OPTIMAL REGION FOR THE TRANSPORT PROBLEM TO THE BOUNDARY

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Abstract. We consider a region $\Omega \subset \mathbb{R}^2$ where a mass f is transported to the boundary and the aim is to find an optimal free transport region E that minimizes the total cost outside E of this transport problem plus a penalization term on E. First, we study the regularity of the transport density σ in this transport problem to the boundary. Then, we show existence of an optimal set E for this shape optimization problem and, we prove regularity on this optimal set E in the case where the penalization term on E is given by the perimeter (or the fractional perimeter) of E.

1. Introduction

In this paper, we study a shape optimization problem where the functional to be minimized will be given by the total cost outside a free transport set E of a transport problem to the boundary. More precisely, we will consider a shape optimization problem of the form:

$$\min\left\{\mathcal{J}(E)+P(E)\ :\ E\subset\Omega\right\}, \tag{1.1}$$

where $\mathcal{J}(E)$ represents the total work outside E of transporting a mass to the boundary, while $P(E)$ is some penalization on E. In order to describe this functional $\mathcal{J}$, we need to introduce first some well known facts, terminology and notations concerning the transport problem to the boundary. Let f be a nonnegative Borel measure on a compact domain $\Omega \subset \mathbb{R}^2$ and assume that we want to transport this mass f to the boundary $\partial\Omega$ paying a transport cost $|x-y|$ for each unit of mass that moves from a point x to a destination $y \in \partial\Omega$ plus an additional boundary cost $g(y)$ at the exit point y, where g is a given continuous function on $\partial\Omega$. In other words, we consider

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g\,\mathrm{d}[(\Pi_y)_{\#}\gamma]:\gamma\in\mathcal{M}^+(\Omega\times\Omega),\ (\Pi_x)_{\#}\gamma=f,\ \mathrm{spt}[(\Pi_y)_{\#}\gamma]\subset\partial\Omega\right\}, \tag{1.2}$$

where $\mathcal{M}^+(\Omega\times\Omega)$ is the set of nonnegative Borel measures on $\Omega\times\Omega$, Π_x and Π_y are the two canonical projections of $\Omega\times\Omega$ onto Ω. We note that this transport problem with boundary cost g has been already considered in [18, 15, 16]. While in [14, 7], the authors studied the same problem but the boundary $\partial\Omega$ was assumed to be a free Dirichlet region, which means a region where transportation is free (i.e. $g=0$).

From now on, we assume that g is $1-$Lipschitz on $\partial\Omega$. Then, one can show that Problem (1.2) has a dual formulation which is the following (see [18, 15]):

$$\sup\left\{\int_\Omega u\,\mathrm{d}f\ :\ u\in\mathrm{Lip}_1(\Omega),\ u=g\ \text{ on }\ \partial\Omega\right\}. \tag{1.3}$$

We note that g is assumed to be $1-$Lip over $\partial\Omega$ since if this is not the case, then clearly there will be no admissible function u for Problem (1.3) (i.e., a $1-$Lip function u on Ω with $u=g$ on $\partial\Omega$).

In fact, it is easy to see that we have the following inequality $\sup(1.3) \leq \min(1.2)$. Indeed, if γ is a transport plan in Problem (1.2) and u is admissible in the dual problem (1.3), then one has

$$\int_\Omega u \,\mathrm{d}f \leq \int_{\Omega\times\Omega}[|x-y|+u(y)]\,\mathrm{d}\gamma = \int_{\Omega\times\Omega}[|x-y|+g(y)]\,\mathrm{d}\gamma = \int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega} g\,\mathrm{d}[(\Pi_y)_\#\gamma].$$

From the duality $\sup(1.3) = \min(1.2)$, we see that if γ is an optimal transport plan in Problem (1.2) and u is a Kantorovich potential (i.e. a maximizer) in Problem (1.3), then we must have

$$u(x) - u(y) = |x-y| \quad \text{for } \gamma-\text{a.e. } (x,y). \tag{1.4}$$

Since u is 1-Lip on Ω, then u must be linear on the line segment $[x,y]$ for $\gamma-$a.e. (x,y). In the sequel, any maximal line segment $[x,y]$ that satisfies the equality (1.4) will be called a *transport ray*. In other words, the optimal transport plan γ moves the mass f onto the boundary through these transport rays. In fact, one can show that u is differentiable in the interior of any transport ray $[x,y]$ and ∇u will be given by the opposite unit direction of the transport ray $[x,y]$ (i.e., $\nabla u(z) = \frac{x-y}{|x-y|}$ for all $z \in]x,y[$). Thanks to this fact, we infer that two different transport rays cannot intersect at an interior point of at least one of them.

For this transport problem, one can see that $\gamma = (Id,T)_\# f$, where T is a Borel selector function of the following multivalued map (notice that $\tilde{T}$ has a closed graph):

$$\tilde{T}(x) := \operatorname{argmin}\{|x-y| + g(y) \,:\, y \in \partial\Omega\}, \quad \text{for all } x \in \Omega,$$

is an optimal transport plan for Problem (1.2) while the Kantorovich potential is given by

$$u(x) = \min\{|x-y| + g(y) \,:\, y \in \partial\Omega\}, \quad \text{for all } x \in \Omega.$$

This follows from the fact that $\sup(1.3) \leq \min(1.2)$, γ and u are admissible in Problems (1.2) & (1.3) respectively and, we have the following:

$$\int_\Omega u\,\mathrm{d}f = \int_\Omega [|x-T(x)| + g(T(x))]\,\mathrm{d}f(x) = \int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega} g\,\mathrm{d}[(\Pi_y)_\#\gamma].$$

Moreover, if γ' is an optimal transport plan for Problem (1.2) then we have $y \in \tilde{T}(x)$, for $\gamma'-$a.e. (x,y), since

$$\int_{\Omega\times\Omega}[|x-y|+g(y)]\,\mathrm{d}\gamma' \geq \int_{\Omega\times\Omega}[|x-T(x)|+g(T(x))]\,\mathrm{d}\gamma' = \int_{\Omega\times\Omega}[|x-y|+g(y)]\,\mathrm{d}\gamma.$$

On the other hand, it is not difficult to see that if g is $\lambda-$Lip with $\lambda < 1$ then $\tilde{T}(x)$ is a singleton at every point x where the Kantorovich potential u is differentiable. This follows immediately from the fact that if $\{y,\, y'\} \subset \tilde{T}(x)$, then we have $\nabla u(x) = \frac{x-y}{|x-y|} = \frac{x-y'}{|x-y'|}$ so that $|g(y)-g(y')| = ||x-y'|-|x-y|| = |y-y'|$, which is a contradiction as soon as $y \neq y'$. But, u is Lipschitz and so, the set of points where u is not differentiable is negligible. Therefore, $\gamma = (Id,T)_\# f$ will be the unique optimal transport plan for Problem (1.2) provided that $f \in L^1(\Omega)$.

In the theory of optimal transport with distance cost, it is classical to associate with an optimal transport plan γ a nonnegative measure σ on Ω (called *transport density*) which represents the amount of transport taking place in each region of Ω. This measure σ is defined as follows:

$$<\sigma,\varphi>= \int_{\Omega\times\Omega}\int_0^1 \varphi((1-t)x+ty)|x-y|\,\mathrm{d}t\,\mathrm{d}\gamma(x,y), \quad \text{for all } \varphi \in C(\Omega). \tag{1.5}$$

The L^p summability of this transport density σ was already studied in [11, 12, 13, 19]. In particular, the authors prove that $\sigma \in L^p(\Omega)$ as soon as $f \in L^p(\Omega)$ and $p < 2$. While in [15], the author has improved this L^p result on σ for all $p \in [1, \infty]$, under the assumptions that $f \in L^p(\Omega)$, Ω satisfies a uniform exterior ball condition (see [15, Definition 3.1]) and, g is a $\lambda-$Lip (with $\lambda < 1$) and semi-concave function on $\partial\Omega$. However, the higher order regularity of this transport density σ is still an open question (but, we will give in Section 2 a partial answer)!

This transport density σ has also applications to some shape optimization problems (see, for instance, [4]). In addition, this σ arises in the following minimal flow formulation (or the so-called Beckmann problem):

$$\min \left\{ \int_\Omega |v| + \int_{\partial\Omega} g \,\mathrm{d}\nu \,:\, v \in L^1(\Omega, \mathbb{R}^2),\, \nu \in \mathcal{M}^+(\partial\Omega),\, \nabla \cdot v = f - \nu \right\}. \tag{1.6}$$

More precisely, consider the flow $v := -\sigma\nabla u$. First, it is easy to check that $\nabla \cdot v = f - T_{\#}f$ in Ω. Indeed, for every $\varphi \in C^1(\Omega)$, one has

$$- \int_\Omega \nabla\varphi \cdot \mathrm{d}v = \int_{\Omega\times\Omega} \int_0^1 [\nabla\varphi \cdot \nabla u]((1-t)x + ty)|x-y| \,\mathrm{d}t \,\mathrm{d}\gamma(x,y)$$

$$= \int_{\Omega\times\Omega} \int_0^1 \nabla\varphi((1-t)x+ty)\cdot[x-y] \,\mathrm{d}t \,\mathrm{d}\gamma(x,y) = \int_{\Omega\times\Omega} [\varphi(x)-\varphi(y)] \,\mathrm{d}\gamma(x,y) = \int_\Omega \varphi \,\mathrm{d}[f - T_{\#}f].$$

Moreover, using that $|\nabla u| = 1 \;\; \sigma -$ a.e., we get

$$\int_\Omega |v| + \int_{\partial\Omega} g \,\mathrm{d}[T_{\#}f] = \sigma(\Omega) + \int_{\partial\Omega} g \,\mathrm{d}[T_{\#}f] = \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega} g \,\mathrm{d}[(\Pi_y)_{\#}\gamma].$$

Hence, $\min$ (1.6) $\le \min$ (1.2). On the other hand, let $v \in L^1(\Omega, \mathbb{R}^2)$ be such that $\nabla \cdot v = f - \nu$, where $\nu \in \mathcal{M}^+(\partial\Omega)$, and u be a smooth $1 -$ Lip function on Ω with $u = g$ on $\partial\Omega$. Then, we have

$$\int_\Omega |v| + \int_{\partial\Omega} g \,\mathrm{d}\nu \ge - \int_\Omega \nabla u \cdot v \,\mathrm{d}x + \int_{\partial\Omega} g \,\mathrm{d}\nu = \int_\Omega u \,\mathrm{d}f.$$

This implies that $\min$ (1.6) $\ge \sup$ (1.3). Hence, the flow v (with the boundary measure $\nu = T_{\#}f$) solves Problem (1.6) and, we have

$$\min\,(1.6) = \sup\,(1.3) = \min\,(1.2).$$

Thanks to [19, Chapter 4], one can also show that this flow v (with the measure $T_{\#}f$) is the unique minimizer for Problem (1.6). In addition, the pair (σ, u) is the unique solution for the following Monge-Kantorovich system:

$$\begin{cases} -\nabla \cdot [\sigma\nabla u] = f & \text{in } \mathring{\Omega}, \\ u = g & \text{on } \partial\Omega, \\ |\nabla u| \le 1 & \text{in } \Omega, \\ |\nabla u| = 1 & \sigma - \text{a.e.} \end{cases} \tag{1.7}$$

We note that this system (1.7) describes the growth of a sandpile on a bounded table, with a wall on the boundary of height g, under the action of a vertical source here modeled by f (see [7, 10]).

