HESSIAN INEQUALITIES AND THE FRACTIONAL LAPLACIAN

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Abstract. We study relations between the k-Hessian energy $\mathcal{E}_k[u] = \int_{\mathbb{R}^n} -u\, F_k[u]dx$, and the fractional Sobolev energy $E_k[u] = \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{k}{k+1}} u\right|^{k+1} dx$, where F_k $(k = 1, \ldots, n)$ is the k-Hessian operator on $\mathbb{R}^n$, and u is a k-convex function vanishing at ∞. We prove that there is a constant $C > 0$ such that
$$C^{-1}\, E_k[u] \le \mathcal{E}_k[u] \le C\, E_k[u],$$
where the lower estimate is obtained under some additional assumptions on u. In general, we show that there exists a function $\tilde{u}$ such that $c^{-1} \le |u/\tilde{u}| \le c$, and $C^{-1}\, E_k[\tilde{u}] \le \mathcal{E}_k[u] \le C\, E_k[\tilde{u}]$, where c, C are positive constants depending only on k and n.

1. Introduction

The fully nonlinear k-Hessian operator F_k $(k = 1, 2, \ldots, n)$ on $\Omega \subseteq \mathbb{R}^n$ is defined by

$$F_k[u] = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.1}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$. It is known that F_k is elliptic on the cone of k-convex functions.

We recall that an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is called k-convex if $F_k[q] \ge 0$ for any quadratic polynomial q such that $u - q$ has a local finite maximum in Ω. A function $u \in C^2(\Omega)$ is k-convex if and only if $F_j[u] \ge 0$ for all $j = 1, \ldots, k$. We refer to Trudinger and Wang [TW1], [TW2] (see also [W2]) where a comprehensive theory of k-convex functions has been developed. In particular, it is shown that, for a k-convex function u, $\mu_k[u] = F_k[u]$ is a positive Borel measure in Ω (the so-called k-Hessian measure associated with u).

The k-Hessian energy is defined by

$$\mathcal{E}_k[u] = \int_\Omega -u\, F_k[u]\, dx, \tag{1.2}$$

for a k-convex function $u \in C^2(\Omega)$. For the sake of convenience, we use the same notation for general k-convex functions u assuming that integration is performed with respect to the k-Hessian measure $F_k[u]$. When $k = 1$, $F_k[u] = \Delta u$, and integration by parts shows that $\mathcal{E}_1[u]$ coincides with the classical energy

$$E[u] = \int_\Omega |\nabla u|^2\, dx, \tag{1.3}$$

for $u \in W_0^{1,2}(\Omega)$.

Following [TW2], we will denote by $\Phi(\Omega)$ the cone of k-convex functions on Ω, and by $\Phi_0(\Omega)$ its subcone which consists of k-convex functions with zero boundary values.

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In this paper we assume for simplicity that $\Omega = \mathbb{R}^n$, so that $\Phi_0(\mathbb{R}^n)$ consists of all k-convex functions vanishing at ∞. We study relations between the Hessian energy, $\mathcal{E}_k[u]$, and the fractional Sobolev energy

$$E_{\alpha,p}[u] = \int_{\mathbb{R}^n} \left|(-\Delta)^{\alpha/2} u\right|^p \, dx, \tag{1.4}$$

where

$$\alpha = \frac{2k}{k+1} \quad \text{and} \quad p = k+1. \tag{1.5}$$

In the classical case $k = 1$, we have $\alpha = 1$ and $p = 2$. When $k = 2, \dots, n$, it follows that $1 < \alpha < 2$ and $p > 2$.

For α and p given by (1.5) we will use the following notation:

$$E_k[u] = E_{\frac{2k}{k+1}, k+1}[u] = \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{k}{k+1}} u\right|^{k+1} \, dx, \tag{1.6}$$

for the fractional Sobolev energy associated with $u \in \Phi_0(\mathbb{R}^n)$.

In Section 3 (Theorem 3.1) we will prove the inequality

$$\mathcal{E}_k[u] \le c_{k,n} E_k[u], \tag{1.7}$$

for all $u \in \Phi_0(\mathbb{R}^n)$, where $c_{k,n}$ is a positive constant depending only on k and n, $1 \le k < \frac{n}{2}$. When $k \ge \frac{n}{2}$ the Hessian energy $\mathcal{E}_k[u]$ is always infinite unless $u = 0$.

In Section 2 (Theorem 2.1), the converse inequality will be established:

$$E_k[u] \le C_{k,n} \mathcal{E}_k[u], \tag{1.8}$$

under the additional assumption

$$(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2} u]^k \ge 0. \tag{1.9}$$

To justify the last assumption we notice that it holds for the fundamental solution of the k-Hessian equation. (See Section 4 for further discussion.) Moreover, for every $u \in \Phi_0(\mathbb{R}^n)$,

$$-(-\Delta)^{\alpha/2} u \ge 0. \tag{1.10}$$

Indeed, if $u \in \Phi_0(\mathbb{R}^n)$, i.e., u is a k-convex function vanishing at ∞, then u is subharmonic in the classical sense, and by the maximum principle, $u \le 0$ on $\mathbb{R}^n$. Furthermore, it follows (see, e.g., [Lan]) that $u = -(-\Delta)^{-1}\mu = -I_2\mu$, where μ is a positive Borel measure on $\mathbb{R}^n$. Here $I_\alpha = (-\Delta)^{-\alpha/2}$ is the Riesz potential of order $\alpha \in (0, n)$ on $\mathbb{R}^n$:

$$I_\alpha \mu(x) = a_{\alpha,n} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}} \, dy, \qquad x \in \mathbb{R}^n, \tag{1.11}$$

where $a_{\alpha,n}$ is a positive normalization constant, and μ is a measure on $\mathbb{R}^n$. (When $d\mu = v\, dx$, we write $I_\alpha v$ in place of $I_\alpha \mu$.)

