SEMI-DISCRETE MODELING OF SYSTEMS OF WEDGE DISCLINATIONS AND EDGE DISLOCATIONS VIA THE AIRY STRESS FUNCTION METHOD

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Abstract. We present a variational theory for lattice defects of rotational and translational type. We focus on finite systems of planar wedge disclinations, disclination dipoles, and edge dislocations, which we model as the solutions to minimum problems for isotropic elastic energies under the constraint of kinematic incompatibility. Operating under the assumption of planar linearized kinematics, we formulate the mechanical equilibrium problem in terms of the Airy stress function, for which we introduce a rigorous analytical formulation in the context of incompatible elasticity. Our main result entails the analysis of the energetic equivalence of systems of disclination dipoles and edge dislocations in the asymptotics of their singular limit regimes. By adopting the regularization approach via core radius, we show that, as the core radius vanishes, the asymptotic energy expansion for disclination dipoles coincides with the energy of finite systems of edge dislocations. This proves that Eshelby's kinematic characterization of an edge dislocation in terms of a disclination dipole is exact also from the energetic standpoint.

Keywords: Wedge Disclinations, Edge Dislocations, Linearized Elasticity, Airy Stress Function.

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Contents

Introduction

The modeling of translational and rotational defects in solids, typically referred to as *dislocations* and *disclinations*, respectively, dates back to the pioneering work of Vito Volterra on the investigation of the equilibrium configurations of multiply connected bodies [95]. Dislocations, possibly the most common lattice defect, are regarded as the main mechanism of ductility and plasticity of metals and elastic crystals [82, 84, 93]. Disclinations appear at the lattice level in metals alloys [41, 94], minerals [31], graphene [14, 97], and virus shells [60, 79]. Despite both being line defects, their behavior is different, both geometrically and energetically. Moreover, the mathematical modeling is mostly available in the mechanical assumption of cylindrical geometry, where the curves on which the defects are concentrated are indeed line segments parallel to the cylinder axis.

Dislocations entail a violation of translational symmetry and are characterized by the so-called *Burgers vector*. Here we consider only *edge dislocations*, namely those whose Burgers vector is perpendicular to the dislocation line. Disclinations arise as a violation of rotational symmetry and are characterized by the so-called *Frank angle*. Disclinations are defined, (see [8, 78]) as the "closure failure of rotation ... for a closed circuit round the disclination centre". Conceptually, a planar wedge disclination can be realized in the following way, see [95]. In an infinite cylinder, remove a triangular wedge of material and restore continuity by glueing together the two surfaces of the cut: this results in a positive wedge disclination; conversely, open a surface with a vertical cut originating at the axis of the infinite cylinder through the surface, insert an additional wedge of material into the cylinder through the opening, and restore continuity of the material: this results in a negative wedge disclination [86]. Because of the cylindrical geometry, we will work in the cross-section of the material, where the disclinations lines are identified by points in the two-dimensional sections. In this setting, the energy of an edge dislocation scales, far away from its center, as the logarithm of the size of the domain, while the energy of a single disclination is non-singular and scales quadratically with the size of the domain [87]. In many observations disclinations appear in the form of dipoles [61, 69, 86], which are pairs of wedge disclinations of opposite Frank angle placed at a close (but finite) distance. This configuration has the effect of screening the mutual elastic strains resulting in significantly lower energy than the one of single, isolated disclinations.

A continuum theory for disclinations in the framework of linearized elasticity has been developed and systematized, among a number of authors, by de Wit in [35] and subsequently in [37, 38, 39]. A non-linear theory of disclinations and dislocations has been developed in [104], to which we refer the interested reader for a historical excursus and a list of references to classical linearized theories, as well as to other early contributions on the foundation of non-linear theories. For more recent modeling approaches, in [3] disclinations are comprised as a special case of *g.disclinations*, a general concept designed to model phase transformations, grain boundaries, and other plastification mechanisms. Qualitative and quantitative comparison between the classical linearized elasticity approach and the g.disclination theory is discussed in details in [102]. The contributions [43] and [92] propose a mesoscale theory for crystal plasticity designed for modeling the dynamic interplay of disclinations and dislocations based on linearized kinematics and written in terms of elastic and plastic curvature tensors. Variational analysis of a discrete model for planar disclinations is performed in [24]. Finally, we point out that the papers [2] and also [98, 99] consider a differential geometry approach for large non-linear deformations.

While the body of work on dislocations is vast both in the mathematics [83, 9, 54, 46, 47, 28, 29] as well as in the physics and chemistry literature [62, 42, 53, 63, 55, 73] due to their relevance in metallurgy and crystal plasticity, the interest on disclinations has been much lower. This disproportion owes to the fact that disclinations are thought to be less predominant in the formation of plastic microstructure. However, a large body of experimental evidence, some of which in recent years, has shown that disclinations, both in single isolated as well as multi-dipole configuration, are in fact very relevant plastification mechanism, so that understanding their energetics and kinematics is crucial to understanding crystal micro-plasticity.

Interesting examples of disclinations can be observed in martensitic microstructures. This is a complex micrometer-scale pattern emerging in classes of elastic crystals undergoing the austenite-to-martensite phase transformation [13, 15]. While in an ideal scenario such transformation entails a purely elastic, fully reversible change of symmetry of the underlying crystal lattice, in many practical realizations non idealities such as dislocations and in particular disclinations emerge, resulting in possible degradation of reversibility of the shape-memory effect. We refer to [69] for the classification of over forty types of disclinations that can be constructed in MgCd alloys undergoing the hexagonal-to-orthorhombic transformation. Among them, we recall examples of beautiful, self-similar, martensitic microstructures containing a dipole of wedge disclinations (see in particular [74] and [75]), for which models and computations are produced in [85, 23] and mathematical theories are derived in [21]. Examples of complex self-similar microstructures incorporating disclinations emerging from the nucleation and evolution of needle-shaped regions occupied by martensitic phase are described in [65, 66] (see also [12, 22] for computations and stochastic models). For more examples of experimental observations and numerical simulations of partial disclinations, see [56, Section 12.3.3] and also [11, 10].

In crystal plasticity, disclinations (with their various configurations, such as isolated wedge disclinations, disclination dipoles, and even quadrupoles) have been recognized to play an important role in the kinematic accommodation of special morphologies of plastic kink bands caused by rotational stretches of the lattice [71, 64, 67, 96]. Modeling and analysis of kinking has recently captured the interest of metallurgists in relation to a novel strengthening mechanism observed in certain classes of Mg-based alloys exhibiting Long-Period Stacking Order (LPSO) phase [68, 1]. Although yet to date in large part not understood, the kink-strengthening mechanism seems to originate from an intricate interplay of elastic and materials instabilities observed in the columnar "mille-feuille" structures of LPSO materials under high compressions in planar geometries [59, 57, 58]. While exact scale-free constructions [64] shed light on the kinematics of the disclination-kinking mechanism, a model based on energy first principles to describe the energetics of systems of disclinations, dislocations and kinks, together with the length scales of their associated plastification patterns and their collective effects on the strenghtening of the LPSO phase, is still unavailable.

With this paper we intend to move a first step in this direction and lay the foundation of a general and comprehensive variational theory suitable to treat systems of rotational and translational defects on a lattice. We focus on three different aspects: we propose a variational model for finite systems of planar wedge disclinations; we study dipoles of disclinations and we identify relevant energy scalings dictated by geometry and loading parameters; finally, we prove the asymptotic energetic equivalence of a dipole of wedge disclinations with an edge dislocation.

Modeling assumptions. We operate under the assumption of plane strain elastic displacements and under the approximation of linearized kinematics so that contributions of individual defects can be added up via superposition. As we are mainly concerned with the modeling of experimental configurations of metals and hard crystals, we restrict our analysis to the case of two-dimensional plain strain geometries, leaving to future work the analysis in the configuration of plane mechanical stresses as in buckled membranes.

We model disclinations and dislocations as point sources of kinematic incompatibility following an approach analogous to [91] and [20]. Alternative approaches according to the stress-couple theory in linearized kinematics are pursued in [35, 43, 92]. Despite their intrinsic limitations, linearized theories have proven useful to describe properties of systems of dislocations both in continuous and discrete models [19, 20, 44, 34, 4, 30, 33, 5, 6, 17, 51, 7] (see also [89, 77, 52, 45] for related nonlinear models for (edge) dislocations). In [85, 23, 21] systems of disclinations have been investigated in linear and finite elasticity models, and qualitative as well as quantitative comparisons have been discussed.

By working in plane strain linearized kinematics, it is convenient to formulate the mechanical equilibrium problem in terms of a scalar potential, the Airy stress function of the system, see, *e.g.*, [76]. This is a classical method in two-dimensional elasticity based on the introduction of a potential scalar function whose second-order derivatives correspond to the components of the stress tensor (see [26, Section 5.7] and [90]). From the formal point of view, by denoting with σ_{ij} the components of the 2×2 mechanical stress tensor, we write

$$\sigma_{11}=\frac{\partial^2 v}{\partial y^2}\,,\qquad \sigma_{12}=\sigma_{21}=-\frac{\partial^2 v}{\partial y\partial x}\,,\qquad \sigma_{22}=\frac{\partial^2 v}{\partial x^2}\,,$$

where $v\colon\mathbb{R}^2\supset\Omega\to\mathbb{R}$ is the Airy stress function. Upon introduction of the Airy potential v, the equation of mechanical equilibrium $\operatorname{Div}\sigma=0$ is identically satisfied and the information on kinematic (in-)compatibility is translated into a loading source problem for a fourth-order elliptic partial differential equation with boundary conditions for the scalar field v. Existence of the Airy stress function and the variational equivalence of the equilibrium problems formulated in terms of strains and stresses, which we refer to as the *laboratory variables*, with the single-equation problem for the Airy potential are proved in [26] in simply connected domains for perfectly compatible (that is, defect-free) elasticity. Although the Airy stress function method has been adopted by a number of authors to model lattice defects, the equivalence of the equilibrium problem formulated in terms of strains and stresses with the problem formulated in term of the Airy function for simply connected domains for the incompatible elasticity, is, to the best of our knowledge, overlooked. In the present contribution, we attack and solve this question, providing a rigorous, analytical structure to the equilibrium problems for systems of disclinations formulated for the Airy stress function. Our systematization of the Airy stress function method is useful also for the general case of compatible elasticity. We investigate a number of analytical questions, such as the equivalence of boundary datum in terms of the laboratory variables and the Airy potential, fine Poincaré and trace inequalities in perforated domains, and density of Airy potentials under non-standard constraints. To make our presentation clear, we gather most of our original results on the analytical aspects of the Airy stress function method in a series of appendices which can be read and referred to separately from the rest of the paper.

Main contributions and novelties of this work. We construct a rigorous variational setting so that the equilibrium problem formulated in terms of the Airy potential is well posed in terms of existence, uniqueness, and regularity of solutions. Although we focus on finite systems of isolated disclinations, our formulation is general and can be applied to treat configurations in linearized planar elasticity in different geometries and regimes. An immediate application of our analysis is in providing rigorous framework for numerical calculations of lattice defects with the Airy potential method (see, *e.g.*, [91, 103]).

From the point of view of the applications in Materials Science, we prove rigorously energy scalings for systems of isolated wedge disclinations, disclination dipoles, and edge dislocations. Starting from the modeling work of Eshelby [40] (see also [39]) aimed at showing kinematic equivalence of a dipole of wedge disclinations with an edge dislocation, we compute energy estimates and characterize the relation between a dipole of disclinations and an edge dislocation in a precise variational sense. Our result is significant because disclination dipoles are fundamental building blocks for kinks as well as grain boundaries [72, 49, 80], which are important configurations in crystals and metals.

The results contained in this paper serve as a bridge across length scales: the smallest one of disclinations, the intermediate one of disclination dipoles, and the larger one of edge dislocations. Starting from the smallest length scale, we progressively zoom out and unveil the different energy scalings that are proper of the three phenomena. These scaling laws suggest the correct energy renormalizations.

Eshelby's kinematic equivalence of disclination dipoles with edge dislocations is established here at the energetic level in a precise quantitative way. Within the formalism of the Airy stress function, we show that the energy of a system of disclination dipoles coincides, in the limit as the dipole distance vanishes and upon renormalization, with the energy of a system of edge dislocations as described in [20] via the core-radius approach. Our analysis complements the work of [20]: we compute the expansion of the renormalized energy for edge dislocations as well as disclination dipoles in the Airy stress function rather than in the laboratory variables.

Outline of the paper and methods. The outline of the paper is as follows. Section 1 is devoted to the presentation of the mechanical equilibrium equations, in terms both of the laboratory variables and of the Airy stress function of the system. Our main results of this section (Propositions 1.3 and 1.6) contain the proof of the equivalence of the mechanical equilibrium problem formulated in terms of the laboratory variables and of the Airy stress function. Our result is based on a crucial characterization of traction-free boundary displacements for the problem formulated in terms of the Airy potential. Such a characterization involves a non-standard tangential boundary condition for the Hessian of the Airy stress function which we are able to characterize in terms of classical Dirichlet-type boundary conditions for the bilaplacian equation (Proposition 1.7).

In Section 2, we focus on the analysis of systems of isolated disclinations performed for the mechanical problem formulated for the Airy potential. In a domain Ω we consider a finite number K of disclinations and we operate under the assumption that their centers $\{y^k\}_{k=1}^K \subset \Omega$ are fixed (hence the term *isolated*). We show that the mechanical equilibrium formulated in terms of the Airy potential is the solution to a non-homogeneous fourth-order elliptic equation where the source term is a finite sum of Dirac deltas, each of which is placed at a disclination site y^k and is modulated by the corresponding Frank angle s^k. Therefore, the existence of non-trivial solutions to the equilibrium problem follows from the presence of a point-source loading term measuring the *charge* of a disclination which is the signature of a rotational mismatch in the lattice. (Here, the term "charge" can be misleading. We intend to make an analogy with electric charges: same-sign charges repel each other, whereas opposite-sign charges attract each other. Incidentally, the same behavior is observed with screw dislocations, see, *e.g.*, [19, 17, 16]. The use of the term "charge" should not be confused with the notion of *topological charge*: dislocations carry one, disclinations do not.) Although the variational problem for isolated disclinations entails regular functionals, the mechanical strains and stresses of wedge disclinations are in fact singular (showing a logarithmic behavior at the disclinations sites), thus violating the requirements of linearized kinematics (see [39, 70]). The Airy potentials corresponding to the singular strains and stresses are the classical solutions for planar wedge disclinations computed in [95] – and correctly recovered by our model – corresponding to the Green's function for the bilaplacian operator. A possible remedy to the unphysical behavior and inconsistency with experimental observations is the smoothening of mechanical strains and stresses by introducing an additional length scale proportional to the disclination core. As the analysis contained in this paper focuses mainly on singular limits for disclination dipoles and dislocations, we ignore regularization of non-singular functionals for isolated disclinations, leaving these issues to future work.

With Section 3 we begin our investigation of systems of disclination dipoles which we then conclude in Section 4. As length scales and mutual distances between disclinations are regarded as model parameters and as we are interested in the asymptotics of such parameters, we call the modeling of Sections 3 and 4 of *interacting* disclinations. The dependence of both the minimizers and the energy scaling regimes on these length scales will be dictated by loading terms for the problem formulated in the Airy variable, and will follow from global minimization of the total energy of the system and not from *a priori* assumptions. We operate by directly computing the limits of energy minima and minimizers; a more general approach via Γ-convergence [18, 32] is not explored in this paper.

We follow Eshelby's [40] derivation aimed at showing that the Burgers vector $b \in \mathbb{R}^2$ of an edge dislocation can be produced by the lattice mismatch caused by a disclination dipole of charge $\pm|b|/h$ at a small dipole distance $h > 0$. Motivated by his proof of their *kinematic* equivalence, we analyze and clarify the relation between a disclination dipole and an edge dislocation from the point of view of their *energies*. Since edge dislocations and wedge disclination dipoles are both characterized by singular mechanical strain and stress as well as singular energies, we make use of the core-radius approach for planar edge dislocations, see [20]. Consequently, we consider a finite collection of disclination dipoles in a domain Ω and we denote by $\varepsilon > 0$ their core radius with the geometry requirement that $0 < h < \varepsilon$. The limits as h and ε vanish are taken one at a time, first as $h \to 0$ and then as $\varepsilon \to 0$. As a consequence of the first limit $h \to 0$, the length scale ε emerges as the core radius of the dislocations. The material responds to continuum theories of elasticity at scales larger than ε, whereas discrete descriptions are better suited at scales smaller then ε, thus establishing the *semi-discrete* nature of our model.

In Section 3, we consider one dipole of disclinations with charges $\pm s$ and we keep ε fixed while taking the limit as $h \to 0$. At this stage, the energy of the dipole behaves asymptotically as $h^2|\log h|$ (see Proposition 3.3). By rescaling the energy by h^2, we prove convergence to a functional that features a surface load and a bulk elastic term (see (3.20)). The latter is the elastic energy of an edge dislocation with core radius of size ε.

Then we provide an expansion of the ε-regularized dislocation energy as $\varepsilon \to 0$. This is tackled in Section 4 where, relying on an additive decomposition between plastic (*i.e.*, determined by the disclinations) and elastic parts of the Airy stress function, in an analogous fashion to [20], we study the limit as $\varepsilon \to 0$ (Theorem 4.3), we compute the renormalized energy of the system (Theorem 4.6), and we finally obtain the energetic equivalence, which is the sought-after counterpart of Eshelby's kinematic equivalence. We show that the minimizer of the ε-regularized energy converges to a limit function which is the distributional solution to a PDE and is not characterized via a variational principle. From the technical point of view, our results rely on a density theorem for traction-free $H^2(\Omega)$ Airy stress functions, which can be locally approximated, close to each singularity, with a sequence of smooth, traction-free functions (Proposition E.1). Our asymptotic expansion of the ε-regularized energy obtained via the Airy stress function formulation (see (4.43)) is in agreement with [20, Theorem 5.1 and formula (5.2)] at all orders. We stress that the results in Section 4 are written for finite systems of disclination dipoles and dislocations. In particular, Theorem 4.6 fully characterizes the energy of a finite system of dislocations: the renormalized energy F in (4.45) contains information on the mutual interaction of the dislocations. In conclusion, we combine in a cascade the converge result of Section 3 for disclination dipoles for vanishing h with the asymptotic expansion of Section 4 of the renormalized energy of edge dislocations for vanishing ε. We compute, via a diagonal argument, the asymptotic expansion of the $\varepsilon(h)$-regularized energy of finite systems of disclination dipoles for vanishing dipole distance h, thus extending the asymptotic analysis of [20] to finite systems of dipoles of wedge disclinations.

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Notation. For $d \in \{2,3\}$, $m \in \mathbb{N}$, and for every $k \in \mathbb{Z}$, let $\mathscr{R}^k(A;\mathbb{R}^m)$ denote the space of k-regular $\mathbb{R}^m$-valued functions defined on an open set $A \subset \mathbb{R}^d$ (we will consider Sobolev spaces like $H^k(A;\mathbb{R}^m)$ or spaces of k-differentiable functions like $C^k(A;\mathbb{R}^m)$, for $k \geq 0$). Now we introduce different curl operators and show relationships among them. For $d = 3$ and $m = 3$ we define $\textsc{curl}\colon \mathscr{R}^k(A;\mathbb{R}^3) \to \mathscr{R}^{k-1}(A;\mathbb{R}^3)$ as

$$\textsc{curl}\, V := (\partial_{x_2} V^3 - \partial_{x_3} V^2; \partial_{x_3} V^1 - \partial_{x_1} V^3; \partial_{x_1} V^2 - \partial_{x_2} V^1)$$

for any $V = (V^1;V^2;V^3) \in \mathscr{R}^k(A;\mathbb{R}^3)$, or, equivalently, $(\textsc{curl}\, V)^i = \varepsilon_{ijk}\partial_{x_j} V^k$, where ε_{ijk} is the Levi-Civita symbol. For $d = 3$ and $m = 3 \times 3$ we define $\mathrm{CURL}\colon \mathscr{R}^k(A;\mathbb{R}^{3\times 3}) \to \mathscr{R}^{k-1}(A;\mathbb{R}^{3\times 3})$ by $(\mathrm{CURL}\, M)_{ij} := \varepsilon_{ipk}\partial_{x_p} M_{jk}$ for every $M \in \mathscr{R}^k(A;\mathbb{R}^{3\times 3})$ and we notice that $(\mathrm{CURL}\, M)_{ij} = (\textsc{curl}\, M_j)^i$, where M_j denotes the j-th row of M. Moreover, we denote by $\mathrm{INC}\colon \mathscr{R}^k(A;\mathbb{R}^{3\times 3}) \to \mathscr{R}^{k-2}(A;\mathbb{R}^{3\times 3})$ the operator defined by $\mathrm{INC} := \mathrm{CURL}\,\mathrm{CURL} \equiv \mathrm{CURL} \circ \mathrm{CURL}$.

For $d = 2$ and $m \in \{2, 2 \times 2\}$, we define the following curl operators: $\mathrm{curl}\colon \mathscr{R}^k(A;\mathbb{R}^2) \to \mathscr{R}^{k-1}(A;\mathbb{R})$ as $\mathrm{curl}\, v := \partial_{x_1} V^2 - \partial_{x_2} V^1$ for any $V = (V^1;V^2) \in \mathscr{R}^k(A;\mathbb{R}^2)$, $\mathrm{Curl}\colon \mathscr{R}^k(A;\mathbb{R}^{2\times 2}) \to \mathscr{R}^{k-1}(A;\mathbb{R}^2)$ as $\mathrm{Curl}\, M := (\mathrm{curl}\, M_1; \mathrm{curl}\, M_2)$ for any $M \in \mathscr{R}^k(A;\mathbb{R}^{2\times 2})$.

Let now $A \subset \mathbb{R}^2$ be open. For every $V = (V^1, V^2) \in \mathscr{R}^k(A;\mathbb{R}^2)$, we can define $\underline{V} \in \mathscr{R}^k(A;\mathbb{R}^3)$ as $\underline{V} := (V^1;V^2;0)$ and we have that

$$\textsc{curl}\, \underline{V} = (0;0;\mathrm{curl}\, V).$$

Analogously, if $M \in \mathscr{R}^k(A;\mathbb{R}^{2\times 2})$, then, defining $\underline{M}\colon A \to \mathbb{R}^{3\times 3}$ by $\underline{M}_{ij} = M_{ij}$ if $i,j \in \{1,2\}$ and $\underline{M}_{ij} = 0$ otherwise, we have that $\underline{M} \in \mathscr{R}^k(A;\mathbb{R}^{3\times 3})$,

$$\mathrm{CURL}\, \underline{M} = \begin{bmatrix} 0 & 0 & \mathrm{curl}\, M_1 \\ 0 & 0 & \mathrm{curl}\, M_2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{CURL}\,\mathrm{CURL}\, \underline{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{curl}\,\mathrm{Curl}\, M \end{bmatrix}.$$

In what follows, $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is the set of the matrices $M\in\mathbb{R}^{2\times 2}$ with $M_{ij}=M_{ji}$ for every $i,j=1,2$. Finally, for every $M\in\mathbb{R}^{2\times 2}$ we denote by $M^\top$ the matrix with entries $(M^\top)_{ij}=M_{ji}$ for every $i,j=1,2$.