In Section 2, we will study the regularity of the transport density σ in (1.5). More precisely, we will show continuity and Lipschitz regularity on σ, under some assumptions on the data f, g and Ω.

Finally, the shape optimization problem that we will consider in Section 3 consists in finding a free transport region E that minimizes the total transportation cost outside E, which is given by the quantity $\sigma(\Omega\backslash E)$ (where σ always denotes the transport density in Problem (1.2)), plus a penalization term $P(E)$ on E, among all subsets $E \subset \Omega$. To be more precise, we minimize

$$\min \left\{ \sigma(\Omega\backslash E) + P(E) \;:\; E \subset \Omega \right\}. \tag{1.8}$$

In fact, two cases of penalizations will be studied in Section 3: the simplest case is when the penalization $P(E)$ involves the perimeter of E; in this situation an optimal region E is shown to exist and a second order regularity on E will be proved thanks to the regularity of the transport density σ that we will prove in Section 2. The second case which is more delicate is when $P(E)$ is given by the "fractional" perimeter of E; here we also prove existence of an optimal region E but the difficulty appears when proving regularity on E. We will be able to prove only a first order regularity on E.

2. Regularity of the transport density in the transport problem to the boundary

In this section, we study the higher order regularity of the transport density σ in Problem (1.2) (or equivalently, in the system (1.7)). The continuity of this transport density σ was already studied in [7] but in the particular case when $g = 0$. More precisely, the authors show that σ is continuous on Ω as soon as $f \in C(\Omega)$ and $\partial\Omega$ is of class C^2. We recall that the nonhomogeneous case (i.e. $g \neq 0$) has been already considered in several works (see, for instance, [18, 10]) but there are no results concerning the regularity of the transport density σ in this case, apart the L^p estimates proved in [15]. In the present paper, we will extend the continuity result of [7] on σ to the case of a general $\lambda-$Lip (with $\lambda < 1$) function g on $\partial\Omega$. Moreover, we will study the Lipschitz regularity of this transport density σ.

First of all, we need to prove some regularity on the transport map T and the Kantorovich potential u (see Section 1). In the sequel, we will denote by Σ the set of points where $\tilde{T}$ is not a singleton (or equivalently, where u is not differentiable). Throughout this section, we will assume that g is $\lambda-$Lip with $\lambda < 1$. Let us start by the following:

Lemma 2.1. *Assume that Ω has boundary of class C^2 and $g \in C^2(\partial\Omega)$. Then, for every $x \in \mathring{\Omega}$, we have*

$$\frac{x - T(x)}{|x - T(x)|} \cdot \mathbf{t}(T(x)) = \partial_{\mathbf{t}} g(T(x))$$

and

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x) \geq 0$$

with

$$d(x) = |x - T(x)|,$$

$$K(x) = \sqrt{1 - \partial_{\mathbf{t}} g(T(x))^2}\,\kappa(T(x)) - \partial^2_{\mathbf{tt}} g(T(x)) - \partial_{\mathbf{n}} g(T(x))\kappa(T(x)),$$

where the vector $\mathbf{n} := \mathbf{n}(T(x))$ denotes the unit interior normal vector to $\partial\Omega$ at $T(x)$ while $\mathbf{t} := \mathbf{t}(T(x))$ is the corresponding tangent vector (the rotation with angle $-\frac{\pi}{2}$ of the normal vector $\mathbf{n}$) and $\kappa(T(x))$ denotes the curvature of the boundary at $T(x)$. Moreover, if $x \notin \overline{\Sigma}$, then one has

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x) > 0.$$

Proof. Fix $x \in \mathring{\Omega}$ and let $\alpha(s)$ be a parametrization of $\partial\Omega$ around $T(x)$ with $|\alpha'(s)| = 1$. We define

$$\mathbf{u}(s) = |x - \alpha(s)| + g(\alpha(s)).$$

Then, we have

$$\mathbf{u}'(s) = \frac{-[x - \alpha(s)] \cdot \alpha'(s)}{|x - \alpha(s)|} + \nabla g(\alpha(s)) \cdot \alpha'(s)$$

and

$$\mathbf{u}''(s) = \left[\frac{\alpha'(s)}{|x - \alpha(s)|} - \frac{(x - \alpha(s)) \otimes (x - \alpha(s))\,\alpha'(s)}{|x - \alpha(s)|^3}\right] \cdot \alpha'(s) - \frac{[x - \alpha(s)] \cdot \alpha''(s)}{|x - \alpha(s)|}$$
$$+\nabla g(\alpha(s)) \cdot \alpha''(s) + D^2 g(\alpha(s))\,\alpha'(s) \cdot \alpha'(s).$$

Let $s^\star$ be such that $\alpha(s^\star) = T(x)$, so $s^\star$ is a minimizer of $\mathbf{u}(s)$. Then, the proof follows immediately thanks to the fact that $\mathbf{u}'(s^\star) = 0$ and $\mathbf{u}''(s^\star) \geq 0$. On the other hand, we note that for all $x \in \Omega$ and $y \in \tilde{T}(x)$, we have $\tilde{T}(z) = \{y\}$ for every $z \in]x, y]$, since one has $|z - y| + g(y) = |x - y| - |x - z| + g(y) \leq |x - y'| - |x - z| + g(y') < |z - y'| + g(y')$, for all $y' \neq y \in \partial\Omega$ (using that g is $\lambda-$Lip with $\lambda < 1$, we recall that if z, y and y' are aligned then $|z - y| + g(y) < |z - y'| + g(y')$). Thanks to this fact, we see that if $x \notin \overline{\Sigma}$ then there will be a point $x_0 \notin \overline{\Sigma}$ such that $x \in]x_0, T(x_0)[$. In particular, we have $T(x) = T(x_0)$. Yet, one has

$$1 - \partial_{\mathbf{t}} g(T(x_0))^2 - K(x_0) d(x_0) \geq 0.$$

But, it is clear that we have $d(x_0) = d(x) + |x - x_0|$ and $K(x_0) = K(x)$. Hence, we get

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x) d(x) \geq K(x)|x - x_0|.$$

Consequently, we infer that

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x) d(x) \geq \max\{K(x)|x - x_0|, 1 - \lambda^2 - K(x) d(x)\} > 0. \tag{2.1}$$ □

Now, we are ready to prove regularity on the transport map T and the Kantorovich potential u. First, we note that T is clearly continuous on $\Omega \backslash \overline{\Sigma}$. Moreover, we have the following:

Proposition 2.2. *Assume $\partial\Omega$ is C^2 and $g \in C^2(\partial\Omega)$. Then, the map T is C^1 on $\Omega \backslash \overline{\Sigma}$ and, we have*

$$DT(x) = \frac{1 - \partial_{\mathbf{t}} g(T(x))^2}{1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x) d(x)}\,\mathbf{t} \otimes \mathbf{t} - \frac{\partial_{\mathbf{t}} g(T(x))\sqrt{1 - \partial_{\mathbf{t}} g(T(x))^2}}{1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x) d(x)}\,\mathbf{t} \otimes \mathbf{n}. \tag{2.2}$$

Moreover, u is C^2 on $\Omega \backslash \overline{\Sigma}$ with

$$D^2 u(x) = \frac{-K(x)}{1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x) d(x)} e(x) \otimes e(x), \tag{2.3}$$

where, for every $x \in \Omega \backslash \overline{\Sigma}$, $e(x)$ denotes the orthogonal vector to $\nabla u(x)$ given by

$$e(x) := \sqrt{1 - \partial_{\mathbf{t}} g(T(x))^2}\,\mathbf{t}(T(x)) - \partial_{\mathbf{t}} g(T(x))\,\mathbf{n}(T(x)).$$

Proof. Fix $x_0 \in \Omega\backslash\overline{\Sigma}$. Let $\alpha(s)$, $s \in (-\varepsilon,\varepsilon)$, be the unit parametrization of $\partial\Omega$ around $T(x_0) = \alpha(0)$ with $\alpha'(s) = \mathbf{t}(\alpha(s))$. For every x in a small neighbourhood of x_0, let $s := s(x) \in (-\varepsilon,\varepsilon)$ be such that $T(x) = \alpha(s(x))$. Recalling the estimates in the proof of Lemma 2.1, we have

$$\frac{-(x-\alpha(s))\cdot\alpha'(s)}{|x-\alpha(s)|} + \nabla g(\alpha(s))\cdot\alpha'(s) = 0. \tag{2.4}$$

Differentiating (2.4) with respect to x, we get

$$\frac{\nabla_x s - \alpha'(s)}{|x-\alpha(s)|} + \frac{(x-\alpha(s))\cdot\alpha'(s)}{|x-\alpha(s)|^3}[(I-\nabla_x s\otimes\alpha'(s))(x-\alpha(s))] - \frac{[x-\alpha(s)]\cdot\alpha''(s)}{|x-\alpha(s)|}\nabla_x s$$
$$+[\nabla g(\alpha(s))\cdot\alpha''(s)]\nabla_x s + [D^2 g(\alpha(s))\alpha'(s)\cdot\alpha'(s)]\nabla_x s = 0.$$

Hence, we have

$$\nabla_x s - \partial_{\mathbf{t}} g(T(x))^2\,\nabla_x s - d(x)\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\,\kappa(T(x))\nabla_x s + d(x)\,\partial_{\mathbf{n}} g(T(x))\kappa(T(x))\nabla_x s$$
$$+d(x)\,\partial^2_{\mathbf{tt}} g(T(x))\nabla_x s = \mathbf{t}(T(x)) - \partial_{\mathbf{t}} g(T(x))[\partial_{\mathbf{t}} g(T(x))\mathbf{t}(T(x)) + \sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\mathbf{n}(T(x))].$$

Thanks to Lemma 2.1 and the fact that T is continuous on $\Omega\backslash\overline{\Sigma}$, this implies that $x \mapsto s(x)$ is C^1 on $\Omega\backslash\overline{\Sigma}$ and, we have

$$\nabla_x s = \frac{1-\partial_{\mathbf{t}} g(T(x))^2}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{t}(T(x)) - \frac{\partial_{\mathbf{t}} g(T(x))\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{n}(T(x)).$$

Consequently, writing $DT(x) = \alpha'(s)\otimes\nabla_x s$ we get (2.2). On the other hand, one has

$$\nabla u(x) = \frac{x-T(x)}{d(x)} = \partial_{\mathbf{t}} g(T(x))\mathbf{t}(T(x)) + \sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\,\mathbf{n}(T(x)). \tag{2.5}$$

Hence, we get

$$D^2 u(x) = \frac{I-DT(x)}{d(x)} - \frac{(x-T(x))\otimes\nabla d(x)}{d(x)^2} = \frac{1}{d(x)}[I - DT(x) - \nabla u(x)\otimes\nabla d(x)].$$

Yet,

$$\nabla d(x) = \nabla u(x) - [DT(x)]^t\nabla g(T(x)).$$

Then, we get

$$D^2 u(x) = \frac{1}{d(x)}[I - DT(x) - \nabla u(x)\otimes\nabla u(x) + \nabla u(x)\otimes[DT(x)]^t\nabla g(T(x))].$$

By (2.5), we have

$$\nabla u(x)\otimes\nabla u(x)$$
$$= \partial_{\mathbf{t}} g(T(x))^2\,\mathbf{t}\otimes\mathbf{t} + [1-\partial_{\mathbf{t}} g(T(x))^2]\,\mathbf{n}\otimes\mathbf{n} + \partial_{\mathbf{t}} g(T(x))\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\,[\mathbf{t}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{t}].$$