It follows that, for every $u \in \Phi_0(\mathbb{R}^n)$,

$$-(-\Delta)^{\alpha/2} u = (-\Delta)^{\alpha/2} \, (-\Delta)^{-1} \mu = I_{2-\alpha}\mu \ge 0, \tag{1.12}$$

by the semigroup property of $(-\Delta)^{\alpha/2}$ and the positivity of the Riesz kernel.

Other related inequalities involving the fractional Laplacian are discussed in Section 4. In particular, we will use the approach of Caffarelli and Silvestre [CaS] to prove that, for $0 < \alpha < 2$, $1 \le p < \infty$, and $f \ge 0$,

$$(-\Delta)^{\alpha/2}(f^p) \le p \, f^{p-1} \cdot (-\Delta)^{\alpha/2} f \quad \text{a.e.}, \tag{1.13}$$

provided $f \in L^p(w)$, where $w(x) = (1+|x|)^{-(n+\alpha)}$. (See Lemma 4.1 below.) The case $\alpha = 2$ is classical.

For the sake of convenience, we prove some of the above inequalities under the condition that f (or u) is a C^2-function. This assumption can be relaxed using an appropriate approximation procedure.

2. A lower bound for the Hessian energy

In this section we prove inequality (1.8) under the assumption (1.9).

Theorem 2.1. *$1 \le k < \frac{n}{2}$, and let $\alpha = \frac{2k}{k+1}$. Suppose $u \in C^2(\mathbb{R}^n)$ is a $k-$convex function on $\mathbb{R}^n$ vanishing at ∞. If*

(i) $-(-\Delta)^{\alpha/2}u \ge 0$,

(ii) $(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \ge 0$,

then there exists a positive constant $C_{k,n}$ such that

$$\int_{\mathbb{R}^n} \left(-(-\Delta)^{\alpha/2}u\right)^{k+1} dx \le C_{k,n} \int_{\mathbb{R}^n} -u\, F_k[u]\, dx. \tag{2.1}$$

Remark 2.2. As was mentioned above, $-(-\Delta)^{\alpha/2}u \ge 0$, for every $u \in \Phi_0(\mathbb{R}^n)$, so condition (i) in the statement of the theorem is redundant.

Proof. By duality

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \left(-(-\Delta)^{\alpha/2}u\right)^{k+1} dx\right)^{\frac{1}{k+1}} &= \sup_{0<\|\phi\|_{L^{1+\frac{1}{k}}}<\infty} \frac{\int_{\mathbb{R}^n} -(-\Delta)^{\alpha/2}u\phi dx}{\|\phi\|_{L^{1+\frac{1}{k}}}} \\ &= \sup_{0<\|\phi\|_{L^{1+\frac{1}{k}}}<\infty} \frac{\left|\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(-u)\phi dx\right|}{\|\phi\|_{L^{1+\frac{1}{k}}}}.\end{aligned} \tag{2.2}$$

In what follows we may assume that $\phi \ge 0$ and $(-\Delta)^{\alpha/2}\phi \ge 0$. Indeed, the extremal function ϕ in the duality relation (2.2) is $\phi = [-(-\Delta)^{\alpha/2}u]^k$. Hence $\phi \ge 0$, and the second inequality, $(-\Delta)^{\alpha/2}\phi \ge 0$, coincides with (ii).

We set $\phi_j = I_\alpha[\chi_{B_j(0)}(-\Delta)^{\alpha/2}\phi]$, where $j = 1, 2, \ldots$, and $\chi_{B_j(0)}$ is the characteristic function of the ball $B_j(0)$ of radius j centered at 0. Then $\phi_j \ge 0$ is an increasing sequence such that $\phi = \lim_{j\to\infty}\phi_j$, and $(-\Delta)^{\alpha/2}\phi_j$ is compactly supported. In particular, $\phi_j(x) \le C_j(1+|x|)^{\alpha-n}$. The latter estimate ensures that $\phi_j \in L^{1+\frac{1}{k}}(\mathbb{R}^n)$ since $k < \frac{n}{2}$.

Thus, it suffices to prove the inequality

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(-u)\, \phi dx \le C_{k,n}\|\phi\|_{L^{1+\frac{1}{k}}} \cdot \left(\int_{\mathbb{R}^n} -u\, F_k[u]\, dx\right)^{\frac{1}{k+1}} \tag{2.3}$$

with ϕ_j in place of ϕ, since passing to the limit as $j \to \infty$ yields (2.1). In other words, without loss of generality we may assume that $(-\Delta)^{\alpha/2}\phi$ is compactly supported and $0 \le \phi(x) \le C_j(1+|x|)^{\alpha-n}$.

By the self-adjointness of $(-\Delta)^{\alpha/2}$ we get

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(-u)\phi dx = \int_{\mathbb{R}^n} -u(-\Delta)^{\alpha/2}\phi dx \tag{2.4}$$

Since $(-\Delta)^{\alpha/2}\phi \ge 0$, we can solve the equation

$$F_k[v] = (-\Delta)^{\alpha/2}\phi,$$

in the viscosity sense, where v is a k-convex function vanishing at ∞ (see [TW2]).

