. For any $e \in V_1$ there exists a compact set $K \subseteq \mathfrak{N}(e)$, and an intrinsically Lipschitz function $\varphi : K \rightarrow \mathbb{V}(e)$ such that the following two conditions hold.*

- (i) $\mathcal{H}^1(\text{graph}(\varphi)) > 0$, where $\text{graph}(\varphi) := \{a \cdot \varphi(a) : a \in K\}$,
- (ii) for any intrinsically Lipschitz map $\tilde{\varphi} : \Omega \rightarrow \mathbb{V}(e)$, where Ω is an open subset of $\mathfrak{N}(e)$, we have

$$\mathcal{H}^1(\text{graph}(\varphi) \cap \text{graph}(\tilde{\varphi})) = 0.$$

As a consequence, there exists no intrinsically Lipschitz map $\psi : \mathfrak{N}(e) \rightarrow \mathbb{V}(e)$ such that $\psi|_K = \varphi$.

0.3.3. *Volume bounds for the quantitative singular strata in noncollapsed RCD spaces.* In this subsection I will state the main result obtained in [21] together with Bruè and Semola about volume bounds for the quantitative singular strata in noncollapsed RCD spaces.

Building upon [90], it is possible to prove that any tangent cone to an N -dimensional $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathcal{H}^N)$ is a metric cone. Letting then $\mathcal{R} \subseteq X$ be the set of those points where the tangent cone is the N -dimensional Euclidean space, following [71] it is possible to introduce a stratification

$$\mathcal{S}^0 \subseteq \dots \subseteq \mathcal{S}^{N-1} = \mathcal{S} = X \setminus \mathcal{R},$$

of the singular set \mathcal{S} , where, for any $k = 0, \dots, N-1$, \mathcal{S}^k is the set of those points where no tangent cone splits a factor \mathbb{R}^{k+1} . Adapting the arguments of [71], in [90] the Hausdorff dimension estimate $\dim_H \mathcal{S}^k \leq k$ was obtained.

In [74] a quantitative and effective counterpart of the above mentioned stratification of the singular set was introduced letting, for any $k = 0, \dots, N-1$ and for any $r, \eta > 0$, $\mathcal{S}_{\eta, r}^k$ be the set of those points $x \in X$ where the scale invariant Gromov-Hausdorff distance between the ball $B_s(x)$ and any ball of the same radius centered at the tip of a metric cone splitting a factor \mathbb{R}^{k+1} is bigger than η for any $r < s < 1$. In particular for any $\eta > 0$ and any $0 < r < 1$, we define the k -effective stratum $\mathcal{S}_{\eta, r}^k$ by

$$\mathcal{S}_{\eta, r}^k := \left\{ y \mid d_{GH}(B_s(y), B_s((0, z^*))) \geq \eta s \quad \text{for all } \mathbb{R}^{k+1} \times C(Z) \quad \text{and all } r \leq s \leq 1 \right\},$$

where $B_s((0, z^*))$ denotes the ball in $\mathbb{R}^{k+1} \times C(Z)$ centered at $(0, z^*)$ with radius s . We recall that $C(Z)$ denotes the metric cone with basis the metric space Z . Hence we prove the following theorem, which can be found in our paper [21, Theorem 2.4]. The following theorem has been used as a key tool in [186], where the boundary of N -dimensional $\text{RCD}(K, N)$ spaces is studied, see also [67]. We recall that $v_{K, N}(r)$ denotes the volume of the ball of radius r in the simply connected model of constant sectional curvature $K/(N-1)$ and dimension N .

Theorem 0.12. *Given $K \in \mathbb{R}$, $N \in [2, +\infty)$, an integer $k \in [0, N)$, and $v, \eta > 0$, there exists a constant $c(K, N, v, \eta) > 0$ such that if $(X, \mathbf{d}, \mathcal{H}^N)$ is an $\text{RCD}(K, N)$ metric measure space satisfying*

$$(15) \quad \frac{\mathcal{H}^N(B_1(x))}{v_{K, N}(1)} \geq v \quad \forall x \in X,$$

then, for all $x \in X$ and $0 < r < 1/2$, it holds

$$(16) \quad \mathcal{H}^N(\mathcal{S}_{\eta, r}^k \cap B_{1/2}(x)) \leq c(K, N, v, \eta) r^{N-k-\eta}.$$

0.3.4. *Area formula for intrinsically differentiable graphs in Carnot groups.* In this subsection I will state one of the main results obtained in [33] together with Merlo. It is an area formula for (almost everywhere) intrinsically differentiable graphs in arbitrary Carnot groups. The other results of [33] are described in Chapter 2. In [33] one may also find rectifiability results for level sets of Lipschitz functions between Carnot groups, but I will skip the treatment of such results in this thesis. For the terminology used in this subsection we refer the reader to Chapter 1.

We start introducing the notion of area factor with respect to a splitting of a Carnot group, and then we give the statement of the theorem.

Lemma 0.13 ([109, Proposition 3.1.5]). *Let \mathbb{V}, \mathbb{L} be two homogeneous complementary subgroups in \mathbb{G} . Let \mathbb{P} be a homogeneous subgroup that is a complementary subgroup of \mathbb{L} . Then there exists a map $\varphi_{\mathbb{P}} : \mathbb{V} \rightarrow \mathbb{L}$ such that $\mathbb{P} = \Phi_{\mathbb{P}}(\mathbb{V}) := \mathbb{V} \cdot \varphi_{\mathbb{P}}(\mathbb{V})$.*

Definition 0.14 (Area factor, [132, Lemma 3.2]). *Let \mathbb{V}, \mathbb{L} be two homogeneous complementary subgroups of a Carnot group \mathbb{G} . Let \mathbb{P} be a homogeneous subgroup that is a complementary subgroup of \mathbb{L} . Take $\varphi_{\mathbb{P}} : \mathbb{V} \rightarrow \mathbb{L}$ as in Lemma 0.13 and let $\Phi_{\mathbb{P}} : v \mapsto v \cdot \varphi_{\mathbb{P}}(v)$ be its graph map. Then the *centered area factor* of \mathbb{P} with respect to the splitting $\mathbb{V} \cdot \mathbb{L}$ is the unique $0 < \mathcal{A}(\mathbb{P}) < +\infty$ such that*

$$(17) \quad \mathcal{C}^h_{\mathbb{L}}\mathbb{P} = \mathcal{A}(\mathbb{P})(\Phi_{\mathbb{P}})_*(\mathcal{C}^h_{\mathbb{L}}\mathbb{V}).$$

Theorem 0.15. *Let \mathbb{V}, \mathbb{L} be two homogeneous complementary subgroups of a Carnot group \mathbb{G} , and let h be the homogeneous dimension of \mathbb{V} . Let Γ be the graph of an intrinsically Lipschitz map $\varphi : A \subseteq \mathbb{V} \rightarrow \mathbb{L}$, with A Borel. Let us assume Γ is an intrinsically differentiable graph at \mathcal{S}^h -almost every $x \in \Gamma$ and let us assume that the Hausdorff tangent $\mathbb{V}(x)$ of Γ at x is complemented by \mathbb{L} at \mathcal{S}^h -almost every $x \in \Gamma$. Then, for every Borel function $\psi : \Gamma \rightarrow [0, +\infty)$, the following area formula holds*

$$(18) \quad \int_{\Gamma} \psi d\mathcal{C}^h_{\mathbb{L}}\Gamma = \int_A \psi(a \cdot \varphi(a)) \mathcal{A}(\mathbb{V}(a \cdot \varphi(a))) d\mathcal{C}^h_{\mathbb{L}}\mathbb{V},$$

where \mathcal{C}^h is the centered Hausdorff measure, $\mathbb{V}(a \cdot \varphi(a))$ is the Hausdorff tangent of Γ at the point $a \cdot \varphi(a) \in \Gamma$, and $\mathcal{A}(\cdot)$ is the centered area factor defined with respect to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$, see Definition 0.14.

Let us remark that (18) extends and strengthens the area formula of [132, Theorem 1.1]. Let us stress that when a Rademacher theorem is available, one can remove the hypothesis about the intrinsic differentiability in Theorem 0.15. Nevertheless, as it will be discussed in Chapter 2, a Rademacher theorem might not hold in arbitrary Carnot groups, see [134].

Let us point out that in the literature one can find many more *analytic* area formulae in Carnot groups, i.e., in which the area element is expressed in terms of properly defined intrinsic derivatives of the map ψ . This is the case of [85, Theorem 1.1 and Theorem 1.2] for low-codimensional $C^1_{\mathbb{H}}$ -submanifolds in Heisenberg groups (cf. also [114, Theorem 2]), which has been extended to intrinsically Lipschitz low-codimensional surfaces in [213, Theorem 1.3] (cf. also [82, Theorem 1.6]); and of [23, Proposition 1.8] for one-codimensional $C^1_{\mathbb{H}}$ -graphs in arbitrary Carnot groups. The latter formulae could be derived from Theorem 0.15 explicitly writing the area element in terms of the intrinsic derivatives of the parametrization map φ . Other geometric area formulae for Euclidean C^1 or $C^{1,1}$ -submanifolds in Carnot groups have been investigated, e.g., in [165–167].

0.3.5. *Analytic characterizations of intrinsically C^1 hypersurfaces in Carnot groups of step 2.* In this subsection I will state the main results obtained in [22, 23] together with Di Donato, Don, and Le Donne about the characterization of intrinsically C^1 regular submanifolds through analytic properties of the graphing function in groups of step 2.

We focus our attention on codimension-one intrinsic graphs. Other results in the arbitrary co-horizontal case are in the works [23, 141]. A codimension-one intrinsic graph Γ inside a Carnot group \mathbb{G} comes with a couple of homogeneous and complementary subgroups \mathbb{W} and \mathbb{L} with \mathbb{L} one-dimensional, and a map $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ such that $\Gamma = \{x \in \mathbb{G} : x = w \cdot \varphi(w), w \in U\}$. It turns out that the regularity of the graph Γ is strictly related to the regularity of φ and its intrinsic gradient $\nabla^{\varphi}\varphi$, see Section 4. However, one can define some different notions of regularity that rely on some φ -dependent operators D^{φ}_W with $W \in \text{Lie}(\mathbb{W})$. We recall that given two homogeneous complementary subgroups \mathbb{W} and \mathbb{L} in a Carnot group \mathbb{G} , and a continuous function $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ defined on an open set U of \mathbb{W} , we define, for every

$W \in \text{Lie}(\mathbb{W})$, the continuous *projected vector field* D_W^φ , by defining its action on smooth functions as follows

$$(19) \quad (D_W^\varphi)|_w(f) := W|_{w \cdot \varphi(w)}(f \circ P_{\mathbb{W}}),$$

for all $w \in U$ and all $f \in C^\infty(\mathbb{W})$, where $P_{\mathbb{W}}$ is the projection on \mathbb{W} relative to the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$.

If an adapted basis of the Lie algebra (X_1, \dots, X_n) is fixed and is such that $\mathbb{L} := \exp(\text{span}\{X_1\})$ and $\mathbb{W} := \exp(\text{span}\{X_2, \dots, X_n\})$, then we denote by D^φ the vector-valued operator $(D_{X_2}^\varphi, \dots, D_{X_m}^\varphi) =: (D_2^\varphi, \dots, D_m^\varphi)$, where m is the rank of \mathbb{G} , i.e., the dimension of the first layer of the stratification of \mathbb{G} . The regularity of Γ is related to the validity of the equation $D^\varphi\varphi = \omega$ in an open subset $U \subseteq \mathbb{W}$, for some $\omega: U \rightarrow \mathbb{R}^{m-1}$, which can be understood in different ways. We briefly present some of them here.

Distributional sense. Since \mathbb{L} is one-dimensional, one can see that $D^\varphi\varphi$ is a well-defined distribution. Thus we could interpret the equality $D^\varphi\varphi = \omega$ in the distributional sense.

Broad sense.* For every $j = 2, \dots, m$ and every point $a \in U$, there exists a C^1 integral curve of $D_{X_j}^\varphi$ starting from a for which the Fundamental Theorem of Calculus with derivative ω holds, see [23] for details.

Broad sense. For every $j = 2, \dots, m$ and every point $a \in U$, all the integral curves of $D_{X_j}^\varphi$ starting from a are such that the Fundamental Theorem of Calculus with derivative ω holds, see [23] for details.

Approximate sense. For every $a \in U$, there exist $\delta > 0$ and a family $\{\varphi_\varepsilon \in C^1(B_\delta(a)) : \varepsilon \in (0, 1)\}$ such that $\varphi_\varepsilon \rightarrow \varphi$ and $D_j^{\varphi_\varepsilon} \varphi_\varepsilon \rightarrow \omega_j$ uniformly on $B_\delta(a)$ as ε goes to zero, for every $j = 2, \dots, m$.

In the two works [23] and [22], When \mathbb{G} has step 2 we prove the following theorem, see [23, Theorem 6.17], and [22, Theorem 4.2]. For the relevant definitions in the following statement that we did not explain above, we refer the reader to Section 4 of Chapter 2.

Theorem 0.16. *Let \mathbb{G} be a Carnot group of step 2 and rank m , and let \mathbb{W} and \mathbb{L} be two homogeneous complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Let $U \subseteq \mathbb{W}$ be an open set, and let $\varphi: U \rightarrow \mathbb{L}$ be a continuous function. Then the following conditions are equivalent.*

- (a) *graph(φ) is a $C^1_{\mathbb{H}}$ -hypersurface with tangents complemented by \mathbb{L} ;*
- (b) *φ is uniformly intrinsically differentiable on U ;*
- (c) *φ is intrinsically differentiable on U and its intrinsic gradient is continuous;*
- (d) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that, for every $a \in U$, there exist $\delta > 0$ and a family of functions $\{\varphi_\varepsilon \in C^1(B_\delta(a)) : \varepsilon \in (0, 1)\}$ such that*

$$\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = \varphi, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} D_j^{\varphi_\varepsilon} \varphi_\varepsilon = \omega_j \quad \text{in } L^\infty(B_\delta(a)),$$

for every $j = 2, \dots, m$;

- (e) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ holds in the broad sense on U ;*
- (f) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ holds in the broad* sense on U .*
- (g) *there exists $\omega \in C(U; \mathbb{R}^{m-1})$ such that $D^\varphi\varphi = \omega$ holds in the distributional sense U .*

Moreover if any of the previous holds, ω is the intrinsic gradient of φ .

The previous Theorem 0.16 is the complete generalization to the case of step-2 Carnot groups of all the results scattered in [16, 58, 59] where the authors study the same problem in the Heisenberg groups, and of [93] where partial results are obtained in the case of step-2 Carnot groups.

0.3.6. *The Rank-One Theorem in RCD spaces.* In this subsection I will state the main result obtained in [19] together with Brena and Pasqualetto, which is the Rank-One Theorem in finite dimensional RCD spaces.

Let Ω be an open subset of \mathbb{R}^n , and let $u \in \text{BV}(\Omega; \mathbb{R}^k)$. By using the Lebesgue–Radon–Nikodým Theorem one can write the distributional derivative of u as

$$Du = D^a u + D^s u,$$

where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure \mathcal{L}^n , and $D^s u$ is the singular part of Du . We denote with $Du/|Du|$ the matrix-valued Lebesgue–Radon–Nikodým density of Du with respect to the total variation $|Du|$.

In 1988 Ambrosio and De Giorgi [18], motivated by the study of some functionals coming from the Mathematical Physics, conjectured the following:

Rank-One property: For every $u \in \text{BV}(\Omega; \mathbb{R}^k)$ the matrix $Du/|Du|$ has rank-one $|Du|^s$ -almost everywhere.

In 1993 Alberti [5] solved in the affirmative the previous conjecture.

Let (X, d, \mathbf{m}) be a metric measure space. The definition of BV function on (X, d, \mathbf{m}) by Miranda and Ambrosio can easily be adapted to give a meaning of the total variation $|DF|$ of an arbitrary $F \in \text{BV}_{\text{loc}}(X, d, \mathbf{m})^k$, while in the general metric measure setting a good notion for the Lebesgue–Radon–Nikodým derivative $DF/|DF|$ is missing.

In the setting of RCD metric measure spaces, the study of calculus has been blossoming very fast in the last decade. In particular, very recently in [91] the authors propose and study the notion of $L^0(\text{Cap})$ -normed $L^0(\text{Cap})$ -module, and the notion of capacitary tangent module $L_{\text{Cap}}^0(TX)$, where Cap denotes the usual 2-Capacity.

A fundamental contribution of [65], building on [91], is the fact that, in the setting of $\text{RCD}(K, N)$ spaces, for an arbitrary set of finite perimeter E with finite mass, one can give a meaning to the unit normal $\nu_E = D\chi_E/|D\chi_E|$ as an element of the capacitary tangent module $L_{\text{Cap}}^0(TX)$ such that the Gauss–Green formula holds, see [65, Theorem 2.4]. The Gauss–Green formula has been then successfully employed, together with the former work by Ambrosio–Bruè–Semola [10], to obtain the $(n - 1)$ -rectifiability of the essential boundary of any set of locally finite perimeter in an RCD space of essential dimension n , see [65, 66].

The Gauss–Green formula in [65, Theorem 2.4] has been generalized by Brena–Gigli in [62] for vector-valued BV functions. We give below the statement of the Gauss–Green formula in [62], where the density $\nu_F = DF/|DF|$ is implicitly defined.

Theorem 0.17 ([62, Theorem 3.13]). *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space, and let $F \in \text{BV}(X, d, \mathbf{m})^k$. Then there exists a unique $\nu_F \in L_{\text{Cap}}^0(TX)^k$, up to $|DF|$ -almost everywhere equality, such that $|\nu_F| = 1$ $|DF|$ -almost everywhere, and*

$$\sum_{j=1}^k \int_X F_j \text{div}(v_j) d\mathbf{m} = - \int_X \pi_{|DF|}(v) \cdot \nu_F d|DF|, \quad \text{for every test vector field } v = (v_1, \dots, v_k).$$

For the notion of divergence of a vector field, the notion of test vector fields, the notion of the projection $\pi_{|DF|}$ and of the norm $|\cdot|$ in $L_{\text{Cap}}^0(TX)^k$, we refer the reader to the preliminary section of [19].

The previous Theorem 0.17 tells us that in the setting of $\text{RCD}(K, N)$ spaces we can give a precise meaning to $DF/|DF|$ for an arbitrary vector-valued BV function F . Hence it is meaningful to ask if $DF/|DF|$ is a rank-one matrix $|DF|^s$ -almost everywhere, where $|DF|^s$ is the singular part of the total variation $|DF|$. Before giving the main result we clarify this last sentence by means of a definition.

Definition 0.18. Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space, let $\nu \in L^0_{\text{Cap}}(TX)^k$, and let $\mu \ll \text{Cap}$ be a Radon measure. We say that

$$\text{Rk}(\nu) = 1 \quad \mu\text{-almost everywhere,}$$

if there exist $\omega \in L^0_{\text{Cap}}(TX)$ and $\lambda_1, \dots, \lambda_k \in L^0(\text{Cap})$ such that for every $i = 1, \dots, k$,

$$\nu_i = \lambda_i \omega \quad \mu\text{-almost everywhere.}$$

Theorem 0.19 (Rank-One Theorem for $\text{RCD}(K, N)$ spaces). *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ space, and let $F \in \text{BV}(X, \mathbf{d}, \mathbf{m})^k$. Then*

$$\text{Rk}(\nu_F) = 1 \quad |DF|^s\text{-almost everywhere,}$$

in the sense of Definition 0.18, where ν_F is defined in Theorem 0.17, and $|DF|^s$ is the singular part of the total variation $|DF|$.

The proof of Theorem 0.19 closely follows the one in the Euclidean setting by Massaccesi–Vittone [173], whose strategy has been used by Don–Massaccesi–Vittone to prove the Rank-One property in some Carnot groups [96]. As far as we know, apart from the result of Don–Massaccesi–Vittone [96] that holds for a special class of Carnot groups, Theorem 0.19 is one of the first instances of the validity of the Rank-One Theorem in a large class of metric measure spaces.

CHAPTER 1

Preliminaries

In this first chapter we discuss preliminary results and notation of this thesis.

In Section 1 we discuss general facts of Measure Theory. In particular in Section 1.1 we define the Hausdorff measures, in Section 1.2 we discuss general definitions and facts about Radon measures, and finally in Section 1.3 we briefly discuss the notion and some results regarding Vitali relations.

In Section 2 we discuss general facts and notation in Carnot groups. In particular, in Section 2.1 we set the basic definitions about Carnot groups and, among other things, we give the definitions of homogeneous subgroups, and left-invariant homogeneous distances and norms. In Section 2.2 we introduce and discuss the properties of the Grassmannian of a Carnot group, which is the set of homogeneous subgroups of a Carnot group. We prove that the Grassmannian is compact, and then we discuss basic properties of the Haar measures of homogeneous subgroups of a Carnot group. In Section 2.3 we introduce and study the splitting projections on complementary subgroups of a Carnot group, and the notions of cones over homogeneous subgroups of a Carnot group.

In Section 3 we discuss the definitions of rectifiable measures in Carnot groups. In particular in Section 3.1 we give the notion of tangent measures to a Radon measure, we consider the class of flat measures on a Carnot group, which are by definition Haar measures on homogeneous subgroups, and we give the definition of \mathcal{P} -rectifiable measure. In Section 3.2 we define a couple of functionals that in some sense quantify the distance of a Radon measure from being flat, and we discuss some of their properties. Finally, in Section 3.3 we prove the measurability of the map that associates to a point in the support of a Radon measure its flat tangent when this tangent is assumed to be unique pointwise almost everywhere.

In Section 4 we introduce several classes of intrinsic regular functions and submanifolds in Carnot groups. In particular, in Section 4.1 we introduce and discuss the notion of intrinsically Lipschitz function. In Section 4.2 we introduce the notion of intrinsically differentiable function and graph, uniformly intrinsically differentiable function, and we explore what happens when the target of the function is a horizontal subgroup. In Section 4.3 we introduce the notion of $C^1_{\mathbb{H}}$ -function between Carnot groups, we give the area formula for Lipschitz functions between Carnot groups, we discuss the definition of intrinsically C^1 rectifiable sets, and we discuss the relation of such a definition with the one of \mathcal{P} -rectifiability. Finally we briefly discuss the class of co-horizontal intrinsically C^1 submanifolds.

1. Measure Theory

In this section we introduce the general notation and the basic facts of Measure Theory we are going to exploit in this thesis. Standard references for the material that we are going to discuss are [12, 102, 174].

Let (X, d) be a metric space. We let $B(x, r) := \{z \in X : d(x, z) < r\}$ be the open metric ball relative to the distance d centred at x and with radius $r > 0$. The closed ball will be denoted with $\bar{B}(x, r) := \{z \in X : d(x, z) \leq r\}$. Moreover, for a subset $E \subseteq X$ and $r > 0$, we denote with $\bar{B}(E, r) := \{z \in X : \text{dist}(z, E) \leq r\}$ the *closed r -tubular neighborhood of E*

and with $B(E, r) := \{z \in X : \text{dist}(z, E) < r\}$ the *open r -tubular neighborhood of E* , where $\text{dist}(z, E) := \inf_{x \in E} d(z, x)$. If $A \subseteq X$, we denote with $\text{diam}A$ the *diameter* of A , i.e., the quantity $\sup_{(x,y) \in A \times A} d(x, y)$.

1.1. Hausdorff measures. In this subsection we introduce the Hausdorff measures.

Definition 1.1 (Hausdorff Measures). Let (X, d) be a metric space. We define the h -dimensional *spherical Hausdorff measure* relative to d as

$$\mathcal{S}^h(A) := \sup_{\delta > 0} \inf \left\{ \sum_{j=1}^{\infty} r_j^h : A \subseteq \bigcup_{j=1}^{\infty} \overline{B}(x_j, r_j), r_j \leq \delta \right\},$$

for every $A \subseteq X$. We define the h -dimensional *Hausdorff measure* relative to d as

$$\mathcal{H}^h(A) := \sup_{\delta > 0} \inf \left\{ \sum_{j=1}^{\infty} 2^{-h} (\text{diam}E_j)^h : A \subseteq \bigcup_{j=1}^{\infty} E_j, \text{diam}E_j \leq \delta \right\},$$

for every $A \subseteq X$. We define the h -dimensional *centered Hausdorff measure* relative to d as

$$\mathcal{C}^h(A) := \sup_{E \subseteq A} \mathcal{C}_0^h(E),$$

for every $A \subseteq X$, where

$$\mathcal{C}_0^h(E) := \sup_{\delta > 0} \inf \left\{ \sum_{j=1}^{\infty} r_j^h : E \subseteq \bigcup_{j=1}^{\infty} \overline{B}(x_j, r_j), x_j \in E, r_j \leq \delta \right\},$$

for every $E \subseteq X$. We stress that \mathcal{C}^h is an outer measure, and thus it defines a Borel regular measure, see [97, Proposition 4.1], and that the measures $\mathcal{S}^h, \mathcal{H}^h, \mathcal{C}^h$ are all equivalent measures, see [102, Section 2.10.2] and [97, Proposition 4.2].

Definition 1.2 (Hausdorff distance). Let (X, d) be a metric space. For every couple of sets $A, B \subseteq X$, we define the *Hausdorff distance* of A from B as

$$d_H(A, B) := \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(A, y) \right\},$$

where

$$\text{dist}(x, A) := \inf_{y \in A} d(x, y),$$

for every $x \in X$ and $A \subseteq X$.

1.2. Densities of Radon measures. In this subsection we discuss general facts about Radon measures. For basic references on Measure Theory in this setting we refer the reader to [12, Chapter 1].

Definition 1.3 (Weak convergence of measures). Let (X, d) be a locally compact separable space. Given a family $\{\phi_i\}_{i \in \mathbb{N}}$ of Radon measures on X , we say that ϕ_i weakly converges to a Radon measure ϕ , and we write $\phi_i \rightharpoonup \phi$, if

$$\int f d\phi_i \rightarrow \int f d\phi, \quad \text{for every } f \in C_c(X),$$

where $C_c(X)$ denotes the space of compactly supported functions on X .

Definition 1.4 (Lower and upper densities). If ϕ is a Radon measure on a locally compact separable metric space (X, d) , and $h \geq 0$, we define

$$\Theta_*^h(\phi, x) := \liminf_{r \rightarrow 0} \frac{\phi(\overline{B}(x, r))}{r^h}, \quad \text{and} \quad \Theta^{h,*}(\phi, x) := \limsup_{r \rightarrow 0} \frac{\phi(\overline{B}(x, r))}{r^h},$$

and we say that $\Theta_*^h(\phi, x)$ and $\Theta^{h,*}(\phi, x)$ are the *lower and upper h -density* of ϕ at the point $x \in X$, respectively. Furthermore, we say that the measure ϕ has *h -density* if

$$0 < \Theta_*^h(\phi, x) = \Theta^{h,*}(\phi, x) < \infty, \quad \text{for } \phi\text{-almost every } x \in X.$$

Definition 1.5 (Asymptotically doubling measure). If ϕ is a Radon measure on a locally compact separable metric space (X, d) , we say that ϕ is *asymptotically doubling* if

$$\limsup_{r \rightarrow 0} \frac{\phi(\overline{B}(x, 2r))}{\phi(\overline{B}(x, r))} < +\infty, \quad \text{for } \phi\text{-almost every } x \in X.$$

Remark 1.6. Let ϕ be a Radon measure on a locally compact separable metric space (X, d) . Suppose there exists $h > 0$ with $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in X$. Hence it is readily seen that ϕ is asymptotically doubling.

In the following proposition we recall that the upper and lower densities are natural under restriction to Borel subsets. We will subsequently obtain a refined version of the following proposition in the setting of Carnot groups, see Proposition 1.55.

Proposition 1.7. *Suppose ϕ is a Radon measure on a locally compact separable metric space (X, d) . Suppose there exists $h \geq 0$ with $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in X$. Then, for every Borel set $B \subseteq X$ and for ϕ -almost every $x \in B$ we have*

$$\Theta_*^h(\phi \llcorner B, x) = \Theta_*^h(\phi, x), \quad \text{and} \quad \Theta^{h,*}(\phi \llcorner B, x) = \Theta^{h,*}(\phi, x).$$

Proof. This is a direct consequence of Lebesgue Differentiation Theorem of [125, page 77], that can be applied since (X, d, ϕ) is a Vitali metric measure space due to [125, Theorem 3.4.3]. \square

Let us introduce a useful split of the support of a Radon measure ϕ on a locally compact separable metric space (X, d) .

Definition 1.8 (Support of a measure). Let ϕ be a Radon measure on a locally compact separable metric space (X, d) . Hence the support of ϕ is

$$\text{supp}(\phi) := \{x \in X : \phi(U) > 0 \text{ for every neighborhood } U \text{ of } x\}.$$

We say that ϕ is *supported* on a Borel set A if $\phi(X \setminus A) = 0$.

Definition 1.9 (The sets $E(\vartheta, \gamma)$). Let ϕ be a Radon measure on a locally compact separable metric space (X, d) , and let us suppose that ϕ is supported on a compact set K . For every $\vartheta, \gamma \in \mathbb{N}$ we define

$$(1.1) \quad E_K(\vartheta, \gamma) := \{x \in K : \vartheta^{-1}r^h \leq \phi(\overline{B}(x, r)) \leq \vartheta r^h \text{ for every } 0 < r < 1/\gamma\}.$$

In the following, we will always write $E(\vartheta, \gamma)$, underlying the dependence on the compact set K .

Proposition 1.10. *Let ϕ be a Radon measure on a locally compact separable metric space (X, d) , and let us suppose that ϕ is supported on a compact set K . For every $\vartheta, \gamma \in \mathbb{N}$, the set $E(\vartheta, \gamma)$ defined in Definition 1.9 is compact.*

Proof. This can be obtained arguing verbatim as in [180, Proposition 1.14]. \square

Proposition 1.11. *Let ϕ be a Radon measure supported on the compact set K of a locally compact separable metric space (X, d) . Assume there exists $h > 0$ such that $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in X$. Then $\phi(X \setminus \bigcup_{\vartheta, \gamma \in \mathbb{N}} E(\vartheta, \gamma)) = 0$.*

Proof. Let $w \in K \setminus \bigcup_{\vartheta, \gamma} E(\vartheta, \gamma)$ and note that this implies that either $\Theta_*^h(\phi, x) = 0$ or $\Theta^{h,*}(\phi, x) = \infty$. Since $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in X$, this concludes the proof. \square

1.3. Vitali relations. In this subsection we recall the notion and some basic facts about Vitali relations associated to a Borel measure. Our main reference is [102].

Definition 1.12 (ϕ -Vitali relation). Let (X, d) be a metric space with a Borel measure ϕ on it and let $\mathcal{B}(X)$ be the family of Borel sets of X . We say that $S \subseteq X \times \mathcal{B}(X)$ is a *covering relation* if

$$S = \{(x, B) : x \in B\}.$$

Furthermore for every $Z \subseteq X$ we let

$$(1.2) \quad S(Z) := \{B : (x, B) \in S \text{ for some } x \in Z\}.$$

Finally a covering S is said to be *fine* at $x \in X$ if

$$\inf\{\text{diam}(B) : (x, B) \in S\} = 0.$$

By a ϕ -Vitali relation we mean a covering relation S that is fine at every point of X and such that the following condition holds

If C is a subset of S and Z is a subset of X such that C is fine at each point of Z , then $C(Z)$ has a countable disjoint subfamily covering ϕ -almost all of Z .

If δ is a nonnegative function on $S(X)$, for every $B \in S(X)$ we define its δ -enlargement as

$$(1.3) \quad \hat{B} := \bigcup\{B' \in S(X) : B' \cap B \neq \emptyset \text{ and } \delta(B') \leq 5\delta(B)\}.$$

We recall the following general result due to Federer: it contains a criterion to show that a fine covering relation is a ϕ -Vitali relation, and a Lebesgue theorem for ϕ -Vitali relations.

Proposition 1.13 ([102, Theorem 2.8.17, Corollary 2.9.9 and Theorem 2.9.11]). *Let X be a metric space, and let ϕ be a Borel regular measure on X that is finite on bounded sets. Let S be a covering relation such that $S(X)$ is a family of bounded closed sets, S is fine at each point of X , and let δ be a nonnegative function on $S(X)$ such that*

$$\limsup_{\varepsilon \rightarrow 0^+} \left\{ \delta(B) + \frac{\phi(\hat{B})}{\phi(B)} : (x, B) \in S, \text{diam}B < \varepsilon \right\} < +\infty,$$

for ϕ -almost every $x \in X$. Then S is a ϕ -Vitali relation.

Moreover, if S is a ϕ -Vitali relation on X , and f is a ϕ -measurable real-valued function with $\int_K |f| d\phi < +\infty$ on every bounded ϕ -measurable K , we have

$$\limsup_{\varepsilon \rightarrow 0^+} \left\{ \frac{\int_B |f(z) - f(x)| d\phi(z)}{\phi(B)} : (x, B) \in S, \text{diam}B < \varepsilon \right\} = 0,$$

for ϕ -almost every $x \in X$. In addition, given $A \subseteq X$, if we define

$$P := \left\{ x \in X : \liminf_{\varepsilon \rightarrow 0^+} \left\{ \frac{\phi(B \cap A)}{\phi(B)} : (x, B) \in S, \text{diam}B < \varepsilon \right\} = 1 \right\},$$

then P is ϕ -measurable and $\phi(A \setminus P) = 0$.

2. Carnot Groups

In this section we briefly introduce some notation on Carnot groups that we will extensively use throughout the thesis. Basic references on Carnot groups are [61, 145].

2.1. General definitions and notation. In this subsection we set the basic definitions and notation about Carnot groups.

A stratifiable group \mathbb{G} of step κ is a connected and simply connected Lie group whose Lie algebra \mathfrak{g} admits a stratification $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$. We say that $V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ is a *stratification* of \mathfrak{g} if $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ as vector spaces, and moreover

$$[V_1, V_i] = V_{i+1}, \quad \text{for every } i = 1, \dots, \kappa - 1, \quad [V_1, V_\kappa] = \{0\}, \quad \text{and } V_\kappa \neq \{0\},$$

where $[A, B] := \text{span}\{[a, b] : a \in A, b \in B\}$. We call V_1 the *horizontal layer* of \mathbb{G} . We denote by n the topological dimension of \mathfrak{g} , by n_j the dimension of V_j for every $j = 1, \dots, \kappa$. Furthermore, we define $\pi_i : \mathfrak{g} \rightarrow V_i$ to be the projection maps on the i -th stratum. We will often shorten the notation to $v_i := \pi_i v$. A Carnot group \mathbb{G} , or stratified group, is a stratifiable group \mathbb{G} on which we fix a stratification. The identity element of \mathbb{G} will be denoted by e , or also by 0 when we are identifying \mathbb{G} with \mathbb{R}^n by means of exponential coordinates, which we are now going to introduce.

For a Carnot group \mathbb{G} , the exponential map $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a global diffeomorphism from \mathfrak{g} to \mathbb{G} . Hence, if we choose a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} , every $p \in \mathbb{G}$ can be written in a unique way as

$$(1.4) \quad p = \exp(p_1 X_1 + \cdots + p_n X_n).$$

This means that we can identify $p \in \mathbb{G}$ with the n -tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and the group \mathbb{G} itself with \mathbb{R}^n endowed with the group operation \cdot determined by the Baker-Campbell-Hausdorff formula. When we say that (X_1, \dots, X_n) is an *adapted basis* of \mathfrak{g} we mean that $\{X_1, \dots, X_n\}$ is a basis of \mathfrak{g} , and $\{X_{1+\sum_{i=1}^{j-1} n_i}, \dots, X_{n_j+\sum_{i=1}^{j-1} n_i}\}$ is a basis of V_j for every $j = 1, \dots, \kappa$ (where, by convention, $\sum_{i=1}^0 n_i := 0$). For every $p \in \mathbb{G}$, we define the left translation $\tau_p : \mathbb{G} \rightarrow \mathbb{G}$ as the map

$$q \mapsto \tau_p q := p \cdot q.$$

The stratification of \mathfrak{g} carries with it a natural family of dilations $\delta_\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$, that are Lie algebra automorphisms of \mathfrak{g} and are defined by

$$\delta_\lambda(v_1, \dots, v_\kappa) := (\lambda v_1, \lambda^2 v_2, \dots, \lambda^\kappa v_\kappa), \quad \text{for every } \lambda \in \mathbb{R} \setminus 0.$$

We will also denote with δ_λ the automorphism on \mathbb{G} defined as $\exp \circ \delta_\lambda \circ \exp^{-1}$.

As already remarked above, the group operation \cdot is determined by the Baker-Campbell-Hausdorff formula, and, in exponential coordinates, it has the form (see [112, Proposition 2.1])

$$p \cdot q = p + q + \mathcal{Q}(p, q), \quad \text{for all } p, q \in \mathbb{R}^n,$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_\kappa) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and the \mathcal{Q}_i 's have the following properties. For every $i = 1, \dots, \kappa$ and every $p, q \in \mathbb{G}$ we have

- (i) $\mathcal{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^i \mathcal{Q}_i(p, q)$,
- (ii) $\mathcal{Q}_i(p, q) = -\mathcal{Q}_i(-q, -p)$,
- (iii) $\mathcal{Q}_1 = 0$ and $\mathcal{Q}_i(p, q) = \mathcal{Q}_i(p_1, \dots, p_{i-1}, q_1, \dots, q_{i-1})$.

Thus, we can represent the product \cdot as

$$(1.5) \quad p \cdot q = (p_1 + q_1, p_2 + q_2 + \mathcal{Q}_2(p_1, q_1), \dots, p_\kappa + q_\kappa + \mathcal{Q}_\kappa(p_1, \dots, p_{\kappa-1}, q_1, \dots, q_{\kappa-1})).$$

We recall that a sub-algebra \mathfrak{h} of \mathfrak{g} is said to be *homogeneous* if it is δ_λ -invariant for every $\lambda > 0$. We recall that, given any sub-algebra $\mathfrak{h} \subseteq \mathfrak{g}$, $\mathfrak{h} = W_1 \oplus \cdots \oplus W_\kappa$ is a *grading* of \mathfrak{h} if $[W_i, W_j] \subseteq W_{i+j}$ for every $1 \leq i, j \leq \kappa$, where we mean that $W_\ell := \{0\}$ for every $\ell > \kappa$. The stratification of the Lie algebra \mathfrak{g} naturally induces a grading on each of its homogeneous Lie sub-algebras \mathfrak{h} , i.e.,

$$(1.6) \quad \mathfrak{h} = V_1 \cap \mathfrak{h} \oplus \cdots \oplus V_\kappa \cap \mathfrak{h}.$$

Definition 1.14 (Homogeneous subgroups). A subgroup \mathbb{V} of \mathbb{G} is said to be *homogeneous* if it is a Lie subgroup of \mathbb{G} that is invariant under the dilations δ_λ . Given \mathbb{V} a homogeneous subgroup of \mathbb{G} we denote with $\text{Lie}(\mathbb{V})$ its Lie algebra.

We recall the following basic terminology: a *horizontal subgroup* of a Carnot group \mathbb{G} is a homogeneous subgroup of it that is contained in $\exp(V_1)$; a *Carnot subgroup* $\mathbb{W} = \exp(\mathfrak{h})$ of a Carnot group \mathbb{G} is a homogeneous subgroup of it such that the first layer $V_1 \cap \mathfrak{h}$ of the grading of \mathfrak{h} inherited from the stratification of \mathfrak{g} is the first layer of a stratification of \mathfrak{h} .

Homogeneous Lie subgroups of \mathbb{G} are in bijective correspondence through \exp with the homogeneous Lie sub-algebras of \mathfrak{g} . For every Lie algebra \mathfrak{h} with grading $\mathfrak{h} = W_1 \oplus \dots \oplus W_\kappa$, we define its *homogeneous dimension* as

$$\dim_{\text{hom}}(\mathfrak{h}) := \sum_{i=1}^{\kappa} i \cdot \dim(W_i).$$

Thanks to (1.6) we infer that, if \mathfrak{h} is a homogeneous Lie sub-algebra of \mathfrak{g} , we have $\dim_{\text{hom}}(\mathfrak{h}) := \sum_{i=1}^{\kappa} i \cdot \dim(\mathfrak{h} \cap V_i)$. We introduce now the class of homogeneous and left-invariant distances.

Definition 1.15 (Homogeneous left-invariant distance). A metric $d : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}$ is said to be homogeneous and left-invariant if for every $x, y \in \mathbb{G}$ we have

- (i) $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$ for every $\lambda > 0$,
- (ii) $d(\tau_z x, \tau_z y) = d(x, y)$ for every $z \in \mathbb{G}$.

We remark that two homogeneous left-invariant distances on a Carnot group are always bi-Lipschitz equivalent, and moreover they induce the manifold topology on \mathbb{G} , see [149]. It is well-known that the Hausdorff dimension (for a definition of Hausdorff dimension see for instance [174, Definition 4.8]) of a graded Lie group \mathbb{G} with respect to an arbitrary left-invariant homogeneous distance coincides with the homogeneous dimension of its Lie algebra. For a reference for the latter statement, see [149, Theorem 4.4].

We recall that a *homogeneous norm* $\|\cdot\|$ on \mathbb{G} is a function $\|\cdot\| : \mathbb{G} \rightarrow [0, +\infty)$ such that $\|x\| = \|x^{-1}\|$ for every $x \in \mathbb{G}$; $\|\delta_\lambda x\| = \lambda \|x\|$ for every $\lambda > 0$ and $x \in \mathbb{G}$; $\|x \cdot y\| \leq \|x\| + \|y\|$ for every $x, y \in \mathbb{G}$; and $\|x\| = 0$ if and only if $x = e$. Given an arbitrary homogeneous norm $\|\cdot\|$ on \mathbb{G} , the homogeneous left-invariant distance d induced by $\|\cdot\|$ is defined as follows

$$d(x, y) := \|x^{-1} \cdot y\|.$$

Vice-versa, given a homogeneous left-invariant distance d , it induces a homogeneous norm through the equality $\|x\| := d(x, e)$ for every $x \in \mathbb{G}$, where e is the identity element of \mathbb{G} .

We introduce now a distinguished homogeneous norm on \mathbb{G} .

Definition 1.16 (Box metric). Let \mathbb{G} be a Carnot group. Let $\mathcal{B} := \{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} , and, for every $i = 1, \dots, \kappa$, let us identify each vector space V_i with a vector subspace of \mathbb{R}^n by means of the exponential map and the coordinates associated to \mathcal{B} . We will denote with $|\cdot|$ the standard Euclidean norms on such vector subspaces. Then there exist $\varepsilon_2, \dots, \varepsilon_\kappa$ depending on the group \mathbb{G} such that, if we define

$$\|g\| := \max\{|g_1|, \varepsilon_2 |g_2|^{1/2}, \dots, \varepsilon_\kappa |g_\kappa|^{1/\kappa}\} \quad \text{for all } g \in \mathbb{G},$$

then $\|\cdot\|$ is a homogeneous norm on \mathbb{G} that induces a left-invariant homogeneous distance. We refer to [112, Section 5] for a proof of this fact.

There is a distinguished class of left-invariant homogeneous distances on Carnot groups, known as *Carnot-Carathéodory distances*. If we fix a norm $\|\cdot\|_1$ on the first stratum V_1 of the Lie algebra \mathfrak{g} of \mathbb{G} , we can extend it left-invariantly to the horizontal bundle

$$(1.7) \quad \mathbb{V}_1(x) := (\tau_x)_* V_1,$$

for $x \in \mathbb{G}$, where τ_x is the left translation by x , and V_1 is seen as a subspace of $T_e\mathbb{G} \equiv \mathfrak{g}$. We say that an absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathbb{G}$ is *horizontal* if

$$\gamma'(t) \in V_1(\gamma(t)), \quad \text{for almost every } t \in [0, 1].$$

We define

$$(1.8) \quad d_{cc}^{\|\cdot\|_1}(x, y) := \inf \left\{ \int_0^1 \|\gamma'(t)\|_1 dt : \gamma(0) = x, \quad \gamma(1) = y, \quad \gamma \text{ horizontal} \right\}.$$

The Chow-Rashevskii theorem states that this distance is finite. It is clearly homogeneous and left-invariant.

2.2. Intrinsic Grassmannian in Carnot groups. In this subsection we introduce the Grassmannian in Carnot groups and we discuss some of its basic properties. This treatment is mainly taken from the work [28], that has been subsequently divided in the two works [31, 32].

Let us recall the definition of the Euclidean Grassmannian, along with some of its properties.

Definition 1.17 (Euclidean Grassmannian). Let k, n be natural numbers such that $k \leq n$. We let $Gr(n, k)$ be the set of the k -vector subspaces of \mathbb{R}^n . We endow $Gr(n, k)$ with the following distance

$$d_{eu}(V_1, V_2) := d_{H,eu} \left(V_1 \cap \overline{B}_{eu}(0, 1), V_2 \cap \overline{B}_{eu}(0, 1) \right),$$

where $\overline{B}_{eu}(0, 1)$ is the (closed) Euclidean unit ball, and $d_{H,eu}$ is the Hausdorff distance between sets induced by the Euclidean distance on \mathbb{R}^n .

Remark 1.18 (Euclidean Grassmannian and convergence). Let k, n be natural numbers such that $k \leq n$. It is well-known that the metric space $(Gr(n, k), d_{eu})$ is compact. Moreover, the following hold

- (i) if $V_n \rightarrow V$, then for every $v \in V$ there exist $v_n \in V_n$ such that $v_n \rightarrow v$;
- (ii) if $V_n \rightarrow V$ and there is a sequence $v_n \in V_n$ such that $v_n \rightarrow v$, then $v \in V$.

We now give the definition of the intrinsic Grassmannian on Carnot groups and introduce the class of complemented homogeneous subgroups.

Definition 1.19 (Intrinsic Grassmannian on Carnot groups). Let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q . We define $Gr(h)$ to be the family of homogeneous subgroups \mathbb{V} of \mathbb{G} that have homogeneous dimension h .

Let us recall that if \mathbb{V} is a homogeneous subgroup of \mathbb{G} , any other homogeneous subgroup \mathbb{L} such that

$$\mathbb{V} \cdot \mathbb{L} = \mathbb{G} \quad \text{and} \quad \mathbb{V} \cap \mathbb{L} = \{e\},$$

is said to be a *complementary subgroup* of \mathbb{V} in \mathbb{G} . We say that a homogeneous subgroup \mathbb{V} is *complemented* if it admits at least one homogeneous complementary subgroup. Finally, we let

- (i) $Gr_c(h)$ to be the subfamily of those $\mathbb{V} \in Gr(h)$ that have a complementary subgroup, and we will refer to $Gr_c(h)$ as the *h -dimensional complemented Grassmannian*,
- (ii) $Gr_{\triangleleft}(h)$ to be the subfamily of those $\mathbb{V} \in Gr_c(h)$ having at least one normal complementary subgroup, and we will refer to $Gr_{\triangleleft}(h)$ as the *h -dimensional co-normal Grassmannian*.

Let us introduce the stratification vector of a homogeneous subgroup, and the notion of \mathfrak{s} -co-normal Grassmannian.

Definition 1.20 (Stratification vector). Let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q . For every $\mathbb{V} \in Gr(h)$ we denote with $\mathfrak{s}(\mathbb{V})$ the vector

$$\mathfrak{s}(\mathbb{V}) := (\dim(V_1 \cap \exp^{-1}(\mathbb{V})), \dots, \dim(V_\kappa \cap \exp^{-1}(\mathbb{V}))),$$

that with abuse of language we call the *stratification*, or the *stratification vector*, of \mathbb{V} . Furthermore, we define

$$\mathfrak{S}(h) := \{\mathfrak{s}(\mathbb{V}) \in \mathbb{N}^\kappa : \mathbb{V} \in Gr(h)\}.$$

We remark that the cardinality of $\mathfrak{S}(h)$ is bounded by $\prod_{i=1}^\kappa (\dim V_i + 1)$ for every $h \in \{1, \dots, Q\}$.

Definition 1.21 (\mathfrak{s} -co-normal Grassmannian). For every $\mathfrak{s} \in \mathfrak{S}(h)$ we let

$$Gr_{\triangleleft}^{\mathfrak{s}}(h) := \{\mathbb{V} \in Gr_{\triangleleft}(h) : \mathfrak{s}(\mathbb{V}) = \mathfrak{s}\},$$

and we will refer to $Gr_{\triangleleft}^{\mathfrak{s}}(h)$ as the \mathfrak{s} -co-normal Grassmannian.

We now prove that the Grassmannian introduced above is compact. Notice that, since the homogeneous subgroups of a Carnot group \mathbb{G} are in bijective correspondence with the homogeneous sub-algebras of \mathfrak{g} , and since the Grassmannian of vector subspaces of \mathfrak{g} has a natural compact topology, the core of the following proposition is to prove that the subset of homogeneous sub-algebras is closed in this topology. Anyway, we are going to explicitly provide a distance $d_{\mathbb{G}}$ on $Gr(h)$, and we will directly prove that $(Gr(h), d_{\mathbb{G}})$ is a compact metric space. We will not prove it explicitly, but we stress that the topology induced by $d_{\mathbb{G}}$ on the Grassmannian of vector subspaces of \mathfrak{g} is the usual topology on the Grassmannian.

Proposition 1.22 (Compactness of the Grassmannian). *Let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q , endowed with a left-invariant homogeneous distance d . Let $\overline{B}(e, 1)$ be the closed ball, in the distance d , of center e and radius 1. The function*

$$d_{\mathbb{G}}(\mathbb{W}_1, \mathbb{W}_2) := d_{H, \mathbb{G}}(\mathbb{W}_1 \cap \overline{B}(e, 1), \mathbb{W}_2 \cap \overline{B}(e, 1)),$$

where $\mathbb{W}_1, \mathbb{W}_2 \in Gr(h)$, and $d_{H, \mathbb{G}}$ is the Hausdorff distance associated to d , is a distance on $Gr(h)$. Moreover $(Gr(h), d_{\mathbb{G}})$ is a compact metric space.

Proof. The fact that $d_{\mathbb{G}}$ is a distance comes from well-known properties of the Hausdorff distance. Let us consider a sequence $\{\mathbb{W}_j\}_{j \in \mathbb{N}} \subseteq Gr(h)$, with $\exp^{-1}(\mathbb{W}_j) = W_{j,1} \oplus \dots \oplus W_{j,\kappa}$, where $W_{j,i} := V_i \cap \exp^{-1}(\mathbb{W}_j)$ for every $j \in \mathbb{N}$ and $1 \leq i \leq \kappa$. By extracting a (non re-labelled) subsequence we can suppose that there exist $\{k_i\}_{i=1, \dots, \kappa}$ natural numbers such that the topological dimension is $\dim W_{j,i} = k_i$ for all $j \in \mathbb{N}$, and for all $1 \leq i \leq \kappa$. In particular the topological dimension of $\exp^{-1}(\mathbb{W}_j)$ is constant. Exploiting the compactness of the Euclidean Grassmannian, see Remark 1.18, we get that up to a (non re-labelled) subsequence,

$$(1.9) \quad W_{j,i} \rightarrow W_i, \quad \text{i.e.} \quad d_{\text{eu}}(W_{j,i}, W_i) \rightarrow 0 \quad \text{for every } 1 \leq i \leq \kappa,$$

where the convergence is meant in the Euclidean Grassmannian $Gr(\dim V_i, k_i)$. As a consequence

$$(1.10) \quad W_j = W_{j,1} \oplus \dots \oplus W_{j,\kappa} \rightarrow W = W_1 \oplus \dots \oplus W_\kappa, \quad \text{i.e.,} \quad d_{\text{eu}}(W_j, W) \rightarrow 0,$$

where the convergence is meant in the Euclidean Grassmannian $Gr(n, \sum_{i=1}^\kappa k_i)$. The previous equality is a consequence of (1.9) and the following observation: if V and W are two orthogonal (according to the standard scalar product) linear subspaces such that $\mathbb{R}^n = V \oplus W$, and A, B are vector subspaces of V , and C, D are vector subspaces of W , then

$$d_{\text{eu}}(A \oplus C, B \oplus D) \leq d_{\text{eu}}(A, B) + d_{\text{eu}}(C, D),$$

where the direct sums above are orthogonal too. Let us notice that, from (1.10) it follows that

$$(1.11) \quad d_{H,\text{eu}}(W_j \cap \overline{B}(0,1), W \cap \overline{B}(0,1)) \rightarrow 0,$$

where we stress that $\overline{B}(0,1) \subseteq \mathfrak{g}$ is the preimage by means of \exp of the closed unit ball, in the homogeneous left-invariant metric d , centered at e . The proof of (1.11) can be reached by contradiction exploiting (1.10) and the fact that $\overline{B}(0,1)$ is compact. We leave the routine details to the reader.

In order to conclude the proof, we need to show that

$$(1.12) \quad d_{H,\mathbb{G}}(W_j \cap \overline{B}(0,1), W \cap \overline{B}(0,1)) \rightarrow 0,$$

where, with a little abuse of notation, we denote with $d_{H,\mathbb{G}}$ the Hausdorff distance on \mathfrak{g} induced by pulling back d with the exponential map \exp . Indeed, on the compact set $\overline{B}(0,1)$, one has $d \leq Cd_{\text{eu}}^{1/\kappa}$ for some constant $C > 0$, see for instance [208, Proposition 2.15]. This means that for subsets contained in $\overline{B}(0,1)$ one has $d_{H,\mathbb{G}} \leq Cd_{H,\text{eu}}^{1/s}$. This last inequality with (1.11) gives (1.12). Finally from (1.12) we get, by the very definition of $d_{\mathbb{G}}$,

$$d_{\mathbb{G}}(\mathbb{W}_j, \mathbb{W}) \rightarrow 0,$$

where $\mathbb{W} := \exp W$. If we show that \mathbb{W} is a homogeneous subgroup of homogeneous dimension h we are done. The homogeneity comes from the fact that W admits a stratification (1.10), while the homogeneous dimension is fixed because it depends on the dimensions of W_i that are all equal to k_i . Let us prove \mathbb{W} is a subgroup. First of all \mathbb{W} is inverse-closed, because $\mathbb{W} = \exp W$, and W is a vector space. Now take $a, b \in \mathbb{W}$. By the first point of Remark 1.18 we find $a_n, b_n \in \mathbb{W}_n$ such that $a_n \rightarrow a$, and $b_n \rightarrow b$. Then, by continuity of the product, $a_n \cdot b_n \rightarrow a \cdot b$, and $a_n \cdot b_n \in \mathbb{W}_n$. Then from the second point of Remark 1.18 we get that $a \cdot b \in \mathbb{W}$. \square

In the following proposition we prove that if the distance between two elements of the intrinsic Grassmannian is sufficiently small, then they have the same stratification vector.

Proposition 1.23. *Let Q be a natural number. Let \mathbb{G} be a Carnot group of homogeneous dimension Q , endowed with a left-invariant homogeneous distance d . There exists a constant $\tilde{h}_{\mathbb{G}} > 0$, depending only on \mathbb{G} , such that if $\mathbb{V}, \mathbb{W} \in Gr(h)$ for some $1 \leq h \leq Q$, and $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) \leq \tilde{h}_{\mathbb{G}}$, then $\mathfrak{s}(\mathbb{V}) = \mathfrak{s}(\mathbb{W})$.*

Proof. Let us fix $1 \leq h \leq Q$. Let us suppose by contradiction that there exist \mathbb{V}_i and \mathbb{W}_i in $Gr(h)$ such that, for every $i \in \mathbb{N}$, the stratification vector of \mathbb{V}_i is different from \mathbb{W}_i and such that $d_{\mathbb{G}}(\mathbb{V}_i, \mathbb{W}_i) \rightarrow 0$. Up to extract two (non re-labelled) subsequences we can assume that the \mathbb{V}_i 's have the same stratification vector for every $i \in \mathbb{N}$, as well as the \mathbb{W}_i 's. Then, by compactness, see the proof of Proposition 1.22, we can assume up to passing to a (non re-labelled) subsequence that $\mathbb{W}_i \rightarrow \mathbb{W}$ where \mathbb{W} has the same stratification of the \mathbb{W}_i 's, and $\mathbb{V}_i \rightarrow \mathbb{V}$ where \mathbb{V} has the same stratification of the \mathbb{V}_i 's. Since $d_{\mathbb{G}}(\mathbb{V}_i, \mathbb{W}_i) \rightarrow 0$ we get that $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) = 0$ and then $\mathbb{V} = \mathbb{W}$ but this is a contradiction since they have different stratification vectors. This proves the existence of a constant \tilde{h} that depends both on \mathbb{G} and h . However, taking the minimum over $1 \leq h \leq Q$ of such \tilde{h} 's, the dependence on h is eliminated. \square

In the following proposition we explicitly characterize the Haar measures on elements of the Grassmannian introduced above.

Proposition 1.24. *Let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q , endowed with a left-invariant homogeneous distance d . Suppose $\mathbb{V} \in Gr(h)$ is a homogeneous subgroup of topological dimension ρ .*

Let us identify \mathbb{G} with \mathbb{R}^n by means of the exponential map, and a choice of a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} . Let $\mathcal{H}_{\text{eu}\perp}^\rho \mathbb{V}$ be the ρ -dimensional Euclidean (on $\mathbb{G} \equiv \mathbb{R}^n$) Hausdorff measure restricted to \mathbb{V} . Then $\mathcal{S}^h \perp \mathbb{V}$, $\mathcal{H}^h \perp \mathbb{V}$, $\mathcal{C}^h \perp \mathbb{V}$ and $\mathcal{H}_{\text{eu}\perp}^\rho \mathbb{V}$ are Haar measures of \mathbb{V} . Furthermore, every Haar measure λ of \mathbb{V} is h -homogeneous in the sense that

$$\lambda(\delta_r(E)) = r^h \lambda(E), \quad \text{for every Borel set } E \subseteq \mathbb{V}.$$

Proof. First, one can show by an explicit computation that the Lebesgue measure \mathcal{L}^ρ restricted to the vector space $\exp^{-1}(\mathbb{V})$ is a Haar measure. Indeed, this last assertion comes from the fact that for every $v \in \mathbb{V}$ the map $p \rightarrow v \cdot p$ has unitary Jacobian determinant when seen as a map from \mathbb{V} to \mathbb{V} , see [110, Lemma 2.20]. Thus since when seen \mathbb{V} as immersed in \mathbb{R}^n we have that the Lebesgue measure of \mathbb{V} coincides with $\mathcal{H}_{\text{eu}\perp}^\rho \mathbb{V}$, we conclude that $\mathcal{H}_{\text{eu}\perp}^\rho \mathbb{V}$ is a Haar measure of \mathbb{V} as well. Moreover, the Hausdorff, the spherical Hausdorff, and the centered Hausdorff measures introduced in Definition 1.2 on \mathbb{V} , are non-zero, locally finite, and invariant under left-translations and thus they are Haar measures of \mathbb{V} .

The last part of the proposition comes from the fact that the property is obvious by definition for the spherical Hausdorff measure, and the fact that all the Haar measures are the same up to a constant. \square

In the following proposition we explore some relations on various Haar measures on the elements of the Grassmannian. The following proposition holds for the distance d induced by the norm introduced in Definition 1.16.

Proposition 1.25. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with the homogeneous norm $\|\cdot\|$ introduced in Definition 1.16. Let $1 \leq h \leq Q$, and $\mathbb{W} \in \text{Gr}(h)$ be a homogeneous subgroup of homogeneous dimension h and of topological dimension ρ . Then*

- (i) *there exists a constant $C_1 := C_1(\mathfrak{s}(\mathbb{W}))$ such that for every $p \in \mathbb{W}$ and every $r > 0$ we have*

$$(1.13) \quad \mathcal{H}_{\text{eu}}^\rho(\overline{B}(p, r) \cap \mathbb{W}) = C_1 r^h,$$

where, for the precise definition of $\mathcal{H}_{\text{eu}}^\rho$ we refer to the statement of Proposition 1.24,

- (ii) *there exists a constant $\beta(\mathbb{W})$ such that $\mathcal{C}^h \perp \mathbb{W} = \beta(\mathbb{W}) \mathcal{H}_{\text{eu}\perp}^\rho \mathbb{W}$,*
(iii) *$\beta(\mathbb{W}) = \mathcal{H}_{\text{eu}\perp}^\rho \mathbb{W}(\overline{B}(0, 1))^{-1}$ and in particular $\beta(\mathbb{W})$ only depends on $\mathfrak{s}(\mathbb{W})$.*

Proof. Thanks to Proposition 1.24, we have

$$\mathcal{H}_{\text{eu}}^\rho(\overline{B}(p, r) \cap \mathbb{W}) = \mathcal{H}_{\text{eu}}^\rho(\overline{B}(0, r) \cap \mathbb{W}) = \mathcal{H}_{\text{eu}}^\rho(\delta_r(\overline{B}(0, 1) \cap \mathbb{W})) = r^h \mathcal{H}_{\text{eu}}^\rho(\overline{B}(0, 1) \cap \mathbb{W}).$$

Furthermore, if \mathbb{V} is another homogeneous subgroup such that $\mathfrak{s}(\mathbb{W}) = \mathfrak{s}(\mathbb{V})$, we can find a linear map T that acts as an orthogonal transformation on each of the V_i 's and that maps \mathbb{W} bijectively to \mathbb{V} . Since we are endowing \mathbb{G} with the box metric, see Definition 1.16, we get that $T(\overline{B}(0, 1) \cap \mathbb{W}) = \overline{B}(0, 1) \cap \mathbb{V}$. Since T is an orthogonal transformation itself, it is an isometry of \mathbb{R}^n and this implies that

$$\mathcal{H}_{\text{eu}}^\rho(\overline{B}(0, 1) \cap \mathbb{W}) = \mathcal{H}_{\text{eu}}^\rho(T(\overline{B}(0, 1) \cap \mathbb{W})) = \mathcal{H}_{\text{eu}}^\rho(\overline{B}(0, 1) \cap \mathbb{V}),$$

and thus item (i) is proved.

Concerning item (ii), thanks to Proposition 1.24, we have that both $\mathcal{C}^h \perp \mathbb{W}$ and $\mathcal{H}_{\text{eu}\perp}^\rho \mathbb{W}$ are Haar measures of \mathbb{W} . This implies that there must exist a constant $\beta(\mathbb{W})$ such that $\beta(\mathbb{W}) \mathcal{H}_{\text{eu}\perp}^\rho \mathbb{W} = \mathcal{C}^h \perp \mathbb{W}$.

Finally, in order to prove item (iii), we prove the following. For every left-invariant homogeneous distance d on \mathbb{G} and every homogeneous subgroup $\mathbb{W} \subseteq \mathbb{G}$ of homogeneous dimension h , we have that

$$(1.14) \quad \mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) = 1,$$

where \mathcal{C}^h is the centered Hausdorff measure relative to the distance d and $\overline{B}(0, 1)$ is the closed ball relative to the distance d .

Indeed, let us fix an $\varepsilon > 0$, let us take $A \subseteq \mathbb{W} \cap \overline{B}(0, 1)$ such that $\mathcal{C}_0^h(A) \geq \mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) - \varepsilon$, $\delta > 0$ and a covering of A with closed balls $B_i := \{\overline{B}(x_i, r_i)\}_{i \in \mathbb{N}}$ centred on $A \subseteq \mathbb{W}$ and with radii $r_i \leq \delta$ such that

$$\sum_{i \in \mathbb{N}} r_i^h \leq \mathcal{C}_0^h(A) + \varepsilon.$$

This implies that

$$\begin{aligned} \mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W})(\mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W}) + \varepsilon) &\geq \mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W})(\mathcal{C}_0^h(A) + \varepsilon) \\ &\geq \sum_{i \in \mathbb{N}} \mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W}) r_i^h = \sum_{i \in \mathbb{N}} \mathcal{C}^h(\overline{B}(x_i, r_i) \cap \mathbb{W}) \geq \mathcal{C}^h(A) \\ &\geq \mathcal{C}_0^h(A) \geq \mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) - \varepsilon, \end{aligned}$$

where the first inequality is true since $\mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W}) \geq \mathcal{C}^h(A) \geq \mathcal{C}_0^h(A)$, and the third equality is true since $x_i \in \mathbb{W}$ and $\mathcal{C}^h \llcorner \mathbb{W}$ is a Haar measure on \mathbb{W} . Thanks to the arbitrariness of ε we finally infer that $\mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) \geq 1$.

On the other hand, thanks to [117, item (ii) of Theorem 2.13 and Remark 2.14], we have that, calling $B_t := \{x \in \mathbb{W} \cap \overline{B}(0, 1) : \Theta^{h,*}(\mathcal{C}^h \llcorner \mathbb{W}, x) > t\}$ for every $t > 0$, we infer that $\mathcal{C}^h(B_t) \geq t \mathcal{C}^h(B_t)$ for every $t > 0$. Thus, for every $t > 1$ we conclude $\mathcal{C}^h(B_t) = 0$ and hence for $\mathcal{C}^h \llcorner \mathbb{W}$ -almost every $x \in \mathbb{W} \cap \overline{B}(0, 1)$ we have that $\Theta^{h,*}(\mathcal{C}^h \llcorner \mathbb{W}, x) \leq 1$. For one of such $x \in \mathbb{W} \cap \overline{B}(0, 1)$ we can write

$$\mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W}) = \limsup_{r \rightarrow 0} \frac{\mathcal{C}^h(\overline{B}(x, r) \cap \mathbb{W})}{r^h} = \Theta^{h,*}(\mathcal{C}^h \llcorner \mathbb{W}, x) \leq 1,$$

where the first equality comes from Proposition 1.24. Thus $\mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) = 1$ and this concludes the proof of the first part of (iii) thanks to item (ii). The fact that $\beta(\mathbb{W})$ depends only on $\mathfrak{s}(\mathbb{W})$ follows from item (i) \square

Remark 1.26. The above proposition can be proved verbatim whenever the distance is a *multiradial distance*, see [165, Definition 8.5].

Remark 1.27. We stress here for future references that in the proof of item (iii) of Proposition 1.25, see (1.14), we proved that whenever \mathbb{G} is endowed with an arbitrary left-invariant homogeneous distance d , then for every homogeneous subgroup $\mathbb{W} \subseteq \mathbb{G}$ of homogeneous dimension h , we have that

$$(1.15) \quad \mathcal{C}^h(\mathbb{W} \cap \overline{B}(0, 1)) = 1.$$

In the next proposition we prove that uniform measures supported on homogeneous subgroups of Carnot groups are Haar measures.

Proposition 1.28. *Suppose that μ is a Borel regular measure on \mathbb{G} supported on a homogeneous subgroup $\mathbb{V} \in Gr(h)$, such that $0 \in \text{supp}(\mu)$, and assume that there exists a constant $C > 0$ such that for every $z \in \text{supp}(\mu)$ and every $s > 0$ we have*

$$\mu(\overline{B}(z, s)) = Cs^h.$$

Then μ is a Haar measure of the subgroup \mathbb{V} .

Proof. Without loss of generality we can assume $C = 1$. Thanks to [117, Theorem 3.1] we thus infer that $\mu = \mathcal{C}^h \llcorner \text{supp}(\mu)$. Moreover, for every $x \in \text{supp}(\mu)$, thanks to Remark 1.27, we have that $\mu(\overline{B}(x, r)) = \mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(x, r))$ for every $r > 0$. If by contradiction $\text{supp}(\mu) \neq \mathbb{V}$,

then there would exist a $p \in \mathbb{V}$ and a $r_0 > 0$ such that $\overline{B}(p, r_0) \cap \text{supp}(\mu) = \emptyset$. This however is impossible since we would have

$$(1.16) \quad \begin{aligned} \mathcal{C}^h_{\mathbb{L}\mathbb{V}}(\overline{B}(0, 2(\|p\| + r_0))) &\geq \mathcal{C}^h(\overline{B}(0, 2(\|p\| + r_0)) \cap \text{supp}(\mu)) + \mathcal{C}^h(\overline{B}(p, r_0) \cap \mathbb{V}) \\ &> \mu(\overline{B}(0, 2(\|p\| + r_0))), \end{aligned}$$

where we have used $\mu = \mathcal{C}^h_{\mathbb{L}\text{supp}(\mu)}$. The last inequality is a contradiction with what we found above, since by assumption $0 \in \text{supp}(\mu)$. \square

Let us now exploit Remark 1.27 to prove next proposition that will be useful in Chapter 3. Let $\|\cdot\|$ be a homogeneous norm on \mathbb{G} . A function $\varphi : \mathbb{G} \rightarrow \mathbb{R}$ is said to be *radially symmetric with respect to $\|\cdot\|$* if there is a function $g : [0, \infty) \rightarrow \mathbb{R}$, called *profile function* such that $\varphi(x) = g(\|x\|)$.

Proposition 1.29. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q , and let $h \in \{1, \dots, Q\}$. Let $\varphi : \mathbb{G} \rightarrow \mathbb{R}$ be a continuous radially symmetric function with respect to a homogeneous norm $\|\cdot\|$ on \mathbb{G} , and let g be its profile function. Let $\mathbb{V} \in \text{Gr}(h)$. Then the following holds*

$$\int \varphi d\mathcal{C}^h_{\mathbb{L}\mathbb{V}} = h \int s^{h-1} g(s) ds.$$

Proof. It suffices to prove the proposition for positive simple functions, since the general result follows by Beppo Levi's convergence theorem. Thus suppose \mathbb{V} has topological dimension ρ and let $\varphi(z) := \sum_{i=1}^N a_i \chi_{\overline{B}(0, r_i)}(z)$ and note that thanks to Remark 1.27 for every $\mathbb{V} \in \text{Gr}(h)$ we have that $\mathcal{C}^h_{\mathbb{L}\mathbb{V}}(\overline{B}(0, r_i)) = r_i^h$, and then

$$\begin{aligned} \int \varphi(z) d\mathcal{C}^h_{\mathbb{L}\mathbb{V}} &= \sum_{i=1}^N a_i \mathcal{C}^h_{\mathbb{L}\mathbb{V}}(\overline{B}(0, r_i)) = \sum_{i=1}^N a_i r_i^h \\ &= h \sum_{i=1}^N a_i \int_0^{r_i} s^{h-1} ds = h \int \sum_{i=1}^N a_i s^{h-1} \chi_{[0, r_i]}(s) ds = h \int s^{h-1} g(s) ds. \end{aligned}$$

\square

The following proposition is a consequence of the choice of the norm in Definition 1.16, since it is based on Proposition 1.25.

Proposition 1.30. *Let \mathbb{G} be a Carnot group endowed with the box norm defined in Definition 1.16. Let $h \in \{1, \dots, Q\}$ and suppose that $\{\mathbb{V}_i\}_{i \in \mathbb{N}}$ is a sequence of homogeneous subgroups in $\text{Gr}(h)$ converging in the Grassmannian metric $d_{\mathbb{G}}$ to some $\mathbb{V} \in \text{Gr}(h)$. Then, $\mathcal{C}^h_{\mathbb{L}\mathbb{V}_i} \rightarrow \mathcal{C}^h_{\mathbb{L}\mathbb{V}}$.*

Proof. First of all notice that Proposition 1.23 implies that there exists a $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ we have that \mathbb{V}_i and \mathbb{V} have the same stratification and thus the same topological dimension ρ . Since the \mathbb{V}_i 's have the same stratification if $i \geq i_0$, Proposition 1.25(iii) implies that $\beta(\mathbb{V}_i) = \beta(\mathbb{V})$ for every $i \geq i_0$. Thus, for every continuous function f with compact support, thanks to Proposition 1.25 we have

$$\lim_{i \rightarrow \infty} \int f d\mathcal{C}^h_{\mathbb{L}\mathbb{V}_i} - \int f d\mathcal{C}^h_{\mathbb{L}\mathbb{V}} = \lim_{i \rightarrow \infty} \beta(\mathbb{V}) \left(\int f d\mathcal{H}^{\rho}_{\text{eu}\mathbb{L}\mathbb{V}_i} - \int f d\mathcal{H}^{\rho}_{\text{eu}\mathbb{L}\mathbb{V}} \right) = 0,$$

where the last identity comes from the fact that $\mathcal{H}^{\rho}_{\text{eu}\mathbb{L}\mathbb{V}_i} \rightarrow \mathcal{H}^{\rho}_{\text{eu}\mathbb{L}\mathbb{V}}$. \square

2.3. Splitting projections and cones over homogeneous subgroups. In this subsection we introduce and discuss the splitting projections on complementary subgroups, and different notions of cones over homogeneous subgroups in Carnot groups. The basic references for this subsection are [28, 110].

We now introduce the projections related to a splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$ of the group. From now on let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q with an arbitrary homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance d .

Definition 1.31 (Projections related to a splitting). For every $\mathbb{V} \in Gr_c(h)$ with a homogeneous complementary subgroup \mathbb{L} , we can find two unique elements $g_{\mathbb{V}} := P_{\mathbb{V}}(g) \in \mathbb{V}$ and $g_{\mathbb{L}} := P_{\mathbb{L}}(g) \in \mathbb{L}$ such that

$$g = P_{\mathbb{V}}(g) \cdot P_{\mathbb{L}}(g) = g_{\mathbb{V}} \cdot g_{\mathbb{L}}.$$

We will refer to $P_{\mathbb{V}}(g)$ and $P_{\mathbb{L}}(g)$ as the *splitting projections*, or simply *projections*, of g onto \mathbb{V} and \mathbb{L} , respectively.

We recall here below a very useful fact on splitting projections.

Proposition 1.32. *Let us fix $\mathbb{V} \in Gr_c(h)$ and \mathbb{L} two complementary homogeneous subgroups of \mathbb{G} .*

Then, for every $x \in \mathbb{G}$ the map $\Psi : \mathbb{V} \rightarrow \mathbb{V}$ defined as $\Psi(z) := P_{\mathbb{V}}(xz)$ is invertible and it has unitary Jacobian. As a consequence $\mathcal{S}^h(P_{\mathbb{V}}(\mathcal{E})) = \mathcal{S}^h(P_{\mathbb{V}}(xP_{\mathbb{V}}(\mathcal{E}))) = \mathcal{S}^h(P_{\mathbb{V}}(x\mathcal{E}))$ for every $x \in \mathbb{G}$ and $\mathcal{E} \subseteq \mathbb{G}$ Borel.

Proof. The first part is a direct consequence of [110, Proof of Lemma 2.20]. For the second part it is sufficient to use the first part and the fact that for every $x, y \in \mathbb{G}$ we have $P_{\mathbb{V}}(xy) = P_{\mathbb{V}}(xP_{\mathbb{V}}y)$. \square

We collect below two useful propositions on the splitting projections.

Proposition 1.33 (Proposition 2.12 and Corollary 2.15 of [110]). *Let $\|\cdot\|$ be a homogeneous norm on \mathbb{G} that induces a homogeneous left-invariant distance d , and let \mathbb{V} and \mathbb{L} be two complementary subgroups. Then there exists a constant $0 < K < 1$ such that for every $g \in \mathbb{G}$ we have*

$$(1.17) \quad \begin{aligned} K\|P_{\mathbb{L}}(g)\| &\leq \text{dist}(g, \mathbb{V}) \leq \|P_{\mathbb{L}}(g)\|, \\ K(\|P_{\mathbb{L}}(g)\| + \|P_{\mathbb{V}}(g)\|) &\leq \|g\| \leq \|P_{\mathbb{L}}(g)\| + \|P_{\mathbb{V}}(g)\|. \end{aligned}$$

Let us define $C_2(\mathbb{V}, \mathbb{L})$ to be the supremum of all the constants K such that inequalities in (1.17) are satisfied, and let $C_3(\mathbb{V}, \mathbb{L})$ be the supremum of all the constants K such that the first inequality in (1.17) is satisfied. Notice that $C_2(\mathbb{V}, \mathbb{L}) \leq C_3(\mathbb{V}, \mathbb{L})$.

Proposition 1.34. *Let h, Q be natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q . For every $\mathbb{V} \in Gr_c(h)$ with complementary subgroup \mathbb{L} there is a constant $C_4(\mathbb{V}, \mathbb{L}) > 0$ such that for every $p \in \mathbb{G}$ and every $r > 0$ we have*

$$\mathcal{S}^h_{\mathbb{L}}\mathbb{V}(P_{\mathbb{V}}(\overline{B}(p, r))) = C_4(\mathbb{V}, \mathbb{L})r^h.$$

Furthermore, for every Borel set $A \subseteq \mathbb{G}$ for which $\mathcal{S}^h(A) < \infty$, we have

$$(1.18) \quad \mathcal{S}^h_{\mathbb{L}}\mathbb{V}(P_{\mathbb{V}}(A)) \leq 2C_4(\mathbb{V}, \mathbb{L})\mathcal{S}^h(A).$$

Proof. The existence of such $C_4(\mathbb{V}, \mathbb{L})$ is yielded by [110, Lemma 2.20].

Suppose $\{\overline{B}(x_i, r_i)\}_{i \in \mathbb{N}}$ is a countable covering of A with closed balls for which $\sum_{i \in \mathbb{N}} r_i^h \leq$

$2\mathcal{S}^h(A)$. Then

$$\mathcal{S}^h(P_{\mathbb{V}}(A)) \leq \mathcal{S}^h\left(P_{\mathbb{V}}\left(\bigcup_{i \in \mathbb{N}} \overline{B}(x_i, r_i)\right)\right) \leq C_4(\mathbb{V}, \mathbb{L}) \sum_{i \in \mathbb{N}} r_i^h \leq 2C_4(\mathbb{V}, \mathbb{L})\mathcal{S}^h(A).$$

□

We now prove a proposition that will be useful in the Chapter 3.

Proposition 1.35. *Let $\mathbb{W} \in Gr_c(h)$ and assume \mathbb{L} is one of the complementary subgroups of \mathbb{W} . Any other homogeneous subgroup $\mathbb{V} \in Gr(h)$ on which $P_{\mathbb{W}}$ is injective, and satisfying the identity $\mathfrak{s}(\mathbb{V}) = \mathfrak{s}(\mathbb{W})$, is in $Gr_c(h)$ and admits \mathbb{L} as a complementary subgroup.*

Proof. Let $P_{\mathbb{W}}$ be the projection related to the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$. The hypothesis $\mathfrak{s}(\mathbb{V}) = \mathfrak{s}(\mathbb{W})$ implies that \mathbb{V} and \mathbb{W} have the same topological dimension. If by contradiction there exists a $0 \neq y \in \mathbb{L} \cap \mathbb{V}$, then

$$P_{\mathbb{W}}(y) = 0 = P_{\mathbb{W}}(0).$$

This however is not possible since we assumed that $P_{\mathbb{W}}$ is injective on \mathbb{V} . The fact that $\mathbb{L} \cap \mathbb{V} = \{0\}$ together with the fact that $\mathfrak{s}(\mathbb{V}) = \mathfrak{s}(\mathbb{W})$ and that \mathbb{W} and \mathbb{L} are complementary subgroups, imply that $(\mathbb{L} \cap V_i) + (\mathbb{V} \cap V_i) = V_i$ for every $i = 1, \dots, \kappa$. This, jointly with the fact that $\mathbb{L} \cap \mathbb{V} = \{0\}$, implies that \mathbb{L} and \mathbb{V} are complementary subgroups in \mathbb{G} due to the triangular structure of the product \cdot on \mathbb{G} , see (1.5). For an alternative proof of the fact that \mathbb{L} and \mathbb{V} are complementary subgroups, see also [132, Lemma 2.7]. □

Given $\mathbb{W} \in Gr(h)$, and $\alpha > 0$, we now introduce the intrinsic cone $C_{\mathbb{W}}(\alpha)$ and the notion of $C_{\mathbb{W}}(\alpha)$ -set, and prove some of their properties.

Definition 1.36 (Intrinsic cone). For every $\alpha > 0$ and $\mathbb{W} \in Gr(h)$, we define the cone $C_{\mathbb{W}}(\alpha)$ as

$$C_{\mathbb{W}}(\alpha) := \{w \in \mathbb{G} : \text{dist}(w, \mathbb{W}) \leq \alpha \|w\|\}.$$

Definition 1.37 ($C_{\mathbb{W}}(\alpha)$ -set). Given $\mathbb{W} \in Gr(h)$, and $\alpha > 0$, we say that a set $E \subseteq \mathbb{G}$ is a $C_{\mathbb{W}}(\alpha)$ -set if

$$E \subseteq p \cdot C_{\mathbb{W}}(\alpha), \quad \text{for every } p \in E.$$

Remark 1.38 (Equivalent intrinsic cones). Let us observe that if $\mathbb{V} \in Gr_c(h)$, \mathbb{L} is a complementary subgroup of \mathbb{V} , and $\alpha < C_2(\mathbb{V}, \mathbb{L})$, then

$$(1.19) \quad C_{\mathbb{V}}(\alpha) \subseteq \left\{ w \in \mathbb{G} : \|P_{\mathbb{L}}(w)\| \leq \frac{\alpha}{C_2 - \alpha} \|P_{\mathbb{V}}(w)\| \right\}.$$

Indeed, let us take an element w in the complement of the set in the right-hand-side above. Thanks to the fact that $\|w\| \leq \|P_{\mathbb{L}}(w)\| + \|P_{\mathbb{V}}(w)\| < C_2(\mathbb{V}, \mathbb{L})\alpha^{-1}\|P_{\mathbb{L}}(w)\|$, and to Proposition 1.33 we have

$$(1.20) \quad \text{dist}(w, \mathbb{V}) \geq C_2 \|P_{\mathbb{L}}(w)\| > \alpha \|w\|.$$

Therefore, every such w is contained in the complement of the left-hand-side of (1.19), and thus we get the sought conclusion. Moreover, for every $\mathbb{V} \in Gr_c(h)$ and every of its complementary subgroup \mathbb{L} , let us show that for every $\alpha > 0$

$$(1.21) \quad C_{\mathbb{V}, \mathbb{L}}(\alpha) := \{w \in \mathbb{G} : \|P_{\mathbb{L}}(w)\| \leq \alpha \|P_{\mathbb{V}}(w)\|\} \subseteq C_{\mathbb{V}}(C_2^{-1}\alpha).$$

Indeed, if w is an element in the left-hand-side above, we can readily see thanks to Proposition 1.33 that

$$\text{dist}(w, \mathbb{V}) \leq \|P_{\mathbb{L}}(w)\| \leq \alpha \|P_{\mathbb{V}}(w)\| \leq \alpha C_2^{-1} \|w\|.$$

All in all we have proved that if $\mathbb{V} \in Gr_c(h)$, \mathbb{L} is one of its complementary subgroups, and $\alpha < C_2$ we have

$$C_{\mathbb{V}, \mathbb{L}}(C_2\alpha) \subseteq C_{\mathbb{V}}(\alpha) \subseteq C_{\mathbb{V}, \mathbb{L}}(\alpha/(C_2 - \alpha)),$$

thus showing that, for a small α , the cones $C_{\mathbb{V}}(\alpha)$ and $C_{\mathbb{V},\mathbb{L}}(\alpha)$ are equivalent.

Let us now prove the following lemma about the comparison of cones.

Lemma 1.39. *For every $\mathbb{W}_1, \mathbb{W}_2 \in Gr(h)$, $\varepsilon > 0$, and $\alpha > 0$, if $d_{\mathbb{G}}(\mathbb{W}_1, \mathbb{W}_2) < \varepsilon/4$, then*

$$C_{\mathbb{W}_1}(\alpha) \subseteq C_{\mathbb{W}_2}(\alpha + \varepsilon).$$

Proof. We prove that every $z \in C_{\mathbb{W}_1}(\alpha)$ is contained in the cone $C_{\mathbb{W}_2}(\alpha + \varepsilon)$. Thanks to the triangle inequality we infer

$$\text{dist}(z, \mathbb{W}_2) \leq d(z, b) + \inf_{w \in \mathbb{W}_2} d(b, w), \quad \text{for every } b \in \mathbb{W}_1.$$

Thus, choosing $b^* \in \mathbb{W}_1$ in such a way that $d(z, b^*) = \text{dist}(z, \mathbb{W}_1)$, and evaluating the previous inequality at b^* we get

$$(1.22) \quad \text{dist}(z, \mathbb{W}_2) \leq \text{dist}(z, \mathbb{W}_1) + \text{dist}(b^*, \mathbb{W}_2) \leq \alpha \|z\| + \text{dist}(b^*, \mathbb{W}_2),$$

where in the second inequality we used $z \in C_{\mathbb{W}_1}(\alpha)$.