Recalling (2.2), we have

$$[DT(x)]^t \nabla g(T(x)) = \frac{\partial_{\mathbf{t}} g(T(x))[1-\partial_{\mathbf{t}} g(T(x))^2]}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{t} - \frac{\partial_{\mathbf{t}} g(T(x))^2\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{n}.$$

Hence,

$$\begin{aligned}&\nabla u(x)\otimes [DT(x)]^t \nabla g(T(x))\\ &= \frac{\partial_{\mathbf{t}} g(T(x))^2[1-\partial_{\mathbf{t}} g(T(x))^2]}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{t}\otimes\mathbf{t} - \frac{\partial_{\mathbf{t}} g(T(x))^2[1-\partial_{\mathbf{t}} g(T(x))^2]}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{n}\otimes\mathbf{n}\\ &+ \frac{\partial_{\mathbf{t}} g(T(x))[1-\partial_{\mathbf{t}} g(T(x))^2]\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{n}\otimes\mathbf{t} - \frac{\partial_{\mathbf{t}} g(T(x))^3\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}\,\mathbf{t}\otimes\mathbf{n}.\end{aligned}$$

Consequently, we get that

$$D^2u(x) = \beta_1\,\mathbf{t}\otimes\mathbf{t} + \beta_2[\mathbf{t}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{t}] + \beta_3\,\mathbf{n}\otimes\mathbf{n},$$

where

$$\beta_1 = \frac{-[1-\partial_{\mathbf{t}} g(T(x))^2]K(x)}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}, \qquad \beta_2 = \frac{\partial_{\mathbf{t}} g(T(x))\sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\,K(x)}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)},$$

and

$$\beta_3 = \frac{-\partial_{\mathbf{t}} g(T(x))^2 K(x)}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)}.$$

Finally, we infer that

$$D^2u(x) = \frac{-K(x)}{1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x)} e(x)\otimes e(x),$$

where

$$e(x) = \sqrt{1-\partial_{\mathbf{t}} g(T(x))^2}\,\mathbf{t}(T(x)) - \partial_{\mathbf{t}} g(T(x))\,\mathbf{n}(T(x)). \quad \square$$

Thanks to Proposition 2.2, one can characterize the closure of the singular set Σ. More precisely, we have the following:

Lemma 2.3. *Under the assumptions that $\partial\Omega$ is C^2 and $g \in C^2(\partial\Omega)$, we have $\overline{\Sigma} \subset \mathring{\Omega}$. In addition, one has*

$$\overline{\Sigma} = \Sigma \cup \{x \in \Omega\backslash\Sigma \,:\, 1-\partial_{\mathbf{t}} g(T(x))^2 - K(x)d(x) = 0\}.$$

Moreover, for all $x \in \Omega$ and $y \in \tilde{T}(x)$, the transport ray $[x,y]$ does not intersect $\overline{\Sigma}$ at an interior point (i.e., we have $]x,y[\cap\overline{\Sigma} = \emptyset$).

Proof. Fix $x \in \Sigma$ and let $\{y,z\} \subset \tilde{T}(x)$. Then, we have $x = y + [u(x)-g(y)]\nabla u(y) = z + [u(x)-g(z)]\nabla u(z)$. Hence,

$$y - z = [u(x)-g(y)][\nabla u(z) - \nabla u(y)] + [g(y)-g(z)]\nabla u(z).$$

Then,

$$|y-z| \le C|x-y||y-z| + \lambda|y-z|.$$

This implies that

$$|x-y| \geq \frac{1-\lambda}{C}.$$

On the other hand, we have

$$|x-y| + g(y) \leq |x-p(x)| + g(p(x)),$$

where $p(x)$ is a projection point of x on the boundary. Thanks to the fact that g is $\lambda-$Lip with $\lambda < 1$, we get

$$|x-y| \leq \frac{1+\lambda}{1-\lambda}\,|x-p(x)|. \tag{2.6}$$

Consequently,

$$d(x, \partial\Omega) \geq \frac{(1-\lambda)^2}{C(1+\lambda)},$$

where $d(\cdot, \partial\Omega)$ denotes the distance to the boundary. Therefore, $\overline{\Sigma} \subset \mathring{\Omega}$. For the second statement: thanks to Lemma 2.1, we clearly have

$$\Sigma \cup \{x \in \Omega\backslash\Sigma\,:\, 1 - \partial_{\mathbf{t}}g(T(x))^2 - K(x)d(x) = 0\} \subset \overline{\Sigma}.$$

Now, fix a point $x \in \overline{\Sigma}\backslash\Sigma$. Let $(x_n)_n \subset \Sigma$ be a sequence of points converging to x. For every $n \in \mathbb{N}$, let $\{y_n, z_n\} \subset \tilde{T}(x_n)$. Then, we have $x_n = y_n + [u(x_n) - g(y_n)]\nabla u(y_n) = z_n + [u(x_n) - g(z_n)]\nabla u(z_n)$ and so,

$$\frac{y_n - z_n}{|y_n - z_n|} + [u(x_n) - g(y_n)]\,\frac{\nabla u(y_n) - \nabla u(z_n)}{|y_n - z_n|} - \frac{g(y_n) - g(z_n)}{|y_n - z_n|}\,\nabla u(z_n) = 0.$$

Then, passing to the limit when $n \to \infty$, we get

$$\mathbf{t} + d(x)D^2u(T(x))\mathbf{t} - \partial_{\mathbf{t}}g(T(x))\nabla u(T(x)) = 0,$$

where we recall that $\mathbf{t}$ denotes the tangent vector to $\partial\Omega$ at $T(x)$. Thanks to (2.5), (2.3) and the fact that $d(T(x)) = 0$, this implies that

$$\mathbf{t} - \frac{K(x)d(x)}{1-\partial_{\mathbf{t}}g(T(x))^2}[e(x)\otimes e(x)]\mathbf{t} - \partial_{\mathbf{t}}g(T(x))[\partial_{\mathbf{t}}g(T(x))\mathbf{t} + \sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}\,\mathbf{n}] = 0.$$

Yet,

$$e(x)\cdot\mathbf{t} = \sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}.$$

Hence, we have

$$\mathbf{t} - \frac{K(x)d(x)}{1-\partial_{\mathbf{t}}g(T(x))^2}\left[[1-\partial_{\mathbf{t}}g(T(x))^2]\,\mathbf{t} - \partial_{\mathbf{t}}g(T(x))\sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}\,\mathbf{n}\right] - \partial_{\mathbf{t}}g(T(x))^2\,\mathbf{t}$$

$$-\partial_{\mathbf{t}}g(T(x))\sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}\,\mathbf{n} = 0.$$

Then, we get

$$[1-\partial_{\mathbf{t}}g(T(x))^2 - K(x)d(x)]\mathbf{t} - \partial_{\mathbf{t}}g(T(x))\sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}\left[-\frac{K(x)d(x)}{1-\partial_{\mathbf{t}}g(T(x))^2} + 1\right]\mathbf{n} = 0.$$

This yields that

$$1-\partial_{\mathbf{t}}g(T(x))^2 - K(x)d(x) = 0.$$

Fix $x \in \Omega$, $y \in \tilde{T}(x)$ and $z \in]x,y[$. We recall that $]x,y[\cap\Sigma = \emptyset$ and so, $z \notin \Sigma$. If $z \in \overline{\Sigma}$, then we must have

$$1 - \partial_{\mathbf{t}} g(T(z))^2 - K(z)d(z) = 0. \tag{2.7}$$

Yet, we have $T(z) = y$, $d(x) = |x - z| + d(z)$ and $K(x) = K(z)$. Then, by Lemma 2.1, we get that

$$0 \le 1 - \partial_{\mathbf{t}} g(y)^2 - K(x)d(x) = -K(x)|x - z|.$$

Consequently, this implies that $K(x) \le 0$. But, this is in contradiction with (2.7) since we get

$$1 - \partial_{\mathbf{t}} g(T(z))^2 - K(z)d(z) \ge 1 - \lambda^2 > 0.$$

This concludes the proof. □

Similarly to [7], let us denote by $\tau(x)$ the distance from a point x along the transport ray containing x to the closure of the singular set $\overline{\Sigma}$, i.e. the map τ is defined as follows (we assume that τ is extended by 0 on $\overline{\Sigma}$):

$$\tau(x) = \min\{t \ge 0 \, : \, x + t\,\nabla u(x) \in \overline{\Sigma}\}, \text{ for all } x \in \Omega\backslash\overline{\Sigma}.$$

In fact, this map τ will play an important role in the proof of regularity of the transport density σ. Notice that, thanks to Lemma 2.1, we have

$$1 - \partial_{\mathbf{t}} g(T(x))^2 - K(x)\tau(x) \ge 0, \text{ for all } x \in \Omega.$$

First, we see that we have the following:

Lemma 2.4. *Assume that $\partial\Omega$ is C^2 and $g \in C^2(\partial\Omega)$. Then, τ is continuous on Ω.*

Proof. Fix $x \in \Omega$ and let $(x_n)_n$ be a sequence of points in $\Omega\backslash\overline{\Sigma}$ converging to x. For every n, let $t_n > 0$ be such that $\tau(x_n) = t_n$. Up to a subsequence, $t_n \to t^\star \ge 0$, $d(x_n) \to d \ge 0$ and $\nabla u(x_n) \to p$ with $|p| = 1$. In particular, we have $x_n + t_n\nabla u(x_n) \in \overline{\Sigma} \to x + t^\star p \in \overline{\Sigma}$ and $x_n - d(x_n)\nabla u(x_n) \in \tilde{T}(x_n + t_n\nabla u(x_n)) \to x - dp \in \tilde{T}(x + t^\star p)$. Thanks to Lemma 2.3, this implies that $\tau(x) = t^\star$ and so, $\tau(x_n) \to \tau(x)$. □

We are ready to prove continuity on the transport density σ.