1. The mechanical model

1.1. **Plane strain elasticity.** Let Ω be an open bounded simply connected subset of $\mathbb{R}^2$ with C^2 boundary. For any displacement $u\in H^1(\Omega;\mathbb{R}^2)$ the associated elastic strain $\epsilon\in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is given by $\epsilon:=\nabla^{\mathrm{sym}}u:=\frac{1}{2}(\nabla u+\nabla^\top u)$, whereas the corresponding stress $\sigma\in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is defined by

$$\sigma:=\mathbb{C}\epsilon:=\lambda\mathrm{tr}(\epsilon)\mathbb{I}_{2\times 2}+2\mu\epsilon\,; \tag{1.1}$$

here $\mathbb{C}$ is the *isotropic elasticity tensor* with *Lamé constants* λ and μ. Notice that

$$\mathbb{C}\text{ is positive definite} \tag{1.2a}$$

if and only if

$$\mu>0\qquad\text{and}\qquad\lambda+\mu>0\,, \tag{1.2b}$$

or, equivalently,

$$E>0\qquad\text{and}\qquad -1<\nu<\frac{1}{2}\,. \tag{1.2c}$$

Here and below, E is the *Young modulus* and ν is the *Poisson ratio*, in terms of which the Lamé constants λ and μ are expressed by

$$\mu=\frac{E}{2(1+\nu)}\qquad\text{and}\qquad\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}\,. \tag{1.3}$$

We will assume (1.2) throughout the paper.

In plain strain elasticity the isotropic elastic energy associated with the displacement u in the body Ω is defined by

$$\mathcal{E}(u;\Omega):=\frac{1}{2}\int_\Omega\sigma:\epsilon\,\mathrm{d}x=\frac{1}{2}\int_\Omega\big(\lambda(\mathrm{tr}(\epsilon))^2+2\mu|\epsilon|^2\big)\,\mathrm{d}x\,; \tag{1.4}$$

we notice that in formula (1.4) the energy $\mathcal{E}(\cdot;\Omega)$ depends only on ϵ so that in the following, with a little abuse of notation, we will denote by $\mathcal{E}(\cdot;\Omega)\colon L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})\to[0,+\infty)$ the energy functional defined in (1.4), considered as a functional of ϵ (and not of u).

Notice that we can write the elastic energy also as a function of the stress σ as

$$\mathcal{F}(\sigma;\Omega):=\frac{1}{2}\frac{1+\nu}{E}\int_\Omega\big(|\sigma|^2-\nu(\mathrm{tr}(\sigma))^2\big)\,\mathrm{d}x=\mathcal{E}(\epsilon;\Omega)\,, \tag{1.5}$$

where we have used (1.1) and (1.3) to deduce that

$$\epsilon_{11}=\frac{1+\nu}{E}\Big((1-\nu)\sigma_{11}-\nu\sigma_{22}\Big)\,,\quad\epsilon_{12}=\frac{1+\nu}{E}\sigma_{12}\,,\quad\epsilon_{22}=\frac{1+\nu}{E}\Big((1-\nu)\sigma_{22}-\nu\sigma_{11}\Big)\,, \tag{1.6}$$

and

$$\lambda\big(\mathrm{tr}(\epsilon)\big)^2+2\mu|\epsilon|^2=\frac{1+\nu}{E}\big(|\sigma|^2-\nu(\mathrm{tr}(\sigma))^2\big)\,. \tag{1.7}$$

Finally, we reformulate the energy (1.5) using the Airy stress function method. This assumes the existence of a function $v\in H^2(\Omega)$ such that

$$\sigma_{11}=\partial^2_{x_2^2}v\,,\quad\sigma_{12}=-\partial^2_{x_1x_2}v\,,\quad\sigma_{22}=\partial^2_{x_1^2}v\,; \tag{1.8}$$

more precisely, we consider the operator $\mathsf{A}\colon\mathscr{R}^k(\Omega)\to\mathscr{R}^{k-2}(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ such that $\sigma=\sigma[v]=\mathsf{A}(v)$ is defined by (1.8). It is immediate to see that the operator A is not injective, since $\mathsf{A}(v)=\mathsf{A}(w)$ whenever v and w differ up to an affine function; its invertibility under suitable boundary conditions will be discussed in Subsection 1.3 (see Proposition 1.6).

Assuming that there exists v such that $\sigma = \sigma[v] = \mathsf{A}(v)$, from (1.8), we can rewrite (1.5) as

$$\mathcal{F}(\sigma[v];\Omega) = \frac{1}{2}\frac{1+\nu}{E}\int_\Omega \Big(|\nabla^2 v|^2 - \nu|\Delta v|^2\Big)\,\mathrm{d}x =: \mathcal{G}(v;\Omega)\,. \tag{1.9}$$

We notice that if the stress σ admits an Airy potential v, i.e., $\sigma = \sigma[v] = \mathsf{A}(v)$, then

$$\mathrm{Div}\,\sigma[v] \equiv 0\,, \tag{1.10}$$

that is, the equilibrium equation $\mathrm{Div}\,\sigma = 0$ is automatically satisfied. In fact, this is the main advantage in using the Airy stress function method.

1.2. **Kinematic incompatibility: dislocations and disclinations.** Let $u \in C^3(\Omega;\mathbb{R}^2)$ and set $\beta := \nabla u$. Clearly,

$$\mathrm{Curl}\,\beta = 0 \qquad \text{in } \Omega\,. \tag{1.11a}$$

We can decompose β as $\beta = \epsilon + \beta^{\mathrm{skew}}$, where $\epsilon := \frac{1}{2}(\beta + \beta^\top)$ and $\beta^{\mathrm{skew}} := \frac{1}{2}(\beta - \beta^\top)$. By construction,

$$\beta^{\mathrm{skew}} = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix},$$

for some function $f \in C^2(\Omega)$, and hence $\mathrm{Curl}\,\beta^{\mathrm{skew}} = \nabla f$. Therefore, the compatibility condition (1.11a) can be rewritten as

$$\mathrm{Curl}\,\epsilon = -\nabla f \qquad \text{in } \Omega\,, \tag{1.11b}$$

which, applying again the curl operator, yields the *Saint-Venant compatibility condition*

$$\mathrm{curl}\,\mathrm{Curl}\,\epsilon = 0 \qquad \text{in } \Omega\,. \tag{1.11c}$$

Viceversa, given $\epsilon \in C^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, the *Saint-Venant principle* [88] states that if (1.11c) holds, then there exists $u \in C^3(\Omega;\mathbb{R}^2)$ such that $\epsilon = \nabla^{\mathrm{sym}} u$.

In order to apply the direct method of the Calculus of Variations for the minimization of the elastic energy (1.4), the natural functional setting for the displacement u is the Sobolev space $H^1(\Omega;\mathbb{R}^2)$. Therefore, a natural question that arises is whether identities (1.11) make sense also when β is just in $L^2(\Omega;\mathbb{R}^{2\times 2})$. The answer to this question is affirmative as shown by the following result proved in [27] (see also [50]).

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, and simply connected set and let $\epsilon \in L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Then,*

$$\mathrm{curl}\,\mathrm{Curl}\,\epsilon = 0 \qquad \textit{in } H^{-2}(\Omega) \tag{1.12}$$

if and only if there exists $u \in H^1(\Omega;\mathbb{R}^2)$ such that $\epsilon = \nabla^{\mathrm{sym}} u$. Moreover, u is unique up to rigid motions.

Notice that, by the Closed Graph Theorem, we have that (1.12) holds true in $H^{-2}(\Omega)$ if and only if it holds in the sense of distributions. Therefore, the generalizations of identities (1.11) when $u \in H^1(\Omega;\mathbb{R}^2)$ are given by

$$\mathrm{Curl}\,\beta = 0 \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^2)\,, \tag{1.13a}$$

$$\mathrm{Curl}\,\epsilon = -\nabla f \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^2)\,, \tag{1.13b}$$

$$\mathrm{curl}\,\mathrm{Curl}\,\epsilon = 0 \qquad \text{in } \mathcal{D}'(\Omega)\,, \tag{1.13c}$$

where f is a function in $L^2(\Omega)$ and the operator ∇ should be understood in the sense of distributions. (Here and below, $\mathcal{D}'(\Omega;\mathbb{R}^2)$ and $\mathcal{D}'(\Omega)$ denote the families of $\mathbb{R}^2$-valued and $\mathbb{R}$-valued, respectively, distributions on Ω.) Clearly, if β is not a gradient, then equations (1.13) are not satisfied anymore. In particular, if the right-hand side of (1.13a) is equal to some $\alpha \in H^{-1}(\Omega;\mathbb{R}^2)$, then (1.13b) becomes

$$\mathrm{Curl}\,\epsilon = \alpha - \nabla f \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^2)\,. \tag{1.14}$$

Moreover, if the right-hand side of (1.13b) is equal to $-\kappa$ where $\kappa\in H^{-1}(\Omega;\mathbb{R}^2)$ is not a gradient, then (1.13c) becomes

$$\operatorname{curl}\operatorname{Curl}\epsilon=-\theta\qquad\text{in }\mathcal{D}'(\Omega)\,,\tag{1.15}$$

where we have set $\theta\coloneqq\operatorname{curl}\kappa$. Finally, when both incompatibilities are present, we have that

$$\operatorname{curl}\operatorname{Curl}\epsilon=\operatorname{curl}\alpha-\theta\qquad\text{in }\mathcal{D}'(\Omega)\,.\tag{1.16}$$

We will focus on the case when α and θ are finite sums of Dirac deltas. More precisely, we will consider $\alpha\in\mathscr{ED}(\Omega)$ and $\theta\in\mathscr{WD}(\Omega)$, where

$$\mathscr{ED}(\Omega)\coloneqq\Big\{\alpha=\sum_{j=1}^{J}b^j\boldsymbol{\delta}_{x^j}\ :\ J\in\mathbb{N}\,,\ b^j\in\mathbb{R}^2\setminus\{0\}\,,\ x^j\in\Omega\,,\ x^{j_1}\neq x^{j_2}\text{ for }j_1\neq j_2\Big\}\,,$$
$$\mathscr{WD}(\Omega)\coloneqq\Big\{\theta=\sum_{k=1}^{K}s^k\boldsymbol{\delta}_{y^k}\ :\ K\in\mathbb{N}\,,\ s^k\in\mathbb{R}\setminus\{0\}\,,\ y^k\in\Omega\,,\ y^{k_1}\neq y^{k_2}\text{ for }k_1\neq k_2\Big\}\,.$$

In this case (1.16) reads

$$\operatorname{curl}\operatorname{Curl}\epsilon=-\sum_{j=1}^{J}|b^j|\partial_{\frac{(b^j)^\perp}{|b^j|}}\boldsymbol{\delta}_{x^j}-\sum_{k=1}^{K}s^k\boldsymbol{\delta}_{y^k}\qquad\text{in }\mathcal{D}'(\Omega)\,,\tag{1.17}$$

where we recall that $b^\perp=(-b_2;b_1)$ for every $b=(b_1;b_2)\in\mathbb{R}^2$. The measure α identifies a system of J edge dislocations with Burgers vectors b^j; the measure θ identifies a system of K wedge disclinations with Frank angles s^k.

Remark 1.2. For the sake of simplicity we will assume that the weights b^j's and s^k's of the singularities of α and θ lie in $\mathbb{R}^2\setminus\{0\}$ and $\mathbb{R}\setminus\{0\}$, respectively. Actually, in the theory of perfect edge dislocations, we have that $b^j\in\mathcal{B}\subset\mathbb{R}^2$, where $\mathcal{B}$ is the *slip system*, i.e., the (discrete) set of the vectors of the crystallographic lattice. Analogously, in the theory of perfect disclinations, $s^k\in\mathcal{S}$, where, in a regular Bravais lattice, $\mathcal{S}$ is given by the integer multiples of the minimal angle s between two adjacent nearest-neighbor bonds of a given point (namely, $s=\pm\frac{\pi}{2}$ in the square lattice and $s=\pm\frac{\pi}{3}$ in the regular triangular lattice). Whenever b^j are not vectors in $\mathcal{B}$ or s^k are not angles in $\mathcal{S}$, the corresponding dislocations and disclinations are referred to as *partial*, see [36, 78]. Since we will focus only on the regime of finite number of edge dislocations and wedge disclinations, the classes $\mathcal{B}$ and $\mathcal{S}$ do not play any role in our analysis.

Let $\alpha\in\mathscr{ED}(\Omega)$ and $\theta\in\mathscr{WD}(\Omega)$. Following [35, 37], for every open set $A\subset\Omega$ with $\partial A\cap(\operatorname{spt}\alpha\cup\operatorname{spt}\theta)=\emptyset$ we define the Frank angle $\omega\llcorner A$, the Burgers vector $\mathbf{b}\llcorner A$, and the *total Burgers vector* $\mathbf{B}\llcorner A$ restricted to A as

$$\omega\llcorner A\coloneqq\theta(A)\,,\qquad\mathbf{b}\llcorner A\coloneqq\alpha(A)\,,\qquad\mathbf{B}\llcorner A\coloneqq\mathbf{b}\llcorner A-\int_A(-x_2;x_1)\,\mathrm{d}\theta\,.$$

We notice that in [35, 37], the Frank angle is indeed a rotation vector $\mathbf{\Omega}\llcorner A$, which in our plane elasticity setting is the vector perpendicular to the cross section given by $\mathbf{\Omega}\llcorner A=(0;0;\omega\llcorner A)$.

For the purpose of illustration, we notice that if $\operatorname{spt}\theta\subset\Omega\setminus A$, then $\omega\llcorner A=0$ and $\mathbf{B}\llcorner A=\mathbf{b}\llcorner A=\alpha(A)$. Now, if $\operatorname{spt}\alpha\subset\Omega\setminus A$ and $\theta=s\boldsymbol{\delta}_y$ for some $y\in A$, then $\omega\llcorner A=\theta(A)=s$, $\mathbf{b}\llcorner A=0$, and $\mathbf{B}\llcorner A=-s(-y_2;y_1)$. This illustrates the different contributions of dislocations and disclinations to the quantities ω, $\mathbf{b}$, and $\mathbf{B}$ just introduced: dislocations only contribute to the Burgers vector but never to the Frank angle, whereas disclinations contribute both to the Frank angle and to the total Burgers vector.

Finally, supposing for convenience that $\operatorname{spt}\alpha\subset\Omega\setminus A$, if $\theta=s\big(\boldsymbol{\delta}_{y+\frac{h}{2}}-\boldsymbol{\delta}_{y-\frac{h}{2}}\big)$ for some $y,h\in\mathbb{R}^2$ with $y\pm\frac{h}{2}\in A$, we have that

$$\omega\llcorner A=0\qquad\text{and}\qquad\mathbf{B}\llcorner A=-s(-h_2;h_1)\,,$$

which shows that a dipole of opposite disclinations does not contribute to the Frank angle but contributes to the total Burgers vector independently of its center y (see Section 3).

1.3. **Disclinations in terms of the Airy stress function.** In this subsection, we rewrite the incompatibility condition in (1.16) in terms of the Airy stress function v introduced in (1.8). To this purpose, assume that $\alpha\equiv 0$, so that (1.16) coincides with (1.15). Here and henceforth we use the symbols n and t to denote the external unit normal and tangent vectors, respectively, such that $t=n^\perp=(-n_2;n_1)$; in this way, the ordered pair $\{n,t\}$ is a right-handed orthonormal basis of $\mathbb{R}^2$.

Consider $v\colon\Omega\to\mathbb{R}$ and let $\sigma=\sigma[v]=\mathsf{A}(v)$ (see (1.8)) and $\epsilon[v]=\mathbb{C}^{-1}\sigma[v]$ (see (1.6)). Then, formally,

$$\operatorname{curl}\operatorname{Curl}\epsilon[v]\equiv\frac{1-\nu^2}{E}\Delta^2 v\,,\tag{1.18a}$$

$$\mathbb{C}\epsilon[v]\,n\equiv\sigma[v]\,n\equiv(\partial^2_{x_2^2}vn_1-\partial^2_{x_1x_2}vn_2;-\partial^2_{x_1x_2}vn_1+\partial^2_{x_1^2}vn_2)\equiv\nabla^2v\,t\,.\tag{1.18b}$$

As customary in mechanics, we refer to the zero-stress boundary condition $\mathbb{C}\epsilon[v]\,n=0$ on $\partial\Omega$ as *traction-free*. With some abuse of notation, we also name traction-free the same boundary condition measured in terms of the tangential component of the Hessian of the Airy potential, that is $\nabla^2v\,t=0$ on $\partial\Omega$.

If ϵ satisfies the equilibrium equations subject to the incompatibility constraint (1.15) for some $\theta\in\mathscr{W}\mathscr{D}(\Omega)$, namely

$$\begin{cases}\operatorname{curl}\operatorname{Curl}\epsilon=-\theta & \text{in }\Omega\\ \operatorname{Div}\mathbb{C}\epsilon=0 & \text{in }\Omega\\ \mathbb{C}\epsilon\,n=0 & \text{on }\partial\Omega\,,\end{cases}\tag{1.19}$$

then, by (1.10) and (1.18), the Airy stress function v satisfies the system

$$\begin{cases}\dfrac{1-\nu^2}{E}\Delta^2v=-\theta & \text{in }\Omega\\ \nabla^2v\,t=0 & \text{on }\partial\Omega\,.\end{cases}\tag{1.20}$$

Recalling that (1.15) holds in the sense of distributions, the study of the regularity of the fields ϵ and σ in the laboratory setting and of the Airy stress function v must be carried out carefully. The reason is the following: the measure of the elastic incompatibility $\theta\in\mathscr{W}\mathscr{D}(\Omega)$ is an element of the space $H^{-2}(\Omega)$, so that it is natural to expect that $\epsilon,\sigma\in L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and that $v\in H^2(\Omega)$. At this level $\mathbb{C}\epsilon\,n|_{\partial\Omega}$ and $\nabla^2v\,t|_{\partial\Omega}$ make sense only as elements of $H^{-\frac12}(\partial\Omega;\mathbb{R}^2)$, so that the definition of the boundary conditions in (1.19) and (1.20) cannot be intended in a pointwise sense, even when the tangent and normal vectors are defined pointwise.

In Propositions 1.3 and 1.6, we establish the equivalence of problems (1.19) and (1.20) and we show that, under suitable assumptions on the regularity of $\partial\Omega$, the boundary conditions hold in the sense of $H^{\frac12}(\partial\Omega;\mathbb{R}^2)$. To this purpose, we introduce the function $\bar v\in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ defined by

$$\bar v(x):=\begin{cases}\dfrac{E}{1-\nu^2}\dfrac{|x|^2}{16\pi}\log|x|^2 & \text{if }x\neq0\\ 0 & \text{if }x=0\end{cases}\tag{1.21}$$

as the fundamental solution to the equation

$$\frac{1-\nu^2}{E}\Delta^2v=\boldsymbol{\delta}_0\quad\text{in }\mathbb{R}^2\,.\tag{1.22}$$

Given $\theta=\sum_{k=1}^K s^k\boldsymbol{\delta}_{y^k}\in\mathscr{W}\mathscr{D}(\Omega)$, for every $k=1,\dots,K$, we let $v^k(\cdot):=-s^k\bar v(\cdot-y^k)\llcorner\Omega$ and define

$$v^p:=\sum_{k=1}^K v^k\,,\quad \sigma^p:=\sigma^p[v^p]=\mathsf{A}(v^p)=\sum_{k=1}^K\mathsf{A}(v^k),\quad \epsilon^p:=\epsilon^p[v^p]=\mathbb{C}^{-1}\sigma^p[v^p]=\mathbb{C}^{-1}\sigma^p\,,\tag{1.23}$$

which we are going to refer to as the *plastic contributions*. Notice that, by construction, v^p is smooth in $\mathbb{R}^2\setminus\operatorname{spt}\theta$ and hence on $\partial\Omega$ and so are σ^p and ϵ^p.

Recalling (1.21) and (1.22), we see that

$$\frac{1-\nu^2}{E}\Delta^2 v^p = -\theta \qquad \text{in } \Omega\,, \tag{1.24}$$

so that, if v solves the equation in (1.20) and we define the function v^e through the additive decomposition

$$v := v^p + v^e\,, \tag{1.25}$$

then, v^e satisfies

$$\begin{cases} \dfrac{1-\nu^2}{E}\Delta^2 v^e = 0 & \text{in } \Omega \\ \nabla^2 v^e\, t = -\nabla^2 v^p\, t & \text{on } \partial\Omega\,. \end{cases} \tag{1.26}$$

Therefore, by (1.24), we can find a solution v to problem (1.20) if and only if we find a solution to problem (1.26). Similarly, by (1.18a),

$$\operatorname{curl}\operatorname{Curl}\epsilon^p = -\theta \qquad \text{in } \Omega\,, \tag{1.27}$$

so that if ϵ solves the equation in (1.19) and we define the field ϵ^e through the additive decomposition

$$\epsilon := \epsilon^p + \epsilon^e\,, \tag{1.28}$$

then we have $\operatorname{curl}\operatorname{Curl}\epsilon^e = 0$ in Ω and $\mathbb{C}\epsilon^e\, n = -\mathbb{C}\epsilon^p\, n$ on $\partial\Omega$. Therefore, by (1.27), we find a solution ϵ to problem (1.19) if and only if we find a solution to problem

$$\begin{cases} \operatorname{curl}\operatorname{Curl}\epsilon^e = 0 & \text{in } \Omega \\ \operatorname{Div}\mathbb{C}\epsilon^e = 0 & \text{in } \Omega \\ \mathbb{C}\epsilon^e\, n = -\mathbb{C}\epsilon^p\, n & \text{on } \partial\Omega\,, \end{cases} \tag{1.29}$$

where we notice that the second equation above is automatically satisfied by (1.10).

We refer to v^e and ϵ^e as to the *elastic contributions* and we notice that they are compatible fields. Upon noticing that the function $\bar{v}$ is smooth in $\mathbb{R}^2 \setminus \{0\}$ and by requiring that the boundary $\partial\Omega$ be smooth enough, we will see that problems (1.26) and (1.29) admit solutions which are regular enough for the boundary conditions to make sense in $H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$.