Let us notice that, given \mathbb{W} an arbitrary homogeneous subgroup of \mathbb{G} , $p \in \mathbb{G}$ an arbitrary point such that $p^* \in \mathbb{W}$ is one of the points at minimum distance from \mathbb{W} to p , then the following inequality holds

$$(1.23) \quad \|p^*\| \leq 2\|p\|.$$

Indeed,

$$\|p^*\| - \|p\| \leq \|(p^*)^{-1} \cdot p\| = \text{dist}(p, \mathbb{W}) \leq \|p\| \Rightarrow \|p^*\| \leq 2\|p\|.$$

Now, by homogeneity, since $b^* \in \mathbb{W}_1$ is one of the points at minimum distance from \mathbb{W}_1 of z , we get that $\delta_{1/\|z\|}(b^*)$ is one of the points at minimum distance from \mathbb{W}_1 of $\delta_{1/\|z\|}(z)$. Thus, since $\|\delta_{1/\|z\|}(z)\| = 1$, from (1.23) we get that $\|\delta_{1/\|z\|}(b^*)\| \leq 2$. Finally we obtain

$$(1.24) \quad \begin{aligned} \text{dist}(b^*, \mathbb{W}_2) &= \|z\| \text{dist}(\delta_{1/\|z\|}(b^*), \mathbb{W}_2) = \|z\| \text{dist}(\delta_{1/\|z\|}(b^*), \mathbb{W}_2 \cap \overline{B}(0, 4)) \leq \\ &\leq \|z\| d_H(\mathbb{W}_1 \cap \overline{B}(0, 4), \mathbb{W}_2 \cap \overline{B}(0, 4)) \\ &= 4\|z\| d_H(\mathbb{W}_1 \cap \overline{B}(0, 1), \mathbb{W}_2 \cap \overline{B}(0, 1)) < \varepsilon \|z\|, \end{aligned}$$

where the first equality follows from the homogeneity of the distance, and the second is a consequence of the fact that $\|\delta_{1/\|z\|}(b^*)\| \leq 2$, and thus, from (1.23), every point at minimum distance of $\delta_{1/\|z\|}(b^*)$ from \mathbb{W}_2 has norm bounded above by 4; the third inequality comes from the definition of Hausdorff distance, the fourth equality is true by homogeneity and the last inequality comes from the hypothesis $d_{\mathbb{G}}(\mathbb{W}_1, \mathbb{W}_2) < \varepsilon/4$. Joining (1.22), and (1.24) we get $z \in C_{\mathbb{W}_2}(\alpha + \varepsilon)$, that was what we wanted. \square

We now prove that if two homogeneous subgroups are complementary one to the other, then there is a cone around each of them that does not intersect the other.

Lemma 1.40. *Let $\mathbb{V} \in Gr_c(h)$, and let \mathbb{L} be a complementary subgroup of \mathbb{V} . Let $\varepsilon_1 := \varepsilon_1(\mathbb{V}, \mathbb{L}) := C_2(\mathbb{V}, \mathbb{L})/2 > 0$. Hence*

$$\mathbb{L} \cap C_{\mathbb{V}}(\varepsilon_1) = \{0\}.$$

Proof. Let us suppose the statement is false. Thus there exists $0 \neq v \in \mathbb{L} \cap C_{\mathbb{V}}(\varepsilon_1)$. From Proposition 1.33 and from the very definition of the cone $C_{\mathbb{V}}(\varepsilon_1)$ we have

$$C_2(\mathbb{V}, \mathbb{L})\|v\| \leq \text{dist}(v, \mathbb{V}) \leq \varepsilon_1 \|v\| = C_2(\mathbb{V}, \mathbb{L})\|v\|/2,$$

which is a contradiction with the fact that $v \neq 0$. \square

Remark 1.41. Let $\mathbb{V} \in Gr_c(h)$ and let \mathbb{L} be a complementary subgroup of \mathbb{V} . Let us notice that if there exists $\alpha > 0$ such that $\mathbb{L} \cap C_{\mathbb{V}}(\alpha) = \{0\}$, then $C_3(\mathbb{V}, \mathbb{L}) \geq \alpha$. Indeed, it is enough to prove that $\alpha \|P_{\mathbb{L}}(g)\| \leq \text{dist}(g, \mathbb{V})$ for every $g \in \mathbb{G}$. If $g \in \mathbb{V}$ the latter in equality is trivially verified. Hence suppose by contradiction that there exists $g \notin \mathbb{V}$ such that $\alpha \|P_{\mathbb{L}}(g)\| > \text{dist}(g, \mathbb{V})$. Since $\text{dist}(g, \mathbb{V}) = \text{dist}(P_{\mathbb{L}}(g), \mathbb{V})$ we conclude that $P_{\mathbb{L}}(g) \in \mathbb{L} \cap C_{\mathbb{V}}(\alpha) = \{0\}$, that is a contradiction since $g \notin \mathbb{V}$.

We now prove that the subset of the elements of the Grassmannian that admit a complementary homogeneous subgroup is open.

Proposition 1.42. *The family of the complemented subgroups $Gr_c(h)$ is an open subset of $Gr(h)$.*

Proof. Fix a $\mathbb{W} \in Gr_c(h)$, let \mathbb{L} be one complementary subgroup of \mathbb{W} and set

$$\varepsilon < \min\{\varepsilon_1(\mathbb{V}, \mathbb{L}), \hbar_{\mathbb{G}}\},$$

where $\hbar_{\mathbb{G}}$ is defined in Proposition 1.23. Then, if $\mathbb{W}' \in Gr(h)$ is such that $d_{\mathbb{G}}(\mathbb{W}, \mathbb{W}') < \varepsilon/4$, Lemma 1.39 implies that $\mathbb{W}' \subseteq C_{\mathbb{W}}(\varepsilon)$ and in particular

$$\mathbb{L} \cap \mathbb{W}' \subseteq \mathbb{L} \cap C_{\mathbb{W}}(\varepsilon) = \{0\}.$$

Moreover, since $\varepsilon < \hbar_{\mathbb{G}}$, from Proposition 1.23, we get that \mathbb{W}' has the same stratification of \mathbb{W} and thus the same topological dimension. This, jointly with the previous equality and the Grassmann formula, means that $(\mathbb{W}' \cap V_i) + (\mathbb{L} \cap V_i) = V_i$ for every $i = 1, \dots, \kappa$. This, jointly with the fact that $\mathbb{L} \cap \mathbb{W}' = \{0\}$, implies that \mathbb{L} and \mathbb{W}' are complementary subgroups in \mathbb{G} due to the triangular structure of the product \cdot on \mathbb{G} , see (1.5). For an alternative proof of the fact that \mathbb{L} and \mathbb{W}' are complementary subgroups, see also [132, Lemma 2.7]. \square

The following two lemmas will play a fundamental role in the proof of the existence of density result in Chapter 2.

Lemma 1.43. *Let $\mathbb{V} \in Gr_c(h)$ and \mathbb{L} be one of its complementary subgroups. For every $0 < \alpha < C_2(\mathbb{V}, \mathbb{L})/2$, let*

$$(1.25) \quad \mathfrak{c}(\alpha) := \alpha / (C_2(\mathbb{V}, \mathbb{L}) - \alpha).$$

Then we have

$$(1.26) \quad \overline{B}(0, 1) \cap \mathbb{V} \subseteq P_{\mathbb{V}}(\overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha)) \subseteq \overline{B}(0, 1/(1 - \mathfrak{c}(\alpha))) \cap \mathbb{V}.$$

Proof. The first inclusion comes directly from the definition of projections and cones. Concerning the second, if $v \in \overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha)$, thanks to Proposition 1.33 we have

$$(1.27) \quad C_2(\mathbb{V}, \mathbb{L}) \|P_{\mathbb{L}}(v)\| \leq \text{dist}(v, \mathbb{V}) \leq \alpha \|v\| \leq \alpha (\|P_{\mathbb{L}}(v)\| + \|P_{\mathbb{V}}(v)\|).$$

This implies in particular that $\|P_{\mathbb{L}}(v)\| \leq \mathfrak{c}(\alpha) \|P_{\mathbb{V}}(v)\|$ and thus

$$1 \geq \|P_{\mathbb{V}}(v) P_{\mathbb{L}}(v)\| \geq \|P_{\mathbb{V}}(v)\| - \|P_{\mathbb{L}}(v)\| \geq (1 - \mathfrak{c}(\alpha)) \|P_{\mathbb{V}}(v)\|.$$

This concludes the proof of the lemma. \square

Lemma 1.44. *Let $\mathbb{V} \in Gr_c(h)$ and \mathbb{L} be one of its complementary subgroups. Suppose Γ is a $C_{\mathbb{V}}(\alpha)$ -set with $\alpha < C_2(\mathbb{V}, \mathbb{L})/2$, and let*

$$(1.28) \quad \mathfrak{C}(\alpha) := \frac{1 - \mathfrak{c}(\alpha)}{1 + \mathfrak{c}(\alpha)},$$

where $\mathfrak{c}(\alpha)$ is defined in (1.25). Then

$$\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, r) \cap \Gamma)) \geq \mathcal{S}^h\left(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma)\right), \quad \text{for every } x \in \Gamma.$$

The same inequality above holds if we substitute \mathcal{S}^h with any other Haar measure on \mathbb{V} , see Proposition 1.24, because all of them are equal up to a constant.

Proof. First of all, let us note that we have

$$(1.29) \quad \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, r) \cap \Gamma)) = \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, r) \cap x^{-1}\Gamma)),$$

where the last equality is true since $\mathcal{S}^h(P_{\mathbb{V}}(\mathcal{E})) = \mathcal{S}^h(P_{\mathbb{V}}(x^{-1}\mathcal{E}))$ for every Borel $\mathcal{E} \subseteq \mathbb{G}$, see Proposition 1.32. We wish now to prove the following inclusion

$$(1.30) \quad P_{\mathbb{V}}(\overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(x^{-1}\Gamma) \subseteq P_{\mathbb{V}}(\overline{B}(0, r) \cap x^{-1}\Gamma).$$

Indeed, fix an element y of $P_{\mathbb{V}}(\overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(x^{-1}\Gamma)$. Thanks to our choice of y there are a $w_1 \in x^{-1}\Gamma$ and a $w_2 \in \overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)$ such that

$$P_{\mathbb{V}}(w_1) = y = P_{\mathbb{V}}(w_2).$$

Furthermore, since Γ is a $C_{\mathbb{V}}(\alpha)$ -set, we infer that $w_1 \in C_{\mathbb{V}}(\alpha)$ and thus with the same computations as in (1.27) we obtain that $\|P_{\mathbb{L}}(w_1)\| \leq \mathfrak{c}(\alpha)\|P_{\mathbb{V}}(w_1)\|$ and thus

$$(1.31) \quad \|w_1\| \leq (1 + \mathfrak{c}(\alpha))\|P_{\mathbb{V}}(w_1)\| \leq (1 + \mathfrak{c}(\alpha))\|y\|.$$

Furthermore, since by assumption $w_2 \in \overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)$, Lemma 1.43 yields

$$(1.32) \quad \|y\| = \|P_{\mathbb{V}}(w_2)\| \leq \mathfrak{c}(\alpha)r/(1 - \mathfrak{c}(\alpha)) = r/(1 + \mathfrak{c}(\alpha)).$$

The bounds (1.31) and (1.32) together imply that $\|w_1\| \leq r$, and thus $w_1 \in \overline{B}(0, r) \cap x^{-1}\Gamma$ and this concludes the proof of the inclusion (1.30). Finally (1.29), (1.30) imply

$$(1.33) \quad \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, r) \cap \Gamma)) \geq \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(x^{-1}\Gamma)).$$

Furthermore, for every Borel subset \mathcal{E} of \mathbb{G} we have $P_{\mathbb{V}}(x\mathcal{E}) = P_{\mathbb{V}}(xP_{\mathbb{V}}(\mathcal{E}))$, since for every $g \in \mathcal{E}$ we have the following simple equality $P_{\mathbb{V}}(xg) = P_{\mathbb{V}}(xP_{\mathbb{V}}g)$. Therefore, by using the latter observation and Proposition 1.32, we get, denoting with Ψ the map $\Psi(v) = P_{\mathbb{V}}(x^{-1}v)$ for every $v \in \mathbb{V}$, that

$$(1.34) \quad \begin{aligned} & \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, \mathfrak{c}(\alpha)r) \cap C_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(x^{-1}\Gamma)) \\ &= \mathcal{S}^h(P_{\mathbb{V}}(x^{-1}P_{\mathbb{V}}(\overline{B}(x, \mathfrak{c}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha))) \cap P_{\mathbb{V}}(x^{-1}P_{\mathbb{V}}(\Gamma))) \\ &= \mathcal{S}^h(\Psi(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{c}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha))) \cap \Psi(P_{\mathbb{V}}(\Gamma))) \\ &= \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{c}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma)). \end{aligned}$$

Joining together (1.33) and (1.34) gives the sought conclusion. \square

We conclude this subsection with a more detailed study of the co-normal Grassmannian. These results will turn out to be fundamental when approaching the Marstrand–Mattila rectifiability criterion in Chapter 3.

Proposition 1.45. *Let $\varepsilon_2 := \varepsilon_2(\mathbb{V}, \mathbb{L}) := C_3(\mathbb{V}, \mathbb{L})/2 > 0$. For every $\mathfrak{s} \in \mathfrak{S}(h)$ the function $\mathfrak{e} : Gr_{\leq}^{\mathfrak{s}}(h) \rightarrow \mathbb{R}$ defined as*

$$(1.35) \quad \mathfrak{e}(\mathbb{V}) := \sup\{\varepsilon_2(\mathbb{V}, \mathbb{L}) : \mathbb{L} \text{ is a normal complementary subgroup of } \mathbb{V}\},$$

is lower semicontinuous. Moreover the following conclusion holds

if $\mathcal{G} \subseteq Gr_{\leq}^{\mathfrak{s}}(h)$ is compact with respect to the distance $d_{\mathbb{G}}$ on the Grassmannian, then there exists a $\mathfrak{e}_{\mathcal{G}} > 0$ such that $\mathfrak{e}(\mathbb{V}) \geq \mathfrak{e}_{\mathcal{G}}$ for every $\mathbb{V} \in \mathcal{G} \subseteq Gr_{\leq}^{\mathfrak{s}}(h)$.

Proof. Let us prove that the function \mathfrak{e} is lower semicontinuous. Since $\varepsilon_2(\mathbb{V}, \mathbb{L}) = C_3(\mathbb{V}, \mathbb{L})/2$, see Lemma 1.40, it is enough to prove the proposition with $2\mathfrak{e}(\mathbb{V})$ instead of $\mathfrak{e}(\mathbb{V})$, and with $C_3(\mathbb{V}, \mathbb{L})$ instead of $\varepsilon_2(\mathbb{V}, \mathbb{L})$. Let us fix $\mathbb{V} \in Gr_{\geq}^{\mathfrak{s}}(h)$ and $0 < \varepsilon < \mathfrak{e}(\mathbb{V})$, and denote with \mathbb{L} one of the normal complementary subgroups of \mathbb{V} for which $C_3(\mathbb{V}, \mathbb{L}) > 2\mathfrak{e}(\mathbb{V}) - \varepsilon$. For every $\mathbb{W} \in Gr_{\geq}^{\mathfrak{s}}(h)$ thanks to Lemma 1.39 we have

$$(1.36) \quad C_{\mathbb{W}}(C_3(\mathbb{V}, \mathbb{L}) - 4d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) - \varepsilon) \subseteq C_{\mathbb{V}}(C_3(\mathbb{V}, \mathbb{L}) - \varepsilon),$$

whenever $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W})$ is small enough. Therefore if $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W})$ is sufficiently small, the latter inclusion and the same proof as in Lemma 1.40 imply that $\mathbb{L} \cap \mathbb{W} \subseteq \mathbb{L} \cap C_{\mathbb{V}}(C_3(\mathbb{V}, \mathbb{L}) - \varepsilon) = \{0\}$. Since $\mathbb{L} \cap \mathbb{W} = \{0\}$, \mathbb{L} and \mathbb{V} are complementary subgroups and \mathbb{V} and \mathbb{W} have the same stratification vector, and thus the same topological dimension, we have that \mathbb{L} is a complementary subgroup of \mathbb{W} for the same argument used in the proof of Proposition 1.35. Thus, taking (1.36) into account we get that $\mathbb{L} \cap C_{\mathbb{W}}(C_3(\mathbb{V}, \mathbb{L}) - 4d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) - \varepsilon) = \{0\}$ and thus, from Remark 1.41, we get that $C_3(\mathbb{W}, \mathbb{L}) \geq C_3(\mathbb{V}, \mathbb{L}) - 4d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) - \varepsilon$ whenever $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W})$ is sufficiently small. This implies that

$$2\mathfrak{e}(\mathbb{W}) \geq C_3(\mathbb{W}, \mathbb{L}) \geq C_3(\mathbb{V}, \mathbb{L}) - 4d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) - \varepsilon \geq 2\mathfrak{e}(\mathbb{V}) - 4d(\mathbb{V}, \mathbb{W}) - 2\varepsilon,$$

whenever $d_{\mathbb{G}}(\mathbb{V}, \mathbb{W})$ is small enough, and thus

$$\liminf_{d_{\mathbb{G}}(\mathbb{W}, \mathbb{V}) \rightarrow 0} \mathfrak{e}(\mathbb{W}) \geq \mathfrak{e}(\mathbb{V}) - \varepsilon,$$

from which the lower semicontinuity follows due to the arbitrariness of ε . The last conclusion follows since $\mathcal{G} \subseteq Gr_{\geq}^{\mathfrak{s}}(h)$ is compact and \mathfrak{e} is lower semicontinuous. \square

Remark 1.46. We observe that in proof of the previous proposition we did not use the fact that \mathbb{L} is normal, but we stated the proposition in this specific case since we are going to use this formulation in this work. The same proof works in the more general case when $\mathbb{V} \in Gr_c(h)$ with a fixed stratification vector \mathfrak{s} , and $\mathfrak{e}(\mathbb{V}) = \sup\{\varepsilon_2(\mathbb{V}, \mathbb{L}) : \mathbb{L} \text{ is a complementary subgroup of } \mathbb{V}\}$.

Proposition 1.47. *Let $C > 0$ and $\mathbb{V} \in Gr_{\geq}^{\mathfrak{s}}(h)$ be such that $\mathfrak{e}(\mathbb{V}) \geq C$. Then there exists a normal complementary subgroup \mathbb{L} of \mathbb{V} such that*

$$(1.37) \quad \|P_{\mathbb{V}}(g)\| \leq (1 + 2/C)\|g\|, \quad \text{and} \quad \|P_{\mathbb{L}}(g)\| \leq (2/C)\|g\|, \quad \text{for all } g \in \mathbb{G},$$

where we recall that $P_{\mathbb{V}}$ and $P_{\mathbb{L}}$ are the projections relative to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$.

Proof. Thanks to the definition of $\mathfrak{e}(\mathbb{V})$, see (1.35), there exists a normal complementary subgroup \mathbb{L} of \mathbb{V} such that $\varepsilon_1(\mathbb{V}, \mathbb{L}) \geq C/2$. Thus, arguing as in Lemma 1.40, we get $\mathbb{L} \cap C_{\mathbb{V}}(C/2) = \{0\}$. This implies, arguing as in Remark 1.41, that for every $g \in \mathbb{G}$ we have

$$(1.38) \quad C\|P_{\mathbb{L}}(g)\|/2 \leq \text{dist}(P_{\mathbb{L}}(g), \mathbb{V}) = \text{dist}(g, \mathbb{V}) \leq \|g\|.$$

Furthermore, thanks to the triangle inequality we have

$$\|g\| \geq \|P_{\mathbb{V}}(g)\| - \|P_{\mathbb{L}}(g)\| \geq \|P_{\mathbb{V}}(g)\| - (2/C)\|g\|,$$

thus concluding the proof of the proposition. \square

Proposition 1.48. *Let $\mathfrak{s} \in \mathfrak{S}(h)$. Let $C > 0$ and $\mathbb{V} \in Gr_{\geq}^{\mathfrak{s}}(h)$ be such that $\mathfrak{e}(\mathbb{V}) \geq C$. Let \mathbb{L} be a normal complementary subgroup of \mathbb{V} as in the statement of Proposition 1.47. Then the projection $P_{\mathbb{V}} : \mathbb{G} \rightarrow \mathbb{V}$ related to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$ is a $(1 + 2/C)$ -Lipschitz homogeneous homomorphism.*

Proof. Thanks to the fact that \mathbb{L} is normal, we have that for every $x, y \in \mathbb{G}$ the following equality holds

$$P_{\mathbb{V}}(xy) = P_{\mathbb{V}}(x_{\mathbb{V}}x_{\mathbb{L}}y_{\mathbb{V}}y_{\mathbb{L}}) = P_{\mathbb{V}}(x_{\mathbb{V}}y_{\mathbb{V}} \cdot y_{\mathbb{V}}^{-1}x_{\mathbb{L}}y_{\mathbb{V}} \cdot y_{\mathbb{L}}) = P_{\mathbb{V}}(x)P_{\mathbb{V}}(y).$$

Since $P_{\mathbb{V}}$ is always an homogeneous map, we infer that $P_{\mathbb{V}}$ is a homogeneous homomorphism. Moreover, from Proposition 1.47 we have that

$$\|P_{\mathbb{V}}(g)\| \leq (1 + 2/C)\|g\|,$$

for every $g \in \mathbb{G}$. Hence from the fact that $P_{\mathbb{V}}$ is a homomorphism we have

$$\|P_{\mathbb{V}}(x)^{-1}P_{\mathbb{V}}(y)\| = \|P_{\mathbb{V}}(x^{-1}y)\| \leq (1 + 2/C)\|x^{-1}y\|,$$

for every $x, y \in \mathbb{G}$ and thus $P_{\mathbb{V}}$ is $(1 + 2/C)$ -Lipschitz. \square

Remark 1.49. Notice that in the proof of the above proposition we proved that whenever \mathbb{L} is normal, then $P_{\mathbb{V}}$ is a homogeneous homomorphism.

Definition 1.50 (Cylinder). Let \mathbb{V}, \mathbb{L} be two complementary subgroups of \mathbb{G} . For every $u \in \mathbb{G}$, and $r > 0$ we define

$$T(u, r) := P_{\mathbb{V}}^{-1}(P_{\mathbb{V}}(\overline{B}(u, r))).$$

In the following proposition we study the structure of cylinders $T(\cdot, \cdot)$ when \mathbb{L} is normal.

Proposition 1.51. *Let $\mathfrak{s} \in \mathfrak{S}(h)$. Let $C > 0$ and $\mathbb{V} \in Gr_{\leq \mathfrak{s}}(h)$ be such that $\mathfrak{e}(\mathbb{V}) \geq C$. Let \mathbb{L} be a normal complementary subgroup of \mathbb{V} as in Proposition 1.47. Thus, for every $u \in \mathbb{G}$ we have $T(u, r) = P_{\mathbb{V}}(u)\delta_r T(0, 1)$. Furthermore, we have*

$$T(u, r) \subseteq P_{\mathbb{V}}(u)\delta_r P_{\mathbb{V}}^{-1}(\overline{B}(0, (1 + 2/C)) \cap \mathbb{V}) = P_{\mathbb{V}}^{-1}(\overline{B}(P_{\mathbb{V}}(u), (1 + 2/C)r) \cap \mathbb{V}).$$

Finally, for every $h \in \mathbb{L}$ we have $\overline{B}(uh, r) \subseteq T(u, r)$.

Proof. First of all, we note that thanks to Proposition 1.48 we have that $w \in P_{\mathbb{V}}(\overline{B}(u, r))$ if and only if there exists a $v \in \overline{B}(0, 1)$ such that $w = P_{\mathbb{V}}(u)\delta_r P_{\mathbb{V}}(v)$. Therefore, given $u \in \mathbb{G}$ and $r > 0$, we have that $y \in T(u, r)$ if and only if $y = P_{\mathbb{V}}(u)\delta_r P_{\mathbb{V}}(v)h$ for some $v \in \overline{B}(0, 1)$ and $h \in \mathbb{L}$. Thus we conclude that $T(u, r) = P_{\mathbb{V}}(u)\delta_r T(0, 1)$ for every $u \in \mathbb{G}$ and $r > 0$.

Secondly, thanks to Proposition 1.48 we infer that $P_{\mathbb{V}}(\overline{B}(0, 1)) \subseteq \mathbb{V} \cap \overline{B}(0, (1 + 2/C))$ and thus combining such inclusion with the first part of the proposition we deduce that

$$T(u, r) \subseteq P_{\mathbb{V}}(u)\delta_r P_{\mathbb{V}}^{-1}(\overline{B}(0, (1 + 2/C)) \cap \mathbb{V}) = P_{\mathbb{V}}^{-1}(\overline{B}(P_{\mathbb{V}}(u), (1 + 2/C)r) \cap \mathbb{V}),$$

where the last equality is true since $P_{\mathbb{V}}$ is a homogeneous homomorphism. Finally, thanks to the first part of the proposition, for every $u \in \mathbb{V}$ and every $h \in \mathbb{L}$ we have

$$\overline{B}(uh, r) \subseteq T(uh, r) = T(u, r),$$

and this concludes the proof of the proposition. \square

3. Rectifiable measures in Carnot groups

In this section we are going to introduce and discuss the notion of \mathcal{P} -rectifiable measure in Carnot groups. This definition was first given in the work [180], and later extensively studied in the papers [28, 33]. Hence, the main references for this section are [28, 33, 180].

We recall that throughout this section \mathbb{G} will be a fixed Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance d , and $1 \leq h \leq Q$.

3.1. \mathcal{P} -rectifiable measures. On a Carnot group we have a natural family of dilations, see Section 2. Hence, we can define the notion of tangent measures to an arbitrary Radon measure. This notion mimics the classical one in the Euclidean setting, see [89, 202].

Definition 1.52 (Tangent measures). Let ϕ be a Radon measure on \mathbb{G} . For every $x \in \mathbb{G}$ and every $r > 0$ we define the measure

$$T_{x,r}\phi(E) := \phi(x \cdot \delta_r(E)), \quad \text{for every Borel set } E.$$

Furthermore, we define $\text{Tan}(\phi, x)$, the *tangent measures to ϕ at x* , to be the collection of the *non-null* Radon measures ν for which there is a sequence $\{r_i\}_{i \in \mathbb{N}}$, with $r_i \rightarrow 0$, and a sequence $\{c_i\}_{i \in \mathbb{N}}$, with $c_i > 0$, such that

$$c_i T_{x,r_i}\phi \rightharpoonup \nu.$$

Moreover, we define $\text{Tan}_h(\phi, x)$, the *h -tangent measures to ϕ at x* , to be the collection of Radon measures ν for which there is a sequence $\{r_i\}_{i \in \mathbb{N}}$, with $r_i \rightarrow 0$, such that

$$r_i^{-h} T_{x,r_i}\phi \rightharpoonup \nu.$$

Remark 1.53. (Zero as a tangent measure) We remark that in our definition, the zero measure could be an element of $\text{Tan}_h(\phi, x)$ as in [89], while in [202] and [178] this is excluded by definition.

Lemma 1.54. *Assume ϕ is a Radon measure on \mathbb{G} and suppose that, for a sequence $r_i \rightarrow 0$, $r_i^{-h} T_{x,r_i}\phi \rightharpoonup \nu$. Then, for every $z \in \text{supp}(\nu)$ there exists a sequence $y_i \in \text{supp}(\phi)$ such that $\delta_{1/r_i}(x^{-1}y_i) \rightarrow z$.*

Proof. A simple argument by contradiction yields the claim. The proof follows verbatim as its Euclidean analogue, see for instance [89, Proof of proposition 3.4]. \square

Let ϕ be a Radon measure on \mathbb{G} . We stress that whenever the h -lower density of ϕ is strictly positive, and the h -upper density of ϕ is finite ϕ -almost everywhere, the set $\text{Tan}_h(\phi, x)$ is nonempty for ϕ -almost every $x \in \mathbb{G}$. More in general, if ϕ is asymptotically doubling, the set $\text{Tan}(\phi, x)$ is nonempty for ϕ -almost every $x \in \mathbb{G}$. The latter two observations are consequences of the argument in [174, Theorem 14.3], see also [202, Theorem 2.5], together with [12, Theorem 1.59]. We briefly recall here the fact that the tangents are local and that the density is natural with respect to restrictions to Borel sets.

Proposition 1.55. *Let ϕ be an asymptotically doubling Radon measure on \mathbb{G} . Then*

- (i) *for every Borel set $B \subseteq \mathbb{G}$ the measure $\phi \llcorner B$ is an asymptotically doubling measure, and we have that for every $h \geq 0$, the following equalities hold for ϕ -almost every $x \in B$*

$$\Theta_*^h(\phi \llcorner B, x) = \Theta_*^h(\phi, x), \quad \text{and} \quad \Theta^{h,*}(\phi \llcorner B, x) = \Theta^{h,*}(\phi, x),$$

- (ii) *for every non-negative $\rho \in L^1(\phi)$, and for ϕ -almost every $x \in \mathbb{G}$, we have*

$$\text{Tan}(\rho\phi, x) = \rho(x)\text{Tan}(\phi, x).$$

More precisely, for ϕ -almost every $x \in \mathbb{G}$ the following holds

$$(1.39) \quad \begin{aligned} &\text{if } r_i \rightarrow 0 \text{ is such that } c\phi(\overline{B}(x, r_i))^{-1} T_{x,r_i}\phi \rightharpoonup \nu, \\ &\text{then } c\phi(\overline{B}(x, r_i))^{-1} T_{x,r_i}(\rho\phi) \rightharpoonup \rho(x)\nu. \end{aligned}$$

- (iii) *Suppose there exists $h \geq 0$ with $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in \mathbb{G}$. Then, for every non-negative $\rho \in L^1(\phi)$, and for ϕ -almost every $x \in \mathbb{G}$, we have $\text{Tan}_h(\rho\phi, x) = \rho(x)\text{Tan}_h(\phi, x)$. More precisely, for ϕ -almost every $x \in \mathbb{G}$ the following holds*

$$(1.40) \quad \text{if } r_i \rightarrow 0 \text{ is such that } r_i^{-h} T_{x,r_i}\phi \rightharpoonup \nu \text{ then } r_i^{-h} T_{x,r_i}(\rho\phi) \rightharpoonup \rho(x)\nu.$$

Proof. The item (i) is a direct consequence of Lebesgue Differentiation Theorem of [125, page 77], that can be applied since (\mathbb{G}, d, ϕ) is a Vitali metric measure space due to [125, Theorem 3.4.3].

Let us pass to the item (ii). Notice that, since ϕ is asymptotically doubling, hence for ϕ -almost every $x \in \mathbb{G}$, and every $\nu \in \text{Tan}(\phi, x)$, there exists $c > 0$ such that

$$c\phi(\overline{B}(x, r_i))^{-1}T_{x, r_i}\phi \rightharpoonup \nu.$$

The latter assertion is a consequence of the very same argument in [174, Remark 14.4(3)]. Now the proof of item (ii) can be concluded arguing verbatim as in [89, Proposition 3.12] and exploiting that ϕ is asymptotically doubling. Indeed, the proof in [89] only relies on the Lebesgue Differentiation Theorem [125, page 77], which is true because (\mathbb{G}, d, ϕ) is a Vitali metric measure space due to [125, Theorem 3.4.3]. The proof of item (iii) is analogous to the proof of item (ii). \square

Before going on, we state here a useful proposition about the structure of Radon measures with positive h -lower density, and finite h -upper density almost everywhere.

Proposition 1.56 (See [180, Proposition 1.17 and Corollary 1.18]). *Let ϕ be a Radon measure supported on a compact set of a Carnot group \mathbb{G} . Assume there exists $h > 0$ such that $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in \mathbb{G}$.*

For every $\vartheta, \gamma \in \mathbb{N}$ we have that $\phi \llcorner E(\vartheta, \gamma)$ is mutually absolutely continuous with respect to $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$, where $E(\vartheta, \gamma)$ is defined in Definition 1.9.

We now introduce the notion of flat measures on a Carnot group.

Definition 1.57 (Flat measures). For every $h \in \{1, \dots, Q\}$ we let $\mathfrak{M}(h)$ to be the family of flat h -dimensional measures in \mathbb{G} , i.e.,

$$\mathfrak{M}(h) := \{\lambda \mathcal{S}^h \llcorner \mathbb{W} : \text{for some } \lambda > 0 \text{ and } \mathbb{W} \in \text{Gr}(h)\}.$$

Furthermore, if G is a subset of the h -dimensional Grassmannian $\text{Gr}(h)$, we let $\mathfrak{M}(h, G)$ to be the set

$$(1.41) \quad \mathfrak{M}(h, G) := \{\lambda \mathcal{S}^h \llcorner \mathbb{W} : \text{for some } \lambda > 0 \text{ and } \mathbb{W} \in G\}.$$

We stress that in the previous definitions we can use any of the Haar measures on \mathbb{W} , see Proposition 1.24, since they are the same up to a constant.

The following definition has been proposed in [180]. The study of the notions given in Definition 1.58 and Definition 1.61 is at the core of Chapter 2 and Chapter 3 of this thesis.

Definition 1.58 (\mathcal{P}_h and \mathcal{P}_h^* -rectifiable measures). Let $h \in \{1, \dots, Q\}$. A Radon measure ϕ on \mathbb{G} is said to be a \mathcal{P}_h -rectifiable measure if for ϕ -almost every $x \in \mathbb{G}$ we have

- (i) $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < +\infty$,
- (ii) there exists a $\mathbb{V}(x) \in \text{Gr}(h)$ such that $\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda \geq 0\}$.

Furthermore, we say that ϕ is \mathcal{P}_h^* -rectifiable if (ii) is replaced with the weaker

$$(ii)^* \quad \text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V} : \lambda \geq 0 \text{ and } \mathbb{V} \in \text{Gr}(h)\}.$$

Remark 1.59. (About $\lambda = 0$ in Definition 1.58) It is readily noticed that, since in Definition 1.58 we are asking $\Theta_*^h(\phi, x) > 0$ for ϕ -almost every x , we can not have the zero measure as a tangent measure. As a consequence, a posteriori, we have that in item (ii) and item (ii)* above we can restrict to $\lambda > 0$. We will tacitly work in this restriction from now on.

On the contrary, if we only know that for ϕ -almost every $x \in \mathbb{G}$ we have

$$(1.42) \quad \Theta^{h,*}(\phi, x) < +\infty, \quad \text{and} \quad \text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda > 0\},$$

for some $\mathbb{V}(x) \in \text{Gr}(h)$, hence $\Theta_*^h(\phi, x) > 0$ for ϕ -almost every $x \in \mathbb{G}$, and the same property holds with the item (ii)* above. Indeed, if at some x for which (1.42) holds we have $\Theta_*^h(\phi, x) =$

0, then there exists $r_i \rightarrow 0$ such that $r_i^{-h}\phi(\overline{B}(x, r_i)) = 0$. Since $\Theta^{h,*}(\phi, x) < +\infty$, up to subsequences (see [12, Theorem 1.60]), we have $r_i^{-h}T_{x, r_i}\phi \rightarrow \lambda\mathcal{S}^h\llcorner\mathbb{V}(x)$, for some $\lambda > 0$. Hence, by applying [12, Proposition 1.62(b)] we conclude that $r_i^{-h}T_{x, r_i}\phi(\overline{B}(0, 1)) \rightarrow \lambda\mathcal{S}^h\llcorner\mathbb{V}(x)(\overline{B}(0, 1)) > 0$, that is a contradiction.

Remark 1.60 (About the rectifiability of Hausdorff measures). We observe that if Γ is a Borel set in \mathbb{G} with $0 < \mathcal{S}^h(\Gamma) < +\infty$, $\mathcal{S}^h\llcorner\Gamma$ is \mathcal{P}_h -rectifiable if and only if $\mathcal{C}^h\llcorner\Gamma$ (or $\mathcal{H}^h\llcorner\Gamma$) is \mathcal{P}_h -rectifiable. This is because $\mathcal{S}^h, \mathcal{H}^h, \mathcal{C}^h$ are equivalent measures (see Definition 1.1), the \mathcal{P}_h -rectifiability implies being asymptotically doubling, and then we can transfer the property of being \mathcal{P}_h -rectifiable from one measure to the other by using Lebesgue–Radon–Nikodým theorem (see [125, page 82]) and the locality of tangents in Proposition 1.55.

Let us briefly comment on why one should be interested in the study of the notion of \mathcal{P} -rectifiability. The definition of \mathcal{P} -rectifiable measure is natural in the setting of Carnot groups. Indeed, we have on \mathbb{G} a family of dilations $\{\delta_\lambda\}_{\lambda>0}$, see Section 2, that we can use to give a good definition of blow-up of a measure. Hence we ask, for a measure to be rectifiable, that the tangents are flat. The natural class of flat spaces, i.e., the analogous of vector subspaces of the Euclidean space, seems to be the class of homogeneous subgroups of \mathbb{G} , see Definition 1.57. This latter assertion is suggested also from the result in [175, Theorem 3.2] according to which on every locally compact group G endowed with dilations and isometric left translations, if a Radon measure μ has a unique (up to multiplicative constants) tangent μ -almost everywhere then this tangent is μ -almost everywhere (up to multiplicative constants) the left Haar measure of a closed dilation-invariant subgroup of G . As a consequence, in the definition of \mathcal{P}_h -rectifiable measure we can equivalently substitute item (ii) of Definition 1.58 with the weaker requirement

$$(ii)' \text{ Tan}_h(\phi, x) \subseteq \{\lambda\nu_x : \lambda > 0\}, \text{ where } \nu_x \text{ is a Radon measure on } \mathbb{G}.$$

Moreover, we stress that if a metric group is locally compact, isometrically homogeneous and admits one dilation, as it is for the class of metric group studied in [175], and moreover the distance is geodesic, then it is a sub-Finsler Carnot group, see [144, Theorem 1.1], and [86, Theorem D] for a generalization and related studies.

We are now going to define different subclasses of \mathcal{P}_h and \mathcal{P}_h^* -rectifiable measures. More precisely we give the following definition.

Definition 1.61 (Subclasses of \mathcal{P}_h and \mathcal{P}_h^* -rectifiable measures). Let $h \in \{1, \dots, Q\}$. In the following we denote by \mathcal{P}_h^c the family of those \mathcal{P}_h -rectifiable measures such that for ϕ -almost every $x \in \mathbb{G}$ we have

$$\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_c(h)),$$

namely at ϕ -almost every point the tangent measures are supported on complemented homogeneous subgroups. Furthermore, the family of those \mathcal{P}_h^* -rectifiable measures ϕ such that for ϕ -almost every $x \in \mathbb{G}$ we have

- (i) $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_c(h))$ is denoted by $\mathcal{P}_h^{*,c}$,
- (ii) $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_{\triangleleft}(h))$ is denoted by $\mathcal{P}_h^{*,\triangleleft}$,
- (iii) $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_{\triangleleft}^{\mathfrak{s}}(h))$, for some $\mathfrak{s} \in \mathfrak{S}(h)$, is denoted by $\mathcal{P}_h^{*,\triangleleft,\mathfrak{s}}$.

We have the following simple lemma.

Lemma 1.62. *Let $h \in \{1, \dots, Q\}$ and assume ϕ is a Radon measure on \mathbb{G} . If $\{r_i\}_{i \in \mathbb{N}}$ is an infinitesimal sequence such that $r_i^{-h}T_{x, r_i}\phi \rightarrow \lambda\mathcal{C}^h\llcorner\mathbb{V}$ for some $\lambda > 0$ and $\mathbb{V} \in Gr(h)$ then*

$$\lim_{i \rightarrow \infty} \phi(\overline{B}(x, r_i))/r_i^h = \lambda.$$

Proof. Since $\mathcal{C}^h \llcorner \mathbb{V}(x)(\partial \overline{B}(0, 1)) = 0$, see e.g., [132, Lemma 3.5], thanks to Remark 1.27 and to [12, Proposition 1.62(b)] we have

$$\lambda = \lambda \mathcal{C}^h \llcorner \mathbb{V}(x)(\overline{B}(0, 1)) = \lim_{i \rightarrow \infty} \frac{T_{x, r_i} \phi(\overline{B}(0, 1))}{r_i^h} = \lim_{i \rightarrow \infty} \frac{\phi(\overline{B}(x, r_i))}{r_i^h},$$

and this concludes the proof. \square

The above lemma has the following immediate consequence.

Corollary 1.63. *Let $h \in \{1, \dots, Q\}$ and assume ϕ is a \mathcal{P}_h^* -rectifiable measure. Then for ϕ -almost every $x \in \mathbb{G}$ we have*

$$\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{C}^h \llcorner \mathbb{W} : \lambda \in [\Theta_*^h(\phi, x), \Theta^{h,*}(\phi, x)] \text{ and } \mathbb{W} \in \text{Gr}(h)\}.$$

We prove the following compactness result that will be of crucial importance in the proof of the co-normal Marstrand–Mattila rectifiability criterion in Chapter 3.

Proposition 1.64. *Let $h \in \{1, \dots, Q\}$ and assume ϕ is a \mathcal{P}_h^* -rectifiable measure. Then, for ϕ -almost all $x \in \mathbb{G}$ the set $\text{Tan}_h(\phi, x)$ is weak* compact.*

Proof. Since the statement of the proposition does not depend on the choice of the left-invariant homogeneous distance on \mathbb{G} , we assume that \mathbb{G} is endowed with the left-invariant homogeneous distance induced by the box norm in Definition 1.16.

Let $x \in \mathbb{G}$ be such that $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ and $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)$. We now prove that for every sequence $\{\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j\}_{j \in \mathbb{N}} \subseteq \text{Tan}_h(\phi, x)$, there are a $\lambda > 0$ and $\mathbb{V} \in \text{Gr}(h)$ such that, up to non-relabelled subsequences we have

$$\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j \rightharpoonup \lambda \mathcal{C}^h \llcorner \mathbb{V}.$$

Indeed, thanks to Corollary 1.63 we have that $\lambda_j \in [\Theta_*^h(\phi, x), \Theta^{h,*}(\phi, x)]$ for every $j \in \mathbb{N}$ and thus we can assume without loss of generality that

$$\lambda_j \rightarrow \lambda \in [\Theta_*^h(\phi, x), \Theta^{h,*}(\phi, x)],$$

up to a non-relabelled subsequence. Furthermore, thanks to Proposition 1.22 there exists a $\mathbb{V} \in \text{Gr}(h)$ such that $\mathbb{V}_j \rightarrow \mathbb{V}$ with respect to the Grassmannian metric $d_{\mathbb{G}}$. Thus, thanks to Proposition 1.30 and a simple computation that we omit, we conclude that

$$\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j \rightharpoonup \lambda \mathcal{C}^h \llcorner \mathbb{V}.$$

Since we assumed $\{\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j\} \subseteq \text{Tan}_h(\phi, x)$ then, for every $j \in \mathbb{N}$ there is an infinitesimal sequence $\{r_\ell(j)\}_{\ell \in \mathbb{N}}$ such that

$$r_\ell(j)^{-h} T_{x, r_\ell(j)} \phi \rightharpoonup \lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j.$$

Thus, the forthcoming Lemma 1.73 implies that $\lim_{\ell \rightarrow \infty} F_{0,1}(r_\ell(j)^{-h} T_{x, r_\ell(j)} \phi, \lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j) = 0$, and in particular for every $j \in \mathbb{N}$ there exists an $\ell_j \in \mathbb{N}$ such that defined $\tau_j := r_{\ell_j}(j)$ we have that $\tau_j \rightarrow 0$, and

$$F_{0,1}(\tau_j^{-h} T_{x, \tau_j} \phi, \lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j) \leq 1/j.$$

Since $\limsup_{j \rightarrow \infty} \tau_j^{-h} T_{x, \tau_j} \phi(\overline{B}(0, r)) \leq \Theta^{h,*}(\phi, x) r^h$ for every $r > 0$, thanks to [12, Corollary 1.60], we can assume without loss of generality that there exists a Radon measure ν such that $\tau_j^{-h} T_{x, \tau_j} \phi \rightharpoonup \nu$. On the other hand, by definition we have that $\nu \in \text{Tan}_h(\phi, x)$ and thus by hypothesis on ϕ there is a $\eta > 0$ and a $\mathbb{W} \in \text{Gr}(h)$ such that $\nu = \eta \mathcal{C}^h \llcorner \mathbb{W}$. This implies

that for every $j \in \mathbb{N}$ we have

$$\begin{aligned} F_{0,1}(\eta\mathcal{C}^h \llcorner \mathbb{W}, \lambda\mathcal{C}^h \llcorner \mathbb{V}) &\leq F_{0,1}(\eta\mathcal{C}^h \llcorner \mathbb{W}, \mathfrak{r}_j^{-h} T_{x,\mathfrak{r}_j} \phi) + F_{0,1}(\mathfrak{r}_j^{-h} T_{x,\mathfrak{r}_j} \phi, \lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j) \\ &\quad + F_{0,1}(\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j, \lambda\mathcal{C}^h \llcorner \mathbb{V}) \\ &\leq F_{0,1}(\eta\mathcal{C}^h \llcorner \mathbb{W}, \mathfrak{r}_j^{-h} T_{x,\mathfrak{r}_j} \phi) + 1/j + F_{0,1}(\lambda_j \mathcal{C}^h \llcorner \mathbb{V}_j, \lambda\mathcal{C}^h \llcorner \mathbb{V}). \end{aligned}$$

The arbitrariness of j and Lemma 1.73 implies that $F_{0,1}(\eta\mathcal{C}^h \llcorner \mathbb{W}, \lambda\mathcal{C}^h \llcorner \mathbb{V}) = 0$ and since flat measures are cones, we conclude that $\eta\mathcal{C}^h \llcorner \mathbb{W} = \lambda\mathcal{C}^h \llcorner \mathbb{V}$. This shows that $\lambda\mathcal{C}^h \llcorner \mathbb{V} \in \text{Tan}_h(\phi, x)$ and then the proof is concluded. \square

3.2. Quantifying the distance of a Radon measure from flat measures. In this subsection we are going to define functionals that in some sense tell how far is a measure from being h -flat around a point $x \in \mathbb{G}$ and at a certain scale $r > 0$.

We first introduce now a way to estimate how far two measures are, cf. [202, 1.9(2)].

Definition 1.65 (Definition of F_K). Given ϕ and ψ two Radon measures on \mathbb{G} , and given $K \subseteq \mathbb{G}$ a compact set, we define

$$(1.43) \quad F_K(\phi, \psi) := \sup \left\{ \left| \int f d\phi - \int f d\psi \right| : f \in \text{Lip}_1^+(K) \right\},$$

where $\text{Lip}_1^+(K)$ is the set of nonnegative 1-Lipschitz functions supported on K . We also write $F_{x,r}$ for $F_{\overline{B}(x,r)}$.

Remark 1.66. With few computations that we omit, it is easy to see that $F_{x,r}(\phi, \psi) = rF_{0,1}(T_{x,r}\phi, T_{x,r}\psi)$. Furthermore, F_K enjoys the triangle inequality. Indeed if ϕ_1, ϕ_2, ϕ_3 are Radon measures and $f \in \text{Lip}_1^+(K)$, then

$$\begin{aligned} \left| \int f d\phi_1 - \int f d\phi_2 \right| &\leq \left| \int f d\phi_1 - \int f d\phi_3 \right| + \left| \int f d\phi_3 - \int f d\phi_2 \right| \\ &\leq F_K(\phi_1, \phi_3) + F_K(\phi_3, \phi_2). \end{aligned}$$

The arbitrariness of f lets us conclude that $F_K(\phi_1, \phi_2) \leq F_K(\phi_1, \phi_3) + F_K(\phi_3, \phi_2)$.

Let us introduce a Gauge for Radon measures on Carnot groups, cf. [202, 1.9(1)].

Definition 1.67 (Definition of F_r). For a given Radon measure ϕ on \mathbb{G} and for $r > 0$, let us define $F_r(\phi) := \int \text{dist}(z, B(0, r)^c) d\phi(z)$.

Lemma 1.68. For every Radon measure ϕ on \mathbb{G} and every $r > 0$ we have that $F_r(\phi) = F_{0,r}(\phi, 0)$.

Proof. It is immediate to see that $F_{0,r}(\phi, 0) \geq F_r(\phi)$ for every $r > 0$. In order to prove the vice-versa, note that for every $f \in \text{Lip}_1^+(\overline{B}(0, r))$ we have that $f|_{\partial B(0, r)} = 0$. Thanks to this observation, for every $y \in \overline{B}(0, r)$ if we let $x \in \partial B(0, r)$ be a point of minimal distance of y from $B(0, r)^c$ we have

$$f(y) = |f(y) - f(x)| \leq d(y, x) = \text{dist}(y, B(0, r)^c),$$

and this finally shows that $F_{0,r}(\phi, 0) = F_r(\phi)$, concluding the proof of the lemma. \square

In the following proposition we show a distance that metrizes the weak* convergence of Radon measures.

Proposition 1.69. Let \mathcal{M} be the set of Radon measures on \mathbb{G} . The function defined on $\mathcal{M} \times \mathcal{M}$ as

$$D(\phi, \psi) := \sum_{p=0}^{\infty} 2^{-p} \min\{1, F_{0,p}(\phi, \psi)\},$$

is a distance, and (\mathcal{M}, D) is a separable metric space. The topology induced by D on \mathcal{M} coincides with the weak* topology.

Moreover, let us assume $\{\phi_i\}_{i \in \mathbb{N}}$ is a sequence of Radon measures such that

$$\limsup_{i \rightarrow \infty} \phi_i(\overline{B}(0, r)) < \infty,$$

for every $r > 0$. Then $\{\phi_i\}_{i \in \mathbb{N}}$ has a converging subsequence with respect to the weak* topology.

Proof. The result is stated in [202, Proposition 1.12] in the Euclidean case, but the proof works verbatim for Radon measures on Carnot groups. \square

Proposition 1.70. *The function $F_{0,1}(\cdot, \cdot)$ is a metric on $\mathfrak{B}(h) := \{\psi \in \mathfrak{M}(h) : F_1(\psi) = 1\}$ and $(\mathfrak{B}(h), F_{0,1})$ is a compact metric space.*

Proof. First of all, we note that for every $\mu, \nu \in \mathfrak{B}(h)$ we have that $F_{0,1}(\mu, \nu) = 0$ if and only if $\mu = \nu$ and this is an immediate consequence of the fact that μ and ν are cones. Symmetry follows directly from the definition, and the triangle inequality follows from Remark 1.66.

We are left to show that $(\mathfrak{B}(h), F_{0,1})$ is a compact metric space. Let Ψ_i be a sequence in $\mathfrak{B}(h)$ and note that since $\mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(0, 1)) = 1$ for every $\mathbb{V} \in Gr(h)$, because of Remark 1.27, we deduce that $\Psi_i = (h+1)\mathcal{C}^h \llcorner \mathbb{V}_i$ for some $\mathbb{V}_i \in Gr(h)$, due to Proposition 1.29. Thus, we can find a (non-reabeled) subsequence of the homogeneous subgroups \mathbb{V}_i that converges to some $\mathbb{V} \in Gr(h)$ in the Hausdorff metric thanks to the compactness of the Grassmannian $Gr(h)$, see Proposition 1.22. Hence, by Proposition 1.30 we infer that $\Psi_i \rightharpoonup (h+1)\mathcal{C}^h \llcorner \mathbb{V} \in \mathfrak{B}(h)$ and therefore the compactness follows. \square

We are now ready to define a functional that in some sense tells us how far is a measure from being h -flat. The following definition is inspired by [202, 2.1(3)].

Definition 1.71 (Definition of $d_{x,r}$). For every $x \in \mathbb{G}$, every $h \in \{1, \dots, Q\}$ and every $r > 0$ we define the functional

$$(1.44) \quad d_{x,r}(\phi, \mathfrak{M}(h)) := \inf_{\substack{\Theta > 0, \\ \mathbb{V} \in Gr(h)}} \frac{F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x \mathbb{V})}{r^{h+1}}.$$

Furthermore, if G is a subset of the h -dimensional Grassmannian $Gr(h)$, we also define

$$d_{x,r}(\phi, \mathfrak{M}(h, G)) := \inf_{\substack{\Theta > 0, \\ \mathbb{V} \in G}} \frac{F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x \mathbb{V})}{r^{h+1}}.$$

Remark 1.72. It is a routine computation to prove that, whenever $h \in \mathbb{N}$ and $r > 0$ are fixed, the function $x \mapsto d_{x,r}(\phi, \mathfrak{M}(h, G))$ is a continuous function. The proof can be reached as in [180, Item (ii) of Proposition 2.2]. Moreover, from the invariance property in Remark 1.66 and Proposition 1.24, if in (1.44) we use the measure $\mathcal{C}^h \llcorner x \mathbb{V}$ instead of $\mathcal{S}^h \llcorner x \mathbb{V}$ we obtain the same definition.

Let us now prove the following criterion of flatness of h -tangents. We first need an auxiliary lemma that is inspired by [202, Proposition 1.11]. The reader might find a complete proof in [180, Proposition 1.10]. Here, we omit it.

Lemma 1.73. *Let $\{\mu_i\}$ be a sequence of Radon measures on \mathbb{G} . Let μ be a Radon measure on \mathbb{G} . The following are equivalent*

- (1) $\mu_i \rightharpoonup \mu$;
- (2) $F_K(\mu_i, \mu) \rightarrow 0$, for every $K \subseteq \mathbb{G}$ compact.

Proposition 1.74. *Let ϕ be a Radon measure on \mathbb{G} . Further, let G be a subfamily of $Gr(h)$ and let $\mathfrak{M}(h, G)$ be the set defined in (1.41). If at a point $x \in \mathbb{G}$ for which item (i) in Definition 1.58 holds we have $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, G)$, then for every $k > 0$ we have*

$$\lim_{r \rightarrow 0} d_{x,kr}(\phi, \mathfrak{M}(h, G)) = 0.$$

Proof. Let us fix $x \in \mathbb{G}$ a point for which $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, G)$ and let us assume by contradiction that there exist $k > 0$ and $r_i \rightarrow 0$ such that, for some $\varepsilon > 0$, we have

$$(1.45) \quad d_{x,kr_i}(\phi, \mathfrak{M}(h, G)) > \varepsilon.$$

Since ϕ and x satisfy the hypotheses in item (i) in Definition 1.58, we can use [12, Proposition 1.62(b)] and then, up to subsequences, there are $\Theta^* > 0$ and $\mathbb{V}^* \in G$ such that

$$(1.46) \quad r_i^{-h} T_{x,r_i} \phi \rightarrow \Theta^* \mathcal{S}^h \llcorner \mathbb{V}^*.$$

Thus,

$$d_{x,kr_i}(\phi, \mathfrak{M}(h, G)) = d_{0,k}(r_i^{-h} T_{x,r_i} \phi, \mathfrak{M}(h, G)) \leq k^{-h-1} F_{0,k}(r_i^{-h} T_{x,r_i} \phi, \Theta^* \mathcal{S}^h \llcorner \mathbb{V}^*) \rightarrow 0,$$

where the first equality follows from the first part of Remark 1.66, and the last convergence follows from (1.46), and Lemma 1.73. This is in contradiction with (1.45). \square

The following proposition is an adaptation of [202, 4.4(4)] and it will be crucial in the proof of Marstrand–Mattila’s rectifiability criterion in Chapter 3.

Proposition 1.75. *Let $h \in \{1, \dots, Q\}$, and $\vartheta, \gamma \in \mathbb{N}$. Let ϕ be a Radon measure supported on a compact set of \mathbb{G} , and let $G \subseteq Gr(h)$. Assume there exists an $x \in E(\vartheta, \gamma)$ (cf. Definition 1.9), a $\sigma \in (0, 2^{-10(h+1)}\vartheta^{-1})$ and a $0 < t < 1/(2\gamma)$ such that*

$$d_{x,t}(\phi, \mathfrak{M}(h, G)) \leq \sigma^{h+4}.$$

Then, there is a $\mathbb{V} \in G$ such that

(i) *whenever $y, z \in \overline{B}(x, t/2) \cap x\mathbb{V}$ and $\sigma t \leq r, s \leq t/2$ we have*

$$\phi(\overline{B}(y, r) \cap \overline{B}(x\mathbb{V}, \sigma^2 t)) \geq (1 - 2^{10(h+1)}\vartheta\sigma)(r/s)^h \phi(\overline{B}(z, s));$$

(ii) *furthermore, let us assume the homogeneous subgroup \mathbb{V} yielded by item (i) above admits a complementary normal subgroup \mathbb{L} , and denote by $P_{\mathbb{V}}$ the splitting projection on \mathbb{V} according to this splitting. Then, for every $k > 0$ with $\sigma k < 2^{-10h}\vartheta^{-1}$, if we define $T_{\mathbb{V}}(0, t/4k) := P_{\mathbb{V}}^{-1}(P_{\mathbb{V}}(\overline{B}(0, t/4k)))$ we have*

$$\phi(\overline{B}(x, t/4) \cap xT_{\mathbb{V}}(0, t/4k)) \leq (1 + 4\sigma(2kh + 1))\mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1)))k^{-h}\phi(\overline{B}(x, t/4)).$$

Proof. First of all, we notice that by the definition of $d_{x,t}(\phi, \mathfrak{M}(h, G))$ there exist $\mathbb{V} \in G$ and $\lambda > 0$ such that

$$F_{x,t}(\phi, \lambda \mathcal{C}^h \llcorner x\mathbb{V}) \leq \sigma^{h+3}t^{h+1}.$$

Proof of (i). The key of the proof of item (i) is to show that for every $w \in \overline{B}(x, t/2) \cap x\mathbb{V}$, every $\tau \in (0, t/2]$ and every $\rho \in (0, \tau]$ we have

$$(1.47) \quad \phi(\overline{B}(w, \tau)) \leq \lambda \mathcal{C}^h \llcorner (x\mathbb{V})(\overline{B}(w, \tau + \rho)) + \sigma^{h+3}t^{h+1}/\rho,$$

$$(1.48) \quad \lambda \mathcal{C}^h \llcorner (x\mathbb{V})(\overline{B}(w, \tau - \rho)) \leq \phi(\overline{B}(w, \tau) \cap \overline{B}(x\mathbb{V}, \rho)) + \sigma^{h+3}t^{h+1}/\rho.$$

Before proving that (1.47) and (1.48) together imply the claim, we need to give a lower bound for λ . Since $x \in E(\vartheta, \gamma)$, with the choice $w = x$, $\tau = t/4$, and $\rho := \sigma^2 t$ we have, from (1.47), that the following inequality holds

$$(1.49) \quad \begin{aligned} \vartheta^{-1}(t/4)^h &\leq \phi(\overline{B}(x, t/4)) \leq \lambda \mathcal{C}^h \llcorner (x\mathbb{V})(\overline{B}(x, (1/4 + \sigma^2)t)) + \sigma^{h+1}t^h \\ &= \lambda(1/4 + \sigma^2)^h t^h + \sigma^{h+1}t^h, \end{aligned}$$

where the last equality comes from Remark 1.27. Since we know that $\sigma \leq 1/(2^{10(h+1)}\vartheta)$, we infer that $\sigma^{h+1} \leq 1/(8^h\vartheta)$, and then from (1.49) we infer

$$(1.50) \quad \vartheta^{-1}4^{-h} \leq \lambda(1/4 + \sigma^2)^h + \sigma^{h+1} \quad \text{and in particular} \quad \lambda \geq \vartheta^{-1}2^{-3h},$$

where we exploited the fact that $1/4 + \sigma^2 < 1$, the fact that $\sigma^{h+1} \leq 1/(8^h\vartheta)$ and the fact that $4^{-h} - 8^{-h} \geq 8^{-h}$.

Let us now prove that (1.47) and (1.48) imply the claim. Since by hypothesis $r, s \geq \sigma t$ with the choice $\rho = \sigma^2 t$ we have $\rho < r, s$. Furthermore since $\sigma t \leq r, s \leq t/2$ and $y, z \in \overline{B}(x, t/2) \cap x\mathbb{V}$, the bounds (1.47) and (1.48) imply

$$\begin{aligned} \frac{\phi(\overline{B}(y, r) \cap \overline{B}(x\mathbb{V}, \rho))}{\phi(\overline{B}(z, s))} &\geq \frac{\lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(\overline{B}(y, r - \rho)) - \sigma^{h+3}t^{h+1}/\rho}{\lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(\overline{B}(z, s + \rho)) + \sigma^{h+3}t^{h+1}/\rho} \\ &= \frac{r^h \lambda(1 - \sigma^2 t/r)^h - \sigma^{h+1}(t/r)^h}{s^h \lambda(1 + \sigma^2 t/s)^h + \sigma^{h+1}(t/s)^h} \\ &\geq \frac{r^h \lambda(1 - \sigma)^h - \sigma^{h+1}(t/r)^h}{s^h \lambda(1 + \sigma)^h + \sigma^{h+1}(t/s)^h} \geq \frac{r^h \lambda(1 - \sigma)^h - \sigma}{s^h \lambda(1 + \sigma)^h + \sigma}, \end{aligned}$$

where the equality in the second line comes from Remark 1.27, and we are using $\sigma t/r \leq 1$, and $\sigma t/s \leq 1$. Since $2h\sigma \leq 1$, we have that $(1 + \sigma)^h \leq 1 + 2h\sigma$, that can be easily proved by induction on h . This together with (1.50) and Bernoulli's inequality $(1 - \sigma)^h \geq 1 - \sigma h$ allows us to finally infer that

$$\frac{\phi(\overline{B}(y, r) \cap \overline{B}(x\mathbb{V}, \rho))}{\phi(\overline{B}(z, s))} \geq \frac{r^h}{s^h} \frac{1 - (\lambda h + 1)\sigma/\lambda}{1 + (2h\lambda + 1)\sigma/\lambda} \geq (1 - 2^{10(h+1)}\vartheta\sigma) \frac{r^h}{s^h},$$

where the last inequality comes from the fact that $\sigma \leq 1/2^{10(h+1)}\vartheta$, from (1.50) and some easy algebraic computations that we omit. An easy way to verify the last inequality is to show that $(1 - \tilde{\alpha}\sigma)/(1 + \tilde{\beta}\sigma) \geq 1 - \tilde{\gamma}\sigma$, where $\tilde{\alpha} := (\lambda h + 1)/\lambda$, $\tilde{\beta} := (2h\lambda + 1)/\lambda$ and $\tilde{\gamma} := 2^{10(h+1)}\vartheta$, and observe that the latter inequality is implied by the fact that $\tilde{\alpha} + \tilde{\beta} - \tilde{\gamma} \leq 0$.

Therefore, we are left to prove (1.47) and (1.48). In order to prove (1.47), we let $g(z) := \min\{1, \text{dist}(z, \mathbb{G} \setminus B(w, \tau + \rho))/\rho\}$ and note that

$$\begin{aligned} \phi(\overline{B}(w, \tau)) &\leq \int g(z) d\phi(z) \leq \int g(z) d\lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(z) + \text{Lip}(g)F_{x,t}(\phi, \lambda\mathcal{C}^h_{\perp}(x\mathbb{V})) \\ &\leq \lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(\overline{B}(w, \tau + \rho)) + \sigma^{h+3}t^{h+1}/\rho. \end{aligned}$$

On the other hand, to prove (1.48) we let

$$h(z) := \min\{1, \text{dist}(z, \mathbb{G} \setminus (B(w, \tau) \cap B(x\mathbb{V}, \rho)))/\rho\},$$

and then

$$\begin{aligned} \lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(\overline{B}(w, \tau - \rho)) &\leq \int h(z) d\lambda\mathcal{C}^h_{\perp}(x\mathbb{V})(z) \\ &\leq \int h(z) d\phi(z) + \text{Lip}(h)F_{x,t}(\phi, \lambda\mathcal{C}^h_{\perp}(x\mathbb{V})) \\ &\leq \phi(\overline{B}(w, \tau) \cap \overline{B}(x\mathbb{V}, \rho)) + \sigma^{h+3}t^{h+1}/\rho. \end{aligned}$$

Proof of (ii): In this proof let us fix $\tau := t/4$ and define the function

$$\ell(z) := \min\{1, \text{dist}(z, \mathbb{G} \setminus B(B(x, \tau) \cap xT(0, \tau/k), \rho))/\rho\},$$

where $0 < \rho < \tau$. With this definition we have the following chain of inequalities

(1.51)

$$\begin{aligned} \phi(\overline{B}(x, \tau) \cap xT(0, \tau/k)) &\leq \int \ell(z) d\phi(z) \leq \int \ell(z) d\lambda \mathcal{C}^h_{\mathbb{L}}(x\mathbb{V})(z) + \text{Lip}(\ell) F_{x,t}(\phi, \lambda \mathcal{C}^h_{\mathbb{L}}(x\mathbb{V})) \\ &\leq \lambda \mathcal{C}^h_{\mathbb{L}}(x\mathbb{V})(\overline{B}(x, \tau + \rho) \cap xT(0, \tau/k + \rho)) + 4^{h+1} \sigma^{h+3} \tau^{h+1} / \rho \\ &\leq \lambda \mathcal{C}^h_{\mathbb{L}} \mathbb{V}(P_{\mathbb{V}}(\overline{B}(0, 1)))(\tau/k + \rho)^h + 4^{h+1} \sigma^{h+3} \tau^{h+1} / \rho, \end{aligned}$$

where the third inequality above comes from the fact that, according to the proof of Proposition 1.48, the projection $P_{\mathbb{V}}$ is a homomorphism, and then the following chain of equalities holds

$$\begin{aligned} (1.52) \quad P_{\mathbb{V}}(\overline{B}(T(0, \tau/k), \rho)) &= P_{\mathbb{V}}(T(0, \tau/k) \overline{B}(0, \rho)) \\ &= P_{\mathbb{V}}(\overline{B}(0, t/k)) P_{\mathbb{V}}(\overline{B}(0, \rho)) = P_{\mathbb{V}}(\overline{B}(0, \tau/k + \rho)). \end{aligned}$$

Putting together (1.48) when specialized to the case $w = x$ and $\tau = t/4$, with (1.51) and Remark 1.27, we infer that

$$(1.53) \quad \frac{\phi(\overline{B}(x, \tau) \cap xT(0, \tau/k))}{\phi(\overline{B}(x, \tau))} \leq \frac{\lambda \mathcal{C}^h_{\mathbb{L}} \mathbb{V}(P_{\mathbb{V}}(\overline{B}(0, 1)))(\tau/k + \rho)^h + 4^{h+1} \sigma^{h+3} \tau^{h+1} / \rho}{\lambda(\tau - \rho)^h - 4^{h+1} \sigma^{h+3} \tau^{h+1} / \rho}.$$

Since $\sigma^2 < 1$ we choose $\rho := \sigma^2 \tau$ and note that since $\sigma k < 2^{-10h} \vartheta^{-1}$, the previous inequality yields

$$\begin{aligned} \frac{\phi(\overline{B}(x, \tau) \cap xT(0, \tau/k))}{\phi(\overline{B}(x, \tau))} &\leq \frac{\lambda \mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1)))(1/k + \sigma^2)^h + 4^{h+1} \sigma^{h+1}}{\lambda(1 - \sigma^2)^h - 4^{h+1} \sigma^{h+1}} \\ &\leq (1 + 4\sigma(2kh + 1)) \mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1))) k^{-h}, \end{aligned}$$

where we omit the straightforward computations that lead to the last inequality but we stress that we need $\mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1))) \geq 1$, that in turns comes from the fact that $P_{\mathbb{V}}(\overline{B}(0, 1)) \supseteq \overline{B}(0, 1) \cap \mathbb{V}$ and $\mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{V}) = 1$, due to Remark 1.27; and also the bound on λ in (1.50). The last inequality concludes the proposition. \square

Now we are going to define an analogous version of the quantity defined in Definition 1.71. This next definition will be useful while dealing with measures for which we only know that they are asymptotically doubling. The following definition is inspired by [202, 2.1(3)].

Definition 1.76 (Definition of $\overline{d}_{x,r}$). For every $x \in \mathbb{G}$, every $h \in \{1, \dots, Q\}$ and every $r > 0$ we define the functional

$$(1.54) \quad \overline{d}_{x,r}(\phi, \mathfrak{M}(h)) := \inf_{\mathbb{V} \in Gr(h)} F_{0,1} \left(\frac{T_{x,r}\phi}{F_1(T_{x,r}\phi)}, (h+1)\mathcal{C}^h_{\mathbb{L}}\mathbb{V} \right).$$

Furthermore, if G is a subset of the h -dimensional Grassmannian $Gr(h)$, we also define

$$(1.55) \quad \overline{d}_{x,r}(\phi, \mathfrak{M}(h, G)) := \inf_{\mathbb{V} \in G} F_{0,1} \left(\frac{T_{x,r}\phi}{F_1(T_{x,r}\phi)}, (h+1)\mathcal{C}^h_{\mathbb{L}}\mathbb{V} \right).$$

Remark 1.77 (About the definition of $\overline{d}_{x,r}$). For every Radon measure ϕ on \mathbb{G} and every $r > 0$ it is immediate to see that $F_1(T_{0,r}\phi) = r^{-1} F_r(\phi)$. Moreover, thanks to the first part of Remark 1.66, by few simple computations we get

$$(1.56) \quad F_{0,1} \left(\frac{T_{x,r}\phi}{F_1(T_{x,r}\phi)}, (h+1)\mathcal{C}^h_{\mathbb{L}}\mathbb{V} \right) = r^{-(h+1)} F_{0,r} \left(\frac{T_{x,1}\phi}{r^{-(h+1)} F_r(T_{x,1}\phi)}, (h+1)\mathcal{C}^h_{\mathbb{L}}\mathbb{V} \right),$$

for all $r > 0$ and $\mathbb{V} \in Gr(h)$. Hence, since $F_1((h+1)\mathcal{C}^h_{\mathbb{L}}\mathbb{V}) = 1$ as a consequence of Proposition 1.29 and Remark 1.27, we notice that the definition in (1.55) agrees with the definition

given in [202, 2.1(3)]. Namely, $\bar{d}_{x,r}(\phi, \mathfrak{M}(h, G)) = \bar{d}_r(T_{x,1}\phi, \mathfrak{M}(h, G)) = \bar{d}_1(T_{x,r}\phi, \mathfrak{M}(h, G))$, where \bar{d}_r is the analogue in the Carnot setting of the functional defined in [202, 2.1(3)] in the Euclidean setting.

For the sake of completeness, and for some benefits toward subsequent calculations, let us give here the precise analogue, in the setting of Carnot groups, of the definition of the function \bar{d} that Preiss gave in the Euclidean setting. Let \mathcal{C} be an arbitrary cone of Radon measures on \mathbb{G} without the origin, that means $0 \notin \mathcal{C}$ and $\mu \in \mathcal{C}$ implies $\lambda T_{0,\nu}\mu \in \mathcal{C}$ for every $\lambda, \nu > 0$. Then, for every $r > 0$ and Radon measure ϕ we define

$$(1.57) \quad \bar{d}_r(\phi, \mathcal{C}) := \inf \left\{ F_{0,r} \left(\frac{\phi}{F_r(\phi)}, \psi \right) : \psi \in \mathcal{C}, F_r(\psi) = 1 \right\}.$$

By the explicit expression and the continuity of $F_r(\cdot)$ with respect to the weak* convergence, one easily verifies that for every $r > 0$ the following implication holds

$$(1.58) \quad \phi_k \rightharpoonup \phi, F_r(\phi) > 0 \Rightarrow \bar{d}_r(\phi_k, \mathcal{C}) \rightarrow \bar{d}_r(\phi, \mathcal{C}),$$

compare [202, 2.1(6)]. Moreover, due to a slight modification of (1.56), we have, for every $r > 0$ and every Radon measure ϕ ,

$$(1.59) \quad \bar{d}_r(\phi, \mathcal{C}) = \bar{d}_1(T_{0,r}\phi, \mathcal{C}).$$

We now adapt some classical results contained in [202] to our context. The aim will be to prove that when a Radon measure on \mathbb{G} has a tangent at a point that is a cone (of measures) with compact basis, then the measure is asymptotically doubling at the point. The following proposition is the analogue of [202, Propostion 2.2].

Proposition 1.78. *Assume that \mathcal{T} is a non-empty cone of Radon measures on \mathbb{G} , i.e., for every $\nu \in \mathcal{T}$ and every $\lambda, \eta > 0$ we have $\eta T_{0,\lambda}\nu \in \mathcal{T}$, and moreover $0 \notin \mathcal{T}$. Then, the following are equivalent*

- (i) *the set $\mathfrak{B}(\mathcal{T}) := \{\nu \in \mathcal{T} : F_1(\nu) = 1\}$ is weak* compact,*
- (ii) *for every sequence $\{\nu_i\}_{i \in \mathbb{N}} \subseteq \mathcal{T}$ such that $\lim_{i \rightarrow \infty} F_1(\nu_i) = 0$, we have $\nu_i \rightharpoonup 0$,*
- (iii) *there is a $q \in (0, \infty)$ such that $\nu(\bar{B}(0, 2r)) \leq q\nu(\bar{B}(0, r))$ for every $r > 0$ and every $\nu \in \mathcal{T}$.*

Proof. Let us first prove that (i) \Rightarrow (ii). Let ν_i be a sequence in \mathcal{T} and let us assume that $\lim_{i \rightarrow \infty} F_1(\nu_i) = 0$. We note that $\nu_i \rightharpoonup 0$ if and only if $F_{0,t}(\nu_i, 0) = F_t(\nu_i) \rightarrow_i 0$ for every $t > 0$. This means that if ν_i does not converge to 0, we infer that there are a $t > 1$ and an $\varepsilon > 0$ such that, up to passing to subsequences, we have $F_t(\nu_i) > \varepsilon$ for every $i \in \mathbb{N}$. We can assume without loss of generality that $F_1(\nu_i) > 0$ for every i . Let us define

$$r_i := \sup\{r \in [1, t] : F_r(\nu_i) \leq F_1(\nu_i) + 1/i\}.$$

It is immediate to see that up to further subsequences $F_1(T_{0,r_i}\nu_i) = r_i^{-1}F_{r_i}(\nu_i) > 0$ and that

$$\lim_{i \rightarrow \infty} \frac{F_{t/r_i}(T_{0,r_i}\nu_i)}{F_1(T_{0,r_i}\nu_i)} = \lim_{i \rightarrow \infty} \frac{F_t(\nu_i)}{F_{r_i}(\nu_i)} > \varepsilon \lim_{i \rightarrow \infty} (F_1(\nu_i) + 1/i)^{-1} = \infty.$$

Thanks to the fact that \mathcal{T} is a cone, we know that $F_1(T_{0,r_i}\nu_i)^{-1}T_{0,r_i}\nu_i \in \mathfrak{B}(\mathcal{T})$ and thus there must exist a converging (non-reabeled) subsequence of r_i and a $\nu \in \mathfrak{B}(\mathcal{T})$ such that $F_1(T_{0,r_i}\nu_i)^{-1}T_{0,r_i}\nu_i \rightharpoonup \nu$. This however implies that

$$\infty = \lim_{i \rightarrow \infty} \frac{F_{t/r_i}(T_{0,r_i}\nu_i)}{F_1(T_{0,r_i}\nu_i)} \leq \lim_{i \rightarrow \infty} \frac{F_t(T_{0,r_i}\nu_i)}{F_1(T_{0,r_i}\nu_i)} = \lim_{i \rightarrow \infty} F_t(F_1(T_{0,r_i}\nu_i)^{-1}T_{0,r_i}\nu_i) = F_t(\nu),$$

that is a contradiction with the fact that ν is a Radon measure.

Secondly, let us show that (ii) \Rightarrow (iii). Since \mathcal{T} is a cone, it suffices to prove that there exists $q \in (0, +\infty)$ such that $\nu(\bar{B}(0, 2)) \leq q\nu(\bar{B}(0, 1))$ for every $\nu \in \mathcal{T}$. Indeed, we thus would

get that for every $\nu \in \mathcal{T}$ and $r > 0$ we have $\nu(\overline{B}(0, 2r)) = T_{0,r}\nu(\overline{B}(0, 2)) \leq qT_{0,r}\nu(\overline{B}(0, 1)) = q\nu(\overline{B}(0, r))$. Suppose by contradiction that there exists a sequence of measures $\nu_i \in \mathcal{T}$ such that $\nu_i(\overline{B}(0, 2)) > i\nu_i(\overline{B}(0, 1))$. Note now that since \mathcal{T} is a cone, the measures $\nu_i(\overline{B}(0, 2))^{-1}\nu_i$ are still in \mathcal{T} and $\lim_{i \rightarrow \infty} F_1(\nu_i(\overline{B}(0, 2))^{-1}\nu_i) = 0$. Thanks to (ii) this shows in particular that

$$(1.60) \quad \nu_i(\overline{B}(0, 2))^{-1}\nu_i \rightharpoonup 0$$

However, since $F_3(\nu_i(\overline{B}(0, 2))^{-1}\nu_i) \geq 1$ for every $i \in \mathbb{N}$, this is a contradiction with (1.60), according to which one should have

$$\lim_{i \rightarrow \infty} F_3(\nu_i(\overline{B}(0, 2))^{-1}\nu_i) = F_3(0) = 0,$$

since F_3 is a weak* continuous operator on Radon measures.

Finally, let us prove the implication (iii) \Rightarrow (i). Let $\{\nu_i\}_{i \in \mathbb{N}}$ be a sequence in $\mathfrak{B}(\mathcal{T})$ and note that for every $i \in \mathbb{N}$ we have

$$\nu_i(\overline{B}(0, 1/2)) \leq 2F_1(\nu_i) = 2,$$

and thus thanks to (iii) we infer that for every $r > 0$ we have $\nu_i(\overline{B}(0, r))$ is uniformly bounded above independently on $i \in \mathbb{N}$. Finally, Proposition 1.69 and the weak* continuity of F_1 conclude the proof. \square

Remark 1.79. Let us notice that if \mathcal{T} is a non-empty cone of Radon measures such that $\mathfrak{B}(\mathcal{T})$ is weak* compact, for every $\lambda > 1$ there is $\tau > 1$ such that $F_{\tau r}(\psi) \leq \lambda F_r(\psi)$ for every $r > 0$ and $\psi \in \mathcal{T}$. The proof follows verbatim from the five lines in [202, (1) \Rightarrow (5) of Proposition 2.2].

Proposition 1.80. *For every Radon measure ϕ on \mathbb{G} and ϕ -almost every $x \in \mathbb{G}$ the set $\text{Tan}(\phi, x)$ is either empty or a cone. Moreover, if ϕ is a Radon measure on \mathbb{G} such that the set $\mathfrak{B}(\phi, x) := \{\nu \in \text{Tan}(\phi, x) : F_1(\nu) = 1\}$ is non-empty and weak* compact for ϕ -almost every $x \in \mathbb{G}$, then ϕ is asymptotically doubling.*

Proof. In order to prove the first part of the statement, let $x \in \text{supp}(\phi)$ be a point where $\text{Tan}(\phi, x)$ is non-empty, choose a $\nu \in \text{Tan}(\phi, x)$ and assume that $r_i \rightarrow 0$ and c_i are two sequences such that

$$c_i T_{x, r_i} \phi \rightharpoonup \nu.$$

To conclude the proof of the claim we need to show that for every $\eta, \lambda > 0$ we have $\eta T_{0, \lambda} \nu \in \text{Tan}(\phi, x)$ and to do this, we just note that

$$\eta c_i T_{x, \lambda r_i} \phi = \eta T_{0, \lambda} (c_i T_{x, r_i} \phi) \rightharpoonup \eta T_{0, \lambda} \nu.$$

This shows that $\eta T_{0, \lambda} \nu \in \text{Tan}(\phi, x)$ and thus $\text{Tan}(\phi, x)$ is a cone.

Fix a point $x \in \mathbb{G}$ where the set $\mathfrak{B}(\phi, x)$ is a compact cone and thanks to Proposition 1.78(iii) we infer there exists a $q > 0$ such that $\nu(\overline{B}(0, 2r)) \leq q\nu(\overline{B}(0, r))$ for every $\nu \in \text{Tan}(\phi, x)$ and every $r > 0$. Let $\mathfrak{d} := \inf\{\text{dist}(z, B(0, 1/2)^c) : z \in \overline{B}(0, 1/4)\} > 0$. We now claim that

$$(1.61) \quad \limsup_{r \rightarrow 0} F_1(T_{x, 2r} \phi) / F_1(T_{x, r} \phi) \leq 2\mathfrak{d}^{-1} q^2.$$

Indeed, if by contradiction r_i is an infinitesimal sequence such that

$$F_1(T_{x, 2r_i} \phi) > 2\mathfrak{d}^{-1} q^2 F_1(T_{x, r_i} \phi),$$

then for every $\nu \in \mathfrak{B}(\phi, x)$ we have

$$(1.62) \quad \begin{aligned} F_{0,1}(T_{x,2r_i}\phi/F_1(T_{x,2r_i}\phi), \nu) &\geq F_{0,1/2}(T_{x,2r_i}\phi/F_1(T_{x,2r_i}\phi), \nu) \\ &\geq F_{1/2}(\nu) - F_{1/2}(T_{x,2r_i}\phi)/F_1(T_{x,2r_i}\phi), \end{aligned}$$

where the last inequality comes from Remark 1.66 and Lemma 1.68. Furthermore, we also have for every $\nu \in \mathfrak{B}(\phi, x)$ that

$$(1.63) \quad F_{1/2}(\nu) = \frac{F_{1/2}(\nu)}{F_1(\nu)} \geq \frac{\mathfrak{d}\nu(\overline{B}(0, 1/4))}{2\nu(\overline{B}(0, 1))} \geq \frac{\mathfrak{d}}{2q^2}.$$

Thanks to the absurd hypothesis and the fact that for every $s > 0$ we have $F_s(T_{x,r}\phi) = sF_1(T_{x,rs}\phi)$, we infer that

$$(1.64) \quad F_{1/2}(T_{x,2r_i}\phi)/F_1(T_{x,2r_i}\phi) = F_1(T_{x,r_i}\phi)/2F_1(T_{x,2r_i}\phi) \leq \mathfrak{d}/4q^2.$$

Putting (1.62), (1.63) and (1.64) together, we conclude that

$$(1.65) \quad F_{0,1}(T_{x,2r_i}\phi/F_1(T_{x,2r_i}\phi), \nu) \geq \mathfrak{d}/4q^2 \geq \min\{\mathfrak{d}/4q^2, 1/2\} =: \varepsilon,$$

for every $\nu \in \mathfrak{B}(\phi, x)$. Let us now denote, for simplicity, $\mathcal{S} := \text{Tan}(\phi, x)$. By taking into account the definition of \bar{d}_1 in (1.57), we get from the previous computations that $\bar{d}_1(T_{x,2r_i}\phi, \mathcal{S}) \geq \varepsilon$ for every i . Let us fix $\nu \in \text{Tan}(\phi, x)$, and take $c_i > 0$ and $s_i \rightarrow 0$ such that $c_i T_{x,s_i}\phi \rightarrow \nu$. Let us note that (1.57) and (1.58) imply that

$$\lim_{i \rightarrow 0} \bar{d}_1(T_{x,s_i}\phi, \mathcal{S}) = \lim_{i \rightarrow \infty} \bar{d}_1(c_i T_{x,s_i}\phi, \mathcal{S}) = \bar{d}_1(\nu, \mathcal{S}) = 0.$$

Thanks to the above chain of identities, for i sufficiently large, we denote by ℓ_i the smallest number among those $\ell \in [0, s_i]$ with the property that $d_1(T_{x,\eta}\phi, \mathcal{S}) < \varepsilon$ for every $\ell < \eta \leq s_i$. Since $\bar{d}_1(T_{x,2r_i}\phi, \mathcal{S}) \geq \varepsilon$ we conclude that $\ell_i > 0$ for i sufficiently large and $\bar{d}_1(T_{x,\ell_i}\phi, \mathcal{S}) = \varepsilon$ by the minimality of ℓ_i and the continuity of the map $\eta \mapsto \bar{d}_1(T_{x,\eta}\phi, \mathcal{S})$.

If, up to subsequences, $\ell_i/s_i \rightarrow t > 0$, we conclude that, thanks to (1.58),

$$\bar{d}_1(T_{0,t}\nu, \mathcal{S}) = \lim_{i \rightarrow +\infty} \bar{d}_1(T_{x,ts_i}\phi, \mathcal{S}) \geq \varepsilon,$$

where the last inequality is true since ts_i is arbitrarily near to ℓ_i for i large enough, and $\bar{d}_1(T_{x,\ell_i}\phi, \mathcal{S}) \geq \varepsilon$. The previous inequality gives a contradiction since $T_{0,t}\nu \in \mathcal{S}$ and hence we should have $\bar{d}_1(T_{0,t}\nu, \mathcal{S}) = 0$. Thus, $\ell_i/s_i \rightarrow 0$. This means that for every $r \geq 1$, taking into account (1.59), we have

$$(1.66) \quad \limsup_{i \rightarrow +\infty} \bar{d}_r(T_{x,\ell_i}\phi, \mathcal{S}) = \limsup_{i \rightarrow +\infty} \bar{d}_1(T_{x,r\ell_i}\phi, \mathcal{S}) \leq \varepsilon,$$

since $\ell_i \leq r\ell_i \leq s_i$ for i sufficiently large. Since $\varepsilon < 1$, we have that $\lambda := 2/(1 + \varepsilon) > 1$, and hence, by Remark 1.79, there exists $\tau > 1$ such that $F_{\tau r}(\psi) \leq \lambda F_r(\psi)$ for every $\psi \in \mathcal{S}$ and for every $r > 0$, since \mathcal{S} has a compact basis. Hence, taking (1.66) into account with τr instead of r , we get that, whenever $r \geq 1$ and i is sufficiently big, there exists $\psi \in \mathcal{S}$ with $F_{\tau r}(\psi) = 1$ and

$$F_{0,\tau r} \left(\frac{T_{x,\ell_i}\phi}{F_{\tau r}(T_{x,\ell_i}\phi)}, \psi \right) \leq \varepsilon/2.$$

As a consequence, whenever $r \geq 1$ and i is sufficiently big, by the triangle inequality for F (cf. Proposition 1.70) and by the fact that $F_{\tau r}(\cdot) \geq F_r(\cdot)$, we get that

$$\frac{F_r(T_{x,\ell_i}\phi)}{F_{\tau r}(T_{x,\ell_i}\phi)} \geq F_r(\psi) - \varepsilon/2 \geq \lambda^{-1} F_{\tau r}(\psi) - \varepsilon/2 \geq 1/2.$$

Hence, iterating, we have shown that there exists $\tau > 1$ such that that for every $r \geq 1$ and every $p \in \mathbb{N}$,

$$\limsup_{i \rightarrow +\infty} \frac{F_{\tau^p r}(T_{x, \ell_i} \phi)}{F_r(T_{x, \ell_i} \phi)} < +\infty.$$

By the arbitrariness of $p \in \mathbb{N}$ and $r \geq 1$, this implies that we are in a position to apply Proposition 1.69 to the sequence $\frac{T_{x, \ell_i} \phi}{F_1(T_{x, \ell_i} \phi)}$, which then converges, up to subsequences, to $\tilde{\nu} \in \mathcal{T}$ with $F_1(\tilde{\nu}) = 1$. But then, by (1.58),

$$\bar{d}_1(\tilde{\nu}, \mathcal{T}) = \lim_{i \rightarrow +\infty} \bar{d}_1(T_{x, \ell_i} \phi, \mathcal{T}) \geq \varepsilon,$$

that is a contradiction since $\bar{d}_1(\tilde{\nu}, \mathcal{T}) = 0$. Hence we finally have proven (1.61).