Proposition 2.5. *Assume $\partial\Omega$ is C^2, $g \in C^2(\partial\Omega)$ with $|\nabla g| \le \lambda < 1$ and, f is continuous in $\mathring{\Omega}$. Then, the transport density σ is continuous in $\mathring{\Omega}$. In addition, $\sigma \in C(\Omega)$ as soon as $f \in C(\Omega)$.*

Proof. Recalling the definition of the transport density σ (see (1.5)): for all $\varphi \in C(\Omega)$, we have

$$< \sigma, \varphi >= \int_\Omega \int_0^1 \varphi((1-t)x + tT(x))|x - T(x)|f(x)\,\mathrm{d}t\,\mathrm{d}x.$$

Yet, we know that $\overline{\Sigma}$ meets each transport ray at exactly one point and so, thanks to [19, Chapter 3], $\overline{\Sigma}$ is negligible. Hence, taking a change of variable $y = (1-t)x + t\,T(x)$ on $\Omega\backslash\overline{\Sigma}$, we get that

$$< \sigma, \varphi >= \int_0^1 \int_{\Omega_t} \varphi(y)\,\frac{d(y)}{1-t}\,f\bigg(\frac{y - tT(y)}{1-t}\bigg)\mathcal{J}_t(y)^{-1}\,\mathrm{d}y\,\mathrm{d}t,$$

where

$$\Omega_t := [(1-t)Id + t\,T](\Omega\backslash\overline{\Sigma}) \quad \text{and} \quad \mathcal{J}_t(y) = \det[(1-t)Id + tDT(x)].$$

This implies that

$$\sigma(y)=\int_0^1 \frac{d(y)}{1-t}\, f\Big(\frac{y-tT(y)}{1-t}\Big)\chi_{\Omega_t}(y)\, \mathcal{J}_t(y)^{-1}\,\mathrm{d}t,\ \text{ for a.e. } y\in\Omega.$$

Yet, it is easy to see that $y\in\Omega_t$ if and only if $0\le t\le 1-\frac{d(y)}{\tau(T(y))}$. Hence, we get that

$$\sigma(y)=\int_0^{1-\frac{d(y)}{\tau(T(y))}} \frac{d(y)}{1-t}\, f\Big(\frac{y-tT(y)}{1-t}\Big)\mathcal{J}_t(y)^{-1}\,\mathrm{d}t,\ \text{ for a.e. } y\in\Omega.$$

Let us compute the Jacobian $\mathcal{J}_t(y)$. Without loss of generality, one can assume that $T(x)=(0,0)$ and $\mathbf{t}(T(x))=<1,0>$. Hence, by (2.2), we have

$$\mathcal{J}_t(y)=\det[(1-t)Id+tDT(x)]$$

$$=\det\begin{bmatrix} 1-t+t\frac{1-\partial_{\mathbf{t}}g(T(x))^2}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)} & t\frac{-\partial_{\mathbf{t}}g(T(x))\sqrt{1-\partial_{\mathbf{t}}g(T(x))^2}}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)} \\ 0 & 1-t\end{bmatrix}$$

$$=(1-t)\Big(1-t+t\frac{1-\partial_{\mathbf{t}}g(T(x))^2}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\Big)=(1-t)\Big(1-t+t\frac{1-\partial_{\mathbf{t}}g(T(y))^2}{1-\partial_{\mathbf{t}}g(T(y))^2-K(y)\frac{d(y)}{1-t}}\Big)$$

$$=(1-t)\Big(\frac{(1-t)(1-\partial_{\mathbf{t}}g(T(y))^2)-(1-t)K(y)d(y)}{(1-t)(1-\partial_{\mathbf{t}}g(T(y))^2)-K(y)d(y)}\Big).$$

For all $y\in\Omega\backslash\overline{\Sigma}$, we get that

$$\sigma(y)=\int_0^{1-\frac{d(y)}{\tau(T(y))}} \frac{d(y)}{(1-t)^2}\, f\Big(\frac{y-tT(y)}{1-t}\Big)\frac{(1-t)(1-\partial_{\mathbf{t}}g(T(y))^2)-K(y)d(y)}{(1-t)(1-\partial_{\mathbf{t}}g(T(y))^2)-(1-t)K(y)d(y)}\,\mathrm{d}t.$$

Set $s=\frac{t\,d(y)}{1-t}$ (so, $t=\frac{s}{s+d(y)}$). Then, using the fact that $\tau(T(y))-d(y)=\tau(y)$, we infer that

$$\sigma(y)=\int_0^{\tau(y)} f(y+s\nabla u(y))\Big[1-s\frac{K(y)}{1-\partial_{\mathbf{t}}g(T(y))^2-K(y)d(y)}\Big]\,\mathrm{d}s. \tag{2.8}$$

Now, fix $y\in\Omega\backslash\overline{\Sigma}$ and let $(y_n)_n$ be a sequence of points in $\Omega\backslash\overline{\Sigma}$ converging to y. Thanks to Lemmas 2.1 & 2.4, we see that $\sigma(y_n)\to\sigma(y)$. Hence, σ is continuous on $\Omega\backslash\overline{\Sigma}$. On the other hand, we have

$$\sigma(y)\le ||f||_\infty\Big[1+\tau(y)\frac{\max\{0,-K(y)\}}{1-\lambda^2}\Big]\tau(y).$$

Consequently, σ is continuous on $\overline{\Sigma}$ as well ($\sigma=0$ on $\overline{\Sigma}$). This concludes the proof that the transport density σ is continuous on Ω. □

Moreover, one can prove Lipschitz regularity on the transport density σ as soon as we show that τ is Lipschitz. We note that the Lipschitz regularity of τ was already proved in [7, Theorem 2.12] but in the particular case $g=0$. In the next proposition, we will extend this result to the case of a general ($\lambda-$Lip with $\lambda<1$) function g on $\partial\Omega$ and so, the steps of the proof here will follow those in the proof of [7, Theorem 2.12].

Proposition 2.6. *Assume that $\partial\Omega$ is $C^{2,1}$, $g\in C^{2,1}(\partial\Omega)$ and, f is locally Lipschitz in $\mathring{\Omega}$. Then, the transport density σ is locally Lipschitz in $\mathring{\Omega}\backslash\overline{\Sigma}$. Moreover, σ is locally Lipschitz in $\Omega\backslash\overline{\Sigma}$ as soon as $f\in \mathrm{Lip}(\Omega)$.*

Proof. First, we prove that τ is locally Lipschitz on $\Omega\backslash\overline{\Sigma}$. We claim that there is a uniform constant C depending only on λ, $\operatorname{diam}(\Omega)$, $||\kappa||_\infty$, $||\nabla\kappa||_\infty$, $||D^2 g||_\infty$ and $||D^3 g||_\infty$ such that for every $x \in \partial\Omega$, there is a neighbourhood $V(x) \subset \partial\Omega$ of x such that

$$\tau(y) \leq \tau(x) + C|y-x|, \quad \text{for all } y \in V(x). \tag{2.9}$$

Fix $x \in \partial\Omega$. Assume that $1-\partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x) = 0$. From (2.6), we have $\tau(x) \leq \frac{(1+\lambda)\operatorname{diam}(\Omega)}{2(1-\lambda)}$ and then, $K(x) \geq \frac{2(1-\lambda)^2}{\operatorname{diam}(\Omega)}$. Hence, there is a small $\varepsilon > 0$ such that $K(y) \geq \frac{(1-\lambda)^2}{\operatorname{diam}(\Omega)}$, for all $y \in \partial\Omega \cap B(x,\varepsilon)$, and so we have

$$\begin{aligned}\tau(y) \leq \frac{1-\partial_{\mathbf{t}} g(y)^2}{K(y)} &= \frac{1-\partial_{\mathbf{t}} g(y)^2}{K(x)} + \frac{[1-\partial_{\mathbf{t}} g(y)^2](K(x)-K(y))}{K(x)K(y)} \\ = \tau(x) + \frac{\partial_{\mathbf{t}} g(x)^2 - \partial_{\mathbf{t}} g(y)^2}{K(x)} &+ \frac{[1-\partial_{\mathbf{t}} g(y)^2](K(x)-K(y))}{K(x)K(y)} \leq \tau(x) + C|x-y|,\end{aligned}$$

where the constant C depends only on λ, $\operatorname{diam}(\Omega)$, $||D^2 g||_\infty$, $||D^3 g||_\infty$, $||\kappa||_\infty$ and $||\nabla\kappa||_\infty$. Now, assume that $1-\partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x) > 0$ (we note that this case is more delicate). Set $\bar{x} = x + \tau(x)\nabla u(x)$. Let us denote by $D^+u(\bar{x})$ the superdifferential of u at $\bar{x}$. We note that u is locally semi-concave in Ω and so, we have $D^+u(\bar{x}) = \operatorname{co}[D^\star u(\bar{x})]$ (the convex hull of the set of limiting gradients $D^\star u(\bar{x})$). Set $e_2 = \nabla u(x)$ and $e_1 = R_{-\frac{\pi}{2}}\, e_2$. We claim that there is a Lipschitz arc $\beta : [-\delta,\delta] \mapsto \Sigma$ (for some $\delta > 0$) such that $\beta(0) = \bar{x}$, $|\beta'| = 1$ and $\beta'(0) = n_1$, where n_1 is a unit normal vector to $[p_1, e_2]$, for some vector $p_1 \in D^\star u(\bar{x})$, with $n_1 \cdot e_1 > 0$. For this aim, we start by proving that e_2 is isolated in the set of limiting gradients $D^\star u(\bar{x})$. Assume that this is not the case, then there will be a sequence $\{p_n\} \subset D^\star u(\bar{x})$ such that $e_2 = \lim_n p_n$. But, it is not difficult to see that, for every $n \in \mathbb{N}$, there is a point $x_n \in \tilde{T}(\bar{x})$ such that $p_n = \nabla u(x_n)$. In particular, we have $\bar{x} = x + [u(\bar{x}) - g(x)]\nabla u(x) = x_n + [u(\bar{x}) - g(x_n)]\nabla u(x_n)$ and so,

$$\frac{x_n - x}{|x_n - x|} + [u(\bar{x}) - g(x_n)]\,\frac{\nabla u(x_n) - \nabla u(x)}{|x_n - x|} - \frac{g(x_n) - g(x)}{|x_n - x|}\,\nabla u(x) = 0.$$

Passing to the limit when $n \to \infty$ and using the fact that $x_n \to x$, we infer that

$$\mathbf{t}(x) + \tau(x) D^2 u(x)\,\mathbf{t}(x) - \partial_{\mathbf{t}} g(x)\nabla u(x) = 0.$$

Hence,

$$\mathbf{t}(x) - \frac{K(x)\tau(x)}{1-\partial_{\mathbf{t}} g(x)^2}[e(x)\otimes e(x)]\mathbf{t}(x) - \partial_{\mathbf{t}} g(x)[\partial_{\mathbf{t}} g(x)\mathbf{t}(x) + \sqrt{1-\partial_{\mathbf{t}} g(x)^2}\,\mathbf{n}(x)] = 0.$$

This implies that

$$[1-\partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x)]\mathbf{t}(x) - \partial_{\mathbf{t}} g(x)\sqrt{1-\partial_{\mathbf{t}} g(x)^2}\left[-\frac{K(x)\tau(x)}{1-\partial_{\mathbf{t}} g(x)^2} + 1\right]\mathbf{n}(x) = 0,$$

which contradicts the fact that $1 - \partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x) > 0$. Assume that $\dim(D^+u(\bar{x})) = 1$. Then, there is a vector $p_1 \neq e_2 \in D^\star u(\bar{x})$ such that $D^+u(\bar{x}) = [p_1, e_2]$. Let n_1 be a unit normal vector to $[p_1, e_2]$ such that $n_1 \cdot e_1 > 0$. Thanks to [1, Lemma 4.5], there exists a Lipschitz arc $\beta : [-\delta,\delta] \mapsto \Sigma$ such that $\beta(0) = \bar{x}$, $|\beta'| = 1$ and $\beta'(0) = n_1$. Now, assume that $\dim(D^+u(\bar{x})) = 2$. As e_2 is isolated in $D^\star u(\bar{x})$, then there exist two vectors $p_1 \neq p_2 \in D^\star u(\bar{x})$ such that the segments $[p_1, e_2]$ and $[p_2, e_2]$ are contained in $\partial D^+u(\bar{x})$. Let n_1 and n_2 be the outward unit normal vectors to $D^+u(\bar{x})$ exposing the faces $[p_1, e_2]$ and $[p_2, e_2]$, respectively. It is easy to see that there exist two numbers λ_1, $\lambda_2 > 0$ such that $e_2 = \lambda_1 n_1 + \lambda_2 n_2$. Hence, we either have $n_1 \cdot e_1 > 0$ or $n_2 \cdot e_1 > 0$ (without loss of generality, assume that $n_1 \cdot e_1 > 0$). Again by [1, Lemma 4.5], we know that there is a Lipschitz arc $\beta : [-\delta, \delta] \mapsto \Sigma$ such that $\beta(0) = \bar{x}$, $|\beta'| = 1$ and $\beta'(0) = n_1$. So, the claim is proved.