We start by proving the following result, which is one implication in the equivalence of problems (1.19) and (1.20).

Proposition 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected, open set, let $\theta \in \mathscr{W}\mathscr{D}(\Omega)$, and let ϵ be a distributional solution to the first two equations in* (1.19)*. If $\partial\Omega$ is of class C^2, then $\epsilon \in L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and $\mathbb{C}\epsilon\, n \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$. As a consequence, a solution ϵ to* (1.19) *is uniquely determined up to a rigid motion.*
Moreover, there exists a function $v \in H^2(\Omega)$ such that $\epsilon = \mathbb{C}^{-1}\mathsf{A}(v)$. Such v is a distributional solution to the first equation in (1.20) *and $\nabla^2 v\, t \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$. As a consequence, a solution v to* (1.20) *is uniquely determined up to an affine function.*

Proof. Observe that, since $\theta \in H^{-2}(\Omega)$, the distributional solution ϵ to the first two equations in (1.19) is indeed in $L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and the incompatibility equation in (1.19) holds in the H^{-2}-sense. Moreover, since the function $\bar{v}$ defined in (1.21) is of class $H^2_{\mathrm{loc}}(\mathbb{R}^2)$, it follows that the matrix-valued functions ϵ^p and σ^p defined in (1.23) are elements of $L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and that equation (1.27) holds in $H^{-2}(\Omega)$. Therefore, the elastic strain ϵ^e defined in (1.28) by difference is itself in $L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and satisfies (1.29).

We define $G\colon H^1(\Omega;\mathbb{R}^2) \to \mathbb{R}$ as

$$G(u) := \int_\Omega \mathbb{C}\nabla^{\mathrm{sym}} u : \nabla^{\mathrm{sym}} u \,\mathrm{d}x + 2\int_\Omega \sigma^p : \nabla^{\mathrm{sym}} u \,\mathrm{d}x\,,$$

and we observe that it is bounded below in $H^1(\Omega;\mathbb{R}^2)$, so that, by applying the direct method of Calculus of Variations, in view of Korn's inequality, it admits a unique minimizer $u^e \in H^1(\Omega;\mathbb{R}^2)$, up to rigid motions. Setting $\epsilon^e := \nabla^{\rm sym} u^e$, we have that ϵ^e satisfies (1.29), which is the Euler-Lagrange equation for G. Notice that, since $\partial\Omega$ is of class C^2 and since ϵ^p is smooth on $\partial\Omega$, by standard regularity results $u^e \in H^2(\Omega;\mathbb{R}^2)$ and hence $\epsilon^e \in H^1(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})$. It follows that $\mathbb{C}\epsilon^e n \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$, so that, in view of (1.28) $\mathbb{C}\epsilon n \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$.

Now we can apply [26, Theorem 5.6-1(a)], and in particular the argument in [26, page 397], which guarantees that a strain field $\epsilon^e \in H^m(\Omega;\mathbb{R}^{2\times2}_{\rm sym})$ admits an Airy stress function $v^e = \mathsf{A}^{-1}(\epsilon^e) \in H^{m+2}(\Omega)$, for every $m \geq 0$. By applying this result with $m = 1$, we obtain that $v^e \in H^3(\Omega)$ and hence $\nabla^2 v^e t \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$; this, together with (1.29) implies that v^e solves (1.26). By taking v^p as in (1.23) and by defining v according to (1.25), we have that $\nabla^2 v\, t \in H^{\frac12}(\partial\Omega;\mathbb{R}^2)$ because v^p is smooth in a neighborhood of $\partial\Omega$. Thanks to (1.18) and (1.24), v solves (1.20). Since affine functions are in the kernel of the Hessian operator ∇^2, the last statement of the theorem follows. □

Remark 1.4. It is easy to check that rigid motions in the laboratory variables correspond to affine functions in the Airy variable.

In order to prove the converse implication of Proposition 1.3, we state the following result, which is an immediate consequence of [48, Theorem 2.20] (applied with $k = 4$, $m = n = p = 2$, and with $f \equiv 0$ and $h_j \in C^\infty$).

Lemma 1.5. *Let $A \subset \mathbb{R}^2$ be a bounded open set with boundary of class C^4 and let $f \in C^\infty(\partial A;\mathbb{R}^2)$. Then, there exists a solution $w \in H^2(A)$ to*

$$\begin{cases} \dfrac{1-\nu^2}{E}\Delta^2 w = 0 & \text{in } A\,,\\ \nabla^2 w\, t = f & \text{on } \partial A\,, \end{cases} \tag{1.30}$$

where the first equation holds in $H^{-2}(A)$ and the second one is meant in $H^{-\frac12}(\partial A;\mathbb{R}^2)$. Moreover, $w \in H^4(A)$ and hence $\nabla^2 w\, t \in H^{\frac32}(\partial A;\mathbb{R}^2)$.

Proposition 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class C^4 and let $\theta \in \mathscr{W}\mathscr{D}(\Omega)$. Then there exists a weak solution $v \in H^2(\Omega)$ to (1.20) and the condition $\nabla^2 v\, t = 0$ on $\partial\Omega$ holds in $H^{\frac32}(\partial\Omega;\mathbb{R}^2)$. Furthermore, the function $\epsilon = \epsilon[v] := \mathbb{C}^{-1}\mathsf{A}(v)$ is a distributional solution to the first two equations in (1.19) and satisfies the boundary condition $\mathbb{C}\epsilon n = 0$ in $H^{\frac32}(\partial\Omega;\mathbb{R}^2)$.*

Proof. Recalling the definition of v^p in (1.23), formula (1.24), and the decomposition in (1.25), it is enough to show that there exists a solution $v^e \in H^4(\Omega)$ to (1.26). Indeed, this follows from Lemma 1.5 applied with $A = \Omega$ and $f = -\nabla^2 v^p t$, since $v^p \in C^\infty_{\rm loc}(\Omega \setminus \operatorname{spt}\theta)$. By taking $\epsilon = \epsilon[v] := \mathbb{C}^{-1}\mathsf{A}(v)$, with $v = v^e + v^p$, we have that $\epsilon \in L^2(\Omega;\mathbb{R}^{2\times2})$ is a weak solution to (1.19) and that the boundary condition $\mathbb{C}\epsilon n = 0$ on $\partial\Omega$ holds in $H^{\frac32}(\partial\Omega;\mathbb{R}^2)$; this last statement follows from (1.28) since $\epsilon^p = \mathbb{C}^{-1}\mathsf{A}(v^p)$ is smooth near the boundary of Ω and $\epsilon^e = \mathbb{C}^{-1}\mathsf{A}(v^e)$ satisfies the condition $\mathbb{C}\epsilon^e n = -\mathbb{C}\epsilon^p n$ on $\partial\Omega$ in $H^{\frac32}(\partial\Omega;\mathbb{R}^2)$, by Lemma 1.5 and (1.18b). □

Now we show how the boundary condition $\nabla^2 v\, t = 0$ on $\partial\Omega$ in (1.19) can be formulated in terms of classical Dirichlet-type boundary conditions.

Proposition 1.7. *Let $A \subset \mathbb{R}^2$ be a bounded open set with boundary of class C^4. Let $\theta \in \mathscr{W}\mathscr{D}(A)$ and let $v \in H^2(A)$ be such that*

$$\frac{1-\nu^2}{E}\Delta^2 v = -\theta \qquad \text{in } A. \tag{1.31}$$

Then, denoting by $\Gamma^0, \Gamma^1, \dots, \Gamma^L$ the connected components of ∂A, we have that

$$\nabla^2 v\, t = 0 \quad \text{on } \partial A \quad \Leftrightarrow \quad v = a^l\,, \quad \partial_n v = \partial_n a^l \quad \text{on } \Gamma^l\,, \quad \text{for every } l = 0, 1, \dots, L\,, \tag{1.32}$$

where $a^0, a^1, \dots, a^L$ are affine functions.

Proof. We start by proving the implication "⇒". Recalling the additive decomposition (1.25), by Lemma 1.5 and since $v^p \in C^\infty_{\rm loc}(A \setminus \operatorname{spt}\theta)$, we have that if $v = v^e + v^p$ satisfies (1.31) and

$$\nabla^2 v\, t = 0 \qquad \text{on } \partial A\,, \tag{1.33}$$

then $v^e \in H^4(A)$. Therefore, by the Rellich–Kondrakov Theorem we also have that $v^e \in C^2(\overline{A})$, so that $v = v^e + v^p$ is of class C^2 in a neighborhood of ∂A. By Proposition A.2, we deduce that the function v has an affine trace on each connected component of ∂A.

Viceversa, assume that v is a solution to (1.31) and satisfies

$$v = a^l\,, \quad \partial_n v = \partial_n a^l \quad \text{on } \Gamma^l\,, \qquad \text{for every } l = 0, 1, \dots, L\,, \tag{1.34}$$

for some affine functions $a^0, a^1, \dots, a^L$. Then, adopting again the additive decomposition (1.25) and recalling (1.24), we have that v^e satisfies

$$\begin{cases} \dfrac{1-\nu^2}{E}\Delta^2 v^e = 0 & \text{in } A \\ v^e = a^l - v^p & \text{on } \Gamma^l, \text{ for every } l = 0, 1, \dots, L \\ \partial_n v^e = \partial_n a^l - \partial_n v^p & \text{on } \Gamma^l, \text{ for every } l = 0, 1, \dots, L\,. \end{cases} \tag{1.35}$$

Therefore, by standard regularity results for higher order problems (see, for instance, [48]), we have that $v^e \in H^4(A)$ and, again by the Rellich-Kondrakov Theorem, $v^e \in C^2(\overline{A})$. It follows that $v = v^e + v^p$ is of class C^2 in a neighborhood of ∂A, and we can apply again Proposition A.2 to deduce that (1.33) holds true. □

2. Finite systems of isolated disclinations

We now study the equilibrium problem for a finite family of isolated disclinations in a body Ω. The natural idea would be to consider the minimum problem for the elastic energy $\mathcal{G}$ defined in (1.9) under the incompatibility constraint (1.20), associated with a measure $\theta \in \mathscr{WD}(\Omega)$; however, this is inconsistent, since one can easily verify that the Euler–Lagrange equation for $\mathcal{G}$ is $\Delta^2 v = 0$.

To overcome this inconsistency, we define a suitable functional which embeds the presence of the disclinations and whose Euler–Lagrange equation is given by (1.20). To this purpose, let $\Omega \subset \mathbb{R}^2$ be a bounded, open, and simply connected set with boundary of class C^4; for every $\theta \in \mathscr{WD}(\Omega)$ let $\mathcal{I}^\theta \colon H^2(\Omega) \to \mathbb{R}$ be the functional defined by

$$\mathcal{I}^\theta(v;\Omega) \coloneqq \mathcal{G}(v;\Omega) + \langle \theta, v\rangle\,, \tag{2.1}$$

and consider the minimum problem

$$\min\left\{\mathcal{I}^\theta(v;\Omega) : v \in H^2(\Omega)\,,\ \nabla^2 v\, t = 0 \text{ on } \partial\Omega\right\}. \tag{2.2}$$

A simple calculation shows that the Euler–Lagrange equation for the functional (2.1), with respect to variations in $H^2_0(\Omega)$, is given by (1.20). By Proposition 1.7, we deduce that the minimum problem in (2.2) is equivalent, up to an affine function, to the minimum problem

$$\min\{\mathcal{I}^\theta(v;\Omega) \,:\, v \in H^2_0(\Omega)\}\,. \tag{2.3}$$

Lemma 2.1. *For every $\theta \in \mathscr{WD}(\Omega)$, the functional $\mathcal{I}^\theta(\cdot;\Omega)$ is strictly convex in $H^2(\Omega)$ and it is bounded below and coercive in $H^2_0(\Omega)$. As a consequence, the minimum problem* (2.3) *has a unique solution.*

Proof. We start by proving that $\mathcal{I}^\theta(\cdot;\Omega)$ is bounded below and coercive in $H^2_0(\Omega)$. To this purpose, we first notice that there exists a constant $C_1 = C_1(\nu, E, \Omega) > 0$ such that for every $v \in H^2_0(\Omega)$

$$\mathcal{G}(v;\Omega) \geq \frac{1}{2}\frac{1-\nu^2}{E}\min\{1-2\nu, 1\}\|\nabla^2 v\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \geq C_1\|v\|^2_{H^2(\Omega)}\,, \tag{2.4}$$

where in the last passage we have used Friedrichs's inequality in $H^2_0(\Omega)$. Notice that the positivity of C_1 is a consequence of (1.2c).

Now, using that $H^2_0(\Omega)$ embeds into $C^0(\Omega)$, we have that there exists a constant $C_2 = C_2(\theta,\Omega)>0$ such that for every $v\in H^2_0(\Omega)$

$$\langle\theta,v\rangle=\sum_{k=1}^K s^k v(x^k)\geq -C_2\|v\|_{H^2(\Omega)}\,. \tag{2.5}$$

By (2.4) and (2.5), we get that for every $v\in H^2_0(\Omega)$

$$\mathcal{I}^\theta(v;\Omega)\geq C_1\|v\|^2_{H^2(\Omega)}-C_2\|v\|_{H^2(\Omega)}\geq -\frac{C_2^2}{4C_1}\,, \tag{2.6}$$

which implies boundedness below and coercivity of $\mathcal{I}^\theta(\cdot;\Omega)$ in $H^2_0(\Omega)$.

Now we show that $\mathcal{G}(\cdot;\Omega)$ is strictly convex in $H^2(\Omega)$, which, together with the linearity of the map $v\mapsto\langle\theta,v\rangle$, implies the strict convexity of $\mathcal{I}^\theta(\cdot;\Omega)$ in $H^2(\Omega)$. To this purpose, let $v,w\in H^2(\Omega)$ with $v\neq w$ and let $\lambda\in(0,1)$; then a simple computation shows that

$$\begin{aligned}\mathcal{G}(\lambda v+(1-\lambda)w;\Omega)&=\lambda\mathcal{G}(v;\Omega)+(1-\lambda)\mathcal{G}(w;\Omega)-\lambda(1-\lambda)\mathcal{G}(v-w;\Omega)\\&<\lambda\mathcal{G}(v;\Omega)+(1-\lambda)\mathcal{G}(w;\Omega)\,,\end{aligned} \tag{2.7}$$

which is the strict convexity condition.

By the direct method of the Calculus of Variations, problem (2.3) has a unique solution. □

Remark 2.2. We highlight that inequality (2.6) shows that $\mathcal{I}^\theta(\cdot;\Omega)$ could be negative. In particular, being $\mathcal{G}$ non-negative, the sign of $\mathcal{I}^\theta$ is determined by the value of the linear contribution $\langle\theta,v\rangle$. It follows that the minimum problem (2.2) and hence (2.3) are non trivial and, as we will see later (see, e.g., (2.11)), the minimum of $\mathcal{I}^\theta(\cdot;\Omega)$ is indeed negative.

Remark 2.3. Notice that the functional $\mathcal{G}^{\frac12}(\cdot;\Omega)$ defines a seminorm on $H^2(\Omega)$ and a norm in $H^2_0(\Omega)$, since $\mathcal{G}(v;\Omega)\equiv\langle v,v\rangle_{\mathcal{G}_\Omega}$ where the product $\langle\cdot,\cdot\rangle_{\mathcal{G}_\Omega}$, defined by

$$\langle v,w\rangle_{\mathcal{G}_\Omega}:=\frac12\frac{1+\nu}{E}\int_\Omega\big(\nabla^2v:\nabla^2w-\nu\Delta v\Delta w\big)\,\mathrm{d}x\,, \tag{2.8}$$

is a bilinear, symmetric, and positive semidefinite form in $H^2(\Omega)$ and positive definite in $H^2_0(\Omega)$. We remark that in $H^2_0(\Omega)$ the norm $\mathcal{G}^{\frac12}(\cdot;\Omega)$ is equivalent to the standard norm $\|\cdot\|_{H^2(\Omega)}$.

In the following lemma, for any given $\xi\in\mathbb{R}^2$ and $R>0$, we compute the minimal value of $\mathcal{I}^\theta(\cdot;B_R(\xi))$ associated with a single disclination located at ξ, corresponding to $\theta=s\boldsymbol{\delta}_\xi$ for some $s\in\mathbb{R}\setminus\{0\}$. The explicit computation is straightforward and is omitted.

Lemma 2.4. *Let $s\in\mathbb{R}\setminus\{0\}$, $\xi\in\mathbb{R}^2$, and $R>0$. The function $v_R\colon\overline{B}_R(\xi)\to\mathbb{R}$ defined by*

$$\begin{aligned}v_R(x):=&-s\bar v(x-\xi)-s\frac{E}{1-\nu^2}\frac{R^2-|x-\xi|^2(1+\log R^2)}{16\pi}\\=&-sR^2\bigg(\bar v\Big(\frac{x-\xi}{R}\Big)+\frac{E}{1-\nu^2}\frac{1}{16\pi}\Big(1-\Big|\frac{x-\xi}{R}\Big|^2\Big)\bigg)\,,\end{aligned}$$

with $\bar v$ as in (1.21), belongs to $H^2(B_R(\xi))\cap C^\infty_{\mathrm{loc}}(B_R(\xi)\setminus\{\xi\})$ and solves

$$\begin{cases}\dfrac{1-\nu^2}{E}\Delta^2v=-s\boldsymbol{\delta}_\xi & \text{in } B_R(\xi)\\ v=\partial_nv=0 & \text{on } \partial B_R(\xi)\,.\end{cases} \tag{2.9}$$

Hence v_R is the only minimizer of problem (2.3) for $\Omega=B_R(\xi)$ and $\theta=s\boldsymbol{\delta}_\xi$. Moreover,

$$\mathcal{G}(v_R;B_R(\xi))=\frac{E}{1-\nu^2}\frac{s^2R^2}{32\pi}\qquad\text{and}\qquad\langle s\boldsymbol{\delta}_\xi,v_R\rangle=-\frac{E}{1-\nu^2}\frac{s^2R^2}{16\pi}\,, \tag{2.10}$$

so that

$$\min_{v\in H^2_0(B_R(\xi))}\mathcal{I}^{s\boldsymbol{\delta}_\xi}(v;B_R(\xi))=\mathcal{I}^{s\boldsymbol{\delta}_\xi}(v_R;B_R(\xi))=-\frac{E}{1-\nu^2}\frac{s^2R^2}{32\pi}\,. \tag{2.11}$$

In view of (1.5) and (1.9), the first equality in (2.10) is the stored elastic energy of a single disclination located at the center of the ball $B_R(\xi)$. Observe that, according to the formulation of the mechanical equilibrium problem in the Airy variable (2.9), the charge contribution displayed in the second equality in (2.10) adds to the total energy functional of the system, but does not correspond to an energy of elastic nature.

3. Dipole of disclinations

In (2.10) we have seen that an isolated disclination in the center of a ball of radius R carries an elastic energy of the order R^2. Here we show that the situation dramatically changes when considering a dipole of disclinations with opposite signs; indeed, when the distance between the disclinations vanishes, a dipole of disclinations behaves like an edge dislocation and its elastic energy is actually of the order $\log R$.

3.1. **Dipole of disclinations in a ball.** For every $h > 0$ let

$$y^{h,\pm} := \pm\frac{h}{2}(1;0) \tag{3.1}$$

and let $\bar{v}_h\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$\bar{v}_h(x) := -s(\bar{v}(x - y^{h,+}) - \bar{v}(x - y^{h,-}))\,, \tag{3.2}$$

where $\bar{v}$ is given in (1.21). By construction, $\bar{v}_h \llcorner B_R(0) \in H^2(B_R(0))$ and

$$\Delta^2 \bar{v}_h = -\theta_h \qquad \text{in } \mathbb{R}^2\,, \tag{3.3}$$

where we have set

$$\theta_h := s\big(\boldsymbol{\delta}_{y^{h,+}} - \boldsymbol{\delta}_{y^{h,-}}\big)\,. \tag{3.4}$$

We start by proving that the H^2 norm of $\bar{v}_h$ in an annulus $A_{r,R}(0) := B_R(0) \setminus \overline{B}_r(0)$ with fixed radii $0 < r < R$ vanishes as $h \to 0$.

Lemma 3.1. *For every $0 < r < R$ there exists a constant $C(r,R)$ such that*

$$\lim_{h\to 0} \frac{1}{h^2}\|\bar{v}_h\|^2_{H^2(A_{r,R}(0))} = C(r,R)s^2\,. \tag{3.5}$$

Proof. Since $\bar{v}_0 \equiv 0$ in $A_{r,R}(0)$, for every $x \in A_{r,R}(0)$, we have that

$$\lim_{h\to 0}\frac{\bar{v}_h(x)}{h} = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0}\bar{v}_h(x) = \frac{E}{1-\nu^2}\frac{s}{8\pi}(x_1\log|x|^2 + x_1) =: \bar{v}'(x)\,. \tag{3.6}$$

Therefore, by the Dominated Convergence Theorem, in order to prove (3.5), it is enough to show that

$$\|\bar{v}'\|_{H^2(A_{r,R}(0))} = C(r,R)\,, \tag{3.7}$$

for some $C(r,R) > 0$. By straightforward computations, we have that

$$\begin{aligned}
\partial_{x_1}\bar{v}'(x) &= \frac{E}{1-\nu^2}\frac{s}{8\pi}\left(\log|x|^2 + 1 + \frac{2x_1^2}{|x|^2}\right),\\
\partial_{x_2}\bar{v}'(x) &= \frac{E}{1-\nu^2}\frac{s}{8\pi}\frac{2x_1x_2}{|x|^2}\,,\\
\partial^2_{x_1^2}\bar{v}'(x) &= \frac{E}{1-\nu^2}\frac{s}{4\pi}\left(\frac{x_1}{|x|^2} + 2\frac{x_1x_2^2}{|x|^4}\right) = \frac{E}{1-\nu^2}\frac{s}{4\pi}\frac{1}{|x|^4}(x_1^3 + 3x_1x_2^2)\,,\\
\partial^2_{x_2^2}\bar{v}'(x) &= \frac{E}{1-\nu^2}\frac{s}{4\pi}\frac{1}{|x|^4}(x_1^3 - x_1x_2^2)\,,\\
\partial^2_{x_1x_2}\bar{v}'(x) &= \frac{E}{1-\nu^2}\frac{s}{4\pi}\frac{1}{|x|^4}(x_2^3 - x_1^2x_2)\,.
\end{aligned} \tag{3.8}$$

Therefore, by (3.6) and (3.8) we deduce that

$$
\begin{aligned}
|\bar{v}'(x)|^2 =& \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{64\pi^2}x_1^2(\log|x|^2+1)^2\,,\\
|\nabla\bar{v}'(x)|^2 =& \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{64\pi^2}\bigg(\Big(\log|x|^2+1\Big)^2+4\log|x|^2\frac{x_1^2}{|x|^2}+8\frac{x_1^2}{|x|^2}\bigg)\,,\\
|\nabla^2\bar{v}'(x)|^2 =& \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{8\pi^2}\frac{1}{|x|^2}\,,
\end{aligned}
$$

which, integrating over $A_{r,R}(0)$ yields (3.7) and, in turn, (3.5). □

The next lemma is devoted to the asymptotic behavior of the elastic energy of $\bar{v}_h$ as $h\to 0$. Its proof is contained in Appendix B.