Hence, from (1.61), we deduce

$$\limsup_{r \rightarrow 0} \frac{\phi(\bar{B}(x, 2r))}{\phi(\bar{B}(x, r))} \leq \limsup_{r \rightarrow 0} \frac{2F_1(T_{x, 4r} \phi)}{2^{-1}F_1(T_{x, r} \phi)} \leq 16\mathfrak{d}^{-2}q^4,$$

whence the conclusion. \square

Let us now prove a simple consequence of the previous proposition.

Proposition 1.81. *Let ϕ be a Radon measure on \mathbb{G} such that for ϕ -almost every $x \in \mathbb{G}$ we have $\text{Tan}(\phi, x) = \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x), \lambda > 0\}$ for some homogeneous subgroup $\mathbb{V}(x)$ of homogeneous dimension $h \in \mathbb{N}$. Then, for ϕ -almost every $x \in \mathbb{G}$, the measure $T_{x, r} \phi / F_1(T_{x, r} \phi)$ weak* converges to $(h+1)\mathcal{C}^h \llcorner \mathbb{V}(x)$ as $r \rightarrow 0$.*

Proof. For ϕ -almost every $x \in \mathbb{G}$ we have that $\mathfrak{B}(\phi, x) = \{(h+1)\mathcal{C}^h \llcorner \mathbb{V}(x)\}$, taking into account Proposition 1.29 and Remark 1.27. Hence $\mathfrak{B}(\phi, x)$ is clearly compact for ϕ -almost every $x \in \mathbb{G}$, and then ϕ is asymptotically doubling, due to Proposition 1.80. Hence for every sequence $r_i \rightarrow 0$ we can extract a subsequence in i such that $T_{x, r_i} \phi / F_1(T_{x, r_i} \phi)$ weak* converges to some $\nu \in \text{Tan}(\phi, x)$, due to the fact that ϕ is asymptotically doubling and thus the hypothesis of Proposition 1.69 is verified. Since $F_1(\nu) = 1$ by continuity of F_1 , we conclude that $\nu = (h+1)\mathcal{C}^h \llcorner \mathbb{V}(x)$. Thus, being the sequence r_i arbitrary, we obtain the thesis. \square

The following proposition, which is inspired by [202, 4.4(4)], will be of crucial importance in the proof of some results in Chapter 2.

Proposition 1.82. *Let $0 < \sigma < 1/5$, ϕ be a Radon measure on \mathbb{G} , $h \in \{1, \dots, Q\}$, and $\bar{d}_{z, t}(\phi, \mathfrak{M}(h, \{\mathbb{V}\})) \leq \sigma^{h+4}$, then*

$$\phi(\bar{B}(y, s) \cap \bar{B}(y\mathbb{V}, \sigma^2 t / (h+1))) \geq (1 - 5\sigma)(s/r)^h \phi(\bar{B}(x, r)),$$

whenever $x, y \in z\mathbb{V} \cap \bar{B}(z, (1-\sigma)t)$, $\sigma t \leq r \leq (1-\sigma)t - \|z^{-1}x\|$, and $\sigma t \leq s \leq (1-\sigma)t - \|z^{-1}y\|$.

Proof. The definition of $\bar{d}_{z, t}(\cdot, \mathfrak{M}(h, \{\mathbb{V}\}))$ implies that

$$F_{0,1}(T_{z, t} \phi / F_1(T_{z, t} \phi), (h+1)\mathcal{C}^h \llcorner \mathbb{V}) \leq \sigma^{h+4}.$$

Up to redefining ϕ we can assume without loss of generality that $z = 0$, $t = 1$ and that $F_1(\phi) = 1$. Thus, let $q := \sigma^2 / (h+1)$, $x \in \mathbb{V}$ and $r > 0$ as in the hypothesis of the proposition. Define

$$g(w) := \min\{1, \text{dist}(w, \mathbb{G} \setminus B(x, r+q)) / q\}.$$

Notice that $\overline{B}(x, r) \Subset \overline{B}(0, 1)$, and thanks to the assumptions on ϕ we infer that, calling $\text{Lip}(g)$ the Lipschitz constant of the function g ,

(1.67)

$$\begin{aligned} \phi(\overline{B}(x, r)) &\leq \int g(w) d\phi(w) \leq (h+1) \int g(w) d\mathcal{C}^h \llcorner \mathbb{V}(w) + \text{Lip}(g) F_{0,1}(\phi, (h+1)\mathcal{C}^h \llcorner \mathbb{V}(w)) \\ &\leq (h+1)\mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(x, r+q)) + \sigma^{h+4}/q = (h+1)(r+q)^h + \sigma^{h+4}/q, \end{aligned}$$

where in the last equality we are exploiting Remark 1.27. With the same argument used above, cf. (1.48), for every y and $s > 0$ as in the hypothesis of the proposition one can also show that

$$(1.68) \quad (h+1)(s-q)^h = (h+1)\mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(y, s-q)) \leq \phi(\overline{B}(y, s) \cap \overline{B}(\mathbb{V}, q)) + \sigma^{h+4}/q.$$

Thus, putting together (1.67) and (1.68) we infer that

$$\begin{aligned} \frac{\phi(\overline{B}(y, s) \cap \overline{B}(\mathbb{V}, q))}{\phi(\overline{B}(x, r))} &\geq \frac{(h+1)(s-q)^h - \sigma^{h+4}/q}{(h+1)(r+q)^h + \sigma^{h+4}/q} = \left(\frac{s}{r}\right)^h \frac{\left(1 - \frac{\sigma^2}{s(h+1)}\right)^h - \frac{\sigma^{h+2}}{s^h}}{\left(1 + \frac{\sigma^2}{r(h+1)}\right)^h + \frac{\sigma^{h+2}}{r^h}} \\ &\geq \left(\frac{s}{r}\right)^h \frac{\left(1 - \frac{\sigma}{h+1}\right)^h - \sigma^2}{\left(1 + \frac{\sigma}{h+1}\right)^h + \sigma^2} \geq \left(\frac{s}{r}\right)^h \frac{1 - h/(h+1)\sigma - \sigma^2}{1 + 2h/(h+1)\sigma + \sigma^2} \\ &\geq \left(\frac{s}{r}\right)^h \frac{1 - 2\sigma}{1 + 3\sigma} \geq (1 - 5\sigma) \left(\frac{s}{r}\right)^h, \end{aligned}$$

where in the third inequality above we are using that $\sigma \leq r$ and $\sigma \leq s$; in the fourth inequality we are using that $(1 - \sigma/(h+1))^h \geq 1 - h/(h+1)\sigma$ by Bernoulli inequality, and $(1 + \sigma/(h+1))^h \leq 1 + 2h/(h+1)\sigma$, which can be easily verified by induction since $2h\sigma/(h+1) \leq 1$. \square

3.3. Measurability of the map "points to tangents". In this subsection we state and prove three measurability results that will play a crucial role in Chapter 2. Roughly speaking, we prove that when a measure has unique tangents (or unique approximate tangents), the map that associates a point $x \in \mathbb{G}$ to its tangent (or approximate tangent) is measurable. The reference for this subsection is the work [33].

Lemma 1.83. *Let ϕ be a Radon measure on \mathbb{G} such that, for ϕ -almost every $x \in \mathbb{G}$, there exists $\tau(\phi, x) \in Gr(h)$ such that*

$$\text{Tan}(\phi, x) = \{\lambda \mathcal{C}^h \llcorner \tau(\phi, x) : \lambda > 0\}.$$

Then the map $x \mapsto \tau(\phi, x)$ is ϕ -measurable as a map from \mathbb{G} to $Gr(h)$.

Proof. First of all, from Proposition 1.80 we get that ϕ is asymptotically doubling. We let $\{\mathbb{V}_\ell\}_{\ell \in \mathbb{N}}$ be a countable dense set in $Gr(h)$, which exists thanks to the compactness of the Grassmannian, see Proposition 1.22. Furthermore, for every $r \in (0, 1) \cap \mathbb{Q}$ every $\varepsilon > 0$, and every $\ell \in \mathbb{N}$ we define the function

$$\begin{aligned} f_{r,\ell,\varepsilon}(x) &:= \phi(B(x, r))^{-1} \phi(\{w \in \overline{B}(x, r) : \text{dist}(x^{-1}w, \mathbb{V}_\ell) \geq \varepsilon \|x^{-1}w\|\}) \\ &=: \phi(B(x, r))^{-1} \phi(I(x, r, \ell, \varepsilon)), \end{aligned}$$

when $\phi(B(x, r)) > 0$ and we set it to be $+\infty$ if $\phi(B(x, r)) = 0$. We claim that the functions $f_{r,\ell,\varepsilon}$ are upper semicontinuous. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points converging to some $x \in \mathbb{G}$. If $\phi(B(x, r)) = 0$ the upper semicontinuity on the sequence $\{x_i\}_{i \in \mathbb{N}}$ is trivially verified by definition of $f_{r,\ell,\varepsilon}$. So let us assume without loss of generality that $\phi(B(x, r)) > 0$. Since $x_i \rightarrow x$ and ϕ is a Radon measure we have, by Fatou's Lemma, that $\phi(B(x, r)) \leq$

$\liminf_{i \rightarrow +\infty} \phi(B(x_i, r))$, and then we can assume without loss of generality that $\phi(B(x_i, r)) > 0$ for every i .

Since the sets $I(x_i, r, \ell, \varepsilon)$ are contained in $\overline{B}(x, 2)$ provided i is sufficiently big, we infer thanks to Fatou's Lemma that

$$(1.69) \quad \begin{aligned} \limsup_{i \rightarrow +\infty} f_{r, \ell, \varepsilon}(x_i) &= \limsup_{i \rightarrow +\infty} \phi(B(x_i, r))^{-1} \int \chi_{I(x_i, r, \ell, \varepsilon)}(z) d\phi(z) \\ &\leq \phi(B(x, r))^{-1} \int \limsup_{i \rightarrow +\infty} \chi_{I(x_i, r, \ell, \varepsilon)}(z) d\phi(z). \end{aligned}$$

Furthermore, since $x_i \rightarrow x$ and the sets $I(x_i, r, \ell, \varepsilon)$ and $I(x, r, \ell, \varepsilon)$ are closed, we have

$$\limsup_{i \rightarrow +\infty} \chi_{I(x_i, r, \ell, \varepsilon)} = \chi_{\limsup_{i \rightarrow +\infty} I(x_i, r, \ell, \varepsilon)} \leq \chi_{I(x, r, \ell, \varepsilon)},$$

where the first equality is true in general. Then, from (1.69), we infer that

$$\begin{aligned} \limsup_{i \rightarrow +\infty} f_{r, \ell, \varepsilon}(x_i) &\leq \phi(B(x, r))^{-1} \int \limsup_{i \rightarrow +\infty} \chi_{I(x_i, r, \ell, \varepsilon)}(z) d\phi(z) \\ &\leq \phi(B(x, r))^{-1} \int \chi_{I(x, r, \ell, \varepsilon)}(z) d\phi(z) = f_{r, \ell, \varepsilon}(x), \end{aligned}$$

and this concludes the proof that $f_{r, \ell, \varepsilon}$ is upper semicontinuous. This shows that for every $\ell \in \mathbb{N}$ and $\varepsilon > 0$, the function

$$f_{\ell, \varepsilon} := \liminf_{r \in \mathbb{Q} \cap (0, 1), r \rightarrow 0} f_{r, \ell, \varepsilon},$$

is ϕ -measurable. Hence also $\tilde{f}_{\ell, \varepsilon} := \sup_{\tilde{\varepsilon} \in \mathbb{Q}, \tilde{\varepsilon} > \varepsilon} f_{\ell, \varepsilon}$ is ϕ -measurable. As a consequence, since $\text{Tan}(\phi, x) = \{\lambda \mathcal{C}^h \llcorner \tau(\phi, x) : \lambda > 0\}$ for ϕ -almost every $x \in \mathbb{G}$, we infer that the set

(1.70)

$$B_{\ell, \varepsilon} := \{x \in \mathbb{G} : \tilde{f}_{\ell, \varepsilon}(x) = 0\} \cap \{x \in \mathbb{G} : \text{there exists } \tau(\phi, x)\} = \{x \in \mathbb{G} : \tau(\phi, x) \subseteq C_{\mathbb{V}_\ell}(\varepsilon)\},$$

is ϕ -measurable as well.

Let us now prove that

$$(1.71) \quad B_{\ell, \varepsilon} = \{x \in \mathbb{G} : \tau(\phi, x) \subseteq C_{\mathbb{V}_\ell}(\varepsilon)\},$$

Let us show the inclusion \subseteq in (1.71). If $\tilde{f}_{\ell, \varepsilon}(x) = 0$, then $f_{\ell, \tilde{\varepsilon}}(x) = 0$ for every $\tilde{\varepsilon} > \varepsilon$, $\tilde{\varepsilon} \in \mathbb{Q}$. Hence, we first get that there exist $r_i \rightarrow 0$ such that $f_{r_i, \ell, \tilde{\varepsilon}}(x) \rightarrow_i 0$. Since ϕ is asymptotically doubling, thanks to Proposition 1.69 we deduce that $\phi(B(x, r_i))^{-1} T_{x, r_i} \phi$ converges, up to subsequences, to some tangent measure $\nu \in \text{Tan}(\phi, x)$, and then from the hypothesis we have $\nu = \lambda \mathcal{C}^h \llcorner \tau(\phi, x)$, for some $\lambda > 0$.

Thanks to [89, Proposition 2.7], we infer that for every $\tilde{\varepsilon} > \varepsilon$ the following holds

(1.72)

$$\begin{aligned} &\nu(\{w \in B(0, 1) : \text{dist}(w, \mathbb{V}_\ell) > \tilde{\varepsilon} \|w\|\}) \\ &\leq \liminf_{i \rightarrow \infty} \phi(B(x, r_i))^{-1} T_{x, r_i} \phi(\{w \in B(0, 1) : \text{dist}(w, \mathbb{V}_\ell) > \tilde{\varepsilon} \|w\|\}) \\ &= \liminf_{i \rightarrow \infty} \phi(B(x, r_i))^{-1} \phi(\{w \in B(x, r_i) : \text{dist}(x^{-1}w, \mathbb{V}_\ell) > \tilde{\varepsilon} \|x^{-1}w\|\}) = 0, \end{aligned}$$

where the last identity comes from the fact that $f_{r_i, \ell, \tilde{\varepsilon}}(x) \rightarrow_i 0$. This shows in particular that

$$\tau(\phi, x) \subseteq \{w \in \mathbb{G} : \text{dist}(w, \mathbb{V}_\ell) \leq \tilde{\varepsilon} \|w\|\} = C_{\mathbb{V}_\ell}(\tilde{\varepsilon}),$$

and thus, taking $\tilde{\varepsilon} \rightarrow \varepsilon$ we get the sought first inclusion in (1.71).

For the other inclusion, if $\tau(\phi, x) \subseteq C_{\mathbb{V}_\ell}(\varepsilon)$, we get that $\tau(\phi, x) \subseteq \{w \in \mathbb{G} : \text{dist}(w, \mathbb{V}_\ell) < \tilde{\varepsilon} \|w\|\}$ for every $\tilde{\varepsilon} > \varepsilon$. Moreover, as a consequence of a routine argument (cf. [174, Remark

14.4(3)) we get that there exists an infinitesimal sequence $r_i \rightarrow 0$, and a $\lambda > 0$, such that $\phi(B(x, r_i))^{-1}T_{x, r_i} \rightharpoonup \lambda \mathcal{C}^h \lrcorner \tau(\phi, x)$. Hence, using [89, Proposition 2.7], we get that for every $\tilde{\varepsilon} > \varepsilon$ the following inequalities hold

$$\begin{aligned}
(1.73) \quad f_{\ell, \tilde{\varepsilon}}(x) &= \liminf_{r \in \mathbb{Q}, r \rightarrow 0} f_{r, \ell, \tilde{\varepsilon}}(x) \leq \liminf_{i \rightarrow \infty} f_{r_i, \ell, \tilde{\varepsilon}}(x) \\
&= \liminf_{i \rightarrow \infty} \phi(B(x, r_i))^{-1} \phi(\{w \in \overline{B}(x, r_i) : \text{dist}(x^{-1}w, \mathbb{V}_\ell) \geq \tilde{\varepsilon} \|x^{-1}w\|\}) \\
&\leq \limsup_{i \rightarrow \infty} \phi(B(x, r_i))^{-1} T_{x, r_i} \phi(\{w \in \overline{B}(0, 1) : \text{dist}(w, \mathbb{V}_\ell) \geq \tilde{\varepsilon} \|w\|\}) \\
&\leq \lambda \mathcal{C}^h \lrcorner \tau(\phi, x)(\{w \in \overline{B}(0, 1) : \text{dist}(w, \mathbb{V}_\ell) \geq \tilde{\varepsilon} \|w\|\}) = 0,
\end{aligned}$$

where the last inequality is true since $\tau(\phi, x) \subseteq C_{\mathbb{V}_\ell}(\varepsilon)$ holds. Hence we have finally proved the claim (1.71).

In order to prove that the map $x \mapsto \tau(\phi, x)$ is ϕ -measurable, it suffices to check that the for every open $\Omega \subseteq Gr(h)$ we have that $\tau^{-1}(\Omega)$ is ϕ -measurable. To show this we note that, thanks to Lemma 1.39, there is a sequence of radii $r_k > 0$ such that

$$\Omega = \bigcup_{\substack{k \in \mathbb{N} \\ \mathbb{V}_k \in \Omega}} \{\mathbb{W} \in Gr(h) : \mathbb{W} \subseteq C_{\mathbb{V}_k}(r_k)\}.$$

This implies that, up to ϕ -null sets, $\tau^{-1}(\Omega) = \bigcup_{k \in \mathbb{N}} B_{k, r_k}$, which thanks to the above discussion is a ϕ -measurable set. \square

Lemma 1.84. *Let ϕ be a \mathcal{P}_h -rectifiable measure. Denote with $\tau(\phi, x)$ the unique element of $Gr(h)$, that exists ϕ -almost everywhere by definition, for which*

$$\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{C}^h \lrcorner \tau(\phi, x) : \lambda > 0\}.$$

Then the map $x \mapsto \tau(\phi, x)$ is ϕ -measurable as a map from \mathbb{G} to $Gr(h)$.

Proof. From a routine argument (cf. [174, Remark 14.4(3)]), we get that

$$\text{Tan}(\phi, x) = \{\lambda \mathcal{C}^h \lrcorner \tau(\phi, x)\},$$

for ϕ -almost every $x \in \mathbb{G}$. Hence we can apply Lemma 1.83 to conclude the proof. \square

The proof of the following lemma follows as the ones above. We omit the details. Notice that the following statement could be also obtained arguing as in [178, Proposition 3.9], after having noticed that, since $2^{-h} \leq \Theta^{h,*}(\mathcal{S}^h \lrcorner E, x) \leq 1$ for $\mathcal{S}^h \lrcorner E$ -almost every $x \in \mathbb{G}$ due to [102, 2.10.19(1) and 2.10.19(5)], the condition (1.74) is equivalent to asking that $\mathbb{V}(x)$ is an approximate tangent plane to E at x in the sense of [178, Equation (3.2)].

Lemma 1.85. *Let $E \subseteq \mathbb{G}$ be a Borel set of positive and finite \mathcal{S}^h -measure, and suppose that for \mathcal{S}^h -almost every $x \in E$ there exists $\mathbb{V}(x) \in Gr(h)$ for which for every $0 < \varepsilon < 1$ and every $0 < \beta < 1$ there exists a $\rho(x, \varepsilon, \beta) > 0$ such that*

$$(1.74) \quad \mathcal{S}^h \lrcorner E(\overline{B}(x, r) \setminus xC_{\mathbb{V}(x)}(\beta)) \leq \varepsilon \mathcal{S}^h \lrcorner E(\overline{B}(x, r)),$$

for every $0 < r < \rho(x, \varepsilon, \beta)$. Then the map $x \mapsto \mathbb{V}(x)$ from E to $Gr(h)$ is $\mathcal{S}^h \lrcorner E$ -measurable.

Remark 1.86. The results in Lemma 1.83, Lemma 1.84, and Lemma 1.85 are readily true also when we allow $\tau(\phi, x)$ (or $\mathbb{V}(x)$) to be in some Borel subset of $Gr(h)$.

4. Intrinsic regular functions and submanifolds

In this section we discuss the various notions of intrinsic regular functions and submanifolds in Carnot groups. We will fix h, Q natural numbers such that $1 \leq h \leq Q$. Let \mathbb{G} be a Carnot group of homogeneous dimension Q equipped with a homogeneous norm $\|\cdot\|$ which induces a left-invariant homogeneous distance d .

4.1. Intrinsically Lipschitz functions and graphs. In this subsection we discuss the definition of intrinsically Lipschitz function. The following Definition 1.87 of intrinsically Lipschitz function is equivalent to the classical one in [110, Definition 11] because the cones in [110, Definition 11] and the cones $C_{\mathbb{V}}(\alpha)$ are equivalent whenever \mathbb{V} admits a complementary subgroup, see Remark 1.88.

Definition 1.87 (Intrinsically Lipschitz functions). Let $\mathbb{W} \in Gr_c(h)$, assume \mathbb{L} is a complementary subgroup of \mathbb{W} , and let $E \subseteq \mathbb{W}$. Let $\alpha > 0$. A function $f : E \rightarrow \mathbb{L}$ is said to be an α -intrinsically Lipschitz function if $\text{graph}(f) := \{v \cdot f(v) : v \in E\}$ is a $C_{\mathbb{W}}(\alpha)$ -set. A function $f : E \rightarrow \mathbb{L}$ is said to be an *intrinsically Lipschitz function* if there exists $\alpha > 0$ such that f is an α -intrinsically Lipschitz function.

Remark 1.88. Let us fix \mathbb{V}, \mathbb{L} two complementary subgroups of \mathbb{G} . Let us recall, from Remark 1.38, that if we define

$$C_{\mathbb{V}, \mathbb{L}}(\alpha) := \{w \in \mathbb{G} : \|P_{\mathbb{L}}(w)\| \leq \alpha \|P_{\mathbb{V}}(w)\|\},$$

then, for every $\alpha < C_2(\mathbb{V}, \mathbb{L})$, the following inclusions hold

$$C_{\mathbb{V}, \mathbb{L}}(C_2\alpha) \subseteq C_{\mathbb{V}}(\alpha) \subseteq C_{\mathbb{V}, \mathbb{L}}(\alpha/(C_2 - \alpha)).$$

Thus, in Definition 1.87 we can equivalently use the cones $C_{\mathbb{V}, \mathbb{L}}$ instead of the cones $C_{\mathbb{V}}$ to give the notion of intrinsically Lipschitz function.

Proposition 1.89. *Let us fix $\mathbb{W} \in Gr_c(h)$ with complementary subgroup \mathbb{L} . If $\Gamma \subseteq \mathbb{G}$ is a $C_{\mathbb{W}}(\alpha)$ -set for some $\alpha \leq \varepsilon_1(\mathbb{W}, \mathbb{L})$, then the map $P_{\mathbb{W}} : \Gamma \rightarrow \mathbb{W}$ is injective. As a consequence Γ is the intrinsic graph of an intrinsically Lipschitz map defined on $P_{\mathbb{W}}(\Gamma)$.*

Proof. Suppose by contradiction that $P_{\mathbb{W}} : \Gamma \rightarrow \mathbb{W}$ is not injective. Then, there exist $p \neq q$ with $p, q \in \Gamma$ such that $P_{\mathbb{W}}(p) = P_{\mathbb{W}}(q)$. Thus $p^{-1} \cdot q \in \mathbb{L}$. Moreover, since Γ is a $C_{\mathbb{W}}(\alpha)$ -set, we have that $p^{-1} \cdot q \in C_{\mathbb{W}}(\alpha)$. Eventually we get

$$p^{-1} \cdot q \in \mathbb{L} \cap C_{\mathbb{W}}(\alpha) \subseteq \mathbb{L} \cap C_{\mathbb{W}}(\varepsilon_1(\mathbb{W}, \mathbb{L})),$$

where the last inclusion follows since $\alpha \leq \varepsilon_1(\mathbb{W}, \mathbb{L})$. The above inclusion, jointly with Lemma 1.40, gives that $p^{-1} \cdot q = 0$ and this is a contradiction. Concerning the last part of the statement, let us notice that the map $P_{\mathbb{L}} \circ ((P_{\mathbb{W}})_{|\Gamma})^{-1}$ is well-defined from $P_{\mathbb{W}}(\Gamma)$ to \mathbb{L} and its intrinsic graph is Γ by definition. Moreover, since Γ is a $C_{\mathbb{W}}(\alpha)$ -set, the latter map is intrinsically Lipschitz by Definition 1.87. \square

Let us now end this section with the definition of the intrinsic translation of a function.

Definition 1.90 (Intrinsic graph of a function). Let \mathbb{W}, \mathbb{L} be complementary subgroups of \mathbb{G} . Let $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, we denote

$$\Phi(U) := \text{graph}(\varphi) := \{\Phi(w) := w \cdot \varphi(w) : w \in U\}.$$

Definition 1.91 (Intrinsic translation of a function). Let \mathbb{W}, \mathbb{L} be complementary subgroups of \mathbb{G} . Given a function $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, we define, for every $q \in \mathbb{G}$,

$$U_q := \{a \in \mathbb{W} : P_{\mathbb{W}}(q^{-1} \cdot a) \in U\},$$

and $\varphi_q: U_q \subseteq \mathbb{W} \rightarrow \mathbb{L}$ by setting

$$(1.75) \quad \varphi_q(a) := (P_{\mathbb{L}}(q^{-1} \cdot a))^{-1} \cdot \varphi(P_{\mathbb{W}}(q^{-1} \cdot a)).$$

4.2. Intrinsically differentiable functions and graphs. In this subsection we discuss the notion of intrinsically differentiable function and graph, see [109, Definition 3.2.1]. From now on let \mathbb{V} and \mathbb{L} be two fixed complementary subgroups in a Carnot group \mathbb{G} endowed with a homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance d .

The notion of intrinsic differentiability was first given in [114, Definition 4.4] and then first studied in [16], see [16, Definition 1.1]. In this last reference the notion of intrinsic differentiability is given in terms of the graph distance. We here give a slightly different definition of intrinsic differentiability that is indeed equivalent to the one in [16], by [208, Proposition 4.76], when \mathbb{V} is a normal subgroup.

Definition 1.92 (Intrinsically linear function). The map $\ell: \mathbb{V} \rightarrow \mathbb{L}$ is said to be *intrinsically linear* if $\text{graph}(\ell)$ is a homogeneous subgroup of \mathbb{G} .

Definition 1.93 (Intrinsically differentiable function). Let $\varphi: U \subseteq \mathbb{V} \rightarrow \mathbb{L}$ be a function with U Borel in \mathbb{V} . Fix a density point $a_0 \in \mathcal{D}(U)$ of U , let $p_0 := \varphi(a_0)^{-1} \cdot a_0^{-1}$ and denote with $\varphi_{p_0}: U_{p_0} \subseteq \mathbb{V} \rightarrow \mathbb{L}$ the shifted function introduced in Definition 1.91. We say that φ is *intrinsically differentiable* at a_0 if there is an intrinsically linear map $d^\varphi \varphi_{a_0}: \mathbb{V} \rightarrow \mathbb{L}$ such that

$$(1.76) \quad \lim_{b \rightarrow e, b \in U_{p_0}} \frac{\|d^\varphi \varphi_{a_0}(b)^{-1} \cdot \varphi_{p_0}(b)\|}{\|b\|} = 0.$$

The function $d^\varphi \varphi_{a_0}$ is called the *intrinsic differential* of φ at a_0 . We say that φ is *intrinsically differentiable* if it is intrinsically differentiable at every point $a_0 \in U$. We also denote by $\text{ID}(U, \mathbb{W}; \mathbb{L})$ the set of intrinsically differentiable functions $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$.

Definition 1.94 (Intrinsically differentiable graph). Let $\varphi: K \subseteq \mathbb{V} \rightarrow \mathbb{L}$ be a continuous function with K compact in \mathbb{V} . Let $a_0 \in K$. We say that $\text{graph}(\varphi)$ is an *intrinsically differentiable graph* at $a_0 \cdot \varphi(a_0)$ if there exists a homogeneous subgroup $\mathbb{V}(a_0)$ such that for every compact set $K' \subseteq \mathbb{G}$, the following holds

$$(1.77) \quad \lim_{\lambda \rightarrow \infty} d_{H, \mathbb{G}} \left(\delta_\lambda((a_0 \cdot \varphi(a_0))^{-1} \cdot \text{graph}(\varphi)) \cap K', \mathbb{V}(a_0) \cap K' \right) = 0,$$

where $d_{H, \mathbb{G}}$ is the Hausdorff distance between closed subsets of \mathbb{G} .

When we want to stress that the homogeneous dimension of \mathbb{V} is h , we sometimes refer to $\text{graph}(\varphi)$ as an *h -dimensional intrinsically differentiable graph* at $a_0 \cdot \varphi(a_0)$.

Let us fix $\varphi: U \subseteq \mathbb{V} \rightarrow \mathbb{L}$ with U open. Whenever the intrinsic differential introduced in Definition 1.93 exists, it is unique: see [109, Theorem 3.2.8]. In [109] the authors prove the following result: a function $\varphi: U \subseteq \mathbb{V} \rightarrow \mathbb{L}$, with U open, is intrinsically differentiable at a_0 if and only if $\text{graph}(\varphi)$ is an intrinsically differentiable graph at $a_0 \cdot \varphi(a_0)$ with the tangent $\mathbb{V}(a_0)$ complemented by \mathbb{L} , see Definition 1.94, and moreover $\mathbb{V}(a_0) = \text{graph}(d^\varphi \varphi_{a_0})$. If we have $\varphi: U \subseteq \mathbb{V} \rightarrow \mathbb{L}$ with U compact and with positive Haar measure in \mathbb{V} , the above equivalence still holds at density points of U . We do not give a proof of this last assertion since it follows by routine modifications of the argument in [109], and moreover we do not need it in this thesis.

Let us now give the definition of uniformly intrinsically differentiable map between complementary subgroups.

Definition 1.95 (Uniformly intrinsic differentiability). Let $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ be a function with U open. For a point $a_0 \in U$, let $p_0 := \varphi(a_0)^{-1} \cdot a_0^{-1}$ and denote by $\varphi_{p_0}: U_{p_0} \subseteq \mathbb{W} \rightarrow \mathbb{L}$ the shifted function defined in Definition 1.91.

We say that φ is *uniformly intrinsically differentiable* at a_0 if, setting $p_a := \varphi(a)^{-1} \cdot a^{-1}$ for every $a \in U$, we have

$$(1.78) \quad \lim_{r \rightarrow 0} \left(\sup \left\{ \frac{\|d^\varphi \varphi_{a_0}(b)^{-1} \cdot \varphi_{p_a}(b)\|}{\|b\|} : a \in U \cap B(a_0, r), b \in U_{p_a} \cap B(a_0, r), a \neq b \right\} \right) = 0.$$

We say that φ is *uniformly intrinsically differentiable* on U if it is uniformly intrinsically differentiable at every $a_0 \in U$. We finally denote by $\text{UID}(U, \mathbb{W}; \mathbb{L})$ the set of uniformly intrinsically differentiable functions $\varphi: U \subseteq \mathbb{W} \rightarrow \mathbb{L}$.

Let us now discuss more in detail the case in which we have a splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$, where \mathbb{L} is a horizontal homogeneous subgroup.

Proposition 1.96 ([93, Proposition 3.4]). *Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} with \mathbb{L} horizontal and k -dimensional, and let $\ell: \mathbb{W} \rightarrow \mathbb{L}$ be an intrinsically linear function. Then ℓ only depends on the horizontal components of the elements in \mathbb{W} , namely on $\mathbb{W}_1 := \mathbb{W} \cap \mathbb{V}_1$, where $\mathbb{V}_1 = \exp(\mathbb{V}_1)$. In particular, if $\pi_{\mathbb{V}_1}$ denotes the projection from \mathbb{G} to \mathbb{V}_1 , i.e., the map $\exp \circ \pi_{\mathbb{V}_1} \circ \exp^{-1}$, one has*

$$\ell(a) = \ell(\pi_{\mathbb{V}_1} a), \quad \forall a \in \mathbb{W}.$$

As a consequence, $\exp^{-1} \circ \ell \circ \exp|_{\text{Lie}(\mathbb{W}) \cap \mathbb{V}_1} : \text{Lie}(\mathbb{W}) \cap \mathbb{V}_1 \rightarrow \text{Lie}(\mathbb{L})$ is linear, and there exists a constant $C := C(\ell) > 0$ such that

$$(1.79) \quad \|\ell(a)\| \leq C \|\pi_{\mathbb{V}_1} a\|, \quad \forall a \in \mathbb{W}.$$

Let us now describe how to represent the intrinsic gradient of a function in coordinates, when \mathbb{L} is horizontal.

Definition 1.97 (Intrinsic gradient). Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} with \mathbb{L} horizontal and k -dimensional, let $U \subseteq \mathbb{W}$ be open, and let $\varphi: U \rightarrow \mathbb{L}$ be intrinsically differentiable at $a_0 \in U$. By Proposition 1.96, the map $\exp^{-1} \circ (d^\varphi \varphi_{a_0}) \circ \exp|_{\text{Lie}(\mathbb{W}) \cap \mathbb{V}_1}$ is linear and thus there exists a linear map $\nabla^\varphi \varphi_{a_0} \in \text{Lin}(\text{Lie}(\mathbb{W}) \cap \mathbb{V}_1; \text{Lie}(\mathbb{L}))$ such that

$$d^\varphi \varphi_{a_0}(\exp W) = \exp(\nabla^\varphi \varphi_{a_0}(W)), \quad \forall W \in \text{Lie}(\mathbb{W}) \cap \mathbb{V}_1.$$

Remark 1.98 (Intrinsic gradient in exponential coordinates). Assume (X_1, \dots, X_n) is an adapted basis of the Lie algebra \mathfrak{g} such that

$$\mathbb{L} = \text{span}\{X_1, \dots, X_k\}, \quad \mathbb{W} = \text{span}\{X_{k+1}, \dots, X_n\},$$

and identify \mathbb{W} and \mathbb{L} with \mathbb{R}^{n-k} and \mathbb{R}^k , respectively, through exponential coordinates associated to $\{X_1, \dots, X_n\}$. Then, by Definition 1.97, with a little abuse of notation, we get a $k \times (n_1 - k)$ matrix $\nabla^\varphi \varphi_{a_0}$ such that, in coordinates, one has

$$d^\varphi \varphi_{a_0}(a) = \left(\nabla^\varphi \varphi_{a_0}(a_{k+1}, \dots, a_m)^T, 0, \dots, 0 \right), \quad \forall a = (a_{k+1}, \dots, a_n) \in \mathbb{W} \cong \mathbb{R}^{n-k}.$$

4.3. Intrinsically C^1 functions, submanifolds and rectifiability. In this subsection we recall the definition of intrinsically C^1 function between Carnot groups. Moreover, we give the definitions of intrinsically C^1 submanifolds and rectifiable sets.

A notion of intrinsically C^1 regular submanifold, $C^1_{\mathbb{H}}$ -submanifold from now on, was firstly introduced and studied in [111] in the setting of Heisenberg groups, and then in [112] in arbitrary Carnot groups \mathbb{G} . Initially, the authors only took hypersurfaces into account. A first step toward a general definition of $C^1_{\mathbb{H}}$ -submanifolds in arbitrary codimensions was done in [115, Definition 3.1, Definition 3.2] in the setting of Heisenberg groups \mathbb{H}^n . Then a general notion of (\mathbb{G}, \mathbb{M}) -regular submanifold, where \mathbb{G} and \mathbb{M} are Carnot groups, was

proposed by Magnani in [168, Definition 3.5]. According to the latter definition, a (\mathbb{G}, \mathbb{M}) -regular submanifold is locally the zero-level set of an \mathbb{M} -valued $C_{\mathbb{H}}^1$ -function defined on an open subset of \mathbb{G} and whose intrinsic Pansu differential df is surjective. For more general definition of $C_{\mathbb{H}}^1$ -submanifold, we refer the reader to [168, Definition 3.1], [169, Definition 10.2] and to [132, Section 2.5].

Below, we present the approach to $C_{\mathbb{H}}^1$ -rectifiability presented in [132]. In [132] the authors give the following definitions of $C_{\mathbb{H}}^1$ -submanifold of a Carnot group and rectifiable sets. We first recall the definition of $C_{\mathbb{H}}^1$ -function along with the area formula for Lipschitz functions between Carnot groups, which will be useful later on.

Before starting, we recall the definition of Carnot homomorphism. We call a Lie group homomorphism $\varphi : \mathbb{G} \rightarrow \mathbb{H}$ between two Carnot groups a *Carnot homomorphism* if

$$\varphi \circ \delta_\lambda = \delta_\lambda \circ \varphi, \quad \forall \lambda > 0.$$

Definition 1.99 (Differentiability and $C_{\mathbb{H}}^1$ -function). Let \mathbb{G} and \mathbb{G}' be two Carnot groups endowed with left-invariant homogeneous distances d and d' , respectively. Let $U \subseteq \mathbb{G}$ be Borel and let $f : U \rightarrow \mathbb{G}'$. Let $x \in U$ be a point of density one for U in \mathbb{G} . We say that f is *Pansu differentiable at $x \in U$* if there exists a Carnot homomorphism $df_x : \mathbb{G} \rightarrow \mathbb{G}'$ such that

$$\lim_{y \rightarrow x} \frac{d'(f(x)^{-1} \cdot f(y), df_x(x^{-1} \cdot y))}{d(x, y)} = 0.$$

Moreover we say that f is of class $C_{\mathbb{H}}^1$ in U if the map $x \mapsto df_x$ is well-defined and continuous from U to the space of Carnot homomorphisms from \mathbb{G} to \mathbb{G}' .

We shall recall the area formula for Lipschitz maps in Carnot groups which is due to Magnani. First we recall Rademacher theorem in this setting, which is due to Pansu. The following statement is in Magnani's work [160, Theorem 3.9].

Theorem 1.100. *Let \mathbb{G} and \mathbb{H} be two Carnot groups. Let us call Q the homogeneous dimension of \mathbb{G} . Then every Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \rightarrow (\mathbb{H}, d_{\mathbb{H}})$, where A is a measurable set, is differentiable \mathcal{H}^Q -a.e., i.e., for \mathcal{H}^Q -a.e. point x of A , there exists a Carnot homomorphism $df_x : \mathbb{G} \rightarrow \mathbb{H}$ such that*

$$(1.80) \quad \lim_{y \in A, y \rightarrow x} \frac{d_{\mathbb{H}}(f(x)^{-1} f(y), df_x(x^{-1}y))}{d_{\mathbb{G}}(x, y)} = 0.$$

Remark 1.101. We discuss here how df_x is defined. From [160, Step 1 and Step 2 of Theorem 3.9], and [160, Equation (3) in Step 1 of Theorem 3.9], it follows that

$$df_x(z) := \lim_{x\delta_t z \in A, t \rightarrow 0} \delta_{1/t} \left(f(x)^{-1} f(x\delta_t z) \right)$$

does exist for every x in a \mathcal{H}^Q -full measure set $A_\omega \subseteq A$, and every z in a countable dense subset of \mathbb{G} . Then, from [160, Step 2 of Theorem 3.9], for $x \in A_\omega$, the map df_x can be extended to all $z \in \mathbb{G}$, by density.

Definition 1.102 (Jacobian of a Lipschitz map). Given any Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \rightarrow (\mathbb{H}, d_{\mathbb{H}})$ we can define the *Jacobian*

$$J_Q(df_x) := \frac{\mathcal{H}^Q(df_x(B(0, 1)))}{\mathcal{H}^Q(B(0, 1))},$$

at every differentiability point x of f .

The following result is proved in [160, Theorem 4.4].

Theorem 1.103. *Given any Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \rightarrow (\mathbb{H}, d_{\mathbb{H}})$, where A is a measurable set, we have*

$$\int_A J_Q(df_x)d\mathcal{H}^Q(x) = \int_{\mathbb{H}} \#(f^{-1}(y) \cap A)d\mathcal{H}^Q(y).$$

Let us now pass to the definition of $C_{\mathbb{H}}^1$ -submanifold and to the definition of intrinsically C^1 rectifiability.

Definition 1.104 ($C_{\mathbb{H}}^1$ -submanifold). Given a Carnot group \mathbb{G} , we say that $\Sigma \subseteq \mathbb{G}$ is a $C_{\mathbb{H}}^1$ -submanifold of \mathbb{G} if there exists a Carnot group \mathbb{G}' such that for every $p \in \Sigma$ there exists an open neighborhood Ω of p and a function $f \in C_{\mathbb{H}}^1(\Omega; \mathbb{G}')$ such that

$$(1.81) \quad \Sigma \cap \Omega = \{g \in \Omega : f(g) = 0\},$$

and $df_p : \mathbb{G} \rightarrow \mathbb{G}'$ is surjective with $\text{Ker}(df_p)$ complemented. In this case we say that Σ is a $C_{\mathbb{H}}^1(\mathbb{G}, \mathbb{G}')$ -submanifold. When $\mathbb{G}' = \mathbb{R}$, we say that Σ is a $C_{\mathbb{H}}^1$ -hypersurface.

Definition 1.105 ($(\mathbb{G}, \mathbb{G}')$ -rectifiable set). Given two arbitrary Carnot groups \mathbb{G} and \mathbb{G}' of homogeneous dimensions Q and Q' , respectively, we say that $\Sigma \subseteq \mathbb{G}$ is a $(\mathbb{G}, \mathbb{G}')$ -rectifiable set if there exist countably many subsets Σ_i of \mathbb{G} that are $C_{\mathbb{H}}^1(\mathbb{G}, \mathbb{G}')$ -submanifolds, such that

$$\mathcal{H}^{Q-Q'} \left(\Sigma \setminus \bigcup_{i=1}^{+\infty} \Sigma_i \right) = 0.$$

The definition of rectifiability in Definition 1.105 is based on the assumption that the good class with which we are choosing to cover our rectifiable set is the class of intrinsically C^1 submanifolds. This approach has been taken to its utmost level of generality through the works [132, 163, 164].

We stress here that the notion of \mathcal{P} -rectifiability is strictly weaker than the notion in Definition 1.105. Indeed, the following result holds. This result is taken from [28, Proposition 1.2], and we will not present the proof here.

Proposition 1.106. *Let us fix \mathbb{G} and \mathbb{G}' two arbitrary Carnot groups of homogeneous dimensions Q and Q' respectively. Let us take $\Sigma \subseteq \mathbb{G}$ a $(\mathbb{G}, \mathbb{G}')$ -rectifiable set. Then $\mathcal{S}^{Q-Q'} \llcorner \Sigma$ is a $\mathcal{P}_{Q-Q'}$ -rectifiable measure with complemented tangents, namely a $\mathcal{P}_{Q-Q'}^c$ -rectifiable measure. Moreover, there exist a Carnot group \mathbb{G} , a Borel set $\Sigma \subseteq \mathbb{G}$, and $1 \leq h \leq Q$ such that $\mathcal{S}^h \llcorner \Sigma$ is a \mathcal{P}_h -rectifiable measure but, for every Carnot group \mathbb{G}' , Σ is not $(\mathbb{G}, \mathbb{G}')$ -rectifiable.*

Remark 1.107. We remark that the proof of Proposition 1.106 in [28] is heavily based on the results in [132, Lemma 3.4 & Corollary 3.6]. The two latter results in the reference are consequences of the area formula [132, Theorem 1.1]. As a consequence the approach in [132] is, in some sense, reversed with respect to our approach to rectifiability. The authors in [132] deal with the category of $C_{\mathbb{H}}^1(\mathbb{G}, \mathbb{G}')$ -regular submanifolds and prove the area formula relying on [132, Proposition 2.2], that ultimately tells that a Borel regular measure μ with positive and finite Federer's density θ with respect to the spherical Hausdorff measure \mathcal{S}^h admits the representation $\mu = \theta \mathcal{S}^h$. Then with this area formula they are able to prove the results that led to the proof of the above Proposition 1.106.

We stress that in the paper [33], together with A. Merlo, we proved an area formula for intrinsically differentiable graphs, see [33, Theorem 1.3], that extends the result of [132, Theorem 1.1].

Remark 1.108 (\mathcal{P} -rectifiability and $(\mathbb{G}, \mathbb{G}')$ -rectifiable sets). From Definition 1.104 and Definition 1.105 it follows that the tangent subgroup \mathbb{W} at almost every point of a $(\mathbb{G}, \mathbb{G}')$ -rectifiable set is normal and complemented. Moreover, from [132, Lemma 2.14, (iv)], every complementary subgroup of \mathbb{W} must be a Carnot subgroup of \mathbb{G} that in addition is isomorphic to \mathbb{G}' . This results in a lack of generality of this approach to rectifiability. Let us give here an

example where the previous phenomenon becomes clear. If we take L an horizontal subgroup in the first Heisenberg group \mathbb{H}^1 , on the one hand $\mathcal{S}^1 \llcorner L$ is \mathcal{P}_1 -rectifiable, on the other hand L is not $(\mathbb{H}^1, \mathbb{G}')$ -rectifiable for every Carnot group \mathbb{G}' since L is not normal.

Let us stress that the second part of Proposition 1.106 is not surprising. Indeed, the approach to rectifiability through intrinsically C^1 submanifolds described above and used in [132] is selecting rectifiable sets whose tangents are complemented normal subgroups of \mathbb{G} , see [132, Section 2.5] for a more detailed discussion. This can be easily understood if one thinks that the parametrizing class of objects is given by $C_{\mathbb{H}}^1$ -regular submanifolds Σ with complemented tangents $\text{Ker}(df_p)$ at $p \in \Sigma$, which are complemented (and normal) subgroups.

In some sense we could say that the approach of [132] is covering, in the utmost generality known up to now, the case of low-codimensional rectifiable sets in a Carnot group \mathbb{G} . It has been clear since the works [115, 178] that, already in the Heisenberg groups \mathbb{H}^n , one should approach the low-dimensional rectifiability in a different way with respect to the low-codimensional one. Indeed, in the low-dimensional case in \mathbb{H}^n , the authors in [115, 178] choose as a parametrizing class of objects the images of $C_{\mathbb{H}}^1$ -regular (or Lipschitz-regular) functions from subsets of \mathbb{R}^d to \mathbb{H}^n , with $1 \leq d \leq n$, see [115, Definition 3.1 & Definition 3.2], and [178, Definition 2.10 and Definition 3.13].

The bridge between the definition of \mathcal{P} -rectifiability and the ones discussed in the above paragraphs is done in [178] in the setting of Heisenberg groups, and in [128] in arbitrary homogeneous groups but only in the case of horizontal tangents. Let us stress that the result in [178, (i) \Leftrightarrow (iv) of Theorem 3.15] shows that in the Heisenberg groups the \mathcal{P} -rectifiability with tangents that are vertical subgroups is equivalent to the rectifiability given in terms of $C_{\mathbb{H}}^1$ -regular submanifolds. Moreover [178, (i) \Leftrightarrow (iv) of Theorem 3.14] shows that in the Heisenberg groups the \mathcal{P} -rectifiability with tangents that are horizontal subgroups is equivalent to the rectifiability given in terms of Lipschitz-regular images.

Moreover, very recently, in [128, Theorem 1.1], the authors prove a generalization of [178, Theorem 3.14] in arbitrary homogeneous groups. Namely they prove that in a homogeneous group the k -rectifiability of a set in the sense of Federer can be characterized with the fact that the tangent measures to the set are horizontal subgroups, or equivalently with the fact that there exists an approximate tangent plane that is a horizontal subgroup almost everywhere. In our setting this implies that the \mathcal{P} -rectifiability with tangents that are horizontal subgroups is equivalent to the rectifiability given in terms of Lipschitz-regular images, which is Federer's one. For results similar to the ones of [33, 128, 178] but in the different setting of the parabolic \mathbb{R}^n and in all the codimensions, we point out the recent [176].

One natural question to ask after the negative result in Proposition 1.106 is whether the notion of \mathcal{P} -rectifiability and the notion of $(\mathbb{G}, \mathbb{G}')$ -rectifiability do coincide in some cases. In [33, Corollary 5.3] we show that this is the case in the co-horizontal setting, namely we prove the following result. Here, we omit the proof, which is based on the slicing result in [33, Proposition 5.1], which is itself ultimately based on the rectifiability result presented in Proposition 2.37.

Proposition 1.109. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q , and let $1 \leq h \leq Q$ be a natural number. Let $\Gamma \subseteq \mathbb{G}$ be a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$. The following are equivalent*

1. $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h^c -rectifiable measure, and at $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$ the support of every tangent measure is complemented by a horizontal subgroup.
2. Γ is $C_{\mathbb{H}}^1(\mathbb{G}, \mathbb{R}^{Q-h})$ -rectifiable.

Thus the following question becomes natural. A positive answer to the following question would imply that, whenever they can agree, the notions of \mathcal{P} -rectifiable set and the notion of intrinsically C^1 rectifiable set in the sense Definition 1.104 do agree.

Question 1. Let ϕ be a \mathcal{P}_h -rectifiable measure on \mathbb{G} such that at ϕ -almost every point the support of every tangent measure is the Haar measure of a given normal complemented subgroup. Understand whether \mathbb{G} can be covered ϕ -almost everywhere with the countable union of $C^1_{\mathbb{H}}(\mathbb{G}, \mathbb{G}')$ -submanifolds (where \mathbb{G}' need not to be unique), see Definition 1.104.

Let us now discuss, with more details, the case of co-horizontal submanifolds in a Carnot group.

Definition 1.110 ($\nabla_{\mathbb{W}}, \nabla_{\mathbb{L}}$). Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} , with \mathbb{L} horizontal and k -dimensional and let $f \in C^1_{\mathbb{H}}(U; \mathbb{R}^k)$. Consider an adapted basis (X_1, \dots, X_n) of the Lie algebra \mathfrak{g} such that $\mathbb{L} = \exp(\text{span}\{X_1, \dots, X_k\})$ and $\mathbb{W} = \exp(\text{span}\{X_{k+1}, \dots, X_n\})$. Then, we define $\nabla_{\mathbb{L}}f$ and $\nabla_{\mathbb{W}}f$ by setting

$$\nabla_{\mathbb{L}}f := \begin{pmatrix} X_1 f^{(1)} & \dots & X_k f^{(1)} \\ \vdots & \ddots & \vdots \\ X_1 f^{(k)} & \dots & X_k f^{(k)} \end{pmatrix}, \quad \nabla_{\mathbb{W}}f := \begin{pmatrix} X_{k+1} f^{(1)} & \dots & X_m f^{(1)} \\ \vdots & \ddots & \vdots \\ X_{k+1} f^{(k)} & \dots & X_m f^{(k)} \end{pmatrix}.$$

In particular, one has that, in exponential coordinates, $\nabla_{\mathbb{H}}f = (\nabla_{\mathbb{L}}f \mid \nabla_{\mathbb{W}}f)$, where $\nabla_{\mathbb{H}}$ is the intrinsic differential, see Definition 1.99, in coordinates.

We recall the notion of co-horizontal $C^1_{\mathbb{H}}$ -submanifold of arbitrary codimension, see [141, Definition 3.3.4]. We stress that we changed the terminology with respect to [141, Definition 3.3.4]. What the author calls co-Abelian submanifold, for us is a co-horizontal submanifold. This is a particular case of Definition 1.104, when $\mathbb{G}' \equiv \mathbb{R}^k$.

Definition 1.111 (co-horizontal $C^1_{\mathbb{H}}$ -submanifold). Let \mathbb{G} be a Carnot group of rank m and let $1 \leq k \leq m$. We say that $\Sigma \subseteq \mathbb{G}$ is a *co-horizontal $C^1_{\mathbb{H}}$ -submanifold of codimension k* if, for every $p \in \Sigma$, there exist a neighborhood U of p and a map $f \in C^1_{\mathbb{H}}(U; \mathbb{R}^k)$ such that

$$(1.82) \quad \Sigma \cap U = \{g \in U : f(g) = 0\},$$

and the Pansu-differential $df_p: \mathbb{G} \rightarrow \mathbb{R}^k$ of f is surjective.

We say that Σ is a codimension k *co-horizontal $C^1_{\mathbb{H}}$ -submanifold with complemented tangents* if, in addition, given a representation around p as in (1.82), the homogeneous subgroup $\text{Ker}(df_p)$ admits a horizontal complement (of dimension k). In this case, we call $\text{Ker}(df_p)$ the *homogeneous tangent space to Σ at p* . This homogeneous subgroup at p is independent of the choice of f , see [169, Theorem 1.7].

A first natural question one could try to answer is whether it is possible to (locally) write a $C^1_{\mathbb{H}}$ -submanifold as an intrinsic graph of a function. The answer to the previous question is affirmative for $C^1_{\mathbb{H}}$ -hypersurfaces. Moreover the graphing function is intrinsically Lipschitz according to Definition 1.87 (actually it is UID, see the forthcoming discussion), while it is in general neither Euclidean Lipschitz nor Lipschitz with respect to any sub-Riemannian distance, see [114, Example 3.3 and Proposition 3.4].

A more general implicit function theorem was proved by Magnani in [169, Theorem 1.4]. This theorem holds for arbitrary (\mathbb{G}, \mathbb{M}) -regular submanifolds with the additional property that $\text{Ker}(df_x)$ has a complementary subgroup in \mathbb{G} , where x is the point around which we want to parametrize the submanifold. From [169, Eq. (1.8)] it follows, also in this case, that the parametrization is intrinsically Lipschitz. The validity of the implicit function theorem leads the way to a very general definition of (\mathbb{G}, \mathbb{M}) -regular sets for \mathbb{G} , where \mathbb{M} is just a homogeneous group, given in [169, Definition 10.2], compare with the above Definition 1.104.

We will not deal with objects at this level of generality, but we refer the interested reader to [169, Sections 10,11,12]. As already pointed out above, the class of intrinsic regular submanifolds is also studied in [132], where area and coarea formulae are proved. For an alternative proof of the implicit function theorem, one can also see [132, Section 2.5].

Coming back to the co-horizontal case, we remark that, if $\Sigma \subseteq \mathbb{G}$ is a co-horizontal $C_{\mathbb{H}}^1$ -submanifold with complemented tangents, then one can use the implicit function theorem, see [113, Theorem 2.1] for the one-codimensional case, and see [169, Theorem 1.4] for the more general statement, to locally represent the submanifold as a graph of a function $\varphi: U \subseteq \mathbb{W} := \text{Ker}(df_p) \rightarrow \mathbb{L}$, with \mathbb{W} and \mathbb{L} complementary subgroups.

A finer study on the regularity of the parametrizing function of a $C_{\mathbb{H}}^1$ -submanifold has been initiated in [16] in the setting of Heisenberg groups \mathbb{H}^n , for the class of $C_{\mathbb{H}}^1$ -hypersurfaces. For this study in arbitrary CC-spaces, see also [81]. In [16], with this aim, the authors introduced the notion of uniform intrinsic differentiability that we gave in Definition 1.95.

Building upon an implicit function theorem, the authors in [16] prove that in \mathbb{H}^n the graphing map φ for a $C_{\mathbb{H}}^1$ -hypersurface is UID. The idea behind this implication is the following: a function $f \in C_{\mathbb{H}}^1$ not only has continuous derivatives, but also its horizontal gradient $\nabla_{\mathbb{H}}f$ uniformly approximates f at first order, see [169, Theorem 1.2], and [132, Proposition 2.4]. This notion is sometimes referred to as strict differentiability. This fact has a strong analogy with the Euclidean setting. Indeed, in the Euclidean framework, a function f with continuous partial derivatives is Fréchet-differentiable, and the proof relies on a use of a mean value inequality, that is exactly what one finds in [169, Theorem 1.2], and [16, Lemma 4.2]. We stress that [16, Lemma 4.2] is an instance of the *stratified mean value theorem* that can be found in [106]. Finally, the uniform differentiability of f translates into the uniform intrinsic differentiability of φ .

The fact that the graphing function is UID was proved in the case of co-horizontal $C_{\mathbb{H}}^1$ -submanifolds in \mathbb{H}^n in [37], and more in general for co-horizontal $C_{\mathbb{H}}^1$ -submanifolds with complemented tangents in every Carnot group, in [93]. The converse implication, i.e., the fact that the graph of a UID function is a $C_{\mathbb{H}}^1$ -submanifold, was firstly shown to be true in [16, 37] in the setting of \mathbb{H}^n , and lately generalized in [93] for arbitrary Carnot groups \mathbb{G} to functions with horizontal target, see the forthcoming Proposition 1.112 for a precise statement. Notice that the lack of generality in the statement, namely, the fact that one restricts the target to be horizontal, is due to the fact that a generalized version of Whitney's extension theorem, beyond the case in which the target is horizontal, is still not known to be true.

The following proposition follows from [93, Theorem 4.1 and Theorem 4.6] and relates level sets of \mathbb{R}^k -valued $C_{\mathbb{H}}^1$ -functions, and ultimately co-horizontal $C_{\mathbb{H}}^1$ -submanifolds with complemented tangents, with uniformly intrinsically differentiable functions.

Proposition 1.112 ([93, Theorem 4.1 and Theorem 4.6]). *Let \mathbb{W} and \mathbb{L} be two complementary subgroups of a Carnot group \mathbb{G} , with \mathbb{L} horizontal and k -dimensional, take $U \subseteq \mathbb{W}$ open and $\varphi \in \text{UID}(U, \mathbb{W}; \mathbb{L})$. Then, for every $a \in U$, there exist a neighborhood V of $a \cdot \varphi(a)$ in \mathbb{G} , and $f \in C_{\mathbb{H}}^1(V; \mathbb{R}^k)$, such that*

$$\text{graph}(\varphi)(U) \cap V =: \Phi(U) \cap V = \{g \in V : f(g) = 0\},$$

and, for every $g \in V$, the Pansu differential $(df_g)_{\mathbb{L}}: \mathbb{L} \rightarrow \mathbb{R}^k$ is bijective. As a consequence $\text{graph}(\varphi)$ is a co-horizontal $C_{\mathbb{H}}^1$ -submanifold of codimension k , with tangents complemented by \mathbb{L} . Moreover, if (X_1, \dots, X_n) is an adapted basis of the Lie algebra \mathfrak{g} such that $\mathbb{L} = \exp(\text{span}\{X_1, \dots, X_k\})$ and $\mathbb{W} = \exp(\text{span}\{X_{k+1}, \dots, X_n\})$, then $\det \nabla_{\mathbb{L}}f \neq 0$ and, in exponential coordinates, one has

$$(1.83) \quad \nabla^{\varphi} \varphi(a) = -(\nabla_{\mathbb{L}}f(\Phi(a)))^{-1} \nabla_{\mathbb{W}}f(\Phi(a)), \quad \forall a \in U.$$

For the definition of $\nabla^\varphi\varphi$, $\nabla_{\mathbb{W}}$ and $\nabla_{\mathbb{L}}$ we refer to Definition 1.97 and Definition 1.110.

On the other hand, if $1 \leq k \leq m$ and Σ is a codimension k co-horizontal $C_{\mathbb{H}}^1$ -submanifold with complemented tangents, then, for every $p \in \Sigma$, there exist two complementary subgroups \mathbb{W} and \mathbb{L} of \mathbb{G} with \mathbb{L} horizontal and k -dimensional, a neighborhood $V \subseteq \mathbb{G}$ of p and $\varphi \in \text{UID}(U, \mathbb{W}; \mathbb{L})$, with $U = P_{\mathbb{W}}(V)$, such that

$$\Sigma \cap V = \text{graph}(\varphi).$$

Remark 1.113. Notice that, in the setting of Proposition 1.112, in the case $k = 1$, up to possibly changing basis, one may assume $X_1 f \neq 0$ on V , and, in coordinates, formula (1.83) reads as

$$(1.84) \quad \nabla^\varphi\varphi(a) = - \left(\frac{X_2 f}{X_1 f}, \dots, \frac{X_m f}{X_1 f} \right) \circ \Phi(a), \quad \forall a \in U.$$

Remark 1.114 (Tangent subgroups to $C_{\mathbb{H}}^1$ -submanifolds). From the previous Proposition 1.112 it directly follows that every co-horizontal $C_{\mathbb{H}}^1$ -submanifold with complemented tangents has Hausdorff tangent everywhere, and moreover such Hausdorff tangent is the homogeneous tangent space as defined in Definition 1.111. For a proof of this property in a more general context one can see [169, Theorem 1.7], or [132, Lemma 2.14, point (iii)]. This convergence is moreover locally uniform: we will not use this information, but this comes from [141, Theorem 3.1.1].

Remark 1.115 (Hausdorff dimension of a co-horizontal submanifold). Let Σ be a co-horizontal $C_{\mathbb{H}}^1$ -submanifold of codimension k with complemented tangents in a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance. Hence the Hausdorff dimension of Σ is $Q - k$. This comes from the implicit function theorem (see e.g., [96, Theorem A.5], and [169, Theorem 1.4]), that allows to locally write the hypersurface as the graph of an intrinsically Lipschitz function. Thus, the local estimate of the Hausdorff measure of the graph of an intrinsically Lipschitz function in [109, Theorem 2.3.7] gives the sought conclusion.

Fine Structure of \mathcal{P} -rectifiable measures

In this chapter we are going to study the fine structure properties of \mathcal{P} -rectifiable measures. The content of this chapter is a selection of the results obtained in [31–33] together with A. Merlo. I stress that the two papers [31, 32] are two companion papers derived from [28] that is available on arXiv as version 2 in the submission history of the file [31].

In this chapter, if not otherwise specified, \mathbb{G} will be a fixed Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a homogeneous left-invariant distance d . Moreover, h will be a natural number in the set $\{1, \dots, Q\}$.

In Section 1 we first prove that the support of an arbitrary \mathcal{P}_h -rectifiable measure on a Carnot group can be covered ϕ -almost everywhere with sets with the cone property with arbitrarily small opening. In the case the tangents of the measure are complemented ϕ -almost everywhere, we show that the support of ϕ can be covered ϕ -almost everywhere with intrinsically Lipschitz graphs with arbitrarily small Lipschitz constants.

In Section 2 we exploit the results in Section 1 to prove that an arbitrary \mathcal{P}_h -rectifiable measure ϕ with complemented tangents has h -density ϕ -almost everywhere.

In Section 3 we slightly improve the results in Section 1, and we prove that the support of a \mathcal{P}_h -rectifiable measure ϕ with complemented tangents can be covered ϕ -almost everywhere with intrinsically Lipschitz graphs with arbitrarily small Lipschitz constant that in addition are intrinsically differentiable graphs almost everywhere.

Finally, in Section 4 we give equivalent properties for a measure $\mathcal{S}^h \llcorner \Gamma$, where $\Gamma \subseteq \mathbb{G}$, and $0 < \mathcal{S}^h(\Gamma) < +\infty$, to be a \mathcal{P}_h -rectifiable measure with complemented tangents, and we prove that the h -density of the centered Hausdorff measure $\mathcal{C}^h \llcorner \Gamma$ is 1 $\mathcal{S}^h \llcorner \Gamma$ -almost everywhere.

1. Covering the support of \mathcal{P} -rectifiable measures with sets with the cone property

In this section we aim at proving the next two theorems, that are two of the main results contained in [28]. In the first result we prove that the support of a \mathcal{P}_h -rectifiable measure ϕ , see Definition 1.58, can be covered ϕ -almost all by sets with the cone property with arbitrarily small opening.

Theorem 2.1. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a \mathcal{P}_h -rectifiable measure on \mathbb{G} .*

Then \mathbb{G} can be covered ϕ -almost everywhere with countably many compact sets with the cone property with arbitrarily small opening. In other words for every $\alpha > 0$ we have

$$\phi\left(\mathbb{G} \setminus \bigcup_{i=1}^{+\infty} \Gamma_i\right) = 0,$$

where Γ_i are compact $C_{\mathbb{V}_i}(\alpha)$ -sets, where \mathbb{V}_i are homogeneous subgroups of \mathbb{G} of homogeneous dimension h .

If we ask that the tangents are complemented subgroups, we can improve the previous result. In particular we can take the Γ_i 's to be intrinsically Lipschitz graphs. For the definition of intrinsically Lipschitz function, we refer the reader to Definition 1.87.

Theorem 2.2. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a \mathcal{P}_h^c -rectifiable measure on \mathbb{G} , i.e., a \mathcal{P}_h -rectifiable measure with tangents that are complemented almost everywhere.*

Then \mathbb{G} can be covered ϕ -almost everywhere with countably many compact graphs of intrinsically Lipschitz functions with arbitrarily small Lipschitz constant. In other words for every $\alpha > 0$ we have

$$\phi\left(\mathbb{G} \setminus \bigcup_{i=1}^{+\infty} \Gamma_i\right) = 0,$$

where $\Gamma_i = \text{graph}(\varphi_i)$ are compact sets, with $\varphi_i : A_i \subseteq \mathbb{V}_i \rightarrow \mathbb{L}_i$ being an intrinsically α -Lipschitz function between a compact subset A_i of \mathbb{V}_i , which is a homogeneous subgroup of \mathbb{G} of homogeneous dimension h , and \mathbb{L}_i , which is a subgroup complementary to \mathbb{V}_i .

1.1. Proof. In this subsection section we prove Theorem 2.1 and Theorem 2.2. Let us recall that for an arbitrary Radon measure ϕ on \mathbb{G} supported on a compact set, and for $\vartheta, \gamma \in \mathbb{N}$, we can define $E(\vartheta, \gamma)$ as in Definition 1.9. Moreover, as ϑ, γ vary, the sets $E(\vartheta, \gamma)$ cover ϕ -almost all of \mathbb{G} . In this subsection ϕ will be an arbitrary Radon measure supported on a compact set $K \subseteq \mathbb{G}$, and ϑ, γ will be arbitrary natural numbers.

The first step in order to prove Theorem 2.1 is to observe the following general property, that can be made quantitative at arbitrary points $x \in E(\vartheta, \gamma)$: if the measure $\mathcal{S}^h \llcorner x\mathbb{V}$, with $\mathbb{V} \in Gr(h)$, is sufficiently near to ϕ in a precise measure theoretic sense at the scale r around x , then in some ball of center x and with radius comparable with r , the points in the set $E(\vartheta, \gamma)$ are not too distant from $x\mathbb{V}$. Roughly speaking, if we denote with $F_{x,r}$ the functional that measures the distance between measures on the ball $\overline{B}(x, r)$, see Definition 1.65, we prove that the following implication holds

$$(2.1) \quad \begin{aligned} & \text{if there exist } \Theta, \delta > 0 \text{ such that } F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x\mathbb{V}) \leq \delta r^{h+1}, \\ & \text{then } E(\vartheta, \gamma) \cap \overline{B}(x, r) \subseteq \overline{B}(x\mathbb{V}, \omega(\delta)r) \text{ where } \omega \text{ is continuous and } \omega(0) = 0. \end{aligned}$$

For the precise statement of the implication in (2.1), see Proposition 2.5. Let us remark that when ϕ is a \mathcal{P}_h -rectifiable measure, then for ϕ -almost every $x \in \mathbb{G}$ the bound on $F_{x,r}$ in the premise of (2.1) is satisfied with $\mathbb{V}(x) \in Gr(h)$, and for arbitrarily small $\delta > 0$, whenever $r < r_0(x, \delta)$. Thus for \mathcal{P}_h -rectifiable measures we deduce that the estimate in the conclusion of (2.1) holds for arbitrarily small δ , and with $r < r_0(x, \delta)$. This latter estimate easily implies, by a very general geometric argument, that $E(\vartheta, \gamma) \cap \overline{B}(x, r) \subseteq xC_{\mathbb{V}(x)}(\alpha)$ for arbitrarily small α and for all $r < r_0(x, \alpha)$. For the latter assertion we refer the reader to Proposition 2.7. The proof of Theorem 2.1 is thus concluded by joining together the previous observations and by the general cone-rectifiability criterion in Proposition 2.9.

In order to prove Theorem 2.2 we follow the path of the proof of Theorem 2.1, which we discussed above, but we have to pay attention to one technical detail. We have to split the subset of the Grassmannian $Gr(h)$ made by the homogeneous subgroups \mathbb{V} that admit at least one complementary subgroup \mathbb{L} into countable subsets according to the value of $\varepsilon_1(\mathbb{V}, \mathbb{L})$. Indeed, if we work in an arbitrary Carnot group \mathbb{G} and one of its homogeneous subgroups \mathbb{V} admits a complementary subgroup \mathbb{L} we already proved that there exists a constant $\varepsilon_1 := \varepsilon_1(\mathbb{V}, \mathbb{L})$ such that every $C_{\mathbb{V}}(\varepsilon_1)$ -set is the intrinsic graph of a function $f : A \subseteq \mathbb{V} \rightarrow \mathbb{L}$, see Proposition 1.89. Then, in order to prove Theorem 2.2, we have to change the argument of

Theorem 2.1 by paying attention to the fact that we want to control the opening of the final $C_{\mathbb{V}_i}(\alpha_i)$ -sets with $\alpha_i < \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i)$. This is what we do in the forthcoming Theorem 2.10: we prove a refinement of Theorem 2.11 in which we further ask that the opening of the cones is controlled above also by some a priori defined function $\mathbf{F}(\mathbb{V}, \mathbb{L})$. Let us now start with some preliminary definitions and results.

Definition 2.3. Let us fix $x \in \mathbb{G}$, $r > 0$ and ϕ a Radon measure on \mathbb{G} . We define $\Pi_\delta(x, r)$ to be the subset of homogeneous subgroups $\mathbb{V} \in Gr(h)$ for which there exists a $\Theta > 0$ such that

$$(2.2) \quad F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x \mathbb{V}) \leq 2\delta r^{h+1}.$$

Definition 2.4. For every $\vartheta \in \mathbb{N}$ we define $\delta_{\mathbb{G}} = \delta_{\mathbb{G}}(h, \vartheta) := \vartheta^{-1} 2^{-(4h+5)}$.

In the following proposition we prove that if ϕ is sufficiently $d_{x,r}$ -near to $\mathfrak{M}(h)$, see Definition 1.71 for the definition of $d_{x,r}$, then $E(\vartheta, \gamma)$ is at a controlled distance from a homogeneous subgroup \mathbb{V} .

Proposition 2.5. Let $x \in E(\vartheta, \gamma)$, fix $\delta < \delta_{\mathbb{G}}$, where $\delta_{\mathbb{G}}$ is defined in Definition 2.4, and set $0 < r < 1/\gamma$. Then for every $\mathbb{V} \in \Pi_\delta(x, r)$, see Definition 2.3, we have

$$(2.3) \quad \sup_{w \in E(\vartheta, \gamma) \cap \overline{B}(x, r/4)} \frac{\text{dist}(w, x\mathbb{V})}{r} \leq 2^{1+1/(h+1)} \vartheta^{1/(h+1)} \delta^{1/(h+1)} =: C_5(\vartheta, h) \delta^{1/(h+1)}.$$

Proof. Let \mathbb{V} be any element of $\Pi_\delta(x, r)$ and suppose $\Theta > 0$ is such that

$$\left| \int f d\phi - \Theta \int f d\mathcal{S}^h \llcorner x \mathbb{V} \right| \leq 2\delta r^{h+1}, \text{ for every } f \in \text{Lip}_1^+(\overline{B}(x, r)).$$

Since the function $g(w) := \min\{\text{dist}(w, \overline{B}(x, r)^c), \text{dist}(w, x\mathbb{V})\}$ belongs to $\text{Lip}_1^+(\overline{B}(x, r))$, we deduce that

$$\begin{aligned} 2\delta r^{h+1} &\geq \int g(w) d\phi(w) - \Theta \int g(w) d\mathcal{S}^h \llcorner x \mathbb{V} \\ &= \int g(w) d\phi(w) \geq \int_{\overline{B}(x, r/2)} \min\{r/2, \text{dist}(w, x\mathbb{V})\} d\phi(w). \end{aligned}$$

Suppose that y is a point in $\overline{B}(x, r/4) \cap E(\vartheta, \gamma)$ furthest from $x\mathbb{V}$ and let $D := \text{dist}(y, x\mathbb{V})$. If $D \geq r/8$, this would imply that

$$\begin{aligned} 2\delta r^{h+1} &\geq \int_{\overline{B}(x, r/2)} \min\{r/2, \text{dist}(w, x\mathbb{V})\} d\phi(w) \\ &\geq \int_{\overline{B}(y, r/16)} \min\{r/2, \text{dist}(w, x\mathbb{V})\} d\phi(w) \geq \frac{r}{16} \phi(\overline{B}(y, r/16)) \geq \frac{r^{h+1}}{\vartheta 16^{h+1}}, \end{aligned}$$

where the last inequality follows from the definition of $E(\vartheta, \gamma)$. The previous inequality would imply $\delta \geq \vartheta^{-1} 2^{-(4h+5)}$, which is not possible since $\delta < \delta_{\mathbb{G}} = \vartheta^{-1} 2^{-(4h+5)}$, see Definition 2.4. This implies that $D \leq r/8$ and as a consequence, we have

$$(2.4) \quad \begin{aligned} 2\delta r^{h+1} &\geq \int_{\overline{B}(x, r/2)} \min\{r/2, \text{dist}(w, x\mathbb{V})\} d\phi(w) \\ &\geq \int_{\overline{B}(y, D/2)} \min\{r/2, \text{dist}(w, x\mathbb{V})\} d\phi(w) \geq \frac{D \phi(\overline{B}(y, D/2))}{2} \geq \vartheta^{-1} \left(\frac{D}{2}\right)^{h+1}, \end{aligned}$$

where the second inequality comes from the fact that $\overline{B}(y, D/2) \subseteq \overline{B}(x, r/2)$. This implies thanks to (2.4), that

$$\sup_{w \in E(\vartheta, \gamma) \cap \overline{B}(x, r/4)} \frac{\text{dist}(w, x\mathbb{V})}{r} \leq \frac{D}{r} \leq 2^{1+1/(h+1)} \vartheta^{1/(h+1)} \delta^{1/(h+1)} = C_5(\vartheta, h) \delta^{1/(h+1)}.$$

□

Remark 2.6. Notice that a priori $\Pi_\delta(x, r)$ in the statement of Proposition 2.5 may be empty. Nevertheless it is easy to notice, by using the definitions, that if $d_{x,r}(\phi, \mathfrak{M}) \leq \delta$ then $\Pi_\delta(x, r)$ is nonempty.

In the following proposition we show that if we are at a point $x \in E(\vartheta, \gamma)$ for which the h -tangents are flat, then locally around x the set $E(\vartheta, \gamma)$ enjoys an appropriate cone property with arbitrarily small opening.

Proposition 2.7. *For every $\alpha > 0$ and every $x \in E(\vartheta, \gamma)$ for which*

$$\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda > 0\},$$

for some $\mathbb{V}(x) \in \text{Gr}(h)$, there exists a $\rho(\alpha, x) > 0$ such that whenever $0 < r < \rho$ we have

$$E(\vartheta, \gamma) \cap \overline{B}(x, r) \subseteq xC_{\mathbb{V}(x)}(\alpha).$$

Proof. Let us fix $\alpha > 0$. Let us fix $x \in E(\vartheta, \gamma)$ and $\mathbb{V}(x) \in \text{Gr}(h)$ such that $\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda > 0\}$. Thus, by using Proposition 1.74, we conclude that

$$\liminf_{r \rightarrow 0} \inf_{\Theta > 0} \frac{F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x \mathbb{V}(x))}{r^{h+1}} = 0.$$

From the previous equality it follows that for every $\varepsilon > 0$ there exists $1/\gamma > r_0(\varepsilon) > 0$ such that

$$(2.5) \quad \inf_{\Theta > 0} F_{x,r}(\phi, \Theta \mathcal{S}^h \llcorner x \mathbb{V}(x)) \leq \varepsilon r^{h+1}, \quad \text{whenever } 0 < r \leq r_0(\varepsilon).$$

Now we aim at proving that, for $\varepsilon > 0$ small enough, $E(\vartheta, \gamma) \cap \overline{B}(x, r_0(\varepsilon)/4) \subseteq xC_{\mathbb{V}(x)}(\alpha)$. In order to prove this we notice that (2.5) and Proposition 2.5 imply that, for $\varepsilon > 0$ sufficiently small, the following inequality holds

$$(2.6) \quad \sup_{p \in E(\vartheta, \gamma) \cap \overline{B}(x, r/4)} \text{dist}(p, x \mathbb{V}(x)) \leq C_5(h, \vartheta) \varepsilon^{1/(h+1)} r, \quad \text{whenever } 0 < r \leq r_0(\varepsilon).$$

Indeed, from (2.5) it follows that $\mathbb{V}(x) \in \Pi_\varepsilon(x, r)$ for every $0 < r \leq r_0$, see Definition 2.3; so that it suffices to choose $\varepsilon < \delta_{\mathbb{G}} = \vartheta^{-1} 2^{-(4h+5)}$, see Definition 2.4, in order to apply Proposition 2.5 and conclude (2.6).

Now let us take $\varepsilon < \delta_{\mathbb{G}}$ so small that the inequality $8C_5(h, \vartheta) \varepsilon^{1/(h+1)} < \alpha$ holds. We finally prove $E(\vartheta, \gamma) \cap \overline{B}(x, r_0(\varepsilon)/4) \subseteq xC_{\mathbb{V}(x)}(\alpha)$. Indeed, let $p \in E(\vartheta, \gamma) \cap \overline{B}(x, r_0(\varepsilon)/4)$, and $k \geq 3$ be such that $r_0 2^{-k} < \|x^{-1} \cdot p\| \leq r_0 2^{-k+1}$. Since $p \in E(\vartheta, \gamma) \cap \overline{B}(x, (r_0 2^{-k+3})/4)$, from (2.6) we get

$$\text{dist}(p, x \mathbb{V}(x)) \leq C_5(h, \vartheta) \varepsilon^{1/(h+1)} r_0 2^{-k+3} \leq 8C_5(h, \vartheta) \varepsilon^{1/(h+1)} \|x^{-1} \cdot p\| \leq \alpha \|x^{-1} \cdot p\|,$$

thus showing the claim. □

We now prove a cone-type rectifiability criterion that will be useful in combination with the previous results in order to split the support of a \mathcal{P}_h or a \mathcal{P}_h^c -rectifiable measures with sets that have the cone property. We will need the following estimate on the norm of the conjugate in a Carnot group.

Lemma 2.8. *There exists a constant $C_6 > 1$ such that for every $x, y \in \overline{B}(0, \ell)$ we have*

$$\|y^{-1} \cdot x \cdot y\| \leq C_6(\ell, \mathbb{G}) \|x\|^{1/\kappa},$$

where κ is the step of the group.

Proof. This follows immediately from [110, Lemma 3.12]. □

Proposition 2.9 (Cone-rectifiability criterion). *Suppose that E is a closed subset of \mathbb{G} for which there exists a countable family $\mathcal{F} \subseteq \text{Gr}(h)$ and a function $\alpha : \mathcal{F} \rightarrow (0, 1)$ such that for every $x \in E$ there exist $\rho(x) > 0$, and $\mathbb{V}(x) \in \mathcal{F}$ for which*

$$(2.7) \quad \overline{B}(x, r) \cap E \subseteq xC_{\mathbb{V}(x)}(\alpha(\mathbb{V}(x))),$$

whenever $0 < r < \rho(x)$. Then, there are countably many compact $C_{\mathbb{V}_i}(3\beta_i)$ -sets Γ_i such that $\mathbb{V}_i \in \mathcal{F}$, $\alpha(\mathbb{V}_i) < \beta_i < 2\alpha(\mathbb{V}_i)$, and

$$(2.8) \quad E = \bigcup_{i \in \mathbb{N}} \Gamma_i.$$

Proof. Let us split E in the following way. Let $i, j, k \in \mathbb{N}$, and let $G(i, j, k)$ be the subset of those $x \in E \cap \overline{B}(0, k)$ for which

$$\overline{B}(x, r) \cap E \subseteq xC_{\mathbb{V}_i}(\alpha(\mathbb{V}_i)),$$

for every $0 < r < 1/j$. Then, from the hypothesis, it follows $E = \bigcup_{i, j, k \in \mathbb{N}} G(i, j, k)$. Since E is closed, it is not difficult to see that $G(i, j, k)$ is closed too. Let us fix $i, j, k \in \mathbb{N}$, some $\beta_i < 1$ with $\alpha(\mathbb{V}_i) < \beta_i < 2\alpha(\mathbb{V}_i)$, and let us prove that $G(i, j, k)$ can be covered with countably many compact $C_{\mathbb{V}_i}(3\beta_i)$ -sets. Since $i, j, k \in \mathbb{N}$ are fixed from now on we assume without loss of generality that $G(i, j, k) = E$ so that we can drop the indices.

Let us take $\{q_\ell\}$ a dense subset of E , and let us define the closed tubular neighbourhood of $q_\ell \mathbb{V}$

$$(2.9) \quad S(\ell) := \overline{B}(q_\ell \mathbb{V}, 2^{-\kappa} j^{-\kappa} C_6(14k, \mathbb{G})^{-\kappa} \beta^\kappa),$$

where we recall that κ is the step of the group, and where C_6 is defined in (2.8). We will now prove that $S(\ell) \cap E$ is a $C_{\mathbb{V}}(3\beta)$ -set, or equivalently that for every $p \in S(\ell) \cap E$ we have

$$(2.10) \quad S(\ell) \cap E \subseteq p \cdot C_{\mathbb{V}}(3\beta).$$

If $q \in S(\ell) \cap E \cap \overline{B}(p, 1/(2j))$, the inclusion (2.10) holds thanks to our assumptions on E . If on the other hand $q \in S(\ell) \cap E \setminus \overline{B}(p, 1/(3j))$, let $p^*, q^* \in \mathbb{V}$ be such that $\text{dist}(p, q_\ell \mathbb{V}) = \|(p^*)^{-1} q_\ell^{-1} p\|$, and $\text{dist}(q, q_\ell \mathbb{V}) = \|(q^*)^{-1} q_\ell^{-1} q\|$. Let us prove that $\|q^*\| \leq 4k$ and $\|p^*\| \leq 4k$. This is due to the fact that

$$\|q^*\| - \|q_\ell\| - \|q\| \leq \|(q^*)^{-1} q_\ell^{-1} q\| = \text{dist}(q, q_\ell \mathbb{V}) \leq 1,$$

where the last inequality follows from the definition of $S(\ell)$, see (2.9). From the previous inequality it follows that $\|q^*\| \leq 2k + 1$, since $q, q_\ell \in \overline{B}(0, k)$. A similar computation proves the bound for $\|p^*\|$ and this implies that

$$\|p^{-1} \cdot q_\ell \cdot p^*\| + \|(p^*)^{-1} \cdot q^*\| \leq \|p^{-1}\| + \|q_\ell\| + 2\|p^*\| + \|q^*\| \leq 14k.$$

The application of Lemma 2.8 and the fact that $(q^*)^{-1} p^*$ and $p^{-1} q_\ell p^*$ are in $\overline{B}(0, 14k)$, due to the previous inequality, imply that

$$(2.11) \quad \begin{aligned} \text{dist}(p^{-1} q, \mathbb{V}) &\leq \|(q^*)^{-1} p^* \cdot p^{-1} q\| = \|(q^*)^{-1} p^* \cdot p^{-1} \cdot q_\ell p^* (p^*)^{-1} q^* (q^*)^{-1} q_\ell^{-1} \cdot q\| \\ &\leq \|(q^*)^{-1} p^* \cdot p^{-1} q_\ell p^* \cdot (p^*)^{-1} q^*\| + \|(q^*)^{-1} q_\ell^{-1} q\| \\ &\leq C_6(14k, \mathbb{G}) \|p^{-1} q_\ell p^*\|^{1/\kappa} + \text{dist}(q, q_\ell \mathbb{V}) \\ &= C_6(14k, \mathbb{G}) \text{dist}(p, q_\ell \mathbb{V})^{1/\kappa} + \text{dist}(q, q_\ell \mathbb{V}). \end{aligned}$$

Finally, thanks to (2.9) and (2.11) we infer

$$\text{dist}(p^{-1} q, \mathbb{V}) \leq \frac{C_6(14k, \mathbb{G}) + 1}{2j C_6(14k, \mathbb{G})} \beta \leq \beta j^{-1} \leq 3\beta \|p^{-1} q\|,$$

thus showing (2.10) in the remaining case. In conclusion we have proved that for every $i, j, k, \ell \in \mathbb{N}$, the sets $G(i, j, k) \cap S(\ell)$ are $C_{\mathbb{V}_i}(3\beta_i)$ -sets. This concludes the proof since

$$E \subseteq \bigcup_{i, j, k, \ell \in \mathbb{N}} G(i, j, k) \cap S(\ell),$$

and on the other hand every $G(i, j, k) \cap S(\ell)$ is a bounded and closed, thus compact, $C_{\mathbb{V}_i}(3\beta_i)$ -set. The fact that the sets $G(i, j, k) \cap S(\ell)$ are contained in E follows by definition, thus concluding the proof of the equality. \square

In the following, with the symbol $\text{Sub}(h)$, we denote the subset of $Gr_c(h) \times Gr_c(Q - h)$ defined by

$$(2.12) \quad \{(\mathbb{V}, \mathbb{L}) : \mathbb{V} \in Gr_c(h) \text{ and } \mathbb{L} \text{ is a hom. subgroup that is a complementary subgroup of } \mathbb{V}\},$$

we fix a function $\mathbf{F} : \text{Sub}(h) \rightarrow (0, 1)$, and for every $\ell \in \mathbb{N}$ with $\ell \geq 2$ let us define

$$Gr_c^{\mathbf{F}}(h, \ell) := \{\mathbb{V} \in Gr_c(h) : \exists \mathbb{L} \text{ compl. subgroup of } \mathbb{V} \text{ s.t. } 1/\ell < \mathbf{F}(\mathbb{V}, \mathbb{L}) \leq 1/(\ell - 1)\}.$$

Observe that $Gr_c^{\mathbf{F}}(h, \ell)$ is separable for every $\ell \in \mathbb{N}$, since $Gr_c^{\mathbf{F}}(h, \ell) \subseteq Gr(h)$ and $(Gr(h), d_G)$ is a compact metric space, see Proposition 1.22. For every $\ell \geq 2$, let

$$(2.13) \quad \mathcal{D}_\ell := \{\mathbb{V}_{i, \ell}\}_{i \in \mathbb{N}},$$

be a countable dense subset of $Gr_c^{\mathbf{F}}(h, \ell)$ and

$$(2.14)$$

for all $i \in \mathbb{N}$, choose a compl. subgroup $\mathbb{L}_{i, \ell}$ of $\mathbb{V}_{i, \ell}$ s.t. $1/\ell < \mathbf{F}(\mathbb{V}_{i, \ell}, \mathbb{L}_{i, \ell}) \leq 1/(\ell - 1)$.