As a consequence, we get using (2.4),

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(-u)\, \phi\, dx = \int_{\mathbb{R}^n} -u\, F_k[v]\, dx. \tag{2.5}$$

We now apply the Hessian Schwarz inequality proved in [V] (Theorem 3.1):

$$\int_{\mathbb{R}^n} -u\, F_k[v]\, dx \le \left(\int_{\mathbb{R}^n} -u\, F_k[u]\, dx\right)^{\frac{1}{k+1}} \cdot \left(\int_{\mathbb{R}^n} -v\, F_k[v]\, dx\right)^{\frac{k}{k+1}}. \tag{2.6}$$

It is known [PhV] that there exist positive constants c_1, c_2 depending only on k and n such that, for any k-convex solution v vanishing at infinity to the equation $\nu = F_k[v]$, where $\nu \ge 0$, $v \in L^1_{\rm loc}(\mathbb{R}^n)$,

$$c_1\, \mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu(x) \le -v(x) \le c_2\, \mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu(x), \qquad x \in \mathbb{R}^n. \tag{2.7}$$

Here Wolff's potential $\mathbf{W}_{\alpha,\, p}\nu$ is defined by

$$\mathbf{W}_{\alpha,\, p}\nu(x) = \int_0^\infty \left[\frac{|B_t(x)|_\nu}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n, \tag{2.8}$$

where $\alpha = \frac{2k}{k+1}$, $p = k+1$, $|B_t(x)|_\nu = \int_{B_t(x)} \nu(y)\, dy$, and $B_t(x)$ is a ball of radius t centered at x. With this choice of α, p, we get

$$\mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu(x) = \int_0^\infty \left[\frac{|B_t(x)|_\nu}{t^{n-2k}}\right]^{\frac{1}{k}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n, \tag{2.9}$$

The above estimates hold as well if ν is a positive locally finite measure on $\mathbb{R}^n$. In that case, $|B_t(x)|_\nu = \int_{B_t(x)} d\nu$ in the definition of Wolff's potential. Note that $\mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu$, and hence a k-convex solution v to the equation $\nu = F_k[u]$, is finite a.e. if and only if

$$\int_1^\infty \left[\frac{|B_t(0)|_\nu}{t^{n-2k}}\right]^{\frac{1}{k}} \frac{dt}{t} < \infty, \tag{2.10}$$

as in the classical case $k = 1$.

Clearly, (2.10) holds if ν is compactly supported and $k < \frac{n}{2}$. Hence, inequalities (2.7) are applicable with $\nu = (-\Delta)^{\alpha/2}\phi$. In particular,

$$0 \le -v(x) \le c_2 \mathbf{W}_{\frac{2k}{k+1},\, k+1}[(-\Delta)^{\alpha/2}\phi](x).$$

We next notice that Wolff's potential $\mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu$ is pointwise bounded by the iterated Riesz potential $I_\alpha(I_\alpha \nu)^{\frac{1}{k}}$, i.e., the Havin–Maz'ya potential (see, e.g., [AH], Sec. 2.6, Proposition 2.6.9, and Sec. 4.5):

$$\mathbf{W}_{\frac{2k}{k+1},\, k+1}\nu(x) \le C_{k,n}\, I_\alpha(I_\alpha \nu)^{\frac{1}{k}}(x), \qquad x \in \mathbb{R}^n. \tag{2.11}$$

From this we obtain the following estimate of the Hessian energy $\mathcal{E}_k[v]$, where $F_k[v] = (-\Delta)^{\alpha/2}\phi$:

$$\begin{aligned} \int_{\mathbb{R}^n} -v\, F_k[v] dx &\le c_2 \int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}\phi \cdot \mathbf{W}_{\frac{2k}{k+1},\, k+1}[(-\Delta)^{\alpha/2}\phi]\, dx \\ &\le C_{k,n} \int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}\phi \cdot I_\alpha(I_\alpha(-\Delta)^{\alpha/2}\phi)^{\frac{1}{k}}\, dx. \end{aligned}$$

Here we used the upper estimate (2.7) in the first inequality, and (2.11) with $\nu = (-\Delta)^{\alpha/2}\phi$ in the second one. However, since $I_\alpha = (-\Delta)^{-\alpha/2}$, it follows using selfadjointness and the semigroup property,

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}\phi \cdot I_\alpha(I_\alpha(-\Delta)^{\alpha/2}\phi)^{\frac{1}{k}}\, dx = \int_{\mathbb{R}^n} \phi\, (\phi)^{\frac{1}{k}}\, dx = ||\phi||^{1+\frac{1}{k}}_{L^{1+\frac{1}{k}}}.$$

Thus,

$$\int_{\mathbb{R}^n} -v\, F_k[v]\, dx \le C_{k,n}\, ||\phi||^{1+\frac{1}{k}}_{L^{1+\frac{1}{k}}}.$$

Combining the preceding estimate with (2.5) and (2.6) we deduce (2.3). □

3. An upper bound for the Hessian energy

Theorem 3.1. *Let $u \in C^2(\mathbb{R}^n)$ be a $k-$convex function vanishing at ∞. Let $\alpha = \frac{2k}{k+1}$, where $1 \le k < \frac{n}{2}$. Then there exists a positive constant $c_{k,n}$ such that*

$$\int_{\mathbb{R}^n} -u\, F_k[u]\, dx \le c_{k,n} \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} u\right|^{k+1} dx. \tag{3.1}$$

Proof. Let $v = F_k[u] \ge 0$ be the k-Hessian measure associated with u. By Wolff's inequality [HW] (see also [AH], Sec. 4.5),

$$||I_\alpha v||^{1+\frac{1}{k}}_{L^{1+\frac{1}{k}}(\mathbb{R}^n)} \le C \int_{\mathbb{R}^n} \mathbf{W}_{\frac{2k}{k+1},\, k+1} v \cdot v\, dx, \tag{3.2}$$

where C depends only on k and n.