Lemma 3.2. *For every $R>0$*

$$
\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{G}(\bar{v}_h;B_R(0))=\frac{E}{1-\nu^2}\frac{s^2}{8\pi}\,. \tag{3.9}
$$

The next proposition shows that the same behavior in (3.9) persists when replacing $\bar{v}_h$ with the minimizer v_h of $\mathcal{I}^{\theta_h}(\cdot;B_R(0))$ in $H^2_0(B_R(0))$, for θ_h given by (3.4).

Proposition 3.3. *For every $0<h<R$, let v_h be the minimizer of $\mathcal{I}^{\theta_h}(\cdot;B_R(0))$ in $H^2_0(B_R(0))$. Then,*

$$
\lim_{h\to 0}\frac{1}{h^2|\log h|}\mathcal{G}(v_h;B_R(0))=\frac{E}{1-\nu^2}\frac{s^2}{8\pi} \tag{3.10}
$$

and

$$
\lim_{h\to 0}\frac{1}{h^2|\log h|}\mathcal{I}^{\theta_h}(v_h;B_R(0))=-\frac{E}{1-\nu^2}\frac{s^2}{8\pi}\,. \tag{3.11}
$$

Proof. We start by noticing that, for every $0<h<R$, the minimizer v_h of $\mathcal{I}^{\theta_h}(\cdot;B_R(0))$ in $H^2_0(B_R(0))$ is unique by Lemma 2.1. Let $w_h\in H^2(B_R(0))$ be defined by the formula $w_h:=v_h-\bar{v}_h\llcorner B_R(0)$, where $\bar{v}_h$ is defined in (3.2). Then, by (3.3), we have that w_h is the unique solution to

$$
\begin{cases}
\Delta^2 w=0 & \text{in } B_R(0)\\
w=-\bar{v}_h & \text{on } \partial B_R(0)\\
\partial_n w=-\partial_n\bar{v}_h & \text{on } \partial B_R(0)\,.
\end{cases} \tag{3.12}
$$

By [48, Theorem 2.16], we have that there exists a constant $C=C(R)>0$ such that

$$
\|w_h\|_{H^2(B_R(0))}\le C\|\bar{v}_h\|_{C^2(\partial B_R(0))}\le C\|\bar{v}_h\|_{H^2(A_{r,R}(0))}\,, \tag{3.13}
$$

where $0<r<R$ is fixed.

By (3.13) and Lemma 3.1 for h small enough we get

$$
\|w_h\|^2_{H^2(B_R(0))}\le C\|\bar{v}_h\|^2_{H^2(A_{r,R}(0))}\le C(r,R)s^2h^2\,, \tag{3.14}
$$

which, together with Lemma 3.2, recalling the definition of $\langle\cdot;\cdot\rangle_{\mathcal{G}_{B_R(0)}}$ in (2.8), yields

$$
\begin{aligned}
\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{G}(v_h;B_R(0)) =& \lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{G}(\bar{v}_h;B_R(0))+\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{G}(w_h;B_R(0))\\
&-2s\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\langle\bar{v}_h;w_h\rangle_{\mathcal{G}_{B_R(0)}}=\frac{E}{1-\nu^2}\frac{s^2}{8\pi}\,,
\end{aligned}
$$

i.e., (3.10). Finally, since

$$
\langle\theta_h,v_h\rangle=\langle\theta_h,\bar{v}_h\rangle+\langle\theta_h,w_h\rangle=-\frac{E}{1-\nu^2}\frac{s^2}{4\pi}h^2|\log h|+w_h(y^{h,+})-w_h(y^{h,-})\,,
$$

using that $w_h \in C^\infty(B_R(0))$ and (3.14), we get

$$\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\langle\theta_h, v_h\rangle = -\frac{E}{1-\nu^2}\frac{s^2}{4\pi} + \lim_{h\to 0}\frac{1}{h|\log h|}\partial_{x_1}w_h(0) = -\frac{E}{1-\nu^2}\frac{s^2}{4\pi}\,,$$

which, added to (3.10), yields (3.11). □

3.2. **Core-radius approach for a dipole of disclinations.** We discuss the convergence of a wedge disclination dipole to a planar edge dislocation. We remind that the kinematic equivalence of a dipole of wedge disclinations with an edge dislocation has been first pointed out in [40] with a geometric construction in a continuum (see [100] for a construction on the hexagonal lattice).

Let $s>0$, $R>0$, $h\in(0,R)$, and let $\theta_h \coloneqq s\boldsymbol{\delta}_{(\frac{h}{2};0)} - s\boldsymbol{\delta}_{(-\frac{h}{2};0)}$. Moreover, let $v_h \in H^2(B_R(0))$ satisfy

$$\begin{cases}\Delta^2 v_h = -\theta_h & \text{in } B_R(0)\\ v_h = \partial_n v_h = 0 & \text{on } \partial B_R(0)\,.\end{cases} \tag{3.15}$$

Then, since $\frac{\theta_h}{h}\to -s\partial_{x_1}\boldsymbol{\delta}_0$ as $h\to 0$, we expect that, formally, $\frac{v_h}{h}\to v$, where v satisfies

$$\begin{cases}\Delta^2 v = s\partial_{x_1}\boldsymbol{\delta}_0 & \text{in } B_R(0)\\ v = \partial_n v = 0 & \text{on } \partial B_R(0)\,,\end{cases} \tag{3.16}$$

namely, v is the Airy function associated with the elastic stress field of an edge dislocation centered at the origin and with Burgers vector $b = se_2$, see (1.17). Notice that the resulting Burgers vector is orthogonal to the direction of the disclination dipole d (directed from the negative to the positive charge), more precisely we can write $\frac{b}{|b|} = \frac{d^\perp}{|d|}$ (see [40] and also [39, formula (7.17)] and [101, formula (7)]).

The convergence of the right-hand side of (3.15) to the right-hand side of (3.16) represents the kinematic equivalence between an edge dislocation and a wedge disclination dipole, obtained in the limit as the dipole distance h tends to zero. We now focus our attention on the investigation of the energetic equivalence of these defects, which we pursue by analyzing rigorously the convergence of the solutions of (3.15) to those of (3.16).

As this analysis entails singular energies, we introduce regularized functionals parameterized by $0<\varepsilon<R$, representing the core radius. To this purpose, we define

$$\mathscr{B}_{\varepsilon,R} \coloneqq \big\{w\in H^2_0(B_R(0)) : w = a \text{ in } B_\varepsilon(0) \text{ for some affine function } a\big\} \tag{3.17}$$

and, recalling (2.1), we introduce, for $h<\varepsilon$, the functional $\widetilde{\mathcal{J}}^s_{h,\varepsilon}\colon \mathscr{B}_{\varepsilon,R}\to\mathbb{R}$ defined by

$$\widetilde{\mathcal{J}}^s_{h,\varepsilon}(w^h) \coloneqq \mathcal{G}(w^h; B_R(0)) + \frac{s}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\Big[w^h\Big(x+\frac{h}{2}e_1\Big) - w^h\Big(x-\frac{h}{2}e_1\Big)\Big]\,\mathrm{d}\mathcal{H}^1(x)\,,$$

associated with a pair of disclinations of opposite charges $\pm s$ placed at $\pm(\frac{h}{2},0)$, respectively. We identify the relevant rescaling for the Airy stress function w^h, parametrized by the dipole distance h, and corresponding to the energy regime of interest. We stress that the energy scalings are dictated by the scaling of w^h and not from *a priori* assumptions. Consequently, we assume $w^h = hw$ and write

$$\widetilde{\mathcal{J}}^s_{h,\varepsilon}(hw) = \mathcal{G}(hw; B_R(0)) + \frac{s}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\Big[hw\Big(x+\frac{h}{2}e_1\Big) - hw\Big(x-\frac{h}{2}e_1\Big)\Big]\,\mathrm{d}\mathcal{H}^1(x)\,. \tag{3.18}$$

It follows that the regularized energy of a disclination dipole of finite charge s is of order $\mathrm{O}(h^2)$. In order to isolate the first non-zero contribution in the limit as $h\to 0$, we divide (3.18) by h^2 and we define $\mathcal{J}^s_{h,\varepsilon}\colon\mathscr{B}_{\varepsilon,R}\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{J}^s_{h,\varepsilon}(w) &\coloneqq \frac{1}{h^2}\widetilde{\mathcal{J}}^s_{h,\varepsilon}(hw)\\ &= \mathcal{G}(w; B_R(0)) + \frac{s}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\frac{w(x+\frac{h}{2}e_1) - w(x-\frac{h}{2}e_1)}{h}\,\mathrm{d}\mathcal{H}^1(x)\,.\end{aligned} \tag{3.19}$$

We show that the minimizers of $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$ converge, as $h\to 0$, to the minimizers in $\mathscr{B}_{\varepsilon,R}$ of the functional $\mathcal{J}^s_{0,\varepsilon}\colon \mathscr{B}_{\varepsilon,R}\to\mathbb{R}$ defined by

$$\mathcal{J}^s_{0,\varepsilon}(w):=\mathcal{G}(w;B_R(0))+\frac{s}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}w\,\mathrm{d}\mathcal{H}^1\,. \tag{3.20}$$

Notice that, by the very definition of $\mathscr{B}_{\varepsilon,R}$ in (3.17),

$$\mathcal{J}^s_{0,\varepsilon}(w)=\mathcal{G}(w;A_{\varepsilon,R}(0))+\frac{s}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}w\,\mathrm{d}\mathcal{H}^1\,. \tag{3.21}$$

We start by showing existence and uniqueness of the minimizers of $\mathcal{J}^s_{h,\varepsilon}$ and $\mathcal{J}^s_{0,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$.

Lemma 3.4. *Let $s\in\mathbb{R}\setminus\{0\}$. For every $0\le h<\varepsilon<R$ there exists a unique minimizer of $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$.*

Proof. The proof relies on the direct method in the Calculus of Variations. We preliminarily notice that the uniqueness of the minimizers follows by the strict convexity of $\mathcal{J}^s_{h,\varepsilon}$ for $h\ge 0$ (see (2.7)).

Let $\{W_{h,\varepsilon,j}\}_{j\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$; for $h>0$, since $W_{h,\varepsilon,j}$ is affine in $B_\varepsilon(0)$ for any $j\in\mathbb{N}$, for any $x\in\partial B_{\varepsilon-h}(0)$ we have that

$$\begin{aligned}\left|\frac{W_{h,\varepsilon,j}\big(x+\frac{h}{2}e_1\big)-W_{h,\varepsilon,j}\big(x-\frac{h}{2}e_1\big)}{h}\right|&=|\partial_{x_1}W_{h,\varepsilon,j}(x)|\le\|\partial_{x_1}W_{h,\varepsilon,j}\|_{L^\infty(B_\varepsilon(0))}\\&\le\frac{1}{\sqrt{\pi}\varepsilon}\|W_{h,\varepsilon,j}\|_{H^2(B_R(0))}\,,\end{aligned}$$

and we notice that the last inequality also holds true for $h=0$. Hence, since the zero function $w=0$ belongs to $\mathscr{B}_{\varepsilon,R}$, by using Friedrich's inequality in $H^2_0(B_R(0))$, we get, for j large enough,

$$\begin{aligned}0=\mathcal{J}^s_{h,\varepsilon}(0)&\ge\mathcal{J}^s_{h,\varepsilon}(W_{h,\varepsilon,j})\\&\ge\frac12\frac{1-\nu^2}{E}\min\{1-2\nu,1\}\|\nabla^2W_{h,\varepsilon,j}\|^2_{L^2(B_R(0);\mathbb{R}^{2\times2})}-\frac{s}{\sqrt{\pi}\varepsilon}\|W_{h,\varepsilon,j}\|_{H^2(B_R(0))}\\&\ge C\|W_{h,\varepsilon,j}\|^2_{H^2(B_R(0))}-\frac{s}{\sqrt{\pi}\varepsilon}\|W_{h,\varepsilon,j}\|_{H^2(B_R(0))}\,,\end{aligned}\tag{3.22}$$

for some constant $C>0$ depending only on R (other than on E and ν). By (3.22), we deduce that $\|W_{h,\varepsilon,j}\|^2_{H^2(B_R(0))}$ is uniformly bounded. It follows that, up to a subsequence, $W_{h,\varepsilon,j}\rightharpoonup W_{h,\varepsilon}$ (as $j\to\infty$) in $H^2(B_R(0))$ for some function $W_{h,\varepsilon}\in H^2_0(B_R(0))$ that is affine in $B_\varepsilon(0)$. By the lower semicontinuity of $\mathcal{J}^s_{h,\varepsilon}$ with respect to the weak H^2-convergence, we get that $W_{h,\varepsilon}$ is a minimizer of $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$. □

We are now in a position to prove the convergence of the minimizers and of the minimal values of $\mathcal{J}^s_{h,\varepsilon}$ to $\mathcal{J}^s_{0,\varepsilon}$ as $h\to 0$.

Proposition 3.5. *Let $s\in\mathbb{R}\setminus\{0\}$. Let $0<\varepsilon<R$ and, for every $0<h<\varepsilon$, let $W^s_{h,\varepsilon}$ be the minimizer of $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$. Then, as $h\to 0$, $W^s_{h,\varepsilon}\to W^s_{0,\varepsilon}$ strongly in $H^2(B_R(0))$, where $w^s_{0,\varepsilon}$ is the minimizer of $\mathcal{J}^s_{0,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$. Moreover, $\mathcal{J}^s_{h,\varepsilon}(W^s_{h,\varepsilon})\to\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})$ as $h\to 0$.*

Proof. For every $0<h<\varepsilon$ let $a^s_{h,\varepsilon}(x):=c^s_{h,\varepsilon,0}+c^s_{h,\varepsilon,1}x_1+c^s_{h,\varepsilon,2}x_2$ with $c^s_{h,\varepsilon,0},c^s_{h,\varepsilon,1},c^s_{h,\varepsilon,2}\in\mathbb{R}$ be such that $W^s_{h,\varepsilon}=a^s_{h,\varepsilon}$ in $B_\varepsilon(0)$. Then, arguing as in (3.22), we get

$$0\ge\mathcal{J}^s_{h,\varepsilon}(W^s_{h,\varepsilon})\ge C\|W^s_{h,\varepsilon}\|^2_{H^2(B_R(0))}-\frac{s}{\sqrt{\pi}\varepsilon}\|W^s_{h,\varepsilon}\|_{H^2(B_R(0))}\,.$$

Therefore, up to a (not relabeled) subsequence, $W^s_{h,\varepsilon}\rightharpoonup\bar W^s_{0,\varepsilon}$ in $H^2(B_R(0))$ for some $\bar W^s_{0,\varepsilon}\in H^2_0(B_R(0))$. Moreover, since the functions $W^s_{h,\varepsilon}$ are affine in $\overline{B}_\varepsilon(0)$, also $\bar W^s_{0,\varepsilon}$ is, and hence there exist $c^s_{0,\varepsilon,0},c^s_{0,\varepsilon,1},c^s_{0,\varepsilon,2}\in\mathbb{R}$ such that $\bar W^s_{0,\varepsilon}(x)=c^s_{0,\varepsilon,0}+c^s_{0,\varepsilon,1}x_1+c^s_{0,\varepsilon,2}x_2$ for every $x\in\overline{B}_\varepsilon(0)$. It follows that $\bar W^s_{0,\varepsilon}\in\mathscr{B}_{\varepsilon,R}$.

Now, since $W^s_{h,\varepsilon} \to \bar W^s_{0,\varepsilon}$ in $H^1(B_R(0))$, we get that $c^s_{h,\varepsilon,j} \to c^s_{0,\varepsilon,j}$ as $h \to 0$, for every $j = 1,2,3$, which implies, in particular, that

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\frac{W^s_{h,\varepsilon}(x+\frac{h}{2}e_1)-W^s_{h,\varepsilon}(x-\frac{h}{2}e_1)}{h}\,\mathrm{d}\mathcal{H}^1(x)=\lim_{h\to 0}c^s_{h,\varepsilon,1}=c^s_{0,\varepsilon,1}\\
&=\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}\bar W^s_{0,\varepsilon}\,\mathrm{d}\mathcal{H}^1 \qquad (3.23)\\
&=\lim_{h\to 0}\frac{1}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\frac{\bar W^s_{0,\varepsilon}(x+\frac{h}{2}e_1)-\bar W^s_{0,\varepsilon}(x-\frac{h}{2}e_1)}{h}\,\mathrm{d}\mathcal{H}^1(x)\,.
\end{aligned}$$

Analogously,
(3.24)
$$\lim_{h\to 0}\frac{1}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\frac{W^s_{0,\varepsilon}(x+\frac{h}{2}e_1)-W^s_{0,\varepsilon}(x-\frac{h}{2}e_1)}{h}\,\mathrm{d}\mathcal{H}^1(x)=\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}W^s_{0,\varepsilon}\,\mathrm{d}\mathcal{H}^1\,.$$

By (3.23) and (3.24), using the lower semicontinuity of $\mathcal{G}$, and taking $W^s_{0,\varepsilon}$ as a competitor for $\mathcal{J}^s_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$, we get

$$\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})\le\mathcal{J}^s_{0,\varepsilon}(\bar W^s_{0,\varepsilon})\le\liminf_{h\to 0}\mathcal{J}^s_{h,\varepsilon}(W^s_{h,\varepsilon})\le\lim_{h\to 0}\mathcal{J}^s_{h,\varepsilon}(W^s_{0,\varepsilon})=\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})\,,$$

so that all the inequalities above are in fact equalities. In particular,

$$\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})=\lim_{h\to 0}\mathcal{J}^s_{h,\varepsilon}(W^s_{h,\varepsilon}) \tag{3.25}$$

and consequently $\bar W^s_{0,\varepsilon}$ is a minimizer of $\mathcal{J}^s_{0,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$. In view of Lemma 3.4, we deduce that $\bar W^s_{0,\varepsilon} = W^s_{0,\varepsilon}$, which, together with (3.23) and (3.25), implies that $\mathcal{G}(W^s_{h,\varepsilon}; B_R(0)) \to \mathcal{G}(W^s_{0,\varepsilon}; B_R(0))$ as $h \to 0$. In view of Remark 2.3, this implies that $W^s_{h,\varepsilon} \to W^s_{0,\varepsilon}$ strongly in $H^2(B_R(0))$ as $h \to 0$. Finally, by the Urysohn property, we get that the whole family $\{W^s_{h,\varepsilon}\}_h$ converges to $W^s_{0,\varepsilon}$ as $h \to 0$. □

We conclude this section by determining the minimizer $w^s_{0,\varepsilon}$ of $\mathcal{J}^s_{0,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$.

Lemma 3.6. *Let $s \in \mathbb{R}\setminus\{0\}$. For every $0<\varepsilon<R$ the function $W^s_{0,\varepsilon} : B_R(0) \to \mathbb{R}$ defined by*

$$W^s_{0,\varepsilon}(x):=\begin{cases}\dfrac{s}{16\pi}\dfrac{E}{1-\nu^2}\Big(\alpha_\varepsilon+\beta_\varepsilon\dfrac{1}{|x|^2}+\gamma_\varepsilon|x|^2+2\log|x|^2\Big)x_1 & \text{if } x\in A_{\varepsilon,R}(0)\\ \dfrac{s}{16\pi}\dfrac{E}{1-\nu^2}\Big(\alpha_\varepsilon+\dfrac{\beta_\varepsilon}{\varepsilon^2}+\varepsilon^2\gamma_\varepsilon+4\log\varepsilon\Big)x_1 & \text{if } x\in B_\varepsilon(0)\,,\end{cases} \tag{3.26}$$

with

$$\alpha_\varepsilon:=2\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}-2\log R^2\,,\quad \beta_\varepsilon:=2\varepsilon^2\frac{R^2}{R^2+\varepsilon^2}\,,\quad \gamma_\varepsilon:=-\frac{2}{R^2+\varepsilon^2}\,, \tag{3.27}$$

is the unique minimizer in $\mathscr{B}_{\varepsilon,R}$ of the functional $\mathcal{J}^s_{0,\varepsilon}$ defined in (3.20)*. Moreover,*

$$\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})=-\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)\,. \tag{3.28}$$

The proof of Lemma 3.6 is postponed to Appendix C, where we also state Corollary C.1, which will be used in Section 4.

Remark 3.7. Let $b\in\mathbb{R}^2\setminus\{0\}$. For any $0<h<\varepsilon<R$ let $\mathcal{J}^b_{h,\varepsilon}\colon \mathscr{B}_{\varepsilon,R}\to\mathbb{R}$ be the functional defined as

$$\mathcal{J}^b_{h,\varepsilon}(w):=\mathcal{G}(w;B_R(0))+\frac{|b|}{2\pi(\varepsilon-h)}\int_{\partial B_{\varepsilon-h}(0)}\frac{w\big(x+\frac{h}{2}\frac{\Pi(b)}{|b|}\big)-w\big(x-\frac{h}{2}\frac{\Pi(b)}{|b|}\big)}{h}\,\mathrm{d}\mathcal{H}^1(x)\,,$$

where $\Pi(b)$ denotes the $\frac{\pi}{2}$ clockwise rotation of the vector b, *i.e.*,

$$\Pi(b)=-b^\perp\,. \tag{3.29}$$

By arguing verbatim as in the proof of Proposition 3.5, we have that, as $h \to 0$, the unique minimizer of $\mathcal{J}^b_{h,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$ converges strongly in $H^2(B_R(0))$ to the unique minimizer in $\mathscr{B}_{\varepsilon,R}$ of the functional $\mathcal{J}^b_{0,\varepsilon}$ defined by

$$\mathcal{J}^b_{0,\varepsilon}(w) := \mathcal{G}(w; B_R(0)) + \frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)} \langle \nabla w, \Pi(b)\rangle \, \mathrm{d}\mathcal{H}^1 .$$

Notice that the minimizer of $\mathcal{J}^b_{0,\varepsilon}$ is given by

$$W^b_{0,\varepsilon}(x) := |b| W^{|b|}_{0,\varepsilon}\Big(\Big\langle \frac{\Pi(b)}{|b|}, x\Big\rangle, \Big\langle \frac{b}{|b|}, x\Big\rangle\Big) , \tag{3.30}$$

where the function $W^s_{0,\varepsilon}$ is defined in Lemma 3.6.