Let us now prove Theorem 2.2. In order to do this, we will prove the following more detailed result, from which Theorem 2.2 will follow as a corollary.

Theorem 2.10. *Let $\mathbf{F} : \text{Sub}(h) \rightarrow (0, 1)$ be a function, where $\text{Sub}(h)$ is defined in (2.12), and for every $\ell \in \mathbb{N}$ define \mathcal{D}_ℓ as in (2.13). Set $\mathcal{F} := \{\mathbb{V}_{i, \ell}\}_{i, \ell \in \mathbb{N}}$, and choose $\mathbb{L}_{i, \ell}$ as in (2.14). Furthermore, let $\beta : \mathbb{N} \rightarrow (0, 1)$ and define $\beta(\mathbb{V}_{i, \ell}) := \beta(\ell)$ for every $i, \ell \in \mathbb{N}$. For the ease of notation we rename $\mathcal{F} := \{\mathbb{V}_k\}_{k \in \mathbb{N}}$. Then the following holds.*

Let ϕ be a \mathcal{P}_h^c -rectifiable measure. There are countably many compact sets Γ_k that are $C_{\mathbb{V}_k}(\min\{\mathbf{F}(\mathbb{V}_k, \mathbb{L}_k), \beta(\mathbb{V}_k)\})$ -sets for some $\mathbb{V}_k \in \mathcal{F}$, and such that

$$\phi\left(\mathbb{G} \setminus \bigcup_{k=1}^{+\infty} \Gamma_k\right) = 0.$$

Proof. Let us notice that without loss of generality, by restricting the measure on balls with integer radius, we can suppose that ϕ has a compact support. Fix $\vartheta, \gamma \in \mathbb{N}$ and let $E(\vartheta, \gamma)$ be the set introduced in Definition 1.9 with respect to ϕ . Furthermore, for every $\ell, i, j \in \mathbb{N}$, we let

$$(2.15)$$

$$F_\ell(i, j) := \{x \in E(\vartheta, \gamma) : \overline{B}(x, r) \cap E(\vartheta, \gamma) \subseteq xC_{\mathbb{V}_{i, \ell}}(6^{-1} \min\{\mathbf{F}(\mathbb{V}_{i, \ell}, \mathbb{L}_{i, \ell}), \beta(\mathbb{V}_{i, \ell})\}) \text{ for every } 0 < r < 1/j\}.$$

It is not hard to prove, since $E(\vartheta, \gamma)$ is compact, see Proposition 1.10, that for every ℓ, i, j the sets $F_\ell(i, j)$ are compact. We claim that

$$(2.16) \quad \phi\left(E(\vartheta, \gamma) \setminus \bigcup_{\ell, i, j \in \mathbb{N}} F_\ell(i, j)\right) = 0.$$

Indeed, let $w \in E(\vartheta, \gamma)$ be such that $\text{Tan}_h(\phi, w) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(w) : \lambda > 0\}$ for some $\mathbb{V}(w) \in Gr_c(h)$; the complement of the set of such w 's is ϕ -null since ϕ is \mathcal{P}_h^c -rectifiable. Let $\ell(w) \in \mathbb{N}$ be the smallest natural number for which there exists \mathbb{L} complementary subgroup of $\mathbb{V}(w)$

with $1/\ell(w) < \mathbf{F}(\mathbb{V}(w), \mathbb{L}) \leq 1/(\ell(w) - 1)$. Then by definition we have $\mathbb{V}(w) \in Gr_c^{\mathbf{F}}(h, \ell(w))$. By density of the family $\mathcal{D}_{\ell(w)}$ in $Gr_c^{\mathbf{F}}(h, \ell(w))$ there exists a homogeneous subgroup $\mathbb{V}_{i,\ell(w)} \in \mathcal{D}_{\ell(w)}$ such that

$$d_{\mathbb{G}}(\mathbb{V}_{i,\ell(w)}, \mathbb{V}(w)) < 30^{-1} \min\{1/\ell(w), \beta(\mathbb{V}_{i,\ell(w)})\};$$

for this last observation to hold it is important that β only depends on $\ell(w)$, as it is true by construction. The previous inequality, jointly with Lemma 1.39, imply that

$$(2.17) \quad \begin{aligned} C_{\mathbb{V}(w)}(30^{-1} \min\{1/\ell(w), \beta(\mathbb{V}_{i,\ell(w)})\}) &\subseteq C_{\mathbb{V}_{i,\ell(w)}}(6^{-1} \min\{1/\ell(w), \beta(\mathbb{V}_{i,\ell(w)})\}) \\ &\subseteq C_{\mathbb{V}_{i,\ell(w)}}(6^{-1} \min\{\mathbf{F}(\mathbb{V}_{i,\ell(w)}, \mathbb{L}_{i,\ell(w)}), \beta(\mathbb{V}_{i,\ell(w)})\}), \end{aligned}$$

where the last inclusion follows from the fact that by definition of the family $\mathcal{D}_{\ell(w)}$ it holds $\mathbf{F}(\mathbb{V}_{i,\ell(w)}, \mathbb{L}_{i,\ell(w)}) > 1/\ell(w)$. Thanks to Proposition 2.7 we can find a $\rho(w) > 0$ such that for every $0 < r < \rho(w)$ we have

$$(2.18) \quad \overline{B}(w, r) \cap E(\vartheta, \gamma) \subseteq wC_{\mathbb{V}(w)}(30^{-1} \min\{1/\ell(w), \beta(\mathbb{V}_{i,\ell(w)})\}).$$

In particular, putting together (2.17) and (2.18) we infer that for ϕ -almost every $w \in E(\vartheta, \gamma)$ there are an $i = i(w) > 0$, an $\ell(w) \in \mathbb{N}$ and a $\rho(w) > 0$ such that whenever $0 < r < \rho(w)$ we have

$$\overline{B}(w, r) \cap E(\vartheta, \gamma) \subseteq wC_{\mathbb{V}_{i,\ell(w)}}(6^{-1} \min\{\mathbf{F}(\mathbb{V}_{i,\ell(w)}, \mathbb{L}_{i,\ell(w)}), \beta(\mathbb{V}_{i,\ell(w)})\}).$$

This concludes the proof of (2.16).

Now, if we fix $\ell, i, j \in \mathbb{N}$, we can apply Proposition 2.9 to the set $F_{\ell}(i, j)$. It suffices to take the family \mathcal{F} in the statement of Proposition 2.9 to be the singleton $\{\mathbb{V}_{i,\ell}\}$ and the function α in the statement of Proposition 2.9 to be $\alpha(\mathbb{V}_{i,\ell}) := 6^{-1} \min\{\mathbf{F}(\mathbb{V}_{i,\ell}, \mathbb{L}_{i,\ell}), \beta(\mathbb{V}_{i,\ell})\}$. As a consequence we can write each $F_{\ell}(i, j)$ as the union of countably many compact $C_{\mathbb{V}_{i,\ell}}(\min\{\mathbf{F}(\mathbb{V}_{i,\ell}, \mathbb{L}_{i,\ell}), \beta(\mathbb{V}_{i,\ell})\})$ -sets. Thus the same holds ϕ -almost everywhere for $E(\vartheta, \gamma)$, allowing i, ℓ, j to vary in \mathbb{N} , since (2.16) holds. Finally, since we have

$$\phi(\mathbb{G} \setminus \cup_{\vartheta, \gamma \in \mathbb{N}} E(\vartheta, \gamma)) = 0,$$

due to Proposition 1.11, we can cover ϕ -almost all of \mathbb{G} with compact $C_{\mathbb{V}_{i,\ell}}(\min\{\mathbf{F}(\mathbb{V}_{i,\ell}, \mathbb{L}_{i,\ell}), \beta(\mathbb{V}_{i,\ell})\})$ -sets for i, ℓ that vary in \mathbb{N} , concluding the proof of the proposition. \square

Proof of Theorem 2.2. Let us define $\mathbf{F}(\mathbb{V}, \mathbb{L}) := \varepsilon_1(\mathbb{V}, \mathbb{L})$, and let us take $\beta : \mathbb{N} \rightarrow (0, 1)$ to be the constant function $\beta(n) \equiv \min\{1, \alpha\} > 0$. Hence an application of Theorem 2.10 together with Proposition 1.89 gives the sought conclusion. \square

Finally, let us state the following theorem, which is a more detailed version of Theorem 2.1.

Theorem 2.11. *There exists a countable subfamily $\mathcal{F} := \{\mathbb{V}_k\}_{k \in \mathbb{N}}$ of $Gr(h)$ such that the following holds. Let ϕ be a \mathcal{P}_h -rectifiable measure. For every $0 < \beta < 1$ there are countably many compact sets Γ_k that are $C_{\mathbb{V}_k}(\beta)$ -sets for some $\mathbb{V}_k \in \mathcal{F}$, and such that*

$$\phi\left(\mathbb{G} \setminus \bigcup_{k=1}^{+\infty} \Gamma_k\right) = 0.$$

Proof. The proof is similar to the one of Theorem 2.10. It suffices to choose, as a family \mathcal{F} , an arbitrary countable dense subset of $Gr(h)$ and then one can argue as in the proof of Theorem 2.10 without the technical effort of introducing the parameter ℓ . We skip the details. \square

Proof of Theorem 2.1. It is an immediate consequence of Theorem 2.11. \square

2. Existence of the density of \mathcal{P} -rectifiable measures when the tangents are complemented

In this section we prove that arbitrary \mathcal{P}_h^c -rectifiable measures have density almost everywhere, which is one of the main results of [28].

Theorem 2.12 (Existence of the density). *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a \mathcal{P}_h^c -rectifiable, i.e., a \mathcal{P}_h -rectifiable measure with tangents that are complemented almost everywhere, see Definition 1.58.*

Then, for ϕ -almost every $x \in \mathbb{G}$ we have

$$0 < \liminf_{r \rightarrow 0} \frac{\phi(\overline{B}(x, r))}{r^h} = \limsup_{r \rightarrow 0} \frac{\phi(\overline{B}(x, r))}{r^h} < +\infty.$$

Moreover, for ϕ -almost every $x \in \mathbb{G}$ we have

$$r^{-h} T_{x,r} \phi \rightharpoonup \Theta^h(\phi, x) \mathcal{C}^h \llcorner \mathbb{V}(x), \quad \text{as } r \text{ goes to } 0,$$

where the map $T_{x,r}$ is defined in Definition 1.52, the convergence is understood in the duality with the continuous functions with compact support on \mathbb{G} , $\Theta^h(\phi, x)$ is the h -density, and $\mathcal{C}^h \llcorner \mathbb{V}(x)$ is the h -dimensional centered Hausdorff measure, restricted to the tangent $\mathbb{V}(x)$, see Definition 1.1.

A way of reading the previous theorem is the following: we prove that whenever a Radon measure on a Carnot group has strictly positive h -lower density and finite h -upper density, and at almost every point all the blow-up measures are supported on the entire same (depending on the point) h -dimensional homogeneous complemented subgroup, then the measure has h -density.

In Euclidean spaces the proof of Theorem 2.12 is an almost immediate consequence of the fact that projections on linear spaces are 1-Lipschitz in conjunction with the area formula. In our context we do not have at our disposal the Lipschitz property of projections. Instead, we have at our disposal an area formula for \mathcal{P}_h -rectifiable measures with complemented tangents, which is obtained as a consequence of the rectifiability results we obtain in [28, 33], see [33, Theorem 4.6]. So the proof of Theorem 2.12 require new ideas.

In order to obtain Theorem 2.12 first of all one reduces to the case of the surface measure on an intrinsically Lipschitz graph with very small Lipschitz constant thanks to the structure result Theorem 2.2 proved above. Secondly, one needs to show that the surface measures of the tangents and their push-forward on the graph are mutually absolutely continuous. For this last point to hold it will be crucial on the one hand that a \mathcal{P}_h -rectifiable measure with complemented tangents can be covered almost everywhere with intrinsic graphs, see Theorem 2.2, and on the other hand that asymptotically doubling intrinsically Lipschitz graphs have big projections on their bases, see Proposition 2.17 below. Third, one exploits the fact that the density exists for the surface measures on the tangents to infer its existence for the original measure.

We further notice that Theorem 2.12 extends the implication in [178, (iv) \Rightarrow (ii) of Theorem 3.15] to the setting of \mathcal{P}_h -rectifiable measures whose tangents are complemented in arbitrary Carnot groups. Indeed, in [178, (iv) \Rightarrow (ii) of Theorem 3.15] the authors prove that if $n + 1 \leq h \leq 2n$, and $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h -rectifiable measure with tangents that are vertical subgroups in the Heisenberg group \mathbb{H}^n , then the h -density of $\mathcal{S}^h \llcorner \Gamma$ exists almost everywhere and the tangent is unique almost everywhere. The analogous property in \mathbb{H}^n , but with \mathcal{P}_h -rectifiable measures with tangents that are horizontal subgroups is obtained in [178, (iv) \Rightarrow (ii) of Theorem 3.14], and in arbitrary homogeneous groups in the recent [128, (iii) \Rightarrow (ii) of Theorem 1.1]. However, in the special horizontal case treated in [178, Theorem 3.14] and

[128, Theorem 1.1] the authors do not assume $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$ since it comes from the existence of an approximate tangent, see [178, Theorem 3.10], while the authors in [128] are able to overcome this issue by adapting [102, Lemma 3.3.6] in [128, Theorem 4.4]. For further discussions on this see the forthcoming Remark 2.40.

2.1. Proof. Throughout this subsection we assume that $\mathbb{V} \in Gr_c(h)$ and that $\mathbb{V} \cdot \mathbb{L} = \mathbb{G}$. In this subsection whenever we deal with $C_{\mathbb{V}}(\alpha)$ -sets we are always assuming that $\alpha < \varepsilon_1(\mathbb{V}, \mathbb{L})$, where ε_1 is defined in Lemma 1.40.

This subsection is devoted to the proof of Theorem 2.12, that is obtained through three different steps. Let Γ be a compact $C_{\mathbb{V}}(\varepsilon_1(\mathbb{V}, \mathbb{L}))$ set, and recall that by Proposition 1.89 we can write $\Gamma = \text{graph}(\varphi)$ with $\varphi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{L}$. Let us denote $\Phi(v) := v \cdot \varphi(v)$ for every $v \in P_{\mathbb{V}}(\Gamma)$ to be the graph map of φ .

We first show that if we assume that $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling, then the push-forward measure $(\Phi)_*(\mathcal{S}^h \llcorner \mathbb{V})$ is mutually absolutely continuous with respect to $\mathcal{S}^h \llcorner \Gamma$, see Proposition 2.18. We remark that in the Euclidean case the analogous statement holds true even without the asymptotically doubling assumption: this is true because in the Euclidean case every Lipschitz graph Γ over a \mathcal{S}^h -positive measured subset of a vector subspace of dimension h is such that $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling, since Γ is a differentiable graph almost everywhere. We also stress that every intrinsically Lipschitz graph over an open subset of a h -dimensional homogeneous subgroups has strictly positive lower h -density almost everywhere, see [110, Theorem 3.9].

As a second step in order to obtain the proof of Theorem 2.12 we prove the following statement that can be made quantitative: if $\mathbb{V} \in Gr_c(h)$, Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with α sufficiently small, and $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h -rectifiable measure with complemented tangents, i.e., a \mathcal{P}_h^c -rectifiable measure, then we can give an explicit lower bound on the ratio of the lower and upper h -densities of $\mathcal{S}^h \llcorner \Gamma$. We refer the reader to Proposition 2.21 for a more precise statement. This result is obtained through a blow-up analysis and a careful use of the mutually absolute continuity property that we discussed above, and which is contained in Proposition 2.18. We stress that in order to differentiate in the proof of Proposition 2.21, we need to use proper $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ and $\mathcal{S}^h \llcorner \mathbb{V}$ -Vitali relations, see Proposition 2.19, and Proposition 2.20, respectively.

As a last step of the proof of Theorem 2.12 we first use the result in Proposition 2.21 in order to prove that Theorem 2.12 holds true for measures of the type $\mathcal{S}^h \llcorner \Gamma$, see Theorem 2.23. Then we conclude the proof for arbitrary measures by reducing ourselves to the sets $E(\vartheta, \gamma)$.

We start this chapter with some preliminary results.

Lemma 2.13. *There exists a constant $K \geq 1$ such that for every $w \in \overline{B}(0, 1/(5K))$, every $y \in \partial \overline{B}(0, 1) \cap C_{\mathbb{V}}(\varepsilon_1(\mathbb{V}, \mathbb{L}))$, and every $z \in \overline{B}(y, 1/(5K))$, we have $w^{-1}z \notin \mathbb{L}$.*

Proof. By contradiction let us assume that we can find sequences $\{w_n\}, \{y_n\} \subseteq \partial \overline{B}(0, 1) \cap C_{\mathbb{V}}(\varepsilon_1)$ and $z_n \in \overline{B}(y_n, 1/n)$ such that w_n converges to 0 and $w_n^{-1}z_n \in \mathbb{L}$. By compactness without loss of generality we can assume that the sequence y_n converges to some $y \in \partial \overline{B}(0, 1) \cap C_{\mathbb{V}}(\varepsilon_1)$. Furthermore, by construction we also have that z_n must converge to y . This implies that $w_n^{-1}z_n$ converges to y and since by hypothesis $w_n^{-1}z_n \in \mathbb{L}$, thanks to the fact that \mathbb{L} is closed we infer that $y \in \mathbb{L}$. This however is a contradiction since y has unit norm and at the same time we should have $y \in C_{\mathbb{V}}(\varepsilon_1) \cap \mathbb{L} = \{0\}$ by Lemma 1.40. \square

Let us denote with $A := A(\mathbb{V}, \mathbb{L}) \geq 1$ the infimum of all the constant $K \geq 1$ such that Lemma 2.13 holds.

Proposition 2.14. *Let $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and suppose Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set. For every $x \in \Gamma$ let $\rho(x)$ to be the supremum of all the numbers $\ell > 0$ satisfying the following condition. For*

every $y \in \overline{B}(x, \ell) \cap \Gamma$ we have

$$P_{\mathbb{V}}(\overline{B}(x, r)) \cap P_{\mathbb{V}}(\overline{B}(y, s)) = \emptyset \text{ for every } r, s < d(x, y)/(5A),$$

where $A = A(\mathbb{V}, \mathbb{L})$ is the constant yielded by Lemma 2.13. Then, the function $x \mapsto \rho(x)$ is positive everywhere on Γ , and it is upper semicontinuous.

Proof. Let $x \in \Gamma$ and suppose by contradiction that there is a sequence of points $\{y_i\}_{i \in \mathbb{N}} \subseteq \Gamma$ converging to x and

$$(2.19) \quad P_{\mathbb{V}}(\overline{B}(x, r_i)) \cap P_{\mathbb{V}}(\overline{B}(y_i, s_i)) \neq \emptyset,$$

for some $r_i, s_i < d(x, y_i)/(5A)$. We note that (2.19) is equivalent to assuming that there are $z_i \in \overline{B}(x, r_i)$ and $w_i \in \overline{B}(y_i, s_i)$ such that

$$(2.20) \quad P_{\mathbb{V}}(w_i) = P_{\mathbb{V}}(z_i).$$

Identity (2.20) implies in particular that for every $i \in \mathbb{N}$ we have $w_i^{-1}z_i \in \mathbb{L}$ and let us denote $\rho_i := d(x, y_i)$. Thanks to the assumptions on y_i, z_i and w_i we have that

- (1) $d(0, \delta_{1/\rho_i}(x^{-1}y_i)) = 1$ and thus we can assume without loss of generality that there exists a $y \in \partial \overline{B}(0, 1)$ such that

$$\lim_{i \rightarrow \infty} \delta_{1/\rho_i}(x^{-1}y_i) = y,$$

- (2) $d(0, \delta_{1/\rho_i}(x^{-1}z_i)) \leq 1/(5A)$ and thus up to passing to a non-relabelled subsequence we can assume that there exists a $z \in \overline{B}(0, 1/5A)$ such that

$$\lim_{i \rightarrow \infty} \delta_{1/\rho_i}(x^{-1}z_i) = z,$$

- (3) $d(\delta_{1/\rho_i}(x^{-1}y_i), \delta_{1/\rho_i}(x^{-1}w_i)) \leq 1/(5A)$ and thus, up to passing to a non re-labelled subsequence, we can suppose that there exists a $w \in \overline{B}(y, 1/5A)$ such that

$$\lim_{i \rightarrow \infty} \delta_{1/\rho_i}(x^{-1}w_i) = w.$$

Since Γ is supposed to be a $C_{\mathbb{V}}(\alpha)$ -set, we have that for every $i \in \mathbb{N}$ the point $x^{-1}y_i$ is contained in the cone $C_{\mathbb{V}}(\alpha)$ and, since $C_{\mathbb{V}}(\alpha)$ is closed, we infer that $y \in C_{\mathbb{V}}(\alpha)$. Since we assumed $\alpha < \varepsilon_1(\mathbb{V}, \mathbb{L})$, we have $y \in \partial \overline{B}(0, 1) \cap C_{\mathbb{V}}(\varepsilon_1(\mathbb{V}, \mathbb{L}))$. Since $\delta_{1/\rho_i}(x^{-1}z_i)$ and $\delta_{1/\rho_i}(x^{-1}w_i)$ converge to z and w , respectively, we have

$$\lim_{i \rightarrow \infty} \delta_{1/\rho_i}(w_i^{-1}z_i) = \lim_{i \rightarrow \infty} \delta_{1/\rho_i}(w_i^{-1}x)\delta_{1/\rho_i}(x^{-1}z_i) = w^{-1}z.$$

Furthermore since $w_i^{-1}z_i \in \mathbb{L}$ for every $i \in \mathbb{N}$, we infer that $w^{-1}z \in \mathbb{L}$ since \mathbb{L} is closed. Applying Lemma 2.13 to y, z, w we see that the fact that $w^{-1}z \in \mathbb{L}$, $z \in \overline{B}(0, 1/(5A))$ and $w \in \overline{B}(y, 1/(5A))$ results in a contradiction. This concludes the proof of the first part of the proposition.

In order to show that ρ is upper semicontinuous we fix an $x \in \Gamma$ and we assume by contradiction that there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \Gamma$ converging to x such that

$$(2.21) \quad \limsup_{i \rightarrow \infty} \rho(x_i) > (1 + \tau)\rho(x),$$

for some $\tau > 0$. Fix an $y \in \overline{B}(x, (1 + \tau/2)\rho(x)) \cap \Gamma$ and assume $s, r < d(x, y)/(5A)$. Thus, thanks to (2.21) and the fact that the x_i converge to x , we infer that there exists a $i_0 \in \mathbb{N}$ such that, up to non re-labelled subsequences, for every $i \geq i_0$ we have $\rho(x_i) > (1 + \tau)\rho(x)$, $d(x_i, x) < \tau\rho(x)/4$ and $s, r + d(x_i, x) < d(x_i, y)/(5A)$. Therefore, for every $i \geq i_0$ we have

$$y \in \overline{B}(x_i, (1 + 3\tau/4)\rho(x)) \subseteq \overline{B}(x_i, \rho(x_i)), \quad \text{and} \quad s, r + d(x_i, x) < d(x_i, y)/(5A).$$

This however, thanks to the definition of $\rho(x_i)$, implies that

$$P_{\mathbb{V}}(\overline{B}(x, r)) \cap P_{\mathbb{V}}(\overline{B}(y, s)) \subseteq P_{\mathbb{V}}(\overline{B}(x_i, r + d(x_i, x))) \cap P_{\mathbb{V}}(\overline{B}(y, s)) = \emptyset.$$

Summing up, we have proved that for every $y \in \overline{B}(x, (1 + \tau/2)\rho(x)) \cap \Gamma$ whenever $r, s < d(x, y)/5A$ we have

$$P_{\mathbb{V}}(\overline{B}(x, r)) \cap P_{\mathbb{V}}(\overline{B}(y, s)) = \emptyset,$$

and this contradicts the maximality of $\rho(x)$. This concludes the proof. \square

Corollary 2.15. *Let us fix $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and suppose that Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set. Let us fix $x \in \Gamma$ and choose $\rho(x) > 0$ as in the statement of Proposition 2.14. Then there is a $0 < r(x) < 1/2$ such that the following holds*

$$(2.22) \quad \begin{aligned} & \text{if } 0 < r < r(x) \text{ and } y \in \Gamma \text{ are such that } P_{\mathbb{V}}(\overline{B}(x, 2r)) \cap P_{\mathbb{V}}(\overline{B}(y, 10r)) \neq \emptyset, \\ & \text{then } y \in \overline{B}(x, \rho(x)) \text{ and } d(x, y) \leq 50Ar, \end{aligned}$$

where $A = A(\mathbb{V}, \mathbb{L})$ is the constant yielded by Lemma 2.13.

Proof. Let us first prove that there exists a constant k such that whenever $y \in \Gamma$ is such that $d(x, y) \geq \rho(x)$ then $d(P_{\mathbb{V}}(x), P_{\mathbb{V}}(y)) \geq k$. Indeed if it is not the case, we have a sequence $\{y_i\}_{i \in \mathbb{N}} \subseteq \Gamma$ such that $d(x, y_i) \geq \rho(x)$ for every $i \in \mathbb{N}$ and $d(P_{\mathbb{V}}(x), P_{\mathbb{V}}(y_i)) \rightarrow 0$ as $i \rightarrow +\infty$. Since Γ is compact we can suppose, up to passing to a non re-labelled subsequence, that $y_i \rightarrow y \in \Gamma$. Moreover since $d(x, y_i) \geq \rho(x)$ and $d(P_{\mathbb{V}}(x), P_{\mathbb{V}}(y_i)) \rightarrow 0$ we conclude that $d(x, y) \geq \rho(x)$, and hence $x \neq y$, and moreover $P_{\mathbb{V}}(x) = P_{\mathbb{V}}(y)$. Then $y^{-1} \cdot x \in \mathbb{L} \cap C_{\mathbb{V}}(\alpha)$ that is a contradiction with Lemma 1.40 because $y \neq x$ and $\alpha < \varepsilon_1$. Let us denote $\tilde{\alpha} := \tilde{\alpha}(\alpha, x)$ the supremum of all the constants k for which the previous property holds.

Since $P_{\mathbb{V}}$ is uniformly continuous on the closed tubular neighborhood $\overline{B}(\Gamma, 1)$, there exists a constant $0 < c \leq 1/10$ depending on $\tilde{\alpha} = \tilde{\alpha}(\alpha, x)$ such that for every $y \in \Gamma$ and every $r < c$, we have

$$(2.23) \quad P_{\mathbb{V}}(\overline{B}(y, 10r)) \subseteq \overline{B}(P_{\mathbb{V}}(y), \tilde{\alpha}/10).$$

Let us denote $r(x)$ the supremum of all the constants $0 < c \leq 1/10$ for which (2.23) holds. Let us show the first part of the statement. It is sufficient to prove that if $r < r(x)$ and $y \in \Gamma$ is such that $d(x, y) \geq \rho(x)$, then $P_{\mathbb{V}}(\overline{B}(x, 2r)) \cap P_{\mathbb{V}}(\overline{B}(y, 10r)) = \emptyset$. Indeed if $d(x, y) \geq \rho(x)$ then $d(P_{\mathbb{V}}(x), P_{\mathbb{V}}(y)) \geq \tilde{\alpha}$. Moreover, from (2.23) we deduce that $P_{\mathbb{V}}(\overline{B}(x, 10r)) \subseteq \overline{B}(P_{\mathbb{V}}(x), \tilde{\alpha}/10)$ and $P_{\mathbb{V}}(\overline{B}(y, 10r)) \subseteq \overline{B}(P_{\mathbb{V}}(y), \tilde{\alpha}/10)$. Since $d(P_{\mathbb{V}}(x), P_{\mathbb{V}}(y)) \geq \tilde{\alpha}$ we conclude that $\overline{B}(P_{\mathbb{V}}(x), \tilde{\alpha}/10) \cap \overline{B}(P_{\mathbb{V}}(y), \tilde{\alpha}/10) = \emptyset$ and then also $P_{\mathbb{V}}(\overline{B}(x, 10r)) \cap P_{\mathbb{V}}(\overline{B}(y, 10r)) = \emptyset$, from which the sought conclusion follows. In order to prove $d(x, y) \leq 50Ar$, once we have $y \in \overline{B}(x, \rho(x))$, the conclusion follows thanks to Proposition 2.14. \square

Lemma 2.16. *Fix some $N \in \mathbb{N}$ and assume that \mathcal{F} is a family of closed balls of \mathbb{G} with uniformly bounded radii. Then we can find a countable disjoint subfamily \mathcal{G} of \mathcal{F} such that*

- (i) if $B, B' \in \mathcal{G}$ then $5^N B$ and $5^N B'$ are disjoint,
- (ii) $\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} 5^{N+1} B$.

We recall that with λB we denote the ball with the same center as B and the radius multiplied by λ .

Proof. If $N = 0$, there is nothing to prove, since it is the classical 5-Vitali's covering Lemma.

Let us assume by inductive hypothesis that the claim holds for $N = k$ and let us prove that it holds for $k + 1$. Let \mathcal{G}_k be the family of balls satisfying (i) and (ii) for $N = k$, and apply the 5-Vitali's covering Lemma to the family of balls $\mathcal{F} := \{5^{k+1} B : B \in \mathcal{G}_k\}$. We

obtain a countable subfamily $\tilde{\mathcal{G}}$ of $\tilde{\mathcal{F}}$ such that if $5^{k+1}B, 5^{k+1}B' \in \tilde{\mathcal{G}}$ then $5^{k+1}B$ and $5^{k+1}B'$ are disjoint, and that satisfies $\bigcup_{B \in \tilde{\mathcal{F}}} B \subseteq \bigcup_{B \in \tilde{\mathcal{G}}} 5B$. Therefore, if we define

$$\mathcal{G}_{k+1} := \{B \in \mathcal{G}_k : 5^{k+1}B \in \tilde{\mathcal{G}}\},$$

point (i) directly follows and thanks to the inductive hypothesis we have

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}_k} 5^{k+1}B \subseteq \bigcup_{B \in \mathcal{G}_{k+1}} 5^{k+2}B,$$

proving the second point of the statement. \square

Proposition 2.17. *Let $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and suppose Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with $0 < \mathcal{S}^h(\Gamma) < +\infty$, such that*

$$(2.24) \quad \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0,$$

for \mathcal{S}^h -almost every $x \in \Gamma$. Then, there exists a constant $C_7 > 0$ depending on \mathbb{V}, \mathbb{L} , such that for \mathcal{S}^h -almost every $x \in \Gamma$ there exists an $R := R(x) > 0$ such that for every $0 < \ell \leq R$ we have

$$(2.25) \quad \mathcal{S}^h(P_{\mathbb{V}}(\Gamma \cap \overline{B}(x, \ell))) \geq C_7 \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)^2 \ell^h.$$

In the same hypotheses above, if instead of having (2.24) we have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, 2r))}{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, r))} < \infty,$$

for \mathcal{S}^h -almost every $x \in \Gamma$, then, there exists a constant $C_8 := C_8(h, \mathbb{V}, \mathbb{L}) > 0$ such that for \mathcal{S}^h -almost every $x \in \Gamma$ there exists an infinitesimal sequence $\{\ell_i(x)\}_{i \in \mathbb{N}}$ such that for every $i \in \mathbb{N}$ we have

$$(2.26) \quad \mathcal{S}^h(P_{\mathbb{V}}(\Gamma \cap \overline{B}(x, \ell_i(x)))) > C_8 \ell_i(x)^h$$

Proof. Let us start with the proof of the first part of the statement. First of all, let us recall that two homogeneous left-invariant distances are always bi-Lipschitz equivalent on \mathbb{G} . Therefore if d_c is a Carnot-Carathéodory distance on \mathbb{G} , which is in particular geodesic, see [145, Section 3.3], there exists a constant $\mathfrak{L}(d, d_c) \geq 1$ such that

$$\mathfrak{L}(d, d_c)^{-1} d_c(x, y) \leq d(x, y) \leq \mathfrak{L}(d, d_c) d_c(x, y) \text{ for every } x, y \in \mathbb{G}.$$

We claim that if for every $\vartheta, \gamma \in \mathbb{N}$ for which $\mathcal{S}^h(E(\vartheta, \gamma)) > 0$ we have that for \mathcal{S}^h -almost every $w \in E(\vartheta, \gamma)$ there exists a $R(w) > 0$ such that

$$(2.27) \quad \mathcal{S}^h(P_{\mathbb{V}}(\Gamma \cap \overline{B}(w, \ell))) \geq \frac{C_4(\mathbb{V}, \mathbb{L}) \ell^h}{8 \cdot 5^{3h} A^h \mathfrak{L}(d, d_c)^{2h} \vartheta^2},$$

whenever $0 < \ell < R(w)$, then the proposition is proved. We recall that $A(\mathbb{V}, \mathbb{L})$ is defined after Lemma 2.13. The latter claim is due to the following reasoning. First of all, thanks to [102, Proposition 2.10.19(5)], we know that $\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x) \leq 1$. Secondly, if we set, for every $k \in \mathbb{N}$, $\Gamma_k := \{w \in \Gamma : 1/(k+1) < \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, w) \leq 1/k\}$, we have that

$$(2.28) \quad \mathcal{S}^h \left(\Gamma \setminus \bigcup_{k \in \mathbb{N}} \Gamma_k \right) = 0.$$

We observe now that if $\mathcal{S}^h(\Gamma_k) > 0$, then \mathcal{S}^h -almost every $w \in \Gamma_k$ belongs to some $E(k+1, \gamma)$ provided γ is big enough, or in other words

$$(2.29) \quad \mathcal{S}^h \left(\Gamma_k \setminus \bigcup_{\gamma \in \mathbb{N}} E(k+1, \gamma) \right) = 0.$$

If our claim (2.27) holds true, whenever $\mathcal{S}^h(E(k+1, \gamma)) > 0$, we have that for $\mathcal{S}^h \llcorner E(k+1, \gamma)$ -almost every w there exists $R(w)$ such that whenever $0 < \ell < R(w)$ the following chain of inequalities holds

$$(2.30) \quad \begin{aligned} \mathcal{S}^h(P_{\mathbb{V}}(\Gamma \cap \overline{B}(w, \ell))) &\geq \frac{C_4(\mathbb{V}, \mathbb{L})\ell^h}{8 \cdot 5^{3h} A^h \mathfrak{L}(d, d_c)^{2h} (k+1)^2} \\ &\geq \frac{C_4(\mathbb{V}, \mathbb{L})\ell^h}{2^5 \cdot 5^{3h} A^h \mathfrak{L}(d, d_c)^{2h} k^2} \geq \frac{C_4(\mathbb{V}, \mathbb{L})\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, w)^2 \ell^h}{2^5 \cdot 5^{3h} A^h \mathfrak{L}(d, d_c)^{2h}} \\ &=: C_7 \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, w)^2 \ell^h. \end{aligned}$$

Identities (2.28) and (2.29) together with (2.30) imply that our claim suffices to prove the proposition. Therefore, in the following we will assume that $\vartheta, \gamma \in \mathbb{N}$ are fixed and such that $\mathcal{S}^h(E(\vartheta, \gamma)) > 0$, and we want to prove (2.27).

Let $N \in \mathbb{N}$ be the unique natural number for which $5^{N-2} \leq A \mathfrak{L}(d, d_c)^2 < 5^{N-1}$ and for every $k \in \mathbb{N}$ and $0 < \delta < 1/2$ we define the following sets, where $\rho(x)$ is defined in Proposition 2.14,

$$\begin{aligned} A_{\vartheta, \gamma}(k) &:= \{x \in E(\vartheta, \gamma) : \rho(x) > 1/k\}, \\ \mathcal{D}_{\vartheta, \gamma}(k) &:= \left\{ x \in A_{\vartheta, \gamma}(k) : \lim_{r \rightarrow 0} \frac{\mathcal{S}^h(\overline{B}(x, r) \cap A_{\vartheta, \gamma}(k))}{\mathcal{S}^h(\overline{B}(x, r) \cap E(\vartheta, \gamma))} = 1 \right\}, \\ \mathcal{F}_{\delta}(k) &:= \left\{ \overline{B}(x, r) : x \in \mathcal{D}_{\vartheta, \gamma}(k) \text{ and } r \leq \frac{\min\{\vartheta^{-1}, \gamma^{-1}, k^{-1}, \delta\}}{1000 A \mathfrak{L}(d, d_c)^2} \right\}. \end{aligned}$$

For every $\vartheta, \gamma \in \mathbb{N}$ the sets $A_{\vartheta, \gamma}(k)$ are Borel since thanks to Proposition 2.14, the function ρ is upper semicontinuous. Before going on we observe that $\mathcal{S}^h \llcorner E(\vartheta, \gamma)(A_{\vartheta, \gamma}(k) \setminus \mathcal{D}_{\vartheta, \gamma}(k)) = 0$. This comes from the fact that the points of $\mathcal{D}_{\vartheta, \gamma}(k)$ are exactly the points of density one of $A_{\vartheta, \gamma}(k)$ with respect to the measure $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$, that is asymptotically doubling at $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$ -almost every point because it has positive lower density and finite upper density at $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$ -almost every point, see Proposition 1.55. Moreover, observe that from Proposition 2.14 we have $\mathcal{S}^h(E(\vartheta, \gamma) \setminus \bigcup_{k=1}^{+\infty} A_{\vartheta, \gamma}(k)) = 0$. Let us apply Lemma 2.16 to N and $\mathcal{F}_{\delta}(k)$, and thus we infer that there exists a subfamily $\mathcal{G}_{\delta}(k)$ such that

- (α) for every $B, B' \in \mathcal{G}_{\delta}(k)$ we have that $5^N B \cap 5^N B' = \emptyset$,
- (β) $\bigcup_{B \in \mathcal{F}_{\delta}(k)} B \subseteq \bigcup_{B \in \mathcal{G}_{\delta}(k)} 5^{N+1} B$.

The point (α) above implies in particular that whenever $\overline{B}(x, r), \overline{B}(y, s) \in \mathcal{G}_{\delta}(k)$ we have $d(x, y) > \mathfrak{L}(d, d_c)^{-2} 5^N (r+s)$, since d is $\mathfrak{L}(d, d_c)$ -Lipschitz equivalent to the geodesic distance d_c , and thanks to the choice of N we deduce that

$$r + s < \frac{d(x, y)}{5A}.$$

Throughout the rest of the proof we fix a $w \in \mathcal{D}_{\vartheta, \gamma}(k)$ and a

$$0 < R(w) < \min\{\vartheta^{-1}, \gamma^{-1}, k^{-1}\}/8,$$

such that

$$(2.31) \quad \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(w, \ell))}{\ell^h} \geq \frac{1}{2\vartheta}, \text{ and } \frac{\mathcal{S}^h \llcorner \mathcal{D}_{\vartheta, \gamma}(k)(\overline{B}(w, \ell))}{\mathcal{S}^h \llcorner \Gamma(\overline{B}(w, \ell))} \geq \frac{1}{2}, \text{ for every } 0 < \ell \leq R(w).$$

For the ease of notation we continue the proof fixing the radius $\ell = R(w) = R$. We stress that the forthcoming estimates are verified, *mutatis mutandis*, also for every $0 < \ell < R$. The first inequality above comes from the definition of $E(\vartheta, \gamma)$, see Definition 1.9, while the second is true, up to choose a sufficiently small $R(w)$, because $\mathcal{S}^h \llcorner \Gamma$ -almost every point of $\mathcal{D}_{\vartheta, \gamma}(k)$ has density one with respect to the asymptotically doubling measure $\mathcal{S}^h \llcorner \Gamma$. Let us stress that if we prove our initial claim for such w and $R(w)$ we are done since $\mathcal{S}^h \llcorner \Gamma$ -every point of $\mathcal{D}_{\vartheta, \gamma}(k)$ satisfies (2.31), $\mathcal{S}^h \llcorner E(\vartheta, \gamma)(A_{\vartheta, \gamma}(k) \setminus \mathcal{D}_{\vartheta, \gamma}(k)) = 0$, and $\mathcal{S}^h(E(\vartheta, \gamma) \setminus \cup_{k=1}^{+\infty} A_{\vartheta, \gamma}(k)) = 0$.

Let us notice that the definition of $\mathcal{F}_\delta(k)$ implies that there must exist a ball $B \in \mathcal{G}_\delta(k)$ such that $w \in 5^{N+1}B$. We now prove that for every couple of closed balls $\overline{B}(x, r), \overline{B}(y, s) \in \mathcal{G}_\delta(k)$ such that $\overline{B}(w, R)$ intersects both $\overline{B}(x, 5^{N+1}r)$ and $\overline{B}(y, 5^{N+1}s)$, we have

$$(2.32) \quad P_{\mathbb{V}}(\overline{B}(x, r)) \cap P_{\mathbb{V}}(\overline{B}(y, s)) = \emptyset.$$

Indeed, suppose that $p \in \overline{B}(x, 5^{N+1}r) \cap \overline{B}(w, R)$ and note that

$$(2.33) \quad \begin{aligned} d(x, w) &\leq d(x, p) + d(p, w) \leq R + 5^{N+1}r \\ &\leq \left(\frac{1}{8} + \frac{5^{N+1}}{1000A\mathfrak{L}(d, d_c)^2} \right) \min\{\vartheta^{-1}, \gamma^{-1}, k^{-1}\} \leq \frac{\min\{\vartheta^{-1}, \gamma^{-1}, k^{-1}\}}{4}, \end{aligned}$$

where the last inequality comes from the choice of N . The bound (2.33) shows in particular that

$$d(x, y) \leq d(x, w) + d(w, y) \leq \frac{\min\{\vartheta^{-1}, \gamma^{-1}, k^{-1}\}}{2} < \rho(x),$$

where the last inequality comes from the fact that by construction x is supposed to be in $\mathcal{D}_{\vartheta, \gamma}(k)$. Thanks to the fact that $r + s < d(x, y)/5A$ and $y \in \overline{B}(x, \rho(x)) \cap E(\vartheta, \gamma)$ we have that Proposition 2.14 implies that (2.32) holds.

In order to proceed with the conclusion of the proof, let us define

$$\begin{aligned} \mathcal{F}_\delta(w, R) &:= \{B \in \mathcal{F}_\delta(k) : 5^{N+1}B \cap \overline{B}(w, R) \cap \mathcal{D}_{\vartheta, \gamma}(k) \neq \emptyset\}, \\ \mathcal{G}_\delta(w, R) &:= \{B \in \mathcal{G}_\delta(k) : 5^{N+1}B \cap \overline{B}(w, R) \cap \mathcal{D}_{\vartheta, \gamma}(k) \neq \emptyset\}, \end{aligned}$$

Thanks to our choice of R , see (2.31), and the definition of $\mathcal{G}_\delta(w, R)$ we have

$$\frac{R^h}{2\vartheta} \leq \mathcal{S}^h \llcorner \Gamma(\overline{B}(w, R)) \leq 2\mathcal{S}^h \llcorner \mathcal{D}_{\vartheta, \gamma}(k)(\overline{B}(w, R)) \leq 2\mathcal{S}^h \llcorner \mathcal{D}_{\vartheta, \gamma}(k) \left(\bigcup_{B \in \mathcal{G}_\delta(w, R)} 5^{N+1}B \right).$$

Let $\mathcal{G}_\delta(w, R) = \{\overline{B}(x_i, r_i)\}_{i \in \mathbb{N}}$ and recall that $x_i \in \mathcal{D}_{\vartheta, \gamma}(k)$ and that $5^{N+1}r_i \leq 1/\gamma$. This implies, thanks to Proposition 1.34, that

$$\begin{aligned} \mathcal{S}^h \llcorner \mathcal{D}_{\vartheta, \gamma}(k) \left(\bigcup_{B \in \mathcal{G}_\delta(w, R)} 5^{N+1}B \right) &\leq 2\vartheta 5^{h(N+1)} \sum_{i \in \mathbb{N}} r_i^h \\ &= 2\vartheta 5^{h(N+1)} C_4(\mathbb{V}, \mathbb{L})^{-1} \sum_{i \in \mathbb{N}} \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x_i, r_i))) \\ &= 2\vartheta 5^{h(N+1)} C_4(\mathbb{V}, \mathbb{L})^{-1} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{i \in \mathbb{N}} \overline{B}(x_i, r_i) \right) \right) \\ &\leq 2\vartheta 5^{h(N+1)} C_4(\mathbb{V}, \mathbb{L})^{-1} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \right), \end{aligned}$$

where the first inequality comes from the subadditivity and the upper estimate that we have in the definition of $E(\vartheta, \gamma)$, see Definition 1.9; while identity in the third line above comes

from (2.32). Summing up, for every $\delta > 0$ we have

$$\frac{C_4(\mathbb{V}, \mathbb{L})R^h}{8 \cdot 5^{h(N+1)}\vartheta^2} \leq \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \right).$$

We now prove that the projection under $P_{\mathbb{V}}$ of the closure of $\bigcup_{B \in \mathcal{F}_\delta(w, R)} B$ converges in the Hausdorff sense to $P_{\mathbb{V}}(\overline{\mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R)})$ as δ goes to 0. Since the set $\bigcup_{B \in \mathcal{F}_\delta(w, R)} B$ is a covering of $\mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R)$ we have that

$$(2.34) \quad \mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R) \subseteq \bigcup_{B \in \mathcal{F}_\delta(w, R)} B.$$

On the other hand, since by definition the balls of $\mathcal{F}_\delta(w, R)$ have radii smaller than $\delta/4$ and center in $\mathcal{D}_{\vartheta, \gamma}(k)$, we also have

$$(2.35) \quad \bigcup_{B \in \mathcal{F}_\delta(w, R)} B \subseteq \overline{B}(\mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R), 5^{N+2}\delta).$$

Putting together (2.34) and (2.35), we infer that the closure of $\bigcup_{B \in \mathcal{F}_\delta(w, R)} B$ converges in the Hausdorff metric to the closure of $\overline{B}(w, R) \cap \mathcal{D}_{\vartheta, \gamma}(k)$. Furthermore, since $P_{\mathbb{V}}$ restricted to the ball $\overline{B}(w, R+1)$ is uniformly continuous, we infer that

$$P_{\mathbb{V}} \left(\overline{\bigcup_{B \in \mathcal{F}_\delta(w, R)} B} \right) \xrightarrow{H} P_{\mathbb{V}} \left(\overline{\mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R)} \right).$$

Thanks to the upper semicontinuity of the Lebesgue measure with respect to the Hausdorff convergence we eventually infer that

$$\begin{aligned} \frac{C_4(\mathbb{V}, \mathbb{L})R^h}{8 \cdot 5^{h(N+1)}\vartheta^2} &\leq \limsup_{\delta \rightarrow 0} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\overline{\bigcup_{B \in \mathcal{F}_\delta(w, R)} B} \right) \right) \\ &\leq \mathcal{S}^h(P_{\mathbb{V}}(\overline{\mathcal{D}_{\vartheta, \gamma}(k) \cap \overline{B}(w, R)})) \leq \mathcal{S}^h(P_{\mathbb{V}}(E(\vartheta, \gamma) \cap \overline{B}(w, R))), \end{aligned}$$

where the last inequality above comes from the fact that by construction $\mathcal{D}_{\vartheta, \gamma}(k) \subseteq E(\vartheta, \gamma)$ and the compactness of $E(\vartheta, \gamma)$. Finally, since $C_7 = 2^{-5}5^{-3h}A^{-h}\mathfrak{L}(d, d_c)^{-2h}C_4(\mathbb{V}, \mathbb{L})$, we infer

$$\mathcal{S}^h(P_{\mathbb{V}}(E(\vartheta, \gamma) \cap \overline{B}(w, R))) \geq \frac{C_4(\mathbb{V}, \mathbb{L})R^h}{8 \cdot 5^{h(N+1)}\vartheta^2} \geq \frac{4C_7R^h}{\vartheta^2},$$

thus showing the claim (2.27) and then the proof.

Let us prove the second part of the statement. Let us assume for the ease of notation that the distance d is geodesic. If not, as done in the first part of the proof, one has to properly change the constants in the proof. We will just sketch the proof, that is an adaptation of the proof above. Let $N \in \mathbb{N}$ be the unique natural number for which $5^{N-2} \leq A < 5^{N-1}$, where A is defined after Lemma 2.13. Notice that, since $2^{-h} \leq \Theta^{h,*}(\mathcal{S}^h \lrcorner \Gamma, x) \leq 1$ for $\mathcal{S}^h \lrcorner \Gamma$ -almost every $x \in \mathbb{G}$ (cf. [102, 2.10.19(1) and 2.10.19(5)]), hence, for $\mathcal{S}^h \lrcorner \Gamma$ -almost every $x \in \mathbb{G}$ there exists an infinitesimal sequence $\{\ell_i(x)\}_{i \in \mathbb{N}}$ such that

$$(2.36) \quad \frac{1}{2^{h+1}} \leq \frac{\mathcal{S}^h(\Gamma \cap \overline{B}(x, \ell_i(x)))}{\ell_i(x)^h} \leq 2.$$

Thus, for every $k \in \mathbb{N}$ and $0 < \delta < 1/2$ we define the following sets

$$\begin{aligned} A(k) &:= \{x \in \Gamma : \rho(x) > 1/k\}, \\ \mathcal{D}(k) &:= \left\{ x \in A(k) : \lim_{r \rightarrow 0} \frac{\mathcal{S}^h(\overline{B}(x, r) \cap A(k))}{\mathcal{S}^h(\overline{B}(x, r) \cap \Gamma)} = 1 \right\}, \\ \mathcal{F}_\delta(k) &:= \left\{ \overline{B}(x, r) : x \in \mathcal{D}(k), r \leq \frac{\min\{k^{-1}, \delta\}}{1000A} \text{ and } \frac{1}{2^{h+1}} \leq \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, 5^{N+1}r))}{(5^{N+1}r)^h} \leq 2 \right\}, \end{aligned}$$

where $\rho(x)$ is the number defined in Proposition 2.14. First of all notice that, thanks to (2.36), $\mathcal{F}_\delta(k)$ is a fine covering of $\mathcal{S}^h \llcorner \Gamma$ -almost all $\mathcal{D}(k)$. Furthermore, for every k the sets $A(k)$ are Borel since thanks to Proposition 2.14, the function ρ is upper semicontinuous and, since by assumption $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling, we also know that $\mathcal{S}^h \llcorner \Gamma(A(k) \setminus \mathcal{D}(k)) = 0$. Finally, from Proposition 2.14 we infer that $\mathcal{S}^h(\Gamma \setminus \cup_{k=1}^{+\infty} A(k)) = 0$. Let us apply Lemma 2.16 to N and $\mathcal{F}_\delta(k)$ and we obtain the disjoint subfamily $\mathcal{G}_\delta(k)$ of $\mathcal{F}_\delta(k)$ such that

- (α) for every $B, B' \in \mathcal{G}_\delta(k)$ we have that $5^N B \cap 5^N B' = \emptyset$,
- (β) $\cup_{B \in \mathcal{F}_\delta(k)} B \subseteq \cup_{B \in \mathcal{G}_\delta(k)} 5^{N+1} B$.

Throughout the rest of the proof we fix a $w \in \mathcal{D}(k)$ such that there exists a sequence $\{\ell_i(w)\}_{i \in \mathbb{N}}$ satisfying (2.36), $\ell_i(w) \leq \min\{k^{-1}, \delta\}/8$, and

$$(2.37) \quad \frac{\mathcal{S}^h \llcorner \mathcal{D}(k)(\overline{B}(w, \ell_i(w)))}{\mathcal{S}^h \llcorner \Gamma(\overline{B}(w, \ell_i(w)))} \geq \frac{1}{2} \quad \text{for every } i \in \mathbb{N},$$

where the inequality follows from the fact that $\mathcal{S}^h \llcorner \Gamma$ -almost every point of $\mathcal{D}(k)$ has density one with respect to the asymptotically doubling measure $\mathcal{S}^h \llcorner \Gamma$. Notice that, according to the previous discussion, the previous choice on w is made in a set of full $\mathcal{S}^h \llcorner \Gamma$ -measure, so that if we prove the estimate (2.26) with such a w we are done. For the ease of notation we continue the proof fixing the radius $\ell_i(w) = R$. We stress that the forthcoming estimates are verified also for every $\ell_i(w)$. As done in the first part of the proof above, one can prove that for every couple of closed balls $\overline{B}(x, r), \overline{B}(y, s) \in \mathcal{G}_\delta(k)$ such that $\overline{B}(w, R)$ intersects both $\overline{B}(x, 5^{N+1}r)$ and $\overline{B}(y, 5^{N+1}s)$, we have

$$(2.38) \quad P_V(\overline{B}(x, r)) \cap P_V(\overline{B}(y, s)) = \emptyset.$$

In order to proceed with the conclusion of the proof, let us define

$$\begin{aligned} \mathcal{F}_\delta(w, R) &:= \{B \in \mathcal{F}_\delta(k) : 5^{N+1}B \cap \overline{B}(w, R) \cap \mathcal{D}(k) \neq \emptyset\}, \\ \mathcal{G}_\delta(w, R) &:= \{B \in \mathcal{G}_\delta(k) : 5^{N+1}B \cap \overline{B}(w, R) \cap \mathcal{D}(k) \neq \emptyset\}, \end{aligned}$$

Thanks to our choice of R , see (2.37), and the definition of $\mathcal{G}_\delta(w, R)$ we have

$$\frac{R^h}{2^{h+1}} \leq \mathcal{S}^h \llcorner \Gamma(\overline{B}(w, R)) \leq 2\mathcal{S}^h \llcorner \mathcal{D}(k)(\overline{B}(w, R)) \leq 2\mathcal{S}^h \llcorner \mathcal{D}(k) \left(\bigcup_{B \in \mathcal{G}_\delta(w, R)} 5^{N+1}B \right).$$

Let $\mathcal{G}_\delta(w, R) := \{\overline{B}(x_i, r_i)\}_{i \in \mathbb{N}}$ and recall that $x_i \in \mathcal{D}(k)$. This implies, thanks to Proposition 1.34, that

$$\begin{aligned} \mathcal{S}^h_{\perp} \mathcal{D}(k) \left(\bigcup_{B \in \mathcal{G}_\delta(w, R)} 5^{N+1} B \right) &\leq 2 \cdot 5^{(N+1)h} \sum_{i \in \mathbb{N}} r_i^h \\ &= 2 \cdot 5^{(N+1)h} C_4(\mathbb{V}, \mathbb{L})^{-1} \sum_{i \in \mathbb{N}} \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x_i, r_i))) \\ &= 2 \cdot 5^{(N+1)h} C_4(\mathbb{V}, \mathbb{L})^{-1} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{i \in \mathbb{N}} \overline{B}(x_i, r_i) \right) \right) \\ &\leq 2 \cdot 5^{(N+1)h} C_4(\mathbb{V}, \mathbb{L})^{-1} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \right), \end{aligned}$$

where the first inequality comes from the subadditivity of the measure and the upper estimate that we have in the definition of $\mathcal{F}_\delta(k)$; while the first identity of the second line above comes from (2.38). Summing up, for every $\delta > 0$ we have

$$\frac{C_4(\mathbb{V}, \mathbb{L}) R^h}{5^{(N+1)h} 2^{h+3}} \leq \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \right).$$

Arguing as above, we get the Hausdorff convergence

$$P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \xrightarrow{H, \delta \rightarrow 0} P_{\mathbb{V}} \left(\overline{\mathcal{D}(k)} \cap \overline{B}(w, R) \right).$$

Thanks to the upper semicontinuity of the Lebesgue measure with respect to the Hausdorff convergence we eventually infer that

$$\begin{aligned} (2.39) \quad \frac{C_4(\mathbb{V}, \mathbb{L}) R^h}{5^{(N+1)h} 2^{h+3}} &\leq \limsup_{\delta \rightarrow 0} \mathcal{S}^h \left(P_{\mathbb{V}} \left(\bigcup_{B \in \mathcal{F}_\delta(w, R)} B \right) \right) \\ &\leq \mathcal{S}^h(P_{\mathbb{V}}(\overline{\mathcal{D}(k)} \cap \overline{B}(w, R))) \leq \mathcal{S}^h(P_{\mathbb{V}}(\Gamma \cap \overline{B}(w, R))), \end{aligned}$$

where the last inequality above comes from the compactness of Γ and the fact that $\mathcal{D}(k) \subseteq \Gamma$. \square

Proposition 2.18. *Let us fix $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$, and suppose Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with $0 < \mathcal{S}^h(\Gamma) < +\infty$ such that*

$$\limsup_{r \rightarrow 0} \frac{\mathcal{S}^h_{\perp} \Gamma(\overline{B}(x, 2r))}{\mathcal{S}^h_{\perp} \Gamma(\overline{B}(x, r))} < \infty,$$

for \mathcal{S}^h -almost every $x \in \Gamma$. Let us set $\varphi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{L}$ the map whose graph is Γ , see Proposition 1.89, and set $\Phi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{G}$ to be the graph map of φ . Let us define $\Phi_* \mathcal{S}^h_{\perp} \mathbb{V}$ to be the measure on Γ such that for every measurable $A \subseteq \Gamma$ we have $\Phi_* \mathcal{S}^h_{\perp} \mathbb{V}(A) := \mathcal{S}^h_{\perp} \mathbb{V}(\Phi^{-1}(A)) = \mathcal{S}^h_{\perp} \mathbb{V}(P_{\mathbb{V}}(A))$. Then $\Phi_* \mathcal{S}^h_{\perp} \mathbb{V}$ is mutually absolutely continuous with respect to $\mathcal{S}^h_{\perp} \Gamma$.

Proof. The fact that $\Phi_* \mathcal{S}^h_{\perp} \mathbb{V}$ is absolutely continuous with respect to $\mathcal{S}^h_{\perp} \Gamma$ is an immediate consequence of Proposition 1.34. Vice-versa, suppose by contradiction that there exists a compact subset C of Γ of positive \mathcal{S}^h -measure such that

$$(2.40) \quad 0 = \Phi_* \mathcal{S}^h_{\perp} \mathbb{V}(C) = \mathcal{S}^h(P_{\mathbb{V}}(C)).$$

Since $\mathcal{S}^h \llcorner C$ is asymptotically doubling by Proposition 1.55 and C has positive and finite \mathcal{S}^h -measure, we infer thanks to the second part of Proposition 2.17 that the set C must have a projection of positive \mathcal{S}^h -measure. This however comes in contradiction with (2.40). \square

In the following propositions we are going to introduce two fine coverings of $P_{\mathbb{V}}(\Gamma)$ and \mathbb{V} , respectively, that will be used in the proof of Proposition 2.21 to differentiate with respect to the measure $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$.

Proposition 2.19. *Let $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and suppose that Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with $0 < \mathcal{S}^h(\Gamma) < +\infty$ such that*

$$\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0,$$

for \mathcal{S}^h -almost every $x \in \Gamma$. As in the statement of Proposition 2.18, let us denote with $\Phi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{G}$ the graph map of $\varphi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{L}$ whose intrinsic graph is Γ . Then the covering relation

$$S_1 := \left\{ (z, P_{\mathbb{V}}(\overline{B}(\Phi(z), r) \cap \Gamma)) : z \in P_{\mathbb{V}}(\Gamma) \text{ and } 0 < r < \min\{1, R(\Phi(z))\} \right\},$$

is a $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ -Vitali relation, where $R(\Phi(z))$ is defined as in the first part of Proposition 2.17 for $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ -almost every $z \in \mathbb{V}$, and it is $+\infty$ on the remaining $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ -null set (cf. Proposition 2.18) where the first part of Proposition 2.17 does not hold.

Proof. First of all, it is readily noticed that S_1 is a fine covering of $P_{\mathbb{V}}(\Gamma)$ since $P_{\mathbb{V}}$ is continuous. Let us prove that S_1 is a $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ -Vitali relation in $(P_{\mathbb{V}}(\Gamma), d)$ with the distance d induced from \mathbb{G} . For $x \in P_{\mathbb{V}}(\Gamma)$ and $r > 0$, define $G(x, r) := P_{\mathbb{V}}(\overline{B}(\Phi(x), r) \cap \Gamma)$. Notice that an arbitrary element of $S_1(P_{\mathbb{V}}(\Gamma))$, see (1.2), is of the form $G(x, r)$ for some $x \in P_{\mathbb{V}}(\Gamma)$ and some $0 < r < \min\{1, R(\Phi(x))\}$. Let $\delta(G(x, r)) := r$ and note that the δ -enlargement, see (1.3), of $G(x, r)$ is

(2.41)

$$\begin{aligned} \hat{G}(x, r) &:= \bigcup \{G(y, s) : y \in P_{\mathbb{V}}(\Gamma), 0 < s < \min\{1, R(\Phi(y))\}, \\ &\quad G(y, s) \cap G(x, r) \neq \emptyset \text{ and } \delta(G(y, s)) \leq 5\delta(G(x, r))\} \\ &= \bigcup \{G(y, s) : y \in P_{\mathbb{V}}(\Gamma), 0 < s < \min\{1, R(\Phi(y))\}, G(y, s) \cap G(x, r) \neq \emptyset, s \leq 5r\}. \end{aligned}$$

Whenever $G(x, r) \cap G(y, s) \neq \emptyset$ we have that $d(\Phi(x), \Phi(y)) \leq r+s$: indeed, since $P_{\mathbb{V}}$ is injective on Γ , see Proposition 1.89, we have $P_{\mathbb{V}}(\overline{B}(\Phi(x), r) \cap \Gamma) \cap P_{\mathbb{V}}(\overline{B}(\Phi(y), s) \cap \Gamma) \neq \emptyset$ if and only if $\overline{B}(\Phi(x), r) \cap \overline{B}(\Phi(y), s) \cap \Gamma \neq \emptyset$. In particular, since $s \leq 5r$ we have $\overline{B}(\Phi(y), s) \subseteq \overline{B}(\Phi(x), 12r)$, and thus $\hat{G}(x, r) \subseteq G(x, 12r)$ for every $x \in P_{\mathbb{V}}(\Gamma)$ and $0 < r < \min\{1, R(\Phi(x))\}$.

Finally, thanks to Proposition 2.17 and Proposition 2.18, for \mathcal{S}^h -almost every $x \in P_{\mathbb{V}}(\Gamma)$ we have

$$\begin{aligned} &\limsup_{\xi \rightarrow 0} \left\{ \delta(G(x, r)) + \frac{\mathcal{S}^h(\hat{G}(x, r))}{\mathcal{S}^h(G(x, r))} : 0 < r < \min\{1, R(\Phi(x))\}, \text{diam}(G(x, r)) \leq \xi \right\} \\ (2.42) \quad &\leq 1 + \limsup_{\xi \rightarrow 0} \frac{\mathcal{S}^h(G(x, 12r))}{\mathcal{S}^h(G(x, r))} \leq 1 + \limsup_{\xi \rightarrow 0} \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(x), 12r)))}{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(x), r) \cap \Gamma))} \\ &\leq 1 + \limsup_{\xi \rightarrow 0} \frac{(12r)^h \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, 1)))}{C_7 \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, \Phi(x))^{2r^h}} = 1 + \frac{12^h \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, 1)))}{C_7 \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, \Phi(x))^2}, \end{aligned}$$

where we explicitly mentioned the set over which we take the supremum only in the first line for the ease of notation, and where the first inequality in the third line follows from the fact that $\mathcal{S}^h(P_{\mathbb{V}}(\mathcal{E})) = \mathcal{S}^h(P_{\mathbb{V}}(x\mathcal{E}))$ for every $x \in \mathbb{G}$ and every Borel set $\mathcal{E} \subseteq \mathbb{G}$, see Proposition 1.32. Thanks to (2.42) we can apply the first part of Proposition 1.13 and thus we infer that S_1 is a $\mathcal{S}^h \llcorner P_{\mathbb{V}}(\Gamma)$ -Vitali relation. \square

Proposition 2.20. *Let $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and let Γ be a compact $C_{\mathbb{V}}(\alpha)$ -set with $0 < \mathcal{S}^h(\Gamma) < +\infty$. As in the statement of Proposition 2.18, let us denote with $\Phi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{G}$ the graph map of $\varphi : P_{\mathbb{V}}(\Gamma) \rightarrow \mathbb{L}$ whose intrinsic graph is Γ . Then for \mathcal{S}^h -almost every $w \in P_{\mathbb{V}}(\Gamma)$ we have*

$$(2.43) \quad \lim_{r \rightarrow 0} \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(w), r) \cap \Phi(w)C_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma))}{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(w), r) \cap \Phi(w)C_{\mathbb{V}}(\alpha)))} = 1.$$

Proof. For every $w \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma)$ we let

$$\rho(w) := \inf\{r \geq 0 : \overline{B}(w, r) \cap P_{\mathbb{V}}(\overline{B}(\Gamma, r^{1/\kappa})) \neq \emptyset\}.$$

It is immediate to see that $\rho(w) \leq \text{dist}(w, P_{\mathbb{V}}(\Gamma))$ and that $\rho(w) = 0$ if and only if $w \in P_{\mathbb{V}}(\Gamma)$. Throughout the rest of the proof we let S be the fine covering of \mathbb{V} given by the couples $(w, G(w, r))$ for which

- (α) if $w \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma)$ then $r \in (0, \min\{\rho(w)/2, 1\})$ and $G(w, r) := \overline{B}(w, r) \cap \mathbb{V}$,
- (β) if $w \in P_{\mathbb{V}}(\Gamma)$ then $r \in (0, 1)$ and $G(w, r) := P_{\mathbb{V}}(\overline{B}(\Phi(w), r) \cap \Phi(w)C_{\mathbb{V}}(\alpha))$.

Furthermore, for every $w \in \mathbb{V}$ we define the function δ on $S(\mathbb{V})$, see (1.2), as

$$(2.44) \quad \delta(G(w, r)) := r.$$

If we prove that S is a $\mathcal{S}^h_{\mathbb{L}}\mathbb{V}$ -Vitali relation, the second part of Proposition 1.13 directly implies that (2.43) holds. If for \mathcal{S}^h -almost every $w \in \mathbb{V}$ we prove that

$$(2.45) \quad \lim_{\xi \rightarrow 0} \sup_{(w, G(w, r)) \in S, \text{diam}(G(w, r)) \leq \xi} \left\{ \delta(G(w, r)) + \frac{\mathcal{S}^h(\hat{G}(w, r))}{\mathcal{S}^h(G(w, r))} \right\} \leq 1 + \limsup_{\xi \rightarrow 0} \frac{\mathcal{S}^h(\hat{G}(w, r))}{\mathcal{S}^h(G(w, r))} < \infty,$$

where we explicitly mentioned the set over which we take the supremum only the first time for the ease of notation, and where $\hat{G}(w, r)$ is the δ -enlargement of $G(w, r)$, see (1.3); thus, thanks to the first part of Proposition 1.13 we would immediately infer that S is a $\mathcal{S}^h_{\mathbb{L}}\mathbb{V}$ -Vitali relation. In order to prove that (2.45) holds, we need to get a better understanding of the geometric structure of the δ -enlargement of $G(w, r)$.

If $w \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma)$, we note that there must exist an $0 < r(w) < \min\{\rho(w)/2, 1\}$ such that for every $0 < r < r(w)$ we have

$$\overline{B}(w, r) \cap P_{\mathbb{V}}(\overline{B}(\Gamma, 5r)) = \emptyset.$$

Indeed, if this is not the case there would exist a sequence $r_i \downarrow 0$ and a sequence $\{z_i\}_{i \in \mathbb{N}}$ such that

$$z_i \in \overline{B}(w, r_i) \cap P_{\mathbb{V}}(\overline{B}(\Gamma, 5r_i)).$$

Since $P_{\mathbb{V}}(\Gamma)$ is compact and $P_{\mathbb{V}}$ is continuous on the closed tubular neighborhood $\overline{B}(\Gamma, 1)$, up to passing to a non re-labelled subsequence we have that the z_i 's converge to some $z \in P_{\mathbb{V}}(\Gamma)$ and on the other hand by construction the z_i 's converge to w which is not contained in $P_{\mathbb{V}}(\Gamma)$, and this is a contradiction. This implies that if $0 < r < r(w)$, we have

$$(2.46) \quad \begin{aligned} \hat{G}(w, r) &= \bigcup \{G(y, s) : y \in \mathbb{V}, s > 0, (y, G(y, s)) \in S, G(y, s) \cap G(w, r) \neq \emptyset, \text{ and } s \leq 5r\} \\ &\subseteq \bigcup \{\overline{B}(y, s) \cap \mathbb{V} : \overline{B}(y, s) \cap \overline{B}(w, r) \cap \mathbb{V} \neq \emptyset \text{ and } s \leq 5r\} \subseteq \overline{B}(w, 11r) \cap \mathbb{V}, \end{aligned}$$

where in the inclusion we are using the fact that if y were in $P_{\mathbb{V}}(\Gamma)$, and $s \leq 5r$, then $G(y, s) \subseteq P_{\mathbb{V}}(\overline{B}(\Gamma, s)) \subseteq P_{\mathbb{V}}(\overline{B}(\Gamma, 5r))$ which would be in contradiction with $G(y, s) \cap G(w, r) \neq \emptyset$, since we chose $0 < r < r(w)$. Summing up, if $w \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma)$ the bound (2.45) immediately follows thanks to (2.46) and the homogeneity of \mathcal{S}^h .

If on the other hand $w \in P_{\mathbb{V}}(\Gamma)$ the situation is more complicated. If $y \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma)$ and $s \leq 5r$ are such that

$$(2.47) \quad G(y, s) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) = \overline{B}(y, s) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset,$$

since by construction of the covering S we also assumed that $0 < s < \rho(y)/2$, we infer that we must have $r \geq s^{1/\kappa}$ for (2.47) to be satisfied. This allows us to infer that, for every $w \in P_{\mathbb{V}}(\Gamma)$ and $0 < r < 1$, we have

$$(2.48) \quad \begin{aligned} \hat{G}(w, r) &= \bigcup \{G(y, s) : y \in \mathbb{V}, s > 0, (y, G(y, s)) \in S, G(y, s) \cap G(w, r) \neq \emptyset, \text{ and } s \leq 5r\} \\ &\subseteq \bigcup \{P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) : y \in P_{\mathbb{V}}(\Gamma), P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset, \text{ and } s \leq 5r\} \cup \\ &\cup \bigcup \{\overline{B}(y, s) \cap \mathbb{V} : y \in \mathbb{V} \setminus P_{\mathbb{V}}(\Gamma), \overline{B}(y, s) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset, s \leq \min\{5r, \rho(y)/2\}\} \\ &\subseteq \bigcup \{P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) : y \in P_{\mathbb{V}}(\Gamma), P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset, \text{ and } s \leq 5r\} \cup \\ &\cup \left(\overline{B}(P_{\mathbb{V}}(\overline{B}(\Phi(w), r)), 3r^\kappa) \cap \mathbb{V} \right), \end{aligned}$$

where in the last inclusion we are using the observation right after (2.47) according to which $s \leq r^\kappa$. We now study independently each of the two terms of the union of the last two lines above. Let us first note that if $w, y \in P_{\mathbb{V}}(\Gamma)$, $s \leq 5r$ and

$$P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset,$$

then $P_{\mathbb{V}}(\overline{B}(\Phi(y), 10r)) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), 2r)) \neq \emptyset$. This observation and Corollary 2.15 imply that if $0 < r < r(w)$ is sufficiently small we have $d(\Phi(w), \Phi(y)) \leq 50Ar$, where the constant $A = A(\mathbb{V}, \mathbb{L})$ is yielded by Lemma 2.13. In particular we deduce that for every $0 < r < r(w)$ sufficiently small

$$\begin{aligned} &\bigcup \{P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) : y \in P_{\mathbb{V}}(\Gamma), P_{\mathbb{V}}(\overline{B}(\Phi(y), s)) \cap P_{\mathbb{V}}(\overline{B}(\Phi(w), r)) \neq \emptyset, \text{ and } s \leq 5r\} \\ &\subseteq P_{\mathbb{V}}(\overline{B}(\Phi(w), 50(A+1)r)). \end{aligned}$$

In order to study the term in the last line of (2.48), we prove the following claim: for every $0 < r < 1$, every $z \in P_{\mathbb{V}}(\overline{B}(\Phi(w), r))$, and every $\Delta \in \overline{B}(0, 3r^\kappa) \cap \mathbb{V}$ we have $z\Delta \in P_{\mathbb{V}}(\overline{B}(\Phi(w), C(\Gamma)r))$, where $C(\Gamma)$ is a constant depending only on Γ . Indeed, since Γ is compact and $P_{\mathbb{L}}$ is continuous, there exists a constant $K' := K'(\Gamma)$ such that whenever $0 < r < 1$, and $z \in P_{\mathbb{V}}(\overline{B}(\Phi(w), r))$, there exists an $\ell \in \mathbb{L}$ such that $z\ell \in \overline{B}(\Phi(w), r)$ and $\|\ell\| \leq K'$. Thus there exists a constant $K := K(\Gamma) > 0$ such that whenever $0 < r < 1$, $z \in P_{\mathbb{V}}(\overline{B}(\Phi(w), r))$, and $\Delta \in \overline{B}(0, 3r^\kappa) \cap \mathbb{V}$, there exists $\ell \in \mathbb{L}$ with $z\ell \in \overline{B}(\Phi(w), r)$ and $\|\Delta\| + \|\ell\| \leq K$. Thus we can estimate

$$d(\Phi(w), z\Delta\ell) \leq d(\Phi(w), z\ell) + d(z\ell, z\Delta\ell) \leq r + C_6(K, \mathbb{G})\|\Delta\|^{1/\kappa} \leq C(\Gamma)r,$$

where the second inequality in the last equation comes from Lemma 2.8. Thus $z\Delta \in P_{\mathbb{V}}(\overline{B}(\Phi(w), C(\Gamma)r))$, and the claim is proved. Summing up, we have proved that whenever $w \in P_{\mathbb{V}}(\Gamma)$ and $0 < r < r(w)$ is sufficiently small we have

$$\hat{G}(w, r) \subseteq P_{\mathbb{V}}(\overline{B}(\Phi(w), 50(A+1)r)) \cup P_{\mathbb{V}}(\overline{B}(\Phi(w), C(\Gamma)r)),$$

and thus (2.45) immediately follows by the latter inclusion, the homogeneity of $\mathcal{S}^h \llcorner \mathbb{V}$, and the fact that $\mathcal{S}^h(P_{\mathbb{V}}(x\mathcal{E})) = \mathcal{S}^h(P_{\mathbb{V}}(\mathcal{E}))$ for every $x \in \mathbb{G}$ and \mathcal{E} a Borel subset of \mathbb{G} , see Proposition 1.32. This concludes the proof of the proposition. \square

We are now ready to prove the main step that will be needed for the proof of Theorem 2.12.

Proposition 2.21. *Let us fix $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$. Suppose Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with $0 < \mathcal{S}^h(\Gamma) < +\infty$, and such that $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable. For \mathcal{S}^h -almost every $x \in \Gamma$ we have*

$$(2.49) \quad (1 - \mathfrak{c}(\alpha))^{2h}(1 + \mathfrak{c}(\alpha))^{-h} \leq \frac{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)}{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)} \leq 1,$$

where $\mathfrak{c}(\alpha)$ is defined in Lemma 1.43.

Proof. Let us preliminarily observe that since $\mathcal{S}^h \llcorner \mathbb{V}$ and $\mathcal{C}^h \llcorner \mathbb{V}$ are both Haar measures on \mathbb{V} , they coincide up to a constant. Since for \mathcal{S}^h -almost every $x \in \Gamma$ we have $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$, the upper bound is trivial. Let us proceed with the lower bound. Thanks to Proposition 2.18 and the Radon-Nikodým Theorem, see [125, page 82], there exists $\rho \in L^1(\Phi_* \mathcal{C}^h \llcorner \mathbb{V})$ such that

- (i) $\rho(x) > 0$ for $\Phi_* \mathcal{C}^h \llcorner \mathbb{V}$ -almost every $x \in \Gamma$,
- (ii) $\mathcal{S}^h \llcorner \Gamma = \rho \Phi_* \mathcal{C}^h \llcorner \mathbb{V}$.

We stress that the following reasoning holds for $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \Gamma$. Let $\{r_i\}_{i \in \mathbb{N}}$ be an infinitesimal sequence such that $r_i^{-h} T_{x, r_i} \mathcal{S}^h \llcorner \Gamma \rightarrow \lambda \mathcal{C}^h \llcorner \mathbb{V}(x)$ for some $\lambda > 0$. First of all, we immediately see that Corollary 1.63 implies that $\lambda \in [\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x), \Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)]$ and that

$$\begin{aligned} 1 &= \lim_{i \rightarrow \infty} \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, r_i))}{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, r_i))} = \lim_{i \rightarrow \infty} \frac{\int_{P_{\mathbb{V}}(\overline{B}(x, r_i) \cap \Gamma)} \rho(\Phi(y)) d\mathcal{C}^h \llcorner \mathbb{V}(y)}{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, r_i))} \\ &= \frac{\rho(x)}{\lambda} \lim_{i \rightarrow \infty} \frac{\mathcal{C}^h \llcorner \mathbb{V}(P_{\mathbb{V}}(\overline{B}(x, r_i) \cap \Gamma))}{r_i^h}, \end{aligned}$$

where the last identity comes from Proposition 2.19, that allows us to differentiate by using the second part of Proposition 1.13, and Lemma 1.62. Thanks to Lemma 1.43, Remark 1.27, and the fact that Γ is a $C_{\mathbb{V}}(\alpha)$ -set, we have

$$(2.50) \quad \begin{aligned} \frac{\lambda}{\rho(x)} &\leq \lim_{i \rightarrow \infty} \frac{\mathcal{C}^h \llcorner \mathbb{V}(P_{\mathbb{V}}(\overline{B}(x, r_i) \cap xC_{\mathbb{V}}(\alpha)))}{r_i^h} = \mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha))) \\ &\leq \frac{\mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(0, 1))}{(1 - \mathfrak{c}(\alpha))^h} = (1 - \mathfrak{c}(\alpha))^{-h}, \end{aligned}$$

where in the second equality we used the homogeneity of \mathcal{C}^h and the fact that $\mathcal{C}^h(P_{\mathbb{V}}(x\mathcal{E})) = \mathcal{C}^h(P_{\mathbb{V}}(\mathcal{E}))$ for every $x \in \mathbb{G}$ and \mathcal{E} a Borel subset of \mathbb{G} , see Proposition 1.32. On the other hand, thanks to Lemma 1.44 we have

$$(2.51) \quad \begin{aligned} \frac{\lambda}{\rho(x)} &= \lim_{i \rightarrow \infty} \frac{\mathcal{C}^h \llcorner \mathbb{V}(P_{\mathbb{V}}(\overline{B}(x, r_i) \cap \Gamma))}{r_i^h} \\ &\geq \lim_{i \rightarrow \infty} \frac{\mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r_i) \cap xC_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma))}{\mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r_i) \cap xC_{\mathbb{V}}(\alpha)))} \frac{\mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r_i) \cap xC_{\mathbb{V}}(\alpha)))}{r_i^h} \\ &= \mathfrak{C}(\alpha)^h \mathcal{C}^h(P_{\mathbb{V}}(\overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha))) \geq \mathfrak{C}(\alpha)^h, \end{aligned}$$

where the first identity in the last line comes from Proposition 2.20, and the inequality comes from Lemma 1.43, Remark 1.27, and $\mathfrak{C}(\alpha)$ is defined in (1.28). Putting together (2.50) and (2.51), we have

$$(2.52) \quad \frac{(1 - \mathfrak{c}(\alpha))^h}{(1 + \mathfrak{c}(\alpha))^h} \leq \frac{\lambda}{\rho(x)} \leq \frac{1}{(1 - \mathfrak{c}(\alpha))^h}.$$

Thanks to the definition of $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)$ and $\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)$ we can find two sequences $\{r_i\}_{i \in \mathbb{N}}$ and $\{s_i\}_{i \in \mathbb{N}}$ such that

$$\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) = \lim_{i \rightarrow \infty} \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, r_i))}{r_i^h}, \quad \text{and} \quad \Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x) = \lim_{i \rightarrow \infty} \frac{\mathcal{S}^h \llcorner \Gamma(\overline{B}(x, s_i))}{s_i^h},$$

and without loss of generality, taking Lemma 1.62 into account, we have that, up to passing to subsequences,

$$r_i^{-h} T_{x, r_i} \mathcal{S}^h \llcorner \Gamma \rightharpoonup \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) \mathcal{C}^h \llcorner \mathbb{V}(x), \quad , \quad s_i^{-h} T_{x, s_i} \mathcal{S}^h \llcorner \Gamma \rightharpoonup \Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x) \mathcal{C}^h \llcorner \mathbb{V}(x).$$

The bounds (2.52) imply therefore that

$$(2.53) \quad \begin{aligned} \frac{(1 - \mathfrak{c}(\alpha))^h}{(1 + \mathfrak{c}(\alpha))^h} &\leq \frac{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)}{\rho(x)} \leq \frac{1}{(1 - \mathfrak{c}(\alpha))^h}, \\ \frac{(1 - \mathfrak{c}(\alpha))^h}{(1 + \mathfrak{c}(\alpha))^h} &\leq \frac{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)}{\rho(x)} \leq \frac{1}{(1 - \mathfrak{c}(\alpha))^h}. \end{aligned}$$

Finally the bounds in (2.53) yield

$$(1 - \mathfrak{c}(\alpha))^{2h} (1 + \mathfrak{c}(\alpha))^{-h} \leq \frac{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)}{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)} \leq 1,$$

and this concludes the proof. \square

We prove now the existence of the density of \mathcal{P}_h^c -rectifiable measures, see Theorem 2.12. We first prove an algebraic lemma, then we prove the existence of the density for measures of the type $\mathcal{S}^h \llcorner \Gamma$, and then we conclude with the proof of the existence of the density for arbitrary \mathcal{P}_h^c -rectifiable measures.