From this and the lower estimate in (2.7),

$$C\, \mathbf{W}_{\frac{2k}{k+1},\, k+1} v(x) \le -u(x), \tag{3.3}$$

it follows:

$$||I_\alpha v||^{1+\frac{1}{k}}_{L^{1+\frac{1}{k}}(\mathbb{R}^n)} \le C \int_{\mathbb{R}^n} -u\, F_k[u]\, dx. \tag{3.4}$$

By duality of fractional Sobolev spaces, it follows that, for $v \ge 0$ (here we can assume that v is a positive measure),

$$\left|\int_{\mathbb{R}^n} u \cdot v\, dx\right| = \left|\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2} u \cdot\, (-\Delta)^{-\alpha/2} v\, dx\right| \le \left\|(-\Delta)^{\alpha/2} u\right\|_{L^{k+1}(\mathbb{R}^n)} \cdot ||I_\alpha v||_{L^{1+\frac{1}{k}}(\mathbb{R}^n)}, \tag{3.5}$$

where $I_\alpha v = (-\Delta)^{-\alpha/2} v$ is the Riesz potential of order $\alpha = \frac{2k}{k+1}$ defined by (1.11).

From (3.4) and (3.5) (with $v = F_k[u]$) we deduce:

$$\int_{\mathbb{R}^n} -u\, F_k[u]\, dx \le C \left\|(-\Delta)^{\frac{\alpha}{2}} u\right\|_{L^{k+1}(\mathbb{R}^n)} \cdot \left(\int_{\mathbb{R}^n} -u\, F_k[u]\, dx\right)^{\frac{k}{k+1}}, \tag{3.6}$$

where C depends only on k and n. This proves (3.1). □

As a consequence of Theorems 2.1 and 3.1, we obtain the following corollary.

Corollary 3.2. *Let $u \in C^2(\mathbb{R}^n)$ be a $k-$convex function vanishing at ∞, where $1 \le k < \frac{n}{2}$. Then there exists $\tilde{u}$ such that $c_1 \le u/\tilde{u} \le c_2$, and*

$$C_1 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}\right|^{k+1} dx \le \int_{\mathbb{R}^n} -u\, F_k[u]\, dx \le C_2 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\alpha}{2}} \tilde{u}\right|^{k+1} dx, \tag{3.7}$$

where the constants of equivalence c_i, C_i $(i = 1, 2)$ depend only on k and n.

To prove Corollary 3.2, one can set $\tilde{u} = -I_\alpha (I_\alpha v)^{\frac{1}{k}}$, where $v = F_k[u]$ is the k-Hessian measure associated with u, and apply Theorems 2.1 and 3.1.

4. Fractional Laplacian inequalities

It is well known and easy to prove that, if $p \ge 1$ and $f \in C^2$ is a positive function, then

$$\Delta f^p(x) \ge p f^{p-1}(x) \cdot \Delta f(x).$$

The following lemma provides an extension of this useful inequality to the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, with a simple proof which relies on the approach of [CaS].

Let $a = 1 - \alpha$. Following [CaS], consider an extension of f to the upper half-space $\mathbb{R}^{n+1}_+$ so that $u(x, 0) = f(x)$, and

$$\Delta_x u + \frac{a}{y} u_y + u_{yy} = 0 \quad \text{in } \mathbb{R}^{\text{n}+1}_+. \tag{4.1}$$

Let $P_a(x,y)$ denote the corresponding Poisson kernel

$$P_a(x,y) = c\frac{y^{1-a}}{(|x|^2+y^2)^{(n+1-a)/2}}, \quad x \in \mathbb{R}^n,\ y>0, \tag{4.2}$$

where c is a positive constant depending on a and n, and making $P_a(x,y)dx$ a probability measure for all $y>0$. Then $u = P_a \star f \geq 0$ if $f \geq 0$. In particular, if $a=0$ then $P_0(x,y)$ is the classical Poisson kernel of $\mathbb{R}^{n+1}_+$, and $u = P_0 \star f$ is the harmonic extension of f.

Similarly, by v we denote an extension of f^p to $\mathbb{R}^{n+1}_+$, $v = P_a \star (f^p)$. Both u and v are well defined by our assumption on f. Note that, for a normalization constant $c>0$ depending on a and n we get, see [CaS]:

$$-c(-\Delta)^{\alpha/2} f(x) = \lim_{y\to 0} \frac{u(x,y)-u(x,0)}{y^{1-a}}, \tag{4.3}$$

$$-c(-\Delta)^{\alpha/2} f^p(x) = \lim_{y\to 0} \frac{v(x,y)-v(x,0)}{y^{1-a}}. \tag{4.4}$$

Lemma 4.1. *Let $0<\alpha<2$, $a = 1-\alpha$, and $1 \leq p < \infty$. Suppose $f \in C^2(\mathbb{R}^n) \cap L^p(w)$ where $w(x) = (1+|x|)^{-(n+\alpha)}$. Let u and v be respectively the Caffarelli-Silvestre extensions of f and f^p to the upper half-space $\mathbb{R}^{n+1}_+$. If $f \geq 0$, or p is an even integer, then*

$$\frac{1}{1-a}\lim_{y\to 0} y^a (v_y(x,y) - (u^p)_y(x,y)) = p\, f^{p-1}\cdot (-\Delta)^{\alpha/2} f - (-\Delta)^{\alpha/2}(f^p) \geq 0 \quad a.e. \tag{4.5}$$

Consequently, if $f \geq 0$, then

$$(-\Delta)^{\alpha/2}(f^p) \leq p\, f^{p-1}\cdot(-\Delta)^{\alpha/2} f \quad \text{a.e.} \tag{4.6}$$

Remark 4.2. Notice that inequality (4.6) for the fractional Laplacian is in line with the stability property of a nonnegative subharmonic function f with respect to raising to a power $p \geq 1$. More precisely, it is well known that if f is nonnegative and subharmonic, then f^p, $p \geq 1$, is still subharmonic. An analogue of this property for the fractional Laplacian is an immediate consequence of (4.6): if $f \geq 0$ and $-(-\Delta)^{\alpha/2} f \geq 0$ then $-(-\Delta)^{\alpha/2}(f^p) \geq 0$.