Furthermore, one can easily check that the same proof of Proposition 3.5 applies also to general domains Ω as well as to a general distribution of dipoles of wedge disclinations

$$\theta_h := \sum_{j=1}^J |b^j| \Big(\boldsymbol{\delta}_{x^j + \frac{h}{2}\frac{\Pi(b^j)}{|b^j|}} - \boldsymbol{\delta}_{x^j - \frac{h}{2}\frac{\Pi(b^j)}{|b^j|}} \Big) \in \mathscr{W}\mathscr{D}(\Omega) , \tag{3.31}$$

(with $b^j \in \mathbb{R}^2 \setminus \{0\}$ and $\min_{\substack{j_1,j_2=1,\dots,J \\ j_1 \neq j_2}} |x^{j_1} - x^{j_2}|, \min_{j=1,\dots,J} \mathrm{dist}(x^j, \partial\Omega) > 2\varepsilon$) approximating the family of edge dislocations $\alpha := \sum_{i=1}^J b^j \boldsymbol{\delta}_{x^j} \in \mathscr{E}\mathscr{D}(\Omega)$. In such a case, one can show that, as $h \to 0$, the unique minimizer $w^{\theta_h}_{h,\varepsilon}$ of the functional

$$\mathcal{I}^{\theta_h}_{h,\varepsilon}(w) := \mathcal{G}(w; \Omega) + \sum_{j=1}^J \frac{|b^j|}{2\pi(\varepsilon - h)} \int_{\partial B_{\varepsilon - h}(x^j)} \frac{w(x + \frac{h}{2}\frac{\Pi(b^j)}{|b^j|}) - w(x - \frac{h}{2}\frac{\Pi(b^j)}{|b^j|})}{h} \, \mathrm{d}\mathcal{H}^1(x) \tag{3.32}$$

in the set

$$\mathscr{B}^\alpha_{\varepsilon,\Omega} := \{ w \in H^2_0(\Omega) : w = a^j \text{ in } B_\varepsilon(x^j) \text{ for some affine functions } a^j ,\, j = 1, \dots, J\} , \tag{3.33}$$

converges strongly in $H^2(\Omega)$ to the unique minimizer $w^\alpha_{0,\varepsilon}$ in $\mathscr{B}^\alpha_{\varepsilon,\Omega}$ of the functional

$$\begin{aligned} \mathcal{I}^\alpha_{0,\varepsilon}(w) &:= \mathcal{G}(w; \Omega) + \sum_{j=1}^J \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(x^j)} \langle \nabla w, \Pi(b^j)\rangle \, \mathrm{d}\mathcal{H}^1 \\ &= \mathcal{G}(w; \Omega_\varepsilon(\alpha)) + \sum_{j=1}^J \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(x^j)} \langle \nabla w, \Pi(b^j)\rangle \, \mathrm{d}\mathcal{H}^1 , \end{aligned} \tag{3.34}$$

where $\Omega_\varepsilon(\alpha) := \Omega \setminus \bigcup_{j=1}^J \overline{B}_\varepsilon(x^j)$.

4. Limits for dislocations

In this section, we obtain the full asymptotic expansion in ε of the singular limit functional $\mathcal{I}^\alpha_{0,\varepsilon}$ introduced in (3.34). We first prove the convergence of the minimizers of $\mathcal{I}^\alpha_{0,\varepsilon}$ in a suitable functional setting (see Theorem 4.3) and then, by showing that all terms of the expansion coincide with the corresponding terms of the renormalized energy of edge dislocations of [20], we finally deduce the asymptotic energetic equivalence of systems of disclination dipoles with the corresponding systems of edge dislocations.

Let $\alpha = \sum_{j=1}^J b^j \boldsymbol{\delta}_{x^j} \in \mathscr{E}\mathscr{D}(\Omega)$. We consider the following minimum problem

$$\min_{w \in \mathscr{B}^\alpha_{\varepsilon,\Omega}} \mathcal{I}^\alpha_\varepsilon(w) , \tag{4.1}$$

where $\mathcal{I}^\alpha_\varepsilon(w) := \mathcal{I}^\alpha_{0,\varepsilon}(w)$ is the functional defined in (3.34) and $\mathscr{B}^\alpha_{\varepsilon,\Omega}$ is defined in (3.33). In order to study the asymptotic behavior of the minimizers and minima of $\mathcal{I}^\alpha_\varepsilon$ as $\varepsilon \to 0$, we first introduce some notation.

Fix $R>0$ such that $\overline{\Omega}\subset B_R(x^j)$ for every $j=1,\dots,J$, and let $\varepsilon>0$ be such that the (closed) balls $\overline{B}_\varepsilon(x^j)$ are pairwise disjoint and contained in Ω, i.e.,

$$\varepsilon < D \coloneqq \min_{j=1,\dots,J}\left\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x^i,x^j)\,,\mathrm{dist}(x^j,\partial\Omega)\right\}. \tag{4.2}$$

We define the function $W^\alpha_\varepsilon\colon\Omega_\varepsilon(\alpha)\to\mathbb{R}$ by

$$W^\alpha_\varepsilon(x)\coloneqq\sum_{j=1}^J W^j_\varepsilon(x)\,,\qquad\text{with}\qquad W^j_\varepsilon(\cdot)\coloneqq W^{b^j}_{0,\varepsilon}(\cdot-x^j) \tag{4.3}$$

(see (3.30)). We highlight that the function W^α_ε depends also on R through the constants defined in (3.27). Notice that any function $w\in\mathscr{B}^\alpha_{\varepsilon,\Omega}$ can be decomposed as

$$w=W^\alpha_\varepsilon+\widetilde{w} \tag{4.4}$$

where $\widetilde{w}\in\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$, with

$$\begin{aligned}\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}\coloneqq\{\widetilde{w}\in H^2_0(\Omega)-W^\alpha_\varepsilon:\widetilde{w}+W^\alpha_\varepsilon=a^j\text{ in }B_\varepsilon(x^j)&\\ \text{for some affine functions }a^j\,,\,j=1,\dots,J\}&\\ \equiv\mathscr{B}^\alpha_{\varepsilon,\Omega}-W^\alpha_\varepsilon\,.&\end{aligned} \tag{4.5}$$

Therefore, in view of the decomposition (4.4), for every $w\in\mathscr{B}^\alpha_{\varepsilon,\Omega}$ we have

$$\mathcal{I}^\alpha_\varepsilon(w)=\mathcal{G}(W^\alpha_\varepsilon;\Omega_\varepsilon(\alpha))+\sum_{j=1}^J\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W^\alpha_\varepsilon,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1+\widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w})\,, \tag{4.6}$$

where

$$\begin{aligned}\widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w})\coloneqq\mathcal{G}(\widetilde{w};\Omega_\varepsilon(\alpha))+&\frac{1+\nu}{E}\sum_{j=1}^J\int_{\Omega_\varepsilon(\alpha)}\Big(\nabla^2W^j_\varepsilon:\nabla^2\widetilde{w}-\nu\Delta W^j_\varepsilon\Delta\widetilde{w}\Big)\,\mathrm{d}x\\ &+\sum_{j=1}^J\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla\widetilde{w},\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\,.\end{aligned} \tag{4.7}$$

Notice that the integration for the bulk term $\mathcal{G}$ above is performed on $\Omega_\varepsilon(\alpha)$ and not on Ω, as the function $\widetilde{w}$ is not, in general, affine in $\bigcup_{j=1}^J B_\varepsilon(x^j)$.

In view of (4.6), as in [20, Theorem 4.1], the minimum problem (4.1) (for w) is equivalent to the following minimum problem (for $\widetilde{w}$)

$$\min_{\widetilde{w}\in\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}}\widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w})\,. \tag{4.8}$$

Lemma 4.1. *For every $\widetilde{w}\in\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ we have*

$$\begin{aligned}\widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w})=&\,\mathcal{G}(\widetilde{w};\Omega_\varepsilon(\alpha))+\frac{1+\nu}{E}\sum_{j=1}^J\bigg(-(1-\nu)\int_{\partial\Omega}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\ &+\int_{\partial\Omega}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1-\nu\int_{\partial\Omega}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\bigg)\\ &+\sum_{j=1}^J\bigg(\frac{1+\nu}{E}\sum_{i=1}^J\bigg((1-\nu)\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1-\int_{\partial B_\varepsilon(x^i)}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1\\ &+\nu\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\bigg)+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla\widetilde{w},\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\bigg).\end{aligned} \tag{4.9}$$

Proof. Let $\widetilde{w}\in\widetilde{\mathscr{B}}^{\alpha}_{\varepsilon,\Omega}$ be fixed. By the Gauss–Green Theorem, for every $j=1,\dots,J$ and for every $0<\varepsilon<D$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon(\alpha)}\nabla^2W^j_\varepsilon:\nabla^2\widetilde{w}\,\mathrm{d}x=&-\int_{\partial\Omega}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1+\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\&+\int_{\partial\Omega}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1-\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1,\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}\int_{\Omega_\varepsilon(\alpha)}\Delta W^j_\varepsilon\Delta\widetilde{w}\,\mathrm{d}x=&-\int_{\partial\Omega}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1+\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\&+\int_{\partial\Omega}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1-\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1,\end{aligned}\tag{4.11}$$

where we have used that $\Delta^2W^j_\varepsilon\equiv0$ in $\Omega_\varepsilon(\alpha)$ for every $j=1,\dots,J$. By (4.10) and (4.11) it follows that

$$\begin{aligned}&\int_{\Omega_\varepsilon(\alpha)}\Big(\nabla^2W^j_\varepsilon:\nabla^2\widetilde{w}-\nu\Delta W^j_\varepsilon\Delta\widetilde{w}\Big)\,\mathrm{d}x\\=&-(1-\nu)\int_{\partial\Omega}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1+\int_{\partial\Omega}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1-\nu\int_{\partial\Omega}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\&+(1-\nu)\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\&-\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1+\nu\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1,\end{aligned}$$

which, in view of the very definition of $\widetilde{\mathcal{I}}^\alpha_\varepsilon$ in (4.7), implies (4.10). □

Remark 4.2. Let $\alpha=\sum_{j=1}^Jb^j\boldsymbol{\delta}_{x^j}\in\mathscr{ED}(\Omega)$. For every $0<r<R$ and for every $j=1,\dots,J$ we have that the plastic functions W^j_ε converge in $C^\infty(A_{r,R}(x^j))$, as $\varepsilon\to0$, to the function W^j_0 defined by

$$W^j_0(x):=\frac{|b^j|}{8\pi}\frac{E}{1-\nu^2}\Big((1-\log R^2)-\frac{|x|^2}{R^2}+\log|x|^2\Big)\Big\langle\frac{\Pi(b^j)}{|b^j|},x-x^j\Big\rangle.\tag{4.12}$$

It follows that $W^\alpha_\varepsilon\to\sum_{j=1}^JW^j_0=:W^\alpha_0$ in $C^\infty(\Omega_r(\alpha))$ and hence in $H^2_{\mathrm{loc}}\big(\Omega\setminus\bigcup_{j=1}^J\{x^j\}\big)$. Therefore, in the spirit of (4.5) we set

$$\widetilde{\mathscr{B}}^\alpha_{0,\Omega}:=\{w\in H^2(\Omega)\,:\,w=-W^\alpha_0\,,\,\partial_nw=-\partial_nW^\alpha_0\text{ on }\partial\Omega\}.\tag{4.13}$$

Now we prove the following theorem, which is the equivalent of [20, Theorem 4.1] in terms of the Airy stress function.

Theorem 4.3. *Let $\alpha=\sum_{j=1}^Jb^j\boldsymbol{\delta}_{x^j}\in\mathscr{ED}(\Omega)$ and let $\mathcal{I}^\alpha_\varepsilon$ be the functional in (4.6) for every $\varepsilon>0$. For $\varepsilon>0$ small enough, the minimum problem (4.1) admits a unique solution w^α_ε. Moreover, $w^\alpha_\varepsilon\to w^\alpha_0$, as $\varepsilon\to0$, strongly in $H^2_{\mathrm{loc}}(\Omega\setminus\bigcup_{j=1}^J\{x^j\})$, where $w^\alpha_0\in H^2_{\mathrm{loc}}(\Omega\setminus\bigcup_{j=1}^J\{x^j\})$ is the unique distributional solution to*

$$\begin{cases}\dfrac{1-\nu^2}{E}\Delta^2w=-\displaystyle\sum_{j=1}^J|b^j|\partial_{\frac{(b^j)^\perp}{|b^j|}}\boldsymbol{\delta}_{x^j}&\text{in }\Omega\\ w=\partial_nw=0&\text{on }\partial\Omega.\end{cases}\tag{4.14}$$

Theorem 4.3 is a consequence of Propositions 4.4 and 4.5 below, which are the analogue of [20, Lemma 4.2] and [20, Lemma 4.3], respectively.

Proposition 4.4. *Let $\alpha \in \mathscr{E}\mathscr{D}(\Omega)$ and let $\varepsilon > 0$ be small enough. For every $\widetilde{w} \in \widetilde{\mathscr{B}}^{\alpha}_{\varepsilon,\Omega}$ we have*

$$C_1\big(\|\widetilde{w}\|^2_{H^2(\Omega_\varepsilon(\alpha))} - \|\widetilde{w}\|_{H^2(\Omega_\varepsilon(\alpha))} - 1\big) \leq \widetilde{\mathcal{I}}^{\alpha}_{\varepsilon}(\widetilde{w}) \leq C_2\big(\|\widetilde{w}\|^2_{H^2(\Omega_\varepsilon(\alpha))} + \|\widetilde{w}\|_{H^2(\Omega_\varepsilon(\alpha))} + 1\big)\,, \tag{4.15}$$

for some constants $0 < C_1 < C_2$ independent of ε. Moreover, problem (4.8) admits a unique solution $\widetilde{w}^{\alpha}_{\varepsilon} \in \widetilde{\mathscr{B}}^{\alpha}_{\varepsilon,\Omega}$ and $\|\widetilde{w}^{\alpha}_{\varepsilon}\|_{H^2(\Omega_\varepsilon(\alpha))}$ is uniformly bounded with respect to ε. Furthermore, there exists $\widetilde{w}^{\alpha}_{0} \in \widetilde{\mathscr{B}}^{\alpha}_{0,\Omega}$ such that as $\varepsilon \to 0$ and up to a (not relabeled) subsequence,

$$\widetilde{w}^{\alpha}_{\varepsilon} \rightharpoonup \widetilde{w}^{\alpha}_{0} \quad \textit{weakly in } H^2(\Omega). \tag{4.16}$$

Proposition 4.5. *Let $\alpha = \sum_{j=1}^{J} b^j \boldsymbol{\delta}_{x^j} \in \mathscr{E}\mathscr{D}(\Omega)$ and let $\varepsilon > 0$ be small enough. Let $\widetilde{w}^{\alpha}_{\varepsilon}$ and $\widetilde{w}^{\alpha}_{0}$ be as in Proposition 4.4. Then, as $\varepsilon \to 0$, the whole sequence $\widetilde{w}^{\alpha}_{\varepsilon}$ converges to $\widetilde{w}^{\alpha}_{0}$, strongly in $H^2_{\mathrm{loc}}\big(\Omega \setminus \bigcup_{j=1}^{J}\{x^j\}\big)$ and $\widetilde{w}^{\alpha}_{0}$ is the unique minimizer in $\widetilde{\mathscr{B}}^{\alpha}_{0,\Omega}$ of the functional $\widetilde{\mathcal{I}}^{\alpha}_{0}$ defined by*

$$\begin{aligned}\widetilde{\mathcal{I}}^{\alpha}_{0}(\widetilde{w}) := \mathcal{G}(\widetilde{w};\Omega) + \frac{1+\nu}{E}\sum_{j=1}^{J}\Big(&-(1-\nu)\int_{\partial\Omega}(\partial_n \Delta W^j_0)\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\ &+\int_{\partial\Omega}\langle\nabla^2 W^j_0 n, \nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1 - \nu\int_{\partial\Omega}\Delta W^j_0 \partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\Big)\,.\end{aligned}$$

Moreover,

$$\Delta^2 \widetilde{w}^{\alpha}_{0} = 0 \qquad \textit{in } \Omega \tag{4.17}$$

and

$$\widetilde{\mathcal{I}}^{\alpha}_{\varepsilon}(\widetilde{w}^{\alpha}_{\varepsilon}) \to \widetilde{\mathcal{I}}^{\alpha}_{0}(\widetilde{w}^{\alpha}_{0}) \qquad \textit{as } \varepsilon \to 0\,. \tag{4.18}$$

Proof of Theorem 4.3. By the additive decomposition in (4.4) and by Propositions 4.4, we have that, for $\varepsilon > 0$ small enough, $w^{\alpha}_{\varepsilon} = W^{\alpha}_{\varepsilon} + \widetilde{w}^{\varepsilon}_{\alpha}$, where W^{α}_{ε} is defined in (4.3) and $\widetilde{w}^{\alpha}_{\varepsilon}$ is the unique solution to the minimum problem in (4.8). Therefore, by Remark 4.2 and by Proposition 4.5, we have that $w^{\alpha}_{\varepsilon} \to W^{\alpha}_{0} + \widetilde{w}^{\alpha}_{0} =: w^{\alpha}_{0}$ in $H^2_{\mathrm{loc}}\big(\Omega \setminus \bigcup_{j=1}^{J}\{x^j\}\big)$ as $\varepsilon \to 0$. Notice that, by (4.17) and by the very definition of w^{α}_{0} (see (4.12)),

$$\frac{1-\nu^2}{E}\Delta^2 w^{\alpha}_{0} = \frac{1-\nu^2}{E}\Delta^2 W^{\alpha}_{0} = -\sum_{j=1}^{J}|b^j|\partial_{\frac{(b^j)^\perp}{|b^j|}}\boldsymbol{\delta}_{x^j} \qquad \text{in } \Omega\,, \tag{4.19}$$

i.e., the first equation in (4.14). Finally, the boundary conditions are satisfied since $\widetilde{w}^{\alpha}_{0} \in \widetilde{\mathscr{B}}^{\alpha}_{0,\Omega}$ (see (4.13)). □

Now we prove Proposition 4.4.

Proof of Proposition 4.4. Let $\alpha = \sum_{j=1}^{J} b^j \boldsymbol{\delta}_{x^j} \in \mathscr{E}\mathscr{D}(\Omega)$ and let $\widetilde{w} \in \widetilde{\mathscr{B}}^{\alpha}_{\varepsilon,\Omega}$. We first prove that for every $j = 1, \dots, J$

$$\begin{aligned}\frac{1+\nu}{E}\sum_{i=1}^{J}\bigg(&(1-\nu)\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1 - \int_{\partial B_\varepsilon(x^i)}\langle\nabla^2 W^j_\varepsilon n, \nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1\\ &+\nu\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\bigg) + \frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla\widetilde{w}, \Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1 = \mathrm{O}(\varepsilon)\,.\end{aligned} \tag{4.20}$$

To this purpose, we recall that, for every $i = 1, \dots, J$, there exists an affine function a^i_ε such that

$$\widetilde{w} = a^i_\varepsilon - W^i_\varepsilon - \sum_{k\neq i} W^k_\varepsilon \qquad \text{on } \partial B_\varepsilon(x^i)\,. \tag{4.21}$$

Moreover, as in (C.9), for every function a which is affine in $B_\varepsilon(x^j)$ we have

$$\begin{aligned}&\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(x^j)}(\partial_n\Delta W^j_\varepsilon)a\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\nu\int_{\partial B_\varepsilon(x^j)}\Delta W^j_\varepsilon\partial_n a\,\mathrm{d}\mathcal{H}^1\\&-\frac{1+\nu}{E}\int_{\partial B_\varepsilon(x^j)}\langle\nabla^2W^j_\varepsilon n,\nabla a\rangle\,\mathrm{d}\mathcal{H}^1\\=&-\frac{|b^j|}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\partial_{\frac{(b^j)^\perp}{|b^j|}}a\,\mathrm{d}\mathcal{H}^1=-\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla a,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\,.\end{aligned}\tag{4.22}$$

Let $j=1,\dots,J$ be fixed. We first focus on the case $i=j$ in (4.20). Recalling that W^j_ε is affine in $B_\varepsilon(x^j)$ and that it is the only minimizer of the total energy in $B_R(x^j)\supset\Omega$, by (4.22) we get

$$\begin{aligned}&\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(x^j)}(\partial_n\Delta W^j_\varepsilon)(a^j_\varepsilon-W^j_\varepsilon)\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\nu\int_{\partial B_\varepsilon(x^j)}\Delta W^j_\varepsilon\partial_n(a^j_\varepsilon-W^j_\varepsilon)\,\mathrm{d}\mathcal{H}^1\\&-\frac{1+\nu}{E}\int_{\partial B_\varepsilon(x^j)}\langle\nabla^2W^j_\varepsilon n,\nabla(a^j_\varepsilon-W^j_\varepsilon)\rangle\,\mathrm{d}\mathcal{H}^1+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla(a^j_\varepsilon-W^j_\varepsilon),\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1=0\,.\end{aligned}\tag{4.23}$$

Furthermore, recalling that w^k_ε is smooth in $B_\varepsilon(x^j)$ for every $k\neq j$, by Taylor expansion we have that

$$W^k_\varepsilon(x)=W^k_\varepsilon(x^j)+\langle\nabla W^k_\varepsilon(x^j),x-x^j\rangle+\mathrm{O}(\varepsilon^2)\qquad\text{for every }x\in B_\varepsilon(x^j)\,,$$

whence, using (4.22) with $a(\cdot):=W^k_\varepsilon(x^j)+\langle\nabla W^k_\varepsilon(x^j),\cdot-x^j\rangle$, we deduce that

$$\begin{aligned}&\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(x^j)}(\partial_n\Delta W^j_\varepsilon)\Big(-\sum_{k\neq j}W^k_\varepsilon\Big)\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\nu\int_{\partial B_\varepsilon(x^j)}\Delta W^j_\varepsilon\partial_n\Big(-\sum_{k\neq j}W^k_\varepsilon\Big)\,\mathrm{d}\mathcal{H}^1\\&\qquad-\frac{1+\nu}{E}\int_{\partial B_\varepsilon(x^j)}\Big\langle\nabla^2W^j_\varepsilon n,\nabla\Big(-\sum_{k\neq j}W^k_\varepsilon\Big)\Big\rangle\,\mathrm{d}\mathcal{H}^1\\&\qquad+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\Big\langle\nabla\Big(-\sum_{k\neq j}W^k_\varepsilon\Big),\Pi(b^j)\Big\rangle\,\mathrm{d}\mathcal{H}^1=\mathrm{O}(\varepsilon)\,.\end{aligned}\tag{4.24}$$