Lemma 2.22. *Let us fix $0 < \varepsilon < 1$ a real number, $\ell, h \in \mathbb{N}$, and let f be the function defined as follows*

$$f: \{(\alpha, C) \in (0, +\infty)^2 : \alpha < C\} \rightarrow (0, +\infty), \quad f(\alpha, C) := \frac{\alpha}{C - \alpha}.$$

Then, there exists $\tilde{\alpha} := \tilde{\alpha}(\varepsilon, \ell, h) > 0$ such that the following implication holds

$$\text{if } 0 < \alpha \leq \tilde{\alpha} \text{ and } C > 1/\ell, \text{ then } \alpha < C \text{ and } (1 - f(\alpha, C))^{2h} (1 + f(\alpha, C))^{-h} \geq 1 - \varepsilon.$$

Proof. Let us choose $0 < \tilde{\varepsilon} := \tilde{\varepsilon}(\varepsilon, h) < 1$ such that

$$(1 - x)^{2h} (1 + x)^{-h} \geq 1 - \varepsilon, \quad \text{for all } 0 \leq x \leq \tilde{\varepsilon}.$$

Let us show that the sought constant $\tilde{\alpha}(\varepsilon, \ell, h)$ can be chosen to be $\tilde{\alpha} := \tilde{\varepsilon}/(\ell(1 + \tilde{\varepsilon}))$. Indeed, if $\alpha \leq \tilde{\alpha}$ and $C > 1/\ell$ we infer that $\alpha < C$ and

$$\alpha \leq \frac{\tilde{\varepsilon}}{\ell(1 + \tilde{\varepsilon})} \leq \frac{C\tilde{\varepsilon}}{1 + \tilde{\varepsilon}}, \quad \text{and then} \quad f(\alpha, C) = \frac{\alpha}{C - \alpha} \leq \tilde{\varepsilon}.$$

This implies that if $\alpha \leq \tilde{\alpha}$ and $C > 1/\ell$, then

$$(1 - f(\alpha, C))^{2h} (1 + f(\alpha, C))^{-h} \geq 1 - \varepsilon,$$

where the last inequality above comes from the choice of $\tilde{\varepsilon}$. This concludes the proof. \square

Theorem 2.23. *Let Γ be a compact subset of \mathbb{G} with $0 < \mathcal{S}^h(\Gamma) < +\infty$, and such that $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h^c -rectifiable measure. Then*

$$0 < \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) = \Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x) < +\infty, \quad \text{for } \mathcal{S}^h \llcorner \Gamma\text{-almost every } x \in \mathbb{G}.$$

Proof. In the following, for every $\varepsilon > 0$, we will construct a measurable set $A_\varepsilon \subseteq \Gamma$ such that $\mathcal{S}^h(\Gamma \setminus A_\varepsilon) = 0$ and

$$(2.54) \quad 1 - \varepsilon \leq \frac{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)}{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)} \leq 1, \quad \text{for every } x \in A_\varepsilon.$$

If (2.54) holds then we are free to choose $\varepsilon = 1/n$ for every $n \in \mathbb{N}$ and then the density of $\mathcal{S}^h \llcorner \Gamma$ exists on the set $\cap_{n=1}^{+\infty} A_{1/n}$, that has full $\mathcal{S}^h \llcorner \Gamma$ -measure. So we are left to construct A_ε as in (2.54). Let us define the function

$$\mathbf{F}(\mathbb{V}, \mathbb{L}) := \varepsilon_1(\mathbb{V}, \mathbb{L}), \quad \text{for all } \mathbb{V} \in \text{Gr}_c(h) \text{ with complement } \mathbb{L}.$$

Let us take the family $\mathcal{F} := \{\mathbb{V}_i\}_{i=1}^{+\infty} \subseteq \text{Gr}_c(h)$ and let us choose \mathbb{L}_i complementary subgroups to \mathbb{V}_i as in the statement of Theorem 2.10. We remark that the choices of the family \mathcal{F} and of the complementary subgroups depend on the function \mathbf{F} previously defined, see the discussion before Theorem 2.10. Let us define

$$\beta: \mathbb{N} \rightarrow (0, 1), \quad \beta(\ell) := \tilde{\alpha}(\varepsilon, \ell, h),$$

where $\tilde{\alpha}(\varepsilon, \ell, h)$ is the constant in Lemma 2.22, and with an abuse of notation let us lift β to a function on \mathcal{F} as we did in the statement of Theorem 2.10. From Theorem 2.10 we conclude that there exist countably many Γ_i 's that are compact $C_{\mathbb{V}_i}(\min\{\varepsilon_1(\mathbb{V}_i, \mathbb{L}_i), \beta(\mathbb{V}_i)\})$ -sets contained in Γ such that

$$(2.55) \quad \mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Let us write, for the ease of notation, $\alpha_i := \min\{\varepsilon_1(\mathbb{V}_i, \mathbb{L}_i), \beta(\mathbb{V}_i)\}$ for every $i \in \mathbb{N}$. Since $\Gamma_i \subseteq \Gamma$ and $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable, we conclude, by exploiting the locality of tangents and the Lebesgue Differentiation Theorem, see Proposition 1.55, that the measures $\mathcal{S}^h \llcorner \Gamma_i$ are \mathcal{P}_h^c -rectifiable as well for every $i \in \mathbb{N}$. Thus, since $\alpha_i \leq \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i)$, we can apply Proposition 2.21 and conclude that, for every $i \in \mathbb{N}$, we have

$$(1 - \mathbf{c}(\alpha_i))^{2h} (1 + \mathbf{c}(\alpha_i))^{-h} \leq \frac{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma_i, x)}{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma_i, x)} \leq 1, \quad \text{for } \mathcal{S}^h \llcorner \Gamma_i\text{-almost every } x \in \mathbb{G},$$

where $\mathbf{c}(\alpha_i) := \alpha_i / (C_2(\mathbb{V}_i, \mathbb{L}_i) - \alpha_i)$. Since it holds that $\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma_i, x) = \Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)$, and $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma_i, x) = \Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)$ for $\mathcal{S}^h \llcorner \Gamma_i$ -almost every $x \in \mathbb{G}$, see Proposition 1.55, for every $i \in \mathbb{N}$ we conclude that

$$(2.56) \quad (1 - \mathbf{c}(\alpha_i))^{2h} (1 + \mathbf{c}(\alpha_i))^{-h} \leq \frac{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)}{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)} \leq 1, \quad \text{for } \mathcal{S}^h \llcorner \Gamma_i\text{-almost every } x \in \mathbb{G}.$$

Let us now fix $i \in \mathbb{N}$ and note there exists a unique $\ell(i) \in \mathbb{N}$ such that

$$1/\ell(i) < \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i) \leq 1/(\ell(i) - 1).$$

Moreover, from the definition of β and \mathcal{F} we see that $\beta(\mathbb{V}_i) = \beta(\varepsilon, \ell(i), h)$. This allows us to infer that

- (1) $\alpha_i \leq \beta(\mathbb{V}_i) = \beta(\varepsilon, \ell(i), h)$, since $\alpha_i := \min\{\varepsilon_1(\mathbb{V}_i, \mathbb{L}_i), \beta(\mathbb{V}_i)\}$,
- (2) $C_2(\mathbb{V}_i, \mathbb{L}_i) > 1/\ell(i)$, since $1/\ell(i) < \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i) = C_2(\mathbb{V}_i, \mathbb{L}_i)/2$, see Lemma 1.40.

Thus we can apply Lemma 2.22 and conclude that

$$(1 - \mathbf{c}(\alpha_i))^{2h} (1 + \mathbf{c}(\alpha_i))^{-h} \geq 1 - \varepsilon.$$

This shows, thanks to (2.56), that for every $i \in \mathbb{N}$, we have

$$1 - \varepsilon \leq \frac{\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x)}{\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x)} \leq 1, \quad \text{for } \mathcal{S}^h \llcorner \Gamma_i\text{-almost every } x \in \mathbb{G}.$$

Thus by taking into account (2.55) and the previous equation we conclude (2.54), that is the sought claim. \square

Remark 2.24. It is a classical result that if $E \subseteq \mathbb{R}^n$ is a h -rectifiable set, with $1 \leq h \leq n$, then $\Theta^h(\mathcal{S}^h \llcorner E, x) = 1$ for \mathcal{S}^h -almost every point $x \in E$, see [102, Theorem 3.2.19]. The converse also holds as it is a special case of Preiss's theorem [202].

We point out that as a consequence of the forthcoming Theorem 2.30, we have that whenever $\Gamma \subseteq \mathbb{G}$ is a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, and $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable, then $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ for \mathcal{C}^h -almost every $x \in \Gamma$.

We thus ask the following two questions, the second one being a simplified version of the analogue of Preiss's theorem for sets on Carnot groups.

Question 2. Understand whether the following holds. Let $\Gamma \subseteq \mathbb{G}$ be a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, and $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable. Then $\Theta^h(\mathcal{S}^h \llcorner \Gamma, x) = 1$ for \mathcal{S}^h -almost every $x \in \Gamma$.

Question 3. Understand whether the following holds. Let $\Gamma \subseteq \mathbb{G}$ be a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, and $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ (or $\Theta^h(\mathcal{S}^h \llcorner \Gamma, x) = 1$) for \mathcal{S}^h -almost every $x \in \Gamma$. Then \mathcal{S}^h is \mathcal{P}_h -rectifiable.

Regarding the last question, very recently Julia–Merlo [131] proved that on every homogeneous group there is a homogeneous left-invariant distance such that if the measure $\mathcal{H}^h \llcorner \Gamma$ has h -density one almost everywhere, then Γ is Euclidean rectifiable.

Proof of Theorem 2.12. We stress that by restricting ourselves on balls of integer radii, by using Proposition 1.55, we can assume that ϕ has compact support. Let us first recall that, by Proposition 1.11, we have

$$(2.57) \quad \phi \left(\mathbb{G} \setminus \bigcup_{\vartheta, \gamma \in \mathbb{N}} E(\vartheta, \gamma) \right) = 0.$$

Let us fix $\vartheta, \gamma \in \mathbb{N}$. From Lebesgue Differentiation Theorem and the locality of tangents, see Proposition 1.55, we deduce that ϕ being \mathcal{P}_h^c -rectifiable implies that $\phi \llcorner E(\vartheta, \gamma)$ is \mathcal{P}_h^c -rectifiable. From Proposition 1.56 we deduce that $\phi \llcorner E(\vartheta, \gamma)$ is mutually absolutely continuous with respect to $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$, and thus, by Radon–Nikodým theorem, see [125, page 82], there exists a positive function $\rho \in L^1(\mathcal{S}^h \llcorner E(\vartheta, \gamma))$ such that $\phi \llcorner E(\vartheta, \gamma) = \rho \mathcal{S}^h \llcorner E(\vartheta, \gamma)$. We stress that we can apply Lebesgue–Radon–Nikodým theorem since $\phi \llcorner E(\vartheta, \gamma)$ is asymptotically doubling because it has positive h -lower density and finite h -upper density almost everywhere. By Lebesgue–Radon–Nikodým theorem, see [125, page 82], and the locality of tangents again, we deduce that $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$ is a \mathcal{P}_h^c -rectifiable measure, since $\phi \llcorner E(\vartheta, \gamma)$ is a \mathcal{P}_h^c -rectifiable measure. Thus we can apply Theorem 2.23 to $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$ and obtain that for every $\vartheta, \gamma \in \mathbb{N}$ we have that

$$0 < \Theta_*^h(\mathcal{S}^h \llcorner E(\vartheta, \gamma), x) = \Theta^{h,*}(\mathcal{S}^h \llcorner E(\vartheta, \gamma), x) < +\infty, \quad \text{for } \mathcal{S}^h \llcorner E(\vartheta, \gamma)\text{-a.e. } x \in \mathbb{G}.$$

Since $\phi \llcorner E(\vartheta, \gamma) = \rho \mathcal{S}^h \llcorner E(\vartheta, \gamma)$ we thus conclude from the previous equality and by Lebesgue–Radon–Nikodým theorem that for every $\vartheta, \gamma \in \mathbb{N}$ we have that

$$0 < \Theta_*^h(\phi \llcorner E(\vartheta, \gamma), x) = \Theta^{h,*}(\phi \llcorner E(\vartheta, \gamma), x) < +\infty, \quad \text{for } \phi \llcorner E(\vartheta, \gamma)\text{-a.e. } x \in \mathbb{G}.$$

The previous equality, jointly with Proposition 1.55 and together with (2.57) allows us to conclude the proof. The last piece of Theorem 2.12 readily comes from the first part and Lemma 1.62. \square

3. Covering the support of \mathcal{P} -rectifiable measures with intrinsically differentiable graphs

In this section we aim at proving the next theorem, which is one of the main results contained in [28]. Namely, we prove that the support of a \mathcal{P}_h^c -rectifiable measure ϕ , see Definition 1.58 and Definition 1.61, can be covered ϕ -almost all by countably many compact graphs that, in addition of being intrinsically Lipschitz with arbitrarily small Lipschitz constant, are intrinsically differentiable almost everywhere. Namely, the following Theorem 2.25 is a refinement of Theorem 2.2.

Roughly speaking we say that the graph of a function between complementary subgroups $\varphi : U \subseteq \mathbb{V} \rightarrow \mathbb{L}$ is intrinsically differentiable at $a_0 \cdot \varphi(a_0)$ if $\text{graph}(\varphi)$ admits a homogeneous subgroup as Hausdorff tangent at $a_0 \cdot \varphi(a_0)$, see Definition 1.94 for details.

Theorem 2.25. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a \mathcal{P}_h^c -rectifiable, i.e., a \mathcal{P}_h -rectifiable measure with tangents that are complemented almost everywhere.*

Then \mathbb{G} can be covered ϕ -almost everywhere with countably many compact graphs that are simultaneously intrinsically Lipschitz with arbitrarily small constant, and intrinsically differentiable almost everywhere. In other words, for every $\alpha > 0$, we can write

$$\phi\left(\mathbb{G} \setminus \bigcup_{i=1}^{+\infty} \Gamma_i\right) = 0,$$

where $\Gamma_i = \text{graph}(\varphi_i)$ are compact sets, with $\varphi_i : A_i \subseteq \mathbb{V}_i \rightarrow \mathbb{L}_i$ being a function between a compact subset A_i of \mathbb{V}_i , which is a homogeneous subgroup of \mathbb{G} of homogeneous dimension h , and \mathbb{L}_i , which is a subgroup complementary to \mathbb{V}_i ; in addition $\text{graph}(\varphi_i)$ is a $C_{\mathbb{V}_i}^1(\alpha)$ -set, and it is an intrinsically differentiable graph at $a \cdot \varphi_i(a)$ for $\mathcal{S}^{h \perp} A_i$ -almost every $a \in \mathbb{V}_i$, see Definition 1.94.

Let us briefly remark that when a Rademacher-type theorem holds, i.e., if an intrinsically Lipschitz function is intrinsically differentiable almost everywhere, the result in Theorem 2.25 would simply be deduced as an immediate corollary of Theorem 2.2. We remark that a Rademacher-type theorem at such level of generality, i.e., between arbitrary complementary subgroups of a Carnot group, is now known to be false, see the counterexample in [134]. On the other hand, some positive results in particular cases have been provided in [29, 109, 116] for intrinsically Lipschitz functions with one-dimensional target in groups in which De Giorgi $C_{\mathbb{H}}^1$ -rectifiability for finite perimeter sets holds (cf. also [147]), and for functions with normal targets in arbitrary Carnot groups. We stress that very recently in [213] the author proves the Rademacher theorem at any codimension in the Heisenberg groups \mathbb{H}^n .

3.1. Proof. In this subsection we provide the proof of Theorem 2.25. The key step for proving the rectifiability with intrinsically differentiable graphs is the following proposition.

Proposition 2.26 (Hausdorff convergence to tangents). *Let ϕ be a \mathcal{P}_h -rectifiable measure. Let K be a compact set such that $\phi(K) > 0$. Then for ϕ -almost every point $x \in K$ there exists $\mathbb{V}(x) \in Gr(h)$ such that*

$$\delta_{1/r}(x^{-1} \cdot K) \rightarrow \mathbb{V}(x), \quad \text{as } r \text{ goes to } 0,$$

in the sense of Hausdorff convergence on compact sets.

First of all, by reducing the measure ϕ to have compact support, e.g., considering the restriction to the balls with integer radii, and then by using Proposition 1.11, we can assume without loss of generality that $K \subseteq E(\vartheta, \gamma)$ for some $\vartheta, \gamma \in \mathbb{N}$. In order to prove the Hausdorff convergence to the homogeneous subgroup $\mathbb{V}(x)$ we need to prove two different things: first, around almost every point x of K , the points of the set K at decreasingly small scales lies

ever closer to the points of $x\mathbb{V}(x)$, and this is exactly what comes from the implication (2.1), see Proposition 2.5. Secondly, we have to prove the converse assertion with respect to the previous one, i.e., that the points of $x\mathbb{V}(x)$ around x at decreasingly small scales are ever closer to the points of K . For this latter assumption to hold we also need to add to the condition in (2.1) the additional control $F_{x,r}(\phi \llcorner K, \Theta \mathcal{S}^h \llcorner x\mathbb{V}) \leq \delta r^{h+1}$, see Proposition 2.28. As a consequence of Proposition 2.26, we can prove Theorem 2.25 for measures of the form $\mathcal{S}^h \llcorner \Gamma$. Finally by the usual reduction to $E(\vartheta, \gamma)$, we can give the proof of Theorem 2.25 for arbitrary measures.

Let us now start with the proof of Proposition 2.26 and Theorem 2.25. Throughout this subsection we let \mathbb{G} to be a Carnot group of homogeneous dimension Q endowed with an arbitrary left-invariant distance, and h an arbitrary natural number with $1 \leq h \leq Q$. Whenever ϕ is a Radon measure supported on a compact set we freely use the notation $E(\vartheta, \gamma)$ introduced in Definition 1.9, for $\vartheta, \gamma \in \mathbb{N}$. We start with some useful definitions and facts.

Definition 2.27. For $1 \leq h \leq Q$ and $\vartheta \in \mathbb{N}$, let us set

$$\eta(h) := 1/(h+1),$$

and then let us define the constant

$$C_9 = C_9(h, \vartheta) := \left(\frac{\eta(1-\eta)^h}{32\vartheta} \right)^{h+2}.$$

Proposition 2.28. *Let ϕ be a Radon measure supported on a compact subset of \mathbb{G} and let K be a Borel subset of $\text{supp}(\phi)$. Let ϑ, γ and $1 \leq h \leq Q$ be natural numbers. Let $x \in E(\vartheta, \gamma)$, $0 < r < 1/\gamma$, and $0 < \delta < C_9$. Assume further that there exist $\Theta > 0$ and $\mathbb{V} \in \text{Gr}(h)$ such that*

$$(2.58) \quad F_{x,r}(\phi \llcorner K, \Theta \mathcal{C}^h \llcorner x\mathbb{V}) + F_{x,r}(\phi, \Theta \mathcal{C}^h \llcorner x\mathbb{V}) \leq 2\delta r^{h+1}.$$

Then for every $w \in \overline{B}(x, r/2) \cap x\mathbb{V}$ we have $\phi(K \cap \overline{B}(w, \delta^{\frac{1}{h+2}}r)) > 0$, and thus in particular $K \cap \overline{B}(w, \delta^{\frac{1}{h+2}}r) \neq \emptyset$.

Proof. From the hypothesis we have that $F_{x,r}(\phi, \Theta \mathcal{C}^h \llcorner x\mathbb{V}) \leq 2\delta r^{h+1}$. Define

$$g(x) := \min\{\text{dist}(x, \overline{B}(0, 1)^c), \eta\},$$

where η is defined in Definition 2.27. From the very definition of the function g and the choice of Θ above we deduce that

$$\begin{aligned} \vartheta^{-1}(1-\eta)^h \eta r^{h+1} - \Theta \eta r^{h+1} &\leq \eta r \phi(\overline{B}(x, (1-\eta)r)) - \eta r \Theta \mathcal{C}^h \llcorner x\mathbb{V}(\overline{B}(x, r)) \\ &\leq \int r g(\delta_{1/r}(x^{-1}z)) d\phi(z) - \Theta \int r g(\delta_{1/r}(x^{-1}z)) d\mathcal{C}^h \llcorner x\mathbb{V}(z) \leq 2\delta r^{h+1}, \end{aligned}$$

where in the first inequality we are using that $x \in E(\vartheta, \gamma)$ and Remark 1.27, and in the last inequality we are using that $rg(\delta_{1/r}(x^{-1}\cdot)) \in \text{Lip}_1^+(\overline{B}(x, r))$. Simplifying and rearranging the above chain of inequalities, we infer that

$$\Theta \geq \vartheta^{-1}(1-\eta)^h - 2\delta/\eta \underset{(A)}{\geq} (2\vartheta)^{-1}(1-\eta)^h \underset{(B)}{=} (2\vartheta)^{-1}(1-1/(h+1))^h,$$

where (A) comes from the fact that $\delta < C_9 < ((1-\eta)^h \eta)/(4\vartheta)$, see Definition 2.27, and (B) comes from the definition of η , see Definition 2.27. Since the function $h \mapsto (1-1/(h+1))^h$ is decreasing and bounded below by e^{-1} , we deduce, from the previous inequality, that $\Theta \geq 1/(2\vartheta e)$.

We now claim that for every λ with $\delta^{1/(h+2)} \leq \lambda < 1/2$ and every $w \in x\mathbb{V} \cap \overline{B}(x, r/2)$ we have $\phi(\overline{B}(w, \lambda r) \cap K) > 0$. This will finish the proof. By contradiction assume there is $w \in x\mathbb{V} \cap \overline{B}(x, r/2)$ such that $\phi(\overline{B}(w, \lambda r) \cap K) = 0$. This would imply that

$$(2.59) \quad \begin{aligned} \Theta \eta (1 - \eta)^h \lambda^{h+1} r^{h+1} &= \Theta \eta \lambda r \mathcal{C}^h \llcorner x\mathbb{V}(\overline{B}(w, (1 - \eta)\lambda r)) \\ &\leq \Theta \int \lambda r g(\delta_{1/(\lambda r)}(w^{-1}z)) d\mathcal{C}^h \llcorner x\mathbb{V}(z) \\ &= \Theta \int \lambda r g(\delta_{1/(\lambda r)}(w^{-1}z)) d\mathcal{C}^h \llcorner x\mathbb{V}(z) - \int \lambda r g(\delta_{1/(\lambda r)}(w^{-1}z)) d\phi \llcorner K(z) \leq 2\delta r^{h+1}, \end{aligned}$$

where the first equality comes from Remark 1.27, and the last inequality comes from the choice of Θ as in the statement, and the fact that

$$\lambda r g(\delta_{1/(\lambda r)}(w^{-1}\cdot)) \in \text{Lip}_1^+(\overline{B}(w, \lambda r)) \subseteq \text{Lip}_1^+(\overline{B}(x, r)),$$

because $\lambda < 1/2$ and $w \in \overline{B}(x, r/2)$. Thanks to (2.59), the choice of λ , and the fact, proved some line above, that $1/(4e\vartheta) < \Theta$, we have that

$$\frac{\delta^{\frac{h+1}{h+2}}}{4e\vartheta} \eta (1 - \eta)^h < \Theta \lambda^{h+1} \eta (1 - \eta)^h \leq 2\delta, \quad \text{and then} \quad \delta^{1/(h+2)} \geq \frac{\eta (1 - \eta)^h}{8e\vartheta},$$

which is a contradiction since $\delta < C_9 = ((\eta(1 - \eta)^h)/(32\vartheta))^{h+2}$, see Definition 2.27. \square

Proof of Proposition 2.26. First of all, by reducing the measure ϕ to have compact support, e.g., considering the restriction on the balls with integer radii, and then by using Proposition 1.11, we can assume without loss of generality that $K \subseteq E(\vartheta, \gamma)$ for some $\vartheta, \gamma \in \mathbb{N}$.

Since ϕ is a \mathcal{P}_h -rectifiable measure, by using the locality of tangents with the density $\rho \equiv \chi_K$, see Proposition 1.55, for ϕ -almost every $x \in K$ we have that the following three conditions hold

- (i) $\text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda > 0\}$, where $\mathbb{V}(x) \in \text{Gr}(h)$,
- (ii) $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < +\infty$,
- (iii) if $r_i \rightarrow 0$ is such that there exists $\Theta > 0$ with $r_i^{-h} T_{x, r_i} \phi \rightarrow \Theta \mathcal{C}^h \llcorner \mathbb{V}(x)$, then $r_i^{-h} T_{x, r_i}(\phi \llcorner K) \rightarrow \Theta \mathcal{C}^h \llcorner \mathbb{V}(x)$.

From now on let us fix a point $x \in K$ for which the three conditions above hold. If we are able to prove the convergence in the statement for such a point then the proof of the proposition is concluded.

Thus, it suffices to show that for every $k > 0$ the following holds

$$(2.60) \quad \lim_{r \rightarrow 0} d_{H, \mathbb{G}}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, k), \mathbb{V}(x) \cap \overline{B}(0, k)) = 0,$$

where $d_{H, \mathbb{G}}$ is the Hausdorff distance between closed subsets in \mathbb{G} . For some compatibility with the statements that we already proved, we are going to prove (2.60) for $k = 1/4$. The proof of (2.60) for an arbitrary $k > 0$ can be achieved by changing accordingly the constants in the statements of Proposition 2.5 and Proposition 2.28, that we are going to crucially use in this proof. We leave this generalization to the reader, as it will be clear from this proof.

Let us fix $\varepsilon < \min\{\delta_{\mathbb{G}}, C_9\}$, where $\delta_{\mathbb{G}}$ is defined in Definition 2.4 and C_9 in Definition 2.27, and let us show that there exist an $r_0 = r_0(\varepsilon)$ and a real function f_1 such that

$$(2.61) \quad d_{H, \mathbb{G}}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4), \mathbb{V}(x) \cap \overline{B}(0, 1/4)) \leq f_1(\varepsilon), \quad \text{for all } 0 < r < r_0(\varepsilon),$$

where

$$(2.62) \quad f_1(\varepsilon) := \max\{C_5 \varepsilon^{1/(h+1)} + f_2(\varepsilon), 3\varepsilon^{1/(h+2)} + f_3(\varepsilon)\},$$

and where the constant C_5 is defined in Proposition 2.5, and the functions f_2, f_3 will be introduced in (2.68) and (2.70), respectively. By the definition of f_1, f_2, f_3 it follows that $f_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and thus, if we prove (2.61), we are done.

In order to reach the proof of (2.61) let us add an intermediate step. We claim that there exists an $r_0 := r_0(\varepsilon) < 1/\gamma$ such that the following holds

(2.63)

$$\text{for } 0 < r < r_0 \text{ there is a } \Theta := \Theta(r) \text{ s.t. } F_{x,r}(\phi \llcorner K, \Theta \mathcal{C}^h \llcorner x \mathbb{V}) + F_{x,r}(\phi, \Theta \mathcal{C}^h \llcorner x \mathbb{V}) \leq 2\varepsilon r^{h+1}.$$

The conclusion in (2.63) follows if we prove that

$$(2.64) \quad \liminf_{r \rightarrow 0} \inf_{\Theta > 0} \frac{F_{x,r}(\phi \llcorner K, \Theta \mathcal{C}^h \llcorner x \mathbb{V}) + F_{x,r}(\phi, \Theta \mathcal{C}^h \llcorner x \mathbb{V})}{r^{h+1}} \rightarrow 0.$$

We prove (2.64) by contradiction. If (2.64) was not true, there would exist an $\tilde{\varepsilon}$ and an infinitesimal sequence $\{r_i\}_{i \in \mathbb{N}}$ such that

$$(2.65) \quad \inf_{\Theta > 0} \left(F_{x,r_i}(\phi \llcorner K, \Theta \mathcal{C}^h \llcorner x \mathbb{V}) + F_{x,r_i}(\phi, \Theta \mathcal{C}^h \llcorner x \mathbb{V}) \right) > \tilde{\varepsilon} r_i^{h+1}, \quad \text{for every } i \in \mathbb{N}.$$

Thus, from items (i) and (ii) above, and from [12, Corollary 1.60], we conclude that, up to a non re-labelled subsequence of r_i , there exists a $\Theta^* > 0$ such that we have $r_i^{-h} T_{x,r_i} \phi \rightarrow \Theta^* \mathcal{C}^h \llcorner \mathbb{V}(x)$ as $r_i \rightarrow 0$. Then by exploiting the item (iii) above we get also that

$$r_i^{-h} T_{x,r_i}(\phi \llcorner K) \rightarrow \Theta^* \mathcal{C}^h \llcorner \mathbb{V}(x),$$

as $r_i \rightarrow 0$. These two conclusions immediately imply, by exploiting Remark 1.66 and Lemma 1.73, that

$$\lim_{i \rightarrow +\infty} r_i^{-(h+1)} \left(F_{x,r_i}(\phi \llcorner K, \Theta^* \mathcal{C}^h \llcorner x \mathbb{V}) + F_{x,r_i}(\phi, \Theta^* \mathcal{C}^h \llcorner x \mathbb{V}) \right) \rightarrow 0,$$

which is a contradiction with (2.65). Thus, the conclusion in (2.63) holds. Let us continue the proof of (2.61).

Taking into account the bound on ε and (2.63) we can apply Proposition 2.5, since $\mathbb{V}(x) \in \Pi_\varepsilon(x, r)$ for all $0 < r < r_0$, and Proposition 2.28 to obtain, respectively, that for all $0 < r < r_0$

$$(2.66) \quad \sup_{p \in K \cap \overline{B}(x, r/4)} \text{dist}(p, x \mathbb{V}(x)) \leq \sup_{p \in E(\vartheta, \gamma) \cap \overline{B}(x, r/4)} \text{dist}(p, x \mathbb{V}(x)) \leq C_5 r \varepsilon^{1/(h+1)},$$

$$\text{and for every } p \in \overline{B}(x, r/2) \cap x \mathbb{V}(x) \text{ we have } \overline{B}(p, \varepsilon^{1/(h+2)} r) \cap K \neq \emptyset.$$

Let us proceed with the proof of (2.61). Fix $0 < r < r_0$ and note that for every $w \in \delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4)$ there exists a point $p \in K \cap \overline{B}(x, r/4)$ such that $w =: \delta_{1/r}(x^{-1} \cdot p)$. From the first line of (2.66) we get that $\text{dist}(x^{-1} \cdot p, \mathbb{V}(x)) \leq C_5 r \varepsilon^{1/(h+1)}$ and thus there exists a $v \in \mathbb{V}(x)$ such that $d(x^{-1} \cdot p, v) \leq C_5 r \varepsilon^{1/(h+1)}$. This in particular means that $d(w, \delta_{1/r} v) \leq C_5 \varepsilon^{1/(h+1)}$ and then, since $w \in \overline{B}(0, 1/4)$, we get also that $\delta_{1/r} v \in \mathbb{V}(x) \cap \overline{B}(0, 1/4 + C_5 \varepsilon^{1/(h+1)})$. Thus, we conclude that

$$(2.67) \quad \text{dist}(w, \mathbb{V}(x) \cap \overline{B}(0, 1/4 + C_5 \varepsilon^{1/(h+1)})) \leq C_5 \varepsilon^{1/(h+1)},$$

for all $w \in \delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4)$. Define the following function

$$(2.68) \quad f_2(\varepsilon) := \sup_{u \in \mathbb{V}(x) \cap (\overline{B}(0, 1/4 + C_5 \varepsilon^{1/(h+1)}) \setminus \overline{B}(0, 1/4))} d(u, \delta_{4^{-1}\|u\|}^{-1} u),$$

and notice that by compactness it is easy to see that $f_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. With the previous definition of f_2 in hands, we can exploit (2.67) and conclude that

$$(2.69) \quad \sup_{w \in \delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4)} \text{dist}(w, \mathbb{V}(x) \cap \overline{B}(0, 1/4)) \leq C_5 \varepsilon^{1/(h+1)} + f_2(\varepsilon).$$

The latter estimate is the first piece of information we need to prove (2.61). Let us now estimate $\text{dist}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4), v)$ for every $v \in \mathbb{V}(x) \cap \overline{B}(0, 1/4)$. If $u \in \mathbb{V}(x) \cap (\overline{B}(0, 1/4) \setminus \overline{B}(0, 1/4 - \varepsilon^{1/(h+2)}))$, then there exists a unique $\mu = \mu(u) > 0$ such that $\delta_{\mu(u)}u \in \mathbb{V}(x) \cap \partial \overline{B}(0, 1/4 - \varepsilon^{1/(h+2)})$. Let us define

$$(2.70) \quad f_3(\varepsilon) := \sup_{u \in \mathbb{V}(x) \cap (\overline{B}(0, 1/4) \setminus \overline{B}(0, 1/4 - \varepsilon^{1/(h+2)}))} d(u, \delta_{\mu(u)}u),$$

and by compactness it is easy to see that $f_3(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let us now fix $v \in \mathbb{V}(x) \cap \overline{B}(0, 1/4)$. Then $x \cdot \delta_r v \in \overline{B}(x, r/4) \cap x\mathbb{V}(x) \subseteq \overline{B}(x, r/2) \cap x\mathbb{V}(x)$. We can use the second line of (2.66) to conclude that there exists $w \in \overline{B}(x \cdot \delta_r v, \varepsilon^{1/(h+2)}r) \cap K$. Thus $\tilde{w} := \delta_{1/r}(x^{-1} \cdot w) \in \overline{B}(v, \varepsilon^{1/(h+2)}) \cap \delta_{1/r}(x^{-1} \cdot K)$. Now we have two cases

- if v was in $\overline{B}(0, 1/4 - \varepsilon^{1/(h+2)})$ we would get $\tilde{w} \in \overline{B}(0, 1/4)$ and then

$$(2.71) \quad \text{dist}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4), v) \leq \varepsilon^{1/(h+2)};$$

- if instead $v \in \mathbb{V}(x) \cap (\overline{B}(0, 1/4) \setminus \overline{B}(0, 1/4 - \varepsilon^{1/(h+2)}))$, we denote $v' := \delta_{\mu(v)}v$ the point that we have defined above and then we still have $x \cdot \delta_r v' \in \overline{B}(x, r/2) \cap x\mathbb{V}(x)$. Thus we can again apply the second line of (2.66) to deduce the existence of $w' \in \overline{B}(x \cdot \delta_r v', \varepsilon^{1/(h+2)}r) \cap K$. Then we conclude $\tilde{w}' := \delta_{1/r}(x^{-1} \cdot w') \in \overline{B}(v', \varepsilon^{1/(h+2)}) \cap \delta_{1/r}(x^{-1} \cdot K)$. Now we can estimate

$$(2.72) \quad \begin{aligned} d(\tilde{w}, \tilde{w}') &= \frac{1}{r} d(w, w') \leq \frac{1}{r} (d(w, x \cdot \delta_r v) + d(x \cdot \delta_r v, x \cdot \delta_r v') + d(x \cdot \delta_r v', w')) \\ &\leq 2\varepsilon^{1/(h+2)} + f_3(\varepsilon). \end{aligned}$$

Moreover, since $v' \in \partial \overline{B}(0, 1/4 - \varepsilon^{1/(h+2)})$ and $\tilde{w}' \in \overline{B}(v', \varepsilon^{1/(h+2)})$ we get that $\tilde{w}' \in \overline{B}(0, 1/4) \cap \delta_{1/r}(x^{-1} \cdot K)$. Then by the triangle inequality and (2.72) we conclude that, in this second case,

$$(2.73) \quad d(\tilde{w}', v) \leq 3\varepsilon^{1/(h+2)} + f_3(\varepsilon),$$

and then

$$(2.74) \quad \text{dist}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4), v) \leq 3\varepsilon^{1/(h+2)} + f_3(\varepsilon).$$

By joining together the conclusion of the two cases, see (2.71) and (2.74), we conclude that

$$(2.75) \quad \sup_{v \in \mathbb{V}(x) \cap \overline{B}(0, 1/4)} \text{dist}(\delta_{1/r}(x^{-1} \cdot K) \cap \overline{B}(0, 1/4), v) \leq 3\varepsilon^{1/(h+2)} + f_3(\varepsilon).$$

The equations (2.69) and (2.75) imply (2.61) by the very definition of Hausdorff distance. Thus the proof is concluded. \square

We prove now that the support of a \mathcal{P}_h^c -rectifiable measure $\mathcal{S}^h \llcorner \Gamma$, where Γ is compact and $0 < \mathcal{S}^h(\Gamma) < +\infty$, can be written as the countable union of almost everywhere intrinsically differentiable graphs.

Theorem 2.29. *For every $1 \leq h \leq Q$, there exist a countable subfamily $\mathcal{F} := \{\mathbb{V}_k\}_{k=1}^{+\infty}$ of $Gr_c(h)$, and \mathbb{L}_k complementary subgroups of \mathbb{V}_k such that the following holds.*

Let Γ be a compact subset of \mathbb{G} such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, and assume $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h^c -rectifiable measure. Then, for every $\alpha > 0$, there are countably many compact Γ_i 's that are intrinsic graphs of functions $\varphi_i : P_{\mathbb{V}_i}(\Gamma_i) \rightarrow \mathbb{L}_i$, and that satisfy the following three conditions: Γ_i are $C_{\mathbb{V}_i}(\alpha)$ -sets, Γ_i are intrinsically differentiable graphs at $a \cdot \varphi_i(a)$ for $\mathcal{S}^h \llcorner P_{\mathbb{V}_i}(\Gamma_i)$ -almost every $a \in P_{\mathbb{V}_i}(\Gamma_i)$, and

$$\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Proof. First of all let

$$\mathbf{F}(\mathbb{V}, \mathbb{L}) := \varepsilon_1(\mathbb{V}, \mathbb{L}), \quad \text{for all } (\mathbb{V}, \mathbb{L}) \in \text{Sub}(h),$$

where $\text{Sub}(h)$ is defined in (2.12). Given the above defined function \mathbf{F} , we construct the family $\mathcal{F} := \{\mathbb{V}_k\}_{k=1}^{+\infty}$ and choose \mathbb{L}_k complementary subgroups of \mathbb{V}_k as discussed above the statement of Theorem 2.10. Notice that this choice is dependent on the function \mathbf{F} that we chose above. We claim that the family for which the statement holds is \mathcal{F} .

Applying Theorem 2.10 with $\beta \equiv \min\{1, \alpha\}$ to the measure $\mathcal{S}^h \llcorner \Gamma$ we get countably many compact sets $\Gamma_i \subseteq \Gamma$ that are $C_{\mathbb{V}_i}(\min\{\mathbf{F}(\mathbb{V}_i, \mathbb{L}_i), \alpha, 1\})$ -sets and such that

$$\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Since $\mathbf{F}(\mathbb{V}_i, \mathbb{L}_i) = \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i)$, we conclude that each Γ_i is also the intrinsic graph of a function $\varphi_i : P_{\mathbb{V}_i}(\Gamma_i) \rightarrow \mathbb{L}_i$, see Proposition 1.89. The fact that Γ_i is a $C_{\mathbb{V}_i}(\alpha)$ is true by how we chose β . It is left to show that, for every $i \in \mathbb{N}$, $\text{graph}(\varphi_i)$ is an intrinsically differentiable graph at $a \cdot \varphi_i(a)$ for $\mathcal{S}^h \llcorner P_{\mathbb{V}_i}(\Gamma_i)$ -almost every $a \in P_{\mathbb{V}_i}(\Gamma_i)$.

Indeed, since $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable, we can apply Proposition 2.26 and, for every $i \in \mathbb{N}$ for which $\mathcal{S}^h(\Gamma_i) > 0$, we conclude that

(2.76)

$$\delta_{1/r}(x^{-1} \cdot \Gamma_i) \rightarrow \mathbb{V}(x), \quad \text{as } r \text{ goes to } 0, \text{ for } \mathcal{S}^h \llcorner \Gamma_i\text{-almost every } x \in \mathbb{G}, \text{ where } \mathbb{V}(x) \in Gr(h),$$

in the sense of Hausdorff convergence on compact sets. Moreover, thanks to Proposition 2.18 and to Lebesgue Differentiation Theorem in Proposition 1.55, we infer that $(\Phi_i)_* \mathcal{S}^h \llcorner \mathbb{V}_i$ is mutually absolutely continuous with respect to $\mathcal{S}^h \llcorner \Gamma_i$, where Φ_i is the graph map of φ_i . Furthermore, since every point $x \in \Gamma_i$ can be written as $x = a \cdot \varphi_i(a)$, with $a \in P_{\mathbb{V}_i}(\Gamma_i)$, we conclude, from (2.76) and the latter absolute continuity, that $\Gamma_i = \text{graph}(\varphi_i)$ is an intrinsically differentiable graph at $a \cdot \varphi_i(a)$ for $\mathcal{S}^h \llcorner P_{\mathbb{V}_i}(\Gamma_i)$ -almost every $a \in P_{\mathbb{V}_i}(\Gamma_i)$, and this concludes the proof. \square

Proof of Theorem 2.25. By restricting on closed balls of integer radii we can assume without loss of generality that ϕ has compact support. Let us fix $\vartheta, \gamma \in \mathbb{N}$. We can infer this corollary by working on $\phi \llcorner E(\vartheta, \gamma)$, that is mutually absolutely continuous with respect to $\mathcal{S}^h \llcorner E(\vartheta, \gamma)$, see Proposition 1.56, and by using the previous Theorem 2.29 together with Proposition 1.11. The resulting strategy is identical to the one in the proof of Theorem 2.12 so we omit the details. \square

4. Equivalent criteria for \mathcal{P} -rectifiable sets with complemented tangents

In Theorem 2.25 we showed that every \mathcal{P}_h^c -rectifiable measure can be covered almost everywhere with compact graphs that are intrinsically differentiable almost everywhere. In this section we prove that for measure of the form $\mathcal{S}^h \llcorner \Gamma$ the latter is actually an equivalence.

We first recall that, while $\text{Tan}_h(\phi, x)$ captures the behaviour of tangent measures obtained rescaling with the h -th power of the scale, see Definition 1.52, the Preiss's tangent $\text{Tan}(\phi, x)$, see Definition 1.52, captures the behaviour of all the possible tangent measures, namely

$$\text{Tan}(\phi, x) := \{\nu : \exists\{c_i\}, \text{ with } c_i > 0, \text{ and } \{r_i\} \text{ with } r_i \rightarrow_i 0 \text{ such that } c_i T_{x, r_i} \phi \rightarrow_i \nu\},$$

where the convergence of measures is meant in the duality with $C_c(\mathbb{G})$, see Definition 1.3. For the reader's convenience we recall here that an intrinsic graph with respect to a splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$ of the group is said to be *intrinsically differentiable* at one of its points if the Hausdorff tangent at that point is a homogeneous subgroup, see Definition 1.94 for a precise definition.

For a generalization of the forthcoming Theorem 2.30 in the broader setting of Radon measures with finite h -upper density almost everywhere, see Remark 2.39.

Theorem 2.30. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance d . Let $h \in \{1, \dots, Q\}$, and Let $\Gamma \subseteq \mathbb{G}$ be a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, where \mathcal{S}^h is the h -dimensional spherical Hausdorff measure. Then the following are equivalent*

- (1) $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h -rectifiable measure with complemented tangents, or, in other words, a \mathcal{P}_h^c -rectifiable measure, see Definition 1.61,
- (2) For $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$ we have

$$\text{Tan}(\mathcal{S}^h \llcorner \Gamma, x) = \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) :$$

$$\lambda > 0, \mathbb{V}(x) \text{ is a complemented hom. subgroup of } \mathbb{G}, \dim_{\text{hom}} \mathbb{V}(x) = h\},$$

- (3) There are countably many compact intrinsic graphs Γ_i that are h -dimensional intrinsically differentiable at \mathcal{S}^h -almost every $x \in \Gamma_i$, that have complemented Hausdorff tangents at \mathcal{S}^h -almost every $x \in \Gamma_i$, and such that

$$\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Moreover, denoting with \mathcal{C}^h the centered Hausdorff measure of dimension h , see Definition 1.1, if any of the previous holds, then $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ exists for $\mathcal{C}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$ and

$$r^{-h} (T_{x, r})_* (\mathcal{C}^h \llcorner \Gamma) \rightarrow \mathcal{C}^h \llcorner \mathbb{V}(x), \quad \text{as } r \rightarrow 0, \text{ for } \mathcal{C}^h \llcorner \Gamma\text{-almost every } x \in \mathbb{G},$$

where the convergence of measures is meant in the duality with $C_c(\mathbb{G})$.

Finally, if any of the previous items holds, for every α there are countably many compact intrinsic graphs Γ_i that are h -dimensional intrinsically differentiable at \mathcal{S}^h -almost every $x \in \Gamma_i$, that have complemented Hausdorff tangents at \mathcal{S}^h -almost every $x \in \Gamma_i$, that are $C_{\mathbb{V}_i}(\alpha)$ -sets for some homogeneous complemented subgroup \mathbb{V}_i of homogeneous dimension h , and such that

$$\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Let us observe that when a Rademacher theorem is available, we can equivalently consider as the building blocks in item (3) of Theorem 2.30 the class of intrinsically Lipschitz graphs over subsets of homogeneous subgroups of homogeneous dimension h , without asking anything a priori on the differentiability, compare e.g., with Corollary 3.3. Let us recall that a Rademacher theorem is proved in [109, 116] in the setting of Carnot groups \mathbb{G} of type \star , i.e., a class strictly larger than Carnot groups of step 2, and for maps $\varphi : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$, where \mathbb{W} and \mathbb{L} are complementary subgroups of \mathbb{G} , with \mathbb{L} horizontal and one-dimensional. Moreover, with the recent results of [147], the latter codimension-one Rademacher theorem can be extended to the groups of type diamond introduced in [147]. Recently, by making use of the theory of currents, the author of [213] has proved the Rademacher theorem for intrinsically

Lipschitz maps between complementary subgroups of any dimension in the Heisenberg groups \mathbb{H}^n , while in [29] we proved the validity of a Rademacher theorem for co-normal intrinsically Lipschitz graphs.

Nevertheless, Rademacher theorem is now known to be false in arbitrary Carnot groups in a very strict sense, i.e., there exists an intrinsically Lipschitz graph in a Carnot group such that at every point of it there exist infinitely many blow-ups and each of these blow-ups is not a homogeneous subgroup, see [134, Theorem 1.1]. This latter result implies that in general in item (3) of Theorem 2.30 one cannot equivalently consider as building blocks of a locally well-behaved definition of rectifiable sets the family of intrinsically Lipschitz graphs. So, in some sense, the result of Theorem 2.30 is sharp also in view of the negative result of [134].

Let us further notice that we do not consider in this work the relations between the three items in Theorem 2.30 and the existence of an approximate tangent in the sense of [174, Definition 15.17] (cf. [178, Definition 3.7]), see also the discussion in Remark 2.40. All in all, taking into account that $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable with co-horizontal tangents if and only if Γ is $C_{\mathbb{H}}^1(\mathbb{G}, \mathbb{R}^{Q-h})$ -rectifiable, see Definition 1.105 and Proposition 1.109, our result in Theorem 2.30 extends and strengthens [178, (i) \Leftrightarrow (ii) \Leftrightarrow (iv) \Leftrightarrow (v) of Theorem 3.15]. Notice also that in the previous chain of equivalences, we can also drop the assumption on the lower density in [178, (iv),(v)]. Moreover, taking into account the Rademacher theorem of [29] in the co-normal case, our result in Theorem 2.30 extends [178, (i) \Leftrightarrow (ii) \Leftrightarrow (iv) \Leftrightarrow (v) of Theorem 3.14] as well. We refer the reader also to the statement of Corollary 3.3. Let us recall, for the reader's convenience, that [178, Theorem 3.15] deals with the characterization of co-horizontal rectifiability in the Heisenberg groups \mathbb{H}^n , while [178, Theorem 3.14] deals with the characterization of horizontal rectifiability in the Heisenberg groups \mathbb{H}^n .

Let us briefly comment on the proof of Theorem 2.30. For what concerns the implications (1) \Rightarrow (2), and (1) \Rightarrow (3), the first is just a matter of routine argument, see [174, Remark 14.4(3)], and the second is a consequence of Theorem 2.25. What one needs to show are the implications (2) \Rightarrow (1), and (3) \Rightarrow (1), both of them non-trivial.

For what concerns the implication (2) \Rightarrow (1), we first use that the hypothesis of flat Preiss's tangents allows to conclude that Γ is \mathcal{S}^h -almost everywhere covered by countably many graphs Γ_i of intrinsically Lipschitz functions defined on subsets of homogeneous subgroups of homogeneous dimension h , namely $\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0$, see Proposition 2.32. Hence we exploit the general fact, that dates back to Preiss's paper (cf. [202, Corollary 2.7]), that a measure with a compact-based tangent at a point is asymptotically doubling at that point. Joining the latter two observations, we deduce that, for every i , the measure $\mathcal{S}^h \llcorner \Gamma_i$ is asymptotically doubling, and then this enables us to prove that Γ_i has big projections on the plane over which Γ_i is a graph, see Proposition 2.17. Finally, the big projections property of Proposition 2.17 allows us to conclude that the h -lower density $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma_i, \cdot)$ is positive $\mathcal{S}^h \llcorner \Gamma_i$ almost everywhere, see Proposition 2.33. Hence, the proof of the implication (2) \Rightarrow (1) is concluded since we can argue, by exploiting Lebesgue Differentiation Theorem, that $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, \cdot)$ is positive $\mathcal{S}^h \llcorner \Gamma$ -almost everywhere, which was the non-trivial missing information to prove (1). Let us stress that in (2) we are not requiring anything a priori on the positivity of the h -lower density of $\mathcal{S}^h \llcorner \Gamma$, otherwise the implication (2) \Rightarrow (1) would have been trivial. Nevertheless, we deduce the positivity of the h -lower density from the fact that the tangents are flat and complemented as we discussed above.

The proof of the implication (3) \Rightarrow (1) relies on the fact that an arbitrary h -dimensional (almost everywhere) intrinsically differentiable graph Γ with complemented Hausdorff tangents has the property that $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable. This is exactly the content of Proposition 2.37. In order to prove the latter, we show that when we have an arbitrary h -dimensional

(almost everywhere) intrinsically differentiable graph Γ with complemented Hausdorff tangents, at ($\mathcal{S}^h_\perp\Gamma$ -almost) every point we have that the graph Γ is, at arbitrarily small scales, contained in a cone with arbitrarily small opening and with basis the Hausdorff tangent at that point. This observation enables us to perform a covering argument and to show directly that $\Theta^h(\mathcal{C}^h_\perp\Gamma, \cdot) = 1$ at $\mathcal{C}^h_\perp\Gamma$ -almost every point. Then the fact that $\mathcal{C}^h_\perp\Gamma$, and hence $\mathcal{S}^h_\perp\Gamma$, is \mathcal{P}^c_h -rectifiable is reached by using a classical argument from the existence of density. Let us notice that in Proposition 2.37 it is essential to work with the centered Hausdorff measure $\mathcal{C}^h_\perp\Gamma$, since we consider coverings with balls centered on Γ . The second part of the statement in Theorem 2.30 is a consequence of the fact that the h -density of $\mathcal{C}^h_\perp\Gamma$ is 1 as a consequence of the previous reasoning, and the fact that $\mathcal{C}^h_\perp\mathbb{V}(B(0,1)) = 1$ for every homogeneous subgroup \mathbb{V} of homogeneous dimension h , see Remark 1.27. The last part of the statement in Theorem 2.30 is a direct consequence of Theorem 2.25.

4.1. Proof. Let us start with the proof of Theorem 2.30. In this subsection, we let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance d . Let $h \in \{1, \dots, Q\}$. We first prove that if Preiss's tangent $\text{Tan}(\phi, x)$ of a measure ϕ at x is the cone over a homogeneous subgroup, then the support of the measure can be covered almost everywhere with sets with the cone property with arbitrarily small opening.

Proposition 2.31. *Suppose ϕ is a Radon measure on \mathbb{G} such that, for ϕ -almost every $x \in \mathbb{G}$, we have $\text{Tan}(\phi, x) = \{\lambda\mathcal{S}^h_\perp\mathbb{V}(x) : \lambda > 0\}$ for some $\mathbb{V}(x) \in \text{Gr}(h)$. Then, for every $\alpha \in (0, 1)$ there exist $\{\mathbb{V}_i\}_{i \in \mathbb{N}} \subseteq \text{Gr}(h)$, and a family of compact $C_{\mathbb{V}_i}(\alpha)$ -sets $\{\Gamma_i\}_{i \in \mathbb{N}}$ such that*

$$\phi(\mathbb{G} \setminus \cup_{i \in \mathbb{N}} \Gamma_i) = 0.$$

Proof. First of all, by Proposition 1.80, the measure ϕ is asymptotically doubling. Up to restricting ϕ to closed balls and by using the locality of tangents and Lebesgue Theorem in Proposition 1.55, we may assume that ϕ is supported on a compact set K and that it is still asymptotically doubling. Let S be dense countable subset of $(\text{Gr}(h), d_{\mathbb{G}})$, that exists thanks to Proposition 1.22. Thanks to Proposition 1.30, we infer that also the countable set $\{(h+1)\mathcal{C}^h_\perp\mathbb{W} : \mathbb{W} \in S\}$ is dense in the metric space $(\{(h+1)\mathcal{C}^h_\perp\mathbb{V} : \mathbb{V} \in \text{Gr}(h)\}, F_{0,1})$.

Let us now fix $\epsilon < 1/10$, $\sigma < 1/100(\epsilon/(3(1+\epsilon)))^h$, $\mathbb{V} \in S$ and let us denote

$$K_{\mathbb{V}} := \{x \in K : F_{0,1}((h+1)\mathcal{C}^h_\perp\mathbb{V}, (h+1)\mathcal{C}^h_\perp\mathbb{V}(x)) < \sigma^{h+4}\},$$

where $\mathbb{V}(x) \in \text{Gr}(h)$ is such that $\text{Tan}(\phi, x) = \{\lambda\mathcal{S}^h_\perp\mathbb{V}(x), \lambda > 0\}$. Since $\{(h+1)\mathcal{C}^h_\perp\mathbb{W} : \mathbb{W} \in S\}$ is dense in the metric space $(\{(h+1)\mathcal{C}^h_\perp\mathbb{V} : \mathbb{V} \in \text{Gr}(h)\}, F_{0,1})$ we conclude that $K = \cup_{\mathbb{V} \in S} K_{\mathbb{V}}$. By Lemma 1.83, one gets that $K_{\mathbb{V}}$ is ϕ -measurable for every $\mathbb{V} \in S$. Thus by Proposition 1.55, we can assume without loss of generality that ϕ is asymptotically doubling and supported on $K_{\mathbb{V}}$ for some $\mathbb{V} \in S$, which from now on we fix.

We now claim that for ϕ -almost every $x \in \mathbb{G}$ the following holds

$$(2.77) \quad \lim_{r \rightarrow 0^+} \bar{d}_{x,r}(\phi, \mathfrak{M}(h, \{\mathbb{V}\})) = F_{0,1}((h+1)\mathcal{C}^h_\perp\mathbb{V}, (h+1)\mathcal{C}^h_\perp\mathbb{V}(x)).$$

Indeed, for ϕ -almost every $x \in \mathbb{G}$ the measure $T_{x,r}\phi/F_1(T_{x,r}\phi)$ converges to $(h+1)\mathcal{C}^h_\perp\mathbb{V}(x)$ as $r \rightarrow 0^+$, see Proposition 1.81, and thus, from the definition of $\bar{d}_{x,r}$, we get that

$$(2.78) \quad \bar{d}_{x,r}(\phi, \mathfrak{M}(h, \{\mathbb{V}\})) = F_{0,1}(T_{x,r}\phi/F_1(T_{x,r}\phi), (h+1)\mathcal{C}^h_\perp\mathbb{V}),$$

from which we deduce the claim (2.77) by using the previous convergence and the continuity of $F_{0,1}$, see Lemma 1.73. Moreover, the function $x \rightarrow \bar{d}_{x,r}(\phi, \mathfrak{M}(h, \{\mathbb{V}\}))$ is continuous in x for every $r > 0$. Indeed, by (2.78) and the continuity of $F_{0,1}$, it is sufficient to see that, for every $r > 0$, the map $x \rightarrow T_{x,r}\phi/F_1(T_{x,r}\phi)$ is continuous from \mathbb{G} to the space of Radon measures equipped with the weak* convergence, which is clear again by the continuity of $F_1(\cdot)$ and by

the continuity of the map $x \rightarrow T_{x,r}\phi$, which is readily verified (see, e.g., the computations at the end of [180, page 22]).

Hence, by using Severini-Egoroff Theorem, we can assume without loss of generality that ϕ is supported on a compact set E such that $\text{diam}(E) < s$ and such that $\bar{d}_{x,r}(\phi, \mathfrak{M}(h, \{\mathbb{V}\})) < \sigma^{h+4}$ whenever $x \in E$ and $r \in (0, 400(h+1)s)$. Let us now fix $\tilde{x}, \tilde{y} \in E$ and denote $a := d(\tilde{x}, \tilde{y})$, $\tilde{t} := 2a(1 + \epsilon)$, $\tilde{r} := a(1 + \epsilon)$ and $\tilde{s} := a\epsilon$.

Let us apply Proposition 1.82 first with the choices $x = y = z = \tilde{y}$, $s = r = \tilde{r}$, $t = \tilde{t}$ and σ as above, that yields

$$(2.79) \quad \phi(\overline{B}(\tilde{y}, \tilde{r}) \cap \overline{B}(\tilde{y}\mathbb{V}, \sigma^2\tilde{t}/(h+1))) \geq (1 - 5\sigma)\phi(\overline{B}(\tilde{y}, \tilde{r})),$$

and secondly with $x = y = z = \tilde{x}$, $r = \tilde{r} + a$, $s = \tilde{s}$, $t = 3a(1 + \epsilon)$ and σ , we get

$$(2.80) \quad \phi(\overline{B}(\tilde{x}, \tilde{s}) \cap \overline{B}(\tilde{x}\mathbb{V}, \sigma^2 \cdot 3a(1 + \epsilon)/(h+1))) \geq (1 - 5\sigma)(\tilde{s}/(\tilde{r} + a))^h \phi(\overline{B}(\tilde{x}, \tilde{r} + a)).$$

Putting together (2.79) and (2.80), we conclude that

$$(2.81) \quad \begin{aligned} & \phi(\overline{B}(\tilde{y}, \tilde{r}) \setminus \overline{B}(\tilde{y}\mathbb{V}, 2a\sigma^2(1 + \epsilon)/(h+1))) = \phi(\overline{B}(\tilde{y}, \tilde{r})) \\ & \quad - \phi(\overline{B}(\tilde{y}, \tilde{r}) \cap \overline{B}(\tilde{y}\mathbb{V}, 2a\sigma^2(1 + \epsilon)/(h+1))) \\ & \leq 5\sigma\phi(\overline{B}(\tilde{y}, \tilde{r})) \leq 5\sigma\phi(\overline{B}(\tilde{x}, \tilde{r} + a)) \leq \frac{5\sigma}{1 - 5\sigma} \left(\frac{2(1 + \epsilon)}{\epsilon} \right)^h \phi(\overline{B}(\tilde{x}, \tilde{s})) \end{aligned}$$

If by contradiction $\overline{B}(\tilde{x}, \epsilon a) \cap \overline{B}(\tilde{y}\mathbb{V}, 2a\sigma^2(1 + \epsilon)/(h+1)) = \emptyset$ then from (2.81) and the fact that $\overline{B}(\tilde{x}, \tilde{s}) \subseteq \overline{B}(\tilde{y}, \tilde{r})$, we infer

$$\phi(\overline{B}(\tilde{x}, \tilde{s})) \leq \frac{5\sigma}{1 - 5\sigma} \left(\frac{2(1 + \epsilon)}{\epsilon} \right)^h \phi(\overline{B}(\tilde{x}, \tilde{s})),$$

that is in contradiction thanks with the choice of σ . Hence, for every $\tilde{x}, \tilde{y} \in E$ we have that $\overline{B}(\tilde{x}, \epsilon a) \cap \overline{B}(\tilde{y}\mathbb{V}, 2a\sigma^2(1 + \epsilon)/(h+1)) \neq \emptyset$ and thus $d(\tilde{x}, \tilde{y}\mathbb{V}) \leq a(\epsilon + 2\sigma^2(1 + \epsilon)/(h+1)) = d(\tilde{x}, \tilde{y})(\epsilon + 2\sigma^2(1 + \epsilon)/(h+1))$. Hence, the compact set E is a $C_{\mathbb{V}}(\epsilon + 2\sigma^2(1 + \epsilon)/(h+1))$ -set. Since it is clear that, for every given $\alpha > 0$, σ and ϵ can be chosen small enough in order to have $\epsilon + 2\sigma^2(1 + \epsilon)/(h+1) < \alpha$, the proof is thus concluded. \square

In the case the tangents are complemented we can give the following improvement of the latter Proposition.

Proposition 2.32. *Let $1 \leq h \leq Q$ be a natural number. There exist $\{\mathbb{V}_i\}_{i \in \mathbb{N}} \subseteq Gr_c(h)$ and \mathbb{L}_i complementary subgroups of \mathbb{V}_i such that the following holds.*

Suppose ϕ is a Radon measure on \mathbb{G} such that, for ϕ -almost every $x \in \mathbb{G}$, we have $\text{Tan}(\phi, x) = \{\lambda \mathcal{S}^h_{\mathbb{L}}\mathbb{V}(x) : \lambda > 0\}$ for some $\mathbb{V}(x) \in Gr_c(h)$. Then, for every $\alpha \in (0, 1)$ there exists a family of compact sets $\{\Gamma_i\}_{i \in \mathbb{N}}$ such that

$$\phi(\mathbb{G} \setminus \cup_{i \in \mathbb{N}} \Gamma_i) = 0,$$

and, for every $i \in \mathbb{N}$, Γ_i is a compact intrinsically Lipschitz graph, which is also a $C_{\mathbb{V}_i}(\alpha)$ -set, of a map $\varphi_i : A_i \subseteq \mathbb{V}_i \rightarrow \mathbb{L}_i$, where A_i is compact.

Proof. The proof follows exactly the same lines as the proof of Proposition 2.31, so we just sketch it underlying the main differences. For every $\ell \in \mathbb{N}$, with $\ell \geq 2$, let us define

$$Gr_c(h, \ell) := \{\mathbb{V} \in Gr_c(h) : \exists \mathbb{L} \text{ compl. subgroup of } \mathbb{V} \text{ s.t. } 1/\ell < \varepsilon_1(\mathbb{V}, \mathbb{L}) \leq 1/(\ell - 1)\}.$$

Observe that Proposition 1.22 implies that $Gr_c(h, \ell)$ is separable for every $\ell \in \mathbb{N}$, since $Gr_c(h, \ell) \subseteq Gr(h)$ and $(Gr(h), d_{\mathbb{G}})$ is a compact metric space. Let

$$(2.82) \quad \mathcal{D}_\ell := \{\mathbb{V}_{i,\ell}\}_{i \in \mathbb{N}},$$

be a countable dense subset of $Gr_c(h, \ell)$ and

for all $i \in \mathbb{N}$, choose a compl. subgroup $\mathbb{L}_{i,\ell}$ of $\mathbb{V}_{i,\ell}$ s.t. $1/\ell < \varepsilon_1(\mathbb{V}_{i,\ell}, \mathbb{L}_{i,\ell}) \leq 1/(\ell - 1)$.

Now, let $S := \{\mathbb{V}_{i,\ell}\}_{i,\ell \in \mathbb{N}}$, which is a dense countable subset of $(Gr_c(h), d_{\mathbb{G}})$ thanks to the definition given above. As in the above Proposition 2.31, we infer that also the countable set $\{(h+1)\mathcal{C}^h \llcorner \mathbb{W} : \mathbb{W} \in S\}$ is dense in the metric space $(\{(h+1)\mathcal{C}^h \llcorner \mathbb{V} : \mathbb{V} \in Gr_c(h)\}, F_{0,1})$. Let us now fix, for every $\ell \in \mathbb{N}$, $\varepsilon_\ell < \min\{1/10, 1/(2\ell), \alpha/2\}$, where α is as in the statement, and $\sigma_\ell < \min\{1/100(\varepsilon_\ell/(3(1+\varepsilon_\ell)))^h, \sigma'_\ell\}$, where σ'_ℓ is chosen small enough such that $\varepsilon_\ell + 2(\sigma'_\ell)^2(1+\varepsilon_\ell)/(h+1) < \min\{\alpha, 1/\ell\}$. Moreover, for every $\mathbb{V}_{i,\ell} \in \mathcal{D}_\ell$, let us denote

$$K_{\mathbb{V}_{i,\ell}} := \{x \in K : F_{0,1}((h+1)\mathcal{C}^h \llcorner \mathbb{V}_{i,\ell}, (h+1)\mathcal{C}^h \llcorner \mathbb{V}(x)) < \sigma_\ell^{h+4}\},$$

where $\mathbb{V}(x)$ is the element of $Gr_c(h)$ for which $\text{Tan}(\phi, x) = \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x), \lambda > 0\}$. Arguing as in Proposition 2.31, being K the compact set on which we can assume ϕ is supported without loss of generality, we have $K = \bigcup_{\ell \in \mathbb{N}} \bigcup_{\mathbb{V}_{i,\ell} \in \mathcal{D}_\ell} K_{\mathbb{V}_{i,\ell}}$. Hence, we can assume without loss of generality that ϕ is supported on $K_{\mathbb{V}_{i,\ell}}$ for some $\mathbb{V}_{i,\ell}$. The computations in Proposition 2.31 can be repeated substituting σ_ℓ with σ accordingly, allowing us to conclude that ϕ -almost every $K_{\mathbb{V}_{i,\ell}}$ can be covered by compact sets that are $C_{\mathbb{V}_{i,\ell}}(\varepsilon_\ell + 2(\sigma_\ell)^2(1+\varepsilon_\ell)/(h+1))$. By the very choice of σ'_ℓ this implies that the latter compact sets are $C_{\mathbb{V}_{i,\ell}}(\min\{\alpha, 1/\ell\})$ -sets, and since $1/\ell < \varepsilon_1(\mathbb{V}_{i,\ell}, \mathbb{L}_{i,\ell})$, we also conclude that they are graphs according to the splitting $\mathbb{G} = \mathbb{V}_{i,\ell} \cdot \mathbb{L}_{i,\ell}$, see Proposition 1.89. \square

4.1.1. From flat tangents to \mathcal{P} -rectifiability. In this subsection we first prove that, in an arbitrary Carnot group, having flat (complemented) tangent measures à la Preiss implies being \mathcal{P} -rectifiable, see Theorem 2.34. Then we will prove a rectifiable criterion, see Proposition 2.37, which will allow us to complete the proof of Theorem 2.30. Throughout this subsection we assume that $\mathbb{V} \in Gr_c(h)$ and that $\mathbb{V} \cdot \mathbb{L} = \mathbb{G}$. We stress that in this subsection, whenever we deal with $C_{\mathbb{V}}(\alpha)$ -sets, we are always assuming that $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$, if not otherwise specified.

Let us begin with a proposition that roughly tells us the following. If Γ is a compact $C_{\mathbb{V}}(\alpha)$ -set with $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$, and moreover we know that the measure $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling, hence the lower h -density of $\mathcal{S}^h \llcorner \Gamma$ is positive almost everywhere, see Proposition 2.33. The latter conclusion eventually leads to the following result: if a set has flat complemented Preiss's tangents, then it is \mathcal{P} -rectifiable with complemented tangents, see Theorem 2.34. We stress that the following Proposition 2.33 gives as a consequence that the hypotheses of the second part of Proposition 2.17 imply the hypotheses of the first part of Proposition 2.17. This is not trivial a priori: indeed, with the second part of Proposition 2.17, one proves Proposition 2.18, which is hence used as a fundamental tool in the proof of the following Proposition 2.33.

Proposition 2.33. *Let $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and suppose Γ is a $C_{\mathbb{V}}(\alpha)$ -set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, and $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling. Then, $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$ for \mathcal{S}^h -almost every $x \in \Gamma$.*

Proof. Assume by contradiction that there exists a compact set $C \subseteq \Gamma$ of positive \mathcal{S}^h -measure such that $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) = 0$ for every $x \in C$. Since by Proposition 2.18 the measures $\mathcal{S}^h \llcorner \Gamma$ and $\Phi_* \mathcal{S}^h \llcorner \mathbb{V}$ are mutually absolutely continuous, the set $P_{\mathbb{V}}(C)$ must have positive \mathcal{S}^h -measure as well. In particular we have thanks to Proposition 2.20, Lemma 1.44, Proposition 1.34, and

Proposition 1.32 that for \mathcal{S}^h -almost every $x \in C$ we have

$$\begin{aligned}
& \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha))) \\
&= \liminf_{r \rightarrow 0} \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma))}{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(w), \mathfrak{C}(\alpha)r) \cap \Phi(w)C_{\mathbb{V}}(\alpha)))} \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(\Phi(w), \mathfrak{C}(\alpha)r) \cap \Phi(w)C_{\mathbb{V}}(\alpha)))}{(\mathfrak{C}(\alpha)r)^h} \\
&= \liminf_{r \rightarrow 0} \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, \mathfrak{C}(\alpha)r) \cap xC_{\mathbb{V}}(\alpha)) \cap P_{\mathbb{V}}(\Gamma))}{(\mathfrak{C}(\alpha)r)^h} \\
&\leq \liminf_{r \rightarrow 0} \frac{\mathcal{S}^h_{\perp \mathbb{V}}(P_{\mathbb{V}}(\overline{B}(x, r) \cap \Gamma))}{(\mathfrak{C}(\alpha)r)^h} \leq 2C_4(\mathbb{V}, \mathbb{L}) \liminf_{r \rightarrow 0} \frac{\mathcal{S}^h_{\perp \Gamma}(\overline{B}(x, r))}{(\mathfrak{C}(\alpha)r)^h} = 0,
\end{aligned}$$

where $\mathfrak{C}(\alpha)$ is the constant introduced in Lemma 1.44. The above computation is in contradiction with the fact that $\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, 1) \cap C_{\mathbb{V}}(\alpha)))$ is positive thus concluding the proof of the proposition. \square

Theorem 2.34. *Let $\Gamma \subseteq \mathbb{G}$ be compact such that $0 < \mathcal{S}^h(\Gamma) < +\infty$. Assume that for $\mathcal{S}^h_{\perp \Gamma}$ -almost every $x \in \mathbb{G}$ we have $\text{Tan}(\mathcal{S}^h_{\perp \Gamma}, x) = \{\lambda \mathcal{S}^h_{\perp \mathbb{V}}(x) : \lambda > 0, \}$, where $\mathbb{V}(x) \in \text{Gr}_c(h)$. Then, $\mathcal{S}^h_{\perp \Gamma}$ is \mathcal{P}_h^c -rectifiable.*

Proof. We have that $\mathcal{S}^h_{\perp \Gamma}$ is asymptotically doubling, see Proposition 1.80. Moreover, from Proposition 2.32, there exist $\{\mathbb{V}_i\}_{i \in \mathbb{N}} \subseteq \text{Gr}_c(h)$, and $\{\mathbb{L}_i\}_{i \in \mathbb{N}}$, such that \mathbb{L}_i and \mathbb{V}_i are homogeneous complementary subgroups, with the property that for every $\alpha > 0$ there exists a family of compact sets $\{\Gamma_i\}$ such that Γ_i is a $C_{\mathbb{V}_i}(\min\{\alpha, \varepsilon_1(\mathbb{V}_i, \mathbb{L}_i)\})$ -set, and

$$(2.83) \quad \mathcal{S}^h(\Gamma \setminus \cup_{i \in \mathbb{N}} \Gamma_i) = 0.$$

Since $\mathcal{S}^h_{\perp \Gamma}$ is asymptotically doubling, then $\mathcal{S}^h_{\perp \Gamma_i}$ is asymptotically doubling for every $i \in \mathbb{N}$, see Proposition 1.55. Hence, we can apply Proposition 2.33 to conclude that $\Theta_*^h(\mathcal{S}^h_{\perp \Gamma_i}, x) > 0$ for every $i \in \mathbb{N}$ and for \mathcal{S}^h -almost every $x \in \Gamma_i$. In addition, from the previous inequality and [102, 2.10.19(5)], for every $i \in \mathbb{N}$, we get that

$$(2.84) \quad 0 < \Theta_*^h(\mathcal{S}^h_{\perp \Gamma_i}, x) \leq \Theta^{h,*}(\mathcal{S}^h_{\perp \Gamma_i}, x) < +\infty, \quad \text{for } \mathcal{S}^h\text{-almost every } x \in \Gamma_i.$$

Moreover, since for \mathcal{S}^h -almost every $x \in \Gamma$ we have $\text{Tan}(\mathcal{S}^h_{\perp \Gamma}, x) = \{\lambda \mathcal{S}^h_{\perp \mathbb{V}}(x) : \lambda > 0\}$ with $\mathbb{V}(x) \in \text{Gr}_c(h)$, we deduce that, for every $i \in \mathbb{N}$, the locality of tangents in Proposition 1.55 ensures that for \mathcal{S}^h -almost every $x \in \Gamma_i$ we have $\text{Tan}(\mathcal{S}^h_{\perp \Gamma_i}, x) = \{\lambda \mathcal{S}^h_{\perp \mathbb{V}}(x) : \lambda > 0\}$. From the previous equality, we conclude that for every $i \in \mathbb{N}$ we have $\text{Tan}_h(\mathcal{S}^h_{\perp \Gamma_i}, x) \subseteq \{\lambda \mathcal{S}^h_{\perp \mathbb{V}}(x) : \lambda > 0\}$. Hence, from the latter conclusion and (2.84) we get that $\mathcal{S}^h_{\perp \Gamma_i}$ is a \mathcal{P}_h^c -rectifiable measure for every $i \in \mathbb{N}$. Finally, from (2.83) and Proposition 1.55 we conclude that $\mathcal{S}^h_{\perp \Gamma}$ is a \mathcal{P}_h^c -rectifiable measure. \square

Remark 2.35. With the very same argument as in Theorem 2.34 we can show that whenever ϕ is a Radon measure on \mathbb{G} such that $\Theta^{h,*}(\phi, x) < +\infty$ at ϕ -almost every $x \in \mathbb{G}$, and $\text{Tan}(\phi, x) = \{\lambda \mathcal{S}^h_{\perp \mathbb{V}}(x) : \lambda > 0\}$ with $\mathbb{V}(x) \in \text{Gr}_c(h)$ for ϕ -almost every $x \in \mathbb{G}$, hence ϕ is \mathcal{P}_h^c -rectifiable.

4.1.2. *From approximate tangent planes to \mathcal{P} -rectifiability.* In this subsection we aim at proving that whenever an approximate (complemented) h -dimensional tangent plane to a set Γ exists almost everywhere (in the sense of the forthcoming Proposition 2.37), then the measure $\mathcal{S}^h_{\perp \Gamma}$ is \mathcal{P}_h^c -rectifiable. First, we need a crucial estimate on projections that will be useful also later on.

Proposition 2.36. *Let $\mathbb{V}, \mathbb{W} \in \text{Gr}_c(h)$ be complemented by the same homogeneous subgroup \mathbb{L} . Then, there exists an increasing function $\Delta : (0, C_2(\mathbb{W}, \mathbb{L})) \rightarrow (0, +\infty)$, depending only on \mathbb{V}, \mathbb{W} , and \mathbb{L} , such that $\lim_{\beta \rightarrow 0} \Delta(\beta) = 0$, and satisfying the following condition.*

For every $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$ and every $C_{\mathbb{V}}(\alpha)$ -set Γ of finite and positive \mathcal{S}^h -measure if there are an $x \in \Gamma$, a $\beta \leq C_2(\mathbb{W}, \mathbb{L})$ and a $\rho > 0$ such that

$$(2.85) \quad \Gamma \cap \overline{B}(x, r) \subseteq xC_{\mathbb{W}}(\beta), \quad \text{for all } 0 < r < \rho.$$

then

$$\left| \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta)) \cap P_{\mathbb{V}}(\Gamma))}{r^h} - \frac{\mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma))}{r^h} \right| \leq \Delta(\beta),$$

for every $0 < r < (1 + \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1})^{-1}C_2(\mathbb{V}, \mathbb{L})\rho =: \varrho(\rho, \alpha)$.

Proof. Let us fix an $x \in \Gamma$, a $0 < \beta \leq C_2(\mathbb{W}, \mathbb{L})$ and a ρ where (2.85) holds. We denote with $P_{\mathbb{V}}, P_{\mathbb{L}}^{\mathbb{V}}$, respectively, the projections associated to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$, and analogously for the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$. For the sake of notation, for every fixed $0 < r < (1 + \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1})^{-1}C_2(\mathbb{V}, \mathbb{L})\rho$ we let

$$A_r := P_{\mathbb{V}}(\overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta)) \cap P_{\mathbb{V}}(\Gamma) \quad \text{and} \quad B_r := P_{\mathbb{V}}(\overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma).$$

Since the inclusion $B_r \subseteq A_r$ is always verified, we want to estimate the measure of those w contained in $A_r \setminus B_r$. If $y \in A_r$, there are $w \in \Gamma$ such that $P_{\mathbb{V}}(w) = y$, and an $\ell \in \mathbb{L}$ such that $y\ell \in \overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta)$. Let us notice that Proposition 1.33 implies that $\|P_{\mathbb{V}}(x^{-1}y)\| = \|P_{\mathbb{V}}(x^{-1}y\ell)\| \leq C_2(\mathbb{V}, \mathbb{L})^{-1}r$. Moreover, since Γ is a $C_{\mathbb{V}}(\alpha)$ -set, we even get that, by exploiting Remark 1.38,

$$(2.86) \quad \begin{aligned} \|P_{\mathbb{L}}^{\mathbb{V}}(x^{-1}w)\| &\leq \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1}\|P_{\mathbb{V}}(x^{-1}w)\| = \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1}\|P_{\mathbb{V}}(x^{-1}y)\| \\ &\leq \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1}C_2(\mathbb{V}, \mathbb{L})^{-1}r. \end{aligned}$$

This implies in particular that

$$(2.87) \quad \begin{aligned} \|x^{-1}w\| &\leq \|P_{\mathbb{V}}(x^{-1}w)\| + \|P_{\mathbb{L}}^{\mathbb{V}}(x^{-1}w)\| = \|P_{\mathbb{V}}(x^{-1}y)\| + \|P_{\mathbb{L}}^{\mathbb{V}}(x^{-1}w)\| \\ &\leq (1 + \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1})C_2^{-1}(\mathbb{V}, \mathbb{L})r. \end{aligned}$$

Hence, from the choice of r , we infer that $(1 + \alpha(C_2(\mathbb{V}, \mathbb{L}) - \alpha)^{-1})C_2(\mathbb{V}, \mathbb{L})^{-1}r < \rho$ and thus we can use the hypothesis in (2.85) applied to w to obtain that $x^{-1}w \in C_{\mathbb{W}}(\beta)$. Thus, by also exploiting Remark 1.38 and the fact that $x^{-1}y\ell \in C_{\mathbb{W}}(\beta)$ we get that

$$(2.88) \quad \begin{aligned} \|P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}y)\ell\| &\leq \beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}\|P_{\mathbb{W}}(x^{-1}y)\| \\ \|P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}y)P_{\mathbb{L}}^{\mathbb{V}}(w)\| &\leq \beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}\|P_{\mathbb{W}}(x^{-1}y)\|, \end{aligned}$$

where the last inequality comes from the fact that

$$P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}w) = P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}yP_{\mathbb{L}}^{\mathbb{V}}(w)) = P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}y)P_{\mathbb{L}}^{\mathbb{V}}(w).$$

Thanks to (2.88) we deduce that

$$(2.89) \quad \begin{aligned} \|\ell^{-1}P_{\mathbb{L}}^{\mathbb{V}}(w)\| &\leq \|P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}y)\ell\| + \|P_{\mathbb{L}}^{\mathbb{W}}(x^{-1}y)P_{\mathbb{L}}^{\mathbb{V}}(w)\| \leq 2\beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}\|P_{\mathbb{W}}(x^{-1}y)\| \\ &= 2\beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}\|P_{\mathbb{W}}(x^{-1}y\ell)\| \\ &\leq 2\beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}C_2(\mathbb{W}, \mathbb{L})^{-1}\|x^{-1}y\ell\| \\ &\leq 2\beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1}C_2(\mathbb{W}, \mathbb{L})^{-1}r. \end{aligned}$$

This in particular implies that

$$(2.90) \quad \begin{aligned} \|x^{-1}w\| &= \|x^{-1}yP_{\mathbb{L}}^{\mathbb{V}}(w)\| \leq r + \|\ell^{-1}P_{\mathbb{L}}^{\mathbb{V}}(w)\| \\ &\leq (1 + 2\beta(C_2(\mathbb{W}, \mathbb{L}) - \beta)^{-1})C_2(\mathbb{W}, \mathbb{L})^{-1}r =: f_2(\beta)r. \end{aligned}$$

The above chain of inequalities, together with the hypothesis (2.85), allows us to conclude that

$$A_r \subseteq P_{\mathbb{V}}(\overline{B}(x, f_2(\beta)r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma).$$

Finally this allows us to infer

$$(2.91) \quad \begin{aligned} \mathcal{S}^h(A_r) - \mathcal{S}^h(B_r) &\leq \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, f_2(\beta)r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma) \setminus P_{\mathbb{V}}(\overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma)) \\ &= \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, f_2(\beta)r) \setminus \overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta) \cap \Gamma)), \end{aligned}$$

where the last identity comes from the injectivity of $P_{\mathbb{V}}$ when restricted to Γ , since $\alpha \leq \varepsilon_1(\mathbb{V}, \mathbb{L})$, see Proposition 1.89. Finally, Proposition 1.32 implies

$$(2.92) \quad \begin{aligned} \mathcal{S}^h(A_r) - \mathcal{S}^h(B_r) &\leq \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(x, f_2(\beta)r) \setminus \overline{B}(x, r) \cap xC_{\mathbb{W}}(\beta))) \\ &= \mathcal{S}^h(P_{\mathbb{V}}(\overline{B}(0, f_2(\beta)) \setminus \overline{B}(0, 1) \cap C_{\mathbb{W}}(\beta)))r^h =: \Delta(\beta)r^h. \end{aligned}$$

The function Δ is easily seen to be increasing and thanks to the continuity from above of the measure, the fact that $\lim_{\beta \rightarrow 0} \Delta(\beta) = 0$ immediately follows too since $f_2(\beta) \rightarrow 1$ as $\beta \rightarrow 0$. \square

Proposition 2.37. *Let Γ be a Borel set of \mathcal{C}^h positive and finite measure such that at \mathcal{C}^h -almost every $x \in \Gamma$ there exists $\mathbb{V}(x) \in Gr_c(h)$ for which for every $0 < \beta < 1$ there exists a $\rho(x, \beta) > 0$ such that*

$$(2.93) \quad \Gamma \cap \overline{B}(x, \rho(x, \beta)) \subseteq xC_{\mathbb{V}(x)}(\beta).$$

Then, the measure $\mathcal{C}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable. In addition we have that $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ and $\text{Tan}_h(\mathcal{C}^h \llcorner \Gamma, x) = \{\mathcal{C}^h \llcorner \mathbb{V}(x)\}$ for $\mathcal{C}^h \llcorner \Gamma$ -almost every x .

Proof. First of all we define the family of sets

$$\mathcal{F} := \{\Gamma \subseteq \mathbb{G} : \Gamma \text{ Borel, } \mathcal{S}^h \llcorner \Gamma \text{ is } \mathcal{P}_h^c\text{-rectifiable, } \Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1 \text{ for } \mathcal{S}^h \llcorner \Gamma\text{-almost every } x\}.$$

By a classical argument, see for example [180, Proposition 1.22], we can write Γ as $\Gamma = \Gamma^r \cup \Gamma^u$ where Γ^r is a Borel set for which there are countable many $\Sigma_k \in \mathcal{F}$ such that $\Gamma^r \subseteq \cup_k \Sigma_k$ and Γ^u is a Borel set such that $\mathcal{S}^h(\Gamma \cap \Sigma) = 0$ for every $\Sigma \in \mathcal{F}$.

Let us prove that $\Gamma^r \in \mathcal{F}$. For every $k \in \mathbb{N}$ we define $\tilde{\Sigma}_k := \Sigma_k \setminus \cup_{1 \leq i \leq k-1} \Sigma_i$. Thanks to Proposition 1.55 the measure $\mathcal{S}^h \llcorner \tilde{\Sigma}_k$ is still \mathcal{P}_h^c -rectifiable, the $\tilde{\Sigma}_k$ are pairwise disjoint and their union still contains Γ^r . Again by Proposition 1.55 we infer that for every $k \in \mathbb{N}$ the measure $\mathcal{S}^h \llcorner (\tilde{\Sigma}_k \cap \Gamma^r)$ is \mathcal{P}_h^c -rectifiable and $\Theta^h(\mathcal{C}^h \llcorner (\tilde{\Sigma}_k \cap \Gamma^r), x) = 1$ for $\mathcal{S}^h \llcorner (\tilde{\Sigma}_k \cap \Gamma^r)$ -almost every x and this finally implies that

$$\Theta^h(\mathcal{C}^h \llcorner \Gamma^r, x) = 1,$$

for $\mathcal{S}^h \llcorner \Gamma^r$ -almost every x . Applying Proposition 1.55 to the measure $\mathcal{S}^h \llcorner \Gamma^r$ and to the Borel set $\tilde{\Sigma}_k$ we infer that $\text{Tan}_h(\mathcal{S}^h \llcorner \Gamma^r, x)$ is unique and flat $\mathcal{S}^h \llcorner \Gamma^r$ -almost everywhere on $\tilde{\Sigma}_k$. Since the $\tilde{\Sigma}_k$ countably cover Γ^r this concludes the proof that $\mathcal{S}^h \llcorner \Gamma^r$ is \mathcal{P}_h^c -rectifiable.

The above argument shows that we can assume by contradiction that Γ is compact set of positive and finite \mathcal{S}^h -measure and that

$$(2.94) \quad \mathcal{S}^h(\Gamma \cap \Sigma) = 0 \text{ for every } \Sigma \in \mathcal{F}.$$

For every $\eta > 0$ we let

$$Gr_c^\eta(h) := \left\{ \mathbb{V} \in Gr_c(h) : \inf_{\mathbb{W} \in Gr(h) \setminus Gr_c(h)} d_{\mathbb{G}}(\mathbb{V}, \mathbb{W}) \geq \eta \right\} \subseteq Gr_c(h).$$

Thanks to Proposition 1.22 it follows that $Gr_c^\eta(h)$ is a closed, thus compact, subset of $Gr(h)$. Thanks to Lemma 1.85, for every $\eta > 0$ the set $\Gamma^\eta := \{x \in \Gamma : (2.93) \text{ holds at } x \text{ and } \mathbb{V}(x) \in Gr_c^\eta(h)\}$

$Gr_c^\eta(h)$ is \mathcal{S}^h -measurable. In addition to this, since $\mathbb{V}(x)$ belongs $\mathcal{S}^h \llcorner \Gamma$ -almost everywhere to $Gr_c(h)$, that is an open set in $Gr(h)$, see Proposition 1.42, we have

$$\mathcal{S}^h \left(\Gamma \setminus \bigcup_{\eta \in \mathbb{Q}^+ \setminus \{0\}} \Gamma^\eta \right) = 0.$$

In particular there exists an $\eta_0 > 0$ such that $\mathcal{S}^h(\Gamma^{\eta_0}) > 0$. In the following we let E be a compact subset of Γ^{η_0} such that

$$\mathcal{S}^h(\Gamma^{\eta_0} \setminus E) < \mathcal{S}^h(\Gamma^{\eta_0})/2.$$

Note further that thanks to Remark 1.46 we have that

$$m(\eta_0) := \min_{\mathbb{V} \in Gr_c^{\eta_0}(h)} \mathbf{e}(\mathbb{V}) > 0.$$

Let $\mathcal{D} := \{\mathbb{V}_j\}_{j \in \mathbb{N}}$ be a countable dense subset of $Gr_c^{\eta_0}(h)$ and

for all $j \in \mathbb{N}$ we choose a compl. subgroup \mathbb{L}_j of \mathbb{V}_j s.t. $\varepsilon_1(\mathbb{V}_j, \mathbb{L}_j) > \mathbf{e}(\mathbb{V}_j)/2 \geq m(\eta_0)/2$.

