FREE-DISCONTINUITY PROBLEMS GENERATED BY SINGULAR PERTURBATION

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Synopsis

We show that some free discontinuity problems can be obtained as a limit of non-convex local functionals with a singular perturbation of higher order

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1. Introduction

The approximation by smooth functionals of functionals with volume and surface energies defined in spaces of discontinuous functions has been an object of recent research, also in view of its possible applications to computer vision (see e.g. [**34**], [**33**]) or fracture mechanics (see [ ]).

Earlier approximations by elliptic functionals have been obtained by Modica and Mortola for the perimeter functional [**32**] and by Ambrosio and Tortorelli for the Mumford-Shah functional in computer vision [**11**]. Recently, non-local approximations have been proposed by Braides and Dal Maso [**17**] and by De Giorgi (see Gobbino [**30**]). As a particular case of more general results, we have for example that, given $g \in L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^n$ open bounded set) the families of functionals

$$\frac{1}{\varepsilon}\int_\Omega \arctan\Big(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u|^2 dy\Big)\, dx + \int_\Omega |u-g|^2\, dx, \tag{1.1}$$

defined in $H^1(\Omega)$, and

$$\frac{1}{\varepsilon^{n+1}}\int_\Omega\int_\Omega \arctan\Big(\frac{(u(x)-u(y))^2}{\varepsilon}\Big) e^{-|x-y|^2/\varepsilon} dy\, dx + \int_\Omega |u-g|^2\, dx, \tag{1.2}$$

defined in $L^1(\Omega)$, both approach as $\varepsilon \to 0+$ in the sense of De Giorgi's Γ-convergence the n-dimensional version of the Mumford-Shah functional

$$\alpha\int_\Omega |\nabla u|^2\, dx + \beta\mathcal{H}^{n-1}(S(u)\cap\Omega) + \int_\Omega |u-g|^2\, dx \tag{1.3}$$

defined in the De Giorgi-Ambrosio's $SBV(\Omega)$ space of special functions of bounded variation (α, β are suitable constants; here $S(u)$ denotes the set of essential discontinuity points of u, and $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure). The methods proposed in [**17**] and [**30**] are suitable also for the approximation of more general surface energies, whose densities may depend also on the traces $u^\pm$ of u and on the local orientation of $S(u)$ (see [ ], [ ]).

In this paper we present a simple one-dimensional study to show that similar "non-locality effects" can be obtained by a singular perturbation with higher derivatives of families of *local* integral functionals. As an example of application we can treat a "perturbed local version" of the family in (1.1), whose limit is trivial when not perturbed. The Γ-limit as $\varepsilon \to 0+$ of

$$\frac{1}{\varepsilon}\int_{(a,b)} \arctan(\varepsilon|u'|^2)\, dx + \varepsilon^3\int_{(a,b)} |u''|^2\, dx + \int_{(a,b)} |u-g|^2\, dx, \tag{1.4}$$

($u \in H^2(a,b)$) turns out to be a functional of the form

$$\int_{(a,b)} |u'|^2\,dx + c\sum_{t\in S(u)} \sqrt{|u^+-u^-|} + \int_{(a,b)} |u-g|^2\,dx, \tag{1.5}$$

with c explicitly computable. The main result of the paper is Theorem 3.1, where a characterization is given of Γ-limits of functionals of the form (1.4) with more general integrands. Note that the functional in (1.4) are trivially lower semicontinuous and coercive, so that this approximation provides approximate solutions to minimum problems involving functionals of the form (1.5) alternative to other ones obtained by discretization or stronger non-locality. An explicit example is also given of a family converging to the Mumford-Shah functional in Section 5.

Finally, we remark that our approach has some contact points with recent results regarding singular perturbations with a well at infinity by Bouchitté, Dubs and Seppecher [**14**], and that it is close in spirit to the analysis of Ericksen's bar by Truskinovsky and Zanzotto [**35**]. Further physical insight can be found in [ ], [ ], [ ], [ ], and [ ].

2. Notation and preliminaries

The notation $\mathcal{A}(\mathbb{R})$ stands for the family of all bounded open subsets of $\mathbb{R}$; $\mathcal{L}_1$ denotes the Lebesgue measure on $\mathbb{R}$, $\#$ stands for the counting measure, and δ_t is the Dirac mass at t. We use equivalently the notation $\int_K g\,d\#$ and $\sum_{t\in K} g(t)$, if K is finite or countable, and g is a positive function. The letter c denotes a positive constant, when it is not necessary to specify its value.

Definition 2.1. Let I be a bounded open subset of $\mathbb{R}$. We recall that the space $SBV(I)$ is defined as the space of functions $u \in L^1(I)$ whose distributional derivative Du can be written as $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t$ for some $f \in L^1(\mathbb{R})$, a (finite or countable) set $S(u) \subset \mathbb{R}$ and a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$. We set $\|u\|_{BV(I)} = \int_I |u|\,dt + \int_I |f|\,dt + \sum_t |a_t|$. It is easy to see that for such functions the left hand-side and right hand-side approximate limits $u^-(t)$, $u^+(t)$ exist at every point, that in the notation above $S(u) = \{t \in \mathbb{R} :\ u^-(t) \neq u^+(t)\}$, and that $a_t = u^+(t) - u^-(t)$. Moreover $f = u'$, the approximate derivative of u, a.e.

The definition above describes a particular case of a SBV-functions space as introduced by De Giorgi and Ambrosio [**25**]. In this space the following compactness result holds (see [**5**], [**2**]).