Remark 4.3. Lemma 4.1 remains valid whenever we consider a convex function ϕ composed with u in place of u^p. In particular, the following generalization of inequality (4.6) holds:

$$(-\Delta)^{\alpha/2}\,(\phi(f)) \leq\ \phi'(f)\cdot(-\Delta)^{\alpha/2} f \quad \text{a.e.}, \tag{4.7}$$

where $f \in C^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \frac{\phi(f)\,dx}{(1+|x|)^{n+\alpha}} < \infty$.

Proof. Consider an extension of f to the upper half-space $\mathbb{R}^{n+1}_+$ so that $u(x,0) = f(x)$, and

$$\Delta_x u + \frac{a}{y}\, u_y + u_{yy} = 0 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{4.8}$$

Let $P_a(x,y)$ denote the corresponding Poisson kernel. Then $u = P_a \star f$ and if $f \geq 0$, then $P_a \star f \geq 0$. Similarly, by v we denote an extension of f^p to $\mathbb{R}^{n+1}_+$, $v = P_a \star (f^p)$. Both u and v are well defined by our assumption on f. Note that, for a normalization constant $c>0$ depending on a and n:

$$-c(-\Delta)^{\alpha/2} f(x) = \lim_{y\to 0} \frac{u(x,y)-u(x,0)}{y^{1-a}}, \tag{4.9}$$

$$-c(-\Delta)^{\alpha/2} f^p(x) = \lim_{y\to 0} \frac{v(x,y)-v(x,0)}{y^{1-a}}. \tag{4.10}$$

By Jensen's inequality, we get that for every $p \geq 1$,

$$0 \leq u^p = (P_a \star f)^p \leq P_a \star f^p = v,$$

whenever p is an even integer, or u is positive.

Hence,

(4.11)

$$
\begin{aligned}
-c(-\Delta)^{\alpha/2}(f^p)(x) &= \lim_{y\to 0}\frac{v(x,y)-v(x,0)}{y^{1-a}} = \lim_{y\to 0}\frac{u^p(x,y)-v(x,0)}{y^{1-a}} + \lim_{y\to 0}\frac{v(x,y)-u^p(x,y)}{y^{1-a}}\\
&= \lim_{y\to 0}\frac{u^p(x,y)-f^p(x)}{y^{1-a}} + \lim_{y\to 0}\frac{v(x,y)-u^p(x,y)}{y^{1-a}}\\
&= pf^{p-1}(x)\lim_{y\to 0}\frac{u(x,y)-f(x)}{y^{1-a}} + \lim_{y\to 0}\frac{v(x,y)-u^p(x,y)}{y^{1-a}}\\
&\geq -pc((-\Delta)^{\alpha/2}f(x))\cdot f(x)^{p-1},
\end{aligned}
$$

since $\lim_{y\to 0} u(x,y) = f(x)$ a.e. and $v \geq u^p$. This proves inequality (4.6).

Notice moreover that

(4.12)

$$
\begin{aligned}
-c(-\Delta)^{\alpha/2}(f^p)(x) &= -pc((-\Delta)^{\alpha/2}f(x))\cdot f(x)^{p-1} + \lim_{y\to 0}\frac{v(x,y)-u^p(x,y)}{y^{1-a}}\\
&= -pc((-\Delta)^{\alpha/2}f(x))\cdot f(x)^{p-1} + \frac{1}{1-a}\lim_{y\to 0} y^a(v_y(x,y)-(u^p)_y(x,y)).
\end{aligned}
$$

The latter limit exists since

$$
-\frac{1}{1-a}\lim_{y\to 0} y^a\, v_y(x,y) = \lim_{y\to 0}\frac{v(x,0)-v(x,y)}{y^{1-a}} = c(-\Delta)^{\alpha/2}(f^p)(x), \tag{4.13}
$$

which is a consequence of the corresponding fact established in [CaS] (see the displayed formula at the bottom of page 1246). In this regard it is worth mentioning that there is a typo in the second equality where $1-a$ should be replaced by $-(1-a)^{-1}$. $\square$

Remark 4.4. Notice that, in this particular case, a comparison principle holds, although a weak maximum principle in general does not. For example we recall the case of the Laplace operator in $\mathbb{R}^{n+1}_+$. Both $q_1 = y$ and $q_2 = 2y$ are positive and harmonic, $q_2 - q_1$ is harmonic and moreover $q_1 = q_2$ on $y = 0$, nevertheless $q_2 > q_1$. Moreover, we remark that u^p is a subsolution of the equation $\Delta_x w^p + a\frac{(w^p)_y}{y} + (w^p)_{yy} = 0$. Indeed