Let $\alpha : [-\varepsilon, \varepsilon] \mapsto \partial\Omega$ be such that $\alpha(0) = x$ and $\alpha'(0) = \mathbf{t}(x) = \sqrt{1 - \partial_{\mathbf{t}} g(x)^2}\, e_1 + \partial_{\mathbf{t}} g(x)\, e_2$. One can see that, for every $s \in [0, \varepsilon]$ (resp. $s \in [-\varepsilon, 0]$), there exists a value $t(s) \in [0, \delta]$ (resp. $t(s) \in [-\delta, 0]$) such that $\alpha(s) \in \tilde{T}(\beta(t(s)))$. In particular, we have

$$[\beta(t(s)) - \alpha(s)] \cdot R_{\frac{\pi}{2}} \nabla u(\alpha(s)) = 0. \tag{2.10}$$

Yet,

$$\beta(t(s)) - \alpha(s) = \tau(x) e_2 + t(s)\beta'(0) - s\,\mathbf{t}(x) + o(s) + o(t(s)). \tag{2.11}$$

Moreover,

$$\nabla u(\alpha(s)) = e_2 + D^2 u(x)(\alpha(s) - x) + o(s).$$

From (2.3), one has

$$D^2 u(x) = \frac{-K(x)}{1 - \partial_{\mathbf{t}} g(x)^2}\, e_1 \otimes e_1.$$

Hence, we get

$$\nabla u(\alpha(s)) = e_2 - \frac{K(x)}{1 - \partial_{\mathbf{t}} g(x)^2}[e_1 \cdot s\,\mathbf{t}(x)] e_1 + o(s) = e_2 - \frac{K(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} s\, e_1 + o(s).$$

Then,

$$R_{\frac{\pi}{2}} \nabla u(\alpha(s)) = -e_1 - \frac{K(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} s\, e_2 + o(s). \tag{2.12}$$

Consequently, by (2.11) & (2.12), we get

$$[\beta(t(s)) - \alpha(s)] \cdot R_{\frac{\pi}{2}} \nabla u(\alpha(s)) = [\tau(x) e_2 + t(s)\beta'(0) - s\,\mathbf{t}(x) + o(s) + o(t(s))] \cdot \left[-e_1 - \frac{s\,K(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} e_2 + o(s)\right]$$

$$= \frac{1 - \partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} s - [\beta'(0) \cdot e_1] t(s) - \frac{K(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} [\beta'(0) \cdot e_2] s\, t(s) + o(s) + o(t(s)).$$

Thanks to (2.10), this yields that the right hand term in the last equality is 0. And so, this implies that

$$t(s) = \frac{1 - \partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}\, [\beta'(0) \cdot e_1]} s + o(s).$$

Now, let $y \in \partial\Omega$ be a point in the neighbourhood of x and $s \in [-\varepsilon, \varepsilon]$ be such that $y = \alpha(s)$. Then, we have

$$\tau(y) = |\beta(t(s)) - \alpha(s)| = [\tau(x) e_2 + t(s)\beta'(0) - s\,\mathbf{t}(x) + o(s)] \cdot \left[e_2 - \frac{s\,K(x)}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}} e_1 + o(s)\right]$$

$$= \tau(x) + t(s)[\beta'(0) \cdot e_2] - \partial_{\mathbf{t}} g(x)\, s + o(s) = \tau(x) + \left[\frac{[1 - \partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x)][\beta'(0) \cdot e_2]}{\sqrt{1 - \partial_{\mathbf{t}} g(x)^2}\, [\beta'(0) \cdot e_1]} - \partial_{\mathbf{t}} g(x)\right] s + o(s)$$

$$\leq \tau(x) + \left[\frac{1 - \partial_{\mathbf{t}} g(x)^2 - K(x)\tau(x)}{\sqrt{1 - \lambda^2}\, [\beta'(0) \cdot e_1]} + \lambda\right] s + o(s). \tag{2.13}$$

Notice that

$$\beta'(0)\cdot e_1 = n_1\cdot e_1 = \frac{[e_2-p_1]}{|e_2-p_1|}\cdot e_2 = \frac{1-p_1\cdot e_2}{|p_1-e_2|} = \frac{|p_1-e_2|}{2}. \tag{2.14}$$

Let $x'\in\tilde{T}(\bar{x})$ be such that $\nabla u(x')=p_1$. So, we have $x':=\bar{x}-[u(\bar{x})-g(x')]p_1$. Then, one has

$$\begin{aligned} x'-x &= -[u(\bar{x})-g(x')]p_1+[u(\bar{x})-g(x)]e_2 \\ &= [u(\bar{x})-g(x)][e_2-p_1]+[g(x')-g(x)]p_1 = \tau(x)[e_2-p_1]+[g(x')-g(x)]p_1. \end{aligned} \tag{2.15}$$

On the other hand, recalling Proposition 2.2, we have

$$|\nabla u(x')-\nabla u(x)-D^2u(x)[x'-x]|\le C|x'-x|^2.$$

Therefore,

$$\left|p_1-e_2+\frac{K(x)}{1-\partial_{\mathbf{t}}g(x)^2}\,e_1\otimes e_1\Big(\tau(x)[e_2-p_1]+[g(x')-g(x)]p_1\Big)\right|\le C|\tau(x)[e_2-p_1]+[g(x')-g(x)]p_1|^2. \tag{2.16}$$

We have

$$\begin{aligned} e_1\otimes e_1\Big(\tau(x)[e_2-p_1]+[g(x')-g(x)]p_1\Big) &= -\tau(x)[e_1\cdot p_1]e_1+[g(x')-g(x)][e_1\cdot p_1]e_1 \\ &= [-\tau(x)+g(x')-g(x)][e_1\cdot p_1]e_1. \end{aligned}$$

Yet,

$$p_1-e_2=[e_1\cdot p_1]e_1+([e_2\cdot p_1]-1)e_2=[e_1\cdot p_1]e_1-\frac{|p_1-e_2|^2}{2}e_2.$$

Then,

$$e_1\otimes e_1\Big(\tau(x)[e_2-p_1]+[g(x')-g(x)]p_1\Big)=[-\tau(x)+g(x')-g(x)]\left[p_1-e_2+\frac{|p_1-e_2|^2}{2}e_2\right].$$

Hence, by (2.16), we get

$$\left|p_1-e_2+\frac{K(x)}{1-\partial_{\mathbf{t}}g(x)^2}\,[-\tau(x)+g(x')-g(x)]\left[p_1-e_2+\frac{|p_1-e_2|^2}{2}e_2\right]\right|\le C|\tau(x)[e_2-p_1]+[g(x')-g(x)]p_1|^2.$$

Then,

$$\begin{aligned} \frac{1-\partial_{\mathbf{t}}g(x)^2-K(x)\tau(x)}{1-\partial_{\mathbf{t}}g(x)^2}|p_1-e_2|-\frac{|K(x)|\tau(x)}{1-\partial_{\mathbf{t}}g(x)^2}\frac{|p_1-e_2|^2}{2}-\frac{|K(x)|}{1-\partial_{\mathbf{t}}g(x)^2}|g(x')-g(x)|\left[|p_1-e_2|+\frac{|p_1-e_2|^2}{2}\right] \\ \le C[|p_1-e_2|^2+|g(x')-g(x)|^2]. \end{aligned}$$

Recalling (2.15), we have

$$|g(x')-g(x)|\le\lambda|x'-x|\le\frac{\lambda}{1-\lambda}\tau(x)|p_1-e_2|.$$

Consequently,

$$\frac{1-\partial_{\mathbf{t}}g(x)^2-K(x)\tau(x)}{1-\partial_{\mathbf{t}}g(x)^2}|p_1-e_2|\le C|p_1-e_2|^2.$$

By (2.14), this yields that

$$\frac{1-\partial_{\mathbf{t}}g(x)^2-K(x)\tau(x)}{\beta'(0)\cdot e_1}\le C,$$

where C is a uniform constant depending only on λ, $\mathrm{diam}(\Omega)$, $||\kappa||_\infty$, $||\nabla\kappa||_\infty$, $||D^2g||_\infty$ and $||D^3g||_\infty$. Recalling (2.13), this concludes the proof of our claim (2.9). Now, fix $x\in\Omega\backslash\overline{\Sigma}$.

Then, for all y in the neighbourhood of x such that $T(y) \in V(T(x))$ (we recall that T is continuous on $\Omega\backslash\overline{\Sigma}$), we have

$$\tau(y)-\tau(x) = \tau(T(y))-\tau(T(x))+d(x)-d(y) \leq C|T(y)-T(x)|+u(x)-u(y)+g(T(y))-g(T(x))$$

$$\leq \frac{C}{\mathrm{dist}(x,\overline{\Sigma})}|x-y|, \tag{2.17}$$

where we used the bound $||DT||_{L^\infty(B(x,\varepsilon))} \leq \frac{C}{\mathrm{dist}(x,\overline{\Sigma})}$, which follows immediately from the estimate (2.1) as well as the proposition 2.2. Thanks to [9, Theorem 7.3], (2.17) implies that the map τ is locally Lipschitz on $\Omega\backslash\overline{\Sigma}$. Finally, recalling (2.8), we have

$$\sigma(x) = \int_0^{\tau(x)} f(x+s\nabla u(x))\left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\mathrm{d}s, \text{ for all } x \in \Omega\backslash\overline{\Sigma}.$$

Hence, one has

$$\nabla\sigma(x) = f(x+\tau(x)\nabla u(x))\left[1-\tau(x)\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\nabla\tau(x)$$

$$+\int_0^{\tau(x)}[I+sD^2u(x)]\nabla f(x+s\nabla u(x))\left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\mathrm{d}s$$

$$-\int_0^{\tau(x)} s\, f(x+s\nabla u(x))\left[\frac{\nabla K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}+\frac{K(x)[\nabla[\partial_{\mathbf{t}}g(T(x))^2]+\nabla[K(x)d(x)]]}{(1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x))^2}\right]\mathrm{d}s.$$

For the first term in $\nabla\sigma(x)$, we have

$$\left|f(x+\tau(x)\nabla u(x))\left[1-\tau(x)\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\nabla\tau(x)\right| \leq ||f||_\infty\left[1+\frac{||K||_\infty}{1-\lambda^2}\mathrm{diam}(\Omega)\right]||\nabla\tau||_\infty.$$

Thanks to the smoothness of u on $\Omega\backslash\overline{\Sigma}$ (see Proposition 2.2), one can bound the second term of $\nabla\sigma(x)$ as follows:

$$\left|\int_0^{\tau(x)}[I+sD^2u(x)]\nabla f(x+s\nabla u(x))\left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\mathrm{d}s\right|$$

$$\leq C\int_0^{\tau(x)}|\nabla f|(x+s\nabla u(x))\left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right]\mathrm{d}s \leq C||\nabla f||_\infty\left[1+\frac{||K^-||_\infty}{1-\lambda^2}\mathrm{diam}(\Omega)\right],$$

where $K^-(x) := \max\{0,-K(x)\}$. To show the last inequality, we have to consider two cases: $K(x) \geq 0$ and $K(x) < 0$. If $K(x) \geq 0$, we have

$$0 \leq \left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right] \leq 1.$$