Theorem 2.2. *Let I be a bounded open subset of $\mathbb{R}$, let (u_j) be a sequence in $SBV(I)$ such that* $\sup\|u_j\|_{BV(I)} < +\infty$. *If there exists an increasing concave function $\varphi : (0,+\infty) \to (0,+\infty)$ with $\lim_{s\to 0+}\varphi(s)/s = +\infty$, such that*

$$\sup_j \Big(\int_I |u_j'|^2\,dt + \int_{S(u_j)} \varphi(|u_j^+ - u_j^-|)d\#\Big) < +\infty\,,$$

then, up to subsequences, u_j converges to some $u \in SBV(I)$ in $L^1(I)$,

$$\int_{S(u)} \varphi(|u^+ - u^-|)d\# \le \liminf_j \int_{S(u)} \varphi(|u_j^+ - u_j^-|)d\#\,,$$

and $u_j' \rightharpoonup u'$ weakly in $L^2(I)$.

Definition 2.3. A family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : L^1(I) \to [0,+\infty]$ is said to Γ-*converge* to a functional $F : L^1(I) \to [0,+\infty]$ (with respect to the $L^1(I)$-convergence) if for every $u \in L^1(I)$ and for every sequence (ε_j) of positive numbers decreasing to 0, the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in $L^1(I)$ we have $F(u) \le \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in $L^1(I)$ such that $F(u) \ge \limsup_j F_{\varepsilon_j}(u_j)$.

If we define the *lower and upper* Γ*-limits* by

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) :\ u_\varepsilon \to u\},$$

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) :\ u_\varepsilon \to u\},$$

respectively then the conditions above are equivalent to $F'(u) = F''(u) = F(u)$ at all u. Note that the functions F' and F'' are lower semicontinuous with respect to the $L^1(I)$-convergence. We refer to [**24**] for an exposition of the main properties of Γ-convergence, and to [**17**] for the relevance of such an approximation in the framework of free discontinuity problems.

We will use the following simple lemma.

Lemma 2.4. *Let $K \subset \mathbb{R}$ be a (finite or) countable set and let $\lambda : \mathcal{A}(\mathbb{R}) \to [0, +\infty)$ be an increasing set function, which is superadditive on disjoint open sets; i.e.,*

$$\lambda(A \cup B) \geq \lambda(A) + \lambda(B) \qquad \text{if } A \cap B = \emptyset.$$

Let J be a set of indices and let

$$\lambda(I) \geq c_j \int_I \phi(t)\, dt + d_j \int_{K \cap I} \psi(t)\, d\#(t) \tag{2.1}$$

for all $j \in J$ and $I \in \mathcal{A}(\mathbb{R})$, where c_j, d_j are non-negative real numbers, and ϕ, ψ are positive Borel functions. Then

$$\lambda(I) \geq c \int_I \phi(t)\, dt + d \int_{K \cap I} \psi(t)\, d\#(t) \tag{2.2}$$

for all $I \in \mathcal{A}(\mathbb{R})$, where $c = \sup_j c_j$, $d = \sup_j d_j$.

Proof. Fixed $\varepsilon > 0$ let $\{t_1, \ldots, t_n\} \subseteq K$ be such that $\sum_{i=1}^N \psi(t_i) \geq \int_K \psi\, d\# - \varepsilon$, and let B_ε be an open set such that $t_i \in B_\varepsilon$ for all $i = 1, \ldots, N$, and $\int_{\overline{B}_\varepsilon} \phi\, dt < \varepsilon$. Set $A_\varepsilon = \mathbb{R} \setminus \overline{B}_\varepsilon$. Then for all $j \in J$

$$\lambda(I \cap A_\varepsilon) \geq c_j \Big(\int_I \phi\, dt - \varepsilon\Big), \qquad \lambda(I \cap B_\varepsilon) \geq d_j \Big(\int_I \psi d\# - \varepsilon\Big),$$

so that $\lambda(I \cap A_\varepsilon) \geq c(\int_I \phi\, dt - \varepsilon)$, $\lambda(I \cap B_\varepsilon) \geq d(\int_I \psi d\# - \varepsilon)$ by the arbitrariness of j. The conclusion follows from the inequality $\lambda(I) \geq \lambda(I \cap A_\varepsilon) + \lambda(I \cap B_\varepsilon)$ and the arbitrariness of ε. ☐

3. The main result

The main result of the paper is the following computation of a Γ-limit. Applications to the convergence of minimum problems are given in Section 4.

Theorem 3.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a lower semicontinuous increasing function, such that $\alpha, \beta \in \mathbb{R}$ exist with*

$$\alpha = \lim_{t \to 0+} \frac{f(t)}{t}, \qquad \beta = \lim_{t \to +\infty} f(t). \tag{3.1}$$

Let I be a bounded open subset of $\mathbb{R}$*, and let* $F_\varepsilon : L^1(I) \to [0,+\infty]$ *be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|^2)\,dt + \varepsilon^3 \int_I |u''|^2\,dt & \textit{if } u \in H^2(I) \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.2}$$

Then, there exists the Γ*-limit* $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon(u) = F(u)$ *with respect to the* $L^1(I)$*-convergence, where*

$$F(u) = \begin{cases} \alpha\displaystyle\int_I |u'|^2\,dx + m(\beta)\int_{S(u)} \sqrt{|u^+ - u^-|}\,d\# & \textit{if } u \in SBV(I) \\ +\infty & \textit{otherwise,} \end{cases} \tag{3.3}$$

and $m(\beta) = \beta^{\frac{3}{4}}\left(2\sqrt{3/2} + \sqrt{2/3}\right)$.

The proof of Theorem 3.1 will be obtained as a consequence of some simpler propositions which deal with lower and upper Γ-limits separately. Before stating and proving them, we define a "localized version" of our functionals, which highlights their behaviour as set functions, by setting

$$F_\varepsilon(u, I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|^2)\,dt + \varepsilon^3 \int_I |u''|^2\,dt & \text{if } u \in H^2(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$F(u, I) = \begin{cases} \alpha\displaystyle\int_I |u'|^2\,dx + m(\beta)\int_{I\cap S(u)} \sqrt{|u^+ - u^-|}\,d\# & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.5}$$

for all $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $I \subseteq \mathbb{R}$ bounded open set.

