On the area of the graph of a singular map from the plane to the plane taking three values

Giovanni Bellettini* Maurizio Paolini†

Abstract

We improve an estimate given by Acerbi and Dal Maso in 1994, concerning the area of the graph of a singular map from the disk of $\mathbb{R}^2$ into $\mathbb{R}^2$, taking only three values, and jumping on three half-lines meeting at the origin in a triple junction.

1 Introduction and statement of the result

Given a bounded open set $\Omega \subset \mathbb{R}^2$ let us define the area functional $\mathbb{A} : L^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$ as

$$\mathbb{A}(v,\Omega) := \begin{cases} \displaystyle\int_\Omega \sqrt{1+|\nabla v_1|^2+|\nabla v_2|^2+\left(\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y}-\frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}\right)^2}\, dxdy & \text{if } v=(v_1,v_2)\in \mathcal{C}^1(\Omega;\mathbb{R}^2), \\ +\infty & \text{if } v \in L^1(\Omega;\mathbb{R}^2)\setminus \mathcal{C}^1(\Omega;\mathbb{R}^2), \end{cases}$$

where $\nabla v_i = (\frac{\partial v_i}{\partial x}, \frac{\partial v_i}{\partial y})$ and $|\nabla v_i|^2 = (\frac{\partial v_i}{\partial x})^2 + (\frac{\partial v_i}{\partial y})^2$, for $i = 1,2$. For a function $v \in \mathcal{C}^1(\Omega;\mathbb{R}^2)$ the value $\mathbb{A}(v,\Omega)$ is the area of the graph of v on Ω, which is a two-codimensional surface embedded in $\mathbb{R}^4$. As it happens in codimension one, also in codimension two it may be of interest to extend the functional $\mathbb{A}$ to nonsmooth functions. We refer to [4] for such an extension to the BV setting in codimension one, and for applications to minimal surfaces. As discussed for instance by Acerbi and Dal Maso in [3], one rather natural way to extend $\mathbb{A}(\cdot,\Omega)$ to nonsmooth functions is to consider the $L^1(\Omega;\mathbb{R}^2)$-lower semicontinuous envelope of $\mathbb{A}(\cdot,\Omega)$, defined as

$$\mathcal{A}(v,\Omega) := \inf\left\{\liminf_{\varepsilon\to 0} \mathbb{A}(v^\varepsilon,\Omega) : \{v^\varepsilon\}\subset \mathcal{C}^1(\Omega;\mathbb{R}^2), v^\varepsilon \to v \text{ in } L^1(\Omega;\mathbb{R}^2)\right\} \tag{1.1}$$

(for simplicity from now on we shorthand $\varepsilon = 1/h$ for $h \in \mathbb{N}$). Once one accepts (1.1) as the definition of area of the graph for a nonsmooth function, it is natural to try to describe the domain $\{v \in L^1(\Omega;\mathbb{R}^2) : \mathcal{A}(v,\Omega) < +\infty\}$ of $\mathcal{A}(\cdot,\Omega)$, and to compute the value $\mathcal{A}(v,\Omega)$ for functions v in this domain. The study of this problem was initiated in [2] in the more general case when $\Omega \subset \mathbb{R}^n$, the functions v take values in $\mathbb{R}^k$, and $\mathbb{A}(v,\Omega) := \int_\Omega f(\nabla v)\, d\mathrm{x}$ is the n-dimensional area in $\mathbb{R}^{n+k}$ of the graph of $v \in \mathcal{C}^1(\Omega;\mathbb{R}^k)$, $n,k \geq 1$. Here $f(\nabla v)$ is the euclidean norm of the vector $\mathcal{M}(\nabla v)$ whose components are the determinants of all minors of the Jacobian matrix ∇v, including the 0×0 minor whose determinant is conventionally equal to 1. Then the following properties hold [2]:

*Dipartimento di Matematica, Università di Roma Tor Vergata, via della Ricerca Scientifica 00133 Roma, Italy, and INFN Laboratori Nazionali di Frascati, Frascati, Italy. E-mail: Giovanni.Bellettini@lnf.infn.it

†Dipartimento di Matematica, Università Cattolica “Sacro Cuore”, via Trieste 17, 25121 Brescia, Italy E-mail: paolini@dmf.unicatt.it

- $\mathcal{A}(v,\Omega) = \mathbb{A}(v,\Omega) = \int_\Omega f(\nabla v)\, dx$ for any $v \in \mathcal{C}^1(\Omega;\mathbb{R}^k)\cap L^1(\Omega;\mathbb{R}^k)$, namely $\mathbb{A}(\cdot,\Omega)$ is $L^1(\Omega;\mathbb{R}^k)$-lower semicontinuous on $\mathcal{C}^1(\Omega;\mathbb{R}^k)\cap L^1(\Omega;\mathbb{R}^k)$, and moreover $\mathcal{A}(v,\Omega) = \int_\Omega f(\nabla v)\, dx$ for any $v \in W^{1,p}(\Omega;\mathbb{R}^k)$, $p \geq \min\{n,k\}$;
- $\{v \in L^1(\Omega;\mathbb{R}^k) : \mathcal{A}(v,\Omega) < +\infty\} \subset BV(\Omega;\mathbb{R}^k)$, and

$$\mathcal{A}(v,\Omega) \geq \int_\Omega f(\nabla v)\, dx + |D^s v|(\Omega), \qquad v \in BV(\Omega;\mathbb{R}^k), \tag{1.2}$$

  where $Dv = \nabla v + D^s v$ is the decomposition of the measure Dv into its absolutely continuous and singular parts with respect to the Lebesgue measure, and $|D^s v|(\Omega)$ is the total variation in Ω of the (matrix-valued) measure $D^s v$ [1];
- if $k \geq 2$ there exists $u \in BV_{\rm loc}(\mathbb{R}^n;\mathbb{R}^k)$ such that the function $\Omega \to \mathcal{A}(u,\Omega)$ is not subadditive (see (1.7) below). In particular, $\mathcal{A}(\cdot,\Omega)$ cannot be written as an integral on the whole of $BV(\Omega;\mathbb{R}^k)$.