If $K(x) < 0$, then one has

$$0 \leq \left[1-s\frac{K(x)}{1-\partial_{\mathbf{t}}g(T(x))^2-K(x)d(x)}\right] \leq \left[1-\frac{K(x)}{1-\lambda^2}\mathrm{diam}(\Omega)\right].$$

In the same way, we bound the third term of $\nabla\sigma(x)$. Consequently, this implies that the transport density σ is locally Lipschitz in $\Omega\backslash\overline{\Sigma}$. $\square$

3. Shape optimization: existence, properties and regularity of optimal sets

In this section, we assume again that $f \in L^1(\Omega)$ represents the distribution of some mass in the region $\Omega \subset \mathbb{R}^2$ that we want to export to the boundary, paying the transport cost plus a boundary tax which will be given by a $\lambda-$Lip function g on $\partial\Omega$ with $\lambda < 1$ (see Problem (1.2)). Let σ be the transport density in this transport problem; we recall that $\sigma(\Omega)$ represents the total transportation cost. Let us assume now that we may have a set $E \subset \Omega$ where the traffic inside E is free of charge. Then, the aim of this section is to find an optimal region $E \subset \Omega$ where the traffic may travel without paying any transport cost. However, since E is a cost-free transportation region then a term $P(E)$ will be added (due to roads improvement, traffic devices, ...) to describe the cost of improving the set E, then penalizing too large free traffic regions. In other words, we study the following shape optimization problem:

$$\min\left\{\sigma(\Omega\backslash E) + P(E) \,:\, E \subset \Omega\right\}. \tag{3.1}$$

In fact, one can also consider a more general version of Problem (3.1) by assuming that the traffic cost in E is not completely free but still less than the traffic cost on $\Omega\backslash E$. In other words, we may study the following problem:

$$\min\left\{\sigma(\Omega\backslash E) + \theta\,\sigma(E) + P(E) \,:\, E \subset \Omega\right\},$$

where $0 \le \theta < 1$. Or more generally, assume that H_1 and H_2 are two continuous functions with $0 \le H_1 \le H_2$, then one can consider instead

$$\min\left\{\int_{\Omega\backslash E} H_2(x)\,\mathrm{d}\sigma(x) + \int_E H_1(x)\,\mathrm{d}\sigma(x) + P(E) \,:\, E \subset \Omega\right\}. \tag{3.2}$$

For simplicity of exposition, we will consider Problem (3.1), but it is not difficult to check that all the results in the next subsections hold true in the general case (3.2).

3.1. Penalization with the perimeter.

In this subsection, we consider the simplest version of Problem (3.1) where the penalization term $P(E)$ involves the perimeter of E. In this case, an optimal region E is shown to exist and some classical properties and regularity results on E will be established. Fix $\Lambda > 0$, then we consider the following problem:

$$\min\left\{\sigma(\Omega\backslash E) + \Lambda\,Per(E) \,:\, E \subset \Omega\right\}, \tag{3.3}$$

where $Per(E)$ denotes the perimeter of the set E in the sense of De Giorgi (see [3]).

Proposition 3.1. *Assume that $f \in L^1(\Omega)$. Then, the shape optimization problem (3.3) reaches a minimum.*

Proof. Let $\{E_n\}_{n\in\mathbb{N}} \subset \Omega$ be a minimizing sequence in Problem (3.3). It is clear that one can assume that there is a uniform constant C such that

$$\sigma(\Omega\backslash E_n) + \Lambda\,Per(E_n) \le C, \quad \text{for every } n \in \mathbb{N}.$$

As $\sigma \ge 0$, the previous immediately gives a uniform bound on the perimeter of the sequence $\{E_n\}$. Moreover, $|E_n| \le |\Omega|$. This in turn implies that the sequence $\{1_{E_n}\}_{n\in\mathbb{N}}$ weakly* converges in $BV(\Omega)$ (and then, strongly converges in $L^1(\Omega)$) to a function φ, which has the form $\varphi = 1_E$ for some measurable set $E \subset \Omega$. By using the latter, the lower semicontinuity of the total variation of the distributional gradient of 1_{E_n} and the fact that $\sigma \in L^1(\Omega)$ (see [19]), we get

$$\sigma(\Omega\backslash E) + \Lambda\,Per(E) \le \liminf_n [\sigma(\Omega\backslash E_n) + \Lambda\,Per(E_n)].$$

This concludes the proof of existence of an optimal set E for Problem (3.3). □

Now, we introduce the notion of $\Omega-$convexity, which coincides by the way with the notion of convexity provided that Ω is convex.

Definition 3.1. *For a subset $E \subset \Omega$, we define the $\Omega-$convex hull of E as the union of all the segments included in Ω with both vertices in E. We say that E is $\Omega-$convex if the $\Omega-$convex hull of E is the set E itself.*

Then, we have the following:

Proposition 3.2. *Let E be an optimal set, then any connected component of E is $\Omega-$convex. Moreover, any connected subset of E is contained in the $\Omega-$convex hull of some connected subset of* $\mathrm{spt}(\sigma)$*.*

Proof. Assume without loss of generality that the set E is connected. Suppose that E is not $\Omega-$convex. Let $\widetilde{E}$ be the $\Omega-$convex hull of E. Then, it is not difficult to see that $Per(\widetilde{E}) < Per(E)$. Hence, we get

$$\sigma(\Omega \backslash \widetilde{E}) + \Lambda\, Per(\widetilde{E}) < \sigma(\Omega \backslash E) + \Lambda\, Per(E),$$

which is a contradiction since E minimizes Problem (3.3). The second statement follows in a similar fashion. $\square$

On the other hand, one can show that the map $\Lambda \mapsto E_\Lambda$, where E_Λ is an optimal set in Problem (3.3), is monotone.

Proposition 3.3. *Let $\Lambda_1 > \Lambda_2 > 0$ and E_{Λ_1}, E_{Λ_2} be two corresponding optimal sets, then we have $E_{\Lambda_1} \subset E_{\Lambda_2}$.*

Proof. From the optimality of E_{Λ_1} and E_{Λ_2} in Problem (3.3), we clearly have the following inequalities:

$$\sigma(\Omega \backslash E_{\Lambda_1}) + \Lambda_1 Per(E_{\Lambda_1}) \leq \sigma(\Omega \backslash (E_{\Lambda_1} \cap E_{\Lambda_2})) + \Lambda_1\, Per(E_{\Lambda_1} \cap E_{\Lambda_2})$$

and

$$\sigma(\Omega \backslash E_{\Lambda_2}) + \Lambda_2\, Per(E_{\Lambda_2}) \leq \sigma(\Omega \backslash (E_{\Lambda_1} \cup E_{\Lambda_2})) + \Lambda_2\, Per(E_{\Lambda_1} \cup E_{\Lambda_2}).$$

Using the inequality

$$Per(E \cup F) + Per(E \cap F) \leq Per(E) + Per(F),$$

we get

$$\frac{1}{\Lambda_1}[\sigma(\Omega \backslash E_{\Lambda_1}) - \sigma(\Omega \backslash (E_{\Lambda_1} \cap E_{\Lambda_2}))] \leq \frac{1}{\Lambda_2}[\sigma(\Omega \backslash (E_{\Lambda_1} \cup E_{\Lambda_2})) - \sigma(\Omega \backslash E_{\Lambda_2})].$$

Hence,

$$\left(\frac{1}{\Lambda_2} - \frac{1}{\Lambda_1}\right)\sigma(E_{\Lambda_1} \backslash E_{\Lambda_2}) \leq 0.$$

Thanks to the fact that $\Lambda_1 > \Lambda_2$, this implies that $\sigma(E_{\Lambda_1} \backslash E_{\Lambda_2}) = 0$ and so, we have $E_{\Lambda_1} \subset E_{\Lambda_2}$ $\sigma-$a.e. $\square$

Moreover, we have the following:

Proposition 3.4. *Let E, F be two optimal sets. Then, $E \cup F$ and $E \cap F$ are also optimal sets. In particular, there exist two optimal sets $E_{\max}$ and $E_{\min}$ such that for any optimal set E, we have $E_{\min} \subset E \subset E_{\max}$.*

Proof. From the optimality of E and F in Problem (3.3), we have obviously the following inequalities

$$\sigma(\Omega\backslash E) + \Lambda\, Per(E) \leq \sigma(\Omega\backslash(E\cap F)) + \Lambda\, Per(E\cap F) \tag{3.4}$$

and

$$\sigma(\Omega\backslash F) + \Lambda\, Per(F) \leq \sigma(\Omega\backslash(E\cup F)) + \Lambda\, Per(E\cup F). \tag{3.5}$$

Yet, we clearly have $\sigma(\Omega\backslash(E\cap F)) + \sigma(\Omega\backslash(E\cup F)) = \sigma(\Omega\backslash E) + \sigma(\Omega\backslash F)$. Hence, taking the sum of (3.4) & (3.5) yields that

$$Per(E) + Per(F) \leq Per(E\cup F) + Per(E\cap F).$$

This implies that

$$Per(E) + Per(F) = Per(E\cup F) + Per(E\cap F).$$

Consequently, the two inequalities in (3.4) & (3.5) are in fact equalities and so, $E\cup F$ and $E\cap F$ minimize Problem (3.3). $\square$

Now, let us study the regularity of optimal sets.

Proposition 3.5. *Let E be an optimal set for Problem (3.3). Then, we have the following statements:*

• *If $f \in L^p_{loc}(\Omega)$ with $p > 2$, then $\partial E \cap \mathring{\Omega}$ is of class C^1. Moreover, ∂E is globally C^1 on Ω as soon as $f \in L^p(\Omega)$ with $p > 2$, Ω satisfies a uniform exterior ball condition with a boundary of class C^1 and, g is semi-concave.*

• *If f is continuous in $\mathring{\Omega}$, $\partial\Omega$ is C^2 and $g \in C^2(\partial\Omega)$, then ∂E is C^2 in the interior of Ω. Moreover, ∂E is globally $C^{1,1}$ on Ω provided that $f \in C(\Omega)$.*

• *If f is locally Lipschitz in $\mathring{\Omega}$, $\partial\Omega \in C^{2,1}$ and $g \in C^{2,1}(\partial\Omega)$, then $\partial E\backslash\overline{\Sigma}$ is $C^{2,1}$ in $\mathring{\Omega}$.*

Proof. The first part of the first statement follows immediately from [17, Theorem 3.2] thanks to the fact that the transport density $\sigma \in L^p_{loc}(\Omega)$ (see [19, Theorem 4.20] and takes into account that the target measure is supported on $\partial\Omega$). Moreover, assume that ∂E is not C^1 at some point $x \in \partial E \cap \partial\Omega$. Let $(s, \alpha(s))$ be a parametrization of ∂E around x. Thanks to the $\Omega-$convexity of E (see Proposition 3.2) and the C^1 regularity of $\partial\Omega$, there exists a $\delta > 0$ such that we either have $\{(s, \alpha(s)) : s \in [-\delta, \delta]\backslash\{0\}\} \subset \mathring{\Omega}$, $\{(s, \alpha(s)) : s \in [-\delta, 0[\} \subset \mathring{\Omega}$ and $\{(s, \alpha(s)) : s \in [0, \delta]\} \subset \partial\Omega$, or $\{(s, \alpha(s)) : s \in]0, \delta]\} \subset \mathring{\Omega}$ and $\{(s, \alpha(s)) : s \in [-\delta, 0]\} \subset \partial\Omega$. In all these three possibilities, one can always assume that after a rotation and translation of axes, $x = (0, 0)$ and $|\alpha'(s)| \geq c > 0$, for a.e. $s \in (\varepsilon^-, \varepsilon^+)$, where $\varepsilon^- \in]-\delta, 0[$ and $\varepsilon^+ \in]0, \delta[$ are small enough such that $\alpha(\varepsilon^+) = \alpha(\varepsilon^-)$; we set $\varepsilon := \varepsilon^+ - \varepsilon^- > 0$. If we denote by $\mathcal{C}$ the part of ∂E between $(\varepsilon^+, \alpha(\varepsilon^+))$ and $(\varepsilon^-, \alpha(\varepsilon^-))$ and $\hat{\mathcal{C}}$ the segment joining these two points, then we see that $\hat{C} \subset \Omega$. Now, let $\hat{E}$ be such that $\partial\hat{E} = (\partial E\backslash\mathcal{C}) \cup \hat{\mathcal{C}}$. Hence, we have

$$Per(\hat{E}) - Per(E) = \varepsilon - \int_{\varepsilon^-}^{\varepsilon^+} \sqrt{1 + \alpha'(s)^2}\, \mathrm{d}s \leq (1 - \sqrt{1 + c^2})\, \varepsilon.$$