Remark 3.2. For all $c > 0$ let

$$m(c, z) = \min_{\eta>0} \min\Big\{2c\eta + \int_{-\eta}^{\eta} |v''|^2\,dt : \ v \in H^2(-\eta,\eta),\ v(\pm\eta) = \pm\frac{z}{2},\ v'(\pm\eta) = 0\Big\};$$

then $m(c, z) = c^{\frac{3}{4}}\left(2\sqrt{3/2} + \sqrt{2/3}\right)\sqrt{z} = m(c)\sqrt{z}$. It suffices to notice that the solution to the inner minimum problem is the third order polynomial satisfying the given boundary conditions.

Proposition 3.3. *Let $f(t)=\min\{at,b\}$ and let (u_ε) be a bounded family in $L^1(I)$ satisfying $\sup_{\varepsilon>0}F_\varepsilon(u_\varepsilon)<+\infty$. Then there exists a family (v_ε) in $SBV(I)$ such that $\lim_{\varepsilon\to 0+}\|u_\varepsilon-v_\varepsilon\|_{L^1(I)}=0$,*

$$a\int_I |v'_\varepsilon|^2\,dt=\frac{1}{\varepsilon}\int_I f(\varepsilon|u'_\varepsilon|^2)\,dt\,, \tag{3.6}$$

$$\int_{S(v_\varepsilon)}\sqrt{|v^+_\varepsilon-v^-_\varepsilon|}\,d\#\le\frac{C}{m(b)}\,, \tag{3.7}$$

and for every sequence of positive numbers (ε_j) converging to 0 there exists a subsequence (not relabeled) and a function $u\in SBV(I)$ such that $v_{\varepsilon_j}\to u$ in $L^1(I)$,

$$v'_{\varepsilon_j}\rightharpoonup u'\quad \textit{weakly in } L^2(I)\,, \tag{3.8}$$

$$\int_{I\cap S(u)}\sqrt{|u^+-u^-|}\,d\#\le\liminf_j\int_{I\cap S(v_{\varepsilon_j})}\sqrt{|v^+_{\varepsilon_j}-v^-_{\varepsilon_j}|}\,d\#\,. \tag{3.9}$$

Proof. Set

$$D_\varepsilon=\{t\in I:\ \varepsilon|u'_\varepsilon|^2>b/a\}.$$

Since D_ε is open, we can write $D_\varepsilon=\bigcup_{k\in\mathbf{N}}I^k_\varepsilon$ as the union of disjoint open intervals $I^k_\varepsilon=(a^k_\varepsilon,b^k_\varepsilon)$. It is not restrictive to suppose that $[a^k_\varepsilon,b^k_\varepsilon]\subset I$ for all k. Note that $\frac{1}{\varepsilon}|D_\varepsilon|b\le F_\varepsilon(u_\varepsilon)\le C$, independently of ε. Consider the family (v_ε) defined by

$$v_\varepsilon(t)=\begin{cases}u_\varepsilon(t) & \text{if } t\in I\setminus D_\varepsilon\\ u_\varepsilon(a^k_\varepsilon)+u'_\varepsilon(a^k_\varepsilon)(t-a^k_\varepsilon) & \text{if } t\in I^k_\varepsilon.\end{cases} \tag{3.10}$$

As $|v'_\varepsilon|^2=b/\varepsilon a$ on D_ε, we have

$$a\int_I|v'_\varepsilon|^2\,dt=a\int_{I\setminus D_\varepsilon}|u'_\varepsilon|^2\,dt+\frac{b}{\varepsilon}|D_\varepsilon|=\frac{1}{\varepsilon}\int_I f(\varepsilon|u'_\varepsilon|^2)\,dt\le C\,;$$

i.e., (3.6).

Note that $v_\varepsilon-u_\varepsilon\to 0$ in $L^1(I)$. In fact, if $t\in I^k_\varepsilon$ we get, using Hölder's inequality,

$$\begin{aligned}|v_\varepsilon(t)-u_\varepsilon(t)|&=\int_{a^k_\varepsilon}^t|v'_\varepsilon-u'_\varepsilon|\,ds\le\int_{a^k_\varepsilon}^t\int_{a^k_\varepsilon}^s|u''_\varepsilon(\tau)|\,d\tau\,ds\\&\le\Big(\int_{I^k_\varepsilon}|u''_\varepsilon|^2\Big)^{1/2}\int_{a^k_\varepsilon}^t\sqrt{s-a^k_\varepsilon}\,ds=\frac{2}{3}\Big(\int_{I^k_\varepsilon}|u''_\varepsilon|^2\Big)^{1/2}(t-a^k_\varepsilon)^{3/2}\,;\end{aligned}$$

hence,

$$\int_{I_\varepsilon^k} |v_\varepsilon - u_\varepsilon|\,dt \le \frac{4}{15}\Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^2\Big)^{1/2} |I_\varepsilon^k|^{5/2}.$$

We obtain eventually

$$\begin{aligned}\|u_\varepsilon - v_\varepsilon\|_{L^1(I)} &= \int_{D_\varepsilon} |v_\varepsilon - u_\varepsilon|\,dt \le \frac{4}{15}\sum_{k\in\mathbf{N}}\Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^2\Big)^{1/2} |I_\varepsilon^k|^{5/2} \\ &\le \frac{4}{15}\Big(\int_{D_\varepsilon} |u_\varepsilon''|^2\Big)^{1/2}\Big(\sum_{k\in\mathbf{N}} |I_\varepsilon^k|^5\Big)^{1/2} \le \frac{4}{15}\frac{\sqrt{C}}{\varepsilon^{3/2}}|D_\varepsilon|^{5/2} \le \frac{4\,C^3}{15\,b^{5/2}}\,\varepsilon\,;\end{aligned}$$

in particular (v_ε) is bounded in $L^1(I)$.