The last property distinguishes the case $k = 2$ from the case $k = 1$ where, instead, the functional $\mathcal{A}(\cdot,\Omega)$ can be written in integral form on $BV(\Omega)$. The example of u exhibited in [2] and suggested in [3], concerns the case $n = k = 2$, and is the following. Take three open non-overlapping angular regions A, B, C of the plane $\mathbb{R}^2$ as in Figure 1 (a); the origin is a so-called triple junction, with the three radii meeting at 120 degrees. Let moreover α, β, γ be the vertices of an equilateral triangle in the target space $\mathbb{R}^2$ having center at the origin of the coordinates. Then $u : \mathbb{R}^2 \to \{\alpha,\beta,\gamma\}$ is the discontinuous $BV_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$ function defined as

$$u(x,y) := \begin{cases} \alpha & \text{if } (x,y) \in A, \\ \beta & \text{if } (x,y) \in B, \\ \gamma & \text{if } (x,y) \in C. \end{cases} \tag{1.3}$$

In [2] the following properties are proven: denoting by $|E|$ the Lebesgue measure of the measurable set $E \subseteq \mathbb{R}^2$, by $B_{\rm r}(0) = B_{\rm r} \subset \mathbb{R}^2$ the open disk of radius $\rm r > 0$ centered at the origin, and by $\overline{B}_{\rm r}$ the closure of $B_{\rm r}$,

- for any $r > 0$, $\rho > 0$ with $\rho \in (0,r)$ we have

$$\mathcal{A}(u, B_r \setminus \overline{B_\rho}) = |B_r \setminus B_\rho| + 3(r-\rho)\ell, \tag{1.4}$$

  where $\ell := |\beta - \alpha|$ is the side of the triangle having vertices α, β, γ (see Figure 1 (b)). Since $|D^s u|(B_r \setminus \overline{B_\rho}) = 3(r-\rho)\ell$, formula (1.4) shows that, if we exclude a disk around the triple point, we get equality in the lower bound (1.2);
- for any $r > 0$

$$\mathcal{A}(u, B_r) \leq |B_r| + 4r\ell; \tag{1.5}$$

- for any $r > 0$ we have

$$\mathcal{A}(u, B_r) > |B_r| + 3r\ell, \tag{1.6}$$

  and moreover there exist $\rho > 0$ and $s > 0$ with $0 < \rho < r < s$ such that

$$\mathcal{A}(u, B_r) > \mathcal{A}(u, B_\rho) + \mathcal{A}(u, B_s \setminus \overline{B_{\rho/2}}). \tag{1.7}$$

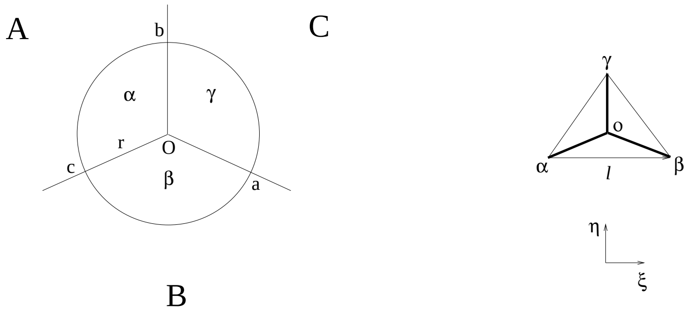

Figure 1: (a): $A = \{\lambda b + \mu c : \lambda, \mu > 0\}$, $B = \{\lambda c + \mu a : \lambda, \mu > 0\}$, $C = \{\lambda a + \mu b : \lambda, \mu > 0\}$. The triple junction inside the disk of radius r, in the source space $\mathbb{R}^2$, has the three angles at 120 degrees. (b): the three vectors α, β, γ are the vertices of an equilateral triangle of side ℓ, in the target space $\mathbb{R}^2$. The unit vectors ξ and η (see step 6 in the proof of Theorem 1.1). The bold segments (of length $\frac{\ell}{\sqrt{3}}$) form the Steiner graph, i.e., the shortest graph connecting α, β, and γ.

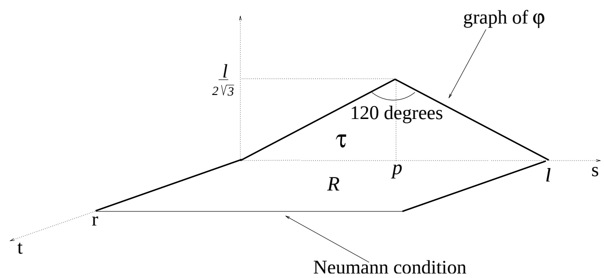

Figure 2: The rectangle $\mathrm{R} = \{(s,t) : s \in [0,\ell], t \in [0,r]\}$ is the domain of the minimizer m. The Dirichlet datum φ is assigned on $\partial_D \mathrm{R}$, which consists of the sides of R but the frontal one; φ is zero on $\{0\} \times [0,r]$ and $\{\ell\} \times [0,r]$; on the side $[0,\ell] \times \{0\}$ the graph of φ is depicted in the figure. The triangle $\mathcal{T}$ corresponds to the triangle with vertices $\alpha, 0, \beta$ in Figure 1 (b).

Inequality (1.5) is an estimate of the area of the graph of u restricted to the disk B_r, which implies that

$$\lim_{r \to 0^+} \mathcal{A}(u, B_r) = 0, \tag{1.8}$$

while (1.7) implies the asserted nonsubadditivity of $\mathcal{A}(u, \cdot)$.

The aim of this paper is to prove a more refined estimate from above of $\mathcal{A}(u, B_r)$ with respect to (1.5), see inequality (1.11) below. Such an estimate requires a better understanding of what we could call the singular part of $\mathcal{A}$, defined in general on an open set $\Omega \subseteq \mathbb{R}^2$ and for a function $v \in BV(\Omega; \mathbb{R}^2)$ as $\mathcal{A}(v, \Omega) - \int_\Omega f(\nabla v)\, d\mathrm{x}$. Our estimate is based on a suitable area-minimizing function m defined on the rectangle

$$\mathrm{R} := [0, \ell] \times [0, r],$$

where the minimization is taken among all functions having a Dirichlet condition on three of the four sides of R (see Figure 2, and Figure 3 for a schematic picture of the graph of m).

More precisely, the result is the following. Set

$$\partial_N \mathrm{R} := [0, \ell] \times \{r\}, \qquad \partial_D \mathrm{R} := \partial \mathrm{R} \setminus \partial_N \mathrm{R}.$$

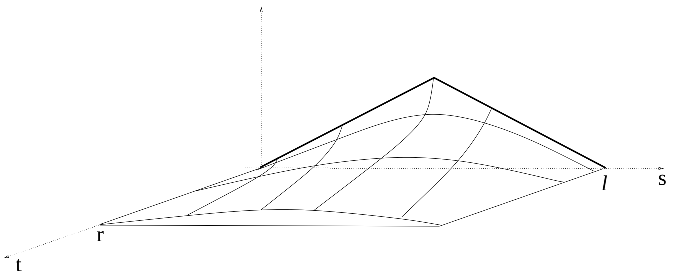

Figure 3: The graph of m, which is a minimal surface satisfying the Neumann condition on the frontal side $\partial_N \mathrm{R} = [0,\ell] \times \{r\}$ of R and Dirichlet conditions on the remaining three sides.