By adding (4.23) and (4.24), in view of (4.21), we get

$$\begin{aligned}&\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(x^j)}(\partial_n\Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\nu\int_{\partial B_\varepsilon(x^j)}\Delta W^j_\varepsilon\partial_n\widetilde{w}\,\mathrm{d}\mathcal{H}^1\\&-\frac{1+\nu}{E}\int_{\partial B_\varepsilon(x^j)}\langle\nabla^2W^j_\varepsilon n,\nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\Big\langle\nabla\widetilde{w},\Pi(b^j)\Big\rangle\,\mathrm{d}\mathcal{H}^1=\mathrm{O}(\varepsilon)\,.\end{aligned}\tag{4.25}$$

Now we focus on the case $i\neq j$ in (4.20). We first notice that, by the Gauss–Green Theorem, for any affine function a there holds

$$\begin{aligned}0=&\int_{B_\varepsilon(x^i)}\Delta W^j_\varepsilon\Delta(-W^i_\varepsilon+a)\,\mathrm{d}x=\int_{B_\varepsilon(x^i)}\Delta^2W^j_\varepsilon(-W^i_\varepsilon+a)\,\mathrm{d}x\\&-\int_{\partial B_\varepsilon(x^i)}(\partial_{(-n)}\Delta W^j_\varepsilon)(-W^i_\varepsilon+a)\,\mathrm{d}\mathcal{H}^1+\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_{(-n)}(-W^i_\varepsilon+a)\,\mathrm{d}\mathcal{H}^1\\=&\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)(-W^i_\varepsilon+a)\,\mathrm{d}\mathcal{H}^1-\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_n(-W^i_\varepsilon+a)\,\mathrm{d}\mathcal{H}^1\,,\end{aligned}\tag{4.26}$$

where the first equality follows from the fact that W^i_ε is affine in $B_\varepsilon(x^i)$ whereas the last one is a consequence of $\Delta^2 W^j_\varepsilon = 0$ in $A_{\varepsilon,R}(x^j)$. Similarly, we have

$$
\begin{aligned}
0 = & \int_{B_\varepsilon(x^i)} \nabla^2 W^j_\varepsilon : \nabla^2(-W^i_\varepsilon + a)\,\mathrm{d}x = \int_{B_\varepsilon(x^i)} \Delta^2 W^j_\varepsilon(-W^i_\varepsilon + a)\,\mathrm{d}x \\
& - \int_{\partial B_\varepsilon(x^i)} (\partial_{(-n)}\Delta W^j_\varepsilon)(-W^i_\varepsilon + a)\,\mathrm{d}\mathcal{H}^1 + \int_{\partial B_\varepsilon(x^i)} \langle \nabla^2 W^j_\varepsilon(-n), \nabla(-W^i_\varepsilon + a)\rangle\,\mathrm{d}\mathcal{H}^1 \\
= & \int_{\partial B_\varepsilon(x^i)} (\partial_n \Delta W^j_\varepsilon)(-W^i_\varepsilon + a)\,\mathrm{d}\mathcal{H}^1 - \int_{\partial B_\varepsilon(x^i)} \langle \nabla^2 W^j_\varepsilon n, \nabla(-W^i_\varepsilon + a)\rangle\,\mathrm{d}\mathcal{H}^1 .
\end{aligned}
\tag{4.27}
$$

Furthermore, as $\varepsilon \to 0$,

$$
\begin{aligned}
& \int_{\partial B_\varepsilon(x^i)} (\partial_n \Delta W^j_\varepsilon)\Big(- \sum_{k\neq i} W^k_\varepsilon\Big)\,\mathrm{d}\mathcal{H}^1 \to 0 \\
& \int_{\partial B_\varepsilon(x^i)} \Delta W^j_\varepsilon \partial_n\Big(- \sum_{k\neq i} W^k_\varepsilon\Big)\,\mathrm{d}\mathcal{H}^1 \to 0 \\
& \int_{\partial B_\varepsilon(x^i)} \Big\langle \nabla^2 W^j_\varepsilon n, \nabla\Big(- \sum_{k\neq i} W^k_\varepsilon\Big)\Big\rangle\,\mathrm{d}\mathcal{H}^1 \to 0 ,
\end{aligned}
\tag{4.28}
$$

since all the integrands are uniformly bounded in ε and the domain of integration is vanishing. Therefore, in view of (4.21), by (4.26), (4.27), (4.28), for any function $\widetilde{w} \in \widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ we have that

$$
\begin{aligned}
& -\nu \sum_{i\neq j} \int_{\partial B_\varepsilon(x^i)} (\partial_n \Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1 + \nu \sum_{i\neq j} \int_{\partial B_\varepsilon(x^i)} \Delta W^j_\varepsilon \partial_n \widetilde{w}\,\mathrm{d}\mathcal{H}^1 \\
& + \sum_{i\neq j} \int_{\partial B_\varepsilon(x^i)} (\partial_n \Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1 - \sum_{i\neq j} \int_{\partial B_\varepsilon(x^i)} \langle \nabla^2 W^j_\varepsilon n, \nabla \widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1 = \mathrm{O}(\varepsilon) ,
\end{aligned}
$$

which, together with (4.25), implies (4.20).

Since the functions W^j_ε (for every $j = 1, \dots, J$) are uniformly bounded with respect to ε on $\partial\Omega$, by the standard trace theorem we get

$$
\begin{aligned}
& \left| -(1-\nu) \int_{\partial\Omega} (\partial_n \Delta W^j_\varepsilon)\widetilde{w}\,\mathrm{d}\mathcal{H}^1 + \int_{\partial\Omega} \langle \nabla^2 W^j_\varepsilon, \nabla\widetilde{w}\rangle\,\mathrm{d}\mathcal{H}^1 - \nu \int_{\partial\Omega} \Delta W^j_\varepsilon \partial_n \widetilde{w}\,\mathrm{d}\mathcal{H}^1 \right| \\
& \leq C\|\widetilde{w}\|_{H^1(\partial\Omega)} \leq C\|\widetilde{w}\|_{H^2(\Omega_\varepsilon(\alpha))} ,
\end{aligned}
\tag{4.29}
$$

where $C > 0$ is a constant that does not depend on ε .

In view of Lemma 4.1, by (4.20) and (4.29) (summing over $j = 1, \dots, J$), for ε small enough, we get

$$
\begin{aligned}
& \left| \frac{1+\nu}{E} \sum_{j=1}^{J} \int_{\Omega_\varepsilon(\alpha)} \big(\nabla^2 W^j_\varepsilon : \nabla^2\widetilde{w} - \nu\Delta W^j_\varepsilon \Delta\widetilde{w}\big)\,\mathrm{d}x + \sum_{j=1}^{J} \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(x^j)} \langle \nabla\widetilde{w}, \Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1 \right| \\
& \leq C\big(\|\widetilde{w}\|_{H^2(\Omega_\varepsilon(\alpha))} + 1\big) ,
\end{aligned}
\tag{4.30}
$$

for some constant $C > 0$ that does not depend on ε .

Now, by applying Proposition D.2 with $f = W^\alpha_\varepsilon$ and by the very definition of $\mathcal{G}$ in (1.9), we deduce the existence of two constants $0 < C_1 < C_2$ independent of ε (but depending on α and Ω) such that

$$
C_1\big(\|\widetilde{w}\|^2_{H^2(\Omega_\varepsilon(\alpha))} - \|W^\alpha_\varepsilon\|^2_{C^\infty(\partial\Omega)}\big) \leq \mathcal{G}(\widetilde{w}; \Omega_\varepsilon(\alpha)) \leq C_2\|\widetilde{w}\|^2_{H^2(\Omega_\varepsilon(\alpha))} ,
\tag{4.31}
$$

for every $\widetilde{w} \in \widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$. Therefore, by (4.30) and (4.31), we deduce (4.15). By (4.15), existence and uniqueness of the solution $\widetilde{w}^\alpha_\varepsilon$ to the minimization problem (4.8) for $\varepsilon > 0$ small enough follows by the direct method in the Calculus of Variations. Furthermore, by (4.15) and by Proposition D.4 applied with $f = W^\alpha_\varepsilon$ and $f^j = \sum_{i\neq j} W^i_\varepsilon$, we have that

$$
C'\|\widetilde{w}^\alpha_\varepsilon\|^2_{H^2(\Omega)} \leq \widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w}^\alpha_\varepsilon) + C'' ,
\tag{4.32}
$$

for some constants $C', C'' > 0$ independent of ε (but depending on α and Ω). Hence, in order to conclude the proof it is enough to construct (for ε small enough) a competitor function $\widehat{w}^\alpha_\varepsilon \in \widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ such that

$$\widetilde{\mathcal{I}}^\alpha_\varepsilon(\widehat{w}^\alpha_\varepsilon) \le C \tag{4.33}$$

for some constant $C > 0$ independent of ε.

We construct $\widehat{w}^\alpha_\varepsilon$ as follows. For every $j = 1, \dots, J$, let $\varphi^j \in C^\infty(\Omega)$ be such that $\varphi^j \equiv 0$ on $\overline{B}_{\frac{D}{4}}(x^j)$, $\varphi^j \equiv 1$ on $\Omega_{\frac{D}{2}}(\alpha)$, and $|\nabla\varphi(x)| \le \frac{C}{|x-x^j|}$ for every $x \in A_{\frac{D}{4},\frac{D}{2}}(x^j)$; for every ε small enough, we define $\widehat{w}^\alpha_\varepsilon \colon \Omega \to \mathbb{R}$ as

$$\widehat{w}^\alpha_\varepsilon := -\sum_{i=1}^J \varphi^j W^j_\varepsilon.$$

By construction,

$$\widehat{w}^\alpha_\varepsilon + W^\alpha_\varepsilon = \sum_{j=1}^J (1-\varphi^j) W^j_\varepsilon \in \widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$$

and

$$\|\widehat{w}^\alpha_\varepsilon\|_{H^2(\Omega_\varepsilon(\alpha))} \le \|\widehat{w}^\alpha_\varepsilon\|_{H^2(\Omega)} \le \sum_{j=1}^J \|\varphi^j W^j_\varepsilon\|_{H^2\left(A_{\frac{D}{4},R}(x^j)\right)} \le C, \tag{4.34}$$

for some constant $C > 0$ independent of ε (but possibly depending on α and on R). By (4.15) and (4.34) we obtain (4.33) and this concludes the proof. □

Proof of Proposition 4.5. We preliminarily notice that, since $\mathcal{G}$ is lower semicontinuous with respect to the weak H^2 convergence,(4.16) yields

$$\mathcal{G}(\widetilde{w}^\alpha_0; \Omega) \le \liminf_{\varepsilon\to 0} \mathcal{G}(\widetilde{w}^\alpha_\varepsilon; \Omega_\varepsilon(\alpha)), \tag{4.35}$$

and hence

$$\widetilde{\mathcal{I}}^\alpha_0(\widetilde{w}^\alpha_0) \le \liminf_{\varepsilon\to 0} \widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w}^\alpha_\varepsilon). \tag{4.36}$$

Here we have used that the boundary integrals on $\partial B_\varepsilon(x^j)$ vanish as $\varepsilon \to 0$ in view of (4.20), and that, by compactness of the trace operator [81, Theorem 6.2, page 103] (see also Remark 4.2), as $\varepsilon \to 0$,

$$\begin{aligned}
\int_{\partial\Omega} (\partial_n \Delta W^j_\varepsilon) \widetilde{w}^\alpha_\varepsilon \, \mathrm{d}\mathcal{H}^1 &\to \int_{\partial\Omega} (\partial_n \Delta W^j_0) \widetilde{w}^\alpha_0 \, \mathrm{d}\mathcal{H}^1 \\
\int_{\partial\Omega} \langle \nabla^2 W^j_\varepsilon n, \nabla \widetilde{w}^\alpha_\varepsilon\rangle \, \mathrm{d}\mathcal{H}^1 &\to \int_{\partial\Omega} \langle \nabla^2 W^j_0 n, \nabla \widetilde{w}^\alpha_0\rangle \, \mathrm{d}\mathcal{H}^1 \\
\int_{\partial\Omega} \Delta W^j_\varepsilon \partial_n \widetilde{w}^\alpha_\varepsilon \, \mathrm{d}\mathcal{H}^1 &\to \int_{\partial\Omega} \Delta W^j_0 \partial_n \widetilde{w}^\alpha_0 \, \mathrm{d}\mathcal{H}^1 .
\end{aligned} \tag{4.37}$$

Moreover, by Proposition E.1 for every $\widehat{w}_0 \in \widetilde{\mathscr{B}}^\alpha_{0,\Omega}$ there exists a sequence $\{\widehat{w}_\varepsilon\}_\varepsilon \subset H^2(\Omega)$ with $\widehat{w}_\varepsilon \in \widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ (for every $\varepsilon > 0$) such that $\widehat{w}_\varepsilon \to \widehat{w}_0$ strongly in $H^2(\Omega)$. It follows that

$$\widetilde{\mathcal{I}}^\alpha_0(\widehat{w}_0) = \lim_{\varepsilon\to 0} \widetilde{\mathcal{I}}^\alpha_\varepsilon(\widehat{w}_\varepsilon), \tag{4.38}$$

which, by the minimality of $\widetilde{w}^\alpha_\varepsilon$ and in view of (4.36), gives

$$\widetilde{\mathcal{I}}^\alpha_0(\widehat{w}_0) = \lim_{\varepsilon\to 0} \widetilde{\mathcal{I}}^\alpha_\varepsilon(\widehat{w}_\varepsilon) \ge \limsup_{\varepsilon\to 0} \widetilde{\mathcal{I}}^\alpha_\varepsilon(\widetilde{w}^\alpha_\varepsilon) \ge \widetilde{\mathcal{I}}^\alpha_0(\widetilde{w}^\alpha_0). \tag{4.39}$$

It follows that $\widetilde{w}^\alpha_0$ is a minimizer of $\widetilde{\mathcal{I}}^\alpha_0$ in $\widetilde{\mathscr{B}}^\alpha_{0,\Omega}$. By convexity (see (2.7)), such a minimizer is unique and, by computing the first variation of $\widetilde{\mathcal{I}}^\alpha_0$ in $\widetilde{w}^\alpha_0$, we have that it satisfies (4.17). Furthermore, by applying (4.39) with $\widehat{w}_0 = \widetilde{w}^\alpha_0$ we get (4.18).

Finally, we discuss the strong convergence of $\widetilde{w}^\alpha_\varepsilon$ in the compact subsets of of $\Omega\setminus\bigcup_{j=1}^J\{x^j\}$. To this purpose, we preliminarily notice that, from (4.18), (4.20), and (4.37), we have that

$$\lim_{\varepsilon\to0}\mathcal{G}(\widetilde{w}^\alpha_\varepsilon;\Omega_\varepsilon(\alpha))=\mathcal{G}(\widetilde{w}^\alpha_0;\Omega)\,.$$

We now want to show that for every (fixed) $r>0$

$$\int_{\Omega_r(\alpha)}|\nabla^2\widetilde{w}^\alpha_\varepsilon-\nabla^2\widetilde{w}^\alpha_0|^2\,\mathrm{d}x\to0\qquad\text{as }\varepsilon\to0\,. \tag{4.40}$$

To this purpose, we will use the weak convergence (4.16) and Remark 2.3; we start by observing that

$$\begin{aligned}&\int_{\Omega_r(\alpha)}|\nabla^2\widetilde{w}^\alpha_\varepsilon-\nabla^2\widetilde{w}^\alpha_0|^2\,\mathrm{d}x-\nu\int_{\Omega_r(\alpha)}|\Delta\widetilde{w}^\alpha_\varepsilon-\Delta\widetilde{w}^\alpha_0|^2\,\mathrm{d}x\\&=\int_{\Omega_r(\alpha)}\big(|\nabla^2\widetilde{w}^\alpha_\varepsilon|^2+|\nabla^2\widetilde{w}^\alpha_0|^2-2\nabla^2\widetilde{w}^\alpha_0:\nabla^2\widetilde{w}^\alpha_\varepsilon\big)\,\mathrm{d}x-\nu\int_{\Omega_r(\alpha)}\big(|\Delta\widetilde{w}^\alpha_\varepsilon|^2+|\Delta\widetilde{w}^\alpha_0|^2-2\Delta\widetilde{w}^\alpha_0\Delta\widetilde{w}^\alpha_\varepsilon\big)\,\mathrm{d}x\,,\end{aligned}$$

whence, thanks to the convergence (4.16), we deduce

$$\int_{\Omega_r(\alpha)}|\nabla^2\widetilde{w}^\alpha_\varepsilon-\nabla^2\widetilde{w}^\alpha_0|^2\,\mathrm{d}x-\nu\int_{\Omega_r(\alpha)}|\Delta\widetilde{w}^\alpha_\varepsilon-\Delta\widetilde{w}^\alpha_0|^2\,\mathrm{d}x\to0\,. \tag{4.41}$$

Since (see the first inequality in (2.4))

$$c(\nu)\int_{\Omega_r(\alpha)}|\nabla^2\widetilde{w}^\alpha_\varepsilon-\nabla^2\widetilde{w}^\alpha_0|^2\,\mathrm{d}x\le\int_{\Omega_r(\alpha)}|\nabla^2\widetilde{w}^\alpha_\varepsilon-\nabla^2\widetilde{w}^\alpha_0|^2\,\mathrm{d}x-\nu\int_{\Omega_r(\alpha)}|\Delta\widetilde{w}^\alpha_\varepsilon-\Delta\widetilde{w}^\alpha_0|^2\,\mathrm{d}x$$

for some constant $c(\nu)>0$ depending only on ν, by (4.41), we get (4.40). Finally, by (4.16), we get that $\widetilde{w}^\alpha_\varepsilon$ converges strongly in $H^1(\Omega)$, as $\varepsilon\to0$, to $\widetilde{w}^\alpha_0$, which together with (4.40), implies that

$$\widetilde{w}^\alpha_\varepsilon\to\widetilde{w}^\alpha_0\qquad\text{strongly in }H^2(\Omega_r(\alpha))\,. \tag{4.42}$$

In conclusion, for any compact set $K\subset\Omega\setminus\bigcup_{j=1}^J\{x^j\}$, there exists $r>0$ such that $K\subset\Omega_r(\alpha)$, which, in view of (4.42), implies the claim and concludes the proof of the proposition. □

We are in a position to discuss the asymptotic expansion of energies and to classify each term of the expansion.

Theorem 4.6. *For every $\varepsilon>0$ small enough, let w^α_ε be the minimizer of $\mathcal{I}^\alpha_\varepsilon$ in $\mathscr{B}^\alpha_{\varepsilon,\Omega}$. Then we have*

$$\mathcal{I}^\alpha_\varepsilon(w^\alpha_\varepsilon)=-\frac{E}{1-\nu^2}\sum_{j=1}^J\frac{|b_j|^2}{8\pi}|\log\varepsilon|+F(\alpha)+f(D,R;\alpha)+\omega_\varepsilon\,, \tag{4.43}$$

where $\omega_\varepsilon\to0$ as $\varepsilon\to0$,

$$\begin{aligned}f(D,R;\alpha)=\sum_{j=1}^J\frac{|b^j|^2}{8\pi}\frac{E}{1-\nu^2}\bigg(&2+\frac{D^2}{R^2}\Big(\frac{D^2}{R^2}-2\Big)-2\log R\\&+\frac{1}{4(1-\nu)}\frac{D^2}{R^2}\Big(\frac{R^2}{D^2}-1\Big)\Big(\frac{D^2}{R^2}\Big(\frac{R^2}{D^2}+1\Big)-2\Big)\bigg)\,,\end{aligned} \tag{4.44}$$

(recall (4.2) for the definition of D) and

$$F(\alpha)=F^{\mathrm{self}}(\alpha)+F^{\mathrm{int}}(\alpha)+F^{\mathrm{elastic}}(\alpha) \tag{4.45}$$

is the renormalized energy defined by

$$F^{\mathrm{self}}(\alpha):=\sum_{j=1}^J\mathcal{G}(W^j_0;\Omega_D(\alpha))+\frac{E}{1-\nu^2}\sum_{j=1}^J\frac{|b_j|^2}{8\pi}\log D\,, \tag{4.46}$$

$$
\begin{aligned}
F^{\mathrm{int}}(\alpha):=&\frac{1+\nu}{E}\sum_{j=1}^{J}\sum_{k\neq j}\Big(-(1-\nu)\int_{\partial\Omega}(\partial_n\Delta W_0^j)W_0^k\,\mathrm{d}\mathcal{H}^1\\
&+\int_{\partial\Omega}\langle\nabla^2W_0^jn,\nabla W_0^k\rangle\,\mathrm{d}\mathcal{H}^1-\nu\int_{\partial\Omega}\Delta W_0^j\partial_nW_0^k\,\mathrm{d}\mathcal{H}^1\Big)\,,
\end{aligned}\tag{4.47}
$$

$$
F^{\mathrm{elastic}}(\alpha):=\widetilde{\mathcal{I}}_0^{\alpha}(\widetilde{w}_0^{\alpha})\,.\tag{4.48}
$$

Remark 4.7. Notice that $F^{\mathrm{self}}(\alpha)$ is independent of D as it can be verified by a simple computation.

Proof. By (4.4) and (4.6), we have that $w_\varepsilon^\alpha=W_\varepsilon^\alpha+\widetilde{w}_\varepsilon^\alpha$, where W_ε^α is defined in (4.3) and $\widetilde{w}_\varepsilon^\alpha$ is the unique minimizer of $\widetilde{\mathcal{I}}_\varepsilon^\alpha$ in $\widetilde{\mathscr{B}}_{\varepsilon,\Omega}^\alpha$ provided by Proposition 4.5. Notice that

$$
\begin{aligned}
&\mathcal{G}(W_\varepsilon^\alpha;\Omega_\varepsilon(\alpha))+\sum_{j=1}^J\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^\alpha,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\\
=&\sum_{j=1}^J\Big(\mathcal{G}(W_\varepsilon^j;\Omega_\varepsilon(\alpha))+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^j,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\Big)\\
&+\sum_{j=1}^J\sum_{k\neq j}\Big(\frac{1+\nu}{E}\int_{\Omega_\varepsilon(\alpha)}\Big(\nabla^2W_\varepsilon^j:\nabla^2W_\varepsilon^k-\nu\Delta W_\varepsilon^j\Delta W_\varepsilon^k\Big)\,\mathrm{d}x\\
&+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^k,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\Big)\\
=:&F_\varepsilon^{\mathrm{self}}(\alpha)+F_\varepsilon^{\mathrm{int}}(\alpha)\,.
\end{aligned}\tag{4.49}
$$

We notice that, for every $j=1,\dots,J$ and for every $0<\varepsilon<r\le D$ with $\varepsilon<1$

$$
\begin{aligned}
&\mathcal{G}(W_\varepsilon^j;\Omega_\varepsilon(\alpha))+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^j,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\\
=&\mathcal{G}(W_\varepsilon^j;\Omega_r(\alpha))+\mathcal{G}(W_\varepsilon^j;A_{\varepsilon,r}(x^j))+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^j,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\,.
\end{aligned}\tag{4.50}
$$

Furthermore, by Corollary C.1, we have that

$$
\begin{aligned}
\mathcal{G}(W_\varepsilon^j;A_{\varepsilon,r}(x^j))+\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W_\varepsilon^j,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1=&-\frac{|b^j|^2}{8\pi}\frac{E}{1-\nu^2}\log\frac{1}{\varepsilon}\\
&+\frac{|b^j|^2}{8\pi}\frac{E}{1-\nu^2}\log r+f_\varepsilon(r,R;|b^j|)\,,
\end{aligned}\tag{4.51}
$$

where $f_\varepsilon(r,R;|b^j|)$ is defined in (C.16).