The functions v_ε are discontinuous only at the points b_ε^k. Set

$$z_\varepsilon^k = |v_\varepsilon^+(b_\varepsilon^k) - v_\varepsilon^-(b_\varepsilon^k)|\,, \qquad w_\varepsilon = u_\varepsilon - v_\varepsilon\,.$$

Since $u_\varepsilon'' = w_\varepsilon''$ on I_ε^k, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{I_\varepsilon^k} f(\varepsilon|u_\varepsilon'|^2)\,dt + \varepsilon^3\int_{I_\varepsilon^k} |u_\varepsilon''|^2\,dt = \frac{b}{\varepsilon}|I_\varepsilon^k| + \varepsilon^3\int_{I_\varepsilon^k} |w_\varepsilon''|^2\,dt \\ &\ge \min\Big\{\frac{b}{\varepsilon}|I_\varepsilon^k| + \varepsilon^3\int_{I_\varepsilon^k} |\varphi''|^2\,dt:\ \varphi\in H^2(I_\varepsilon^k),\ \varphi(a_\varepsilon^k)=0,\ \varphi(b_\varepsilon^k)=z_\varepsilon^k, \\ &\qquad\qquad \varphi'(a_\varepsilon^k)=\varphi'(b_\varepsilon^k)=0\Big\} \qquad (3.11)\\ &\ge \min_{\eta>0}\min\Big\{2\eta b + \int_{I_\varepsilon^k} |\psi''|^2\,dt:\ \psi\in H^2(-\eta,\eta),\ \psi(\pm\eta)=\pm\frac{z_\varepsilon^k}{2}, \psi'(\pm\eta)=0\Big\} \\ &= m(b)\sqrt{z_\varepsilon^k}\,;\end{aligned}$$

the last equality being shown in Remark 3.2. Hence,

$$\sum_{k\in\mathbf{N}} \sqrt{|v_\varepsilon^+(b_\varepsilon^k) - v_\varepsilon^-(b_\varepsilon^k)|} \le \frac{C}{m(b)}\,. \qquad (3.12)$$

By (3.6) and (3.12) we get that $v_\varepsilon \in SBV(I)$ and $\sup_{\varepsilon>0}\|v_\varepsilon\|_{BV(I)} < +\infty$; moreover, $S(v_\varepsilon) = (b_\varepsilon^k)_k$, (3.7) holds, and

$$\sup_{\varepsilon>0}\Big(\int_I |v_\varepsilon'|^2\,dt + \int_{I\cap S(v_\varepsilon)} \sqrt{|v_\varepsilon^+ - v_\varepsilon^-|}\,d\#\Big) < +\infty\,.$$

Applying Theorem 2.2 to the sequence (v_{ε_j}) we obtain the existence of $u\in SBV(I)$ such that, up to passing to a subsequence, $v_{\varepsilon_j}\to u$ in $L^1(I)$ and (3.8) and (3.9) hold. □

Proposition 3.4. *If $f(t)=\min\{at,b\}$ then for all $u\in L^1_{\rm loc}(\mathbb{R})$ we have*

$$F(u,I)\le \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u,I)\,,$$

for all bounded open sets $I\subset\mathbb{R}$, where F_ε is defined in (3.4) *and F in* (3.5)*, with a,b in the place of α,β.*

Proof. It is not restrictive to suppose that $\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)<+\infty$. Let (ε_j) be a sequence of positive numbers converging to 0 and let $u_\varepsilon\to u$ in $L^1_{\rm loc}(\mathbb{R})$ be such that the limit $\lim_j F_{\varepsilon_j}(u_{\varepsilon_j},I)$ exists and equals the $\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)$. We can suppose then that $C:=\sup_j F_{\varepsilon_j}(u_{\varepsilon_j},I)<+\infty$.

Let v_ε be defined by (3.10). Note that $v_\varepsilon\to u$ in $L^1_{\rm loc}(\mathbb{R})$. Fixed $\delta\in(0,1)$, by (3.6) and (3.7) (applied with δb in the place of b) we obtain

$$\begin{aligned}F_{\varepsilon_j}(u_{\varepsilon_j},I)&\ge(1-\delta)\frac{1}{\varepsilon_j}\int_I f(\varepsilon_j|u'_{\varepsilon_j}|^2)\,dt+\delta\frac{1}{\varepsilon_j}\int_I f(\varepsilon_j|u'_{\varepsilon_j}|^2)\,dt+{\varepsilon_j}^3\int_I|u''_{\varepsilon_j}|^2\,dt\\&\ge(1-\delta)a\int_I|v'_{\varepsilon_j}|^2\,dt+m(\delta b)\int_{I\cap S(v_{\varepsilon_j})}\sqrt{|v^+_{\varepsilon_j}-v^-_{\varepsilon_j}|}\,d\#.\end{aligned}$$

By (3.8) and (3.9) we deduce that

$$\liminf_{\varepsilon_j\to 0+}F_{\varepsilon_j}(u_{\varepsilon_j},I)\ge(1-\delta)a\int_I|u'|^2\,dt+m(\delta b)\int_{I\cap S(u)}\sqrt{|u^+-u^-|}\,d\#\,.$$

After noting that the open set function $\lambda(I)=\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)$ is superadditive on disjoint open sets, we can apply Lemma 2.4 with $\phi(t)=|u'(t)|^2$, $\psi(t)=\sqrt{|u^+(t)-u^-(t)|}$, $K=S(u)$, $J=(0,1)$, $c_j=(1-j)a$, $d_j=m(jb)$, and obtain the thesis. $\square$

Proposition 3.5. *Under the hypotheses of Theorem* 3.1 *we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)\ge F(u,I)$$

for all $u\in L^1_{\rm loc}(\mathbb{R})$, and for all bounded open sets $I\subset\mathbb{R}$.