Let us define the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ as

$$\varphi(s,t) := \frac{1}{2\sqrt{3}} \left(\ell - |2s - \ell|\right), \qquad (s,t) \in \mathbb{R}^2. \tag{1.9}$$

The function φ is Lipschitz, piecewise affine, it is independent of t, it is nonnegative on R and vanishes on the two parallel sides $\{0\} \times [0,r]$, $\{\ell\} \times [0,r]$ of $\partial_D \mathrm{R}$, and its graph on the last side $[0,\ell] \times \{r\}$ of $\partial_D \mathrm{R}$ consists of the two bold segments of the triangle $\mathcal{T}$ having vertices $\alpha, 0, \beta$ in Figure 1. The graph of φ on R is depicted in Figure 4.
Let m be the solution of the Dirichlet-Neumann minimum problem

$$\min \left\{ \int_{\mathrm{R}} \sqrt{1 + |\nabla f|^2}\, dsdt : \ f \in W^{1,1}(\mathrm{R}), \ f = \varphi \text{ on } \partial_D \mathrm{R} \right\}. \tag{1.10}$$

It is well known that m is analytic in the interior of R. However, m is *not* Lipschitz, because its gradient blows up around the point $p = (\ell/2, 0)$; this fact is source of some technical difficulties in the proof of Theorem 1.1.
Define

$$\mathfrak{A}_{\min} := \int_{\mathrm{R}} \sqrt{1 + |\nabla m|^2}\, dsdt.$$

Note that $\mathfrak{A}_{\min}$ depends nonlinearly on r, and

$$|\mathrm{R}| = r\ell < \mathfrak{A}_{\min} < |\mathrm{R}| + |\mathcal{T}| = r\ell + \frac{\ell^2}{4\sqrt{3}}.$$

Our result is the following.

Theorem 1.1. *Let $u \in BV(B_r; \{\alpha, \beta, \gamma\})$ be the function defined in* (1.3). *Then*

$$\mathcal{A}(u, B_r) \le |B_r| + 3\mathfrak{A}_{\min}. \tag{1.11}$$

Remark 1.2. Observe that $\mathfrak{A}_{\min} < \frac{2r\ell}{\sqrt{3}}$, since $\frac{2r\ell}{\sqrt{3}}$ is the area of the "roof surface" composed of two rectangles having sides r and $\frac{\ell}{\sqrt{3}}$ in Figure 4, hence

$$3\mathfrak{A}_{\min} < 2\sqrt{3} r\ell < 4r\ell. \tag{1.12}$$

Inequalities (1.11) and (1.12) improve the estimate (1.5) given in [2]. Note also that $3\mathfrak{A}_{\min} > 3r\ell$, consistently with (1.6).

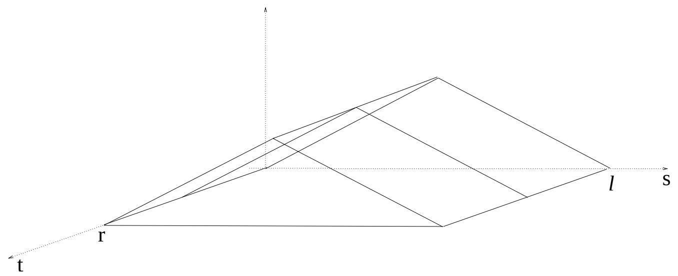

Figure 4: This surface represents the graph of φ on R, and its area is $\frac{2r\ell}{\sqrt{3}}$, and it is larger than $\mathfrak{A}_{\min}$.

Remark 1.3. We believe (1.11) to be an *equality*, but we miss the proof of this assertion.

The proof of (1.11) consists in exhibiting a sequence $\{v^\varepsilon\} \subset \mathcal{C}^1(B_r;\mathbb{R}^2)$ converging to u in $L^1(\Omega;\mathbb{R}^2)$ and such that

$$\lim_{\varepsilon\to 0} \mathcal{A}(v^\varepsilon, B_r) = |B_r| + 3\mathfrak{A}_{\min}.$$

Note that, if equality holds in (1.11), then the sequence $\{v^\varepsilon\}$ is optimal. Note also that if the equality holds in (1.11), the nonlocality nature of $\mathcal{A}(u,\cdot)$ proved in [2], becomes transparent.

We conclude this introduction with the following observation. As shown by inequality (1.8), there is no concentration of two-dimensional area on the triple junction; from the explicit construction given in the proof of Theorem 1.1, it turns out that what concentrates on the triple junction is the Steiner graph depicted in bold in Figure 1 (b), which is one-dimensional and does not contribute to the computation of $\mathcal{A}(u, B_r)$.

2 Proof of Theorem 1.1

Let us preliminarly show that (1.10) has a unique solution. Given an open set $E \subseteq \mathbb{R}^2$ and a function $f \in W^{1,1}(E)$, recall that the integral

$$\int_E \sqrt{1+|\nabla f|^2}\, dsdt$$

is the area of the graph of f on E, and it is $L^1(E)$-lower semicontinuous. Moreover, its $L^1(E)$-lower semicontinuous envelope is naturally defined on the whole of $BV(E)$, and can be expressed using a distributional formulation [1], [4]; for a function $f \in BV(E)$ it is denoted by

$$\int_E \sqrt{1+|Df|^2}.$$

Define the doubled rectangle $\widehat{\mathrm{R}}$ as

$$\widehat{\mathrm{R}} := [0,\ell] \times [0, 2r],$$

and observe that $\varphi : \widehat{\mathrm{R}} \to [0,+\infty)$ is symmetric with respect to the line $\{t = r\}$. From [4, Theorem 15.9] the minimum problem with Dirichlet condition

$$\min\left\{\int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla f|^2}\, dsdt : \ f \in W^{1,1}(\widehat{\mathrm{R}}),\ f = \varphi \text{ on } \partial\widehat{\mathrm{R}}\right\} \tag{2.1}$$

has a solution, that we denote by $\widehat{m}$. Moreover, $\widehat{m}$ is unique, and is analytic in the interior of $\widehat{\mathrm{R}}$ [4]. Let us denote by m the restriction of $\widehat{m}$ to R. Then m is the unique solution of the Dirichlet-Neumann minimum problem (1.10). Observe that

$$0 \leq m \leq \|\varphi\|_{L^\infty(\partial \mathrm{R})} \qquad \text{on R.} \tag{2.2}$$

In what follows we indicate by $D \subset \mathbb{R}^2$ an open disk containing the closure of $\widehat{\mathrm{R}}$.