(4.14)

$$
\begin{aligned}
&\Delta_x u^p + a\frac{(u^p)_y}{y} + (u^p)_{yy}\\
&= p(p-1)u^{p-2}\mid \nabla_x u\mid^2 + pu^{p-1}\Delta_x u + apu^{p-1}\frac{u_y}{y} + p(p-1)u^{p-2}(u_y)^2 + pu^{p-1}u_{yy}\\
&= pu^{p-1}(\Delta_x u + \frac{a}{y}\,u_y + u_{yy}) + p(p-1)(u^{p-2}\mid \nabla_x u\mid^2 + u^{p-2}(u_y)^2) = p(p-1)u^{p-2}\mid \nabla u\mid^2 \geq 0.
\end{aligned}
$$

Thus u^p is a subsolution of v whenever $p \geq 1$ is even. Moreover

$$
\Delta_x(u^p - v) + a\frac{(u^p-v)_y}{y} + (u^p - v)_{yy} \geq 0.
$$

Thus, applying the strong maximum principle (see [GT]), we deduce that either $u^p - v$ is constant or $u^p - v$ does not achieve any maximum in the interior of $\mathbb{R}^{n+1}_+$. If the second case occurs then either $\max_{\mathbb{R}^{n+1}_+}(u^p - v) \geq 0$, or $u^p - v$ does not attain any maximum in $\mathbb{R}^{n+1}_+$. Notice that here we cannot use a weak maximum principle because the domain is unbounded.

Remark 4.5. Applying Lemma 4.1 with $f = -(-\Delta)^{\alpha/2}u$ and $p = k$ we arrive at the following inequality for the iterated fractional Laplacian which appears in condition (1.9):

$$
(-\Delta)^{\alpha/2}\left[-(-\Delta)^{\alpha/2}u\right]^k \leq k\,\left(-(-\Delta)^{\alpha/2}u\right)^{k-1}\cdot(-(-\Delta)^{\alpha}u)\,. \tag{4.15}
$$

It would be interesting to know in relation to Theorem 2.1, under which assumptions the converse inequality may hold (with a possibly different positive constant in place of k) when $\alpha = \frac{2k}{k+1}$, and u is k-convex. This would help verify condition (ii) in Theorem 2.1.

The following corollary of Lemma 4.1 is useful in applications to weighted norm inequalities and some classes of nonlinear partial differential equations (see [KV]).

Corollary 4.6. *Let $1 \le p < \infty$ and $0 < \alpha \le 2$. Suppose μ is a positive Borel measure on $\mathbb{R}^n$. Let $I_\alpha \mu = (-\Delta)^{-\alpha/2}\mu$ be the Riesz potential of μ defined by (1.11). Then*

$$(I_\alpha \mu)^p \le p\, I_\alpha[(I_\alpha \mu)^{p-1}\, d\mu]. \tag{4.16}$$

Remark 4.7. The "integration by parts" inequality (4.16) is known to hold for all $\alpha \in (0, n)$, but with a constant $C(\alpha, p, n)$ in place of p. (See [VW], and also [KV] where such an inequality is used for general quasimetric kernels.)

Proof. Without loss of generality we may assume that μ is compactly supported, and that the right-hand side of (4.16) is finite. Let $f = I_\alpha \mu$. Then $(-\Delta)^{\alpha/2} f = \mu$ in the sense of distributions. Denote by μ_ϵ a mollification of μ so that $\mu_\epsilon = (-\Delta)^{\alpha/2} f_\epsilon$. By Lemma 4.1 with f_ϵ in place of f,

$$(-\Delta)^{\alpha/2}(I_\alpha \mu_\epsilon)^p \le p\,(I_\alpha \mu_\epsilon)^{p-1}\mu_\epsilon.$$

Applying I_α to both sides of the preceding inequality we obtain

$$(I_\alpha \mu_\epsilon)^p \le p\, I_\alpha[(I_\alpha \mu_\epsilon)^{p-1}\mu_\epsilon].$$

Passing to the limit as $\epsilon \to 0$ completes the proof. □

The following corollary concerns certain two-weight inequalities with explicit constants. They are deduced from Corollary 4.6 using the method developed in [VW], [KV], where applications are given for semilinear equations of the type

$$(-\Delta)^{\alpha/2} u = \mu\, u^q + \omega \qquad \text{on } \mathbb{R}^n, \tag{4.17}$$

where $q > 1$, and μ, ω are arbitrary positive measurable functions (or measures) on $\mathbb{R}^n$.

Corollary 4.8. *Let $1 \le p < \infty$ and $0 < \alpha \le 2$. Suppose ω and μ are positive Borel measure on $\mathbb{R}^n$. Let $d\nu = (I_\alpha \mu)^q d\mu$, where $\frac{1}{p} + \frac{1}{q} = 1$. Suppose there exists a positive constant C such that*

$$I_\alpha[(I_\alpha \omega)^q d\mu](x) \le C\, I_\alpha \omega(x) < +\infty \quad d\mu \text{ a.e.} \tag{4.18}$$

Then the following two inequalities hold:

$$\|I_\alpha(f d\mu)\|_{L^p(d\omega)} \le p\, C^{1-\frac{1}{p}}\, \|f\|_{L^p(d\mu)}, \tag{4.19}$$

$$\|I_\alpha(f d\mu)\|_{L^p(d\nu)} \le p\, C\, \|f\|_{L^p(d\mu)}, \tag{4.20}$$

for all $f \in L^p(d\mu)$.

Remark 4.9. Condition (4.18) is necessary and sufficient (with another constant) for the existence of nonnegative solutions to equation (4.17) which vanish at infinity (see [BC], [KV]).