Notice moreover that $f_\varepsilon(r,R;|b^j|)\to f(r,R;|b^j|)$ (as $\varepsilon\to0$) with $f(r,R;|b^j|)$ defined by

$$
\begin{aligned}
f(r,R;|b^j|):=&\frac{|b^j|^2}{8\pi}\frac{E}{1-\nu^2}\Big(2+\frac{r^2}{R^2}\Big(\frac{r^2}{R^2}-2\Big)-2\log R\Big)\\
&+\frac{|b^j|^2}{32\pi}\frac{E}{(1-\nu)^2(1+\nu)}\frac{r^2}{R^2}\Big(\frac{R^2}{r^2}-1\Big)\Big(\frac{r^2}{R^2}\Big(\frac{R^2}{r^2}+1\Big)-2\Big)\,.
\end{aligned}\tag{4.52}
$$

By Remark 4.2, summing over $j=1,\dots,J$ formulas (4.50), (4.51) and (4.52), for $r=D$ we obtain

$$
F_\varepsilon^{\mathrm{self}}(\alpha)=-\sum_{j=1}^J\frac{|b^j|^2}{4\pi}\frac{E}{1-\nu^2}|\log\varepsilon|+F^{\mathrm{self}}(\alpha)+f(D,R;\alpha)+\omega_\varepsilon\,,\tag{4.53}
$$

where $\omega_\varepsilon\to0$ as $\varepsilon\to0$ and $f(D,R;\alpha):=\sum_{j=1}^J f(D,R;|b^j|)$.

We now focus on $F^{\rm int}_\varepsilon(\alpha)$. By arguing as in the proof of Lemma 4.1, for every $j,k=1,\dots,J$ with $k\neq j$, we have that

$$\begin{aligned}&\int_{\Omega_\varepsilon(\alpha)}\Big(\nabla^2W^j_\varepsilon:\nabla^2W^k_\varepsilon-\nu\Delta W^j_\varepsilon\Delta W^k_\varepsilon\Big)\,\mathrm{d}x\\
=&-(1-\nu)\int_{\partial\Omega}(\partial_n\Delta W^j_\varepsilon)W^k_\varepsilon\,\mathrm{d}\mathcal{H}^1+\int_{\partial\Omega}\langle\nabla^2W^j_\varepsilon n,\nabla W^k_\varepsilon\rangle\,\mathrm{d}\mathcal{H}^1-\nu\int_{\partial\Omega}\Delta W^j_\varepsilon\partial_nW^k_\varepsilon\,\mathrm{d}\mathcal{H}^1\\
&+(1-\nu)\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}(\partial_n\Delta W^j_\varepsilon)W^k_\varepsilon\,\mathrm{d}\mathcal{H}^1\\
&-\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\langle\nabla^2W^j_\varepsilon n,\nabla W^k_\varepsilon\rangle\,\mathrm{d}\mathcal{H}^1+\nu\sum_{i=1}^J\int_{\partial B_\varepsilon(x^i)}\Delta W^j_\varepsilon\partial_nW^k_\varepsilon\,\mathrm{d}\mathcal{H}^1\,,\end{aligned}$$

which, in view of (4.22), (4.26), and (4.28), and using Remark 4.2, implies

$$F^{\rm int}_\varepsilon(\alpha)=F^{\rm int}(\alpha)+\omega_\varepsilon\,,\tag{4.54}$$

where $\omega_\varepsilon\to0$ as $\varepsilon\to0$.

Finally, by (4.49), (4.50), (4.53), and (4.54), we get

$$\begin{aligned}&\mathcal{G}(W^\alpha_\varepsilon;\Omega_\varepsilon(\alpha))+\sum_{j=1}^J\frac{1}{2\pi\varepsilon}\int_{\partial B_\varepsilon(x^j)}\langle\nabla W^\alpha_\varepsilon,\Pi(b^j)\rangle\,\mathrm{d}\mathcal{H}^1\\
&=-\sum_{j=1}^J\frac{|b^j|^2}{4\pi}\frac{E}{1-\nu^2}|\log\varepsilon|+F^{\rm self}(\alpha)+f(D,R;\alpha)+F^{\rm int}(\alpha)+\omega_\varepsilon\,,\end{aligned}$$

which, by (4.6) together with Propositions 4.4 and 4.5, allows us to conclude the proof. □

We conclude by showing, via a diagonal argument, that the asymptotic behavior in Theorem 4.6 remains valid also for systems of disclination dipoles, that is, when the finite system $\alpha\in\mathscr{ED}(\Omega)$ of edge dislocations is replaced with the approximating system of disclination dipoles.

Theorem 4.8. *Let $J\in\mathbb{N}$, let $b^1,\dots,b^J\in\mathbb{R}^2\setminus\{0\}$, and let $x^1,\dots,x^J$ be distinct points in Ω. For every $h>0$, let $\theta_h\in\mathscr{WD}(\Omega)$ be the measure defined in (3.31). Then,*

$$\theta_h\stackrel{*}{\rightharpoonup}\alpha:=\sum_{j=1}^Jb^j\boldsymbol{\delta}_{x^j}\in\mathscr{ED}(\Omega)\qquad\text{as }h\to0\,.\tag{4.55}$$

Let $D>0$ be as in (4.2); for every $0<h<\varepsilon<D$ let $w^{\theta_h}_{h,\varepsilon}$ be the unique minimizer in $\mathscr{B}^\alpha_{\varepsilon,\Omega}$ of the functional $\mathcal{I}^{\theta_h}_{h,\varepsilon}$ defined in (3.32). Then there exists a function $\varepsilon\colon\mathbb{R}^+\to\mathbb{R}^+$ with $\varepsilon(h)>h$ and $\varepsilon(h)\to0$ as $h\to0$ such that $w^{\theta_h}_{h,\varepsilon(h)}\to w^\alpha_0$ in $H^2_{\rm loc}\big(\Omega\setminus\bigcup_{j=1}^J\{x^j\}\big)$ as $h\to0$, where w^α_0 is the function provided by Theorem 4.3. Moreover,

$$\mathcal{I}^{\theta_h}_{h,\varepsilon(h)}\big(w^{\theta_h}_{h,\varepsilon(h)}\big)=-\frac{E}{1-\nu^2}\sum_{j=1}^J\frac{|b_j|^2}{8\pi}|\log\varepsilon(h)|+F(\alpha)+f(D,R;\alpha)+\omega_h\,,\tag{4.56}$$

where $F(\alpha)$ and $f(D,R;\alpha)$ are defined in (4.45) and (4.44), respectively, and $\omega_h\to0$ as $h\to0$.

Proof. Convergence (4.55) is obvious. Let now $0<\varepsilon<D$ be fixed. By Remark 3.7, there exists $\bar h<\varepsilon$ such that, for every $h<\bar h$,

$$\|w^{\theta_h}_{h,\varepsilon}-w^\alpha_{0,\varepsilon}\|_{H^2(\Omega)}<\varepsilon\,,\tag{4.57}$$

where $w^\alpha_{0,\varepsilon}$ is the unique minimizer of (3.34) in $\mathscr{B}^\alpha_{\varepsilon,\Omega}$. Choose such an h, call it $h(\varepsilon)$, and notice that this choice can be made in a strictly monotone fashion. Let now $0<r<D$; by (4.57) and Theorem 4.3, we get

$$\big\|w^{\theta_{h(\varepsilon)}}_{h(\varepsilon),\varepsilon}-w^\alpha_0\big\|_{H^2(\Omega_r(\alpha))}\le\big\|w^{\theta_{h(\varepsilon)}}_{h(\varepsilon),\varepsilon}-w^\alpha_{0,\varepsilon}\big\|_{H^2(\Omega_r(\alpha))}+\|w^\alpha_{0,\varepsilon}-w^\alpha_0\|_{H^2(\Omega_r(\alpha))}<\varepsilon+\mathrm{o}_\varepsilon\,,$$

where $o_\varepsilon \to 0$ as $\varepsilon \to 0$. By the arbitrariness of r we get that $w^{\theta_{h(\varepsilon)}}_{h(\varepsilon),\varepsilon} \to w^\alpha_0$ in $H^2_{\mathrm{loc}}(\Omega\setminus\bigcup_{j=1}^J\{x^j\})$, and hence, by the strict monotonicity of the map $\varepsilon \mapsto h(\varepsilon)$ the first part of the claim follows. Finally, (4.56) is an immediate consequence of Theorem 4.6. □

Appendix A. Equivalence of boundary conditions

Here we show that if A is a domain of class C^2 and $v \in C^2(\overline{A})$, then the boundary condition $\nabla^2 v\, t = 0$ on ∂A is equivalent to requiring that $v|_\Gamma$ is the trace of an affine function on every connected component Γ of ∂A. To this end, we first state and prove the following geometric lemma.

Lemma A.1. *Let $A \subset \mathbb{R}^2$ be a bounded, open, simply connected set with C^2 boundary and set $\ell := |\partial A|$. Let $\gamma \in C^2([0,\ell];\mathbb{R}^2)$ be the arc-length parametrization of ∂A and let $\vartheta \in C^1([0,\ell])$ such that $\gamma'(\xi) = (-\sin\vartheta(\xi);\cos\vartheta(\xi))$. Set $\varkappa(\xi) := \vartheta'(\xi)$ for every $\xi \in [0,\ell]$. Let $v \in C^2(\overline{A})$ and let $g_D, g_N \colon [0,\ell] \to \mathbb{R}$ be the functions defined by $g_D := v\circ\gamma$ and $g_N := \partial_n v \circ \gamma$. Then*

$$\begin{cases} g''_D(\xi) = \langle \nabla^2 v(\gamma(\xi))\,\gamma'(\xi), \gamma'(\xi)\rangle - \varkappa(\xi) g_N(\xi) \\ g'_N(\xi) = \langle \nabla^2 v(\gamma(\xi))\,\gamma'(\xi), -(\gamma'(\xi))^\perp\rangle + \varkappa(\xi) g'_D(\xi) \end{cases} \quad \textit{for every } \xi \in [0,\ell]\,. \tag{A.1}$$

Proof. By definition, the unit tangent vector is $t(\gamma(\xi)) = \gamma'(\xi) = (-\sin\vartheta(\xi);\cos\vartheta(\xi))$ and the outer unit normal vector is $n(\gamma(\xi)) = (-\gamma'(\xi))^\perp = (\cos\vartheta(\xi);\sin\vartheta(\xi))$, so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi} t(\gamma(\xi)) &= \gamma''(\xi) = -\varkappa(\xi) n(\gamma(\xi)) = -\varkappa(\xi)(-\gamma'(\xi))^\perp \\ \frac{\mathrm{d}}{\mathrm{d}\xi} n(\gamma(\xi)) &= (-\gamma''(\xi))^\perp = \varkappa(\xi) t(\gamma(\xi)) = \varkappa(\xi)\gamma'(\xi)\,, \end{aligned}$$

and hence

$$\begin{aligned} g'_D(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi} v(\gamma(\xi)) = \langle \nabla v(\gamma(\xi)), \gamma'(\xi)\rangle\,, \\ g'_N(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\langle \nabla v(\gamma(\xi)), n(\gamma(\xi))\rangle = \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), (-\gamma'(\xi))^\perp)\rangle + \langle \nabla v(\gamma(\xi)), (-\gamma''(\xi))^\perp\rangle \\ &= \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), (-\gamma'(\xi))^\perp)\rangle + \varkappa(\xi)\langle \nabla v(\gamma(\xi)), \gamma'(\xi)\rangle \\ &= \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), (-\gamma'(\xi))^\perp)\rangle + \varkappa(\xi) g'_D(\xi) \\ g''_D(\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\langle \nabla v(\gamma(\xi)), \gamma'(\xi)\rangle = \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), \gamma'(\xi)\rangle + \langle \nabla v(\gamma(\xi)), \gamma''(\xi)\rangle \\ &= \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), \gamma'(\xi)\rangle - \varkappa(\xi)\langle \nabla v(\gamma(\xi)), n(\gamma(\xi))\rangle \\ &= \langle \nabla^2 v(\gamma(\xi))\gamma'(\xi), \gamma'(\xi)\rangle - \varkappa(\xi) g_N(\xi)\,, \end{aligned}$$

that is (A.1). □

We are now in a position to prove the main result of this section on the equivalence of the boundary conditions.

Proposition A.2. *Let $A \subset \mathbb{R}^2$ be an open and bounded set with boundary of class C^2. Let $v \in C^2(\overline{A})$. Then, for every connected component Γ of ∂A we have that*

$$\nabla^2 v\, t = 0 \textit{ on } \Gamma \qquad \Leftrightarrow \qquad v = a\,, \quad \partial_n v = \partial_n a \textit{ on } \Gamma\,, \tag{A.2}$$

for some affine function a.

Proof. Let Γ be a connected component of ∂A, set $\ell := |\Gamma|$ and let $\gamma\colon [0,\ell] \to \mathbb{R}^2$ be the arc-length parametrization of Γ, so that the unit tangent vector is $t(\gamma(\xi)) = \gamma'(\xi)$ and the outer unit normal vector is $n(\gamma(\xi)) = \big(-\gamma'(\xi)\big)^\perp$. Moreover, let $\vartheta \in C^1([0,\ell])$ be such that

$$\gamma'(\xi) = (-\sin\vartheta(\xi);\cos\vartheta(\xi)) \tag{A.3}$$

and set $\varkappa(\xi) := \vartheta'(\xi)$ for every $\xi \in [0,\ell]$. Recalling that $\{n(\gamma(\xi)), t(\gamma(\xi))\}$ is an orthonormal basis of $\mathbb{R}^2$ for every $\xi \in [0,\ell]$, we have $\nabla^2 v\, t = 0$ on Γ if and only if

$$\langle \nabla^2 v(\gamma(\xi))\,\gamma'(\xi), \gamma'(\xi)\rangle = \langle \nabla^2 v(\gamma(\xi))\,\gamma'(\xi), -(\gamma'(\xi))^\perp\rangle = 0 \text{ for every } \xi \in [0,\ell]\,. \tag{A.4}$$

Furthermore, letting $g_D, g_N\colon [0,\ell] \to \mathbb{R}$ be the functions defined by $g_D := v\circ\gamma$ and $g_N := \partial_n v \circ \gamma$, Lemma A.1 and (A.4), imply that (A.2) is equivalent to

$$\begin{cases} g_D''(\xi) = -\varkappa(\xi) g_N(\xi) \\ g_N'(\xi) = \varkappa(\xi) g_D'(\xi) \end{cases} \quad \text{for every } \xi \in [0,\ell] \quad \text{if and only if} \quad \begin{cases} v = a & \text{on } \Gamma \\ \partial_n v = \partial_n a & \text{on } \Gamma\,, \end{cases} \tag{A.5}$$

for some affine function $a\colon \mathbb{R}^2 \to \mathbb{R}^2$, namely, for a function a of the form

$$a(x) = c_0 + c_1 x_1 + c_2 x_2\,, \tag{A.6}$$

with $c_0, c_1, c_2 \in \mathbb{R}$. If $v = a$ and $\partial_n v = \partial_n a$ on Γ, by straightforward computations for every $\xi \in [0,\ell]$ we get

$$\begin{aligned} g_N(\xi) =& c_1 \cos\vartheta(\xi) + c_2 \sin\vartheta(\xi)\,, \\ g_D'(\xi) =& -c_1 \sin\vartheta(\xi) + c_2\cos\vartheta(\xi)\,, \\ g_N'(\xi) =& \vartheta'(\xi)\big(-c_1 \sin\vartheta(\xi) + c_2 \cos\vartheta(\xi)\big) = \varkappa(\xi) g_D'(\xi)\,, \\ g_D''(\xi) =& -\vartheta'(\xi)\big(c_1\cos\vartheta(\xi) + c_2 \sin\vartheta(\xi)\big) = -\varkappa(\xi) g_N(\xi)\,, \end{aligned}$$

which proves one implication in (A.5). To prove the opposite implication, we study the ODE system in the left-hand side of (A.5), that, setting $z^1 := g_D'$ and $z^2 := g_N$, can be conveniently rewritten in the form

$$\begin{cases} (z^1)' = -\varkappa z^2 \\ (z^2)' = \varkappa z^1\,. \end{cases} \tag{A.7}$$

By the classical theory of ordinary differential equations, since $\varkappa$ is a continuous function, for any given initial datum $z_0 = (z_0^1; z_0^2) \in \mathbb{R}^2$, the Cauchy problem associated with the system (A.7) with initial condition $(z^1(0); z^2(0)) = z(0) = z_0$ admits a unique solution $z \in C^1([0,\ell];\mathbb{R}^2)$. Furthermore, letting $\bar\vartheta$ denote a primitive of $\varkappa$, we observe that the functions $\xi \mapsto \bar z(\xi) := (-\sin\bar\vartheta(\xi); \cos\bar\vartheta(\xi))$ and $\xi \mapsto \hat z(\xi) := (\cos\bar\vartheta(\xi); \sin\bar\vartheta(\xi))$ provide a basis of solutions to (A.7). Therefore, since $\bar\vartheta$ and ϑ differ by a constant, any solution to (A.7) is of the form

$$(z^1; z^2) = (-c_1 \sin\vartheta + c_2\cos\vartheta; c_1\cos\vartheta + c_2\sin\vartheta)\,,$$

so that, recalling the definitions of z^1 and z^2 and using (A.3), we get

$$\begin{aligned} g_N(\xi) =& c_1\cos\vartheta(\xi) + c_2\sin\vartheta(\xi) = \langle (c_1; c_2), \gamma'(\xi)\rangle \\ g_D(\xi) =& g_D(0) + \int_0^\xi \big(-c_1\sin\vartheta(\zeta) + c_2\cos\vartheta(\zeta)\big)\,\mathrm{d}\zeta =: c_0 + \langle (c_1; c_2), \gamma(\xi)\rangle\,, \end{aligned} \tag{A.8}$$

where we have set $c_0 := g_D(0) - c_1\gamma^1(0) - c^2\gamma^2(0)$. By (A.8) and the definitions of g_D and g_N, we get that $v = a$ and $\partial_n v = \partial_n a$ on Γ, for a certain function a as in (A.6). This concludes the proof of the converse inequality and hence of the whole proposition. □

Appendix B. Proof of Lemma 3.2

This section in devoted to the proof of Lemma 3.2.