Remark 2.1. From [4, Theorem 15.9] it follows that $\widehat{m}$ solves the following minimum problem in $BV(D)$:

$$\min \left\{ \int_{\widehat{\mathrm{R}}} \sqrt{1+|Df|^2} + \int_{\partial\widehat{\mathrm{R}}} |f-\varphi| \, d\mathcal{H}^1 : \ f \in BV(D), \ f = \varphi \text{ on } D \setminus \widehat{\mathrm{R}} \right\}, \tag{2.3}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, and the boundary integral in (2.3) involves the trace of f on $\partial\widehat{\mathrm{R}}$, which is well defined $\mathcal{H}^1$-almost everywhere.

Let us now begin the proof of Theorem 1.1.

Step 1: reduction to a sequence of Lipschitz maps. We claim that, in order to prove (1.11), it is sufficient to construct a sequence $\{u^\varepsilon\} \subset \mathrm{Lip}(B_r; \mathbb{R}^2)$ converging to u in $L^1(B_r; \mathbb{R}^2)$ such that

$$\lim_{\varepsilon \to 0} \mathcal{A}(u^\varepsilon, B_r) = |B_r| + 3\mathfrak{A}_{\min}, \tag{2.4}$$

where we recall (see the Introduction) that, on a Lipschitz map $v = (v_1, v_2) \in \mathrm{Lip}(\Omega; \mathbb{R}^2)$, the relaxed functional $\mathcal{A}(v, \Omega)$ defined in (1.1) has still the usual expression

$$\mathcal{A}(v, \Omega) = \int_\Omega \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + \left(\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x} \right)^2} \, dxdy. \tag{2.5}$$

To prove the claim it is enough to show that for any $v \in L^1(\Omega; \mathbb{R}^2)$ we have

$$\mathcal{A}(v, \Omega) = \inf \left\{ \liminf_{\varepsilon \to 0} \mathcal{A}(v^\varepsilon, \Omega) : \{v^\varepsilon\} \subset \mathrm{Lip}(\Omega; \mathbb{R}^2), v^\varepsilon \to v \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}. \tag{2.6}$$

To prove (2.6) let $\{v^\varepsilon\} \subset \mathrm{Lip}(\Omega; \mathbb{R}^2)$ be such that $\lim_{\varepsilon \to 0} \mathcal{A}(v^\varepsilon, \Omega)$ equals the right hand side of (2.6). Take a standard convolution kernel $\rho_\delta : \mathbb{R}^2 \to [0, +\infty)$ supported in the disk of radius $\delta > 0$ centered at the origin, and define $v_{\varepsilon,\delta} := v^\varepsilon * \rho_\delta \in \mathcal{C}^1(\Omega; \mathbb{R}^2)$. Then the dominated convergent theorem implies that $\lim_{\delta \to 0^+} \mathbb{A}(v_{\varepsilon,\delta}, \Omega) = \mathcal{A}(v^\varepsilon, \Omega)$. Therefore a diagonal argument implies that $\mathcal{A}(v, \Omega)$ is smaller than or equal to the right hand side of (2.6); on the other hand the converse inequality is immediate.

We now pass to define the sequence $\{u^\varepsilon\}$: to do this, we need to specify various subsets of B_r. Define S^b_ε as

$$S^b_\varepsilon := \left\{ (x, y) \in \overline{B}_r : |x| \leq \frac{\varepsilon}{2}, \ y \geq \frac{\varepsilon}{2\sqrt{3}} \right\}, \tag{2.7}$$

and let S^c_ε (resp. S^a_ε) be the counterclockwise rotation of S^b_ε of $2\pi/3$ (resp. of $4\pi/3$), see Figure 6 (a). Denote by T_ε the open equilateral triangle of side ε having the baricenter at the origin as in Figure 6 (b). Let

$$A_\varepsilon := A \setminus \left(S^b_\varepsilon \cup T_\varepsilon \cup S^c_\varepsilon \right), \quad B_\varepsilon := B \setminus \left(S^a_\varepsilon \cup T_\varepsilon \cup S^c_\varepsilon \right), \quad C_\varepsilon := C \setminus \left(S^a_\varepsilon \cup T_\varepsilon \cup S^b_\varepsilon \right).$$

Step 2: definition of u^ε on $A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon$. We set

$$u^\varepsilon := \begin{cases} \alpha & \text{in } A_\varepsilon, \\ \beta & \text{in } B_\varepsilon, \\ \gamma & \text{in } C_\varepsilon. \end{cases} \tag{2.8}$$

Note that $\mathcal{A}(u^\varepsilon, (A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon) \cap B_r) = |A_\varepsilon \cap B_r| + |B_\varepsilon \cap B_r| + |C_\varepsilon \cap B_r|$, hence

$$\lim_{\varepsilon \to 0^+} \mathcal{A}(u^\varepsilon, (A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon) \cap B_r) = |B_r|. \tag{2.9}$$

Before passing to the next step a comment is in order. The construction of the sequence $\{u^\varepsilon\}$ on the three cygar-shaped sets S^b_ε, S^c_ε, S^a_ε makes use of a rescaled version of the minimizer m (where the rescaling has also a correction which takes into account that m is defined on a rectangle): however, m is not Lipschitz, and therefore the resulting sequence would not be in $\mathrm{Lip}(B_r; \mathbb{R}^2)$, and step 1 would be inapplicable. We need therefore a further smoothing argument with a new positive parameter σ, which we now describe, and which shows that m can be approximated by a Lipschitz function that will be denoted by $m_{\sigma,\varphi}$, with the property that

$$\left| \int_{\mathrm{R}} \sqrt{1 + |\nabla m|^2}\, dsdt - \int_{\mathrm{R}} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt \right|$$

becomes as small as we want, provided σ tends to zero (see step 5 below).

We begin by smoothing the function φ in a neighbourhood of the points where it is not differentiable. Given $\sigma \in (0, \ell/4)$ let $\varphi_\sigma : \mathbb{R}^2 \to \mathbb{R}$ be a function with the following properties:

- $\varphi_\sigma \in \mathcal{C}^2(\mathbb{R}^2)$, and φ_σ is symmetric with respect to the line $\{t = r\}$;
- $\varphi_\sigma = \varphi$ on $\widehat{\mathrm{R}} \setminus (B_\sigma(p) \cup B_\sigma(\widehat{p}))$, where $B_\sigma(p)$ (resp. $B_\sigma(\widehat{p})$) is the disk of radius σ centered at $p = (\ell/2, 0)$ (resp. at $\widehat{p} := (\ell/2, 2r)$), and $\varphi_\sigma \leq \varphi$;
- $\|\varphi - \varphi_\sigma\|_{L^1(\widehat{\mathrm{R}})} = O(\sigma)$.