Proof. For all $s>0$ let

$$\widetilde f(s)=\max\{y:\ \frac{yt}{s}\wedge y\le f(t)\ \ \forall\, t>0\}\,.$$

Note that we have

$$\lim_{s\to 0+}\frac{\widetilde{f}(s)}{s}=\alpha \qquad \lim_{s\to+\infty}\widetilde{f}(s)=\beta\,.$$

¿From Proposition 3.4 we have that $\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)$ is finite only if $F(u,I)$ is finite, and that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+}F_\varepsilon(u,I)\geq\frac{\widetilde{f}(s)}{s}\int_I|u'|^2\,dt+m(\widetilde{f}(s))\int_{I\cap S(u)}\sqrt{|u^+-u^-|}d\#.$$

The thesis again follows as above from Lemma 2.4 taking now $J=(0,+\infty)$, $c_j=\widetilde{f}(j)/j$, and $d_j=m(\widetilde{f}(j))$. □

In the sequel $u'(t\pm)$ denote the right-hand side and left-hand side limits of u' at t

Proposition 3.6. *Let $u\in SBV(I)$ satisfy $\#(S(u))<+\infty$, $u\in H^2(I\setminus S(u))$, and $u'(t\pm)=0$ on $S(u)$. Then there exists a family (u_ε) converging to u in $L^1(I)$ such that $\limsup_{\varepsilon\to 0+}F_\varepsilon(u_\varepsilon)\leq F(u)$.*

Proof. Since the construction of u_ε will modify u only in a small neighbourhood of $S(u)$, we can suppose that $I=(-1,1)$ and $S(u)=\{0\}$. Let $z=u^+(0)-u^-(0)$, and let η and v be the minimizing pair in the definition of $m(\beta,z)$. If we set $v_\varepsilon(x)=v(x/\varepsilon)$ then we have $v_\varepsilon(\pm\varepsilon\eta)=\pm z/2$, $v'_\varepsilon(\pm\varepsilon\eta)=0$, and

$$F_\varepsilon(v_\varepsilon,(-\varepsilon\eta,\varepsilon\eta))\leq 2\eta\beta+\varepsilon^3\int_{-\eta}^{\eta}|v''_\varepsilon|^2\,dt=m(\beta,z).$$

We define then

$$u_\varepsilon(x)=\begin{cases}\dfrac{u^+(0)+u^-(0)}{2}+v_\varepsilon(x) & \text{if } x\in(-\varepsilon\eta,\varepsilon\eta)\\ u(x+\varepsilon\eta) & \text{if } x\leq-\varepsilon\eta\\ u(x-\varepsilon\eta) & \text{if } x\geq\varepsilon\eta,\end{cases}$$

so that $u_\varepsilon\in H^2(I)$, $u_\varepsilon\to u$ in $L^1(I)$, and

$$F_\varepsilon(u_\varepsilon)\leq\frac{1}{\varepsilon}\int_I f(\varepsilon|u'|^2)\,dt+\varepsilon^3\int_I|u''|^2\,dt+m(\beta,z).$$

Note that $f(\varepsilon|u'|^2)/\varepsilon\leq K|u'|^2$ for some constant K, and that $f(\varepsilon|u'|^2)/\varepsilon\to\alpha|u'|^2$ a.e. on I; hence, after applying Lebesgue's Dominated Convergence Theorem, we obtain

$$\limsup_{\varepsilon\to 0+}F_\varepsilon(u_\varepsilon)\leq\alpha\int_I|u'|^2\,dt+m(\beta,z)=F(u),$$

and the thesis. □

Proposition 3.7. *Let $u \in SBV(I)$ satisfy $\#(S(u)) < +\infty$ and $u' \in L^2(I)$. Then there exists a sequence (u_j) in $SBV(I)$ such that $S(u_j) \subseteq S(u)$, $u_j \in H^2(I \setminus S(u))$, $u_j'(t\pm) = 0$ on $S(u)$, $u_j \to u$ in $L^1(I)$, and $u_j' \to u'$ in $L^2(I)$.*

Proof. It is not restrictive to suppose $I = (a,b)$. Let $S(u) = \{x_1, \dots, x_N\}$, with $x_i < x_{i+1}$, and set $x_0 = a$, $x_{N+1} = b$. For all $j \in \mathbf{N}$, and $i \in \{0, 1, \dots, N\}$, let u_j^i be the solution to the minimum problem

$$\min\Big\{ \int_{x_i}^{x_{i+1}} |v'|^2\,dt + j \int_{x_i}^{x_{i+1}} |u - v|^2\,dt : \ v \in H^1(x_i, x_{i+1}) \Big\}.$$

Note that u_j^i is also a classical solution of the Euler equation $v'' = j(v - u)$ with the Neumann conditions $v'(x_i) = v'(x_{i+1}) = 0$. The function u_j defined by $u_j = u_j^i$ on (x_i, x_{i+1}) satisfies the required conditions. □

The following proposition concludes the proof of Theorem 3.1.

Proposition 3.8. *We have $\Gamma\text{-}\limsup\limits_{\varepsilon\to0+} F_\varepsilon(u_\varepsilon) \le F(u)$ for all $u \in SBV(I)$.*

Proof. We use the notation $F'' = \Gamma\text{-}\limsup_{\varepsilon\to0+} F_\varepsilon$, and we suppose without loss of generality that $I = (a,b)$.