On the other hand, by [15], we have $\sigma \in L^p(\Omega)$ as soon as $f \in L^p(\Omega)$, Ω satisfies a uniform exterior ball condition and g is $\lambda-$Lip (with $\lambda < 1$) and semi-concave. Then, we have

$$\sigma(\Omega\backslash\hat{E}) - \sigma(\Omega\backslash E) = \int_{E\backslash\hat{E}} \sigma \leq ||\sigma||_{L^p(E\backslash\hat{E})} |E\backslash\hat{E}|^{\frac{1}{q}} = ||\sigma||_{L^p} \left[\int_{\varepsilon^-}^{\varepsilon^+} [\alpha(\varepsilon^+) - \alpha(s)]\mathrm{d}s\right]^{\frac{1}{q}} \leq C||\sigma||_{L^p(E\backslash\hat{E})}\, \varepsilon^{\frac{2}{q}},$$

where q is the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$. Consequently, we get

$$\sigma(\Omega\backslash\hat{E})+\Lambda Per(\hat{E})-[\sigma(\Omega\backslash E)+\Lambda Per(E)]\leq\left[\Lambda(1-\sqrt{1+c^2})+C||\sigma||_{L^p(E\backslash\hat{E})}\,\varepsilon^{\frac{2}{q}-1}\right]\varepsilon.$$

But, this yields to a contradiction with the optimality of E provided that $\varepsilon>0$ is small enough and $p\geq 2$.

For the second statement, let $x\in\partial E$ be a fixed point inside Ω. Let $(s,\alpha(s))$ be a parametrization of ∂E around x. Fix $\varepsilon>0$, then it is clear that α minimizes

$$\min\left\{\int_{-\varepsilon}^{\varepsilon}\int_0^{\gamma(s)}\sigma(s,t)\,\mathrm{d}t\,\mathrm{d}s+\Lambda\int_{-\varepsilon}^{\varepsilon}\sqrt{1+\gamma'(s)^2}\,\mathrm{d}s\,:\,\gamma(\varepsilon)=\alpha(\varepsilon),\,\gamma(-\varepsilon)=\alpha(-\varepsilon)\right\}.$$

From the optimality conditions on α and thanks to the continuity of the transport density σ (see Proposition 2.5), we get that

$$\left[\frac{\alpha'(s)}{\sqrt{1+\alpha'(s)^2}}\right]'=\frac{1}{\Lambda}\,\sigma(s,\alpha(s)).$$

This implies that the optimal region E has boundary of class C^2 in the interior of Ω and the curvature k of ∂E is given by

$$k=\frac{\sigma}{\Lambda}. \tag{3.6}$$

Now, assume that $x\in\partial E\cap\partial\Omega$ and that $\partial\Omega$ is the graph of a smooth function β. Then, α solves

$$\min\left\{\int_{-\varepsilon}^{\varepsilon}\int_0^{\gamma(s)}\sigma(s,t)\,\mathrm{d}t\,\mathrm{d}s+\Lambda\int_{-\varepsilon}^{\varepsilon}\sqrt{1+\gamma'(s)^2}\mathrm{d}s\,:\,\gamma(\varepsilon)=\alpha(\varepsilon),\,\gamma(-\varepsilon)=\alpha(-\varepsilon),\,\gamma\geq\beta\right\}.$$

The optimality conditions on α as well as the fact that $\alpha\geq\beta$ and β is C^2 yield that the curvature k of ∂E satisfies

$$-||\kappa||_\infty\leq k\leq\frac{\sigma}{\Lambda},$$

where κ denotes the curvature of $\partial\Omega$. Finally, the last statement follows immediately from the estimate (3.6) and the proposition 2.6. □

3.2. **Penalization with the fractional perimeter.** In this subsection, we consider another version of Problem (3.1) which is somehow more complicated than the one considered in Subsection 3.1, where the penalization term $P(E)$ will be given now by the fractional perimeter of E (we note that this penalization was already used for the fractional Cheeger problem in [5]). Fix $s\in(0,1)$, then we consider the following problem:

$$\min\left\{\sigma(\Omega\backslash E)+Per_s(E)\,:\,E\subset\Omega\right\}, \tag{3.7}$$

where for every Borel set $E\subset\mathbb{R}^2$, we define its $s-$perimeter as the $W^{s,1}$ semi-norm of the characteristic function of E:

$$Per_s(E)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|1_E(x)-1_E(y)|}{|x-y|^{2+s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We note that the $s-$perimeter of E is somehow an interpolation between the perimeter and the Lebesgue measure of E. More precisely, we have the following inequality (see [5, Corollary 4.4]):

$$Per_s(E)\leq C\,Per(E)^s\,|E|^{1-s}.$$

Proposition 3.6. *Assume that $f\in L^1(\Omega)$. Then, the fractional shape optimization problem (3.7) has a solution.*

Proof. Let $\{E_n\}_{n\in\mathbb{N}} \subset \Omega$ be a minimizing sequence in Problem (3.7). It is clear that one can assume that there is a constant C such that

$$Per_s(E_n) \le C, \text{ for every } n \in \mathbb{N}.$$

Thanks to the fact that $E_n \subset \Omega$, for all $n \in \mathbb{N}$, we infer that the $W^{s,1}$ norm of 1_{E_n} is uniformly bounded and so, up to a subsequence, $\{1_{E_n}\}_{n\in\mathbb{N}}$ converges strongly in $L^1(\Omega)$ to a function 1_E, for some measurable set $E \subset \Omega$. By using the latter, the lower semicontinuity of the $W^{s,1}$ seminorm and the fact that $\sigma \in L^1(\Omega)$, we get that

$$\sigma(\Omega\backslash E) + Per_s(E) \le \liminf_n [\sigma(\Omega\backslash E_n) + Per_s(E_n)]. \quad \square$$

In order to study the regularity of optimal sets in Problem (3.7), we will follow the same technique that is already used in [5] to prove regularity on the fractional Cheeger set. First, we define

$$L_s(A,B) = \int_A \int_B \frac{1}{|x-y|^{2+s}}\, \mathrm{d}x\, \mathrm{d}y, \quad \text{for all Borel sets } A,\, B \subset \mathbb{R}^2.$$

Then, for every $E \subset \mathbb{R}^2$, we set

$$J_s(E,\Omega) := L_s(E\cap\Omega, E^c) + L_s(E\backslash\Omega, E^c \cap \Omega).$$

Notice that if $E \subset \Omega$, then we have $J_s(E,\Omega) = L_s(E,E^c) = \frac{1}{2}Per_s(E)$. Now, we introduce the notion of almost minimality for the functional J_s (which extends the notion introduced by Almgren for the perimeter; see [2]) as follows:

Definition 3.2. *Let $\delta > 0$ and $\rho : (0,\delta) \mapsto \mathbb{R}^+$ a modulus of continuity. We say that a measurable set E is $(J_s,\rho,\delta)-$minimal in Ω (or simply that E is almost $s-$minimal in Ω) if for all $x_0 \in \partial E$ and any measurable set F such that $F\Delta E \subset B(x_0,r)$ for some $r < \min\{\delta, d(x_0,\partial\Omega)\}$, we have*

$$J_s(E,\Omega) \le J_s(F,\Omega) + \rho(r)r^{2-s}.$$

Moreover, we will say that E is a $s-$minimal set in Ω if for any set F with $F\backslash\Omega = E\backslash\Omega$, we have

$$J_s(E,\Omega) \le J_s(F,\Omega).$$

In order to prove regularity on the optimal regions of Problem (3.7), we will introduce some results on the almost $s-$minimal sets that generalize those given in [6] where the authors considered instead the $s-$minimal sets (i.e. $\rho = 0$). In fact, some of these results have already been proven in [8] and so, we will omit some details. First, we start by the following:

Lemma 3.7. *Assume G is a $(J_s,\rho,\delta)-$minimal set in $B_1 := B(0,1)$ and $0 \in \partial G$. For every $n \in \mathbb{N}$, set $G_n := nG$. Then, G_n is $(J_s,\rho_n,n\delta)$-minimal in B_1 with $\rho_n(t) = \rho(\frac{t}{n})$, for all n. Moreover, $G_n \to C$ in $L^1_{loc}(\mathbb{R}^2)$ and, C is a $s-$minimal cone (i.e. $tC = C$ for all $t > 0$) in B_{r_0}, for $r_0 < \min\{\delta, 1\}$.*

Proof. First, it is easy to see that

$$J_s(G_n,B_1) = n^{2-s}\, J_s(G, B_{\frac{1}{n}}), \text{ for all } n \in \mathbb{N}.$$

Now, fix $n \in \mathbb{N}$. Let $x_n \in \partial G_n$ and F_n is a set such that $F_n\Delta G_n \subset B(x_n,r)$, for some $r < \min\{n\delta, n-|x_n|\}$. Set $F := \frac{1}{n}F_n$ and $x_0 = \frac{1}{n}x_n$. So, it is clear that $F\Delta G \subset B(x_0, \frac{r}{n})$. Thus, we have

$$J_s(G_n,B_1) = n^{2-s}\, J_s(G,B_{\frac{1}{n}}) \le n^{2-s}\left[J_s(F,B_{\frac{1}{n}}) + \rho\left(\frac{r}{n}\right)\left(\frac{r}{n}\right)^{2-s}\right] = J_s(F_n,B_1) + \rho_n(r)\, r^{2-s}.$$

Hence, G_n is a $(J_s, \rho_n, n\delta)-$minimal set in B_1, for all n. Let us show that C is $s-$minimal in some neighborhood of the origin. Let F be a set such that $F \Delta C \subset B_{r_0}$, for some $r_0 < \min\{\delta, 1\}$. For every $n \in \mathbb{N}$, set

$$F_n = [F \cap B_{r_0}] \cup [G_n \backslash B_{r_0}].$$

Since G_n is $(J_s, \rho_n, n\delta)-$minimal in B_1 (and so, in B_{r_0} as $B_{r_0} \subset B_1$), then we have

$$J_s(G_n, B_{r_0}) \leq J_s(F_n, B_{r_0}) + \rho_n(r_0) r_0^{2-s}.$$

Moreover, it is not difficult to check that

$$|J_s(F_n, B_{r_0}) - J_s(F, B_{r_0})| \leq L_s(B_{r_0}, (G_n \Delta C) \backslash B_{r_0}).$$