The graph of φ_σ over the segment $[0, \ell] \times \{0\}$ coincides with the graph of φ out of an interval centered at p with length 2σ and it is, roughly speaking, smoothed by a sort of arc of circle around the edge.

We now smoothen the rectangle $\widehat{\mathrm{R}}$. Denote by $\widehat{\mathrm{R}}_\sigma \subset \widehat{\mathrm{R}}$ the $\mathcal{C}^2$ convex set, symmetric with respect to the point $(\ell/2, r)$, the boundary of which is obtained by smoothing the four vertices of $\partial \mathrm{R}$ in a disk of radius σ centered at each vertex, and with $\partial \widehat{\mathrm{R}}_\sigma = \partial \widehat{\mathrm{R}}$ out of the four disks.

Step 3: definition of $f_{\sigma,\varphi}$. Let $\widehat{f}_\sigma$ be the solution of

$$\inf \left\{ \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1 + |\nabla f|^2}\, dsdt : \ f \in \mathrm{Lip}(\widehat{\mathrm{R}}_\sigma), \ f = \varphi_\sigma \text{ on } \partial \widehat{\mathrm{R}}_\sigma \right\}. \tag{2.10}$$

The existence of the *Lipschitz* function $\widehat{f}_\sigma$ is guaranteed by [4, Theorem 12.10], since $\widehat{\mathrm{R}}_\sigma$ is convex of class $\mathcal{C}^2$ and φ_σ is of class $\mathcal{C}^2$.

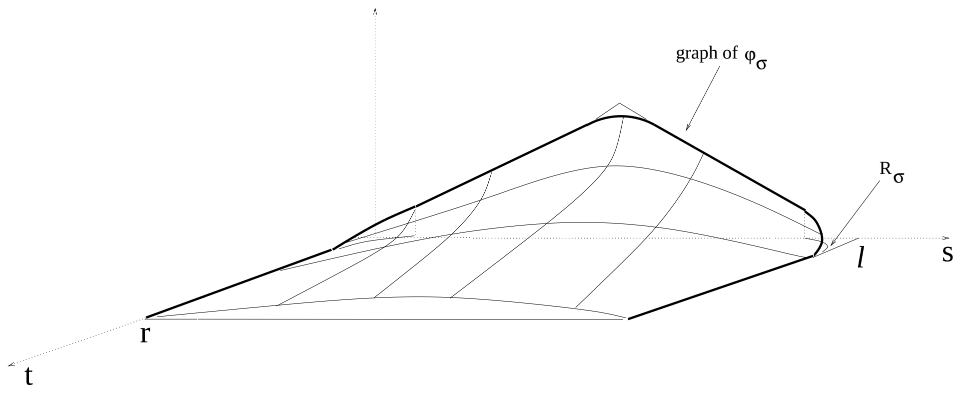

Figure 5: The smoothed sets $R_\sigma \subset R$, and (in bold) the graph of the smoothed boundary datum φ_σ on ∂R_σ but the frontal side. We also depict the graph of the Lipschitz minimizer f_σ on R_σ.

We now use the fact that $\widehat{m}$ solves (2.3), by comparing the area of the graph of $\widehat{m}$ with the area of the competitor function $\widehat{f}_{\sigma,\varphi}$ defined as follows:

$$\widehat{f}_{\sigma,\varphi} := \begin{cases} \widehat{f}_\sigma & \text{in } \widehat{\mathrm{R}}_\sigma, \\ \varphi & \text{in } D \setminus \widehat{\mathrm{R}}_\sigma. \end{cases}$$

Observe that $\widehat{f}_{\sigma,\varphi} \in BV(D)$ is discontinuous in a neighbourhood of the points p and $\widehat{p}$. Since $\widehat{m}$ solves (2.3), it follows that

$$\int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt \leq \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{f}_{\sigma,\varphi}|^2}\, dsdt + \int_{\partial\widehat{\mathrm{R}}} |\widehat{f}_{\sigma,\varphi} - \varphi|\, d\mathcal{H}^1.$$

We have

$$\int_{\partial\widehat{\mathrm{R}}} |\widehat{f}_{\sigma,\varphi} - \varphi|\, d\mathcal{H}^1 = \int_{\partial\widehat{\mathrm{R}} \cap (B_\sigma(p) \cup B_\sigma(\widehat{p}))} |\widehat{f}_{\sigma,\varphi} - \varphi|\, d\mathcal{H}^1 = O(\sigma).$$

Hence

$$\int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt \leq \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{f}_{\sigma,\varphi}|^2}\, dsdt + O(\sigma). \tag{2.11}$$

We now look for a converse inequality. From [4, Theorem 15.9] it follows that $\widehat{f}_\sigma$ solves the following minimum problem in $BV(D)$:

$$\min\left\{ \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|Df|^2} + \int_{\partial\widehat{\mathrm{R}}_\sigma} |f-\varphi_\sigma|\, d\mathcal{H}^1 : \ f \in BV(D),\ f = \varphi_\sigma \text{ on } D \setminus \widehat{\mathrm{R}}_\sigma \right\}. \tag{2.12}$$

We compare the area of the graph of $\widehat{f}_\sigma$ with the area of the competitor function defined as

$$\begin{cases} \widehat{m} & \text{in } \widehat{\mathrm{R}}_\sigma, \\ \varphi_\sigma & \text{in } D \setminus \widehat{\mathrm{R}}_\sigma. \end{cases} \tag{2.13}$$

Observe that the function in (2.13) is discontinuous in a neighbourhood of the points p and $\widehat{p}$, and along the four arcs $\partial\widehat{\mathrm{R}}_\sigma \setminus (\partial\widehat{\mathrm{R}} \cap \partial\widehat{\mathrm{R}}_\sigma)$.

Since $\widehat{f}_\sigma$ solves the minimum problem (2.12), it follows that

$$\begin{aligned}\int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{f}_\sigma|^2}\, dsdt \le & \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt + \int_{\partial\widehat{\mathrm{R}}_\sigma} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1 \\ \le & \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt + \int_{\partial\widehat{\mathrm{R}}_\sigma} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1.\end{aligned} \tag{2.14}$$

We have

$$\begin{aligned}\int_{\partial\widehat{\mathrm{R}}_\sigma} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1 =& \int_{\partial\widehat{\mathrm{R}}_\sigma \cap (B_\sigma(p)\cup B_\sigma(\widehat{p}))} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1 + \int_{\partial\widehat{\mathrm{R}}_\sigma \setminus (\partial\widehat{\mathrm{R}}\cap\partial\widehat{\mathrm{R}}_\sigma)} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1 \\ =& O(\sigma) + \int_{\partial\widehat{\mathrm{R}}_\sigma \setminus (\partial\widehat{\mathrm{R}}\cap\partial\widehat{\mathrm{R}}_\sigma)} |\widehat{m}-\varphi_\sigma|\, d\mathcal{H}^1 \le O(\sigma),\end{aligned} \tag{2.15}$$

where the last inequality is a consequence of (2.2). Hence from (2.14) it follows that

$$\int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{f}_\sigma|^2}\, dsdt \le \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt + O(\sigma). \tag{2.16}$$