If $u \in SBV(I)$ with $\#(S(u)) < +\infty$ and $u' \in L^2(I)$, let (u_j) be given by Proposition 3.7. By Proposition 3.6 we have that $F''(u_j) \le F(u_j)$ for all j, and that $\lim_j F(u_j) = F(u)$. By the lower semicontinuity of F'' we obtain then

$$F''(u) \le \liminf_j F''(u_j) \le \liminf_j F(u_j) = F(u). \tag{3.13}$$

Let now $u \in SBV(I)$ satisfy $F(u) < +\infty$ and $\#(S(u)) = +\infty$. If $S(u) = \{x_1, x_2, \dots\}$, set $z_i = u^+(x_i) - u^-(x_i)$, and

$$u_k = u - \sum_{j=k+1}^{\infty} z_i \chi_{(x_i,b)}.$$

We have $u_k' = u'$, $S(u_k) = \{x_1, \dots, x_k\}$, $u_k^+(x_i) - u_k^-(x_i) = z_i$ on $S(u_k)$, and $\lim_k F(u_k) = F(u)$. By (3.13) we have $F''(u_k) \le F(u_k)$. Using again the lower semicontinuity of F'' we obtain the required inequality.

□

4. Some corollaries and generalizations

We first show that the Γ-limit in Theorem 3.1 is trivial if we replace the exponent 3 by a different one.

Corollary 4.1. *Let f satisfy the hypotheses of Theorem 3.1 and let $\gamma \in \mathbb{R}$. Let I be a bounded open subset of $\mathbb{R}$, and let $F^\gamma_\varepsilon : L^1(I) \to [0, +\infty]$ be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_I f(\varepsilon |u'|^2)\, dt + \varepsilon^\gamma \int_I |u''|^2\, dt & \text{if } u \in H^2(I) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Then, there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) = F^\gamma(u)$ with respect to the $L^1(I)$-convergence, where $F^\gamma(u) = 0$ for all $u \in L^1(I)$ if $\gamma > 3$, $F^\gamma = F$ as in Theorem 3.1 if $\gamma = 3$, and

$$F^\gamma(u) = \begin{cases} \alpha \displaystyle\int_I |u'|^2\, dx & \text{if } u \in H^1(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2}$$

if $\gamma < 3$.

Proof. The case $\gamma = 3$ is dealt with in Theorem 3.1. Fixed $\gamma > 3$, take $\varepsilon_0 > 0$; for all positive $\varepsilon < \varepsilon_0$ we have

$$F^\gamma_\varepsilon(u) \le \varepsilon_0^{\gamma-3} \Big(\frac{1}{\varepsilon} \int_I \varepsilon_0^{3-\gamma} f(\varepsilon |u'|^2)\, dt + \varepsilon^3 \int_I |u''|^2\, dt \Big).$$

We can apply Theorem 3.1 with $\varepsilon_0^{3-\gamma} f$ in the place of f to obtain for all $u \in SBV(I)$

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) &\le \varepsilon_0^{\gamma-3} \Big(\varepsilon_0^{3-\gamma} \alpha \int_I |u'|^2\, dt + \varepsilon_0^{3(3-\gamma)/4} m(\beta) \int_{S(u)} \sqrt{|u^+ - u^-|}\, d\# \Big) \\ &= \alpha \int_I |u'|^2\, dt + \varepsilon_0^{(\gamma-3)/4} m(\beta) \int_{S(u)} \sqrt{|u^+ - u^-|}. \end{aligned}$$

Letting $\varepsilon_0 \to 0+$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) \le \alpha \int_I |u'|^2\, dt$$

for all $u \in SBV(I)$; in particular for every piecewise constant function v we have Γ-$\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon(v) = 0$. By a density argument and by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon$ we deduce (4.1).

Let now $\gamma < 3$. Arguing as above we deduce that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) = +\infty \text{ if } u \in L^1(I)\setminus SBV(I)$$

and for all $\varepsilon_0 > 0$

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) \geq \alpha\int_I |u'|^2\,dt + \varepsilon_0^{(\gamma-3)/4} m(\beta)\int_{S(u)}\sqrt{|u^+ - u^-|}$$

if $u \in SBV(I)$. Letting $\varepsilon_0 \to 0+$ we have then that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) = +\infty \text{ if } u \in L^1(I)\setminus H^1(I)$$

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) \geq \alpha\int_I |u'|^2\,dt \text{ if } u \in H^1(I)\,.$$

On the other hand, if $u \in H^2(I)$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) \leq \limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon(u) = \alpha\int_I |u'|^2\,dt.$$

By a density argument and by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F^\gamma_\varepsilon$ we deduce (4.2). □

Corollary 4.2. *Let $g \in L^2(I)$ and $\lambda > 0$ be fixed. Then for all $\varepsilon > 0$ there exists a minimum point u_ε for the problem*

$$\min\Big\{\frac{1}{\varepsilon}\int_I f(\varepsilon|v'|^2)\,dt + \varepsilon^3\int_I |v''|^2\,dt + \lambda\int_I |v-g|^2\,dt :\ v \in H^2(I)\Big\}, \tag{4.3}$$

and for every sequence (ε_j) of positive numbers converging to 0 there exists a subsequence (not relabeled) such that u_{ε_j} converges in $L^1(I)$ to a function $u \in SBV(I)$, which minimizes

$$\min\Big\{\alpha\int_I |v'|^2 + m(\beta)\int_{S(v)}\sqrt{|v^+ - v^-|}\,d\# + \lambda\int_I |v-g|^2\,dt :\ v \in SBV(I)\Big\}. \tag{4.4}$$

Moreover the minimum values (4.3) converge to (4.4).