Proof. Suppose (4.18) holds. Without loss of generality we may assume that $f \ge 0$ is bounded and has compact support. Then applying Corollary 4.6 with $f d\mu$ in place of μ, together with Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} [I_\alpha(f d\mu)]^p\, d\omega &\le p \int_{\mathbb{R}^n} I_\alpha[(I_\alpha(f d\mu))^{p-1} f\, d\mu]\, d\omega = p \int_{\mathbb{R}^n} (I_\alpha(f d\mu))^{p-1} (I_\alpha \omega)\, f\, d\mu \\ &\le p\, \|f\|_{L^p(d\mu)}\, \|I_\alpha(f d\mu)\|^{p-1}_{L^p(d\nu)}.\end{aligned} \tag{4.21}$$

Using the preceding inequality with $d\nu$ in place of $d\omega$, we deduce

$$\int_{\mathbb{R}^n} [I_\alpha(f d\mu)]^p \, d\nu \le p \int_{\mathbb{R}^n} I_\alpha[(I_\alpha f)^{p-1} f \, d\mu] \, d\nu = p \int_{\mathbb{R}^n} (I_\alpha f)^{p-1} (I_\alpha \nu) \, f \, d\mu \tag{4.22}$$
$$\le p \, \|f\|_{L^p(d\mu)} \, \|I_\alpha(f d\mu)\|^{p-1}_{L^p(d\nu_1)},$$

where $d\nu_1 = (I_\alpha \nu)^q d\mu$. Notice that by (4.18), it follows that $d\nu_1 \le C^q \, d\nu$, and hence

$$\|I_\alpha(f d\mu)\|_{L^p(d\nu_1)} \le C^{\frac{1}{p-1}} \, \|I_\alpha(f d\mu)\|_{L^p(d\nu)}.$$

Thus,

$$\int_{\mathbb{R}^n} [I_\alpha(f d\mu)]^p \, d\nu \le p \, C \, \|f\|_{L^p(d\mu)} \, \|I_\alpha(f d\mu)\|^{p-1}_{L^p(d\nu)}. \tag{4.23}$$

From this it follows

$$\|I_\alpha(f d\mu)\|_{L^p(d\nu)} \le p \, C \, \|f\|_{L^p(d\mu)}, \tag{4.24}$$

which proves (4.19). Estimating the right-hand side of (4.21) by employing the preceding estimate, we deduce (4.20). □

5. A Liouville theorem for α-subharmonic functions

In this section we prove a Liouville type theorem for the fractional Laplacian as an application of the results of the previous section. Our idea originates in the following proof of the Liouville theorem for subharmonic functions.

Proposition 5.1. *Let $f \in C^2(\mathbb{R}^n)$ be a nonnegative subharmonic function such that for some $p \in \mathbb{N}$, $p > 1$,*

$$\lim_{R \to \infty} \int_{\partial B_R(0)} f^{p-1} \langle \nabla f, n \rangle \, d\mathcal{H}^{n-1} = 0.$$

Then f has to be constant.

Proof. **Step 1.** For every $p \ge 1$, and for every $x \in \mathbb{R}^n$,

$$\Delta f^p = p(p-1) f^{p-2} \mid \nabla f \mid^2 + p f^{p-1} \Delta f \ge 0.$$

Moreover we get also that

$$\begin{aligned} \int_{B_R(0)} \Delta f^p dx &= \int_{B_R(0)} \left(p(p-1) f^{p-2} \mid \nabla f \mid^2 + p f^{p-1} \Delta f \right) dx \\ &= p(p-1) \int_{B_R(0)} f^{p-2} \mid \nabla f \mid^2 - p(p-1) \int_{B_R(0)} f^{p-2} \mid \nabla f \mid^2 \, dx \\ &+ p \int_{\partial B_R(0)} f^{p-1} \langle \nabla f, n \rangle d\mathcal{H}^{n-1}. \end{aligned} \tag{5.1}$$

Hence

$$\int_{B_R(0)} \Delta f^p dx = p \int_{\partial B_R(0)} f^{p-1} \langle \nabla f, n \rangle d\mathcal{H}^{n-1},$$

and

$$\int_{\mathbb{R}^n} \Delta f^p dx = \lim_{R \to \infty} p \int_{\partial B_R(0)} f^{p-1} \langle \nabla f, n \rangle d\mathcal{H}^{n-1} = 0.$$

It is worth noting that the first step holds without any assumption on the sign of f whenever $D^\gamma f \in L^q(\mathbb{R}^n)$, $\mid \gamma \mid = 2$, $q \ge 1$ and $f \in L^{\frac{q(p-1)}{q-1}}(\mathbb{R}^n)$.

Step 2. We know, by hypotheses, that $p > 1$, $f \ge 0$ and $\Delta f \ge 0$. Thus

$$\int_{B_R(0)} \Delta f^p dx = \int_{B_R(0)} \left(p(p-1) f^{p-2} \mid \nabla f \mid^2 + p f^{p-1} \Delta f \right) dx \ge 0 \tag{5.2}$$

As a consequence, recalling Step 1, we get:

$$0=\int_{\mathbb{R}^n}\Delta f^p dx \geq \int_{\mathbb{R}^n}(p(p-1)f^{p-2}\mid \nabla f\mid^2 + pf^{p-1}\Delta f)dx \geq 0,$$

i.e.

$$\int_{\mathbb{R}^n} f^{p-1}\Delta f dx = 0,$$

and

$$\int_{\mathbb{R}^n} f^{p-2}\mid \nabla f\mid^2 dx = 0.$$

In particular we deduce $f^{p-1}\Delta f = 0$ in $\mathbb{R}^n$, and $f^{p-2}\mid \nabla f\mid^2 = 0$ in all of $\mathbb{R}^n$. Hence either $f = 0$ or $\nabla f = 0$, and this implies that $\nabla f = 0$ in all of $\mathbb{R}^n$. In particular f has to be constant. $\square$

Remark 5.2. Roughly speaking, Proposition 5.1 says, for example, that if a subharmonic function $f \geq 0$ vanishes at infinity, and the gradient is not too large, then f has to be zero. On the other hand, if the gradient of f vanishes at infinity and f is not too large then f has to be constant.