From now on we let ε be a fixed positive number in $(0, m(\eta_0)/10)$ such that

$$(2.95) \quad 1 - 3m(\eta_0)^{-1}\varepsilon(1 + 3m(\eta_0)^{-2}\varepsilon)/(m(\eta_0) - \varepsilon) > 0,$$

which we can do taking ε small enough. The previous estimate will play a role later on. For every $p, q \in \mathbb{N}$ we define the set

$$(2.96) \quad F(p, q) := \{x \in E : \overline{B}(x, 1/q) \cap \Gamma \subseteq xC_{\mathbb{V}_p}(\varepsilon/6)\},$$

and we claim that

$$(2.97) \quad \mathcal{S}^h \left(E \setminus \bigcup_{p, q \in \mathbb{N}} F(p, q) \right) = 0.$$

By density of the family \mathcal{D} in $Gr_c^{\eta_0}(h)$ and since by construction for every $x \in \Gamma^{\eta_0}$ we have $\mathbb{V}(x) \in Gr_c^{\eta_0}(h)$, we deduce that there must exist a plane $\mathbb{V}_p \in \mathcal{D}$ such that $d_{\mathbb{G}}(\mathbb{V}_p, \mathbb{V}(x)) < 30^{-1}\varepsilon$. This, jointly with Lemma 1.39, implies that

$$(2.98) \quad C_{\mathbb{V}(x)}(30^{-1}\varepsilon) \subseteq C_{\mathbb{V}_p}(6^{-1}\varepsilon).$$

Since by definition of Γ^{η_0} , (2.93) holds at every point $x \in E$, we can find a $\rho(x) > 0$ such that for every $0 < r < \rho(x)$ we have

$$(2.99) \quad \overline{B}(x, r) \cap \Gamma \subseteq xC_{\mathbb{V}(x)}(30^{-1}\varepsilon).$$

In particular, putting together (2.98) and (2.99) we infer that for $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in E$ there are a $p = p(x) > 0$ and a $\rho(x) > 0$ such that whenever $0 < r < \rho(x)$ we have

$$\overline{B}(x, r) \cap \Gamma \subseteq xC_{\mathbb{V}_p}(6^{-1}\varepsilon),$$

and this concludes the proof of (2.97). Thanks to Proposition 2.9 and Proposition 1.89, we get that there are countably many $\mathbb{V}_j \in Gr_c^{\eta_0}(h)$ complemented by some \mathbb{L}_j , compact subsets K_j of \mathbb{V}_j and intrinsically Lipschitz functions $\varphi_j : K_j \subset \mathbb{V}_j \rightarrow \mathbb{L}_j$ such that

- (1) for every $z \in K_j$ we have $\Gamma_j = \{w\varphi_j(w) : w \in K_j\} \subseteq z\varphi_j(z)C_{\mathbb{V}_j}(\varepsilon)$, and $\Gamma_j \subseteq E$
- (2) $\mathcal{S}^h(E \setminus \bigcup_j \Gamma_j) = 0$.

Thanks to [97, Corollary 4.17] we know that $\Theta^{h,*}(\mathcal{C}^h \llcorner E, x) \leq 1$ for $\mathcal{C}^h \llcorner E$ -almost every x and now we wish to prove that $\Theta_*^h(\mathcal{C}^h \llcorner E, x) \geq 1$ for $\mathcal{C}^h \llcorner E$ -almost every x .

Fix a $j \in \mathbb{N}$, and an $x \in \Gamma_j$ such that the conclusion in Proposition 2.20 holds. Notice that such a choice of x can be made in a set of $\mathcal{C}^h \llcorner \Gamma_j$ -full measure in Γ_j . Suppose that r_k is an infinitesimal sequence such that

$$\Theta_*^h(\mathcal{C}^h \llcorner \Gamma_j, x) = \lim_{k \rightarrow \infty} r_k^{-h} \mathcal{C}^h \llcorner \Gamma_j(\overline{B}(x, r_k)).$$

Thanks to item (1) above and to Proposition 2.36 one infers that for every $k \in \mathbb{N}$ we get

$$(2.100) \quad \left| \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap P_{\mathbb{V}_j}(\Gamma_j))}{r_k^h} - \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap \Gamma_j)}{r_k^h} \right| \leq \Delta_j(\varepsilon),$$

where Δ_j was introduced in the statement of Proposition 2.36 and depends only on the split $\mathbb{V}_j \cdot \mathbb{L}_j = \mathbb{G}$. In addition to this, for every $j \in \mathbb{N}$ the definitions of $\varepsilon_1(\cdot, \cdot)$ and of \mathbb{L}_j imply that

$$(2.101) \quad C_2(\mathbb{V}_j, \mathbb{L}_j) = 2\varepsilon_1(\mathbb{V}_j, \mathbb{L}_j) > \mathfrak{e}(\mathbb{V}_j) \geq m(\eta_0),$$

and in turn this means that $\Delta_j(\varepsilon)$ can be estimated with

$$(2.102) \quad \begin{aligned} \Delta_j(\varepsilon) &= \mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, f_2(\varepsilon)) \setminus \overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon))) \\ &= \mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1 + 2\varepsilon(C_2(\mathbb{V}_j, \mathbb{L}_j) - \varepsilon)^{-1} C_2(\mathbb{V}_j, \mathbb{L}_j)^{-1}) \setminus \overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon))) \\ &\leq \mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1 + 3m(\eta_0)^{-2}\varepsilon) \setminus \overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon))), \end{aligned}$$

where the last inequality above comes from (2.101) and the fact that $\varepsilon \in (0, m(\eta_0)/10)$. From (2.100), the invariance properties in Proposition 1.32, the fact that $x \in \Gamma_j$ was chosen in such a way that Proposition 2.20 holds, and the homogeneity of $\mathcal{C}^h \llcorner \mathbb{V}$, we infer that

$$\begin{aligned} &\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon))) \limsup_{k \rightarrow +\infty} \left(\frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap P_{\mathbb{V}_j}(\Gamma_j))}{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)))} \right. \\ &\quad \left. - \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap \Gamma_j)}{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)))} \right) \\ &= \mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon))) \left(1 - \liminf_{k \rightarrow +\infty} \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap \Gamma_j)}{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)))} \right) \leq \Delta_j(\varepsilon). \end{aligned}$$

This implies that, for every $0 \leq \delta \leq 1/100$, up to passing to a non-relabelled subsequence in k , we can assume without loss of generality that for every $k \in \mathbb{N}$ we have

$$(2.103) \quad \begin{aligned} 1 - \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap \Gamma_j))}{r_k^h \mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon)))} &= 1 - \frac{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)) \cap \Gamma_j)}{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(x, r_k) \cap xC_{\mathbb{V}_j}(\varepsilon)))} \\ &\leq \frac{\delta + \Delta_j(\varepsilon)}{\mathcal{C}^h(P_{\mathbb{V}_j}(\overline{B}(0, 1) \cap C_{\mathbb{V}_j}(\varepsilon)))}. \end{aligned}$$

Now, let us fix a $k \in \mathbb{N}$ sufficiently large such that $|\mathcal{C}^h \llcorner \Gamma_j(\overline{B}(x, r_k))/r_k^h - \Theta_*^h(\mathcal{C}^h \llcorner \Gamma_j, x)| \leq \delta$, and let $\Gamma'_j \subseteq \Gamma_j$ be a Borel set such that $|\mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) - \mathcal{C}_0^h(\overline{B}(x, r_k) \cap \Gamma'_j)| \leq \delta r_k^h$. Finally, we choose a covering with balls $\{\overline{B}(y_\ell, s_\ell)\}_{\ell \in \mathbb{N}}$ of $\Gamma'_j \cap \overline{B}(x, r_k)$, with $y_\ell \in \Gamma'_j$, such that $|\sum_{\ell \in \mathbb{N}} s_\ell^h - \mathcal{C}_0^h(\overline{B}(x, r_k) \cap \Gamma'_j)| \leq \delta r_k^h$. This implies in particular that

$$(2.104) \quad \begin{aligned} |\mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) - \sum_{\ell \in \mathbb{N}} s_\ell^h| &\leq |\mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) - \mathcal{C}_0^h(\overline{B}(x, r_k) \cap \Gamma'_j)| \\ &\quad + |\sum_{\ell \in \mathbb{N}} s_\ell^h - \mathcal{C}_0^h(\overline{B}(x, r_k) \cap \Gamma'_j)| \leq 2\delta r_k^h. \end{aligned}$$

The above inequalities together imply in particular that for such a $k \in \mathbb{N}$ we have

$$\begin{aligned}
 (2.105) \quad 1 - \frac{\delta + \Delta_j(\varepsilon)}{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} &\leq \frac{\mathcal{C}^h(P_{V_j}(\overline{B}(x, r_k) \cap \Gamma_j))}{r_k^h \mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \\
 &\leq \frac{\mathcal{C}^h(P_{V_j}(\overline{B}(x, r_k) \cap \Gamma'_j)) + C_4(\mathbb{V}_j, \mathbb{L}_j) \delta r_k^h}{r_k^h \mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \leq \frac{\mathcal{C}^h(P_{V_j}(\bigcup_{\ell \in \mathbb{N}} \overline{B}(y_\ell, s_\ell) \cap y_\ell C_{V_j}(\varepsilon))) + C_4 \delta r_k^h}{r_k^h \mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \\
 &\leq r_k^{-h} \sum_{\ell \in \mathbb{N}} s_\ell^h + \frac{C_4 \delta}{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))},
 \end{aligned}$$

where in the second inequality we used

$$\begin{aligned}
 (2.106) \quad \mathcal{C}^h(P_{V_j}(\overline{B}(x, r_k) \cap \Gamma_j)) - \mathcal{C}^h(P_{V_j}(\overline{B}(x, r_k) \cap \Gamma'_j)) &= \mathcal{C}^h(P_{V_j}(\overline{B}(x, r_k) \cap \Gamma_j \setminus \overline{B}(x, r_k) \cap \Gamma'_j)) \\
 &\leq C_4 \mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j \setminus \overline{B}(x, r_k) \cap \Gamma'_j) = C_4 \left(\mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) - \mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma'_j) \right) \\
 &\leq C_4 \left(\mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) - \mathcal{C}_0^h(\overline{B}(x, r_k) \cap \Gamma'_j) \right) \leq C_4 \delta r_k^h,
 \end{aligned}$$

that is true taking into account Proposition 1.34, the fact that P_{V_j} is injective on Γ_j , see Proposition 1.89, and the fact that $\mathcal{S}^h \leq \mathcal{C}^h$ by definition. Hence putting together (2.104) and (2.105) we deduce that, for k large enough,

$$1 - \frac{(1 + C_4)\delta + \Delta_j(\varepsilon)}{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \leq r_k^{-h} \sum_{\ell \in \mathbb{N}} s_\ell^h \leq \mathcal{C}^h(\overline{B}(x, r_k) \cap \Gamma_j) / r_k^h + 2\delta \leq \Theta_*^h(\mathcal{C}^h \llcorner \Gamma_j, x) + 3\delta.$$

Thanks to the arbitrariness of δ , this implies that for $\mathcal{C}^h \llcorner \Gamma_j$ -almost every x we have, by making use of (2.102)

$$\begin{aligned}
 (2.107) \quad 1 - \mathfrak{T}(\varepsilon, j) &:= 1 - \frac{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1 + 3m(\eta_0)^{-2}\varepsilon) \setminus \overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))}{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \\
 &\leq 1 - \frac{\Delta_j(\varepsilon)}{\mathcal{C}^h(P_{V_j}(\overline{B}(0, 1) \cap C_{V_j}(\varepsilon)))} \leq \Theta_*^h(\mathcal{C}^h \llcorner \Gamma_j, x) \leq \Theta_*^h(\mathcal{C}^h \llcorner E, x).
 \end{aligned}$$

We now wish to get a bound from above of $\mathfrak{T}(\varepsilon, j)$ that does not depend on j . In order to do this, we first of all let $\rho_1(\varepsilon) := 1 + 3m(\eta_0)^{-2}\varepsilon$ and $\rho_2(\varepsilon) := 1 - 3m(\eta_0)^{-1}\varepsilon\rho_1(\varepsilon)/(m(\eta_0) - \varepsilon) > 0$, thanks to (2.95). We claim that the following inclusion holds

$$(2.108) \quad P_{V_j}(\overline{B}(0, \rho_1(\varepsilon)) \setminus \overline{B}(0, 1) \cap C_{V_j}(\varepsilon)) \subseteq P_{V_j}(\overline{B}(0, \rho_1(\varepsilon)) \cap C_{V_j}(\varepsilon)) \setminus P_{V_j}(\overline{B}(0, \rho_2(\varepsilon)) \cap C_{V_j}(\varepsilon)).$$

By definition, if $y \in P_{V_j}(\overline{B}(0, \rho_1(\varepsilon)) \setminus \overline{B}(0, 1) \cap C_{V_j}(\varepsilon))$, there exists an $\ell_1 \in \mathbb{L}_j$ such that $y\ell_1 \in \overline{B}(0, \rho_1(\varepsilon)) \setminus \overline{B}(0, 1) \cap C_{V_j}(\varepsilon)$. Notice that if $\ell \in \mathbb{L}_j$ is such that $y\ell \in C_{V_j}(\varepsilon)$, by Proposition 1.33 we have

$$m(\eta_0)\|\ell\| \leq C_4(\mathbb{V}_j, \mathbb{L}_j)\|\ell\| \leq \text{dist}(y\ell, \mathbb{V}_j) \leq \varepsilon\|y\ell\| \leq \varepsilon\|y\| + \varepsilon\|\ell\|,$$

and then $\|\ell\| \leq \varepsilon\|y\|/(m(\eta_0) - \varepsilon)$. This implies that in order to prove that y does not belong to $P_{V_j}(\overline{B}(0, \rho_2(\varepsilon)) \cap C_{V_j}(\varepsilon))$ we just need to show that $y\ell \notin \overline{B}(0, \rho_2(\varepsilon)) \cap C_{V_j}(\varepsilon)$ for every $\ell \in \overline{B}(0, \varepsilon\|y\|/(m(\eta_0) - \varepsilon)) \cap \mathbb{L}_j$. This however, follows from the following computation

$$\|y\ell\| \geq \|y\ell_1\| - \|\ell_1^{-1}\ell\| \geq 1 - \frac{2\varepsilon\|y\|}{m(\eta_0) - \varepsilon} > \rho_2(\varepsilon),$$

where the last inequality comes from the fact that $\|\ell_1^{-1}\ell\| \leq \|\ell_1^{-1}\| + \|\ell\| \leq 2\varepsilon\|y\|/(m(\eta_0) - \varepsilon)$, and $\|y\| \leq C_4(\mathbb{V}_j, \mathbb{L}_j)^{-1}\|y\ell_1\| \leq m(\eta_0)^{-1}\rho_1(\varepsilon)$. This proves (2.108) and in turn the inequality

$\mathfrak{T}(\varepsilon, j) \leq \rho_1(\varepsilon)^h - \rho_2(\varepsilon)^h$, by homogeneity of \mathcal{C}^h . Furthermore, since on the right-hand side of the previous inequality we have an expression independent on j we conclude that, by exploiting (2.107), for \mathcal{C}^h -almost every $x \in \Gamma_j$ we have

$$1 - (\rho_1(\varepsilon)^h - \rho_2(\varepsilon)^h) \leq \Theta_*^h(\mathcal{C}^h \llcorner E, x).$$

Thanks to the arbitrariness of j and to the fact that $\mathcal{C}^h(E \setminus \cup_j \Gamma_j) = 0$, we deduce that the previous inequality holds for \mathcal{C}^h -almost every $x \in E$. Since ε can be chosen arbitrarily small, we conclude that $\Theta_*^h(\mathcal{C}^h \llcorner E, x) \geq 1$, and then $\Theta^h(\mathcal{C}^h \llcorner E, x) = 1$ for \mathcal{C}^h -almost every $x \in E$.

Eventually, Lemma 1.54 together with (2.93) concludes that for \mathcal{C}^h -almost every $x \in E$ and for every $\nu \in \text{Tan}_h(\mathcal{C}^h \llcorner E, x)$ the support of ν is contained in $\mathbb{V}(x)$. In addition to this, from the existence of the density, the argument in [89, Proposition 3.4], and Proposition 1.28, we have that for $\mathcal{C}^h \llcorner E$ -almost every $x \in \mathbb{G}$ we have $\text{Tan}_h(\mathcal{C}^h \llcorner E, x) = \{\mathcal{C}^h \llcorner \mathbb{V}(x)\}$. This concludes the proof of the fact that $\mathcal{C}^h \llcorner E$ is \mathcal{P}_c^h -rectifiable and this comes in contradiction with the fact that $E \subseteq \Gamma$ has positive $\mathcal{S}^h \llcorner \Gamma$ -measure by construction and (2.94). \square

Let us now verify that an intrinsically differentiable graph satisfies the hypothesis of Proposition 2.37.

Lemma 2.38. *Let $\varphi : A \subseteq \mathbb{V} \rightarrow \mathbb{L}$ be a map such that $\Gamma := \text{graph}(\varphi)$ is an intrinsically differentiable graph at $w \in \Gamma$ with tangent $\mathbb{V}(w)$. Then, for every β there exists $\rho = \rho(\beta)$ such that*

$$\Gamma \cap \overline{B}(w, \rho) \subseteq wC_{\mathbb{V}(w)}(\beta).$$

Proof. We first claim that for every $\varepsilon > 0$ there exists $r_0 := r_0(\varepsilon)$ such that

$$(2.109) \quad \sup_{p \in \Gamma \cap \overline{B}(w, r)} \text{dist}(p, w\mathbb{V}(w)) \leq \varepsilon r, \quad \text{for all } 0 < r \leq r_0.$$

Indeed, this follows just by taking $K' := \overline{B}(0, 1)$ in the definition (1.77) and by exploiting the very definition of Hausdorff distance.

Now let us take $\varepsilon \leq \beta/2$. We claim that $\Gamma \cap \overline{B}(w, r_0(\varepsilon)) \subseteq wC_{\mathbb{V}(w)}(\beta)$. Indeed, let $p \in \Gamma \cap \overline{B}(w, r_0(\varepsilon))$, and $k \geq 1$ be such that $r_0 2^{-k} < \|w^{-1} \cdot p\| \leq r_0 2^{-k+1}$. Since $p \in \Gamma \cap \overline{B}(w, r_0 2^{-k+1})$, from (2.109) we get

$$\text{dist}(p, w\mathbb{V}(w)) \leq \varepsilon r_0 2^{-k+1} \leq 2\varepsilon \|w^{-1} \cdot p\| \leq \beta \|w^{-1} \cdot p\|,$$

thus showing the claim. \square

We are now ready to give the proof of Theorem 2.30.

Proof of Theorem 2.30. We prove different implications in separate points.

1. \Rightarrow 2. If $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable, then $\mathcal{S}^h \llcorner \Gamma$ is asymptotically doubling. Hence, by a routine argument (cf. [174, Remark 14.4(3)]) we get that, for $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, every element in $\text{Tan}(\mathcal{S}^h \llcorner \Gamma, x)$ is a constant multiple of an element of $\text{Tan}_h(\mathcal{S}^h \llcorner \Gamma, x)$, which is by hypothesis of the form $\lambda \mathcal{S}^h \llcorner \mathbb{V}(x)$ with $\mathbb{V}(x) \in Gr_c(h)$, whence the conclusion.
2. \Rightarrow 1. It follows from Theorem 2.34 by approximating the Borel set Γ from within by compact sets.
1. \Rightarrow 3. It is a consequence of Theorem 2.25, and the fact that the Hausdorff tangent at $\mathcal{S}^h \llcorner \Gamma_i$ -almost every x of Γ_i is complemented since it coincides almost everywhere with the subgroup on which it is supported the tangent measure.

3. \Rightarrow 1. Since, for every $i \in \mathbb{N}$, Γ_i is an intrinsically differentiable graph at $\mathcal{S}^h \llcorner \Gamma_i$ -almost every point of it, by Lemma 2.38 we conclude that the hypothesis of Proposition 2.37 is verified. Hence, for every $i \in \mathbb{N}$, $\mathcal{S}^h \llcorner \Gamma_i$ is \mathcal{P}_h^c -rectifiable. Hence, since $\mathcal{S}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0$, by a routine argument involving the locality of tangents and the Lebesgue Differentiation Theorem, see Proposition 1.55, we conclude that $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable as well.

Let us show the second part of the statement. Let us assume item (3) holds. Since, for every $i \in \mathbb{N}$, Γ_i is intrinsically differentiable, arguing as above we can apply Proposition 2.37 and then conclude that $\Theta^h(\mathcal{C}^h \llcorner \Gamma_i, x) = 1$ for $\mathcal{C}^h \llcorner \Gamma_i$ -almost every $x \in \mathbb{G}$. Hence, from the Lebesgue Differentiation Theorem in Proposition 1.55, we conclude that, for every $i \in \mathbb{N}$, $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ for $\mathcal{C}^h \llcorner \Gamma_i$ -almost every $x \in \mathbb{G}$, and hence the same conclusion holds for $\mathcal{C}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$ since $\mathcal{C}^h(\Gamma \setminus \cup_{i=1}^{+\infty} \Gamma_i) = 0$. The convergence in the second part of the statement is a direct consequence of the fact that the density is 1 and Lemma 1.62. The last part of the statement is an immediate consequence of Theorem 2.25. \square

Remark 2.39. With the argument in the proof of Theorem 2.30, taking also into account Remark 2.35, we might prove the same equivalence as in Theorem 2.30 but for Radon measures ϕ on \mathbb{G} with $\Theta^{h,*}(\phi, x) < +\infty$ at ϕ -almost every $x \in \mathbb{G}$, instead of $\mathcal{S}^h \llcorner \Gamma$. Clearly, the density $\Theta^h(\phi, x)$ might not be 1 ϕ -almost everywhere.

We recall the definition of approximate tangent to a set. Let $\Gamma \subseteq \mathbb{G}$ be a Borel set of finite and positive \mathcal{S}^h -measure, and let $\mathbb{V} \in Gr(h)$. We say that \mathbb{V} is an *approximate tangent plane of Γ at x* if

- (1) we have $\Theta^{h,*}(\mathcal{S}^h \llcorner \Gamma, x) > 0$;
- (2) we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{S}^h \llcorner \Gamma(B_r(x) \setminus C_{\mathbb{V}}(\alpha))}{r^h} \rightarrow 0,$$

for every $\alpha > 0$.

Remark 2.40. By properly adapting the argument in Proposition 2.37, one might prove that if $\Gamma \subseteq \mathbb{G}$ is a Borel set with \mathcal{S}^h positive and finite measure, $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$ for $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, and moreover Γ has an approximate complemented tangent at $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, hence the measure $\mathcal{S}^h \llcorner \Gamma$ is \mathcal{P}_h^c -rectifiable. This means that in the statement of Theorem 2.30 one can also add the item

- $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$ for $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, and moreover Γ has an approximate complemented tangent at $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$.

One thus might ask the following question.

Question 4. Is it possible to remove the hypothesis $\Theta_*^h(\mathcal{S}^h \llcorner \Gamma, x) > 0$ in the item above, and still get the equivalence of Theorem 2.30?

In the special case of tangents that are complemented by at least one normal subgroup, the answer to the previous question should be positive, by adapting the arguments of [178], and [128], which deal with the co-horizontal case. Thus one could add the following item

- Γ has an approximate tangent that is complemented by at least one normal subgroup, at $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$.

in the forthcoming Corollary 3.3.

Marstrand–Mattila rectifiability criterion for \mathcal{P} -rectifiable measures

In this chapter we prove a generalization of Marstrand–Mattila rectifiability criterion in the setting of Carnot groups. For the classical Marstrand–Mattila rectifiability criterion in \mathbb{R}^n , we refer the reader to [174]. The content of this chapter comes from a work in collaboration with A. Merlo [32].

In this introductory part we give the statement of the theorem, see the forthcoming Theorem 3.1. We also present a couple of corollaries that can be readily deduced from it. Namely, the one-dimensional Preiss’s theorem in the first Heisenberg group \mathbb{H}^1 endowed with the Korányi distance, see Theorem 3.2, and an enhanced characterization of the \mathcal{P} -rectifiability in the co-normal case, see Corollary 3.3.

In Section 1 we give the complete proof of Theorem 3.1. First, in Section 1.1 we show that in a Carnot group of homogeneous dimension Q , every \mathcal{P}_h^* -rectifiable measure (see Definition 1.58), with $1 \leq h \leq Q$, is such that, at almost every point, the possible tangents, even if different, share the same stratification vector. Moreover the function that associates to each point the (unique) stratification vector of the tangents at that point is measurable, see Proposition 3.12. Finally, in Section 1.2 we complete the technical part of the proof, whose strategy follows the lines of Preiss’s work [202].

In this chapter we aim at proving the following, which is one of the main results of [28].

Theorem 3.1 (Co-normal Marstrand–Mattila rectifiability criterion). *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a $\mathcal{P}_h^{*,\triangleleft}$ -rectifiable, i.e., a \mathcal{P}_h^* -rectifiable measure with tangents that ϕ -almost everywhere admit at least one normal complementary subgroup (see Definition 1.61).*

Then ϕ is a \mathcal{P}_h^c -rectifiable measure, see Definition 1.61. Moreover, there are countably many homogeneous Carnot subgroups \mathbb{V}_i of homogeneous dimension h , and Lipschitz maps $\Phi_i : A_i \subseteq \mathbb{V}_i \rightarrow \mathbb{G}$, where A_i ’s are compact, such that

$$\phi\left(\mathbb{G} \setminus \bigcup_{i \in \mathbb{N}} \Phi_i(A_i)\right) = 0.$$

Notice that passing from a \mathcal{P}^* -rectifiable measure to a \mathcal{P} -rectifiable measure is by far non-trivial. Indeed, we recall that in the definition of \mathcal{P}^* -rectifiability we just ask that the tangents are *flat*, but not necessarily *unique* almost everywhere. On the contrary, in the definition of \mathcal{P} -rectifiability we also ask that, almost everywhere, the tangent is *unique*.

Let us notice that a converse of Theorem 3.1 holds as well. Namely if ϕ is a Radon measure on \mathbb{G} with positive h -lower density and finite h -upper density ϕ -almost everywhere, and there are countably many homogeneous Carnot subgroups \mathbb{V}_i of homogeneous dimension h , and Lipschitz maps $\Phi_i : A_i \subseteq \mathbb{V}_i \rightarrow \mathbb{G}$, where A_i ’s are compact, such that

$$\phi\left(\mathbb{G} \setminus \bigcup_{i \in \mathbb{N}} \Phi_i(A_i)\right) = 0,$$

hence ϕ is \mathcal{P}_h -rectifiable (and a fortiori \mathcal{P}_h^* -rectifiable). The proof is reached first by a classical reduction to measures of the type $\mathcal{S}^h \llcorner \Gamma$, and hence using the Rademacher theorem, and the area formula, which hold for the maps Φ_i . The resulting reasoning is exactly the same as in the last part of the proof of Theorem 3.4.

Let us further notice that the conclusion of Theorem 3.1 also implies that the support of the measure ϕ is Pauls's rectifiable, i.e., it is \mathcal{H}^h -almost everywhere covered by the countable union of Lipschitz images of subsets of Carnot groups of homogenous dimension h , see Definition 4.8 for additional information.

A Marstrand–Mattila rectifiability criterion for codimension-one rectifiable measures in arbitrary Carnot groups has been proved by Merlo in [180]. The techniques used in [180] are likely to be adapted to show the same result in the more general co-horizontal case. Apart from these cases, and the result in Theorem 3.1, we presently do not know if a Marstrand–Mattila rectifiability criterion holds in the generality of \mathcal{P}^* -rectifiable measures with complemented tangents. We believe that such a result could be really challenging to prove because of the lack of regularity of the projection maps onto complemented subgroups in the general case.

Question 5. Determine whether a Marstrand–Mattila type rectifiability criterion holds in the generality of \mathcal{P}^* -rectifiable measures with complemented tangents.

We remark that we are able to prove Theorem 3.1 because of the following two key observations: whenever \mathbb{V} admits a normal complementary subgroup \mathbb{L} , then the projection $P_{\mathbb{V}} : \mathbb{G} \rightarrow \mathbb{V}$ related to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$ is a Lipschitz homogeneous homomorphism, see Proposition 1.48, and moreover \mathbb{V} is a Carnot subgroup, see [29, Remark 2.1]. This allows us to adapt Preiss's machinery in [202] not without some difficulties that are essentially due to the fact that, on the contrary with respect to the Euclidean setting, we do not have a canonical choice of a normal complementary subgroup of \mathbb{V} when there is at least one. We also stress that, for the Marstrand–Mattila rectifiability criterion, the assumption on the strictly positive lower density is necessary already in the Euclidean case, see [202, 5.9].

The hypotheses of Theorem 3.1 are satisfied whenever we have a \mathcal{P}_h^* -rectifiable measure with horizontal tangents. Hence, the previous Marstrand–Mattila rectifiability criterion can be used to prove the following Preiss-type theorem for one-dimensional rectifiable measures in \mathbb{H}^1 endowed with the Korányi norm. For the sake of clarity, let us recall that if we identify $\mathbb{H}^1 \cong \mathbb{R}^3 = \{(x, t) : x \in \mathbb{R}^2, t \in \mathbb{R}\}$ through exponential coordinates, then the *Korányi norm* is $\|(x, t)\| := (\|x\|_{\text{eu}}^4 + t^2)^{1/4}$, where $\|\cdot\|_{\text{eu}}$ is the standard Euclidean norm.

Theorem 3.2 (One-dimensional Preiss's theorem in \mathbb{H}^1). *Let \mathbb{H}^1 be the first Heisenberg group endowed with the Korányi norm. Let ϕ be a Radon measure on \mathbb{H}^1 such that the one-density $\Theta^1(\phi, x)$ exists positive and finite for ϕ -almost every $x \in \mathbb{H}^1$.*

Then \mathbb{H}^1 can be covered ϕ -almost all with countably many images $\Phi_i(A_i)$ of Lipschitz functions $\Phi_i : A_i \subseteq \mathbb{R} \rightarrow \mathbb{H}^1$, and moreover ϕ is absolutely continuous with respect to the one-dimensional Hausdorff measure \mathcal{H}^1 .

Proof of Theorem 3.2. From the fact that the one-density exists at ϕ -almost every $x \in \mathbb{H}^1$ we deduce that at ϕ -almost every $x \in \mathbb{H}^1$ the tangent measures are uniform measures. The argument to obtain the latter assertion is classical, see [89, Proposition 3.4], or cf. [181, Proposition 2.2]. Then from [80, Theorem 1.3] we get that the tangent measures, at ϕ -almost every $x \in \mathbb{H}^1$, are $\mathcal{S}^1 \llcorner L$, where L is a horizontal line. Finally from Theorem 3.1, since every horizontal line admits a normal complementary subgroup, we get the first part of the sought conclusion. The absolute continuity is a consequence of Proposition 1.56. \square

Let us notice that Theorem 3.2 is one of the few cases in which Preiss’s theorem [202] is nowadays known to hold beyond the Euclidean space. The characterization of the k -rectifiability of a measure through the existence of the k -density in Euclidean spaces was one of the great achievement of Geometric Measure Theory, see [202]. Another Preiss’s type result has been proved by A. Lorent [159] in ℓ_∞^3 . Recently, Merlo has accomplished to prove the analogue of Theorem 3.2 for the three-density, which requires a deeper understanding of three-uniform measures in the first Heisenberg group \mathbb{H}^1 , see [180, 181].

A result related to Theorem 3.2 in the broad generality of metric spaces is contained in [203]. Nevertheless we stress that here we prove Theorem 3.2 in the general setting of Radon measures and we ask no bound on the density, just its existence: namely, we prove that whenever the one-density of a Radon measure exists on \mathbb{H}^1 endowed with the Korányi norm, hence we have that it is one-rectifiable à la Federer. We remark that, even if we take advantage of the fact that the classification of the one-uniform measures on \mathbb{H}^1 was known from [80], the result in Theorem 3.2 is non-trivial by using our approach, since it requires the Marstrand–Mattila rectifiability criterion in Theorem 3.1.

We stress that very recently David Bate in [46] has proved a general Marstrand–Mattila rectifiability criterion to deduce the rectifiability à la Federer of a set in a complete metric space by means of the existence of possibly rotating tangents that are bi-Lipschitz to Euclidean spaces. As kindly pointed out to the author of this thesis by Bate, this criterion will give the one-rectifiability Preiss’s theorem in arbitrary complete metric spaces, see [43], thus generalizing Theorem 3.2.

Let us end this introductory part, by giving a complete picture of measures with density in the first Heisenberg group \mathbb{H}^1 . If in \mathbb{H}^1 endowed with the Korányi norm we have a Radon measure ϕ such that there exists $\alpha \geq 0$ for which the α -density $\Theta^\alpha(\phi, x)$ exists positive and finite for ϕ -almost every $x \in \mathbb{H}^1$ we first get that α is an integer, see [77, Theorem 1.1]. Thus the only non-trivial cases are

- $\alpha = 1$. In this case ϕ is \mathcal{P}_1 -rectifiable, see Theorem 3.1. Moreover we can cover ϕ -almost all of \mathbb{H}^1 with countably many images of Lipschitz maps from subsets of \mathbb{R} to \mathbb{H}^1 .

Notice that we can improve the latter conclusion. Indeed, we can cover ϕ -almost all of \mathbb{H}^1 with countably many images of $C_{\mathbb{H}}^1$ -functions defined on open subsets of \mathbb{R} to \mathbb{H}^1 . This last improvement comes from Pansu–Rademacher theorem for Lipschitz maps between Carnot groups, see [200], and the Whitney extension theorem proved in [130, Theorem 6.5].

- $\alpha = 2$. In this case ϕ is \mathcal{P}_2 -rectifiable, see [180, Theorem 3.7]. This means that the tangent measure is ϕ -almost everywhere unique and it is a Haar measure of the vertical line in \mathbb{H}^1 .
- $\alpha = 3$. In this case ϕ is \mathcal{P}_3 -rectifiable, see [181], and [180, Theorem 4]. Moreover we can cover ϕ -almost all of \mathbb{H}^1 with countably many $C_{\mathbb{H}}^1$ -hypersurfaces, see [180, Theorem 4].

As it is clear from the list above, an interesting line of investigation could be a finer study of the structure of \mathcal{P}_2 -rectifiable measures in \mathbb{H}^1 .

We end this introductory part by stressing here the following immediate corollary of Theorem 2.30, Theorem 3.1, and the main result in [29], which is a Rademacher theorem for intrinsically Lipschitz functions with normal target. The following Corollary 3.3 gives a rather complete picture of the rectifiability in the co-normal case in arbitrary Carnot groups, that in turn becomes very similar to the Euclidean one. Compare also with Remark 2.40.

Corollary 3.3. *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance d . Let $h \in \{1, \dots, Q\}$, and let $\Gamma \subseteq \mathbb{G}$ be a Borel set such that $0 < \mathcal{S}^h(\Gamma) < +\infty$, where \mathcal{S}^h is the h -dimensional spherical Hausdorff measure. Then the following are equivalent*

- (1) $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h^* -rectifiable measure with tangents complemented by at least one normal subgroup, or, in other words, a $\mathcal{P}_h^{*, \triangleleft}$ -rectifiable measure, see Definition 1.61,
- (2) $\mathcal{S}^h \llcorner \Gamma$ is a \mathcal{P}_h -rectifiable measure with tangents complemented by at least one normal subgroup,
- (3) For $\mathcal{S}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$ we have

$$\text{Tan}(\mathcal{S}^h \llcorner \Gamma, x) = \{\lambda \mathcal{S}^h \llcorner \mathbb{V}(x) : \lambda > 0, \mathbb{V}(x) \text{ is a homogeneous subgroup}$$

of \mathbb{G} that admits at least one normal compl. subgroup, $\dim_{\text{hom}} \mathbb{V}(x) = h\}$,

- (4) There are countably many compact intrinsic graphs Γ_i that are h -dimensional intrinsically differentiable graphs at \mathcal{S}^h -almost every $x \in \Gamma_i$ (see Definition 1.94), that have Hausdorff tangents complemented by at least one normal subgroup at \mathcal{S}^h -almost every $x \in \Gamma_i$, and such that

$$\mathcal{S}^h(\Gamma \setminus \bigcup_{i=1}^{+\infty} \Gamma_i) = 0,$$

- (5) There are countably many compact intrinsic graphs Γ_i that are graphs of intrinsically Lipschitz functions $\varphi_i : U_i \subseteq \mathbb{V}_i \rightarrow \mathbb{L}_i$, where \mathbb{L}_i is a normal subgroup, $\mathbb{V}_i, \mathbb{L}_i$ are homogeneous complementary subgroup, $\dim_{\text{hom}} \mathbb{V}_i = h$, and such that

$$\mathcal{S}^h(\Gamma \setminus \bigcup_{i=1}^{+\infty} \Gamma_i) = 0.$$

Moreover, denoting with \mathcal{C}^h the centered Hausdorff measure of dimension h , see Definition 1.1, if any of the previous holds, then $\Theta^h(\mathcal{C}^h \llcorner \Gamma, x) = 1$ exists for $\mathcal{C}^h \llcorner \Gamma$ -almost every $x \in \mathbb{G}$, the tangent $\mathbb{V}(x)$ is unique \mathcal{S}^h -almost everywhere, and

$$r^{-h}(T_{x,r})_*(\mathcal{C}^h \llcorner \Gamma) \rightarrow \mathcal{C}^h \llcorner \mathbb{V}(x), \quad \text{as } r \rightarrow 0, \text{ for } \mathcal{C}^h \llcorner \Gamma\text{-almost every } x \in \mathbb{G},$$

where the convergence of measures is meant in the duality with $C_c(\mathbb{G})$.

1. Proof

This section is devoted to the proof of the following result, which is a restatement of the main result in Theorem 3.1. From now on, if not otherwise specified, \mathbb{G} will be a fixed Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces a homogeneous left-invariant distance d . Moreover, h will be a natural number in the set $\{1, \dots, Q\}$.

Theorem 3.4 (Co-normal Marstrand–Mattila rectifiability criterion). *Let \mathbb{G} be a Carnot group of homogeneous dimension Q endowed with a homogeneous norm $\|\cdot\|$ that induces an arbitrary left-invariant homogeneous distance. Let $h \in \{1, \dots, Q\}$, and let ϕ be a $\mathcal{P}_h^{*, \triangleleft}$ -rectifiable measure. Then there are countably many $\mathbb{W}_i \in \text{Gr}_{\triangleleft}(h)$, that are in addition Carnot subgroups, compact sets $K_i \subseteq \mathbb{W}_i$, and Lipschitz functions $f_i : K_i \rightarrow \mathbb{G}$ such that*

$$\phi \left(\mathbb{G} \setminus \bigcup_{i \in \mathbb{N}} f_i(K_i) \right) = 0.$$

As a consequence, ϕ is \mathcal{P}_h^c -rectifiable.

We stress again, as done right after Theorem 3.1, that passing from a \mathcal{P}^* -rectifiable measure to a \mathcal{P} -rectifiable measure is by far non-trivial. We now briefly discuss the strategy of the proof of Theorem 3.4, which is ultimately an adaptation of Preiss's technique in [202, Section 4.4(4), Lemma 5.2, Theorem 5.3, and Corollary 5.4] in our setting, see Proposition 1.75,

Proposition 3.15, and Proposition 3.18, respectively. In particular we show that whenever a Radon measure satisfies precise structure conditions, see the hypotheses of Proposition 3.15, that are always verified whenever ϕ is \mathcal{P}_h^* -rectifiable with tangents that admit at least one normal complementary subgroup, see Proposition 3.17, then it is possible to find a Lipschitz function $f : K \subseteq \mathbb{V} \rightarrow \mathbb{G}$, with a Carnot subgroup $\mathbb{V} \in Gr_{\leq}(h)$, such that $\phi(f(K)) > 0$. Then, by a classical measure theoretic argument, this implies that \mathbb{G} can be covered ϕ -almost all with $\cup_{i \in \mathbb{N}} f_i(K_i)$, where $f_i : K_i \subseteq \mathbb{V}_i \rightarrow \mathbb{G}$ are Lipschitz functions, and \mathbb{V}_i are Carnot subgroups in $Gr_{\leq}(h)$, see the first part of the proof of Theorem 3.4.

The last part of Theorem 3.4 is reached from the first part and the following key observation: if a homogeneous subgroup of a Carnot group admits a normal complementary subgroup, then it is a Carnot subgroup, see [29, Remark 2.1]. Thus the maps f_i are Lipschitz maps between Carnot groups and we can apply Pansu–Rademacher theorem, see [200], Magnani’s area formula, see [161], and a classical argument (compare with [178]) to conclude that $\mathcal{S}^h \llcorner f_i(K_i)$ is a \mathcal{P}_h^c -rectifiable measure, see the last part of the proof of Theorem 3.4. From this latter observation, the proof of Theorem 3.4 is concluded.

Proof of Theorem 3.1. It is an immediate consequence of Theorem 3.4. \square

Memorandum: Throughout all this chapter we let \mathbb{G} be a Carnot group of homogeneous dimension Q equipped with the box norm introduced in Definition 1.16. This does not result in a loss of generality since our aim is to prove Theorem 3.4 that is clearly independent on the choice of the particular homogeneous norm $\|\cdot\|$ that induces a left-invariant homogeneous distance on \mathbb{G} , since all left-invariant homogeneous distances are bi-Lipschitz equivalent on \mathbb{G} . So, from now on, we may suppose that \mathbb{G} is endowed with the left-invariant homogeneous distance induced by the box norm introduced in Definition 1.16.

1.1. Rigidity of the stratification of \mathcal{P}_h^* -rectifiable measures. In this section we show that in a Carnot group of homogeneous dimension Q , every \mathcal{P}_h^* -rectifiable measure, with $1 \leq h \leq Q$, is such that, at almost every point, the possible tangents, even if different, all share the same stratification vector.

We let $\varphi : \mathbb{G} \rightarrow [0, 1]$ be a positive, smooth, radially symmetric function with respect to $\|\cdot\|$ (see the discussion before Proposition 1.29), supported in $\overline{B}(0, 2)$, and such that $\varphi \equiv 1$ on $\overline{B}(0, 1)$. We shall denote by g its profile function, that is defined right above the statement of Proposition 1.29. Let us recall that $\mathfrak{s}(\cdot)$ denotes the stratification vector, see Definition 1.20, and $\mathfrak{S}(h)$ denotes all the possible stratification vectors of the homogeneous subgroups of homogeneous dimension h .

Proposition 3.5. *For every $h \in \{1, \dots, Q\}$ there exists a constant $\mathfrak{J}(\varphi, \mathbb{G}, h) = \mathfrak{J} > 0$ such that for every $\mathbb{V} \in Gr(h)$ and every $\mathfrak{s} \in \mathfrak{S}(h) \setminus \{\mathfrak{s}(\mathbb{V})\}$, we have*

$$\inf_{\substack{\mathbb{W} \in Gr(h) \\ \mathfrak{s}(\mathbb{W}) = \mathfrak{s}}} \int \varphi(z) \text{dist}(z, \mathbb{W}) d\mathcal{C}^h \llcorner \mathbb{V} > \mathfrak{J},$$

where the stratification vector $\mathfrak{s}(\cdot)$ was introduced in Definition 1.20.

Proof. Suppose by contradiction this is not the case. Thus there are two sequences $\{\mathbb{W}_i\} \subseteq Gr(h)$ and $\{\mathbb{V}_i\} \subseteq Gr(h)$ such that for every $i \in \mathbb{N}$ we have $\mathfrak{s}(\mathbb{W}_i) \neq \mathfrak{s}(\mathbb{V}_i)$ and

$$(3.1) \quad \int \varphi(z) \text{dist}(z, \mathbb{W}_i) d\mathcal{C}^h \llcorner \mathbb{V}_i \leq 1/i.$$

Thanks to the pigeonhole principle and the fact that $\mathfrak{S}(h)$, see Definition 1.20, is a finite set we can assume up to passing to a non re-labelled subsequence that

$$\mathfrak{s}(\mathbb{W}_i) = \mathfrak{s}_1 \neq \mathfrak{s}_2 = \mathfrak{s}(\mathbb{V}_i), \quad \text{for every } i \in \mathbb{N}.$$

Furthermore, thanks to Proposition 1.22, we can also assume, up to passing to a non re-labelled subsequence, that

$$\mathbb{W}_i \xrightarrow{d_{\mathbb{G}}} \mathbb{W} \in Gr(h), \quad \text{and} \quad \mathbb{V}_i \xrightarrow{d_{\mathbb{G}}} \mathbb{V} \in Gr(h).$$

Furthermore, thanks to Proposition 1.23, we also deduce that

$$\mathfrak{s}(\mathbb{W}) = \mathfrak{s}_1 \neq \mathfrak{s}_2 = \mathfrak{s}(\mathbb{V}).$$

In order to conclude the proof of the proposition we first note for every $\mathbb{U} \in Gr(h)$ and every $R > 0$, if $z \in \overline{B}(0, R)$, then every element $u \in \mathbb{U}$ for which $\text{dist}(z, \mathbb{U}) = d(u, z)$ is contained in $\overline{B}(0, 2R)$. Hence, the same argument as in (1.22) and (1.24) allows us to conclude that for every $z \in \overline{B}(0, 2)$ the following inequality holds

$$(3.2) \quad \text{dist}(z, \mathbb{W}_i) \geq \text{dist}(z, \mathbb{W}) - 8d_{\mathbb{G}}(\mathbb{W}, \mathbb{W}_i), \quad \text{for all } i \in \mathbb{N}.$$

Putting together (3.1) and (3.2) thanks to Proposition 1.29 we infer

$$(3.3) \quad \begin{aligned} 1/i &\geq \int \varphi(z) \text{dist}(z, \mathbb{W}_i) d\mathcal{C}^h \llcorner \mathbb{V}_i \geq \int \varphi(z) \text{dist}(z, \mathbb{W}) d\mathcal{C}^h \llcorner \mathbb{V}_i - 8d_{\mathbb{G}}(\mathbb{W}, \mathbb{W}_i) \int \varphi(z) d\mathcal{C}^h \llcorner \mathbb{V}_i \\ &= \int \varphi(z) \text{dist}(z, \mathbb{W}) d\mathcal{C}^h \llcorner \mathbb{V}_i - 8d_{\mathbb{G}}(\mathbb{W}, \mathbb{W}_i) h \int s^{h-1} g(s) ds. \end{aligned}$$

Therefore, since $\varphi(z) \text{dist}(z, \mathbb{W})$ is a continuous function with compact support, thanks to Proposition 1.30 and sending i to $+\infty$ in the previous inequality we conclude

$$\int \varphi(z) \text{dist}(z, \mathbb{W}) d\mathcal{C}^h \llcorner \mathbb{V} = 0.$$

In particular $\text{dist}(z, \mathbb{W}) = 0$ for $\mathcal{S}^h \llcorner \mathbb{V}$ -almost every $z \in \mathbb{V}$, and since both $\text{Lie}(\mathbb{V})$ and $\text{Lie}(\mathbb{W})$ are vector subspaces of $\text{Lie}(\mathbb{G})$ we have $\mathbb{V} \subseteq \mathbb{W}$. On the one hand this allows us to infer that

$$\dim(V_i \cap \mathbb{V}) \leq \dim(V_i \cap \mathbb{W}), \quad \text{for every } i \in \{1, \dots, \kappa\},$$

and on the other hand, since $\mathfrak{s}(\mathbb{V}) \neq \mathfrak{s}(\mathbb{W})$, there must exist an $\ell \in \{1, \dots, \kappa\}$ such that $\dim(V_\ell \cap \mathbb{V}) < \dim(V_\ell \cap \mathbb{W})$. This however contradicts the fact that $\mathbb{W} \in Gr(h)$, indeed

$$h = \dim_{\text{hom}} \mathbb{V} = \sum_{i=1}^{\kappa} i \cdot \dim(V_i \cap \mathbb{V}) < \sum_{i=1}^{\kappa} i \cdot \dim(V_i \cap \mathbb{W}) = \dim_{\text{hom}}(\mathbb{W}).$$

□

Proposition 3.6. *Let $\mathfrak{s} \in \mathfrak{S}(h)$. For every Radon measure ψ we define*

$$\mathcal{F}_{\mathfrak{s}}(\psi) := \inf_{\substack{\mathbb{W} \in Gr(h) \\ \mathfrak{s}(\mathbb{W}) = \mathfrak{s}}} \int \varphi(z) \text{dist}(z, \mathbb{W}) d\psi.$$

Then, the functional $\mathcal{F}_{\mathfrak{s}}$ on Radon measures is continuous with respect to the weak topology in the duality with the functions with compact support on \mathbb{G} .*

Proof. Let $\psi_i \rightharpoonup \psi$ and note that for every $\mathbb{V} \in Gr(h)$ for which $\mathfrak{s}(\mathbb{V}) = \mathfrak{s}$, we have

$$(3.4) \quad \lim_{i \rightarrow +\infty} \int \varphi(z) \text{dist}(z, \mathbb{V}) d\psi_i = \int \varphi(z) \text{dist}(z, \mathbb{V}) d\psi,$$

since $\varphi(z) \text{dist}(z, \mathbb{V})$ is a continuous function with compact support. Let us first prove that

$$\mathcal{F}_{\mathfrak{s}}(\psi) \leq \liminf_{i \rightarrow \infty} \mathcal{F}_{\mathfrak{s}}(\psi_i).$$

Indeed, if by contradiction $\mathcal{F}_{\mathfrak{s}}(\psi) > \liminf_{i \rightarrow \infty} \mathcal{F}_{\mathfrak{s}}(\psi_i)$, up to passing to a non re-labelled subsequence in i that realizes the lim inf and up to choosing a quasi-minimizer for $\mathcal{F}_{\mathfrak{s}}(\psi_i)$, we can find $\delta > 0$, and $\mathbb{W}_i \in Gr(h)$ with $\mathfrak{s}(\mathbb{W}_i) = \mathfrak{s}$ such that

$$(3.5) \quad \mathcal{F}_{\mathfrak{s}}(\psi) > \int \varphi(z) \text{dist}(z, \mathbb{W}_i) d\psi_i + \delta, \quad \text{for all } i \in \mathbb{N}.$$

We can assume that $\mathbb{W}_i \rightarrow \mathbb{W} \in Gr(h)$, with $\mathfrak{s}(\mathbb{W}) = \mathfrak{s}$, up to a non re-labelled subsequence, see Proposition 1.22 and Proposition 1.23. Thus since $\psi_i \rightarrow \psi$ passing to the limit the right hand side of (3.5)¹ we obtain $\mathcal{F}_{\mathfrak{s}}(\psi) > \int \varphi(z) \text{dist}(z, \mathbb{W}) d\psi$, that is a contradiction with the definition of $\mathcal{F}_{\mathfrak{s}}$. The proof of the proposition is concluded if we prove that

$$\mathcal{F}_{\mathfrak{s}}(\psi) \geq \limsup_{i \rightarrow \infty} \mathcal{F}_{\mathfrak{s}}(\psi_i).$$

In order to prove the previous inequality let us fix $\varepsilon > 0$ and $\mathbb{V}_{\varepsilon} \in Gr(h)$ with $\mathfrak{s}(\mathbb{V}_{\varepsilon}) = \mathfrak{s}$ such that

$$(3.6) \quad \int \varphi(z) \text{dist}(z, \mathbb{V}_{\varepsilon}) d\psi - \varepsilon \leq \mathcal{F}_{\mathfrak{s}}(\psi).$$

Putting together (3.4) and (3.6), we infer

$$(3.7) \quad \begin{aligned} \limsup_{i \rightarrow \infty} \mathcal{F}_{\mathfrak{s}}(\psi_i) - \varepsilon &\leq \limsup_{i \rightarrow \infty} \int \varphi(z) \text{dist}(z, \mathbb{V}_{\varepsilon}) d\psi_i - \varepsilon \\ &= \int \varphi(z) \text{dist}(z, \mathbb{V}_{\varepsilon}) d\psi - \varepsilon \leq \mathcal{F}_{\mathfrak{s}}(\psi). \end{aligned}$$

The arbitrariness of ε concludes the limsup inequality and the proof of the proposition. \square

Definition 3.7. For every $\mathcal{T} \subseteq \mathfrak{M}(h)$, where we recall that $\mathfrak{M}(h)$ is the set of h -flat measures, we define $\mathfrak{s}(\mathcal{T})$ to be the set

$$\mathfrak{s}(\mathcal{T}) := \{\mathfrak{s}(\mathbb{V}) : \text{there exists a non-null Haar measure of } \mathbb{V} \text{ in } \mathcal{T}\}.$$

Namely we are considering all the possible stratification vectors of the homogeneous subgroups that are the support of some element of \mathcal{T} .

In the following theorem we prove that whenever we have a \mathcal{P}_h^* -rectifiable measure, at almost every point the tangents have the same stratification vector.

Theorem 3.8. *Assume ϕ is a \mathcal{P}_h^* -rectifiable measure. Then, for ϕ -almost every $x \in \mathbb{G}$ the set $\mathfrak{s}(\text{Tan}_h(\phi, x)) \subseteq \mathfrak{S}(h)$ is a singleton.*

Remark 3.9. In the notation of the above proposition, since for ϕ -almost every $x \in \mathbb{G}$ we have $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)$, the set $\mathfrak{s}(\text{Tan}_h(\phi, x))$ is well defined ϕ -almost everywhere.

Proof. Suppose by contradiction there exists a point $x \in \mathbb{G}$ where

- (i) $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$,
- (ii) $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)$,
- (iii) there are $\mathbb{V}_1, \mathbb{V}_2 \in Gr(h)$ with $\mathfrak{s}(\mathbb{V}_1) \neq \mathfrak{s}(\mathbb{V}_2)$ and $\lambda_1, \lambda_2 \geq 0$ such that

$$\lambda_1 \mathcal{C}_h^h \llcorner \mathbb{V}_1, \lambda_2 \mathcal{C}_h^h \llcorner \mathbb{V}_2 \in \text{Tan}_h(\phi, x).$$

Assume that $\{r_i\}_{i \in \mathbb{N}}$ and $\{s_i\}_{i \in \mathbb{N}}$ are two infinitesimal sequences such that $r_i \leq s_i$ and for which

$$\frac{T_{x, r_i} \phi}{r_i^h} \rightarrow \lambda_1 \mathcal{C}_h^h \llcorner \mathbb{V}_1, \quad \text{and} \quad \frac{T_{x, s_i} \phi}{s_i^h} \rightarrow \lambda_2 \mathcal{C}_h^h \llcorner \mathbb{V}_2.$$

¹Setting $f_i(z) := \varphi(z) \text{dist}(z, \mathbb{W}_i)$ and $f(z) := \varphi(z) \text{dist}(z, \mathbb{W})$ we notice that $f_i \rightarrow f$ uniformly on $\overline{B}(0, 2)$ since $\mathbb{W}_i \rightarrow \mathbb{W}$. Thus $|\int f d\psi - \int f_i d\psi_i| \leq |\int f d\psi - \int f d\psi_i| + |\int f d\psi_i - \int f_i d\psi_i|$ and the limit is zero because $\psi_i \rightarrow \psi$, $\sup_i \psi_i(\overline{B}(0, 2)) < +\infty$ and $f_i \rightarrow f$ uniformly on $\overline{B}(0, 2)$.

Note that thanks to Lemma 1.62, we have in particular that $\Theta_*^h(\phi, x) \leq \lambda_1, \lambda_2 \leq \Theta^{h,*}(\phi, x)$. Throughout the rest of the proof we let $\mathfrak{s} := \mathfrak{s}(\mathbb{V}_1)$ and we define

$$f(r) := \inf_{\substack{\mathbb{W} \in Gr(h) \\ \mathfrak{s}(\mathbb{W}) = \mathfrak{s}}} \int \varphi(z) \text{dist}(z, \mathbb{W}) d \frac{T_{x,r}\phi}{r^h}.$$

Thanks to Proposition 3.5 and Proposition 3.6 we infer that the function f is continuous on $(0, \infty)$ and that

$$\lim_{i \rightarrow \infty} f(r_i) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} f(s_i) > \mathfrak{J}\lambda_2 \geq \mathfrak{J}\Theta_*^h(\phi, x).$$

Let us choose, for i sufficiently large, $\sigma_i \in [r_i, s_i]$ in such a way that $f(\sigma_i) = \mathfrak{J}\Theta_*^h(\phi, x)/2$ and $f(r) \leq \mathfrak{J}\Theta_*^h(\phi, x)/2$ for every $r \in [r_i, \sigma_i]$. Up to passing to a non re-labelled subsequence, since ϕ is \mathcal{P}_h^* -rectifiable, we can assume that $\sigma_i^{-h} T_{x, \sigma_i} \phi \rightarrow \lambda_3 \mathcal{C}^h \llcorner \mathbb{V}_3$ for some $\lambda_3 > 0$ and some $\mathbb{V}_3 \in Gr(h)$. Thanks to Lemma 1.62, we infer that $\Theta_*^h(\phi, x) \leq \lambda_3 \leq \Theta^{h,*}(\phi, x)$ and thanks to the continuity of the functional $\mathcal{F}_\mathfrak{s}$ in Proposition 3.6, we conclude that

$$(3.8) \quad \mathfrak{J}\Theta_*^h(\phi, x)/2 = \lim_{i \rightarrow \infty} f(\sigma_i) = \lim_{i \rightarrow \infty} \mathcal{F}_\mathfrak{s}(\sigma_i^{-h} T_{x, \sigma_i} \phi) = \lambda_3 \mathcal{F}_\mathfrak{s}(\mathcal{C}^h \llcorner \mathbb{V}_3).$$

The chain of identities (3.8) together with the bounds on λ_3 imply

$$(3.9) \quad 0 < \mathfrak{J}\Theta_*^h(\phi, x)/2\Theta^{h,*}(\phi, x) \leq \mathcal{F}_\mathfrak{s}(\mathcal{C}^h \llcorner \mathbb{V}_3) \leq \mathfrak{J}/2.$$

Since $\mathbb{V}_3 \in Gr(h)$, (3.9) on the one hand implies by means of Proposition 3.5 that $\mathfrak{s}(\mathbb{V}_3) = \mathfrak{s}$. On the other hand, since $\mathcal{F}_\mathfrak{s}(\mathcal{C}^h \llcorner \mathbb{V}_3) > 0$, we have that $\mathfrak{s}(\mathbb{V}_3) \neq \mathfrak{s}$, resulting in a contradiction. \square

Definition 3.10. Assume ϕ is a \mathcal{P}_h^* -rectifiable measure. Recall that κ is the step of the group \mathbb{G} . For every $x \in \mathbb{G}$ we define the map $\mathfrak{s}(\phi, x) \in \mathbb{N}^\kappa$ in the following way

$$\mathfrak{s}(\phi, x) := \begin{cases} \mathfrak{s} & \text{if } \text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h) \text{ and } \mathfrak{s}(\text{Tan}_h(\phi, x)) \text{ is the singleton } \{\mathfrak{s}\}, \\ 0^\kappa & \text{otherwise,} \end{cases}$$

where 0^κ denotes the κ -tuple vector with 0 entries.

Remark 3.11. The map $\mathfrak{s}(\phi, \cdot)$ is well defined and different from 0^κ ϕ -almost everywhere thanks to Theorem 3.8.

Proposition 3.12. Assume ϕ is a \mathcal{P}_h^* -rectifiable measure. Then, the map $x \mapsto \mathfrak{s}(\phi, x)$ is ϕ -measurable, where on \mathbb{N}^κ we consider the discrete topology.

Proof. Let $\tilde{h}_\mathbb{G}$ be the constant introduced in Proposition 1.23. Let us first prove that there exists $\tilde{\alpha} := \tilde{\alpha}(\mathbb{G})$ such that the following assertion holds

$$(3.10) \quad \text{for every } 1 \leq h \leq Q \text{ and for every } \mathbb{V}, \mathbb{W} \in Gr(h), \text{ if } \mathbb{V} \subseteq C_{\mathbb{W}}(\tilde{\alpha}), \text{ then } d_\mathbb{G}(\mathbb{V}, \mathbb{W}) \leq \tilde{h}_\mathbb{G}.$$

Indeed, if this was not the case, we can find an $1 \leq h \leq Q$ and sequences $\{\mathbb{V}_i\}, \{\mathbb{W}_i\}$ in $Gr(h)$ such that $\mathbb{V}_i \subseteq C_{\mathbb{W}_i}(i^{-1})$ and for which $d_\mathbb{G}(\mathbb{V}_i, \mathbb{W}_i) > \tilde{h}_\mathbb{G}$, for all $i \in \mathbb{N}$. Thus, up to non re-labelled subsequences, we can assume that $\mathbb{V}_i \rightarrow \mathbb{V}$ and $\mathbb{W}_i \rightarrow \mathbb{W}$, for some $\mathbb{V}, \mathbb{W} \in Gr(h)$, thanks to Proposition 1.22. Thanks to the aforementioned convergences and the fact that $\mathbb{V}_i \subseteq C_{\mathbb{W}_i}(i^{-1})$ for every $i \in \mathbb{N}$ we deduce that $\mathbb{V} \subseteq \mathbb{W}$ and thus $\mathbb{V} = \mathbb{W}$ since they both have homogeneous dimension h . But this latter equality is readily seen to be in contradiction with the fact that $d_\mathbb{G}(\mathbb{V}_i, \mathbb{W}_i) > \tilde{h}_\mathbb{G}$, for all $i \in \mathbb{N}$, since $\mathbb{W}_i \rightarrow \mathbb{W}$ and $\mathbb{V}_i \rightarrow \mathbb{V}$.

The proof will be similar to the proof of Lemma 1.83, so we will not give complete details. Let $\{\mathbb{V}_\ell\}_{\ell=1, \dots, N}$ be a finite $\tilde{\alpha}/3$ -dense set in $Gr(h)$, where $\tilde{\alpha}$ is defined above. For every $r \in (0, 1) \cap \mathbb{Q}$ and $\ell = 1, \dots, N$ we define the functions on \mathbb{G}

$$f_{r, \ell}(x) := r^{-h} \phi(\{w \in \overline{B}(x, r) : \text{dist}(x^{-1}w, \mathbb{V}_\ell) \geq \tilde{\alpha} \|x^{-1}w\|\}) =: r^{-h} \phi(I(x, r, \ell)).$$

We claim that the functions $f_{r,\ell}$ are upper semicontinuous. Indeed, one can run the very same argument starting with (1.69) in Lemma 1.83. This implies that the function

$$f_\ell := \liminf_{r \in \mathbb{Q}, r \rightarrow 0} f_{r,\ell},$$

is ϕ -measurable and as a consequence, since $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)$ for ϕ -almost every $x \in \mathbb{G}$, we infer that the set

$$B_\ell := \{x \in \mathbb{G} : f_\ell(x) = 0\} \cap \{x \in \mathbb{G} : \text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)\},$$

is ϕ -measurable as well. If we prove that for ϕ -almost every $x \in B_\ell$ there exists a non-zero Haar measure ν in $\text{Tan}_h(\phi, x)$ relative to a homogeneous subgroup \mathbb{V} of \mathbb{G} such that $d_{\mathbb{G}}(\mathbb{V}, \mathbb{V}_\ell) \leq \tilde{h}_{\mathbb{G}}$, we infer that

$$(3.11) \quad \mathfrak{s}(\text{Tan}_h(\phi, x)) = \{\mathfrak{s}(\mathbb{V}_\ell)\}, \quad \text{for } \phi\text{-almost every } x \in B_\ell,$$

and thus $\mathfrak{s}(\phi, x) = \mathfrak{s}(\mathbb{V}_\ell)$ for ϕ -almost every $x \in B_\ell$. Indeed, if we are able to find such a measure ν relative to \mathbb{V} , (3.11) is an immediate consequence of the fact that if $d_{\mathbb{G}}(\mathbb{V}, \mathbb{V}_\ell) \leq \tilde{h}_{\mathbb{G}}$, Proposition 1.23 implies that \mathbb{V} and \mathbb{V}_ℓ have the same stratification vector; and the fact that, from Theorem 3.8, ϕ -almost everywhere the tangent subgroups have the same stratification.

In order to construct such a non-zero Haar measure ν , we fix a point $x \in B_\ell$ in the ϕ -full-measure subset of B_ℓ such that the following conditions hold

- (i) $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$,
- (ii) $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h)$,

and we let $\{r_i\}_{i \in \mathbb{N}}$ be an infinitesimal sequence of rational numbers such that $\lim_{i \rightarrow \infty} f_{r_i, \ell}(x) = 0$.

Thanks to item (i) above and the compactness of measures, see [12, Proposition 1.59], we can find a non re-labelled subsequence of r_i such that

$$r_i^{-h} T_{x, r_i} \phi \rightharpoonup \nu.$$

Such a ν belongs by definition to $\text{Tan}_h(\phi, x)$ and thus there is a $\lambda > 0$ and a $\mathbb{V} \in Gr(h)$ such that $\nu = \lambda \mathcal{C}^h \llcorner \mathbb{V}$. Thanks to [89, Proposition 2.7], and arguing as in (1.72), we conclude in particular that

$$\mathbb{V} \subseteq \{w \in \mathbb{G} : \text{dist}(w, \mathbb{V}_\ell) \leq \tilde{\alpha} \|w\|\} = C_{\mathbb{V}_\ell}(\tilde{\alpha}),$$

and then, from (3.10) we conclude that $d_{\mathbb{G}}(\mathbb{V}, \mathbb{V}_\ell) \leq \tilde{h}_{\mathbb{G}}$, that was what we wanted to prove.

An immediate consequence of (3.11) is that

$$(3.12) \quad \text{if } \ell, m \in \{1, \dots, N\} \text{ and } \mathfrak{s}(\mathbb{V}_\ell) \neq \mathfrak{s}(\mathbb{V}_m) \text{ then } \phi(B_\ell \cap B_m) = 0.$$

On the other hand, the B_ℓ 's cover ϕ -almost all \mathbb{G} . To prove this latter assertion, we note that since ϕ is \mathcal{P}_h^* -rectifiable, for ϕ -almost all $x \in \mathbb{G}$ there is an infinitesimal sequence $r_i \rightarrow 0$, a $\lambda > 0$ and a $\mathbb{V} \in Gr(h)$ such that $r_i^{-h} T_{x, r_i} \phi \rightharpoonup \lambda \mathcal{C}^h \llcorner \mathbb{V}$. Since the set $\{\mathbb{V}_\ell : \ell = 1, \dots, N\}$ is $\tilde{\alpha}/3$ -dense in $Gr(h)$, there must exist an $\ell \in \{1, \dots, N\}$ such that

$$(3.13) \quad \mathbb{V} \subseteq \{w \in \mathbb{G} : \text{dist}(w, \mathbb{V}_\ell) < \tilde{\alpha} \|w\|\}.$$

This last inclusion follows since there exists ℓ such that $d_{\mathbb{G}}(\mathbb{V}, \mathbb{V}_\ell) \leq \tilde{\alpha}/3$ and the observation that every point in $\partial \overline{B}(0, 1) \cap \mathbb{V}$ is such that every point at minimum distance of it from \mathbb{V}_ℓ is in $\overline{B}(0, 2) \cap \mathbb{V}_\ell$. The previous inclusion, jointly with [89, Proposition 2.7], and arguing with (1.73), implies that $f_\ell(x) = 0$. This proves that $x \in B_\ell$ and as a consequence that the B_ℓ 's cover ϕ -almost all \mathbb{G} .

We are ready to prove the measurability of the map $x \mapsto \mathfrak{s}(\phi, x)$. Fix an $\mathfrak{s} \in \mathfrak{S}(h)$ and let $D(\mathfrak{s}) := \{x \in \mathbb{G} : \mathfrak{s}(\phi, x) = \mathfrak{s}\} \cap \bigcup_{\ell=1}^N B_\ell$. Since by the previous step the B_ℓ 's cover ϕ -almost

all \mathbb{G} we know that $\{x \in \mathbb{G} : \mathfrak{s}(\phi, x) = \mathfrak{s}\} \setminus \bigcup_{\ell=1}^N B_\ell$ is ϕ -null and thus it is ϕ -measurable. Furthermore, thanks to (3.11) and (3.12) we know that up to ϕ -null sets we have

$$D(\mathfrak{s}) = \bigcup_{\mathfrak{s} \in \mathfrak{S}(h)} \{B_\ell : \mathfrak{s}(\mathbb{V}_\ell) = \mathfrak{s}\}.$$

Since the sets B_ℓ are ϕ -measurable, this concludes the proof that $\{x \in \mathbb{G} : \mathfrak{s}(\phi, x) = \mathfrak{s}\}$ is ϕ -measurable for every $\mathfrak{s} \in \mathfrak{S}(h)$, taking also into account that $\mathfrak{s}(\phi, \cdot)^{-1}(0)$ is ϕ -null. \square

1.2. Proof of Theorem 3.4. This long and technical subsection is devoted to the core of the proof of Theorem 3.4. The main ingredient for the proof is Proposition 3.18, which is proved exploiting Proposition 3.15 and Proposition 3.17, together with Proposition 3.16.

Definition 3.13. Let $C > 0$ be a real number. Through the rest of this subsection we let

$$C_{10}(C) := 1 + 2/C,$$

and

$$C_{11}(C) := (10(1 + C_{10}))^{2(Q+10)}.$$

Remark 3.14. Let $h \in \{1, \dots, Q\}$, and let $\mathfrak{s} \in \mathfrak{S}(h)$ be fixed, let $C > 0$, and let $\mathbb{V} \in G_{\leq}^{\mathfrak{s}}(h)$ with $\mathfrak{e}(\mathbb{V}) \geq C$, where \mathfrak{e} is defined in (1.35). Let \mathbb{L} be a complementary subgroup of \mathbb{V} and $P := P_{\mathbb{V}}$ the projection on \mathbb{V} related to this splitting. Note that with the previous choices of C_{10} and C_{11} , thanks to Proposition 1.47 and Remark 1.27, we have

$$2(1 + C_{10})^h \mathcal{C}^h(P(\overline{B}(0, 1))) < C_{11}/2^{h+3},$$

since $\mathcal{C}^h(P(\overline{B}(0, 1))) \leq \mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(0, C_{10})) = C_{10}^h$.

Proposition 3.15. Let $h \in \{1, \dots, Q\}$, $\mathfrak{s} \in \mathfrak{S}(h)$, and let \mathcal{G} be a subset of $Gr_{\leq}^{\mathfrak{s}}(h)$ such that there exists a constant $C > 0$ for which

$$\mathfrak{e}(\mathbb{V}) \geq C \text{ for all } \mathbb{V} \in \mathcal{G},$$

where we recall that \mathfrak{e} was defined in (1.35). Further let $r > 0$, $\varepsilon \in (0, 5^{-h-5}C_{11}^{-3h}]$, $r_1 := (1 - \varepsilon/h)r$, and $\mu := 2^{-7}h^{-3}C_{11}^{-5h}\varepsilon^2$, where C_{10} and C_{11} are defined in terms of C in Definition 3.13.

Let ϕ be a Radon measure and let $z \in \text{supp}(\phi)$. We define $Z(z, r_1)$ to be the set of the triplets $(x, s, \mathbb{V}) \in \overline{B}(z, C_{11}r_1) \times (0, C_{11}r] \times Gr_{\leq}^{\mathfrak{s}}(h)$ such that

$$(3.14) \quad \phi(\overline{B}(y, t)) \geq (1 - \varepsilon)(t/C_{11}r)^h \phi(\overline{B}(z, C_{11}r)),$$

whenever $y \in \overline{B}(x, C_{11}s) \cap x\mathbb{V}$ and $t \in [\mu s, C_{11}s]$. We moreover ask that we can find a compact subset E of $\overline{B}(z, C_{11}r_1)$ such that $z \in E$,

$$(3.15) \quad \phi(\overline{B}(z, C_{11}r_1) \setminus E) \leq \mu^{h+1}C_{11}^{-h}\phi(\overline{B}(z, C_{11}r_1)),$$

and such that for every $x \in E$ and every $s \in (0, C_{11}r - d(x, z)]$ there is a $\mathbb{V} \in Gr_{\leq}^{\mathfrak{s}}(h)$ such that $(x, s, \mathbb{V}) \in Z(z, r_1)$. Furthermore we assume that there exists $\mathbb{W} \in \mathcal{G}$ such that $(z, r, \mathbb{W}) \in Z(z, r_1)$, and let us fix \mathbb{L} a normal complementary subgroup of \mathbb{W} such that Proposition 1.48 holds. Let us denote $P := P_{\mathbb{W}}$ the projection on \mathbb{W} related to the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{L}$.

Let us recall that with the notation $T(u, r)$ we mean the cylinder with center $u \in \mathbb{G}$ and radius $r > 0$ related to the projection $P = P_{\mathbb{W}}$, see Definition 1.50. For every $u \in P(\overline{B}(z, r_1))$ let $s(u) \in [0, r]$ be the infimum of the numbers λ with the following property: for every $\lambda < s \leq r$ we have

- (1) $E \cap T(u, s/4h) \neq \emptyset$, and
- (2) $\phi(\overline{B}(z, C_{11}r) \cap T(u, C_{10}s)) \leq \mu^{-h}(s/C_{11}r)^h \phi(\overline{B}(z, C_{11}r))$.

Finally, we define

$$\begin{aligned} (\alpha) \quad A &:= \{u \in P(\overline{B}(z, r_1)) : s(u) = 0\}, \\ (\beta) \end{aligned}$$

$$\begin{aligned} A_1 &:= \left\{ u \in P(\overline{B}(z, r_1)) : s(u) > 0, \text{ and } \phi(\overline{B}(z, C_{11}r) \cap T(u, C_{10}s(u))) \right. \\ &\quad \left. \geq \varepsilon^{-1}(s(u)/C_{11}r)^h \phi(\overline{B}(z, C_{11}r)) \right\} \end{aligned}$$

$$\begin{aligned} (\gamma) \quad A_2 &:= \left\{ u \in P(\overline{B}(z, r_1)) : s(u) > 0, \text{ and } \phi((\overline{B}(z, C_{11}r) \setminus E) \cap T(u, s(u)/4h)) \geq \right. \\ &\quad \left. 2^{-1}(s(u)/4hC_{11}r)^h \phi(\overline{B}(z, C_{11}r)) \right\}. \end{aligned}$$

Then we have

- (i) $s(u) \leq C_{11}h\mu r$ for every $u \in P(\overline{B}(z, r_1))$,
- (ii) The function $u \mapsto s(u)$ is lower semicontinuous on $P(\overline{B}(z, r_1))$ and as a consequence A is compact,
- (iii) $P(\overline{B}(z, r_1)) \subseteq A \cup A_1 \cup A_2$,
- (iv) $\mathcal{C}^h(P(\overline{B}(z, r)) \setminus A) \leq 5^{h+3}C_{11}^{3h}\mathcal{C}^h(P(\overline{B}(0, 1)))\varepsilon r^h$,
- (v) $P(E \cap P^{-1}(A)) = A$, $\mathcal{S}^h(E \cap P^{-1}(A)) > 0$ and there is a constant $\mathfrak{C} > 1$ such that

$$\mathfrak{C}^{-1}\mathcal{S}^h(E \cap P^{-1}(A)) \leq \phi(E \cap P^{-1}(A)) \leq \mathfrak{C}\mathcal{S}^h(E \cap P^{-1}(A)).$$

Proof. We prove each point of the proposition in a separate paragraph. For the sake of notation we write $Z := Z(z, r_1)$, and without loss of generality we will always assume that $z = 0$, since $P_{\mathbb{W}}$ is a homogeneous homomorphism, see Proposition 1.48, and thus the statement is left-invariant. Since it will be used here and there in the proof, we estimate $\phi(\overline{B}(0, C_{11}r) \setminus \overline{B}(0, C_{11}r_1))$. Since $(0, r, \mathbb{W}) \in Z$, we infer that

$$\phi(\overline{B}(0, C_{11}r_1)) \geq (1 - \varepsilon)(r_1/r)^h \phi(\overline{B}(0, C_{11}r)).$$

This implies that

$$\begin{aligned} (3.16) \quad \phi(\overline{B}(0, C_{11}r) \setminus \overline{B}(0, C_{11}r_1)) &= \phi(\overline{B}(0, C_{11}r)) - \phi(\overline{B}(0, C_{11}r_1)) \\ &\leq \phi(\overline{B}(0, C_{11}r))(1 - (1 - \varepsilon)(r_1/r)^h) \\ &= \phi(\overline{B}(0, C_{11}r))(1 - (1 - \varepsilon)(1 - \varepsilon/h)^h) \leq 2\varepsilon\phi(\overline{B}(0, C_{11}r)), \end{aligned}$$

where in the last inequality we used that $h \mapsto (1 - \varepsilon/h)^h$ is increasing.

Proof of (i): Let $u \in P(\overline{B}(0, r_1))$ and let $C_{11}\mu hr < s \leq r$. Then

$$\phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s)) \leq \phi(\overline{B}(0, C_{11}r)) \leq \mu^{-h}(s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

where the last inequality comes from the fact that $C_{11}\mu hr < s$. Defined $v := u\delta_{\mu}(u^{-1})$, we immediately note that $v \in \mathbb{W}$ and that, from Proposition 1.47, $d(v, u) = \mu d(u, 0) \leq C_{10}\mu r$. Furthermore, for every $\Delta \in \overline{B}(0, \mu r)$ we have

$$\begin{aligned} (3.17) \quad d(0, u\delta_{\mu}(u^{-1})\Delta) &\leq \mu\|u\| + \|u\| + \|\Delta\| \leq \mu C_{10}r_1 + C_{10}r_1 + \mu r \\ &\leq (C_{10}(1 + \mu) + 2\mu)r_1 \leq C_{11}r_1, \end{aligned}$$

where in the inequality above we used the fact that $r_1 > r/2$, and $C_{11} > 2(C_{10} + 1) > C_{10}(1 + \mu) + 2\mu$. Thus, on the one hand we have $\overline{B}(v, \mu r) \subseteq \overline{B}(u, (1 + C_{10})\mu r)$ and on the other, thanks to (3.17), we deduce that

$$(3.18) \quad \overline{B}(v, \mu r) \subseteq \overline{B}(0, C_{11}r_1),$$

Since $(0, r, \mathbb{W}) \in Z$, this implies thanks to the definition of Z and E that

$$(3.19) \quad \phi(\overline{B}(v, \mu r)) \geq (1 - \varepsilon)\mu^h C_{11}^{-h} \phi(\overline{B}(0, C_{11}r_1)) > \phi(\overline{B}(0, C_{11}r_1) \setminus E).$$

Furthermore, thanks to (3.18), (3.19) and the definition of $T(\cdot, \cdot)$, we also infer that

$$\emptyset \neq E \cap \overline{B}(v, \mu r) \subseteq E \cap \overline{B}(u, (1 + C_{10})\mu r) \subseteq E \cap T(u, s/4h),$$

where the last inclusion is true since $(1 + C_{10})\mu r \leq C_{11}\mu r/4 < s/(4h)$.