Proof. The existence of minimum points for (4.3) is assured by an application of the direct methods of the calculus of variations. In fact the functional in (4.3) is coercive in $H^2(I)$ and lower semicontinuous (see e.g. [31]). Note that the value in (4.3) is less than or equal to $\lambda\int_I |g|^2\,dt$. Hence, we have $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$ and $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2(I)} < +\infty$. Hence by Proposition 3.3 from every sequence (ε_j) of positive numbers converging to $0+$ we can extract a further subsequence (not relabeled) such that u_{ε_j} converges in $L^1(I)$ to some $u \in SBV(I)$. The minimality of u and the convergence of minimum values are standard consequences of Γ-convergence. □

Proposition 4.3. *Let f satisfy the hypotheses of Theorem 3.1. Let I be a bounded open subset of $\mathbb{R}$, and let $p > 1$. Let $F_\varepsilon : L^1(I) \to [0, +\infty]$ be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_I f(\varepsilon|u'|^2)\,dt + \varepsilon^{2p-1} \int_I |u''|^p\,dt & \text{if } u \in W^{2,p}(I) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.5}$$

Then, there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon(u) = F(u)$ with respect to the $L^1(I)$-convergence, where

$$F(u) = \begin{cases} \alpha \displaystyle\int_I |u'|^2\,dx + m_p(\beta) \int_{S(u)} \sqrt{|u^+ - u^-|}\,d\# & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.6}$$

and

$$m_p(c) = \min_{\eta>0} \min\Big\{ 2c\eta + \int_{-\eta}^{\eta} |v''|^p\,dt :\ v \in W^{2,p}(-\eta,\eta),\ v(\pm\eta) = \pm\frac{1}{2},\ v'(\pm\eta) = 0 \Big\} \tag{4.7}$$

for all $c > 0$.

Proof. We argue as in the proof of Theorem 3.1, taking into account that

$$\min_{\eta>0} \min\Big\{ 2c\eta + \int_{-\eta}^{\eta} |v''|^p\,dt :\ v \in W^{2,p}(-\eta,\eta),\ v(\pm\eta) = \pm\frac{z}{2},\ v'(\pm\eta) = 0 \Big\} = m_p(c)\sqrt{|z|},$$

if $z \in \mathbb{R}$. This equality easily follows setting (for $z \neq 0$)

$$\eta = \delta\sqrt{|z|}, \qquad v(x) = z\,w(x/\sqrt{|z|}),$$

with w a test function for the problem defining $m_p(c)$. □

5. An application: approximation of the Mumford-Shah functional

We show that it is possible to approximate the Mumford-Shah functional using a slight variation of Theorem 3.1. For all $\varepsilon > 0$ let $a_\varepsilon \geq 1$ with

$$\lim_{\varepsilon\to 0+} a_\varepsilon = +\infty \qquad \text{and} \qquad \lim_{\varepsilon\to 0+} \varepsilon a_\varepsilon = 0.$$

Fixed $K > 0$ set

$$C_\varepsilon = \frac{K}{4}\,\frac{1}{\sqrt{\varepsilon\,a_\varepsilon}}, \tag{5.1}$$

and

$$f_\varepsilon(z) = \begin{cases} z & \text{if } z \leq 1 \\ a_\varepsilon & \text{if } 1 < z < (1+C_\varepsilon)^2 \\ 0 & \text{if } z \geq (1+C_\varepsilon)^2. \end{cases} \tag{5.2}$$

Theorem 5.1. *Let I be a bounded open subset of $\mathbb{R}$. The functionals $F_\varepsilon : L^1(I) \to [0,+\infty]$ given by*

$$F_\varepsilon(u) = \begin{cases} \displaystyle\frac{1}{\varepsilon}\int_I f_\varepsilon(\varepsilon|u'|^2)\,dt + \varepsilon^3\int_I |u''|^2\,dt & \text{if } u \in H^2(I) \\ +\infty & \text{otherwise} \end{cases} \tag{5.3}$$

Γ-converge in the $L^1(I)$-topology as $\varepsilon \to 0+$ to the functional $F : L^1(I) \to [0,+\infty]$ given by

$$F(u) = \begin{cases} \displaystyle\int_I |u'|^2\,dt + K\#(S(u)) & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.4}$$

Proof. We first prove the lower semicontinuity inequality. Let $u_\varepsilon \to u$ in $L^1(I)$ with $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon, I) < +\infty$, and let $D_\varepsilon = \{\varepsilon|u_\varepsilon'|^2 > 1\} = \bigcup_{k\in\mathbf{N}} I_\varepsilon^k$, in the notation of Proposition 3.4. Again, it is not restrictive to suppose that $\overline{I}_\varepsilon^k \subset I$ for all k. Note that for all I_ε^k we have the estimate

$$\frac{1}{4\sqrt{\varepsilon}}(I_\varepsilon^k)^2 \le \int_{I_\varepsilon^k} |u_\varepsilon|\,dx \le c,$$

so that $(I_\varepsilon^k)^2 \le C\sqrt{\varepsilon}$.

We divide D_ε into two sets:

$$D_\varepsilon^1 = \bigcup\{I_\varepsilon^k :\ I_\varepsilon^k \subset D_\varepsilon :\ \varepsilon|u_\varepsilon'|^2 < (1+C_\varepsilon)^2 \text{ on } I_\varepsilon^k\}, \qquad D_\varepsilon^2 = D_\varepsilon \setminus D_\varepsilon^1.$$

If $I_\varepsilon^k \subset D_\varepsilon^2$ we have

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{I_\varepsilon^k} f(\varepsilon|u_\varepsilon'|^2)\,dt + \varepsilon^3\int_{I_\varepsilon^k} |u_\varepsilon''|^2\,dt \\ &\ge 2\min_{\eta>0}\min\Big\{\frac{1}{\varepsilon}\eta a_\varepsilon + \varepsilon^3\int_0^\eta |u''|^2\,dt : u'(0)=0,\ u'(\eta) = C_\varepsilon/\sqrt{\varepsilon}\Big\} \end{aligned} \tag{5.5}$$