But, one can show that we have (see the proof of [6, Theorem 3.3]):

$$\lim_{n \to \infty} L_s(B_{r_0}, (G_n \Delta C) \backslash B_{r_0}) = 0.$$

Hence, we get

$$\limsup_{n \to \infty} J_s(G_n, B_{r_0}) \leq J_s(F, B_{r_0}).$$

On the other hand, by [6, Proposition 3.1], $J_s(\cdot, B_{r_0})$ is lower semicontinuous and, since $G_n \to C$ in $L^1_{loc}(\mathbb{R}^2)$, then we have

$$J_s(C, B_{r_0}) \leq \liminf_{n \to \infty} J_s(G_n, B_{r_0}).$$

Consequently, we get

$$J_s(C, B_{r_0}) \leq J_s(F, B_{r_0}).$$

This yields that C is $s-$minimal in B_{r_0}. In order to show that C is a cone, we need a monotonicity formula for (almost) $s-$minimal sets that generalize the classical one for minimal sets. In fact, it is well known that if E is a minimal set in some neighborhood of $0 \in \partial E$, then the functional

$$\phi_E(r) := \frac{\mathcal{H}^1(\partial E \cap B_r)}{r}$$

is monotone increasing (i.e. $[\phi_E(r)]' \geq 0$) and, it is constant as soon as E is a cone. In [6, Section 7] and [8, Section 7], the authors extend this monotonicity formula to the (almost) $s-$minimal sets. Let E be a $(J_s, \rho, \delta)-$minimal set in B_1 with $0 \in \partial E$. So, we define the extension $\tilde{u}_E : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ of the function $u_E := 1_E - 1_{E^c}$ as the solution of

$$\begin{cases} \nabla \cdot [z^{1-s} \nabla \tilde{u}] = 0 & \text{in } \ \mathbb{R}^2 \times \mathbb{R}_+, \\ \tilde{u} = u_E & \text{on } \ \{z = 0\}. \end{cases}$$

Now, we introduce as in [6, 8] the functional Φ_E as follows (where $B_r^+ := B_r \cap \{z > 0\}$):

$$\Phi_E(r) := \frac{\int_{B_r^+} z^{1-s} |\nabla \tilde{u}_E|^2}{r^{2-s}} + (2-s) \int_0^r \rho(t) t^{1-s} \, \mathrm{d}t.$$

Then, one can show that Φ_E is monotone increasing in r (see [6, Theorem 8.1] or [8, Lemma 7.3]). Using [6, Proposition 9.1], since $G_n \to C$ in $L^1_{loc}(\mathbb{R}^2)$ then we have $\Phi_{G_n}(r) \to \Phi_C(r)$, for every r. Yet, it is easy to check that

$$\Phi_{G_n}(r) = \Phi_G\left(\frac{r}{n}\right) - (2-s) \int_0^{\frac{r}{n}} \rho(t) t^{1-s} \, \mathrm{d}t + (2-s) \int_0^r \rho_n(t) t^{1-s} \, \mathrm{d}t. \tag{3.8}$$

Passing to the limit in (3.8) when $n \to \infty$, we get the following:

$$\Phi_C(r) = \lim_{\varepsilon \to 0} \Phi_G(\varepsilon), \ \text{ for all } \ r.$$

In other words, this means that Φ_C is constant. Consequently, by [6, Corollary 8.2], we infer that C is a cone. $\square$

Hence, we have the following (see also [6, Theorem 9.4] and [8, Theorem 7.4]):

Lemma 3.8. *If the set G is $(J_s, \rho, \delta)-$minimal in B_1 and $0 \in \partial G$, then ∂G is C^1 in a neighborhood of the origin.*

Proof. Thanks to Lemma 3.7, we know that $nG \to C$ in $L^1_{loc}(\mathbb{R}^2)$ and C is a $s-$minimal cone. But, by [20, Theorem 1], we infer that C is a half-plane. This concludes the proof. $\square$

Moreover, we get

Lemma 3.9. *If E is $(J_s, \rho, \delta)-$minimal in Ω, then ∂E is C^1 in the interior of Ω.*

Proof. Fix $x_0 \in \partial E \cap \mathring{\Omega}$. Let $r_0 > 0$ be small enough so that $B(x_0, r_0) \subset \Omega$. So, E is $(J_s, \rho, \delta)-$minimal in $B(x_0, r_0)$. Now, set $G = \frac{E - x_0}{r_0}$. Then, it is easy to see that G is $(J_s, \rho_0, \delta_0)-$minimal in B_1 with $\rho_0 = \rho(r_0 r)$ and $\delta_0 = \frac{\delta}{r_0}$, since we have

$$J_s(G, B_1) = \frac{1}{r_0^{2-s}} J_s(E, B_{r_0}).$$

Thanks to Lemma 3.8, ∂G is C^1 in a neighborhood of the origin. By scaling and translating back, we infer that ∂E is C^1 in a neighborhood of x_0. $\square$

Finally, we are ready to state our regularity results.

Proposition 3.10. *Let E be a minimizer for Problem (3.7). Assume that g is $\lambda-$Lip with $\lambda < 1$ and $f \in L^p_{loc}(\Omega)$ with $p > \frac{2}{s}$. Then, $\partial E \cap \mathring{\Omega}$ is C^1.*

Proof. Thanks to Lemma 3.9, it is sufficient to prove that for every point $x_0 \in \partial E \cap \mathring{\Omega}$, the set E is $(J_s, \rho, \delta)-$minimal in $B(x_0, r_0)$, for some $0 < r_0 < d(x_0, \partial\Omega)$, $\delta > 0$ and, a modulus of continuity ρ. For all $x \in \partial E \cap B(x_0, r_0)$, $r < r_0 - |x - x_0|$ and, F such that $E \Delta F \subset B(x, r)$, we have $F \subset \Omega$ and then thanks to the minimality of E in Problem (3.7), we get that

$$Per_s(E) - \int_E \sigma \leq Per_s(F) - \int_F \sigma.$$

Hence,

$$\begin{aligned} Per_s(E) &\leq Per_s(F) + \int_{E \cap B(x,r)} \sigma - \int_{F \cap B(x,r)} \sigma \\ &\leq Per_s(F) + C||\sigma||_{L^p(B(x_0,r_0))}\, r^{\frac{2}{q}}. \end{aligned}$$

This implies that E is $(J_s, \rho, \delta)-$minimal in $B(x_0, r_0)$ with $\rho(r) = C||\sigma||_{L^p(B(x_0,r_0))}\, r^{s-\frac{2}{p}}$ (we recall that $\sigma \in L^p_{loc}(\Omega)$ thanks to the fact that f belongs to $L^p_{loc}(\Omega)$ and the target measure is concentrated on $\partial\Omega$; see [19]), since we have

$$J_s(E, \Omega) \leq J_s(F, \Omega) + \rho(r) r^{2-s}. \quad \square$$

Moreover, one can prove regularity on the optimal set E at points touching $\partial\Omega$ provided that $\partial\Omega$ is smooth.

Proposition 3.11. *Let E be a minimizer for Problem (3.7). Assume that $f \in L^p(\Omega)$ with $p > \frac{2}{s}$, g is $\lambda-$Lip with $\lambda < 1$ and semi-concave, $\partial\Omega$ is $C^{1,\alpha}$ and Ω satisfies a uniform exterior ball condition. Then, ∂E is C^1 on Ω.*

Proof. From Proposition 3.10, we know that $\partial E \cap \mathring{\Omega}$ is C^1. Now, fix $x \in \partial E \cap \partial\Omega$ and $\varepsilon > 0$. We show that E is $(J_s, \rho, \delta)-$minimal in $B(x, \varepsilon)$, for some $\delta > 0$ and a modulus of continuity ρ. Let $x_0 \in \partial E \cap B(x, \varepsilon)$, $r < d(x_0, \partial B(x, \varepsilon))$ and F be such that $E \Delta F \subset B(x_0, r)$. We note that F is not necessarily contained in Ω. But anyway, $F \cap \Omega$ is admissible in Problem (3.7) and so, we have

$$Per_s(E) - \int_E \sigma \leq Per_s(F \cap \Omega) - \int_{F\cap\Omega} \sigma.$$

Thanks to [15], we get that

$$\begin{aligned} Per_s(E) &\leq Per_s(F \cap \Omega) + \int_{E\cap B(x_0,r)} \sigma - \int_{F\cap\Omega\cap B(x_0,r)} \sigma \\ &\leq Per_s(F \cap \Omega) + C||\sigma||_{L^p}\, r^{\frac{2}{q}}. \end{aligned}$$

Yet, we have $J_s(E, \Omega) = \frac{1}{2}Per_s(E)$ and $L_s(F \cap \Omega, (F \cap \Omega)^c) = \frac{1}{2}Per_s(F \cap \Omega)$. Hence, we get that

$$J_s(E, \Omega) \leq L_s(F \cap \Omega, (F \cap \Omega)^c) + \rho(r)r^{2-s}, \tag{3.9}$$

where $\rho(r) = C||\sigma||_{L^p}\, r^{s-\frac{2}{p}}$. But, $J_s(F, \Omega) = L_s(F \cap \Omega, F^c) + L_s(F\backslash\Omega, F^c \cap \Omega)$. Hence, by (3.9), we have

$$J_s(E, \Omega) \leq J_s(F, \Omega) + L_s(F \cap \Omega, F^c \cup \Omega^c) - L_s(F \cap \Omega, F^c) - L_s(F\backslash\Omega, F^c \cap \Omega) + \rho(r)r^{2-s}.$$

On the other hand, one has

$$\begin{aligned} &L_s(F \cap \Omega, F^c \cup \Omega^c) - L_s(F \cap \Omega, F^c) - L_s(F\backslash\Omega, F^c \cap \Omega) \\ &= L_s(F \cap \Omega, F^c) + L_s(F \cap \Omega, F \cap \Omega^c) - L_s(F \cap \Omega, F^c) - L_s(F\backslash\Omega, F^c \cap \Omega) \\ &= L_s(F \cap \Omega, F \cap \Omega^c) - L_s(F\backslash\Omega, F^c \cap \Omega) \leq L_s(\Omega, B(x_0, r) \cap \Omega^c). \end{aligned}$$

Yet, thanks to [8, Section 3] and the fact that $\partial\Omega$ is $C^{1,\alpha}$, we have the following estimate:

$$L_s(\Omega, B(x_0, r) \cap \Omega^c) = \int_\Omega \int_{B(x_0,r)\cap\Omega^c} \frac{1}{|x-y|^{2+s}}\, \mathrm{d}x\, \mathrm{d}y \leq C\, r^{2-s+\alpha}.$$

This implies that

$$J_s(E, \Omega) \leq J_s(F, \Omega) + \tilde{\rho}(r)r^{2-s}$$

with $\tilde{\rho}(r) = Cr^\beta$ and $\beta = \min\{\alpha, s - \frac{2}{p}\}$. Hence, E is $(J_s, \tilde{\rho}, \delta)-$minimal in $B(x, \varepsilon)$ and so, ∂E is C^1 inside $B(x, \varepsilon)$. □

We conclude this paper by the following:

Remark 3.1. *In fact, it seems difficult to prove higher regularity (for instance, C^2) on an optimal set E of Problem (3.7) and so, the second order regularity of optimal set E is still an open question! On the other hand, it is not easy to go beyond $C^{2,1}$ regularity on an optimal region E for Problem (3.3), since this requires to show some smoothness on the transport density σ (and then, on the map τ) which seems to be tricky.*

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