Proof of (ii): Let $u \in P(\overline{B}(0, r_1))$ and let $0 < s \leq s(u)$. By definition of $s(u)$, up to eventually increasing s such that it still holds $0 < s \leq s(u)$, there are two cases. Either

$$(3.20) \quad \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s)) > (1 + \tau)^h \mu^{-h} (s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

for some $\tau > 0$ or

$$(3.21) \quad E \cap T(u, s/4h) = \emptyset.$$

If $v \in P(\overline{B}(0, r_1))$ is sufficiently close to u then $s + C_{10}^{-1}d(u, v) \leq (1 + \tau)s$ and $s + C_{10}^{-1}d(u, v) \leq r$, since $s(u) \leq r$ thanks to point (i). If (3.20) holds, this implies that

$$(3.22) \quad \begin{aligned} \phi(\overline{B}(0, C_{11}r) \cap T(v, C_{10}(s + C_{10}^{-1}d(u, v)))) &> \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s)) \\ &\geq (1 + \tau)^h \mu^{-h} (s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)) \\ &\geq \mu^{-h} ((s + C_{10}^{-1}d(u, v))/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)), \end{aligned}$$

where the last inequality is true provided $d(u, v)$ is suitably small. On the other hand, if (3.21) holds, then

$$(3.23) \quad E \cap T(v, (s - 4hd(u, v))/4h) \subseteq E \cap T(u, s/4h) = \emptyset.$$

Taking into account (3.22) and (3.23), this shows that

$$s(v) \geq \min\{s - 4hd(u, v), s + C_{10}^{-1}d(u, v)\} = s - 4hd(u, v),$$

provided v is sufficiently close to u . This implies that $\liminf_{v \rightarrow u} s(v) \geq s$ for every $s \leq s(u)$ for which at least one between (3.20) and (3.21) holds. In particular, from the definition of $s(u)$, we deduce that there exists a sequence $s_i \rightarrow s(u)^-$ such that at each s_i at least one between (3.20) and (3.21) holds. In conclusion we infer

$$\liminf_{v \rightarrow u} s(v) \geq s(u).$$

Proof of (iii): Suppose that $u \in P(\overline{B}(0, r_1)) \setminus (A \cup A_1)$. Since $u \notin A \cup A_1$, then $s(u) > 0$ and

$$(3.24) \quad \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s(u))) < \varepsilon^{-1} (s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)).$$

Thanks to the definition of $s(u)$, for every $0 < s < s(u)$, up to eventually increasing s in such a way that it still holds $0 < s < s(u)$, we have either

$$(3.25) \quad \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s)) > \mu^{-h} (s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

or

$$(3.26) \quad E \cap T(u, s/4h) = \emptyset.$$

Let us assume that (3.26) does not hold for some $s < s(u)$. Then (3.26) does not hold for any t such that $s \leq t < s(u)$. Thus, in this case, we deduce the existence of $t_i < s(u)$ such

that $t_i \rightarrow s(u)$ for which (3.25) holds. Thus we have

$$\begin{aligned}
(3.27) \quad \mu^{-h}(s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)) &= \lim_{i \rightarrow +\infty} \mu^{-h}(t_i/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)) \\
&\leq \limsup_{i \rightarrow +\infty} \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}t_i)) \\
&\leq \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s(u))) \\
&\leq \varepsilon^{-1}(s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),
\end{aligned}$$

that is a contradiction thanks to the choice of μ and ε . This proves that for every $0 < \rho < s(u)$ we have $E \cap T(u, \rho/4h) = \emptyset$ and thus

$$E \cap \text{int}(T(u, s(u)/4h)) = \emptyset.$$

Let us now define the constants

$$s := 16hs(u)/\varepsilon, \quad \text{and} \quad \sigma := (2h-1)\varepsilon/32h^2.$$

Thanks to item (i), from which $s(u) \leq C_{11}h\mu r$, and from the very definition of μ , we deduce that

$$(3.28) \quad 0 < s(u) \leq s = 16hs(u)/\varepsilon \leq r - r_1, \quad \text{and} \quad \mu \leq \sigma \leq 1.$$

Thanks to the compactness of E and the definition of $s(u)$ we have that

$$E \cap T(u, s(u)/4h) \neq \emptyset.$$

Let us fix $x \in E \cap T(u, s(u)/4h)$ and assume $\mathbb{V} \in Gr_{\leq}^s(h)$ to be such that $(x, s, \mathbb{V}) \in Z$. We claim that

$$(3.29) \quad \|P(x^{-1}y)\| \geq \sigma \|x^{-1}y\|, \quad \text{for every } y \in x\mathbb{V}.$$

Assume by contradiction that there is a $y \in x\mathbb{V}$ such that $\|x^{-1}y\| = 1$ and for which $\|P(x^{-1}y)\| < \sigma$. Let us fix $w \in \overline{B}(0, \sigma s)$ and let $t \in \mathbb{R}$ be such that $|t| \leq C_{10}s(u)/(4h\sigma)$. Then, we have

$$(3.30) \quad d(0, x\delta_t(x^{-1}y)w) \leq d(0, x) + |t|\|x^{-1}y\| + \sigma s \leq d(0, x) + \frac{C_{10}s(u)}{4h\sigma} + \sigma s.$$

Thanks to the choice of the constants and item (i), according to which $s(u) \leq C_{11}h\mu r$, we infer that

$$\begin{aligned}
(3.31) \quad \frac{C_{10}s(u)}{4h\sigma} + \sigma s &\leq C_{10}s(u)(1 - 1/2h + 8h/((2h-1)\varepsilon)) \\
&\leq C_{10}2^{-7}h^{-2}\varepsilon^2r(1 - 1/2h + 8h/((2h-1)\varepsilon)) \leq C_{10}\varepsilon r/h,
\end{aligned}$$

where in the first inequality above we are using the fact that $C_{10} \geq 1$, and in the second we are using the explicit expression $\mu = 2^{-7}h^{-3}C_{11}^{-5h}\varepsilon^2$ and the fact that $C_{11}^{-5h+1} < 1$. Hence, since $x \in \overline{B}(0, C_{11}r_1)$ putting together (3.30) and (3.31) we infer that

$$(3.32) \quad d(0, x\delta_t(x^{-1}y)w) \leq C_{11}r_1 + C_{10}\varepsilon r/h < C_{11}r,$$

where the second inequality comes from the definition of r_1 and the fact that $C_{11} \geq C_{10}$. As a consequence of the previous computations we finally deduce that

$$\overline{B}(x\delta_t(x^{-1}y), \sigma s) \subseteq \overline{B}(0, C_{11}r), \quad \text{for every } |t| \leq C_{10}s(u)/(4h\sigma).$$

We now prove that for every $|t| \leq C_{10}s(u)/(4h\sigma)$ and every $w \in \overline{B}(0, \sigma s)$, we have

$$(3.33) \quad x\delta_t(x^{-1}y)w \in T(u, C_{10}s(u)).$$

Indeed, thanks to Proposition 1.48, we have that

$$P(x\delta_t(x^{-1}y)w) = P(x)\delta_t(P(x^{-1}y))P(w),$$

and thus since $x \in T(u, s(u)/4h)$ by means of Proposition 1.51 we infer that

$$d(u, P(x)) \leq C_{10}s(u)/4h.$$

Thanks to this, and together with the fact that $\|P(w)\| \leq C_{10}\sigma s$ due to Proposition 1.47, we can estimate

$$\begin{aligned} d(u, P(x)\delta_t(P(x^{-1}y))P(w)) &\leq d(u, P(x)) + |t|\|P(x^{-1}y)\| + C_{10}\sigma s \\ &\leq \frac{C_{10}s(u)}{4h} + \frac{C_{10}s(u)}{4h} + C_{10}\sigma s \\ &\leq \frac{C_{10}s(u)}{2h} + C_{10}\left(1 - \frac{1}{2h}\right)s(u) \leq C_{10}s(u), \end{aligned}$$

where in the second inequality of the last line we are using $\sigma s = s(u)(1 - 1/(2h))$. Summing up, the above computations yield that

$$(3.34) \quad \overline{B}(x\delta_t(x^{-1}y), \sigma s) \subseteq \overline{B}(0, C_{11}r) \cap T(u, C_{10}s(u)), \quad \text{for every } |t| \leq C_{10}s(u)/(4h\sigma).$$

Now we are in a position to write the following chain of inequalities

$$\begin{aligned} (3.35) \quad \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s(u))) &\geq (2\sigma s)^{-1} \int_{-s(u)/4h\sigma}^{s(u)/4h\sigma} \phi(\overline{B}(x\delta_t(x^{-1}y), \sigma s)) dt \\ &\geq (2\sigma s)^{-1}(s(u)/2h\sigma)(1 - \varepsilon)(\sigma s/rC_{11})^h \phi(\overline{B}(0, C_{11}r)) \\ &= (1 - \varepsilon)(1 - 1/2h)^h 16h^2(2h - 1)^{-2} \varepsilon^{-1}(s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)) \\ &\geq \varepsilon^{-1}(s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)), \end{aligned}$$

where the first inequality is true by applying Fubini theorem to the function $F(t, z) := \chi_{\overline{B}(0, \sigma s)}(\delta_t(y^{-1}x)x^{-1}z)$ on the domain $[-s(u)/(4h\sigma), s(u)/(4h\sigma)] \times \mathbb{G}$, and by noticing that when $|t| \leq s(u)/(4h\sigma)$ we have (3.34); the second inequality is true since $x \in E$ and then $(x, s, \mathbb{V}) \in Z$ for some $\mathbb{V} \in Gr_{\leq}^s(h)$; and the last inequality is true since $(1 - \varepsilon)(1 - 1/(2h))^h 16h^2(2h - 1)^{-2} \geq 1$. Since (3.35) is a contradiction with the assumption $u \notin A_1$ we get that (3.29) holds and thus $P|_{\mathbb{V}}$ is injective, since it is also a homomorphism. Furthermore, since \mathbb{V} has the same stratification as \mathbb{W} , Proposition 1.35 implies that $\mathbb{V} \cdot \mathbb{L} = \mathbb{G}$, where \mathbb{L} is the chosen normal complementary subgroup of \mathbb{W} . Thanks to [109, Proposition 3.1.5], there exists an intrinsically linear function $\ell : \mathbb{W} \rightarrow \mathbb{L}$ such that $\mathbb{V} = \text{graph}(\ell)$ and thus $P|_{\mathbb{V}}$ is also surjective. In particular we can find a $w \in x\mathbb{V}$ in such a way that $P(w) = u$ and, by using (3.29) and $d(u, P(x)) \leq C_{10}s(u)/4h$, that follows from Proposition 1.51, and the fact that P is a homogeneous homomorphism, we conclude that the following inequality holds

$$(3.36) \quad \|x^{-1}w\| \leq \sigma^{-1}\|P(x)^{-1}P(w)\| = \sigma^{-1}\|P(x)^{-1}u\| \leq \frac{C_{10}s(u)}{4h\sigma}.$$

We now claim that the inclusion

$$(3.37) \quad B(w, s(u)/4h) \subseteq (\overline{B}(0, C_{11}r) \setminus E) \cap \text{int}(T(u, s(u)/4h)),$$

concludes the proof of item (iii). Indeed, we have $(x, s, \mathbb{V}) \in Z$, and since $w \in \overline{B}(x, C_{11}s) \cap x\mathbb{V}$, see (3.36), and we have $\mu s \leq s(u)/4h \leq C_{11}s$, we infer, by approximation and using the hypothesis, that

$$(3.38) \quad \phi(B(w, s(u)/4h)) \geq (1 - \varepsilon)(s(u)/4hC_{11}r)^h \phi(\overline{B}(0, C_{11}r)).$$

Putting together (3.37) and (3.38) we deduce that

$$\phi((\overline{B}(0, C_{11}r) \setminus E) \cap \text{int}(T(u, s(u)/4h))) \geq (1 - \varepsilon)(s(u)/4hC_{11}r)^h \phi(\overline{B}(0, C_{11}r)).$$

and thus $u \in A_2$, which proves item (iii). In order to prove the inclusion (3.37) we note that since $\|x^{-1}w\| \leq C_{10}s(u)/(4h\sigma)$, see (3.36), we have thanks to the same computation we performed in (3.30), (3.31), and (3.32), that $\overline{B}(w, s(u)/(4h)) \subseteq \overline{B}(0, C_{11}r)$. Furthermore, since $P(w) = u$ the inclusion (3.37) follows thanks to the fact that $\overline{B}(w, s(u)/4h) \subseteq T(u, s(u)/4h)$, see Proposition 1.51, and the fact that $\text{int}(T(u, s(u)/4h)) \cap E = \emptyset$.

Proof of (iv): Let $\tau > 1$. Thanks to [102, Theorem 2.8.4], we deduce that there exists a countable set $D \subseteq A_1$ such that the following two hold

(1) the family $\{\overline{B}(w, C_{10}^2s(w)) \cap \mathbb{W} : w \in D\}$ is a disjointed subfamily of

$$\{\overline{B}(w, C_{10}^2s(w)) \cap \mathbb{W} : w \in A_1\},$$

(2) for every $w \in A_1$ there exists a $u \in D$ such that $\overline{B}(w, C_{10}^2s(w)) \cap \overline{B}(u, C_{10}^2s(u)) \cap \mathbb{W} \neq \emptyset$ and $s(w) \leq \tau s(u)$.

Furthermore, if we define for every $u \in A_1$ the set

$$(3.39) \quad \hat{B}(u, C_{10}^2s(u)) := \bigcup \{\overline{B}(w, C_{10}^2s(w)) \cap \mathbb{W} : w \in A_1, \\ \overline{B}(u, C_{10}^2s(u)) \cap \overline{B}(w, C_{10}^2s(w)) \cap \mathbb{W} \neq \emptyset, s(w) \leq \tau s(u)\},$$

we have, thanks to [102, Corollary 2.8.5], that

$$A_1 \subseteq \bigcup_{u \in A_1} \overline{B}(u, C_{10}^2s(u)) \cap \mathbb{W} \subseteq \bigcup_{u \in D} \hat{B}(u, C_{10}^2s(u)).$$

An easy computation based on the triangle inequality, which we omit, leads to the following inclusion

$$(3.40) \quad \hat{B}(u, C_{10}^2s(u)) \subseteq \mathbb{W} \cap \overline{B}(u, (1+2\tau)C_{10}^2s(u)), \quad \text{for every } u \in A_1.$$

Since $D \subseteq A_1$, and since $T(u, C_{10}s(u)) \subseteq P^{-1}(\overline{B}(u, C_{10}^2s(u)) \cap \mathbb{W})$ for every $u \in A_1$, see Proposition 1.51, we conclude, by exploiting the fact that

$$\{\overline{B}(w, C_{10}^2s(w)) \cap \mathbb{W} : w \in D\},$$

is a disjointed family, the following inequality

$$\phi(\overline{B}(0, C_{11}r)) \geq \sum_{u \in D} \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s(u))) \geq \varepsilon^{-1} \sum_{u \in D} (s(u)/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

where the last inequality above comes from the fact that $D \subseteq A_1$. The above inequality can be rewritten as $\sum_{u \in D} s(u)^h \leq C_{11}^h \varepsilon r^h$. In particular, thanks to Remark 1.27, and (3.40) we infer that

$$(3.41) \quad \mathcal{C}^h(A_1) \leq \sum_{u \in D} \mathcal{C}^h(\overline{B}(u, (1+2\tau)C_{10}^2s(u)) \cap \mathbb{W}) \\ = C_{10}^{2h} (1+2\tau)^h \sum_{u \in D} s(u)^h \leq C_{10}^{2h} C_{11}^h (1+2\tau)^h \varepsilon r^h.$$

With a similar argument we used to prove the existence of D , we can construct a countable set $D' \subseteq A_2$ such that the family $\{\overline{B}(u, C_{10}s(u)/4h) \cap \mathbb{W} : u \in D'\}$ is disjointed and the family $\{\hat{B}(u, C_{10}s(u)/4h) : u \in D'\}$, constructed as in (3.39), covers A_2 . In a similar way as in (3.40) we have $\hat{B}(u, C_{10}s(u)/(4h)) \subseteq \mathbb{W} \cap \overline{B}(u, (1+2\tau)C_{10}s(u)/4h)$ for every $u \in A_2$. Moreover, since

$$T(u, s(u)/4h) \subseteq P^{-1}(\overline{B}(u, C_{10}s(u)/4h) \cap \mathbb{W}),$$

for every $u \in A_2$, see Proposition 1.51, we conclude by exploiting the fact that

$$\{\overline{B}(u, C_{10}s(u)/(4h)) \cap \mathbb{W} : u \in D'\},$$

is a disjointed family, the following inequality

$$(3.42) \quad \begin{aligned} \phi(\overline{B}(0, C_{11}r) \setminus E) &\geq \sum_{u \in D'} \phi((\overline{B}(0, C_{11}r) \setminus E) \cap T(u, s(u)/4h)) \\ &\geq 2^{-1} \phi(\overline{B}(0, C_{11}r)) \sum_{u \in D'} (s(u)/4h C_{11}r)^h, \end{aligned}$$

where the last inequality holds since $D' \subseteq A_2$. From the previous inequality, (3.16), and the fact that $0 \in E$, we infer that

$$(3.43) \quad \begin{aligned} \sum_{u \in D'} (s(u)/4h C_{11}r)^h &\leq \frac{2\phi(\overline{B}(0, C_{11}r) \setminus E)}{\phi(\overline{B}(0, C_{11}r))} \\ &\leq 2 \cdot \frac{\phi(\overline{B}(0, C_{11}r) \setminus \overline{B}(0, C_{11}r_1)) + \phi(\overline{B}(0, C_{11}r_1) \setminus E)}{\phi(\overline{B}(0, C_{11}r))} \\ &\leq 2 \cdot \frac{2\varepsilon\phi(\overline{B}(0, C_{11}r)) + \mu^{h+1}C_{11}^{-h}\phi(\overline{B}(0, C_{11}r))}{\phi(\overline{B}(0, C_{11}r))} \leq 10\varepsilon. \end{aligned}$$

Consequently, we deduce that

$$(3.44) \quad \begin{aligned} \mathcal{C}^h(A_2) &\leq \sum_{u \in D'} \mathcal{C}^h(\mathbb{W} \cap \overline{B}(u, (1+2\tau)C_{10}s(u)/4h)) \\ &= (1+2\tau)^h C_{10}^h \sum_{u \in D'} (s(u)/4h)^h \leq 10(1+2\tau)^h C_{10}^h C_{11}^h \varepsilon r^h. \end{aligned}$$

Finally, putting together (3.41), (3.44), item (iii) of this proposition, and Remark 1.27, we conclude the following inequality

$$\begin{aligned} \mathcal{C}^h(P(\overline{B}(0, r)) \setminus A) &\leq \mathcal{C}^h(P(\overline{B}(0, r)) \setminus P(\overline{B}(0, r_1))) + \mathcal{C}^h(A_1) + \mathcal{C}^h(A_2) \\ &\leq \mathcal{C}^h(P(\overline{B}(0, 1)))r^h(1 - (1 - \varepsilon/h)^h) + C_{10}^{2h}C_{11}^h(1+2\tau)^h \varepsilon r^h \\ &\quad + 10(1+2\tau)^h C_{10}^h C_{11}^h \varepsilon r^h \\ &\leq 50(1+2\tau)^h C_{11}^{3h} \mathcal{C}^h(P(\overline{B}(0, 1)))\varepsilon r^h, \end{aligned}$$

where in the last inequality we used that $1 \leq C_{10} \leq C_{11}$, and that $\mathcal{C}^h(P(\overline{B}(0, 1))) \geq 1$ since $P(\overline{B}(0, 1)) \supseteq \overline{B}(0, 1) \cap \mathbb{W}$ and $\mathcal{C}^h(\overline{B}(0, 1) \cap \mathbb{W}) = 1$, thanks to Remark 1.27. With the choice $\tau = 2$, item (iv) follows.

Proof of (v): Let $u \in A$ and note that since $s(u) = 0$, for every $s > 0$ we have that

$$E \cap T(u, s/4h) \neq \emptyset.$$

Since the sets $E \cap T(u, s/4h)$ are compact we infer the following equality thanks to the finite intersection property

$$\emptyset \neq E \cap \bigcap_{s>0} T(u, s/4h) = E \cap P^{-1}(u).$$

This implies that $u \in P(E \cap P^{-1}(u))$ for every $u \in A$, and as a consequence $A \subseteq P(E \cap P^{-1}(A))$. Since the inclusion $P(E \cap P^{-1}(A)) \subseteq A$ is obvious we finally infer that $A = P(E \cap P^{-1}(A))$. Moreover, thanks to item (iv) and to the choice of $\varepsilon < 5^{-h-5}C_{11}^{-3h}$, we conclude that $\mathcal{S}^h(A) > 0$ thanks to the fact that $\mathcal{C}^h \llcorner \mathbb{W}$ and $\mathcal{S}^h \llcorner \mathbb{W}$ are equivalent, see Proposition 1.24, and thanks to the following chain of inequalities

$$\begin{aligned} \mathcal{C}^h(A) &\geq \mathcal{C}^h(P(\overline{B}(0, r))) - \mathcal{C}^h(P(\overline{B}(0, r)) \setminus A) \\ &\geq \mathcal{C}^h(P(\overline{B}(0, 1)))r^h - 5^{h+3}C_{11}^{3h}\mathcal{C}^h(P(\overline{B}(0, 1)))\varepsilon r^h \geq \frac{24}{25}r^h. \end{aligned}$$

Thanks to the fact that P is C_{10} -Lipschitz, see Proposition 1.48, we further infer that

$$0 < \mathcal{S}^h(A) = \mathcal{S}^h(P(E \cap P^{-1}(A))) \leq C_{10}^h \mathcal{S}^h(E \cap P^{-1}(A)).$$

For every s sufficiently small and $u \in A$, by definition of $s(u)$ and A , we have the following chain of inequalities

$$\phi(\overline{B}(x, C_{10}s)) \leq \phi(\overline{B}(0, C_{11}r) \cap T(u, C_{10}s)) \leq \mu^{-h}(s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

whenever $x \in E \cap P^{-1}(u)$, where the first inequality comes from the fact that $x \in E \subseteq \overline{B}(0, C_{11}r_1)$, and Proposition 1.51. Finally by [102, 2.10.17(2)] and the previous inequality we infer

$$(3.45) \quad \phi_{\perp}(E \cap P^{-1}(A)) \leq C_{10}^{-h} C_{11}^{-h} \mu^{-h} \frac{\phi(\overline{B}(0, C_{11}r))}{r^h} \mathcal{S}^h_{\perp}(E \cap P^{-1}(A)).$$

On the other hand, if we assume $x \in E$ and s sufficiently small, we have $(x, s, \mathbb{V}) \in Z$ for some $\mathbb{V} \in Gr_{\leq}^s(h)$. This implies that, by using the very definition of Z , that

$$\phi(\overline{B}(x, s)) \geq (1 - \varepsilon)(s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

and thus by [102, 2.10.19(3)], we have

$$(3.46) \quad \phi_{\perp}E \geq (1 - \varepsilon) \frac{\phi(\overline{B}(0, C_{11}r))}{(C_{11}r)^h} \mathcal{S}^h_{\perp}E.$$

Putting together (3.45) and (3.46), we conclude the proof of item (v). \square

Proposition 3.16. *Let ϕ be a $\mathcal{P}_h^{*, \triangleleft}$ -rectifiable measure such that there exists an $\mathfrak{s} \in \mathbb{N}^k$ for which for ϕ -almost every $x \in \mathbb{G}$ we have*

$$(3.47) \quad \text{Tan}_h(\phi, x) \subseteq \{\lambda \mathcal{S}^h_{\perp} \mathbb{V} : \lambda > 0 \text{ and } \mathbb{V} \in Gr_{\leq}^{\mathfrak{s}}(h)\}.$$

Then, the set

$$(3.48) \quad \mathcal{G}(x) := \{\mathbb{V} \in Gr_{\leq}^{\mathfrak{s}}(h) : \text{there exists } \Theta > 0 \text{ such that } \Theta \mathcal{S}^h_{\perp} \mathbb{V} \in \text{Tan}_h(\phi, x)\},$$

is a compact subset of $Gr_{\leq}^{\mathfrak{s}}(h)$ for all $x \in \mathbb{G}$ for which (3.47) holds, and the sets

$$(3.49) \quad \mathcal{G}_C := \{x \in \mathbb{G} : \mathfrak{c}(\mathbb{V}) \in (C, \infty) \text{ for every } \mathbb{V} \in \mathcal{G}(x)\},$$

where \mathfrak{c} is defined in (1.35), are ϕ -measurable for every $C > 0$.

Proof. The fact that $\mathcal{G}(x)$ is compact is an immediate consequence of Proposition 1.64, the compactness of the Grassmannian in Proposition 1.22, and the convergence result in Proposition 1.30. For every $\lambda, k, r > 0$ define the function $\mathcal{M}_{\lambda, k, r}(x, \mathbb{V}) : \mathbb{G} \times Gr_{\leq}^{\mathfrak{s}}(h) \rightarrow \mathbb{R}$ as

$$\mathcal{M}_{\lambda, k, r}(x, \mathbb{V}) := F_{0, k}(r^{-h} T_{x, r} \phi, \lambda \mathcal{C}^h_{\perp} \mathbb{V}),$$

where $F_{0, k}$ is defined in Definition 1.65. We claim that, for any choice of the parameters, the function $\mathcal{M}_{\lambda, k, r}$ is continuous when $\mathbb{G} \times Gr_{\leq}^{\mathfrak{s}}(h)$ is endowed with respect to the topology induced by the metric $d + d_{\mathbb{G}}$, where $d_{\mathbb{G}}$ is the metric on the Grassmannian. Indeed, assume $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathbb{G}$ and $\{\mathbb{V}_i\} \subseteq Gr_{\leq}^{\mathfrak{s}}(h)$ are two sequences converging to $x \in \mathbb{G}$ and $\mathbb{V} \in Gr_{\leq}^{\mathfrak{s}}(h)$

respectively. Thanks to the triangle inequality we have

$$\begin{aligned}
\limsup_{i \rightarrow \infty} |\mathcal{M}_{\lambda,k,r}(x, \mathbb{V}) - \mathcal{M}_{\lambda,k,r}(x_i, \mathbb{V}_i)| &\leq \limsup_{i \rightarrow \infty} \left(|\mathcal{M}_{\lambda,k,r}(x, \mathbb{V}) - \mathcal{M}_{\lambda,k,r}(x_i, \mathbb{V})| \right. \\
&\quad \left. + |\mathcal{M}_{\lambda,k,r}(x_i, \mathbb{V}) - \mathcal{M}_{\lambda,k,r}(x_i, \mathbb{V}_i)| \right) \\
&\leq \limsup_{i \rightarrow \infty} F_{0,k}(r^{-h}T_{x,r}\phi, r^{-h}T_{x_i,r}\phi) + \limsup_{i \rightarrow \infty} F_{0,k}(\lambda\mathcal{C}^h \llcorner \mathbb{V}, \lambda\mathcal{C}^h \llcorner \mathbb{V}_i) \\
&\leq \limsup_{i \rightarrow \infty} r^{-(h+1)}d(x, x_i)\phi(\overline{B}(x, kr + d(x, x_i))) \\
&\quad + \limsup_{i \rightarrow \infty} F_{0,k}(\lambda\mathcal{C}^h \llcorner \mathbb{V}, \lambda\mathcal{C}^h \llcorner \mathbb{V}_i) = 0,
\end{aligned}$$

where the inequality in the fourth line comes from a simple computation that we omit and the last identity comes from Proposition 1.30. This in particular implies that the function

$$\mathcal{M}(x, \mathbb{V}) := \sup_{\substack{k > 0 \\ k \in \mathbb{Q}}} \inf_{\substack{\lambda > 0 \\ \lambda \in \mathbb{Q}}} \liminf_{\substack{r \rightarrow 0 \\ r \in \mathbb{Q}}} \frac{\mathcal{M}_{\lambda,k,r}(x, \mathbb{V})}{k^{h+1}},$$

is Borel measurable.

We now claim that for ϕ -almost every $x \in \mathbb{G}$ we have that $\mathbb{V} \in \mathcal{G}(x)$ if and only if $\mathcal{M}(x, \mathbb{V}) = 0$. Indeed if $\mathbb{V} \in \mathcal{G}(x)$, there is a $\lambda > 0$ and an infinitesimal sequence $\{r_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} F_{0,k}(r_i^{-h}T_{x,r_i}\phi, \lambda\mathcal{C}^h \llcorner \mathbb{V}) = 0$ for every $k > 0$, see Lemma 1.73. However, by the scaling properties of F , see Remark 1.66, we can choose another infinitesimal sequence $\{s_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Q}$ such that $r_i/s_i \rightarrow 1$, and then $\lim_{i \rightarrow \infty} F_{0,k}(s_i^{-h}T_{x,s_i}\phi, \lambda\mathcal{C}^h \llcorner \mathbb{V}) = 0$ for every $k > 0$ as well, proving the first half of the claim. Vice-versa, if $\mathcal{M}(x, \mathbb{V}) = 0$, then for every $j \in \mathbb{N}$ there exists a $\lambda_j > 0$, with $\lambda_j \in \mathbb{Q}$, and an infinitesimal sequence $\{r_i(j)\} \subseteq \mathbb{Q}$ such that $\lim_{i \rightarrow \infty} F_{0,1}(r_i(j)^{-h}T_{x,r_i(j)}\phi, \lambda_j\mathcal{C}^h \llcorner \mathbb{V}) \leq 1/j$. Since $0 < \Theta_*^h(\phi, x) \leq \Theta^{h,*}(\phi, x) < \infty$ for ϕ -almost every $x \in \mathbb{G}$, we can argue as in the last part of the proof of Proposition 1.64 and hence we can assume without loss of generality that λ_j converge to some non-zero λ and that, for every $j \in \mathbb{N}$, there exists $i_j \in \mathbb{N}$ such that $r_{i_j}(j)$ is an infinitesimal sequence and $r_{i_j}(j)^{-h}T_{x,r_{i_j}(j)}\phi \rightarrow \lambda\mathcal{C}^h \llcorner \mathbb{V}$. This eventually concludes the proof of the claim.

Furthermore, since ϵ , by Proposition 1.45, is lower semicontinuous on $Gr_{\leq}^s(h)$, we know that for every $C > 0$ the set $\mathbb{G} \times \{\mathbb{W} \in Gr_{\leq}^s(h) : \epsilon(\mathbb{W}) \leq C\}$ is closed in $\mathbb{G} \times Gr_{\leq}^s(h)$ and in particular, the set

$$(3.50) \quad \mathcal{M}^{-1}(0) \cap \mathbb{G} \times \{\mathbb{W} \in Gr_{\leq}^s(h) : \epsilon(\mathbb{W}) \leq C\} = \{(x, \mathbb{V}) \in \mathbb{G} \times Gr_{\leq}^s(h) \text{ such that } \mathcal{M}(x, \mathbb{V}) = 0 \text{ and } \epsilon(\mathbb{V}) \leq C\},$$

is Borel. Now, since the projection on the first component of the above set is an analytic set, by the very definition of analytic sets, and since every analytic set is universally measurable, see for example [98, Section 2.2.4], we get that the set

$$\{x \in \mathbb{G} \text{ such that there exists } \mathbb{V} \in Gr_{\leq}^s(h) \text{ with } \mathcal{M}(x, \mathbb{V}) = 0 \text{ and } \epsilon(\mathbb{V}) \leq C\},$$

is ϕ -measurable. In particular its complement, that is \mathcal{G}_C up to ϕ -null sets - since $\mathcal{M}(x, \mathbb{V}) = 0$ if and only if $\mathbb{V} \in \mathcal{G}(x)$ for ϕ -almost every $x \in \mathbb{G}$ - is ϕ -measurable as well. \square

Proposition 3.17. *Let $h \in \{1, \dots, Q\}$, $\mathfrak{s} \in \mathfrak{S}(h)$, and ϕ be a $\mathcal{P}_h^{*,\triangleleft}$ -rectifiable measure supported on a compact set K and for which for ϕ -almost every $x \in \mathbb{G}$ we have*

$$(3.51) \quad \text{Tan}_h(\phi, x) \subseteq \{\lambda\mathcal{S}^h \llcorner \mathbb{V} : \lambda > 0 \text{ and } \mathbb{V} \in Gr_{\leq}^s(h)\}.$$

Let us further assume that there exists a constant $C > 0$ such that $\phi(\mathbb{G} \setminus \mathcal{G}_C) = 0$, where \mathcal{G}_C is defined in (3.49). Throughout the rest of the statement and the proof we will always assume

that C_{10} and C_{11} are the constants introduced in Definition 3.13 in terms of C . Furthermore, let $\varepsilon \in (0, 5^{-10(h+5)}C_{11}^{-3h}]$ and $\mu := 2^{-7}h^{-3}C_{11}^{-5h}\varepsilon^2$.

Then, there are $\vartheta, \gamma \in \mathbb{N}$, a ϕ -positive compact subset E of $E(\vartheta, \gamma)$ (see Definition 1.9), and a point $z \in E \cap \mathcal{G}_C$ such that

- (i) There exists a $\rho_z > 0$ for which $\phi(\overline{B}(z, C_{11}\rho) \setminus E) \leq \mu^{h+1}C_{11}^{-h}\phi(\overline{B}(z, C_{11}\rho))$ for every $0 < \rho < \rho_z$;
- (ii) There exists an $r_0 \in (0, 5^{-10(h+5)}C_{11}^{-3h}\gamma^{-1}]$ such that for every $w \in E$ and every $0 < \rho \leq C_{11}r_0$ we can find a $\mathbb{V}_{w,\rho} \in Gr_{\leq}^5(h)$ such that $\mathfrak{e}(\mathbb{V}_{w,\rho}) \geq C$, see (1.35), and
 - (1) $F_{w,4C_{11}\rho}(\phi, \Theta C^h \mathbb{L}_w \mathbb{V}_{w,\rho}) \leq (4^{-1}\vartheta^{-1}C_{11}^{-1}\mu)^{(h+3)} \cdot (4C_{11}\rho)^{h+1}$ for some $\Theta > 0$,
 - (2) whenever $y \in \overline{B}(w, C_{11}\rho) \cap w\mathbb{V}_{w,\rho}$ and $t \in [\mu\rho, C_{11}\rho]$ we have $\phi(\overline{B}(y, t)) \geq (1 - \varepsilon)(t/C_{11}\rho)^h\phi(\overline{B}(w, C_{11}\rho))$,
 - (3) There exists a normal complementary subgroup $\mathbb{L}_{w,\rho}$ of $\mathbb{V}_{w,\rho}$ as in Proposition 1.47 such that

$$(1 - \varepsilon)\phi(\overline{B}(w, C_{11}\rho) \cap wT_{\mathbb{V}_{w,\rho}}(0, \rho)) \leq C_{11}^{-h}C^h(P(\overline{B}(0, 1)))\phi(\overline{B}(w, C_{11}\rho)),$$

where $T_{\mathbb{V}_{w,\rho}}$ is the cylinder related to the splitting $\mathbb{G} = \mathbb{V}_{w,\rho} \cdot \mathbb{L}_{w,\rho}$, see Definition 1.50, and where $P := P_{\mathbb{V}_{w,\rho}}$ denotes the projection relative to the splitting $\mathbb{G} = \mathbb{V}_{w,\rho} \cdot \mathbb{L}_{w,\rho}$.

- (iii) There exists an infinitesimal sequence $\{\rho_i(z)\}_{i \in \mathbb{N}} \subseteq (0, \min\{r_0, \rho_z\}]$ such that for every $i \in \mathbb{N}$, every $w \in E$ and every $\rho \in (0, C_{11}\rho_i(z)]$ we have $\phi(\overline{B}(w, C_{11}\rho)) \geq (1 - \varepsilon)(\rho/\rho_i(z))^h\phi(\overline{B}(z, C_{11}\rho_i(z)))$.

Proof. For every positive $a, b \in \mathbb{R}$ we define $F(a, b)$ to be the set of those points in K for which

$$br^h \leq \phi(\overline{B}(x, r)), \quad \text{for every } r \in (0, a).$$

One can prove, with the same argument used in the proof of Proposition 1.10, see [180, Proposition 1.14], that the sets $F(a, b)$ are compact. As a consequence, this implies that the sets

$$\tilde{F}(a, b) := \bigcap_{p=1}^{\infty} F(C_{11}a, (1 - \varepsilon)b) \setminus F(C_{11}a/p, b),$$

are Borel. Since ϕ is \mathcal{P}_h^* -rectifiable, \mathbb{G} can be covered ϕ -almost all by countably many sets $\tilde{F}(a, b)$. Indeed, $\phi(\mathbb{G} \setminus \cup_{a,b \in \mathbb{Q}^+} \tilde{F}(a, b)) = 0$ since $0 < \Theta_*^h(\phi, x) < +\infty$ holds ϕ -almost everywhere. In particular thanks to Proposition 1.11 we can find $a, b \in \mathbb{R}$ and $\vartheta, \gamma \in \mathbb{N}$ such that $\phi(\tilde{F}(a, b) \cap E(\vartheta, \gamma)) > 0$. Since $\tilde{F}(a, b) \cap E(\vartheta, \gamma)$ is measurable, there must exist a ϕ -positive compact subset of $\tilde{F}(a, b) \cap E(\vartheta, \gamma)$ that we denote with F . Notice that since $\phi(\mathbb{G} \setminus \mathcal{G}_C) = 0$ the set $F \cap \mathcal{G}_C$ is measurable and ϕ -positive as well.

Let us denote by $Gr_{\leq}^{5,C}(h)$ the set $\{\mathbb{V} \in Gr_{\leq}^5(h) \text{ such that } \mathfrak{e}(\mathbb{V}) \geq C\}$. Since by the very definition of \mathcal{G}_C we have $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_{\leq}^{5,C}(h))$ for ϕ -almost every $x \in F \cap \mathcal{G}_C$, we infer that Proposition 1.74 together with Severini-Egoroff theorem, that can be applied since the functions $x \rightarrow d_{x,kr}(\phi, \mathfrak{M}(h, Gr_{\leq}^{5,C}(h)))$ are continuous in x for every $k, r > 0$ - see Remark 1.72 - yield a ϕ -positive compact subset E of $F \cap \mathcal{G}_C$ and an $r_0 \leq 5^{-10(h+5)}C_{11}^{-3h}\gamma^{-1}$ such that

(3.52)

$$d_{x,4C_{11}\rho}(\phi, \mathfrak{M}(h, Gr_{\leq}^{5,C}(h))) \leq (4^{-1}\vartheta^{-1}C_{11}^{-1}\mu)^{(h+4)} \text{ for every } x \in E \text{ and every } 0 < \rho \leq C_{11}r_0.$$

Let us fix z to be a density point of E with respect to ϕ , and let us show that E and z satisfy the requirements of the proposition. First, by construction E is ϕ -positive and contained in $E(\vartheta, \gamma)$. Second, since z is a density point of E , item (i) follows if we choose ρ_z small enough. Moreover, the bound (3.52) directly implies item (ii.1). Let us prove the remaining items.

Since $E \subseteq E(\vartheta, \gamma)$, $4C_{11}^2 r_0 < \gamma/2$ and $4^{-1}\vartheta^{-1}C_{11}^{-1}\mu \leq 2^{-10(h+1)}\vartheta$, Proposition 1.75(i) implies that for every $w \in E$ and every $0 < \rho < C_{11}r_0$ - choosing $\sigma = 4^{-1}\vartheta^{-1}C_{11}^{-1}\mu$ and $t = 4C_{11}\rho$ in Proposition 1.75 - there exists a $\mathbb{V}_{w,\rho} \in Gr_{\leq}^{\sigma,C}(h)$ such that

$$\phi(\overline{B}(y, r) \cap \overline{B}(w\mathbb{V}, 4^{-1}C_{11}^{-1}\vartheta^{-2}\mu^2\rho)) \geq (1 - 2^{10(h+1)}4^{-1}C_{11}^{-1}\mu)(r/s)^h \phi(\overline{B}(v, s)),$$

whenever $y, v \in \overline{B}(w, 2C_{11}\rho) \cap w\mathbb{V}_{w,\rho}$ and $\vartheta^{-1}\mu\rho \leq r, s \leq 2C_{11}\rho$. Since

$$2^{10(h+1)}4^{-1}C_{11}^{-1}\mu \leq \varepsilon,$$

with the choices $s = C_{11}\rho$ and $v = w$, we finally infer

$$\phi(\overline{B}(y, r)) \geq (1 - \varepsilon)(r/C_{11}\rho)^h \phi(\overline{B}(w, C_{11}\rho)),$$

for every $\mu\rho \leq r \leq C_{11}\rho$ and every $y \in \overline{B}(w, C_{11}\rho) \cap w\mathbb{V}_{w,\rho}$, and this proves item (ii.2). For every $w \in E$ and every $0 < \rho < C_{11}r_0$ we choose one normal complementary subgroup $\mathbb{L}_{w,\rho}$ of $\mathbb{V}_{w,\rho}$ as in Proposition 1.47, and we denote with $P := P_{\mathbb{V}_{w,\rho}}$ the projection relative to this splitting. Eventually, Proposition 1.75(ii), with the choice $k := C_{11}$, implies that for every $0 < \rho < C_{11}r_0$ we have

$$(3.53) \quad \begin{aligned} \phi(\overline{B}(w, C_{11}\rho) \cap wT_{\mathbb{V}_{w,\rho}}(0, \rho)) &\leq (1 + (2C_{11}h + 1)\vartheta^{-1}C_{11}^{-1}\mu)C_{11}^{-h}C^h(P(\overline{B}(0, 1)))\phi(\overline{B}(w, C_{11}\rho)) \\ &\leq (1 + \varepsilon)C_{11}^{-h}C^h(P(\overline{B}(0, 1)))\phi(\overline{B}(w, C_{11}\rho)), \end{aligned}$$

where the last inequality comes from the fact that $(2C_{11}h + 1)\vartheta^{-1}C_{11}^{-1}\mu < \varepsilon$. Hence also item (ii.3) is verified. In order to verify item (iii), note that since $z \in E \subseteq \tilde{F}(a, b)$ on the one hand then there is an infinitesimal sequence $\{\rho_i(z)\}_{i \in \mathbb{N}}$ such that

$$(3.54) \quad \frac{\phi(\overline{B}(z, C_{11}\rho_i(z)))}{(C_{11}\rho_i(z))^h} \leq b.$$

On the other hand for every $w \in E$, and every $0 < \rho < a$ we have

$$(3.55) \quad b \leq \frac{1}{1 - \varepsilon} \frac{\phi(\overline{B}(w, C_{11}\rho))}{(C_{11}\rho)^h}.$$

Putting together (3.54) and (3.55) we finally infer that for every $i \in \mathbb{N}$, every $w \in E$ and every $\rho \in (0, a)$ we have

$$\frac{\phi(\overline{B}(z, C_{11}\rho_i(z)))}{\rho_i(z)^h} \leq \frac{1}{1 - \varepsilon} \frac{\phi(\overline{B}(w, C_{11}\rho))}{\rho^h},$$

concluding the proof of item (iii) and thus of the proposition. \square

Let us now exploit Proposition 3.15, Proposition 3.16, and Proposition 3.17 to show the following result which is at the core of the proof of Theorem 3.4.

Proposition 3.18. *Assume ϕ is a $\mathcal{P}_h^{*,\triangleleft}$ -rectifiable measure supported on a compact set K . Then, there exists a Carnot subgroup $\mathbb{W} \in Gr_{\leq}(h)$, a compact set $K' \subseteq \mathbb{W}$, and a Lipschitz function $f : K' \rightarrow \mathbb{G}$ such that $\phi(f(K')) > 0$.*

Proof. Theorem 3.8 implies that for ϕ -almost every $x \in \mathbb{G}$ the elements of $\text{Tan}_h(\phi, x)$ all share the same stratification vector. Furthermore, thanks to Proposition 3.12, for every $\mathfrak{s} \in \mathfrak{S}(h)$ the set $\mathcal{T}_{\mathfrak{s}} := \{x \in K : \mathfrak{s}(\phi, x) = \mathfrak{s}\}$ is ϕ -measurable. Thus, if we prove that for every $\mathfrak{s} \in \mathfrak{S}(h)$ there exists a Lipschitz function as in the thesis of the proposition whose image has positive $\phi_{\perp} \mathcal{T}_{\mathfrak{s}}$ -measure, the proposition is proved since the sets $\mathcal{T}_{\mathfrak{s}}$ cover ϕ -almost all K and since the locality of tangents hold, see Proposition 1.55. Thanks to this argument, we can assume without loss of generality that there exists a $\mathfrak{s} \in \mathfrak{S}(h)$ such that for ϕ -almost every $x \in K$ we have $\mathfrak{s}(\phi, x) = \mathfrak{s}$.

Let us further reduce ourselves to the setting in which there exists a constant $C > 0$ such that $\phi(\mathbb{G} \setminus \mathcal{G}_C) = 0$, where \mathcal{G}_C is defined in (3.49). Thanks to Proposition 3.16, we know that for ϕ -almost every $x \in \mathbb{G}$ the set $\mathcal{G}(x)$ defined in (3.48) is compact. Hence, taking item (i) of Proposition 1.45 into account, for ϕ -almost every $x \in \mathbb{G}$ there exists a constant $C(x) > 0$ such that $\mathfrak{e}(\mathbb{V}) \geq C(x)$ for every $\mathbb{V} \in \mathcal{G}(x)$. This readily implies that

$$\phi(\mathbb{G} \setminus \cup_{n \in \mathbb{N}} \mathcal{G}_{1/n}) = 0.$$

Hence, since $\mathcal{G}_{1/n}$ is ϕ -measurable for every $n \in \mathbb{N}$, see Proposition 3.16, we can reduce, with the same argument used in the previous paragraph, to deal with the case in which there exists $C > 0$ such that $\phi(\mathbb{G} \setminus \mathcal{G}_C) = 0$.

Let $C_{10} := C_{10}(C)$ and $C_{11} := C_{11}(C)$ be defined as in Definition 3.13, and let $\tilde{\varepsilon} \leq 5^{-10(h+5)} C_{11}^{-3h}$, and $\tilde{\mu} := 2^{-7} h^{-3} C_{11}^{-5h} \tilde{\varepsilon}^2$. Let $E \subseteq K$ be the compact set and let $z \in E \cap \mathcal{G}_C$ be a point yielded by Proposition 3.17 with respect to $\tilde{\varepsilon}, \tilde{\mu}$. Furthermore let $\tilde{\varepsilon} \leq \varepsilon \leq 5^{-h-5} C_{11}^{-3h}$, and $\mu := 2^{-7} h^{-3} C_{11}^{-5h} \varepsilon^2$ such that $(1 - \tilde{\varepsilon})^2 \geq (1 - \varepsilon)$. We define

$$r := \rho_1(z), \quad \text{and} \quad r_1 := (1 - \varepsilon/h)r,$$

where $\rho_1(z)$ is the first term of the sequence $\{\rho_i(z)\}_{i \in \mathbb{N}}$ yielded by item (iii) of Proposition 3.17.

Let us check that the compact set $E \cap \overline{B}(z, C_{11}r_1)$ satisfies the hypothesis of Proposition 3.15 with respect to the choices ε, μ, r . First of all, since $r < \rho_z$, item (i) of Proposition 3.17 implies that (3.15) holds since $\tilde{\mu} \leq \mu$. Secondly, since $r \leq r_0$, item (ii.2) of Proposition 3.17 implies that for every $w \in E$ and every $0 < \rho < C_{11}r$ there exists a $\mathbb{V}_{w,\rho} \in Gr_{\leq}^5(h)$ such that whenever $y \in \overline{B}(w, C_{11}r) \cap w\mathbb{V}_{w,\rho}$ and $t \in [\mu\rho, C_{11}\rho]$ we have

$$\phi(\overline{B}(y, t)) \geq (1 - \tilde{\varepsilon})(t/C_{11}\rho)^h \phi(\overline{B}(w, C_{11}\rho)).$$

Furthermore, since $r = \rho_1(z)$, thanks to item (iii) of Proposition 3.17 we finally infer that for every $w \in E$ and every $0 < \rho < C_{11}r$ we have

$$(3.56) \quad \begin{aligned} \phi(\overline{B}(y, t)) &\geq (1 - \tilde{\varepsilon})(t/C_{11}\rho)^h \phi(\overline{B}(w, C_{11}\rho)) \geq (1 - \tilde{\varepsilon})^2 (t/C_{11}r)^h \phi(\overline{B}(z, C_{11}r)) \\ &\geq (1 - \varepsilon)(t/C_{11}r)^h \phi(\overline{B}(z, C_{11}r)), \end{aligned}$$

whenever $y \in \overline{B}(w, C_{11}r) \cap w\mathbb{V}_{w,\rho}$ and $t \in [\mu\rho, C_{11}\rho]$. The above paragraph shows that the hypotheses of Proposition 3.15 are satisfied by z and $E \cap \overline{B}(z, C_{11}r_1)$ with the choices of r, r_1, ε, μ as above.

Throughout the rest of the proof, for the sake of readability, E will stand for $E \cap \overline{B}(z, C_{11}r_1)$, and in order to conclude the argument we will need to use the other two pieces of information yielded by Proposition 3.17. Indeed, since $r < C_{11}r_0$, item (ii.3) of Proposition 3.17 implies that

$$(3.57) \quad (1 - \varepsilon)\phi(zT_{\mathbb{V}_{z,r}}(0, r) \cap \overline{B}(z, C_{11}r)) \leq \mathcal{C}^h(P(\overline{B}(0, 1)))C_{11}^{-h}\phi(\overline{B}(z, C_{11}r)),$$

where T is the cylinder related to the splitting $\mathbb{G} = \mathbb{V}_{z,r} \cdot \mathbb{L}_{z,r}$, and $\mathbb{L}_{z,r}$ is one normal complementary subgroup of $\mathbb{V}_{z,r}$ chosen as in item (ii.3) of Proposition 3.17. Furthermore, thanks to item (ii.1) of Proposition 3.17 and the fact that $r < r_0$ we know that there exists $\Theta > 0$ such that

$$(3.58) \quad F_{z, 4C_{11}r}(\phi, \Theta \mathcal{C}^h \mathbb{L} z \mathbb{V}_{z,r}) \leq (4^{-1} \vartheta^{-1} C_{11}^{-1} \mu)^{h+3} \cdot (4C_{11}r)^{h+1}.$$

The bound (3.58) together with Proposition 2.5, that we can apply since $4C_{11}r \leq \gamma^{-1}$, and $2^{-1}(4^{-1}\vartheta^{-1}C_{11}^{-1}\mu)^{h+3} \leq \delta_{\mathbb{G}}$, where $\delta_{\mathbb{G}}$ was introduced in Definition 2.4, imply that

$$(3.59) \quad \sup_{w \in E \cap \overline{B}(z, C_{11}r)} \frac{\text{dist}(w, z\mathbb{V}_{z,r})}{4C_{11}r} \leq 2^{1+1/(h+1)} \vartheta^{1/(h+1)} (2^{-1}(4^{-1}\vartheta^{-1}C_{11}^{-1}\mu)^{h+3})^{\frac{1}{h+1}} \leq 2C_{11}^{-1}\mu.$$

The above bound shows that the set E inside the ball $\overline{B}(z, C_{11}r)$ is very squeezed around the plane $\mathbb{V}_{z,r}$. From now on we should denote $\mathbb{W} := \mathbb{V}_{z,r}$, $P := P_{\mathbb{V}_{z,r}}$, $\mathbb{L} := \mathbb{L}_{z,r}$, and $T(\cdot, r) := T_{\mathbb{W}}(\cdot, r)$. In order to simplify the notation, since all the statements are invariant up to substituting ϕ with $T_{z,1}\phi$, we can assume that $z = 0$. Let us recall once more that $\epsilon(\mathbb{V}_{z,r}) \geq C$ from item (ii) of Proposition 3.17.

Since it will turn out to be useful later on, we estimate the distance of the points w of $E \cap T(0, r_1)$ from 0. Thanks to Proposition 1.51 and the fact that $w \in T(0, r_1)$, we have $\|P_{\mathbb{W}}(w)\| \leq C_{10}r_1$. On the other hand, (1.38) and (3.59) imply that

$$\|P_{\mathbb{L}}(w)\| \leq C_{10}\text{dist}(w, \mathbb{W}) \leq 8C_{10}\mu r_1.$$

This in particular implies that

$$d(0, w) \leq \|P_{\mathbb{W}}(w)\| + \|P_{\mathbb{L}}(w)\| \leq C_{10}r_1 + 8C_{10}\mu r_1 \leq 2C_{10}r_1,$$

showing that

$$(3.60) \quad E \cap T(0, r_1) \subseteq \overline{B}(0, 2C_{10}r_1).$$

In the following A , A_1 and A_2 are the sets inside $P(\overline{B}(0, r_1))$ constructed in the statement of Proposition 3.15 with respect to the 0 and the plane \mathbb{W} . Now, let \tilde{A} be the set of those $u \in A$ for which there exists $\rho(u) > 0$ such that

$$(3.61) \quad \phi(\overline{B}(0, C_{11}r) \cap T(u, s)) \leq 2(1 - \epsilon)^4 (s/C_{11}r)^h \mathcal{C}^h(P(\overline{B}(0, 1))) \phi(\overline{B}(0, C_{11}r)),$$

for all $0 < s < \rho(u)$. We claim that \tilde{A} is a Borel set. To prove this, we note that

$$\tilde{A} = \bigcup_{k \in \mathbb{N}} \{u \in A : (3.61) \text{ holds for every } 0 < s < 1/k\} =: \bigcup_{k \in \mathbb{N}} \tilde{A}_k.$$

Let us show that \tilde{A}_k is a compact set for every $k \in \mathbb{N}$, and in order to do this, let us assume $\{u_i\}_{i \in \mathbb{N}}$ is a sequence of points of \tilde{A}_k . Since $\tilde{A}_k \subseteq A$, and A is compact, we can suppose that, up to a non re-labelled subsequence, u_i converges to some $u \in A$. Thus, we have that for every $0 < s < 1/k$ the following chain of inequality holds

$$\begin{aligned} \phi(\overline{B}(0, C_{11}r) \cap T(u, s)) &\leq \limsup_{i \rightarrow \infty} \phi(\overline{B}(0, C_{11}r) \cap T(u_i, s + d(u, u_i))) \\ &\leq 2(1 - \epsilon)^4 \mathcal{C}^h(P(\overline{B}(0, 1))) (s/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)). \end{aligned}$$

This concludes the proof of the fact that \tilde{A}_k is compact and thus \tilde{A} is an F_σ set, and thus Borel.

Let us notice that, since $r_1 < r$, by a compactness argument one finds that there exists a $\tilde{s} := \tilde{s}(r_1, r)$ such that whenever $u \in P(\overline{B}(0, r_1))$, then $P(\overline{B}(u, \tilde{s})) \subseteq P(\overline{B}(0, r))$. The family

$$\mathcal{B} := \{P(\overline{B}(u, s)) : u \in A \setminus \tilde{A}, \text{ and } s \leq \tilde{s} \text{ does not satisfy (3.61)}\}$$

is a fine cover of $A \setminus \tilde{A}$ by the very definition of \tilde{A} . Thus [102, 2.8.17] with a routine argument implies that \mathcal{B} is a \mathcal{S}^h -Vitali relation ([102, 2.8.16]). Therefore, the set $A \setminus \tilde{A}$ can be covered \mathcal{S}^h -almost all by a sequence of disjointed projected balls $\{P(\overline{B}(u_k, s_k))\}_{k \in \mathbb{N}}$ such that $u_k \in A \setminus \tilde{A}$ and

$$\phi(\overline{B}(0, C_{11}r) \cap T(u_k, s_k)) > 2(1 - \epsilon)^4 \mathcal{C}^h(P(\overline{B}(0, 1))) (s_k/C_{11}r)^h \phi(\overline{B}(0, C_{11}r)),$$

for every $k \in \mathbb{N}$. Note that since $T(u_k, s_k) = P^{-1}(P(\overline{B}(u_k, s_k)))$, see Proposition 1.51, we get that $\{T(u_k, s_k)\}_{k \in \mathbb{N}}$ is a disjointed family of cylinders. Moreover, from the very definition of

\tilde{s} , since $u_k \in P(\overline{B}(0, r_1))$ and $s_k \leq \tilde{s}$, we have that $P(\overline{B}(u_k, s_k)) \subseteq P(\overline{B}(0, r))$. This implies that

$$(3.62) \quad \begin{aligned} \phi(T(0, r) \cap \overline{B}(0, C_{11}r)) &\geq \sum_{k \in \mathbb{N}} \phi(\overline{B}(0, C_{11}r) \cap T(u_k, s_k)) \\ &> 2(1 - \varepsilon)^4 \mathcal{C}^h(P(\overline{B}(0, 1))) C_{11}^{-h} r^{-h} \phi(\overline{B}(0, C_{11}r)) \sum_{k \in \mathbb{N}} s_k^h. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{C}^h(A \setminus \tilde{A}) &= \sum_{k \in \mathbb{N}} \mathcal{C}^h(P(\overline{B}(u_k, s_k))) \leq \mathcal{C}^h(P(\overline{B}(0, 1))) \sum_{k \in \mathbb{N}} s_k^h \\ &< 2^{-1} (1 - \varepsilon)^{-4} \frac{\phi(T(0, r) \cap \overline{B}(0, C_{11}r)) C_{11}^h r^h}{\phi(\overline{B}(0, C_{11}r))} \leq 2^{-1} (1 - \varepsilon)^{-5} \mathcal{C}^h(P(\overline{B}(0, 1))) r^h \\ &\leq \frac{27}{50} \mathcal{C}^h(P(\overline{B}(0, 1))) r^h, \end{aligned}$$

where the second inequality on the second line above follows from (3.57). Furthermore, from the previous inequality and from item (iv) of Proposition 3.15 we deduce that

$$\begin{aligned} \mathcal{C}^h(\tilde{A}) &= \mathcal{C}^h(P(\overline{B}(0, r))) - \mathcal{C}^h(P(\overline{B}(0, r)) \setminus A) - \mathcal{C}^h(A \setminus \tilde{A}) \\ &> \mathcal{C}^h(P(\overline{B}(0, 1))) r^h - 5^{h+3} C_{11}^{3h} \varepsilon \mathcal{C}^h(P(\overline{B}(0, 1))) r^h - \mathcal{C}^h(P(\overline{B}(0, 1))) \frac{27}{50} r^h \\ &\geq (1 - 1/25 - 27/50) \mathcal{C}^h(P(\overline{B}(0, 1))) r^h > \frac{2}{5} \mathcal{C}^h(P(\overline{B}(0, 1))) r^h. \end{aligned}$$

Since \tilde{A} is measurable, we can find a compact set $\hat{A} \subseteq \tilde{A}$ and a $\delta \in (0, \varepsilon r/h)$ such that $\mathcal{C}^h(\hat{A}) > 0$ and (3.61) holds for every $u \in \hat{A}$ and $s \in (0, \delta)$. This can be done by taking an interior approximation with compact sets of \tilde{A} .

Thanks to item (v) of Proposition 3.15 we know that

$$(3.63) \quad \hat{A} \subseteq A = P(E \cap P^{-1}(A)),$$

and thus for every $u \in \hat{A}$ we can find a $x \in E$ such that $P(x) = u$. We claim that for every $x \in E$ for which $P(x) \in \hat{A}$, every $s < \min\{r/4, \delta/(1 + C_{11})\}$ and every $w \in \mathbb{V}_{x,s}$ we have

$$(3.64) \quad \|P(w)\| > \|w\|/2C_{11}.$$

Suppose by contradiction that there are an $s < \min\{r/4, \delta/(1 + C_{11})\}$ and a $w \in \mathbb{V}_{x,s}$ with $\|w\| = 1$ such that $\|P(w)\| \leq 1/2C_{11}$. This would imply that for every $k = 0, \dots, \lfloor C_{11}/4 \rfloor - 1$ and every $p \in \overline{B}(0, s/2)$ we have, by exploiting $P(x) = u$ and that P is a homogeneous homomorphism, that

$$(3.65) \quad \begin{aligned} d(P(x\delta_{2ks}(w)p), u) &= d(\delta_{2ks}(P(w))P(p), 0) \leq \|\delta_{2ks}(P(w))\| + \|P(p)\| \\ &\leq 2ks\|P(w)\| + \|P(p)\| \leq ks/C_{11} + C_{10}s \leq (1 + C_{10})s. \end{aligned}$$

Since $u \in \hat{A} \subseteq A \subseteq P(\overline{B}(0, r_1))$, and since $P(x) = u$, we conclude that $x \in T(0, r_1)$. Hence, taking into account that $r_1 < r$, thanks to the inclusion (3.60), we have

$$(3.66) \quad d(x\delta_{2ks}(w)p, 0) \leq \|x\| + 2ks + s \leq 2C_{10}r + (2k + 1)s < 2C_{10}r + 3C_{11}r/4 < C_{11}r.$$

Putting together (3.65) and (3.66), we infer that for every $k = 0, \dots, \lfloor C_{11}/4 \rfloor - 1$ we have

$$\overline{B}(x\delta_{2ks}(w), s/2) \subseteq T(u, (1 + C_{10})s) \cap \overline{B}(0, C_{11}r).$$

Furthermore, since $x \in E$, $\overline{B}(x\delta_{ks}(w), s/2)$ are disjoint and contained in $\overline{B}(x, C_{11}s)$, we have by items (ii.2) and (iii) of Proposition 3.17 that

$$\begin{aligned}
(3.67) \quad \phi\left(\overline{B}(0, C_{11}r) \cap T(u, (1 + C_{10})s)\right) &\geq \sum_{k=1}^{\lfloor C_{11}/4 \rfloor - 1} \phi(\overline{B}(x\delta_{ks}(w), s/2)) \\
&\geq \frac{(1 - \varepsilon)C_{11}}{8} \left(\frac{s/2}{C_{11}s}\right)^h \phi(\overline{B}(x, C_{11}s)) \\
&\geq \frac{(1 - \varepsilon)^2 C_{11}}{8} \left(\frac{s/2}{C_{11}r}\right)^h \phi(\overline{B}(0, C_{11}r)) \\
&= (1 - \varepsilon)^2 \frac{C_{11}}{2^{h+3}} \left(\frac{s}{C_{11}r}\right)^h \phi(\overline{B}(0, C_{11}r)).
\end{aligned}$$

Since by assumption $u \in \hat{A} \subseteq \tilde{A}$ and $(1 + C_{10})s < \delta$, we infer thanks to (3.67) and the definition of \hat{A} that

$$\begin{aligned}
(3.68) \quad (1 - \varepsilon)^2 \frac{C_{11}}{2^{h+3}} \left(\frac{s}{C_{11}r}\right)^h \phi(\overline{B}(0, C_{11}r)) &\leq \phi\left(\overline{B}(0, C_{11}r) \cap T(u, (1 + C_{10})s)\right) \\
&\leq 2(1 - \varepsilon)^4 (1 + C_{10})^h \left(\frac{s}{C_{11}r}\right)^h \mathcal{C}^h(P(\overline{B}(0, 1))) \phi(\overline{B}(0, C_{11}r)).
\end{aligned}$$

The chain of inequalities (3.68) is however in contradiction with the choice of C_{11} thanks to Remark 3.14, and thus the claim (3.64) is proved.

Since P restricted to $E \cap P^{-1}(A)$ is surjective on \hat{A} as remarked after (3.63), thanks to the axiom of choice there exists a function $f : \hat{A} \rightarrow E \cap P^{-1}(A)$ such that $P(f(u)) = u$. We claim that for ϕ -almost every $x \in f(\hat{A})$ there exists a $\mathfrak{r}(x) > 0$ such that for every $y \in f(\hat{A}) \cap \overline{B}(x, \mathfrak{r}(x))$ we have

$$(3.69) \quad \|P(x)^{-1}P(y)\| = \|P(x^{-1}y)\| > C_{11}^{-2}\|x^{-1}y\| = C_{11}^{-2}\|f(P(x))^{-1}f(P(y))\|,$$

where the last identity comes from the fact that f is bijective on its image and thus the left and right inverse must coincide. In order to prove the latter claim, assume by contradiction that there exists an $x \in f(\hat{A})$ such that $\text{Tan}_h(\phi, x) \subseteq \mathfrak{M}(h, Gr_{\leq}^s(h))$ and a sequence $\{y_i\}_{i \in \mathbb{N}} \subseteq f(\hat{A})$, with $y_i \rightarrow x$, such that

$$(3.70) \quad \|P(x^{-1}y_i)\| \leq C_{11}^{-2}\|x^{-1}y_i\|, \quad \text{for every } i \in \mathbb{N}.$$

Defined $\rho_i := \|x^{-1}y_i\|$, thanks to the hypothesis on x and the definitions of y_i and ρ_i we can assume without loss of generality that

- (1) for every $i \in \mathbb{N}$ we have $\rho_i \leq \min\{r/4, \delta/(1 + C_{11})\}$,
- (2) the points $g_i := \delta_{1/\rho_i}(x^{-1}y_i)$ converge to some $y \in \partial\overline{B}(0, 1)$ such that $\|P(y)\| \leq C_{11}^{-2}$,
- (3) $\rho_i^{-h}T_{x, \rho_i}\phi \rightarrow \lambda\mathcal{C}^h \llcorner \mathbb{V}$ for some $\lambda > 0$ and $\mathbb{V} \in Gr_{\leq}^s(h)$.

Since $\mathcal{C}^h \llcorner \mathbb{V}(\partial\overline{B}(p, s)) = 0$, see e.g., [132, Lemma 3.5], for every $p \in \mathbb{G}$ and every $s \geq 0$, thanks to [89, Proposition 2.7] we infer that

$$\begin{aligned}
\lambda\mathcal{C}^h \llcorner \mathbb{V}(\overline{B}(y, \rho)) &= \lim_{i \rightarrow \infty} T_{x, \rho_i}\phi(\overline{B}(y, \rho))/\rho_i^h \geq \lim_{i \rightarrow \infty} T_{x, \rho_i}\phi(\overline{B}(g_i, \rho - d(g_i, y)))/\rho_i^h \\
&\geq \lim_{i \rightarrow \infty} \phi(\overline{B}(y_i, \rho_i\rho/2))/\rho_i^h \geq \vartheta^{-1}(\rho/2)^h > 0,
\end{aligned}$$

where we stress that in the second inequality in the second line we are using that there exists $\vartheta, \gamma \in \mathbb{N}$ such that $E \subseteq E(\vartheta, \gamma)$, since E is provided by Proposition 3.17. The above computation shows that the (contradiction) assumption (3.70) implies that at x there is a flat tangent measure whose support \mathbb{V} contains an element $y \in \partial\overline{B}(0, 1)$ such that $\|P(y)\| \leq C_{11}^{-2}$. Let us prove that if

(HC) there exists a suitably big $i_0 \in \mathbb{N}$ such that we can find a $q_{i_0} \in \mathbb{V}_{x, \rho_{i_0}}$ such that $d(y, q_{i_0}) \leq \mu$,

then we achieve a contradiction with (3.64), and thus we prove the claim (3.69). Indeed, the claim (HC) above would imply thanks to the definition of μ , (3.64), Proposition 1.47, and Proposition 1.48, that

$$\begin{aligned} (4C_{11})^{-1} &< (1 - \mu)/2C_{11} \leq (\|y\| - \|y^{-1}q_{i_0}\|)/2C_{11} \leq \|q_{i_0}\|/2C_{11} \\ &< \|P(q_{i_0})\| \leq \|P(y)\| + \|P(y^{-1}q_{i_0})\| \leq C_{11}^{-2} + C_{10}\mu < 2C_{11}^{-2}, \end{aligned}$$

which is a contradiction since $C_{11} > 10^Q$.

In this paragraph we prove the claim (HC), which is sufficient to conclude the proof of the claim (3.69). Let Θ_i be the positive numbers yielded by item (ii.1) of Proposition 3.17 with the choices $\rho := \rho_i$ around the point x , and notice that

$$\begin{aligned} \limsup_{i \rightarrow \infty} F_{0,4C_{11}}(\lambda \mathcal{C}^h \llcorner \mathbb{V}, \Theta_i \mathcal{C}^h \llcorner \mathbb{V}_{x, \rho_i}) &\leq \limsup_{i \rightarrow \infty} F_{0,4C_{11}}\left(\frac{T_{x, \rho_i} \phi}{\rho_i^h}, \lambda \mathcal{C}^h \llcorner \mathbb{V}\right) \\ &+ \limsup_{i \rightarrow \infty} F_{0,4C_{11}}\left(\frac{T_{x, \rho_i} \phi}{\rho_i^h}, \Theta_i \mathcal{C}^h \llcorner \mathbb{V}_{x, \rho_i}\right) \\ (3.71) \quad &= \limsup_{i \rightarrow \infty} F_{0,4C_{11}}\left(\frac{T_{x, \rho_i} \phi}{\rho_i^h}, \Theta_i \mathcal{C}^h \llcorner \mathbb{V}_{x, \rho_i}\right) \\ &= \limsup_{i \rightarrow \infty} \frac{F_{x, 4C_{11} \rho_i}(\phi, \Theta_i \mathcal{C}^h \llcorner x \mathbb{V}_{x, \rho_i})}{\rho_i^{h+1}} \\ &\leq (\vartheta^{-1} \mu)^{(h+3)}, \end{aligned}$$

where the identity in the third line above comes from Lemma 1.73, the second identity from the scaling property in Remark 1.66 and the last inequality from item (ii.1) of Proposition 3.17 and some algebraic computations that we omit. Defined $g(w) := (\min\{1, 2 - \|w\|\})_+$ by Proposition 1.29 for every $\mathbb{V}' \in Gr(h)$ we have

$$\begin{aligned} \int g d\mathcal{C}^h \llcorner \mathbb{V}' &= h \int s^{h-1} (\min\{1, 2 - |s|\})_+ ds \\ (3.72) \quad &= h \left(\int_0^1 s^{h-1} + \int_1^2 s^{h-1} (2 - s) ds \right) = \frac{2^{h+1} - 1}{h + 1}. \end{aligned}$$

Therefore, since $\text{supp}(g) \subseteq \overline{B}(0, 4C_{11})$, thanks to (3.71) we infer that

$$\begin{aligned} \limsup_{i \rightarrow \infty} |\lambda - \Theta_i| &= \limsup_{i \rightarrow \infty} \frac{|\lambda \int g d\mathcal{C}^h \llcorner \mathbb{V} - \Theta_i \int g d\mathcal{C}^h \llcorner \mathbb{V}_{x, \rho_i}|}{\int g d\mathcal{C}^h \llcorner \mathbb{V}} \\ (3.73) \quad &\leq \limsup_{i \rightarrow \infty} (h + 1) \frac{F_{0,4C_{11}}(\lambda \mathcal{C}^h \llcorner \mathbb{V}, \Theta_i \mathcal{C}^h \llcorner \mathbb{V}_{x, \rho_i})}{2^{h+1} - 1} \leq \frac{(h + 1)(\vartheta^{-1} \mu)^{(h+3)}}{2^{h+1} - 1} \\ &\leq 2(\vartheta^{-1} \mu)^{(h+3)}. \end{aligned}$$

Let $p \in \mathbb{V} \cap \overline{B}(0, 1)$ and $\ell(w) := (\mu \|p\| - \|p^{-1}w\|)_+$. The function ℓ is a positive 1-Lipschitz function whose support is contained in $\overline{B}(0, (1 + \mu)\|p\|)$ and therefore, thanks to Remark 1.66,

we deduce that

(3.74)

$$\begin{aligned}
\liminf_{i \rightarrow \infty} \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} &\geq \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - \limsup_{i \rightarrow \infty} \lambda \left| \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \right| \\
&\geq \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - \limsup_{i \rightarrow \infty} |\lambda - \Theta_i| \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \\
&\quad - \limsup_{i \rightarrow \infty} \left| \int \ell(w) d\lambda \mathcal{C}^h_{\perp \mathbb{V}} - \int \ell(w) d\Theta_i \mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \right| \\
&\geq \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - \limsup_{i \rightarrow \infty} |\lambda - \Theta_i| \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \\
&\quad - \limsup_{i \rightarrow \infty} F_{0, (1+\mu)\|p\|}(\lambda \mathcal{C}^h_{\perp \mathbb{V}}, \Theta_i \mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}}).
\end{aligned}$$

Let us bound separately the two last terms in the last line above. Thanks to the triangle inequality the points $q_i \in \mathbb{V}_{x, \rho_i}$ of minimal distance of p from \mathbb{V}_{x, ρ_i} are contained in $\overline{B}(0, 2\|p\|)$. This, together Remark 1.27, implies that

$$(3.75) \quad \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \leq \mu \|p\| \mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}}(\overline{B}(q_i, (3+2\mu)\|p\|)) \leq (3+2\mu)^{h+1} \|p\|^{h+1}.$$

On the other hand, thanks to Remark 1.66 and the fact that $\mathcal{C}^h_{\perp \mathbb{V}}$ and $\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}}$ are invariant under rescaling, we infer that

$$(3.76) \quad F_{0, (1+\mu)\|p\|}(\lambda \mathcal{C}^h_{\perp \mathbb{V}}, \Theta_i \mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}}) = \left(\frac{(1+\mu)\|p\|}{4C_{11}} \right)^{h+1} F_{0, 4C_{11}}(\lambda \mathcal{C}^h_{\perp \mathbb{V}}, \Theta_i \mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}}).$$

Putting together (3.71), (3.73), (3.74), (3.75) and (3.76) we finally infer that

(3.77)

$$\begin{aligned}
\liminf_{i \rightarrow \infty} \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} &\geq \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - 2(\vartheta^{-1}\mu)^{(h+3)}(3+2\mu)^{h+1} \|p\|^{h+1} \\
&\quad - \left(\frac{(1+\mu)\|p\|}{4C_{11}} \right)^{h+1} (\vartheta^{-1}\mu)^{(h+3)} \\
&\geq \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} - (\vartheta^{-1}\mu)^{(h+3)} \|p\|^{h+1} (2(3+2\mu)^{h+1} + 1).
\end{aligned}$$

Finally, Lemma 1.62 and the fact that $x \in E(\vartheta, \gamma)$ imply that $\lambda \geq \vartheta^{-1}$. This together with a simple computation that we omit, based on Proposition 1.29, shows that

$$(3.78) \quad \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}} \geq \vartheta^{-1} (\mu \|p\|)^{h+1} / (h+1).$$

Putting together (3.77) and (3.78) we eventually infer that

$$\liminf_{i \rightarrow \infty} \lambda \int \ell(w) d\mathcal{C}^h_{\perp \mathbb{V}_{x, \rho_i}} \geq \vartheta^{-1} (\mu^{h+1} / (h+1) - 2^{2(h+2)} \mu^{(h+3)}) \|p\|^{h+1} > 0,$$

proving that for every $p \in \overline{B}(0, 1) \cap \mathbb{V}$ we have $\overline{B}(p, \mu\|p\|) \cap \mathbb{V}_{x, \rho_i} \neq \emptyset$ provided i is chosen suitably big. Thus the claim (HC) is proved taking $p = y$.

Let us conclude the proof of the proposition exploiting the claim (3.69) that we have proved. Defined \mathcal{B} to be the set of full measure in $f(\hat{A})$ on which (3.69) holds, we note that since $\overline{B}(P(x), r) \subseteq P(\overline{B}(x, r))$, the (3.69) implies the following one: for every $u \in P(\mathcal{B})$ there exists a $\tau(u) > 0$ such that

$$(3.79) \quad \|f(u)^{-1} f(w)\| \leq C_{11}^2 \|u^{-1} w\|, \quad \text{whenever } w \in \hat{A} \cap \overline{B}(u, \tau(u)).$$

Furthermore, note that thanks to the proof of item (v) of Proposition 3.15 and recalling that $f(\hat{A}) \subseteq E \cap P^{-1}(A)$, we deduce that $\mathcal{S}^h \llcorner f(\hat{A})$ is mutually absolutely continuous with respect to ϕ and by Proposition 1.34 we finally infer that

$$\mathcal{S}^h(\hat{A} \setminus P(\mathcal{B})) = \mathcal{S}^h(P(f(\hat{A}) \setminus \mathcal{B})) = 0,$$

where the first equality above comes from the fact that $f : \hat{A} \rightarrow f(\hat{A})$ is bijective.

We now prove that if $\tau(u)$ is chosen to be the biggest radius for which (3.79) holds, then the map $u \mapsto \tau(u)$ is upper semicontinuous on \hat{A} . Indeed, assume $\{u_i\}_{i \in \mathbb{N}}$ is a sequence in \hat{A} such that $u_i \rightarrow u \in \hat{A}$ and $\limsup_{i \rightarrow \infty} \tau(u_i) = r_0 \geq 0$. If $r_0 = 0$, then the inequality $\limsup_{i \rightarrow \infty} \tau(u_i) \leq \tau(u)$ is trivially satisfied. Thus, we can assume that $r_0 > 0$, and, without loss of generality, also that the lim sup is actually a lim. For every fixed $0 < s < r_0$ there exists an $i_0 \in \mathbb{N}$ such that

$$s + d(u, u_i) < \tau(u_i) \quad \text{for every } i \geq i_0.$$

As a consequence $\overline{B}(u, s) \subseteq \overline{B}(u_i, \tau(u_i))$ and thus for every $y \in \hat{A} \cap \overline{B}(u, s)$ and $i \geq i_0$ we have

$$(3.80) \quad \|f(u)^{-1}f(y)\| \leq \|f(u)^{-1}f(u_i)\| + \|f(u_i)^{-1}f(y)\| \leq C_{11}^2 \|u_i^{-1}u\| + C_{11}^2 \|u_i^{-1}y\|.$$

Sending i to $+\infty$, thanks to (3.80) we conclude that for every $y \in \overline{B}(u, s) \cap \hat{A}$ we have $\|f(u)^{-1}f(y)\| \leq C_{11}^2 \|u^{-1}y\|$ and thus $s \leq \tau(u)$. The arbitrariness of s concludes that τ is upper semicontinuous and thus for every $j \in \mathbb{N}$ the sets

$$L_j := \{w \in \hat{A} : \tau(w) \geq 1/j\},$$

are Borel. Furthermore, since $\tau(u) > 0$ everywhere on $P(\mathcal{B})$, we infer that $P(\mathcal{B}) \subseteq \cup_{j \in \mathbb{N}} L_j$. This, jointly with the fact that $\mathcal{S}^h(\hat{A}) > 0$, and that $\mathcal{S}^h(\hat{A} \setminus P(\mathcal{B})) = 0$ tells us that we can find a $j \in \mathbb{N}$ and compact subset \mathcal{A} of L_j such that $\mathcal{S}^h(\mathcal{A}) > 0$ and $\text{diam}(\mathcal{A}) < 1/2j$.

Let us conclude the proof by showing that f is Lipschitz on \mathcal{A} and that $\phi(f(\mathcal{A})) > 0$. The fact that $f(\mathcal{A})$ is ϕ -positive follows from Proposition 1.47, item (v) of Proposition 3.15 and the following computation

$$0 < \mathcal{S}^h(\mathcal{A}) = \mathcal{S}^h(P(f(\mathcal{A}))) \leq C_{10}^h \mathcal{S}^h(f(\mathcal{A})).$$

On the other hand, for every $u, v \in \mathcal{A}$ we have $d(u, v) \leq 1/2j$ and since $u, v \in L_j$ then

$$\|f(u)^{-1}f(v)\| \leq C_{11}^2 \|u^{-1}v\|.$$

This eventually concludes the proof of the proposition. \square

Let us now conclude the chapter with the proof of Theorem 3.4.

Proof of Theorem 3.4. If we prove the result for $\phi \llcorner \overline{B}(0, k)$ for every $k \in \mathbb{N}$, the general case follows taking into account the locality of tangents and the Lebesgue Differentiation Theorem in Proposition 1.55. Therefore, we can assume without loss of generality that ϕ is supported on a compact set. Let us set

$$(3.81)$$

$$\mathcal{F} := \{\cup_{i \in \mathbb{N}} f_i(K_i) : K_i \text{ compact subset of } \mathbb{W}_i \in Gr_{\leq}(h), \text{ and } f_i : K_i \rightarrow \mathbb{G} \text{ is Lipschitz}\}.$$

Let $m := \inf_{F \in \mathcal{F}} \{\phi(\mathbb{G} \setminus F)\}$. We claim that if $m = 0$ the proof of the proposition is concluded. Indeed, if $m = 0$ we can take $F_n \in \mathcal{F}$ such that $\phi(\mathbb{G} \setminus F_n) < 1/n$ and then $\phi(\mathbb{G} \setminus \cup_{n \in \mathbb{N}} F_n) = 0$. Let us prove that $m = 0$. Indeed, if by contradiction $m > 0$, we can take, as before, $F'_n \in \mathcal{F}$ such that $0 < \phi(\mathbb{G} \setminus \cup_{n \in \mathbb{N}} F'_n) \leq m$. Since $F' := \cup_{n \in \mathbb{N}} F'_n$ is Borel, we have, thanks to the locality of tangents and to the Lebesgue Differentiation Theorem in Proposition 1.55, that $\phi \llcorner F'$ is a $\mathcal{P}_h^{*, \leq}$ -rectifiable measure with compact support. Thus we can apply Proposition 3.18 to conclude that there exists $\mathbb{W} \in Gr_{\leq}(h)$, K a compact subset

of \mathbb{W} , and a Lipschitz function $f : K \rightarrow \mathbb{G}$ such that $\phi_{\perp} F'(f(K)) > 0$. Thus we get that $\phi(\mathbb{G} \setminus (f(K) \cup F')) < m$, that is a contradiction with the definition of m .

In order to prove the last part of the theorem, let us notice that, thanks to the locality of tangents and to the Lebesgue Differentiation Theorem in Proposition 1.55, we can reduce to $\phi_{\perp} E(\vartheta, \gamma)$, thanks also to Proposition 1.11. Moreover, taking into account that $\mathcal{S}^h_{\perp} E(\vartheta, \gamma)$ is mutually absolutely continuous with respect to $\phi_{\perp} E(\vartheta, \gamma)$, see Proposition 1.56, we can finally reduce to prove that $\mathcal{S}^h_{\perp} f(K)$ is a \mathcal{P}_h^c -rectifiable measure whenever K is a compact subset of $\mathbb{W} \in Gr_{\leq}(h)$ and $f : K \rightarrow \mathbb{G}$ is a Lipschitz function. The fact that $\mathcal{S}^h_{\perp} f(K)$ is a \mathcal{P}_h^c -rectifiable measure follows from the following claim: if K is a compact subset of $\mathbb{W} \in Gr_{\leq}(h)$ and $f : K \rightarrow \mathbb{G}$ is a Lipschitz function, then for $\mathcal{S}^h_{\perp} f(K)$ -almost every $x \in \mathbb{G}$ we have that there exists $\mathbb{W}(x) \in Gr(h)$ such that the following convergence of measures holds

$$(3.82) \quad r^{-h} T_{x,r} \mathcal{S}^h_{\perp} f(K) \rightharpoonup \mathcal{S}^h_{\perp} \mathbb{W}(x), \quad \text{as } r \text{ goes to } 0.$$

Let us finally sketch the proof of (3.82). Since $\mathbb{W} \in Gr_{\leq}(h)$, i.e., it admits a normal complementary subgroup, we get that \mathbb{W} is a Carnot subgroup of \mathbb{G} , see [29, Remark 2.1]. Thus we can apply Pansu–Rademacher theorem to $f : K \subseteq \mathbb{W} \rightarrow \mathbb{G}$, see [161, Theorem 3.4.11], to obtain that f is Pansu-differentiable \mathcal{S}^h -almost everywhere, with Pansu differential df , and the area formula holds, see [161, Corollary 4.3.6]. The proof of (3.82) with $\mathbb{W}(x) := df(x)(\mathbb{W})$ for $\mathcal{S}^h_{\perp} f(K)$ -almost every x is now just a routine task, building on [161, Proposition 4.3.1 and Proposition 4.3.3], and by using the area formula in [161, Corollary 4.3.6]. We do not give all the details as the proof follows verbatim as in the argument contained in [178, pages 716-717], with the obvious substitutions taking into account that the authors in [178] only deal with Heisenberg groups \mathbb{H}^n in the case \mathbb{W} is horizontal. \square

Pauls's rectifiability and intrinsically C^1 rectifiability

In this chapter we are going to discuss the relationship between the intrinsically C^1 rectifiability and Pauls's rectifiability in Carnot groups. The content of this chapter comes from a work in collaboration with my PhD advisor E. Le Donne [25].

In the following introductory part we introduce the definition of Pauls's rectifiability and we state the two theorems that we will prove throughout this chapter, namely Theorem 4.1 and Theorem 4.2.

In Section 1 we introduce and discuss several notions of rectifiability which generalize the notions proposed by Pauls and Cole–Pauls in [84, 201]. In Section 2 we construct a Carnot algebra of topological dimension 8 that has uncountably many pairwise non-isomorphic Carnot sub-algebras of topological dimension 7. Hence, in Section 3 we exploit the construction in Section 2 to build the example that proves Theorem 4.1. Finally, in Section 4 we prove Theorem 4.2. In particular in Section 4.1 we introduce some notation on Carnot groups of step 2, and in Section 4.2 we prove a length-comparison result for curves on intrinsically Lipschitz graphs in Carnot groups of step 2. In Section 4.3 we use the latter estimate to prove the equivalence between the intrinsic and the induced distance on intrinsically Lipschitz graphs in \mathbb{H}^n , with $n \geq 2$. Finally, in Section 4.4 we complete the proof of Theorem 4.2.

At the beginning of 2000 S. Pauls proposed a notion of rectifiability in Carnot groups that is different from the one discussed in Chapter 2, see [201, Definition 4.1]. According to his definition, given a Carnot group \mathbb{G} of homogeneous dimension Q , a subset E of another Carnot group is \mathbb{G} -rectifiable if it can be covered \mathcal{H}^Q -a.e. by countably many Lipschitz images of subsets of \mathbb{G} . The relation between the notion of rectifiability of Pauls and the intrinsically C^1 rectifiability firstly introduced by Franchi, Serapioni and Serra Cassano and taken to its utmost level of generality in Definition 1.104, is not so well understood nowadays. Notice that in [84, Definition 3] the authors propose another definition of rectifiability in which they allow \mathbb{G} in the previous definition to be a homogeneous subgroup of a Carnot group.

One of the queries that has been left open is whether a k -dimensional $C^1_{\mathbb{H}}$ -submanifold in a Carnot group is Lipschitz (or better bi-Lipschitz) parametrizable by subsets of k -dimensional homogeneous subgroups of a Carnot group. One positive result in this direction has been obtained in [84] in which the authors proved that every C^1 -hypersurface in \mathbb{H}^1 is N -rectifiable, where N is a vertical plane in \mathbb{H}^1 , and the maps used for the parametrization could be even defined on open sets. Then this result was improved by Bigolin and Vittone in [60] showing that every non-characteristic point of a C^1 -hypersurface in \mathbb{H}^1 admits a neighbourhood U and a bi-Lipschitz chart between an open subset of N and U . In [60] the authors also provided a partial negative answer to the query: they showed the existence of a $C^1_{\mathbb{H}}$ -hypersurface in \mathbb{H}^1 that has a point with no bi-Lipschitz map from an open subset of N and any of its neighbourhoods.

Recently, using some ideas coming from the theory of quantitative differentiability, in [92] the authors showed that every intrinsically $C^{1,\alpha}$ hypersurface, with $\alpha > 0$, in the n -th Heisenberg group \mathbb{H}^n is Lipschitz parametrizable with subsets of an arbitrary vertical homogeneous subgroup, and actually it has big pieces of bi-Lipschitz images of an arbitrary

vertical homogeneous subgroup, see [92, Theorem 1.6]. The intrinsically $C^{1,\alpha}$ condition can be relaxed in the first Heisenberg group \mathbb{H}^1 just asking for an intrinsically Lipschitz condition together with some extra Hölder regularity along the vertical direction, see [92, Theorem 1.11]. Anyway, as far as we know nowadays the following question is still open, as pointed out in [92].

Question 7. Determine whether a codimension-one intrinsically Lipschitz graph (or a $C^1_{\mathbb{H}}$ -hypersurface) in \mathbb{H}^1 is almost everywhere covered by (bi)-Lipschitz images of subsets of codimension-one subgroups of \mathbb{H}^1 . In the opposite direction determine whether a (bi)-Lipschitz image of a subset of a codimension-one subgroup of \mathbb{H}^1 is almost everywhere covered by intrinsically Lipschitz hypersurfaces (or even better by $C^1_{\mathbb{H}}$ -hypersurfaces).

We stress that in the n -th Heisenberg group \mathbb{H}^n , with $n \geq 2$, the second question above has a positive answer. Indeed, as a consequence of the same reasoning in the final part of the proof of Theorem 3.1, we have that every Lipschitz image of a subset of a codimension-one subgroup in \mathbb{H}^n , with $n \geq 2$, is \mathcal{P}_h^c -rectifiable. Hence an application of Theorem 2.25, together with [180, Proposition B.11], implies that such a Lipschitz image can be covered almost everywhere by $C^1_{\mathbb{H}}$ -hypersurfaces. Another way of seeing this is to use [201, Theorem 4.3] together with the criterion in [178, Theorem 3.15]. We stress that a more detailed study of the so-called Rickman rugs, i.e., bi-Lipschitz images of codimension-one subgroups in \mathbb{H}^1 , is contained in [198]. We point out that some results about quantitative rectifiability in the Heisenberg groups are contained also in [76, 79]. See [101] or [177, Section 9.7] for a survey on the subject.

We stress that recently, in a slightly different direction, Le Donne and Young in [151] proved that a sub-Riemannian manifold with constant Gromov-Hausdorff tangents \mathbb{G} , is countably \mathbb{G} -rectifiable, where \mathbb{G} is a Carnot group. This result gives a possible way to show that smooth hypersurfaces in Carnot groups - sufficiently smooth in order to carry a sub-Riemannian structure - are \mathbb{G} -rectifiable for some \mathbb{G} . This is exactly what we do in the second part of this chapter with smooth non-characteristic hypersurfaces in \mathbb{H}^n with $n \geq 2$.

In this chapter we prove the following theorem, according to which in arbitrary Carnot groups a smooth non-characteristic hypersurface, which is in particular a $C^1_{\mathbb{H}}$ -hypersurface, might not be rectifiable according to Pauls's definition. We recall that a point on a smooth submanifold in a Carnot group is said to be non-characteristic if the horizontal bundle at the point is not contained in the Euclidean tangent at the point, see Definition 4.17.

Theorem 4.1. *There exist a Carnot group \mathbb{G} and an analytic non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is not Pauls Carnot rectifiable, see Definition 4.8.*

Pauls Carnot rectifiability is a generalization of Pauls rectifiability defined in [201, Definition 4.1] in which we allow countably many Carnot groups models, see Definition 4.8. Our result shows that even very regular objects, such as analytic non-characteristic hypersurfaces, which for sure are intrinsically C^1 rectifiable, may not be Pauls Carnot rectifiable.

In this chapter, we also show that such an example does not exist in the n -th Heisenberg group \mathbb{H}^n with $n \geq 2$ (see Theorem 4.46 and Remark 4.47 for a more exhaustive statement). We stress that the following Theorem 4.2 could also be obtained as a consequence of the main result of the paper [92], which appeared after our work [25], and which is indeed more general. Anyway our proof is slightly different than the one in [92].