It is easy to compute such a minimum remarking that it is equivalent to

$$\begin{aligned} &\min_{\eta>0}\min\Big\{\eta a_\varepsilon + \int_0^\eta |u''|^2\,dt : u'(0)=0,\ u'(\eta) = C_\varepsilon\sqrt{\varepsilon}\Big\} \\ &= \min_{\eta>0}\min\Big\{\eta a_\varepsilon + \int_0^\eta |w'|^2\,dt : w(0)=0,\ w(\eta) = C_\varepsilon\sqrt{\varepsilon}\Big\}, \end{aligned}$$

and see, by our choice of C_ε, that the minimizing pair in (5.5) is

$$\eta = \frac{\varepsilon K}{4a_\varepsilon}, \qquad u(x) = \frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}x^2,$$

(up to an additive constant for u) which gives

$$\frac{1}{\varepsilon}\int_{I_\varepsilon^k} f(\varepsilon|u_\varepsilon'|^2)\,dt + \varepsilon^3\int_{I_\varepsilon^k}|u_\varepsilon''|^2\,dt \geq K.$$

This shows that $\#(\{k : I_\varepsilon^k \subset D_\varepsilon^2\})$ is equibounded. We can suppose that $\#(\{k : I_\varepsilon^k \subset D_\varepsilon^2\}) = N$, independent of ε. Since $|I_\varepsilon^k| \to 0$ as $\varepsilon \to 0+$, we can suppose also that D_ε^2 shrinks to a finite set H. Of course, $\#(H) \leq N$.

For all $t > 0$ let $I_t = \{x \in I : \operatorname{dist}(x, H) > t\}$. For $j \in \mathbf{N}$, we define

$$f^j(s) = \begin{cases} s & \text{if } s \leq 1 \\ j & \text{if } s > 1 \end{cases}$$

and F_ε^j as in (3.4) with f^j in the place of f; then, for fixed t and j, we have, for ε small enough $F_\varepsilon(u_\varepsilon, I_t) \geq F_\varepsilon^j(u_\varepsilon, I_t)$. By applying Proposition 3.5 we get that $u \in SBV(I_t)$ and

$$\liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon, I_t) \geq \int_{I_t}|u'|^2 + m(j,1)\int_{I_t\cap S(u)}\sqrt{|u^+ - u^-|}d\#.$$

By the arbitrariness of j we have then that $u \in H^1(I_t)$ for all $t > 0$, and

$$\liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon, I_t) \geq \int_{I_t}|u'|^2. \tag{5.6}$$

By the arbitrariness of t we obtain that $u \in SBV(I)$ and $S(u) \subset H$. On the other hand, we clearly have, fixed t

$$\liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon, I\setminus \overline{I}_t) \geq \liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon, D_\varepsilon^2) \geq KN \geq K\#(S(u)). \tag{5.7}$$

By (5.6), (5.7), and the arbitrariness of t we get

$$\liminf_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2\,dt + K\#(S(u)).$$

It suffices now to construct a recovery sequence in the case $u \in SBV(-1,1)$ with $S(u) = \{0\}$, $u \in H^2((-1,1)\setminus\{0\})$ and $u'(0\pm) = 0$. The general case follows from Propositions 3.7 and 3.8.

We suppose without loss of generality that $u^+(0) > u^-(0)$. Let $z = u^+(0) - u^-(0)$, and let

$$\eta_\varepsilon = \frac{z\sqrt{\varepsilon}}{2(1+C_\varepsilon)} + \frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{2\sqrt{a_\varepsilon}}.$$

For ε small enough the following function is well defined and belongs to $H^2(-1,1)$:

$$u_\varepsilon(x)=\begin{cases} u(x+\eta_\varepsilon) & \text{if } x\le -\eta_\varepsilon\\ u^-(0)+\frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}\,(x+\eta_\varepsilon)^2 & \text{if } -\eta_\varepsilon<x<-\eta_\varepsilon+\frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}}\\ u^-(0)+\frac{(1+C_\varepsilon)^2\varepsilon}{2\sqrt{a_\varepsilon}}+\frac{1+C_\varepsilon}{\sqrt{\varepsilon}}(x+\eta_\varepsilon-\frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}}) & \text{if } |x|\le\eta_\varepsilon-\frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}}\\ u^+(0)-\frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}\,(x-\eta_\varepsilon)^2 & \text{if } \eta_\varepsilon-\frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}}<x<\eta_\varepsilon\\ u(x-\eta_\varepsilon) & \text{if } x\ge\eta_\varepsilon. \end{cases}$$

The function u_ε is obtained "filling the gap of u" with two minimizers for the minimum problem in (5.5) joined by a steep affine function, with slope $(1+C_\varepsilon)/\sqrt{\varepsilon}$ so that it gives no contribution to the integrals. A direct computation gives

$$\limsup_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon)=\int_I |u'|^2\,dt+K=F(u),$$

and the proof is concluded. □

Corollary 5.2. *Let $g\in L^2(I)$, $\lambda>0$, and let (ε_j) be a sequence of positive numbers converging to 0. Then, for every sequence (u_j) of minimizers of the problems*

$$\min\Big\{\frac{1}{\varepsilon_j}\int_I f_{\varepsilon_j}(\varepsilon_j|u'|^2)\,dt+\varepsilon_j^3\int_I |u''|^2\,dt+\lambda\int_I |u-g|^2\,dt:\ u\in H^2(I)\Big\}$$

there exists a subsequence converging in $L^1(I)$ to a minimizer of the problem

$$\min\Big\{\int_I |u'|^2\,dt+\#(S(u))+\lambda\int_I |u-g|^2\,dt:\ u\in SBV(I)\Big\},$$

and we have also convergence of the minimum values.

Proof. Since we have $\sup\|u_j-g\|_{L^2(I)}<+\infty$ the sequence (u_j) is bounded in $L^1(I)$. Repeating the reasoning of the first part of the proof of Theorem 5.1 we get that up to subsequences u_j converges in $L^1(I)$ to some $u\in SBV(I)$. The minimality of u and the convergence of minimum values are standard consequences of Γ-convergence. □

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