Observe now that

$$\begin{aligned}\int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{f}_{\sigma,\varphi}|^2}\, dsdt =& \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{f}_\sigma|^2}\, dsdt + \int_{\widehat{\mathrm{R}}\setminus\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \varphi|^2}\, dsdt \\ \le & \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{f}_\sigma|^2}\, dsdt + O(\sigma)\sqrt{1+(\mathrm{lip}(\varphi))^2} \\ =& \int_{\widehat{\mathrm{R}}_\sigma} \sqrt{1+|\nabla \widehat{f}_\sigma|^2}\, dsdt + O(\sigma).\end{aligned} \tag{2.17}$$

Therefore, using (2.16), from (2.17) we deduce

$$\int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{f}_{\sigma,\varphi}|^2}\, dsdt \le \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt + O(\sigma). \tag{2.18}$$

From (2.11) and (2.18) it follows that

$$\left| \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{f}_{\sigma,\varphi}|^2}\, dsdt - \int_{\widehat{\mathrm{R}}} \sqrt{1+|\nabla \widehat{m}|^2}\, dsdt \right| \le O(\sigma),$$

hence by symmetry

$$\left| \int_{\mathrm{R}} \sqrt{1+|\nabla f_{\sigma,\varphi}|^2}\, dsdt - \int_{\mathrm{R}} \sqrt{1+|\nabla m|^2}\, dsdt \right| \le O(\sigma), \tag{2.19}$$

where

$$f_{\sigma,\varphi} := \begin{cases} \widehat{f}_\sigma & \text{in } \mathrm{R}_\sigma, \\ \varphi & \text{in } \mathrm{R}\setminus\mathrm{R}_\sigma. \end{cases}$$

Step 4: definition of $m_{\sigma,\varphi}$. We define the function $m_{\sigma,\varphi} : \mathrm{R} \to \mathbb{R}$ as follows:

$$m_{\sigma,\varphi}(s,t) := \begin{cases} f_{\sigma,\varphi}\left(s, \dfrac{(t-\sigma)r}{r-\sigma}\right) & \text{if } (s,t) \in \mathrm{R},\ t \geq \sigma, \\ \\ \dfrac{t}{\sigma}\varphi_\sigma(s,t) + \left(1 - \dfrac{t}{\sigma}\right)\varphi(s,t) & \text{if } (s,t) \in \mathrm{R},\ t \leq \sigma. \end{cases} \tag{2.20}$$

Observe that $m_{\sigma,\varphi} \in \mathrm{Lip}(\mathrm{R})$.

Step 5. We have

$$\left| \int_{\mathrm{R}} \sqrt{1 + |\nabla m|^2}\, dsdt - \int_{\mathrm{R}} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt \right| \leq O(\sigma). \tag{2.21}$$

We define $N_\sigma := \{(s,t) \in \mathrm{R} : t \leq \sigma\}$, and we split

$$\int_{\mathrm{R}} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt = \int_{\mathrm{R}\setminus N_\sigma} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt + \int_{N_\sigma} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt. \tag{2.22}$$

Let us first estimate the second integral on the right hand side of (2.22). Note that $\|\frac{\partial}{\partial s} m_{\sigma,\varphi}\|_{L^\infty(N_\sigma)} = O(1)$ since φ and φ_σ are Lipschitz on R. Moreover $\|\frac{\partial}{\partial t} m_{\sigma,\varphi}\|_{L^\infty(N_\sigma)} = O(1) + \frac{1}{\sigma} O(\|\varphi - \varphi_\sigma\|_{L^\infty(N_\sigma)}) = O(1)$. It follows that

$$\int_{N_\sigma} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt = O(\sigma), \tag{2.23}$$

since $|N_\sigma| = O(\sigma)$.
On the other hand it is not difficult to prove that there exist two positive constants C_1^σ, C_2^σ such that $C_1^\sigma \leq C_2^\sigma$, $C_1^\sigma = 1 + O(\sigma)$, $C_2^\sigma = 1 + O(\sigma)$, and

$$C_1^\sigma \int_{\mathrm{R}} \sqrt{1 + |\nabla f_{\sigma,\varphi}|^2}\, dsdt \leq \int_{\mathrm{R}\setminus N_\sigma} \sqrt{1 + |\nabla m_{\sigma,\varphi}|^2}\, dsdt \leq C_2^\sigma \int_{\mathrm{R}} \sqrt{1 + |\nabla f_{\sigma,\varphi}|^2}\, dsdt. \tag{2.24}$$

Then (2.21) follows from (2.23), (2.24) and (2.19).

Step 6: definition of $u^{\varepsilon,\sigma}$ on $S_\varepsilon^b \cup S_\varepsilon^c \cup S_\varepsilon^a$. We set

$$\xi = (\xi_1, \xi_2) := \frac{\beta - \alpha}{\ell} \in \mathbb{S}^1, \qquad \eta = (\eta_1, \eta_2) := \xi^\perp,$$

where $^\perp$ denotes the counterclockwise rotation of $\pi/2$.
Let $\psi_\varepsilon : \left[\frac{\varepsilon}{2\sqrt{3}}, r\right] \to [0, r]$ be the unique increasing affine function mapping $\left[\frac{\varepsilon}{2\sqrt{3}}, r\right]$ into $[0, r]$. Note that for any $y \in \left[\frac{\varepsilon}{2\sqrt{3}}, r\right]$ we have

$$\psi'(y) = \frac{r}{r - \frac{\varepsilon}{2\sqrt{3}}} =: \kappa_\varepsilon, \qquad \lim_{\varepsilon \to 0} \kappa_\varepsilon = 1. \tag{2.25}$$

Recalling the definition of $m_{\sigma,\varphi}$ in (2.20) we set

$$u^{\varepsilon,\sigma}(x,y) := \alpha + \left(\frac{1}{2} + \frac{x}{\varepsilon}\right) \ell\xi + m_{\sigma,\varphi}\left(\frac{\ell}{2} + \frac{\ell x}{\varepsilon}\ ,\ \psi_\varepsilon(y)\right) \eta, \qquad (x,y) \in S_\varepsilon^b. \tag{2.26}$$