We are now in a position to treat the case of the fractional Laplace operator.

Theorem 5.3. *Let $0 < \alpha < 2$ and $q > 1$. Let $f \in C^2(\mathbb{R}^n) \cap L^{\frac{q(p-1)}{q-1}}((1+\mid x\mid)^{-(n+\alpha)})(\mathbb{R}^n)$, and $D^\gamma f \in C^2(\mathbb{R}^n) \cap L^q((1+\mid x\mid)^{-(n-2+\alpha)})(\mathbb{R}^n)$, $\mid \gamma\mid = 2$, for some $p \in \mathbb{N}$, $p \geq 1$, and $\frac{q(p-1)}{q-1} \geq 1$. Suppose that $f \geq 0$ and $-(-\Delta)^{\alpha/2} f \geq 0$ in $\mathbb{R}^n$. If*

$$\lim_{R\to\infty}\int_{\partial B_R(0)} f^{p-1}\langle \nabla f, n\rangle\, d\mathcal{H}^{n-1} = 0,$$

then f is constant.

Proof. It is enough to prove that our hypotheses force $\Delta f^p = 0$. Then by the usual Liouville theorem it follows that f has to be constant. Indeed, we will prove that $-(-\Delta)^{\alpha/2} f^p = 0$ in all of $\mathbb{R}^n$, which yields $\Delta f^p = 0$ for $\alpha \leq 2$.

Let us recall that

$$-(-\Delta)^{\alpha/2} f(x) = c_{n,\alpha}\int_{\mathbb{R}^n}\frac{\Delta f(y)}{\mid x-y\mid^{n-2+\alpha}}dy.$$

For every $s, R > 0$ we get from Fubini's theorem that

(5.3)

$$c_{n,\alpha}\int_{B_s(0)}\left(\int_{B_R(x)\setminus B_\epsilon(x)}\frac{\Delta f^p}{\mid x-y\mid^{n-2+\alpha}}dy\right)dx = c_{n,\alpha}\int_{B_s(0)}\left(\int_{B_R(0)\setminus B_\epsilon(0)}\frac{\Delta f^p(x-z)}{\mid z\mid^{n-2+\alpha}}dz\right)dx$$

$$= c_{n,\alpha}\int_{B_R(0)\setminus B_\epsilon(0)}\frac{1}{\mid z\mid^{n-2+\alpha}}\left(\int_{B_s(0)}\Delta f^p(x-z)dx\right)dz.$$

Hence

$$c_{n,\alpha}\int_{B_R(0)\setminus B_\epsilon(0)}\frac{1}{\mid z\mid^{n-2+\alpha}}\left(\int_{\mathbb{R}^n}\Delta f^p(x-z)dx\right)dz = 0.$$

Thus recalling Step 1 of the previous Proposition 5.1 we conclude, taking in account our hypotheses, that

$$c_{n,\alpha}\int_{\mathbb{R}^n}\int_{B_R(x)\setminus B_\epsilon(x)}\frac{\Delta f^p}{\mid x-y\mid^{n-2+\alpha}}dydx = 0.$$

Letting $\epsilon \to 0$, and $R \to \infty$, we deduce using the dominated convergence theorem that

$$\int_{\mathbb{R}^n}(-\Delta)^{\alpha/2}(f^p)(x)\, dx = 0.$$

Assume now that $f \geq 0$,

$$-\int_{\mathbb{R}^n} (-\Delta)^{\alpha/2}(f^p)(x)\, dx = 0,$$

and

$$-(-\Delta)^{\alpha/2} f(x) \geq 0.$$

Then (4.11) and (4.12), together with our assumptions on f, force

$$-\int_{\mathbb{R}^n} p\,(-\Delta)^{\alpha/2} f(x)\, f(x)^{p-1}\, dx = 0, \tag{5.4}$$

and

$$\int_{\mathbb{R}^n} \lim_{y\to 0} y^a (v_y(x,y) - (u^p)_y(x,y))\, dx = 0. \tag{5.5}$$

Thus, from (5.4) it follows

$$-p\, f^{p-1}(x)\, (-\Delta)^{\alpha/2} f(x) = 0.$$

Then either $(-\Delta)^{\alpha/2} f(x) = 0$ in $\{f > 0\}$ or $f = 0$ whenever $(-\Delta)^{\alpha/2} f(x) > 0$. Moreover we get from (5.5) that

$$\lim_{y\to 0} y^a (v_y(x,y) - (u^p)_y(x,y)) = 0.$$

As a consequence

$$-(-\Delta)^{\alpha/2} f^p(x) = \lim_{y\to 0} y^a (u^p)_y(x,y) = -p f^{p-1}(-\Delta)^{\alpha/2} f(x) = 0,$$

and this yields the result because $-(-\Delta)^{\alpha/2} f^p(x) = 0$. □

Remark 5.4. It is worth remarking that such a result reinforces, in a sense, the validity of the assumption (ii) in Theorem 2.1. Indeed, if we were dealing with functions such that

$$(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2} u]^k \leq 0,$$

(i.e. condition (ii) is never verified, namely $-(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2} u]^k \geq 0$), then, for example considering $k = 1$, recalling that $-(-\Delta)^{\alpha/2} u \geq 0$, and making the finite energy assumption, from our Liouville type theorem we deduce that $-(-\Delta)^{\alpha/2} u = 0$ so that u itself should be zero.

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