Theorem 4.2. *Let S be a C^∞ -hypersurface in the n -th Heisenberg group \mathbb{H}^n with $n \geq 2$. Then S is $\mathbb{H}^{n-1} \times \mathbb{R}$ -rectifiable according to Pauls's definition of rectifiability, even with bi-Lipschitz maps.*

Let us briefly comment on the proofs of the two theorems above. To prove Theorem 4.1, whose proof is in Section 3, we will show the existence of an analytic hypersurface S - of

Hausdorff dimension 12 in a Carnot group of topological dimension 8 - that cannot be \mathcal{H}^{12} -a.e. covered by countably many Lipschitz images of subsets of Carnot groups of Hausdorff dimension 12.

We will actually show a more general property for the set S that we propose: for every Carnot group \mathbb{G} of Hausdorff dimension 12, every Lipschitz map $f : U \subseteq \mathbb{G} \rightarrow S$ satisfies $\mathcal{H}^{12}(f(U)) = 0$, see the proof of Theorem 4.1. We will call this property *purely Pauls Carnot unrectifiability* (Definition 4.8), which implies that S is not Pauls Carnot rectifiable, see Remark 4.11. The key property for the proof of the previous result is that every \mathcal{H}^{12} -positive subset of S has uncountably many points with pairwise non-isomorphic Carnot groups as tangents, see the statement and the proof of Theorem 4.24, and Theorem 4.31.

The idea to build such a hypersurface is the following: at first, in Proposition 4.16, we show the existence of a Carnot algebra \mathfrak{g} of dimension 8 that has uncountably many pairwise non-isomorphic Carnot sub-algebras of dimension 7. This is done by exploiting the existence of an uncountable family \mathcal{F} of Carnot algebras of dimension 7 that are known to be pairwise non-isomorphic, see [119]. Notice that 7 is the minimal dimension for which this fact holds. Indeed, there are, up to isomorphisms, only finitely many Carnot algebras of dimension ≤ 6 , see again [119]. Then we construct an example of a smooth non-characteristic hypersurface S , see the proof of Theorem 4.24, in the Carnot group whose Lie algebra is \mathfrak{g} , with the property that the tangent spaces of S form an uncountable subfamily of \mathcal{F} , and such that every \mathcal{H}^{12} -positive subset of S has uncountably many points with pairwise non-isomorphic Carnot groups as tangents. Similar examples were previously constructed in [151, 211].

Having in our hands the pathological example S , we prove our main result in Theorem 4.1. We do it via a blow up analysis and using the area formula for Lipschitz maps between Carnot groups proved by Magnani in [160].

We point out that we also construct, in every Carnot group \mathbb{G} , a smooth non-characteristic hypersurface that has every subgroup of \mathbb{G} of codimension-one as tangent, see Lemma 4.21.

We also prove a variant of Theorem 4.1. Namely, we show in Corollary 4.25 that our example S is not *bi-Lipschitz homogeneous rectifiable*, see Definition 4.5. More precisely, it is impossible to \mathcal{H}^{12} -a.e. cover S by countably many bi-Lipschitz images of subsets of metric spaces of Hausdorff dimension 12 that have bi-Lipschitz equivalent tangents. Actually, again, we prove more: we show that S is *purely bi-Lipschitz homogeneous unrectifiable* according to Definition 4.5, after having provided a general criterion for *purely bi-Lipschitz homogeneous unrectifiability* (Lemma 4.7).

Notice that, from this last result, it follows that S is not rectifiable according to the countable bi-Lipschitz variant of the definition given in [84, Definition 3], that is, the one that allows the parametrizing spaces to be homogeneous subgroups of Carnot groups, see also Remark 4.6. Nevertheless, we are still not able to prove that our counterexample is not rectifiable according to [84, Definition 3], see Remark 4.26.

We remark here that, from how we are going to construct the example S , it follows that every tangent to S is a Carnot group. Consequently, together with the previously discussed results, we immediately deduce that S is also an example of metric space that cannot be Lipschitz parametrized by countably many of its tangents, see Remark 4.30.

To prove Theorem 4.2, we will use [210, Theorem 1.1], [82, Proposition 3.8], and [151, Theorem 2]. The proof is contained in Section 4.

The idea is the following: first we show that every smooth non-characteristic hypersurface S in \mathbb{H}^n , with $n \geq 2$, carries a structure of polarized manifold (Proposition 4.45). Indeed, we show that the intersection of the horizontal bundle of \mathbb{H}^n with the tangent bundle of S

is a step-2 bracket generating distribution (Proposition 4.42). This was already known from [210, Theorem 1.1], but we give a different proof based on simple explicit computations.

Before going on, let us notice that Proposition 4.45 is very likely to hold for $C^{1,1}$ non-characteristic hypersurfaces. The reason for which we stated it in the C^∞ -category is merely technical. Indeed, Proposition 4.42 is stated for C^2 non-characteristic hypersurfaces, but its proof can be adapted to work in the $C^{1,1}$ case. Moreover, in the proof of Proposition 4.45, we use the fundamental results in [185], and [52] (see also [129]), which require C^∞ -regularity, but can be very likely adapted to $C^{1,1}$ -regularity in our case. The serious difficult point seems to pass from this $C^{1,1}$ -regularity to $C_{\mathbb{H}}^1$, that would probably require a completely different argument.

In order to conclude the proof we show that every sub-Riemannian structure on the polarized manifolds S gives rise to a distance that is locally bi-Lipschitz equivalent to the distance on S seen as subset of \mathbb{H}^n (Proposition 4.43). We will call these distances the *intrinsic distance* and the *induced distance*, respectively. The equivalence is due to the general fact that in \mathbb{H}^n , with $n \geq 2$, the intrinsic distance and the induced distance on the graph of an intrinsically Lipschitz function are equivalent (Proposition 4.41). This tells us also that in Proposition 4.43, we are merely using the fact that S is locally the graph of an intrinsically Lipschitz function. The proof of Proposition 4.41 was suggested to us by Fässler and Orponen, and it is reminiscent of the result already known from [82, Proposition 3.8].

Eventually we use the fundamental tool [151, Theorem 2] and the key fact that the tangents to the hypersurface are all isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$ (Lemma 4.44). With these three steps we conclude the proof of Theorem 4.2.

1. Notions of rectifiability

In this section we are going to introduce the general notion of (\mathcal{F}, μ) -rectifiability, see Definition 4.3, and a specialization of it, namely the notion of bi-Lipschitz homogeneous rectifiability, see Definition 4.5. Finally, we are going to introduce and discuss the notion of Pauls Carnot rectifiability, see Definition 4.8.

Definition 4.3 ((\mathcal{F}, μ) -rectifiability). Given a family \mathcal{F} of metric spaces we say that a metric space (X, d) , with an outer measure μ on it, is (\mathcal{F}, μ) -rectifiable if there exist countably many bi-Lipschitz embeddings $f_i : U_i \subseteq (X_i, d_i) \rightarrow (X, d)$ where $(X_i, d_i) \in \mathcal{F}$, $i \in \mathbb{N}$, and

$$\mu \left(X \setminus \bigcup_{i \in \mathbb{N}} f_i(U_i) \right) = 0.$$

We say that a metric space (X, d) is *purely (\mathcal{F}, μ) -unrectifiable* if for every $(X', d') \in \mathcal{F}$ and every bi-Lipschitz embedding $f : U \subseteq (X', d') \rightarrow (X, d)$ it holds

$$\mu(f(U)) = 0.$$

In Definition 4.5, we are going to specialize the notion of (\mathcal{F}, μ) -rectifiability by taking \mathcal{F} as the class of metric spaces that are locally compact, locally doubling and with bi-Lipschitz equivalent tangents. In Remark 4.6 we discuss further specializations of this notion.

Then we shall give the notion of Pauls Carnot rectifiability in Definition 4.8, generalizing the definition given in [201, Definition 4.1]. In Remark 4.10 we briefly discuss some Lipschitz variants of (\mathcal{F}, μ) -rectifiability, for specific families \mathcal{F} .

We stress here that from now on every metric space (X, d) will be separable. We also remark that if (X, d) is locally complete we can equivalently ask each set U_i in Definition 4.3 to be closed. Indeed, in this case every bi-Lipschitz map $f_i : U_i \rightarrow (X, d)$ extends, locally to

the closure $\overline{U_i}$, to a bi-Lipschitz map. We will freely use this last observation throughout the chapter.

Remark 4.4. Having a look at Definition 4.3, assuming we have $\mu(X) > 0$, which will be always in our case, we see that one necessary condition for the (\mathcal{F}, μ) -rectifiability of (X, d) is the existence of at least one bi-Lipschitz map $f : U \subseteq (X', d') \rightarrow (X, d)$, where $(X', d') \in \mathcal{F}$ and $\mu(f(U)) > 0$. So if a metric space (X, d) , with an outer measure μ on it such that $\mu(X) > 0$, is (\mathcal{F}, μ) -purely unrectifiable then it cannot be (\mathcal{F}, μ) -rectifiable.

Before giving Definition 4.5, first we recall two definitions. Given a separable metric space (X, d) and a Borel regular measure μ on X that is finite on bounded sets, we say that μ is *locally doubling* if for each $a \in X$ there exists $R_a > 0$ and $C_a > 0$ such that

$$0 < \mu(B(x, 2r)) \leq C_a \mu(B(x, r)) < +\infty, \quad \forall x \in B(a, R_a), \quad \forall 0 \leq r \leq R_a.$$

In this case we say that (X, d, μ) is a *locally doubling metric measure space*.

For a locally compact locally doubling metric measure space, as a consequence of the Gromov compactness theorem, we know that for every $x \in X$, the set of Gromov-Hausdorff tangents $\text{Tan}(X, d, x)$ is nonempty. Indeed, for every sequence of positive numbers $\lambda_i \rightarrow 0$, up to subsequences it holds

$$(X, \lambda_i^{-1}d, x) \rightarrow (X_\infty, d_\infty, x_\infty),$$

in the pointed Gromov-Hausdorff convergence. For general definitions and theory about (pointed) Gromov-Hausdorff convergence one can see [212, Chapter 27], [68, Chapters 7, 8], and [118]. We say that the metric space (X, d_X) is *bi-Lipschitz equivalent* to the metric space (Y, d_Y) if there exists a bijective map $f : X \rightarrow Y$ such that

$$\frac{1}{C} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2), \quad \text{for all } x_1, x_2 \in X.$$

Definition 4.5 (bi-Lipschitz homogeneous rectifiability). Let (X, d) be a metric space of Hausdorff dimension k . Set $\mathcal{T}_k := \{(X_i, d_i)\}_{i \in I}$ to be the family of all the metric spaces (X_i, d_i) such that:

- $(X_i, d_i, \mathcal{H}^k)$ is a locally compact locally doubling metric measure space, with $k = \dim_H X_i$;
- every two tangent spaces, at every two points of X_i , are bi-Lipschitz equivalent.

We say that (X, d) is *bi-Lipschitz homogeneous rectifiable* if it is $(\mathcal{T}_k, \mathcal{H}^k)$ -rectifiable according to Definition 4.3. We say that (X, d) is *purely bi-Lipschitz homogeneous unrectifiable* if it is purely $(\mathcal{T}_k, \mathcal{H}^k)$ -unrectifiable according to Definition 4.3.

Remark 4.6. The family \mathcal{T}_k defined in Definition 4.5 is very rich. For example it contains all homogeneous Lie groups \mathbb{G} equipped with a left-invariant homogeneous distance $d_{\mathbb{G}}$, with Hausdorff dimension k . Indeed, by homogeneity, every tangent space at every point of such a group \mathbb{G} is isometric to $(\mathbb{G}, d_{\mathbb{G}})$, moreover $(\mathbb{G}, d_{\mathbb{G}})$ is locally compact and k -Ahlfors-regular [149, Theorem 4.4, (iii)], and then \mathcal{H}^k is a doubling measure on it. We remark here that the larger class of self-similar metric Lie groups of Hausdorff dimension k , whose definition is in [149], is still a subclass of \mathcal{T}_k . Going beyond Lie groups, we remark that in \mathcal{T}_k one has all those Carnot-Carathéodory spaces whose nilpotentization is constantly equal to a fixed Carnot group of homogeneous dimension k . This last statement is a consequence of Mitchell's theorem (see [185], [52], and [129]) and the bi-Lipschitz equivalence of left-invariant homogeneous distances on Carnot groups.

In the very rich class of homogeneous Lie groups we distinguish homogeneous subgroups of homogeneous dimension k of arbitrary Carnot groups, with the restricted distance, and

obviously also Carnot groups of homogeneous dimension k endowed with arbitrary homogeneous left-invariant distances. We can then give different notions of rectifiability for each of these subfamilies of \mathcal{T}_k .

Notice that if we take the subfamily of \mathcal{T}_k made of arbitrary homogeneous subgroups, of dimension k , of Carnot groups endowed with arbitrary homogeneous left-invariant distances, we obtain a notion of rectifiability that is a variation of [84, Definition 3] where we now allow countably many homogeneous subgroups but we require bi-Lipschitz maps. Similarly, if we only consider Carnot groups, we obtain a similar variation of [201, Definition 4.1].

We give next a criterion for purely bi-Lipschitz homogeneous unrectifiability.

Lemma 4.7. *Let (X, d, \mathcal{H}^k) be a locally compact locally doubling metric measure space, with $k = \dim_H X$. If every \mathcal{H}^k -positive measure subset of X contains two points that have two tangent spaces that are not bi-Lipschitz equivalent, then (X, d) is purely bi-Lipschitz homogeneous unrectifiable (according to Definition 4.5).*

Proof. We prove that there is no bi-Lipschitz map $f : U \subseteq (X', d') \rightarrow (X, d)$, where $\mathcal{H}^k(f(U)) > 0$ and $(X', d') \in \mathcal{T}_k$. As (X, d) is locally compact, we can restrict ourselves to consider U to be closed.

If there exists such a map, first of all notice that $\mathcal{H}^k(U) > 0$ because f is bi-Lipschitz. Now we can restrict ourselves to the set of the points of density one of U with respect to \mathcal{H}^k , say W , and W is a set of full \mathcal{H}^k -measure in U as a consequence of Lebesgue Differentiation Theorem [125, page 77] that can be applied due to [125, Theorem 3.4.3]. Then, by the fact that f is bi-Lipschitz, the set $f(W)$ has full \mathcal{H}^k -measure in $f(U)$. The set Z of points in $f(W)$ of density one of $f(U)$ with respect to \mathcal{H}^k , is still a set of full \mathcal{H}^k -measure in $f(U)$ because it is the intersection of two sets of full \mathcal{H}^k -measure in $f(U)$. Then it holds $\mathcal{H}^k(Z) > 0$ since $\mathcal{H}^k(f(U)) > 0$.

By hypothesis there exist two points $x, y \in W$ and $p = f(x), q = f(y) \in Z$ with two non-bi-Lipschitz tangent spaces T_p and T_q . Because of the fact that we are dealing with points of density one, we can say that $\text{Tan}(U, d', x) = \text{Tan}(X', d', x)$ and $\text{Tan}(f(U), d, p) = \text{Tan}(X, d, p)$ and the same holds with y and q , see [143, Proposition 3.1]. Passing to the tangents in p and x we get, as in [151, Section 5.2], some induced bi-Lipschitz map between T_p and one element of $\text{Tan}(X', d', x)$. In the same way we get a bi-Lipschitz map between T_q and one element of $\text{Tan}(X', d', y)$. By hypothesis each element of $\text{Tan}(X', d', x)$ is bi-Lipschitz equivalent to each element of $\text{Tan}(X', d', y)$, so that T_p is bi-Lipschitz equivalent to T_q , which is a contradiction. \square

Let us point out that in Definition 4.3 we require the parametrizing maps to be bi-Lipschitz while for the classical definitions of rectifiability one may just ask for the maps to be Lipschitz. We next give the Lipschitz counterpart of Definition 4.3 for the family of Carnot groups.

Definition 4.8 (Pauls Carnot rectifiability). Let (X, d) be a metric space of Hausdorff dimension k . We say that (X, d) is *Pauls Carnot rectifiable* if there exist countably many Carnot groups \mathbb{G}_i of homogeneous dimension k endowed with homogeneous left-invariant distances d_i , and Lipschitz maps $f_i : U_i \subseteq (\mathbb{G}_i, d_i) \rightarrow (X, d)$ such that

$$\mathcal{H}^k \left(X \setminus \bigcup_{i \in \mathbb{N}} f_i(U_i) \right) = 0.$$

We say that (X, d) is *purely Pauls Carnot unrectifiable* if for every Carnot group \mathbb{G} of homogeneous dimension k endowed with a left-invariant homogeneous distance $d_{\mathbb{G}}$, every Lipschitz

map $f : U \subseteq (\mathbb{G}, d_{\mathbb{G}}) \rightarrow (X, d)$ satisfies

$$\mathcal{H}^k(f(U)) = 0.$$

Remark 4.9. The definition given in Definition 4.8 is a generalization of [201, Definition 4.1] where it was considered only one Carnot group for the parametrization of X . The definition of *purely \mathbb{G} -unrectifiability*, with one Carnot group \mathbb{G} , was already given in [162, Definition 3.1]. That is, given a Carnot group \mathbb{G} of homogeneous dimension k , we say that a metric space (X, d) is *purely \mathbb{G} -unrectifiable* if every Lipschitz map $f : U \subseteq \mathbb{G} \rightarrow X$ satisfies $\mathcal{H}^k(f(U)) = 0$.

Remark 4.10. In this paper we will not focus on the Lipschitz counterpart to Definition 4.5. Restricting to the subfamily of \mathcal{T}_k made of homogeneous subgroups of Carnot groups, such Lipschitz counterpart would lead to a variant of [84, Definition 3] allowing countably many possibly different subgroups. We think there are pathological examples and more easy-to-ask questions that we are not able to answer up to now.

For example Peano’s curve tells that the Euclidean plane \mathbb{R}^2 can be Lipschitz rectified with $(\mathbb{R}, \|\cdot\|^{1/2})$. Notice that $(\mathbb{R}, \|\cdot\|^{1/2})$ is isometric to the vertical line in the Heisenberg group.

Question 8. Forcing the topological dimension to be the same, we wonder whether there exists a Lipschitz map

$$f : U \subseteq (\mathbb{R}^3, \|\cdot\|^{3/4}) \rightarrow \mathbb{H}^1,$$

with $\mathcal{H}^4(f(U)) > 0$.

Remark 4.11. As in Remark 4.4, if (X, d) has Hausdorff dimension k and $\mathcal{H}^k(X) > 0$, it holds that if (X, d) is purely Pauls Carnot unrectifiable then it is not Pauls Carnot rectifiable.

2. A Carnot algebra with uncountably many non-isomorphic Carnot sub-algebras

In this section we prove that there exists a Carnot algebra \mathfrak{g} of dimension 8 that has uncountably many pairwise non-isomorphic Carnot sub-algebras of dimension 7. The Lie algebra \mathfrak{g} is constructed in Definition 4.14, and in Proposition 4.16 we prove the claimed result.

Definition 4.12. Given $\mu \in \mathbb{R}$, we denote by \mathfrak{g}_{μ} the Carnot algebra of step 3 and dimension 7 given by

$$\mathfrak{g}_{\mu} := V_{\mu}^1 \oplus V_{\mu}^2 \oplus V_{\mu}^3,$$

where

$$V_{\mu}^1 := \text{span}\{X_1, X_2, X_3\}, \quad V_{\mu}^2 := \text{span}\{X_4, X_5, X_6\}, \quad V_{\mu}^3 := \text{span}\{X_7\},$$

with the following relations

$$(4.1) \quad \begin{aligned} [X_1, X_2] &= X_4, & [X_1, X_3] &= -X_6, & [X_2, X_3] &= X_5; \\ [X_1, X_5] &= -X_7, & [X_2, X_6] &= \mu X_7, & [X_3, X_4] &= (1 - \mu)X_7, \end{aligned}$$

where all the other commutators between two vectors of the basis $\{X_1, \dots, X_7\}$ that are not listed above are zero.

Remark 4.13. The family $\{\mathfrak{g}_{\mu}\}_{\mu \in \mathbb{R}}$ in Definition 4.12 contains a subfamily that consists of uncountably many pairwise non-isomorphic Carnot algebras, which are called of type 147E, see [119]. Indeed, if $\mu_1, \mu_2 \notin \{0, 1\}$, the Lie algebra \mathfrak{g}_{μ_1} is isomorphic to \mathfrak{g}_{μ_2} if and only if $I(\mu_1) = I(\mu_2)$, where

$$I(\mu) := \frac{(1 - \mu + \mu^2)^3}{\mu^2(\mu - 1)^2}.$$

Our plan is to add a direction X_0 in the first stratum of a specific Carnot algebra given by Definition 4.12, namely the one with $\mu = 0$. Then, we show the existence of uncountably many pairwise non-isomorphic Carnot sub-algebras of dimension 7 in this new Carnot algebra of dimension 8.

Definition 4.14. In this section we denote by \mathfrak{g} the Carnot algebra of step 3 and dimension 8 given by

$$\mathfrak{g} := V^1 \oplus V^2 \oplus V^3,$$

where

$$V^1 := \text{span}\{X_0, X_1, X_2, X_3\}, \quad V^2 := \text{span}\{X_4, X_5, X_6\}, \quad V^3 := \text{span}\{X_7\},$$

with the following bracket relations

$$(4.2) \quad \begin{aligned} [X_1, X_2] &= X_4, & [X_1, X_3] &= -X_6, & [X_1, X_0] &= -X_4, & [X_2, X_3] &= X_5; \\ [X_1, X_5] &= -X_7, & [X_3, X_4] &= X_7, & [X_0, X_6] &= X_7, \end{aligned}$$

and all the other commutators between two elements of the basis $\{X_0, X_1, \dots, X_7\}$ that are not listed above are 0.

Remark 4.15. Let us show that the one defined in Definition 4.14 is a Lie algebra. It suffices to verify Jacobi identity on triples of pairwise different vectors of the basis. Since the step of the stratification is equal to 3, it suffices to show the Jacobi identity on vectors in the first stratum V^1 . Then, as we are extending \mathfrak{g}_0 in Definition 4.12, we just have to check the Jacobi identity on the triples $\{X_1, X_2, X_0\}$, $\{X_2, X_3, X_0\}$ and $\{X_1, X_3, X_0\}$. A simple computation yields

$$(4.3) \quad \begin{aligned} [X_1, [X_2, X_0]] + [X_2, [X_0, X_1]] + [X_0, [X_1, X_2]] &= 0 + [X_2, X_4] + [X_0, X_4] = 0, \\ [X_2, [X_3, X_0]] + [X_3, [X_0, X_2]] + [X_0, [X_2, X_3]] &= 0 + 0 + [X_0, X_5] = 0, \\ [X_1, [X_3, X_0]] + [X_3, [X_0, X_1]] + [X_0, [X_1, X_3]] &= 0 + [X_3, X_4] - [X_0, X_6] = X_7 - X_7 = 0, \end{aligned}$$

which is what we wanted.

Now we are ready for the main proposition of this section.

Proposition 4.16. *If \mathfrak{g} is the Carnot algebra of dimension 8 and step 3 in Definition 4.14, then there exist uncountably many Carnot sub-algebras of dimension 7 of \mathfrak{g} that are pairwise non-isomorphic.*

Proof. We present explicitly an uncountable family of Carnot sub-algebras of dimension 7 of \mathfrak{g} , indexed by $\lambda \in \mathbb{R}$, that are isomorphic to \mathfrak{g}_λ in Definition 4.12 if $\lambda \neq 1$. Then by Remark 4.13 we get the conclusion.

Given $\lambda \in \mathbb{R}$, with $\lambda \neq 1$, let us define the following vector in $V^1 \subseteq \mathfrak{g}$,

$$(4.4) \quad Y_2 := X_2 + \lambda X_0.$$

Then $\{X_1, Y_2, X_3\}$ are linearly independent vectors of V^1 . By explicit computations, using the relations in (4.2), we have

$$(4.5) \quad \begin{aligned} [X_1, Y_2] &= (1 - \lambda)X_4 =: Y_4, \\ [X_1, X_3] &= -X_6, \\ [Y_2, X_3] &= X_5; \end{aligned}$$

$$\begin{aligned}
(4.6) \quad & [X_1, Y_4] = 0, \\
& [X_1, X_5] = -X_7, \\
& [X_1, X_6] = 0, \\
& [Y_2, Y_4] = 0, \\
& [Y_2, X_5] = 0, \\
& [Y_2, X_6] = \lambda X_7, \\
& [X_3, Y_4] = (1 - \lambda)X_7, \\
& [X_3, X_5] = 0, \\
& [X_3, X_6] = 0,
\end{aligned}$$

and all the other commutators between two elements of the linearly independent vectors $\{X_1, Y_2, X_3, Y_4, X_5, X_6, X_7\}$, that are not listed above, vanish. Then in view of (4.5) and (4.6), if $\lambda \neq 1$, the subspace $\mathfrak{w}^1 := \text{span}\{X_1, Y_2, X_3\}$ generates a Carnot sub-algebra of step 3 and dimension 7 in \mathfrak{g} , that is isomorphic to \mathfrak{g}_λ in Definition 4.12. \square

3. Proof of the main results

In this section we construct the example that satisfies Theorem 4.1. We build the hypersurface S in the Carnot group \mathbb{G} whose Lie algebra \mathfrak{g} is as in Definition 4.14. Before that, let us discuss some notions related to hypersurfaces.

Definition 4.17 (Characteristic points). Let \mathbb{G} be a Carnot group. Let S be a Euclidean C^1 -hypersurface in $\mathbb{G} \cong \mathbb{R}^n$. We say that $x \in S$ is a *characteristic point* of S if

$$(4.7) \quad \mathbb{V}_1(x) \subseteq T_x S,$$

where $\mathbb{V}_1(x)$ is the horizontal bundle at x , see (1.7), and $T_x S$ is the Euclidean tangent of S , i.e., the tangent space of S seen as submanifold of $\mathbb{G} \cong \mathbb{R}^n$. We shall use the term Euclidean in contrast with the intrinsic sub-Riemannian one.

We will say that a C^1 -hypersurface S is *non-characteristic* if it does not have characteristic points as in (4.7).

Remark 4.18. We identify \mathbb{G} with \mathbb{R}^n by means of exponential coordinates as in (1.4) associated to a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} , and we call m the dimension of the first stratum of the Lie algebra. If we take $f \in C^1(\mathbb{G})$ we will denote with $\nabla f|_x$ the *full gradient* of f at x , i.e., the vector $\sum_{i=1}^n (\partial_{x_i} f)(x) \partial_{x_i}|_x$, and with $\nabla_{\text{H}} f|_x$ the *horizontal gradient* of f at x , i.e., the vector $\sum_{i=1}^m (X_i f)(x) X_i|_x$.

If S is a Euclidean C^1 -hypersurface in \mathbb{G} , for every point $p \in S$ there exist an open neighbourhood U_p of it and $f \in C^1(U_p)$ such that

$$(4.8) \quad S \cap U_p = \{x \in U_p : f(x) = 0\},$$

with $\nabla f \neq 0$ on $S \cap U_p$. The Euclidean tangent space of S at an arbitrary point $x \in S \cap U_p$ is

$$(4.9) \quad T_x S := \{v : \langle v, \nabla f|_x \rangle_x = 0\},$$

where $\langle \cdot, \cdot \rangle_x$ is the usual inner product, i.e., $\langle \partial_{x_i}|_x, \partial_{x_j}|_x \rangle_x = \delta_{ij}$, and $v = \sum_{i=1}^n v_i \partial_{x_i}|_x$. Then $x \in U_p$ is a characteristic point (4.7) if and only if (see (4.9)) it holds that $X_i f(x) = 0$ for all $i = 1, \dots, m$.

Thus a Euclidean C^1 -hypersurface S in \mathbb{G} with non-characteristic points is C_{H}^1 , because we have the representation in (4.8) with $(X_1 f, \dots, X_m f) \neq 0$ on U_p , see Definition 1.104 and the coordinates representation in Definition 1.110.

Given a $C^1_{\mathbb{H}}$ -hypersurface S , a point $p \in S$ and a representative f around p as in (1.81), we stress that the homogeneous tangent subgroup of S at p (see Definition 1.104, Definition 1.111 for this notion, and Remark 1.114 for properties), that we will also call *tangent group*, or the *intrinsic tangent* of S at p , has the following representation in exponential coordinates

$$(4.10) \quad T_p^I S := \left\{ v \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^m v_i X_i f(p) = 0 \right\}.$$

Let us recall that the set defined in (4.10) is the intrinsic tangent of S at p in the local Hausdorff topology, compare with Remark 1.114.

Now we give the definition of *vertical surface*. Loosely speaking, a vertical surface in a Carnot group \mathbb{G} is a C^1 -surface that depends only on the horizontal coordinates.

Definition 4.19. Let \mathbb{G} be a Carnot group identified with \mathbb{R}^n by means of exponential coordinates associated to a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} . Let m be the dimension of the first stratum of the Lie algebra. A *vertical surface* V is

$$V := \{x \in \Omega \times \mathbb{R}^{n-m} : f(x_1, \dots, x_m) = 0\},$$

where $f : \Omega \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$, with Ω open, is a C^1 -function with $\nabla f \neq 0$ on the set $\{\omega \in \Omega : f(\omega_1, \dots, \omega_m) = 0\}$. Moreover, if f is linear we say that V is a *vertical subgroup of codimension one*.

Remark 4.20. An arbitrary vertical surface as in Definition 4.19 is a C^1 -hypersurface with no characteristic points, i.e., points that satisfy (4.7). This is due to the fact that, if $1 \leq i \leq m$, in exponential coordinates we have

$$X_i = \partial_{x_i} + r_i(x),$$

where $r_i(x)$ is a polynomial combination of $\partial_{x_{i+1}}, \dots, \partial_{x_n}$, see [112, Proposition 2.2], and then, for all $x \in \omega$,

$$X_i f(x) = \partial_{x_i} f(x),$$

as f depends only on the first m variables. Thus, from Remark 4.18, a vertical surface is also a $C^1_{\mathbb{H}}$ -hypersurface.

Before going on let us show that, in arbitrary Carnot groups, we always have a surface that has every vertical subgroup of codimension one as tangent.

Lemma 4.21. *Given a Carnot group \mathbb{G} , there exists a vertical surface V such that for every vertical subgroup \mathbb{W} of codimension one in \mathbb{G} there exists $p \in V$ such that $T_p^1 V = \mathbb{W}$.*

Proof. Let us consider

$$V := \left\{ x \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^m x_i^2 = 1 \right\},$$

where m is the rank of \mathbb{G} . At an arbitrary point $p = (x_1, \dots, x_m, x_{m+1}, \dots, x_n)$, we have that, by Equation (4.10) and Remark 4.20,

$$T_p^I V = \left\{ v \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^m v_i x_i = 0 \right\},$$

and then, as every linear function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ can be written as $f(v) = \sum_{i=1}^m v_i x_i$ for a vector (x_1, \dots, x_m) of norm 1, we get the desired conclusion. \square

Let us now pass to the proof of Theorem 4.1. First of all let us identify \mathbb{G} with \mathbb{R}^8 by using exponential coordinates and the ordered basis (X_0, X_1, \dots, X_7)

$$(4.11) \quad \begin{aligned} x &= (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \rightarrow \\ &\rightarrow \exp(x_0 X_0 + x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4 + x_5 X_5 + x_6 X_6 + x_7 X_7). \end{aligned}$$

In these coordinates we can express the left-invariant vector fields $X_0(x), X_1(x), X_2(x), X_3(x)$ that extend X_0, X_1, X_2, X_3 , in this way, see [112, Proposition 2.2]:

$$(4.12) \quad \begin{aligned} X_0(x) &= \partial_{x_0} + r_0(x), \\ X_1(x) &= \partial_{x_1} + r_1(x), \\ X_2(x) &= \partial_{x_2} + r_2(x), \\ X_3(x) &= \partial_{x_3} + r_3(x), \end{aligned}$$

where $r_0(x), r_1(x), r_2(x), r_3(x)$ are combinations, with polynomial coefficients of the coordinates, of $\partial_{x_4}, \partial_{x_5}, \partial_{x_6}, \partial_{x_7}$. Now we are ready to state and prove one of the main results of this chapter.

Proposition 4.22. *There exist a Carnot group \mathbb{G} and an analytic non-characteristic hypersurface $S \subseteq \mathbb{G}$ with uncountably many pairwise non-isomorphic tangent groups.*

Proof. Let us consider the Carnot algebra \mathfrak{g} in Definition 4.14 and $\mathbb{G} := \exp \mathfrak{g}$ identified with \mathbb{R}^8 by means of the exponential coordinates in (4.11). Let us consider the vertical surface

$$(4.13) \quad S = \left\{ x \in \mathbb{G} \equiv \mathbb{R}^8 : f(x) := \frac{1}{3}x_2^3 + x_0 = 0 \right\}.$$

By Remark 4.20 this is an analytic non-characteristic hypersurface. From easy computations due to the particular form of X_i 's in (4.12) and from the expression of the tangent group in (4.10), it follows that

$$\text{Lie}(T_x^1 S) = \text{span}\{X_1, X_2 - x_2^2 X_0, X_3, X_4, X_5, X_6, X_7\},$$

and then $\text{Lie}(T_x^1 S)$ is isomorphic to the Carnot algebra generated by \mathfrak{w}^1 defined at the end of the proof of Proposition 4.16, where the λ there is now equal to $-x_2^2$. Then, the Lie algebra $\text{Lie}(T_x^1 S)$ is isomorphic to $\mathfrak{g}_{-x_2^2}$ defined in Definition 4.12. Because of the fact that given any $\lambda \leq 0$ there is always a point in S satisfying $\lambda = -x_2^2$, Remark 4.13 grants us that the family $\{\text{Lie}(T_x^1 S)\}_{x \in S}$ contains uncountably many pairwise non-isomorphic Carnot algebras and then the family $\{T_x^1 S\}_{x \in S}$ contains uncountably many pairwise non-isomorphic Carnot groups. \square

Remark 4.23. In particular, every S as in Proposition 4.22 is not bi-Lipschitz equivalent to an open set in a Carnot group. This follows from a blow-up argument and Pansu's differentiability theorem [200]. The argument will be made clear in the proof of the forthcoming Theorem 4.24. We stress that even for some sub-Riemannian manifolds the constancy of the tangent may not give bi-Lipschitz local equivalence with the tangent Carnot group, see [150].

We are now ready to give the first negative result about rectifiability for a hypersurface as in Proposition 4.22.

Theorem 4.24. *There exist a Carnot group \mathbb{G} and an analytic non-characteristic hypersurface $S \subseteq \mathbb{G}$, of Hausdorff dimension 12, such that on every \mathcal{H}^{12} -positive measure subset of it there are two points with non-isomorphic tangents. Moreover, one can find such an S in such a way that S is in addition purely bi-Lipschitz homogeneous unrectifiable according to Definition 4.5.*

Proof. Let us consider \mathfrak{g} the Carnot algebra in Definition 4.14. Let us identify $\mathbb{G} := \exp \mathfrak{g}$ with \mathbb{R}^8 by means of the exponential coordinates in (4.11), and let us fix a left-invariant homogeneous distance d on \mathbb{G} . Let us consider S as in the proof of Proposition 4.22.

We claim that

$$(4.14) \quad \mathcal{H}^{12}(S \cap \{x_2 = \lambda\}) = 0, \quad \forall \lambda \in \mathbb{R}.$$

Indeed, we know that $S \cap \{x_2 = \lambda\}$ is the intersection of two $C_{\mathbb{H}}^1$ -hypersurfaces. Moreover the tangent subgroup to $S \cap \{x_2 = \lambda\}$ at an arbitrary point x is

$$\mathbb{W} := \{v \in \mathbb{G} \equiv \mathbb{R}^8 : x_2^2 v_2 + v_0 = 0\} \cap \{v \in \mathbb{G} \equiv \mathbb{R}^8 : v_2 = 0\} = \{v \in \mathbb{G} \equiv \mathbb{R}^8 : v_0 = v_2 = 0\}.$$

Since \mathbb{W} is complemented by the horizontal subgroup $\mathbb{H} := \{\exp(tX_0 + sX_2) : t, s \in \mathbb{R}\}$, we can apply [96, Theorem A.5] to get that $S \cap \{x_2 = \lambda\}$ is locally the graph of an intrinsically Lipschitz function defined on \mathbb{W} with values in \mathbb{H} . Notice that \mathbb{H} is a subgroup because $[X_0, X_2] = 0$. Since \mathbb{W} has homogeneous and thus Hausdorff dimension 11 with respect to the distance d , then by the estimate on the Hausdorff measure in [109, Theorem 2.3.7] we get (4.14).

Now we claim that each subset U of S that satisfies $\mathcal{H}^{12}(U) > 0$ has at least two points with two non-bi-Lipschitz Gromov-Hausdorff tangents. Indeed, the equation (4.14) tells us that for each $U \subseteq S$ with $\mathcal{H}^{12}(U) > 0$, the coordinate function x_2 takes on U uncountably many values. This, according to the fact that $T_x^1 S$ is a Carnot group isomorphic to the one with Lie algebra $\mathfrak{g}_{-x_2^2}$ (see the proof of Proposition 4.22), immediately tells that there are in U at least two points with two non-isomorphic (because of Remark 4.13) Carnot groups as tangent. By Pansu's version of Rademacher theorem, see Theorem 1.100, two non-isomorphic Carnot groups cannot be bi-Lipschitz equivalent, so the claim follows. Now the proof is completed by using the criterion shown in Lemma 4.7. \square

From Remark 4.4 we have the following consequence to Theorem 4.24.

Corollary 4.25. *There exist a Carnot group \mathbb{G} and an analytic non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is not bi-Lipschitz homogeneous rectifiable according to Definition 4.5*

Remark 4.26. Notice that from Corollary 4.25 it follows that S is not rectifiable according to the countable bi-Lipschitz variant of [84, Definition 3], see Remark 4.6 for details. We notice here that we still are not able to prove that our counterexample is not rectifiable according to [84, Definition 3], see Remark 4.10 for further discussions. Nevertheless, in the forthcoming proof of Theorem 4.1, we show that the same S as in Theorem 4.24 is not rectifiable according to [201, Definition 4.1].

Hence, we are now ready for the proof of Theorem 4.1

Proof of Theorem 4.1. Let us take S and \mathbb{G} as in the proof of Proposition 4.22. Let us fix on \mathbb{G} a homogeneous left-invariant distance d . Then from Remark 1.115 we get that the Hausdorff dimension of S is 12, because the homogeneous dimension of \mathbb{G} is 13. We will show there is no Lipschitz map $f : U \subseteq \hat{\mathbb{G}} \rightarrow (S, d)$, with $\hat{\mathbb{G}}$ a Carnot group of homogeneous dimension 12, and $\mathcal{H}^{12}(f(U)) > 0$.

Suppose by contradiction there is such a map. We can assume U closed, because S is complete. By composing the map f with the inclusion $i : S \hookrightarrow \mathbb{G}$ we get a Lipschitz map $\tilde{f} : U \subseteq \hat{\mathbb{G}} \rightarrow \mathbb{G}$.

Let us call $U_{\text{ND}} \subseteq U$ the set of points where \tilde{f} is non-differentiable, $U_{\text{I}} \subseteq U$ the set of differentiability points x of \tilde{f} for which $d\tilde{f}_x : \hat{\mathbb{G}} \rightarrow \mathbb{G}$ is injective and $U_{\text{NI}} \subseteq U$ the set of differentiability points x of \tilde{f} for which $d\tilde{f}_x$ is not injective. We thus have $U = U_{\text{ND}} \sqcup U_{\text{I}} \sqcup U_{\text{NI}}$, $\tilde{f}(U) = \tilde{f}(U_{\text{ND}}) \cup \tilde{f}(U_{\text{I}}) \cup \tilde{f}(U_{\text{NI}})$ and we know, from Rademacher theorem, see Theorem 1.100, and the fact that \tilde{f} is Lipschitz, that $\mathcal{H}^{12}(\tilde{f}(U_{\text{ND}})) = \mathcal{H}^{12}(U_{\text{ND}}) = 0$.

We claim that $\mathcal{H}^{12}(\tilde{f}(U_{\text{I}})) > 0$. Indeed, the homogeneous dimension of $\hat{\mathbb{G}}$ is 12. Thus, for $x \in U_{\text{NI}}$, we get that $d\tilde{f}_x(\hat{\mathbb{G}})$ is a homogeneous subgroup of \mathbb{G} of homogeneous dimension at most 11, see Lemma 4.28 below. Then

$$J_{12}(d\tilde{f}_x) = \frac{\mathcal{H}^{12}(d\tilde{f}_x(B(0, 1)))}{\mathcal{H}^{12}(B(0, 1))} = 0,$$

and from Theorem 1.103 applied to $\tilde{f} : U_{\text{NI}} \rightarrow \mathbb{G}$ we get $\mathcal{H}^{12}(\tilde{f}(U_{\text{NI}})) = 0$. Now we conclude the proof of the claim:

$$\begin{aligned} \mathcal{H}^{12}(\tilde{f}(U_I)) &= \mathcal{H}^{12}(\tilde{f}(U_I)) + \mathcal{H}^{12}(\tilde{f}(U_{\text{NI}})) + \mathcal{H}^{12}(\tilde{f}(U_{\text{ND}})) \\ &\geq \mathcal{H}^{12}(\tilde{f}(U)) > 0. \end{aligned}$$

For every point z in U_I there exists an injective Carnot homomorphism $d\tilde{f}_z : \hat{\mathbb{G}} \rightarrow \mathbb{G}$. For how it is constructed the differential $d\tilde{f}_z$ (see Remark 1.101) we know that for ω in a dense subset Ω of $\hat{\mathbb{G}}$ we have

$$d\tilde{f}_z(\omega) = \lim_{z\delta_t\omega \in U, t \rightarrow 0} \delta_{1/t} \left(\tilde{f}(z)^{-1} \tilde{f}(z\delta_t\omega) \right).$$

From the very construction we thus get that $d\tilde{f}_z(\omega)$ is in the Hausdorff tangent of S at $\tilde{f}(z)$. Then from the discussion slightly before Definition 4.19 we get that $d\tilde{f}_z(\omega)$ takes values in $T_{\tilde{f}(z)}^1 S$ for $\omega \in \Omega$. Now taking into account that $d\tilde{f}_z$ is defined on all of $\hat{\mathbb{G}}$ by density (see Remark 1.101) and considering that $T_{\tilde{f}(z)}^1 S$ is closed, we get that $d\tilde{f}_z$ takes values in $T_{\tilde{f}(z)}^1 S$, which is a Carnot subgroup of \mathbb{G} of homogeneous dimension 12, thanks to the explicit expression of the tangent in the proof of Proposition 4.22 and [149, Theorem 4.4, (iii)]. Thus as $\hat{\mathbb{G}}$ has homogeneous dimension 12 itself and $d\tilde{f}_z$ is injective, we get that $d\tilde{f}_z$ is an isomorphism and so $\hat{\mathbb{G}}$ is isomorphic to $T_{\tilde{f}(z)}^1 S$ for every $z \in U_I$.

In order to conclude, we notice that in the proof of Theorem 4.24 we showed that on every \mathcal{H}^{12} -positive measure subset of S there are at least two non-isomorphic tangent spaces, so that, because $\mathcal{H}^{12}(\tilde{f}(U_I)) > 0$ holds, we should have at least two non-isomorphic tangent spaces on $\tilde{f}(U_I)$. But we proved that all of them are isomorphic to $\hat{\mathbb{G}}$, thus we get a contradiction.

Hence, we proved that there is no Lipschitz map $f : U \subseteq \hat{\mathbb{G}} \rightarrow (S, d)$, with $\hat{\mathbb{G}}$ a Carnot group of homogeneous dimension 12, and $\mathcal{H}^{12}(f(U)) > 0$. From Remark 4.11 we thus get the conclusion of the proof. \square

We shall prove the auxiliary Lemma 4.28 that has been exploited in the proof of Theorem 4.1. Let us start with a remark.

Remark 4.27. Every Carnot homomorphism induces a linear map $\varphi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ that is a Lie algebra homomorphism such that $\varphi_* \circ \delta_\lambda = \delta_\lambda \circ \varphi_*$. From this property it easily follows that for every $1 \leq i \leq \kappa$, where κ is the step of the group \mathbb{G} , we get $\varphi_*(V_i^{\mathfrak{g}}) \subseteq V_i^{\mathfrak{h}}$, where $V_i^{\mathfrak{g}}$ and $V_i^{\mathfrak{h}}$ are the i -th strata of \mathfrak{g} and \mathfrak{h} , respectively.

Lemma 4.28. *Let $\varphi : \mathbb{G} \rightarrow \mathbb{H}$ be a Carnot homomorphism between two Carnot groups. If φ is not injective then the homogeneous dimension of \mathbb{G} is strictly greater than the homogeneous dimension on $\varphi(\mathbb{G})$.*

Proof. By definition of Carnot homomorphism we get that $\text{Ker}\varphi$ is a homogeneous subgroup of \mathbb{G} and $\varphi(\mathbb{G})$ is a homogeneous subgroup of \mathbb{H} . If an element $g \in \mathfrak{g}$ is in $V_i^{\mathfrak{g}}$ for some i , we say that i is the degree of g and write $\deg g = i$. We take $\{e_1, \dots, e_l, e_{l+1}, \dots, e_n\} \subseteq \cup_{i=1}^{\kappa} V_i$ a basis of \mathfrak{g} , such that $\{e_1, \dots, e_l\}$ is a basis of $\text{Ker}\varphi$, and κ is the step of the group \mathbb{G} . Then $\{\varphi_*(e_{l+1}), \dots, \varphi_*(e_n)\}$ is a basis of the Lie algebra of $\varphi(\mathbb{G})$. By the fact that φ_* preserves the stratification (see Remark 4.27), we get

$$\deg \varphi_*(e_i) = \deg e_i$$

for each $l + 1 \leq i \leq n$. Then by [149, Theorem 4.4, (iii)] and the previous equation we get

$$\dim_{\text{hom}} \varphi(\mathbb{G}) = \sum_{i=l+1}^n \deg \varphi_*(e_i) = \sum_{i=l+1}^n \deg e_i < \sum_{i=1}^n \deg e_i = \dim_{\text{hom}} \mathbb{G}.$$

where we used in the strict inequality that $l > 0$ being φ not injective. \square

Remark 4.29. The example of Theorem 4.1 is actually a $C^1_{\mathbb{H}}$ -hypersurface because it is analytic and non-characteristic, see Remark 4.18. Thus they are rectifiable in the sense of Franchi, Serapioni and Serra Cassano (see also Definition 1.105) but we proved they are not in the sense of [201, Definition 4.1]. Indeed, the definition of Pauls Carnot rectifiability we are adopting here is a generalization of [201, Definition 4.1], see Remark 4.9.

Remark 4.30. We notice that every tangent group to S as in the proof of Theorem 4.24 is a Carnot group. So S is an example of a Euclidean non-characteristic hypersurface in a Carnot group that cannot be Lipschitz parametrizable by countably many subsets of its intrinsic tangents.

We state here as a theorem something we already proved in Theorem 4.24.

Theorem 4.31. *There exists a locally compact and locally doubling metric measure space (X, d, \mathcal{H}^k) , where k is the Hausdorff dimension of X , that satisfies the following two properties:*

- (1) *For each $x \in X$, there exists (up to isometry) only one element in $\text{Tan}(X, d, x)$ and it is a Carnot group;*
- (2) *For each $U \subseteq X$ with $\mathcal{H}^k(U) > 0$ there exists an uncountable family $\{x_i\}_{i \in I} \subseteq U$ of points such that the tangent spaces at these points are pairwise non-bi-Lipschitz equivalent.*

Proof. The example and the proof are exactly the same as in the proof of Theorem 4.24. \square

Remark 4.32. Another example (a sub-Riemannian manifold) that satisfies Theorem 4.31 was presented in [151, Proposition 16].

4. Pauls rectifiability of C^∞ -hypersurfaces in Heisenberg groups

In this section we prove that C^∞ -hypersurfaces in the n -th Heisenberg group \mathbb{H}^n , with $n \geq 2$, are rectifiable according to [84, Definition 3], see Theorem 4.46 and Remark 4.47 for details. We start with a Lipschitz-type estimate, which more generally holds for Carnot groups of step 2, see Proposition 4.38.

We remark here that, in order to prove the main result of this section, we will use Proposition 4.38 only for Heisenberg groups, but we prove it in the general case of Carnot groups of step 2. We point out that the result in Proposition 4.41 requires the latter Lipschitz-type estimate, plus a connectivity argument that makes that proof work only for \mathbb{H}^n , with $n \geq 2$. For arbitrary Carnot groups of step 2, the intrinsic distance $d^{\mathbb{F}}$ in the statement of Proposition 4.41 might not even be finite.

Then in Section 4.3 we show the equivalence of the intrinsic distance and the induced distance for intrinsically Lipschitz graphs in Heisenberg groups. This statement has been suggested to us by Fässler and Orponen, adapting an argument of [82]. After that, in Section 4.4, we prove that C^∞ non-characteristic hypersurfaces in \mathbb{H}^n , with $n \geq 2$, carry a sub-Riemannian structure, see Proposition 4.42 and [210, Theorem 1.1]. Therefore, we show that the sub-Riemannian distance is locally equivalent to the distance induced from \mathbb{H}^n , see Proposition 4.43. By means of [151, Theorem 2] we are able to conclude the result: see Theorem 4.46.

4.1. Carnot Groups of step 2. In this subsection we recall the structure of step-2 groups in exponential coordinates. We stress here that sometimes we use Einstein notation: we do not use the summand symbol Σ and we remind that, in this case, we are tacitly taking the sum over the repeated indices. Every Carnot group of step-2 arises as follows in coordinates.

Let $(B_{jl}^1), \dots, (B_{jl}^n)$ be n linearly independent skew-symmetric $m \times m$ matrices with $j, l = 1, \dots, m$. Consider the homogeneous group $(\mathbb{R}^m \times \mathbb{R}^n, \cdot, \delta_\lambda)$, which turns out to be a Carnot group, where the operation is

$$(4.15) \quad (x_1, \dots, x_m, y_1, \dots, y_n) \cdot (\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n) := \left(x_1 + \tilde{x}_1, \dots, x_m + \tilde{x}_m, y_1 + \tilde{y}_1 + \frac{1}{2}B_{j_1 l}^1 \tilde{x}_{j_1} \tilde{x}_{l_1}, \dots, y_n + \tilde{y}_n + \frac{1}{2}B_{j_l}^n \tilde{x}_{j_l} \tilde{x}_{l_l} \right),$$

and the dilations are

$$(4.16) \quad \delta_\lambda(x_1, \dots, x_m, y_1, \dots, y_n) := (\lambda x_1, \dots, \lambda x_m, \lambda^2 y_1, \dots, \lambda^2 y_n),$$

for every $\lambda > 0$. We shall endow the previous Carnot group with an arbitrary homogeneous norm that induces a homogeneous left-invariant distance.

In the Carnot groups defined above we call X_j , with $j = 1, \dots, m$, the left-invariant vector fields that agrees with ∂_{x_j} at the origin. We call Y_k , with $k = 1, \dots, n$, the left-invariant vector fields that agrees with ∂_{y_k} at the origin. It holds, by simple computations,

$$(4.17) \quad \begin{aligned} X_j &= \partial_{x_j} + \frac{1}{2}B_{j_l}^k x_l \partial_{y_k}, \\ Y_k &= \partial_{y_k}. \end{aligned}$$

We shall consider the following two homogeneous subgroups

$$(4.18) \quad \mathbb{L} := \{(x_1, 0, \dots, 0)\}, \quad \mathbb{W} := \{(0, x_2, \dots, x_m, y_1, \dots, y_n)\}.$$

For what concerns this section, we say that $f : U \subseteq \mathbb{W} \rightarrow \mathbb{L}$ is *L-intrinsically Lipschitz* in U , with $L > 0$, if

$$C_{\mathbb{W}, \mathbb{L}} \left(p, \frac{1}{L} \right) \cap \text{graph}(f) = \{p\}, \quad \text{for all } p \in \text{graph}(f),$$

where $C_{\mathbb{W}, \mathbb{L}} \left(p, \frac{1}{L} \right)$ is defined in (1.21).

In what follows $\tilde{\varphi} : \mathbb{W} \rightarrow \mathbb{L}$ will be an intrinsically L Lipschitz function and $\varphi : \mathbb{R}^{m+n-1} \rightarrow \mathbb{R}$ is defined as

$$(4.19) \quad \tilde{\varphi}(0, x_2, \dots, x_m, y_1, \dots, y_n) = (\varphi(x_2, \dots, x_m, y_1, \dots, y_n), 0, \dots, 0).$$

Let us introduce the following family of vector fields. We stress that whenever φ is C^1 , the vector field D^φ applied to φ gives precisely the intrinsic gradient $\nabla^\varphi \varphi$ of φ in coordinates, compare with Remark 1.98.

Definition 4.33. Given $\tilde{\varphi}$ and φ as in (4.19) we define, for $j = 2, \dots, m$, the vector fields on \mathbb{W} at $\bar{x} := (0, x_2, \dots, x_m, y_1, \dots, y_n)$ as

$$(4.20) \quad D_j^\varphi|_{\bar{x}} := X_j|_{\bar{x}} + \varphi(x_2, \dots, x_m, y_1, \dots, y_n) B_{j_1}^k Y_k|_{\bar{x}}.$$

Definition 4.34. We will say that an absolutely continuous curve $\tilde{\gamma} : I \rightarrow \mathbb{W}$ is *horizontal* for the family of vector fields $\{D_j^\varphi\}_{j=2, \dots, m}$, if there exist $(a_2(t), \dots, a_m(t)) \in L^1(I; \mathbb{R}^{m-1})$ such that

$$(4.21) \quad \tilde{\gamma}'(t) = a_j(t) D_j^\varphi|_{\tilde{\gamma}(t)}, \quad \text{for a.e. } t \in I.$$

Then the φ -length of $\tilde{\gamma}$ is defined as

$$(4.22) \quad \ell_\varphi(\tilde{\gamma}) := \int_I \sqrt{a_2(s)^2 + \cdots + a_m(s)^2} ds.$$

Remark 4.35. Notice that, due to the specific form of X_j , Y_k and D_j^φ in (4.17) and (4.20) respectively, if

$$(4.23) \quad \tilde{\gamma}(t) := (0, x_2(t), \dots, x_m(t), y_1(t), \dots, y_n(t)),$$

then

$$(4.24) \quad \ell_\varphi(\tilde{\gamma}) = \int_I \sqrt{x_2'(s)^2 + \cdots + x_m'(s)^2} ds.$$

Remark 4.36. Using the notation in Section 4.1, the Heisenberg group $\mathbb{H}^{\bar{n}}$ is obtained when $m = 2\bar{n}$, $n = 1$ and $B_{ij}^1 = 1$ if and only if $i = j + \bar{n}$, otherwise it is zero.

4.2. Length comparison for Carnot groups of step 2. In this subsection we shall exploit the notation introduced in Section 4.1. We shall show that for Carnot groups of step 2, the length of the curve $\tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})$ measured with a left-invariant homogeneous distance d in the group \mathbb{G} is controlled from above by $\ell_\varphi(\tilde{\gamma})$ up to a multiplicative constant.

Remark 4.37. For the general theory of sub-Riemannian manifolds, including the Finsler case, one can check [146, Chapter 3], or also [2]. We recall that in \mathbb{G} we have two interpretations for the length of an absolutely continuous curve. Indeed, as in every Carnot-Carathéodory space, if the distance d on \mathbb{G} is induced by a norm $\|\cdot\|$ on the horizontal bundle \mathbb{V}_1 of \mathbb{G} , cf. (1.8), the length of an absolutely continuous curve $\gamma : I \rightarrow \mathbb{G}$ equals the following values

$$(4.25) \quad \text{length}(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(s_{i-1}), \gamma(s_i)) \right\} = \int_I \|\gamma'(t)\| dt,$$

where the sup is over the partitions $\sqcup_{i=0}^n [s_i, s_{i+1}]$ of I .

The proof of the forthcoming proposition was pointed out to us by Fässler and Orponen in the Heisenberg group and it is substantially contained in [82, Proposition 3.8]. We present here a general proof for step-2 groups.

Proposition 4.38. *Let \mathbb{G} be a step-2 Carnot group with the choice of coordinates as in Section 4.1 and \mathbb{W} and \mathbb{L} as in (4.18). Assume \mathbb{G} is endowed with a Carnot-Carathéodory distance d coming from the choice of a norm $\|\cdot\|$ on the horizontal layer of the Lie algebra of \mathbb{G} . Let $\tilde{\varphi} : \mathbb{W} \rightarrow \mathbb{L}$ be intrinsically L Lipschitz. Set $\varphi, D_j^\varphi, \ell_\varphi$ as in (4.19), (4.20), and (4.22), respectively. If $\tilde{\gamma} : I \rightarrow \mathbb{W}$ is horizontal with respect to $\{D_j^\varphi\}_{j=2, \dots, m}$, then*

$$\text{length}(\tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})) \leq C \cdot \ell_\varphi(\tilde{\gamma}),$$

where $C = C(\mathbb{G}, L)$.

Moreover, if the L^1 -norm of the controls $a_j(t)$ of $\tilde{\gamma}$ as in Definition 4.34 is bounded by K , the projection on the first component of the curve $s \mapsto \tilde{\varphi}(\tilde{\gamma}(s))$ is L' -Lipschitz, with $L' = L'(L, K, \mathbb{G})$.

Proof. Set $\gamma : I \rightarrow \mathbb{R}^{m+n-1}$ to be the curve

$$\gamma(t) := (x_2(t), \dots, x_m(t), y_1(t), \dots, y_n(t)),$$

where we use the notation (4.23). By the fact that $\tilde{\gamma}$ is horizontal with respect to $\{D_j^\varphi\}_{j=2, \dots, m}$ we get by easy computations that for each $k = 1, \dots, n$

$$(4.26) \quad y_k'(t) = x_j'(t) \left(\frac{1}{2} B_{ji}^k x_l(t) + \varphi(\gamma(t)) B_{j1}^k \right), \quad \text{for a.e. } t \in I,$$

where we sum over j and l from 2 to m . Now we consider the curve $\tilde{\gamma}$ between two intermediary times $t < t_1$ and we claim that

$$(4.27) \quad \sum_{i=2}^m |x_i(t_1) - x_i(t)| \leq C_1 \ell_\varphi \left(\tilde{\gamma}|_{[t, t_1]} \right),$$

with $C_1 = C_1(m)$. Indeed, this is a consequence of the fundamental theorem of calculus, Cauchy-Schwarz and (4.24).

Set $\Phi(\tilde{\gamma}(t)) := \tilde{\gamma}(t) \cdot \tilde{\varphi}(\tilde{\gamma}(t))$. By the definition of length it suffices to show that for all $[t, t_1] \subseteq I$ there exists a constant $C = C(L, \mathbb{G})$ such that

$$(4.28) \quad d(\Phi(\tilde{\gamma}(t)), \Phi(\tilde{\gamma}(t_1))) \leq C \cdot \ell_\varphi(\tilde{\gamma}|_{[t, t_1]}).$$

By the fact that $\tilde{\varphi}$ is intrinsically Lipschitz and [109, Proposition 2.3.4] one has that, setting $\|\cdot\|$ the homogeneous norm on \mathbb{G} associated to d , there exists a constant $C_0 = C_0(L)$ such that

$$(4.29) \quad d(\Phi(\tilde{\gamma}(t)), \Phi(\tilde{\gamma}(t_1))) \leq C_0 \left\| \pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(t_1)) \right) \right\|.$$

Then we leave to the reader to verify the algebraic equality, which depends on the fact that \mathbb{W} is a normal subgroup,

$$(4.30) \quad \pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(t_1)) \right) = \tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t)).$$

By exploiting the formula for the group law, it holds

$$(4.31) \quad \begin{aligned} & \tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t)) = \\ & = (0, x_2(t_1) - x_2(t), \dots, x_m(t_1) - x_m(t), \sigma_1(t_1, t), \dots, \sigma_n(t_1, t)), \end{aligned}$$

where for each $k = 1, \dots, n$ we have

$$(4.32) \quad \sigma_k(t_1, t) := y_k(t_1) - y_k(t) + B_{1j}^k \varphi(\gamma(t))(x_j(t_1) - x_j(t)) - \frac{1}{2} B_{jl}^k x_j(t_1) x_l(t),$$

where the sums on indices j and l run from 2 to m .

Then by (4.31) and the fact that $\|\cdot\|$ is equivalent to any other homogeneous norm on \mathbb{G} , we have that

$$(4.33) \quad \|\tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t))\| \sim \sum_{i=2}^m |x_i(t_1) - x_i(t)| + \sum_{k=1}^n \sqrt{|\sigma_k(t_1, t)|}.$$

Using (4.26) and that $B_{jl}^k x_j(t) x_l(t) = 0$ by skew-symmetry of B^k we can rewrite $\sigma_k(t_1, t)$ as follows

$$(4.34) \quad \begin{aligned} \sigma_k(t_1, t) &= \left(\int_t^{t_1} y'_k(\xi) d\xi \right) + B_{1j}^k \varphi(\gamma(t))(x_j(t_1) - x_j(t)) - \frac{1}{2} B_{jl}^k x_j(t_1) x_l(t) \\ &= \int_t^{t_1} \left(x'_j(\xi) B_{j1}^k (\varphi(\gamma(\xi)) - \varphi(\gamma(t))) + \frac{1}{2} x'_j(\xi) B_{jl}^k (x_l(\xi) - x_l(t)) \right) d\xi. \end{aligned}$$

Set

$$f(t_1, t) := \sup_{\xi \in [t, t_1]} |\varphi(\gamma(\xi)) - \varphi(\gamma(t))|.$$

It follows from (4.34), (4.27) and Cauchy-Schwarz inequality that

$$(4.35) \quad |\sigma_k(t_1, t)| \leq C_2 \left(\ell_\varphi(\tilde{\gamma}|_{[t, t_1]}) + f(t_1, t) \right) \ell_\varphi(\tilde{\gamma}|_{[t, t_1]}),$$

where $C_2 = C_2(m, B)$ and $B := \max |B_{j_l}^k|$. Now for each $\xi \in [t, t_1]$ we get by the fact φ is intrinsically Lipschitz, (4.30), (4.33), (4.27) and (4.35) with ξ instead of t_1 , that

$$\begin{aligned}
|\varphi(\gamma(\xi)) - \varphi(\gamma(t))| &\leq L \left\| \pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(\xi)) \right) \right\| \\
&= \|\tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(\xi) \cdot \tilde{\varphi}(\tilde{\gamma}(t))\| \\
(4.36) \quad &\sim \sum_{i=1}^m |x_i(\xi) - x_i(t)| + \sum_{k=1}^n \sqrt{|\sigma_k(\xi, t)|} \\
&\leq C_3 \left(\ell_\varphi(\tilde{\gamma}|_{[t, \xi]}) + \sqrt{f(\xi, t)} \sqrt{\ell_\varphi(\tilde{\gamma}|_{[t, \xi]})} \right),
\end{aligned}$$

where $C_3 = C_3(m, B, L)$. Now passing to the supremum as $\xi \in [t, t_1]$ in both sides of (4.36) we get

$$f(t_1, t) \leq C_3 \left(\ell_\varphi(\tilde{\gamma}|_{[t, t_1]}) + \sqrt{f(t_1, t)} \sqrt{\ell_\varphi(\tilde{\gamma}|_{[t, t_1]})} \right),$$

from which there exists $C_4 = C_4(m, B, L)$ such that

$$(4.37) \quad f(t_1, t) \leq C_4 \ell_\varphi(\tilde{\gamma}|_{[t, t_1]}).$$

Finally by chaining (4.29), (4.30), (4.33), (4.27), (4.35), and (4.37), we get (4.28) which was what we wanted. For the second part of the lemma we just chain (4.36) and (4.37) with ξ instead of t_1 , and use the fact that $\ell_\varphi(\tilde{\gamma}|_{[t, \xi]})$ is bounded from above by $C(K, m)|\xi - t|$ by the definition of ℓ_φ in (4.22). \square

4.3. Equivalence of intrinsic distance and induced distance on intrinsically Lipschitz graphs in the Heisenberg groups. In this subsection we shall prove that on intrinsically Lipschitz codimension-one graphs on \mathbb{H}^n , with $n \geq 2$, the intrinsic distance is globally equivalent to the distance induced by \mathbb{H}^n .

Definition 4.39. Given $\tilde{\varphi}$ and φ as in (4.19) we define the intrinsic length distance on the graph $\Gamma := \text{graph}(\tilde{\varphi}) = \{w \cdot \tilde{\varphi}(w) : w \in \mathbb{W}\} \subseteq \mathbb{G}$ as follows

$$(4.38) \quad d^\Gamma(x, y) := \inf \{ \text{length}(\gamma) \mid \gamma : [0, 1] \rightarrow \Gamma, \gamma(0) = x, \gamma(1) = y, \quad \gamma \text{ horizontal} \}.$$

Remark 4.40. Up to a globally bi-Lipschitz change of distance, we can suppose to work with a left-invariant homogeneous distance d on \mathbb{G} coming from a scalar product \mathbf{g} on the horizontal bundle \mathbb{V}_1 . Notice that if Γ is a smooth submanifold of \mathbb{G} and the horizontal bundle \mathbb{V}_1 intersects the tangent bundle of Γ in a bracket generating distribution, then the distance $d^\Gamma(x, y)$ is exactly the sub-Riemannian distance, let us call it $d_{\text{int}}(x, y)$, associated to the sub-Riemannian structure $(\Gamma, \mathbb{V}_1 \cap T\Gamma, \mathbf{g}|_{(\mathbb{V}_1 \cap T\Gamma) \times (\mathbb{V}_1 \cap T\Gamma)})$.

We now stress that in the specific case of the Heisenberg groups the distance d induced from \mathbb{H}^n is bi-Lipschitz equivalent to d^Γ defined in (4.38). The proof was suggested to us by Fässler and Orponen.

Proposition 4.41. *With the same assumptions and notation as in Proposition 4.38, if $\mathbb{G} = \mathbb{H}^n$ with $n \geq 2$, then*

$$(4.39) \quad d(x, y) \sim d^\Gamma(x, y), \quad \forall x, y \in \Gamma,$$

where d^Γ is defined in (4.38) and d is the induced distance, restriction of the one in \mathbb{H}^n . We recall that with (4.39) we mean that there exists a constant $C > 1$ such that $C^{-1}d(x, y) \leq d^\Gamma(x, y) \leq Cd(x, y)$ for every $x, y \in \Gamma$.

Proof. First of all notice that, using the notation in (4.20) and taking into account Remark 4.36, if $\mathbb{G} = \mathbb{H}^n$, then $D_j^\varphi|_{\bar{x}} = X_j|_{\bar{x}}$ for all $j = 2, \dots, 2n$ and $j \neq n+1$, while $D_{n+1}^\varphi|_{\bar{x}} = X_{n+1}|_{\bar{x}} + \varphi(x_2, \dots, x_{2n}, y_1)Y_1|_{\bar{x}}$. We also have $[X_j, X_{n+j}] = Y_1$ for every $j = 1, \dots, n$ and all the other commutators are zero. By definition of d^Γ (4.38), exploiting the definition of length (4.25) and the triangle inequality, we get

$$d(x, y) \leq d^\Gamma(x, y), \quad \forall x, y \in \Gamma.$$

Now we want to prove the opposite inequality up to a multiplicative constant. First of all, by a left translation, we can assume $x = 0$ and $y = w \cdot \tilde{\varphi}(w)$ for $w \in \mathbb{W}$. It holds that

$$(4.40) \quad d(x, y) = d(0, y) = \|w \cdot \tilde{\varphi}(w)\|_d \geq C_0 \|w\|_d,$$

where $\|\cdot\|_d$ is the homogeneous norm associated to d and $C_0 = C_0(\mathbb{W}, \mathbb{L})$, see [109, Proposition 2.2.2]. From now on, in this proof, we will set $\|\cdot\|_d := \|\cdot\|$.

We claim that we conclude the proof if we show that for each $w \in \mathbb{W}$ there exists $\tilde{\gamma} \subseteq \mathbb{W}$, connecting 0 to w , horizontal for $\{D_j^\varphi\}_{j=2, \dots, 2n+1}$, such that

$$(4.41) \quad \ell_\varphi(\tilde{\gamma}) \leq C_1 \|w\|,$$

for some constant C_1 independent on w . Indeed, if (4.41) holds, then from the first part of Proposition 4.38 and (4.40) we get that, setting $\Phi(\tilde{\gamma}) := \tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})$,

$$\text{length}(\Phi(\tilde{\gamma})) \leq C_2 d(x, y),$$

where C_2 is a constant independent on w , and $\Phi(\tilde{\gamma})$ is a curve contained in Γ connecting $x = 0$ to $y = w \cdot \tilde{\varphi}(w)$. Since the length of $\Phi(\tilde{\gamma})$ is finite, we get that it is a horizontal curve [146, Theorem 2.4.5] and then we get

$$d^\Gamma(x, y) \leq C_2 d(x, y),$$

that finishes the proof.

Now we show the existence of $\tilde{\gamma}$, with the required properties, such that (4.41) holds. We concatenate two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, horizontal for $\{D_j^\varphi\}_{j=2, \dots, 2n+1}$, to reach $w := (0, x_2, \dots, x_{2n}, y_1)$ from 0. Due to the fact that φ is continuous, because of [109, Theorem 2.3.6], Peano's theorem [122, Theorem 1.1] ensures that there exists a local solution to the continuous ODE

$$(4.42) \quad \begin{cases} \tau'(s) &= \varphi(0, \dots, 0, s, 0, \dots, 0, \tau(s)), \\ \tau(0) &= 0, \end{cases}$$

where s in the $(n+1)$ -th coordinate. Set

$$\tilde{\gamma}_1(s) := (0, \dots, 0, s, 0, \dots, 0, \tau(s)),$$

the curve with values in $\mathbb{W} \subseteq \mathbb{H}^n$, with s in the $(n+1)$ -th coordinate. By (4.42) it holds, whenever $\tilde{\gamma}_1(s)$ is defined,

$$(4.43) \quad \tilde{\gamma}'_1(s) = D_{n+1}^\varphi|_{\tilde{\gamma}_1(s)}.$$

We show that $\tau(s)$ is defined globally on \mathbb{R} , arguing as in [78, (4.1) and after]. Indeed, whenever $\tau(s)$ exists,

$$(4.44) \quad \tau(s) = \int_0^s \tau'(\xi) d\xi = \int_0^s \tilde{\varphi}(\tilde{\gamma}_1(\xi)) d\xi.$$

Notice that here there is a little abuse of notation: by $\tilde{\varphi}(\tilde{\gamma}_1(\xi))$, that a priori has values in \mathbb{L} , we mean the projection of it on the first coordinate in \mathbb{H}^n . By (4.43) and the second part

of Proposition 4.38 we have that $s \mapsto \tilde{\varphi}(\tilde{\gamma}_1(s))$ is L' -Lipschitz, with $L' = L'(L)$. Then, by (4.44) and the fact that $\varphi(0) = 0$, because we are assuming $x = 0$, we have

$$(4.45) \quad |\tau(s)| \leq \frac{1}{2}L's^2.$$

Thus, as every solution to (4.42) escapes every compact set [122, Theorem 2.1], we get from (4.45) that $\tau(s)$ is globally defined. Then $\tau(s)$ is defined up to $s = x_{n+1}$ and by the previous argument

$$(4.46) \quad |\tau(x_{n+1})| \leq \frac{1}{2}L'x_{n+1}^2.$$

We notice that we can identify the arbitrary point $(x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)$ with a point in \mathbb{H}^{n-1} . Thus we can connect the point $(0, \dots, 0, \tau(x_{n+1}))$, where we just removed the first and the $(n+1)$ -th coordinate from $\tilde{\gamma}_1(x_{n+1})$, to the point $(x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)$, by using a horizontal geodesic in \mathbb{H}^{n-1} with respect to the Carnot-Carathéodory distance $d_{\mathbf{g}}$ induced, on \mathbb{H}^{n-1} , by the scalar product \mathbf{g} that makes $X_2, \dots, X_n, X_{n+2}, \dots, X_{2n}$ orthonormal. We set $\tilde{\gamma}_2 : I \rightarrow \mathbb{W}$ to be the lifting of this horizontal geodesic in \mathbb{H}^n , where the $(n+1)$ -th coordinate of $\tilde{\gamma}_2$ is constantly equal to x_{n+1} . We notice that it is horizontal with respect to the family $\{D_j^\varphi\}_{j=2, \dots, n, n+2, \dots, 2n}$, because $D_j^\varphi = X_j$ for $j = 2, \dots, 2n$ and $j \neq n+1$. Then we have

$$(4.47) \quad \begin{aligned} \ell_\varphi(\tilde{\gamma}_2) &= d_{\mathbf{g}}((0, \dots, 0, \tau(x_{n+1})), (x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)) \\ &\leq C_3 \left(|y_1 - \tau(x_{n+1})|^{1/2} + \sum_{i=2, i \neq n+1}^{2n} |x_i| \right) \\ &\leq C_4 \left(|y_1|^{1/2} + \sum_{i=2}^{2n} |x_i| \right) \leq C_5 \|w\|, \end{aligned}$$

where the first equality follows by the definition of ℓ_φ (4.22) and the fact that $\tilde{\gamma}_2$, restricted to the copy of \mathbb{H}^{n-1} made of points with zero in the first coordinate and x_{n+1} in the $(n+1)$ -th coordinate, is a $d_{\mathbf{g}}$ -geodesic; the second is true because every two homogeneous norms are equivalent, the third one is true because of (4.46), and the last one again by the fact that every two homogeneous norms are equivalent. Now we have

$$(4.48) \quad \ell_\varphi(\tilde{\gamma}_1|_{[0, x_{n+1}]}) = |x_{n+1}| \leq |y_1|^{1/2} + \sum_{i=2}^{2n} |x_i| \leq C_6 \|w\|,$$

where the first equality is true by the definition of ℓ_φ and (4.43) and the third is true again because of the equivalence of homogeneous norms. Now if we set $\tilde{\gamma} := \tilde{\gamma}_1|_{[0, x_{n+1}]} \star \tilde{\gamma}_2$ the concatenation of the two curves, we get that $\tilde{\gamma}$ is horizontal and connects 0 to w . Summing (4.47) and (4.48) we get (4.41) with $C_1 := C_5 + C_6$, which was what was left to prove. \square

4.4. Sub-Riemannian structure of a C^∞ non-characteristic hypersurface in the Heisenberg groups. In this subsection we finally prove in four steps the main theorem in Theorem 4.2.

4.4.1. *The restriction of the horizontal bundle is bracket generating.* Now we are going to prove that for non-characteristic C^2 -hypersurfaces (see Definition 4.17) in \mathbb{H}^n , with $n \geq 2$, the intersection between the horizontal bundle of \mathbb{H}^n and the tangent bundle of S is bracket generating. This result was already known and it is a consequence of a more general one [210, Theorem 1.1]. Nevertheless we give here a simple proof by making explicit computations.

Proposition 4.42. *Consider in \mathbb{H}^n , with $n \geq 2$, a C^2 -hypersurface S . If S has no characteristic points, then the bundle*

$$(4.49) \quad x \mapsto \mathcal{D}_x := \mathbb{V}_1(x) \cap T_x S,$$

gives a step-2 bracket generating distribution on the hypersurface S .

Proof. We refer, for the notation, to Section 4.1. In particular, for the Heisenberg groups \mathbb{H}^n , see Remark 4.36. We need to prove that

$$\forall x \in S, \quad \mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x = T_x S.$$

Let us give the proof first for $n = 2$. We work locally around $x \in S$ so that we can assume that there exists $f \in C^2(\mathbb{H}^n)$ such that

$$S = \{x \in \mathbb{H}^n : f(x) = 0\}.$$

We define locally the vector fields

$$\begin{aligned} Z_1 &:= -(X_2 f)X_1 + (X_1 f)X_2 - (X_4 f)X_3 + (X_3 f)X_4, \\ Z_2 &:= -(X_3 f)X_1 + (X_4 f)X_2 + (X_1 f)X_3 - (X_2 f)X_4, \\ Z_3 &:= -(X_4 f)X_1 - (X_3 f)X_2 + (X_2 f)X_3 + (X_1 f)X_4. \end{aligned}$$

We have that for each $x \in S$, the linear space \mathcal{D}_x is a three-dimensional subspace of $T_x S$, because x is a non-characteristic point (Definition 4.17). Then, because $Z_1|_x$, $Z_2|_x$ and $Z_3|_x$ are linearly independent and are in \mathcal{D}_x , we have

$$(4.50) \quad \mathcal{D}_x = \text{span}\{Z_1|_x, Z_2|_x, Z_3|_x\}, \quad \forall x \in S.$$

Now by doing the computations exploiting the definition of Z_i , and using that $[X_1, X_3] = [X_2, X_4] = Y_1$, we can show that

$$\begin{aligned} [Z_1, Z_2] &= \star_1 - 2(X_1 f X_2 f + X_3 f X_4 f)Y_1, \\ [Z_1, Z_3] &= \star_2 + ((X_1 f)^2 + (X_3 f)^2 - (X_2 f)^2 - (X_4 f)^2)Y_1, \\ [Z_2, Z_3] &= \star_3 + 2(X_1 f X_4 f - X_2 f X_3 f)Y_1, \end{aligned}$$

where $\star_1, \star_2, \star_3$ are some combinations of X_1, X_2, X_3, X_4 with function coefficients.

It is easy to check that it is not possible to have, at some point $x \in S$,

$$\begin{aligned} X_1 f(x)X_2 f(x) + X_3 f(x)X_4 f(x) &= 0, \\ (X_1 f)^2(x) + (X_3 f)^2(x) - (X_2 f)^2(x) - (X_4 f)^2(x) &= 0, \\ X_1 f(x)X_4 f(x) - X_2 f(x)X_3 f(x) &= 0, \end{aligned}$$

because otherwise $X_1 f(x) = X_2 f(x) = X_3 f(x) = X_4 f(x) = 0$, which is impossible because there are no characteristic points. Then, for every $x \in S$, at least one among $[Z_1, Z_2]|_x$, $[Z_1, Z_3]|_x$, and $[Z_2, Z_3]|_x$ has a component along $Y_1|_x$. Thus, as $X_1|_x, X_2|_x, X_3|_x, X_4|_x$ and $Y_1|_x$ are linearly independent, this means that there exists at least one among $[Z_1, Z_2]|_x$, $[Z_1, Z_3]|_x$, and $[Z_2, Z_3]|_x$ which is not in \mathcal{D}_x . Then as $[\mathcal{D}, \mathcal{D}]_x \subseteq T_x S$, and it holds that there exists an element in $[\mathcal{D}, \mathcal{D}]_x$ which is not in \mathcal{D}_x , we get the conclusion.

If $n > 2$ we can argue exactly in the same way. Indeed, because there are no characteristic points, for every $x \in S$ there exists i with $1 \leq i \leq 2n$ such that $X_i f(x) \neq 0$ and one runs the same computations substituting X_1, X_2, X_3, X_4 with $X_i, X_j, X_{i+n}, X_{j+n}$ with $j \neq i$. \square

4.4.2. *Local equivalence of the sub-Riemannian distance and the induced distance.* Let S be a smooth non-characteristic hypersurface in the Heisenberg group \mathbb{H}^n , $n \geq 2$. From Proposition 4.42 we have a bracket generating distribution \mathcal{D} in the Euclidean tangent bundle TS of S . Hence S has the structure of sub-Riemannian manifold: we fix a scalar product \mathbf{g} on \mathbb{V}_1 , the horizontal bundle of \mathbb{H}^n , which induces a scalar product on \mathcal{D} . This scalar product defines a sub-Riemannian distance on S by taking the infimum of the length - measured with the norm $\|\cdot\|_{\mathbf{g}}$ associated to \mathbf{g} - of all the horizontal - according to \mathcal{D} - curves in S . We will call this distance d_{int} , the *intrinsic distance* on S . We can also equip S with the restriction of the distance of \mathbb{H}^n , which we will call induced distance, and with a little abuse of notation we denote it by d .

Proposition 4.43. *Let (\mathbb{H}^n, d) , with $n \geq 2$, be the Heisenberg group equipped with the sub-Riemannian distance coming from a scalar product on the horizontal distribution. Let S be a C^∞ non-characteristic hypersurface in \mathbb{H}^n . For each $p \in S$ there exists an open neighbourhood U_p of p such that*

$$(4.51) \quad d(x, y) \sim d_{\text{int}}(x, y) \quad \forall x, y \in U_p.$$

We recall that with (4.51) we mean that there exist $C \geq 1$ such that $C^{-1}d(x, y) \leq d_{\text{int}}(x, y) \leq Cd(x, y)$ for every $x, y \in U_p$.

Proof. By Remark 4.18, S is a $C_{\mathbb{H}}^1$ -hypersurface. Then, by the implicit function theorem, we get that locally around $p \in S$ the hypersurface S is the graph Γ of a globally defined intrinsically Lipschitz function on the tangent group $\mathbb{W} := T_x^1 S$. By changing coordinates if necessary (see also Lemma 4.44), we can assume \mathbb{W} as in (4.18). Then by Proposition 4.41 we get that $d^\Gamma \sim d$ and from Remark 4.40 we get that, in a neighbourhood of p , $d_{\text{int}} = d^\Gamma$, so that we get the result. \square

4.4.3. *Tangents of C^∞ non-characteristic hypersurfaces.* Now we know that a C^∞ non-characteristic hypersurface in the Heisenberg groups \mathbb{H}^n , $n \geq 2$, carries a sub-Riemannian structure. With the aim of using the rectifiability result from [151, Theorem 2], we calculate the possible tangents of S . We recall this well-known lemma, see for example [115, Lemma 3.26].

Lemma 4.44. *Every vertical subgroup of codimension one in \mathbb{H}^n , $n \geq 2$, is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$, which is a Carnot group.*

Proposition 4.45. *Let S be a C^∞ -hypersurface in \mathbb{H}^n , $n \geq 2$, with no characteristic points. Let \mathcal{D} be as in (4.49) and \mathbf{g} be a scalar product on the horizontal bundle \mathbb{V}_1 of \mathbb{H}^n .*

Then the triple $(S, \mathcal{D}, \mathbf{g}|_{\mathcal{D} \times \mathcal{D}})$ is an equiregular sub-Riemannian manifold with Hausdorff dimension $2n+1$. At each point $x \in S$ we have that the Gromov-Hausdorff tangent is unique and it is isometric the Carnot group $\mathbb{H}^{n-1} \times \mathbb{R}$ endowed with some Carnot-Carathéodory distance.

Proof. Because of the fact that S is non-characteristic it follows that \mathcal{D}_x has dimension $2n-1$ at each point $x \in S$. Also it is a direct consequence of Proposition 4.42 that, for each $x \in S$, the linear space $\mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x$ has dimension $2n$. Then $(S, \mathcal{D}, \mathbf{g}|_{\mathcal{D} \times \mathcal{D}})$ is an equiregular sub-Riemannian manifold with weights $(2n-1, 1)$. Then the Hausdorff dimension of S with respect to the sub-Riemannian distance d_{int} is $2n+1$, since d_{int} is equivalent to d , see Proposition 4.43.

By [52] (see also [129, Theorem 2.5], [129, page 25]) it follows, as we are in the equiregular case, that the Gromov-Hausdorff tangent at every point $x \in S$ is isometric to the Carnot group, endowed with some Carnot distance, that has Lie algebra

$$\mathbb{V}_x := \mathcal{D}_x \oplus ((\mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x)/\mathcal{D}_x),$$

with the bracket operation inherited by the brackets in the Heisenberg group. Then \mathbb{V}_x is isomorphic to a vertical subgroup of \mathbb{H}^n of codimension one and thus it is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$ by Lemma 4.44. \square

4.4.4. *Carnot-rectifiability of C^∞ -hypersurfaces.* We conclude with the main result of this section.

Theorem 4.46. *Let (\mathbb{H}^n, d) , with $n \geq 2$, be the n -th Heisenberg group equipped with a left-invariant homogeneous distance d . If S is a C^∞ -hypersurface in \mathbb{H}^n , then the metric space (S, d) has Hausdorff dimension $2n + 1$ and it is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1})$ -rectifiable according to the Definition 4.3.*

Proof. The fact that (S, d) has Hausdorff dimension $2n + 1$ follows from Remark 1.115. Let us assume first that S has no characteristic points. In this case it directly follows from [151, Theorem 2] and Proposition 4.45 that the metric space (S, d_{int}) has Hausdorff dimension $2n + 1$ and it is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}_{d_{\text{int}}}^{2n+1})$ -rectifiable according to Definition 4.3. Then by Proposition 4.43 we obtain that (S, d) is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}_d^{2n+1})$ -rectifiable.

In the general case, calling Σ_S the set of characteristic points, we know that $\mathcal{H}^{2n+1}(\Sigma_S) = 0$ by [40, Theorem 1.1] (see also [168, Theorem 2.16]). Moreover if $x \in S$ is a non-characteristic point, there exists U_x open subset of S containing x such that U_x is a smooth non characteristic hypersurface. Then we can use the previous argument to conclude that (U_x, d) is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1})$ -rectifiable and by covering $S \setminus \Sigma_S$ with countably many U_x 's we get the conclusion. \square

Remark 4.47. By Theorem 4.46 it follows that every smooth hypersurface S in \mathbb{H}^n , $n \geq 2$, is $\mathbb{H}^{n-1} \times \mathbb{R}$ -rectifiable according to the bi-Lipschitz variant of Pauls's definition [84, Definition 3], see Remark 4.6 for more details about this definition.

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