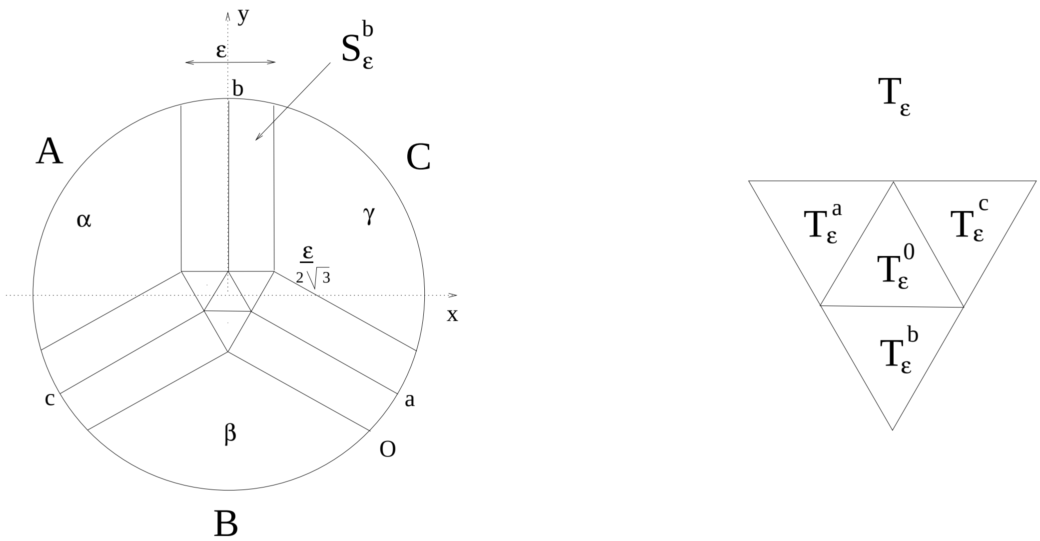

Figure 6: (a): the "cygar-shaped" set S_ε^b defined in (2.7), and its rotated S_ε^c, S_ε^a. The central triangle T_ε of side ε, further subdivided into four triangles T_ε^a, T_ε^b, T_ε^c, T_ε^0, is depicted also in (b) on a larger scale.

Observe that $u^{\varepsilon,\sigma} = (u_1^{\varepsilon,\sigma}, u_2^{\varepsilon,\sigma}) \in \mathcal{C}^\infty(S_\varepsilon^b;\mathbb{R}^2)$, $u^{\varepsilon,\sigma} = \alpha$ on $\{(x,y) \in S_\varepsilon^b : x = -\varepsilon/2\}$, and $u^{\varepsilon,\sigma} = \beta$ on $\{(x,y) \in S_\varepsilon^b : x = \varepsilon/2\}$. Write for simplicity

$$\widetilde{m} = m_{\sigma,\varphi}.$$

We have

$$\nabla u_1^{\varepsilon,\sigma} = \left(\frac{\ell\xi_1}{\varepsilon} + \frac{\widetilde{m}_s}{\varepsilon}\ \ell\eta_1\ ,\ \widetilde{m}_t\ \kappa_\varepsilon\ \eta_1\right), \qquad \nabla u_2^{\varepsilon,\sigma} = \left(\frac{\ell\xi_2}{\varepsilon} + \frac{\widetilde{m}_s}{\varepsilon}\ \ell\eta_2\ ,\ \widetilde{m}_t\ \kappa_\varepsilon\ \eta_2\right),$$

where $\widetilde{m}_s, \widetilde{m}_t$ denote the partial derivatives of $\widetilde{m}$ with respect to s and t respectively, and are evaluated at $(\frac{\ell}{2} + \frac{\ell x}{\varepsilon}, \psi_\varepsilon(y))$. Hence

$$\begin{aligned} |\nabla u_1^{\varepsilon,\sigma}|^2 + |\nabla u_2^{\varepsilon,\sigma}|^2 &= \frac{1}{\varepsilon^2}\left\{\ell^2|\xi|^2 + \left(\widetilde{m}_s\right)^2\ell^2|\eta|^2 + 2\widetilde{m}_s\ell^2\left(\xi_1\eta_1 + \xi_2\eta_2\right)\right\} + \left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2\ |\eta|^2 \\ &= \frac{1}{\varepsilon^2}\left\{\ell^2 + \left(\widetilde{m}_s\right)^2\ell^2\right\} + \left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2, \end{aligned} \tag{2.27}$$

where we have used $|\xi| = |\eta| = 1$ and $\xi_1\eta_1 + \xi_2\eta_2 = 0$.
Moreover

$$\left(\frac{\partial u_1^{\varepsilon,\sigma}}{\partial x}\frac{\partial u_2^{\varepsilon,\sigma}}{\partial y} - \frac{\partial u_1^{\varepsilon,\sigma}}{\partial y}\frac{\partial u_2^{\varepsilon,\sigma}}{\partial x}\right)^2 = \frac{1}{\varepsilon^2}\left(\ell\widetilde{m}_t\ \kappa_\varepsilon\left(\xi_1\eta_2 - \xi_2\eta_1\right)\right)^2 = \frac{1}{\varepsilon^2}\left(\ell\widetilde{m}_t\ \kappa_\varepsilon\right)^2, \tag{2.28}$$

where again $\widetilde{m}_s, \widetilde{m}_t$ are evaluated at $(\frac{\ell x}{\varepsilon} + \frac{\ell}{2}, \psi_\varepsilon(y))$, and we have used $\xi_1\eta_2 - \xi_2\eta_1 = 1$. Therefore

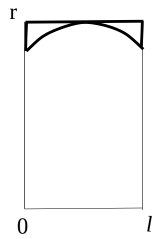

Figure 7: The set P_ε is bounded by the bold contour.

from (2.27) and (2.28) we obtain

$$\begin{aligned}
&1+|\nabla u_1^{\varepsilon,\sigma}|^2+|\nabla u_2^{\varepsilon,\sigma}|^2+\left(\frac{\partial u_1^{\varepsilon,\sigma}}{\partial x}\frac{\partial u_2^{\varepsilon,\sigma}}{\partial y}-\frac{\partial u_1^{\varepsilon,\sigma}}{\partial y}\frac{\partial u_2^{\varepsilon,\sigma}}{\partial x}\right)^2\\
=&1+\left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2+\frac{\ell^2}{\varepsilon^2}\left(1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2\right)=1+\frac{\ell^2}{\varepsilon^2}\left(1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon^2}{\ell^2}\right)\right).
\end{aligned}$$

As a consequence

$$\begin{aligned}
\mathbb{A}(u^\varepsilon,S_\varepsilon^b)&=\frac{\ell}{\varepsilon}\int_{S_\varepsilon^b}\sqrt{1+\left[\widetilde{m}_s\left(\frac{\ell}{2}+\frac{\ell x}{\varepsilon},\psi_\varepsilon(y)\right)\right]^2+\left[\widetilde{m}_t\left(\frac{\ell}{2}+\frac{\ell x}{\varepsilon},\psi_\varepsilon(y)\right)\right]^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon^2}{\ell^2}\right)+O(\varepsilon^2)}\,dxdy\\
&=\frac{1}{\kappa_\varepsilon}\int_{\mathrm{R}\setminus P_\varepsilon}\sqrt{1+\left[\widetilde{m}_s\left(s,t\right)\right]^2+\left[\widetilde{m}_t\left(s,t\right)\right]^2\kappa_\varepsilon^2\left(1+\frac{\varepsilon^2}{\ell^2}\right)+O(\varepsilon^2)}\,dsdt,
\end{aligned} \tag{2.29}$$

where the last equality follows by making the change of variables

$$\Phi:(s,t)\in\mathrm{R}\to\Phi(s,t):=\left(\frac{\varepsilon}{\ell}\left(s-\frac{\ell}{2}\right),\psi_\varepsilon^{-1}(t)\right)=(x,y)\in\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\times\left[\frac{\varepsilon}{2\sqrt{3}},r\right]\supset S_\varepsilon^b,$$

and $P_\varepsilon:=\mathrm{R}\setminus\Phi^{-1}(S_\varepsilon^b)$ (see Figure 7). Hence, recalling also the second equality in (2.25),

$$\lim_{\varepsilon\to0}\mathbb{A}(u^{\varepsilon,\sigma},S_\varepsilon^b)=\int_{\mathrm{R}}\sqrt{1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2}\,dsdt. \tag{2.30}$$

We recall that from (2.21) it follows that

$$\int_{\mathrm{R}}\sqrt{1+\left(\widetilde{m}_s\right)^2+\left(\widetilde{m}_t\right)^2}\,dsdt=\mathfrak{A}_{\min}+O(\sigma). \tag{2.31}$$

Hence, employing the same construction used in step 6 in the strips S_ε^c and S_ε^a, and using (2.31) we obtain

$$\lim_{\varepsilon\to0}\mathbb{A}(u^{\varepsilon,\sigma},S_\varepsilon^b\cup S_\varepsilon^c\cup S_\varepsilon^a)=3\mathfrak{A}_{\min}+O(\sigma). \tag{2.32}$$

Step 7: definition of u^ε on T_ε. We divide T_ε into four closed equilateral triangles T_ε^a, T_ε^b, T_ε^c and T_ε^0 as in Figure 6 (b). We set the value of u^ε on T_ε^0 as the baricenter of α, β, γ, namely

$$u^\varepsilon:=0\qquad\text{in } T_\varepsilon^0. \tag{2.33}$$

We define u^ε on T_ε^b so that:

(i) the value of u^ε on the bottom vertex of T^b_ε is β;

(ii) the value of u^ε on the top side of T^b_ε is the baricenter of α, β, γ (the zero vector);

(iii) u^ε is affine.

Note that u^ε does not depend on x. We make the similar constructions on T^a_ε and on T^c_ε. We compute

$$\mathcal{A}(u^\varepsilon, T_\varepsilon) = \mathcal{A}(u^\varepsilon, T^0_\varepsilon) + \mathcal{A}(u^\varepsilon, T^a_\varepsilon) + \mathcal{A}(u^\varepsilon, T^b_\varepsilon) + \mathcal{A}(u^\varepsilon, T^c_\varepsilon) = O(\varepsilon^2) + O(\varepsilon),$$

since on T^0_ε the integrand is 1, and on $T_\varepsilon \setminus T^0_\varepsilon$ the integrand is $O(\varepsilon^{-1})$. Hence

$$\lim_{\varepsilon\to 0} \mathcal{A}(u^\varepsilon, T_\varepsilon) = 0. \tag{2.34}$$

Finally, let us define u^ε as in (2.8) on $A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon$, as in step 7 on T_ε, and on $S^b_\varepsilon \cup S^c_\varepsilon \cup S^a_\varepsilon$ let

$$u^\varepsilon := u^{\varepsilon,\sigma_\varepsilon},$$

where $\{\sigma_\varepsilon\} \subset (0, +\infty)$ is a sequence such that

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon = 0. \tag{2.35}$$

From (2.8), (2.26), (2.33), and (i)-(iii) it follows that

$$\{u^\varepsilon\} \subset \mathrm{Lip}(B_r; \mathbb{R}^2), \qquad \lim_{\varepsilon\to 0} \int_{B_r} |u^\varepsilon - u| \; dxdy = 0. \tag{2.36}$$

Moreover

$$\mathcal{A}(u^\varepsilon, B_r) = \mathcal{A}(u^\varepsilon, (A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon) \cap B_r) + \mathcal{A}(u^\varepsilon, S^b_\varepsilon) + \mathcal{A}(u^\varepsilon, S^c_\varepsilon) + \mathcal{A}(u^\varepsilon, S^a_\varepsilon) + \mathcal{A}(u^\varepsilon, T_\varepsilon). \tag{2.37}$$

Then (1.11) follows from (2.36), step 1, (2.37), (2.9), (2.32), (2.35) and (2.34). $\square$

Remark 2.2. Theorem 1.1 is still valid with a similar statement, and easy modifications in the proof, under less restrictive hypotheses on u (however always under the assumption $n = k = 2$). In the following two specific cases (of increasing generality):

- $u : \mathbb{R}^2 \to \{\alpha, \beta, \gamma\}$ is a function jumping only along three different radii of B_r meeting at the origin with arbitrary angles;

- $u : \mathbb{R}^2 \to \{\alpha, \beta, \gamma\}$ is a function jumping only along three curves of class $\mathcal{C}^1$ meeting only at the origin, each curve being without self-intersections and connecting the origin with ∂B_r, and provided the three curves have equal length,

we expect the theorem to be true and, *in addition*, the corresponding sequence $\{u^\varepsilon\}$ to be optimal, namely the value $\lim_{\varepsilon\to 0^+} \mathcal{A}(u^\varepsilon, B_r)$ to be equal to the $L^1(B_r; \mathbb{R}^2)$-lower semicontinuous envelope of $\mathbb{A}(\cdot, B_r)$ evaluated at u.

References

[1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Mathematical Monographs, Oxford University Press, 2000.

[2] E. Acerbi, G. Dal Maso, *New lower semicontinuity results for polyconvex integrals*, Calc. Var. Partial Differential Equations **2**, 329–371, 1994

[3] E. De Giorgi, *On the relaxation of functionals defined on cartesian manifolds*, in Developments in Partial Differential Equations and Applications in Mathematical Physics (Ferrara 1992), Plenum Press, New York 1992.

[4] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.