Proof. By straightforward computations we have

$$\partial^2_{x_1^2}\bar v_h(x) = -\frac{E}{1-\nu^2}\frac{s}{16\pi}\left(2\log\frac{\big(x_1-\frac{h}{2}\big)^2 + x_2^2}{\big(x_1+\frac{h}{2}\big)^2+x_2^2} + 4\left(\frac{\big(x_1-\frac{h}{2}\big)^2}{\big(x_1-\frac{h}{2}\big)^2+x_2^2} - \frac{\big(x_1+\frac{h}{2}\big)^2}{\big(x_1+\frac{h}{2}\big)^2+x_2^2}\right)\right);$$

$$\partial^2_{x_2^2}\bar v_h(x) = -\frac{E}{1-\nu^2}\frac{s}{16\pi}\left(2\log\frac{\big(x_1-\frac{h}{2}\big)^2 + x_2^2}{\big(x_1+\frac{h}{2}\big)^2+x_2^2} + 4x_2^2\left(\frac{1}{(x_1-\frac{h}{2})^2+x_2^2} - \frac{1}{(x_1+\frac{h}{2})^2+x_2^2}\right)\right);$$

$$\partial^2_{x_1\,x_2}\bar v_h(x) = -\frac{E}{1-\nu^2}\frac{s}{16\pi}\bigg(4x_2\Big(\frac{x_1-\frac h2}{\big(x_1-\frac h2\big)^2+x_2^2}-\frac{x_1+\frac h2}{\big(x_1+\frac h2\big)^2+x_2^2}\Big)\bigg)\,.$$

Moreover,

$$\begin{aligned}\frac{\big(x_1-\frac h2\big)^2}{\big(x_1-\frac h2\big)^2+x_2^2}-\frac{\big(x_1+\frac h2\big)^2}{\big(x_1+\frac h2\big)^2+x_2^2} &= \frac{-2x_2^2x_1h}{\Big(\big(x_1-\frac h2\big)^2+x_2^2\Big)\Big(\big(x_1+\frac h2\big)^2+x_2^2\Big)}\,,\\ x_2^2\Big(\frac{1}{\big(x_1-\frac h2\big)^2+x_2^2}-\frac{1}{\big(x_1+\frac h2\big)^2+x_2^2}\Big) &= \frac{2x_2^2x_1h}{\Big(\big(x_1-\frac h2\big)^2+x_2^2\Big)\Big(\big(x_1+\frac h2\big)^2+x_2^2\Big)}\,,\end{aligned}\tag{B.1}$$

whence we deduce that

$$\begin{aligned}|\nabla^2\bar v_h(x)|^2 = \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{256\pi^2}\Bigg(&8\log^2\frac{\big(x_1-\frac h2\big)^2+x_2^2}{\big(x_1+\frac h2\big)^2+x_2^2}\\ &+128\,\frac{h^2\,x_2^4\,x_1^2}{\bigg(\Big(\big(x_1-\frac h2\big)^2+x_2^2\Big)\Big(\big(x_1+\frac h2\big)^2+x_2^2\Big)\bigg)^2}\\ &+32x_2^2\Big(\frac{x_1-\frac h2}{(x_1-\frac{\xi}{2})^2+x_2^2}-\frac{x_1+\frac h2}{\big(x_1+\frac h2\big)^2+x_2^2}\Big)^2\Bigg)\,,\end{aligned}\tag{B.2}$$

and

$$|\Delta\bar v_h(x)|^2 = \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{16\pi^2}\log^2\frac{\big(x_1-\frac h2\big)^2+x_2^2}{\big(x_1+\frac h2\big)^2+x_2^2}\,.\tag{B.3}$$

For every open set $A\subset\mathbb{R}^2$ we set:

$$\mathcal{F}^1_h(A) := \int_A \log^2\frac{\big(x_1-\frac h2\big)^2+x_2^2}{\big(x_1+\frac h2\big)^2+x_2^2}\,\mathrm{d}x\,,\tag{B.4}$$

$$\mathcal{F}^2_h(A) := h^2\int_A \frac{x_2^4\,x_1^2}{\bigg(\Big(\big(x_1-\frac h2\big)^2+x_2^2\Big)\Big(\big(x_1+\frac h2\big)^2+x_2^2\Big)\bigg)^2}\,\mathrm{d}x\,,\tag{B.5}$$

$$\begin{aligned}\mathcal{F}^3_h(A) &:= \int_A x_2^2\Big(\frac{x_1-\frac h2}{(x_1-\frac h2)^2+x_2^2}-\frac{x_1+\frac h2}{\big(x_1+\frac h2\big)^2+x_2^2}\Big)^2\mathrm{d}x\\ &= h^2\int_A \frac{x_2^2\big(\frac{h^2}{4}+x_2^2-x_1^2\big)^2}{\bigg(\Big(\big(x_1-\frac h2\big)^2+x_2^2\Big)\Big(\big(x_1+\frac h2\big)^2+x_2^2\Big)\bigg)^2}\,\mathrm{d}x\,,\end{aligned}\tag{B.6}$$

so that, in view of (B.2) and (B.3), it holds

$$\begin{aligned}\int_A|\nabla^2\bar v_h|^2\,\mathrm{d}x &= \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{32\pi^2}\Big(\mathcal{F}^1_h(A)+16\mathcal{F}^2_h(A)+4\mathcal{F}^3_h(A)\Big)\,,\\ \int_A|\Delta\bar v_h|^2\,\mathrm{d}x &= \frac{E^2}{(1-\nu^2)^2}\frac{s^2}{16\pi^2}\mathcal{F}^1_h(A)\,.\end{aligned}\tag{B.7}$$

We start by proving that

$$\lim_{h\to0}\frac{1}{h^2\log\frac Rh}\mathcal{G}(\bar v_h;A_{h,R}(0)) = \frac{E}{1-\nu^2}\frac{s^2}{8\pi}\,.\tag{B.8}$$

To this end, by the very definition of $\mathcal{G}$ in (1.9) and in view of (B.7), it is enough to show that

$$\lim_{h\to0}\frac{1}{h^2\log\frac Rh}\mathcal{F}^1_h(A_{h,R}(0)) = 4\pi\,,\tag{B.9}$$

$$\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^2_h(A_{h,R}(0)) = \frac{\pi}{8}\,, \tag{B.10}$$

$$\lim_{h\to 0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^3_h(A_{h,R}(0)) = \frac{\pi}{2}\,. \tag{B.11}$$

To this purpose, for every $0<h<R$, we set $N_h:=\lceil\frac{\log\frac{R}{h}}{\log 2}\rceil$, so that $2^{N_h-1}h\le R\le 2^{N_h}h$. We start by proving (B.9). By using the change of variable $x=2^{n-1}hy$ for every $n=1,\dots,N_h$, we have

$$\begin{aligned} h^2\sum_{n=1}^{N_h-1}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\le\mathcal{F}^1_h(A_{h,R}(0))\\ \le h^2\sum_{n=1}^{N_h}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\,. \end{aligned}\tag{B.12}$$

We start by discussing the limit of the left-hand side integral in (B.12). Let $K\le N_h$ and let h be sufficiently small; then we have

$$\begin{aligned} &\sum_{n=1}^{N_h-1}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\ge\sum_{n=\lfloor\frac{N_h}{K}\rfloor}^{N_h-1}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\\ &\ge\sum_{n=\lfloor\frac{N_h}{K}\rfloor}^{N_h-1}\int_{A_{1,2}(0)}\frac{4y_1^2}{|y|^4}\,\mathrm{d}y-C\ge N_h\Big(1-\frac{1}{K}\Big)4\pi\log 2-C\,, \end{aligned}$$

for some universal constant $C>0$, where in the second inequality we have used the Taylor expansion of the logarithm. By the very definition of N_h and by (B.12), we thus have that

$$\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^1_h(A_{h,R}(0))\ge\Big(1-\frac{1}{K}\Big)4\pi+\omega(h)\,, \tag{B.13}$$

where $\omega(h)\to 0$ as $h\to 0$. By sending first $h\to 0$ and then $K\to+\infty$ in (B.13) we get the inequality "$\ge$" in (B.9). As for the inequality "$\le$" in (B.9), we notice that

$$\begin{aligned} &\sum_{n=1}^{N_h}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\\ &\le C_0+C_1\Big\lceil\frac{N_h}{K}\Big\rceil+\sum_{n=\lceil\frac{N_h}{K}\rceil}^{N_h}\frac{2^{2n}}{4}\int_{A_{1,2}(0)}\log^2\frac{(y_1-2^{-n})^2+y_2^2}{(y_1+2^{-n})^2+y_2^2}\,\mathrm{d}y\\ &\le C_0+C_1\Big\lceil\frac{N_h}{K}\Big\rceil+\sum_{n=\lceil\frac{N_h}{K}\rceil}^{N_h}\int_{A_{1,2}(0)}\frac{4y_1^2}{|y|^4}\,\mathrm{d}y+C_2\le C_0+C_1\Big\lceil\frac{N_h}{K}\Big\rceil+N_h\Big(1-\frac{1}{K}\Big)4\pi\log 2+C_2\,, \end{aligned}$$

for some universal constants $C_0,C_1,C_2>0$, where we have used that $\log^2(1+t)\le 2t^2$ for $t>-1/2$. Therefore, by the very definition of N_h and by (B.12), we get

$$\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^1_h(A_{h,R}(0))\le\frac{C_1}{K}+\Big(1-\frac{1}{K}\Big)4\pi+\omega(h)\,,$$

where $\omega(h)\to 0$ as $h\to 0$. Now, sending first $h\to 0$ and then $K\to+\infty$, this implies also the inequality "$\le$" in (B.9). In order to prove (B.10), we notice that

$$\begin{aligned} \mathcal{F}^2_h(A_{h,R}(0)) =&h^2\int_0^{2\pi}\mathrm{d}\vartheta\sin^4\vartheta\cos^2\vartheta\int_h^R\frac{\rho^7}{\Big(\big(\rho^2-h\rho\cos\vartheta+\frac{h^2}{4}\big)\big(\rho^2+h\rho\cos\vartheta+\frac{h^2}{4}\big)\Big)^2}\,\mathrm{d}\rho\\ =&h^2\int_0^{2\pi}\mathrm{d}\vartheta\sin^4\vartheta\cos^2\vartheta\int_h^R\frac{\rho^7}{\Big(\rho^4-\rho^2\frac{h^2}{2}\cos(2\vartheta)+\frac{h^4}{16}\Big)^2}\,\mathrm{d}\rho\,, \end{aligned}$$

so that

$$
\begin{aligned}
&\frac{1}{\log\frac{R}{h}}\int_0^{2\pi}\mathrm{d}\vartheta\,\sin^4\vartheta\cos^2\vartheta\int_h^R\frac{\rho^7}{\big(\rho^2+\frac{h^2}{4}\big)^4}\,\mathrm{d}\rho\le\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^2_h(A_{h,R}(0))\\
\le&\frac{1}{\log\frac{R}{h}}\int_0^{2\pi}\mathrm{d}\vartheta\,\sin^4\vartheta\cos^2\vartheta\int_h^R\frac{\rho^7}{\big(\rho^2-\frac{h^2}{4}\big)^4}\,\mathrm{d}\rho\,.
\end{aligned}
\tag{B.14}
$$

By the change of variable $t=\frac{\rho}{h}$, we have that

$$
\int_h^R\frac{\rho^7}{\big(\rho^2\mp\frac{h^2}{4}\big)^4}\,\mathrm{d}\rho=\int_1^{\frac{R}{h}}\frac{t^7}{\big(t^2\mp\frac14\big)^4}\,\mathrm{d}t\,,
$$

and, by de l'Hôpital's rule, we get

$$
\lim_{h\to0}\frac{1}{\log\frac{R}{h}}\int_1^{\frac{R}{h}}\frac{t^7}{\big(t^2\mp\frac14\big)^4}\,\mathrm{d}t=\lim_{N\to+\infty}\frac{N^8}{\big(N^2\mp\frac14\big)^4}=1\,. \tag{B.15}
$$

Now, since

$$
\int_0^{2\pi}\sin^4\vartheta\cos^2\vartheta\,\mathrm{d}\vartheta=\frac{\pi}{8}\,, \tag{B.16}
$$

in view of (B.14) and (B.15), we obtain

$$
\lim_{h\to0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^2_h(A_{h,R}(0))=\frac{\pi}{8}\,, \tag{B.17}
$$

i.e., (B.10).

Finally, we prove that also (B.11) holds true. To this purpose, by using the change of variable $x=2^{n-1}hy$ for every $n=1,\dots,N_h$, we have

$$
\begin{aligned}
&h^2\sum_{n=1}^{N_h-1}\int_{A_{1,2}(0)}\frac{y_2^2\big(y_2^2-y_1^2+2^{-2n}\big)^2}{\Big(\Big(\big(y_1-2^{-n}\big)^2+y_2^2\Big)\Big(\big(y_1-2^{-n}\big)^2+y_2^2\Big)\Big)^2}\,\mathrm{d}y\le\mathcal{F}^3_h(A_{h,R}(0))\\
\le&h^2\sum_{n=1}^{N_h}\int_{A_{1,2}(0)}\frac{y_2^2\big(y_2^2-y_1^2+2^{-2n}\big)^2}{\Big(\Big(\big(y_1-2^{-n}\big)^2+y_2^2\Big)\Big(\big(y_1-2^{-n}\big)^2+y_2^2\Big)\Big)^2}\,\mathrm{d}y\,.
\end{aligned}
$$

Therefore, by arguing as in the proof of (B.9) we have that there exist two functions ω_1,ω_2 with $\omega_j(h)\to0$ as $h\to0$ and a constant $C>0$ such that

$$
\begin{aligned}
&\frac{1}{\log\frac{R}{h}}\Big(1-\frac{1}{K}\Big)N_h\frac{\pi}{2}\log2+\omega_1(h)\le\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^3_h(A_{h,R}(0))\\
\le&\frac{C}{\log\frac{R}{h}}\frac{N_h}{K}+\frac{1}{\log\frac{R}{h}}\Big(1-\frac{1}{K}\Big)N_h\frac{\pi}{2}\log2+\omega_2(h)\,,
\end{aligned}
$$

whence (B.11) follows by sending first $h\to0$ and then $K\to+\infty$.

Now we show that

$$
\lim_{h\to0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{G}(\bar v_h;B_h(0))=0\,. \tag{B.18}
$$

By the very definition of $\mathcal{G}$ in (1.9) and in view of (B.7), it is enough to prove that

$$
\lim_{h\to0}\frac{1}{h^2\log\frac{R}{h}}\mathcal{F}^k_h(B_h(0))=0\qquad\text{for every }k=1,2,3\,. \tag{B.19}
$$

Notice that

$$
0\le\mathcal{F}^1_h(B_h(0))\le2\pi\int_0^h\rho\log^2\frac{\big(\rho+\frac{h}{2}\big)^2}{\big(\rho-\frac{h}{2}\big)^2}\,\mathrm{d}\rho\,,
$$

so that using the change of variable $t = \frac{\rho}{h}$, we get

$$0 \le \frac{1}{h^2 \log \frac{R}{h}} \mathcal{F}_h^1(B_h(0)) \le 2\pi \frac{1}{\log \frac{R}{h}} \int_0^1 t \log^2 \frac{\big(t+\frac12\big)^2}{\big(t-\frac12\big)^2}\, \mathrm{d}t\,,$$

whence the claim (B.19) for $k = 1$ follows since

$$\int_0^1 t \log^2 \frac{\big(t+\frac12\big)^2}{\big(t-\frac12\big)^2}\, \mathrm{d}t < +\infty\,.$$

Now we show that

$$\lim_{h\to 0} \frac{1}{h^2} \mathcal{F}_h^2(B_h(0)) = 0\,. \tag{B.20}$$

To this purpose, we notice that, by the very definition of $\mathcal{F}_h^2$ in (B.5), by passing to polar coordinates (ρ, ϑ) and by using the change of variable $t = \frac{\rho}{h}$, we can write

$$\frac{1}{h^2}\mathcal{F}_h^2(B_h(0)) = \int_0^1 \mathrm{d}t \int_0^{2\pi} \frac{t^7 \sin^4\vartheta \cos^2\vartheta}{\Big(\big(t^2-\frac14\big)^2 + t^2 \sin^2\vartheta\Big)^2}\, \mathrm{d}\vartheta \le \int_0^1 \mathrm{d}t \int_0^{2\pi} \frac{t^7 \sin^4\vartheta}{\Big(\big(t^2-\frac14\big)^2 + t^2 \sin^2\vartheta\Big)^2}\, \mathrm{d}\vartheta\,;$$

since the integrand above is π-periodic and bounded when ϑ is far away from 0, π, and 2π, in order to obtain (B.20) it is enough to show that for $\varepsilon > 0$ small enough we have

$$\int_0^1 \mathrm{d}t \int_0^{\varepsilon} \frac{t^7 \sin^4\vartheta}{\Big(\big(t^2-\frac14\big)^2 + t^2 \sin^2\vartheta\Big)^2}\, \mathrm{d}\vartheta < +\infty\,. \tag{B.21}$$

By a first-order Taylor approximation, the integral above is equivalent to

$$\begin{aligned}
&\int_0^1 \mathrm{d}t \int_0^{\varepsilon} \frac{t^7 \vartheta^4}{\Big(\big(t^2-\frac14\big)^2 + t^2 \vartheta^2\Big)^2}\, \mathrm{d}\vartheta \\
=&\frac12 \int_0^1 \bigg(\varepsilon t^3\Big(\frac{\big(t^2-\frac14\big)^2}{(t^2-\frac14)^2 + t^2\varepsilon^2} + 2\Big) - 3t^2\Big(t^2 - \frac14\Big) \arctan \frac{\varepsilon t}{t^2-\frac14}\bigg)\, \mathrm{d}t < +\infty\,,
\end{aligned}$$

which proves (B.21) and hence (B.20). Analogously, by the very definition of $\mathcal{F}_h^3$ in (B.6), by passing to polar coordinates (ρ, ϑ) and by using the change of variable $t = \frac{\rho}{h}$, we have that

$$\frac{1}{h^2}\mathcal{F}_h^3(B_h(0)) = \int_0^1 \mathrm{d}t \int_0^{2\pi} \frac{t^3 \sin^2\vartheta \big(\frac14 - t^2\cos 2\vartheta\big)^2}{\Big(\big(t^2-\frac14\big)^2 + t^2 \sin^2\vartheta\Big)^2}\, \mathrm{d}\vartheta < +\infty\,,$$

where the boundedness can be proved by arguing as in the proof of (B.21). Indeed, by a first-order Taylor approximation, the integral above close to $\vartheta = 0, \pi, 2\pi$ is equivalent to

$$\int_0^1 \mathrm{d}t \int_0^{\varepsilon} \frac{t^3 \vartheta^2 \big(\frac14 - t^2 + 2t^2\vartheta^2\big)^2}{\Big(\big(t^2-\frac14\big)^2 + t^2 \vartheta^2\Big)^2}\, \mathrm{d}\vartheta\,,$$

which can be proved to be finite by a straightforward computation. This proves (B.19) also for $k = 3$, so that, by (B.8) and (B.18), the proof is concluded. □

Appendix C. Proof of Lemma 3.6

This section is devoted to the proof of Lemma 3.6.

Proof of Lemma 3.6. We preliminarily show that the function $W^s_{0,\varepsilon}$ defined in (3.26) is in $\mathscr{B}_{\varepsilon,R}$. To this purpose, we first notice that

$$\alpha_\varepsilon + \frac{\beta_\varepsilon}{R^2} + \gamma_\varepsilon R^2 + 2\log R^2 = 2\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2} - 2\log R^2 + 2\frac{\varepsilon^2}{R^2+\varepsilon^2} - 2\frac{R^2}{R^2+\varepsilon^2} + 2\log R^2 = 0\,,$$

and hence

$$W^s_{0,\varepsilon}=0 \qquad \text{on } \partial B_R(0)\,. \tag{C.1}$$

We define the function $\widehat{W}_{0,\varepsilon}\colon A_{\varepsilon,R}(0)\to\mathbb{R}$ as

$$\widehat{W}_{0,\varepsilon}(x):=\Big(\alpha_\varepsilon+\beta_\varepsilon\frac{1}{|x|^2}+\gamma_\varepsilon|x|^2+2\log|x|^2\Big)x_1\,, \tag{C.2}$$

and we notice that

$$W^s_{0,\varepsilon}\equiv\frac{s}{16\pi}\frac{E}{1-\nu^2}\widehat{W}_{0,\varepsilon} \qquad \text{in } A_{\varepsilon,R}(0)\,. \tag{C.3}$$

For every $x\in A_{\varepsilon,R}(0)$

$$\nabla\widehat{W}_{0,\varepsilon}(x)=\begin{pmatrix}\alpha_\varepsilon+\beta_\varepsilon\dfrac{x_2^2-x_1^2}{|x|^4}+\gamma_\varepsilon(|x|^2+2x_1^2)+2\log|x|^2+4\dfrac{x_1^2}{|x|^2}\\ -2\beta_\varepsilon\dfrac{x_1x_2}{|x|^4}+2\gamma_\varepsilon x_1x_2+4\dfrac{x_1x_2}{|x|^2}\end{pmatrix}$$

whence, for $x\in\partial B_R(0)$, we deduce that

$$\begin{aligned}\partial_n\widehat{W}_{0,\varepsilon}(x)=&\frac{x_1}{R}\Big(\alpha_\varepsilon+\beta_\varepsilon\frac{x_2^2-x_1^2}{|x|^4}+\gamma_\varepsilon(|x|^2+2x_1^2)+2\log|x|^2+4\frac{x_1^2}{|x|^2}\Big)\\&+\frac{x_2}{R}\Big(-2\beta_\varepsilon\frac{x_1x_2}{|x|^4}+2\gamma_\varepsilon x_1x_2+4\frac{x_1x_2}{|x|^2}\Big)\\=&\frac{x_1}{R}\Big(\alpha_\varepsilon-\frac{\beta_\varepsilon}{|x|^2}+3\gamma_\varepsilon|x|^2+2\log|x|^2+4\Big)\\=&\frac{x_1}{R}\Big(\alpha_\varepsilon-\frac{\beta_\varepsilon}{R^2}+3\gamma_\varepsilon R^2+2\log R^2+4\Big)\\=&\frac{x_1}{R}\Big(2\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}-2\frac{\varepsilon^2}{R^2+\varepsilon^2}-6\frac{R^2}{R^2+\varepsilon^2}+4\Big)=0\end{aligned}$$

and, consequently,

$$\partial_nW^s_{0,\varepsilon}=0 \qquad \text{on } \partial B_R(0)\,. \tag{C.4}$$

Moreover, it is immediate to see that $W^s_{0,\varepsilon}\in C^0(B_R(0))$. Furthermore, the inner and outer traces of $\partial_tW^s_{0,\varepsilon}$ at $\partial B_\varepsilon(0)$ are continuous so that $W^s_{0,\varepsilon}\in H^1(B_R(0))$. Therefore, in order to check that $W^s_{0,\varepsilon}\in H^2(B_R(0))$ it is enough to show that

$$\partial_t\nabla\hat{w}_{0,\varepsilon}=0 \qquad \text{on } \partial B_\varepsilon(0). \tag{C.5}$$

To this end, we observe

$$\begin{aligned}&\nabla^2\widehat{W}_{0,\varepsilon}(x)\\=&\begin{pmatrix}-2\beta_\varepsilon\dfrac{x_1(3x_2^2-x_1^2)}{|x|^6}+6\gamma_\varepsilon x_1+4x_1\dfrac{x_1^2+3x_2^2}{|x|^4} & 2\beta_\varepsilon\dfrac{x_2(3x_1^2-x_2^2)}{|x|^6}+2\gamma_\varepsilon x_2+4x_2\dfrac{x_2^2-x_1^2}{|x|^4}\\ 2\beta_\varepsilon\dfrac{x_2(3x_1^2-x_2^2)}{|x|^6}+2\gamma_\varepsilon x_2+4x_2\dfrac{x_2^2-x_1^2}{|x|^4} & -2\beta_\varepsilon\dfrac{x_1(x_1^2-3x_2^2)}{|x|^6}+2\gamma_\varepsilon x_1+4x_1\dfrac{x_1^2-x_2^2}{|x|^4}\end{pmatrix}\end{aligned}$$

so that, for $x\in\partial B_\varepsilon(0)$, we have

$$\begin{aligned}\partial_{x_1}\nabla\widehat{W}_{0,\varepsilon}\cdot t=&-\frac{x_2}{\varepsilon}\Big(-2\beta_\varepsilon\frac{x_1(3x_2^2-x_1^2)}{|x|^6}+6\gamma_\varepsilon x_1+4x_1\frac{|x|^2+2x_2^2}{|x|^4}\Big)\\&+\frac{x_1}{\varepsilon}\Big(2\beta_\varepsilon\frac{x_2(3x_1^2-x_2^2)}{|x|^6}+2\gamma_\varepsilon x_2+4x_2\frac{x_2^2-x_1^2}{|x|^4}\Big)\\=&\frac{1}{\varepsilon}x_1x_2\Big(4\frac{\beta_\varepsilon}{|x|^4}-4\gamma_\varepsilon-\frac{8}{|x|^2}\Big)=\frac{1}{\varepsilon}x_1x_2\Big(4\frac{\beta_\varepsilon}{\varepsilon^4}-4\gamma_\varepsilon-\frac{8}{\varepsilon^2}\Big)\\=&\frac{8}{\varepsilon}x_1x_2\Big(\frac{R^2}{\varepsilon^2(R^2+\varepsilon^2)}+\frac{1}{R^2+\varepsilon^2}-\frac{1}{\varepsilon^2}\Big)=0\end{aligned} \tag{C.6}$$

and, analogously,

$$\partial_{x_2}\nabla\widehat{W}_{0,\varepsilon}\cdot t=\frac{2}{\varepsilon}(x_2^2-x_1^2)\Big(\frac{\beta_\varepsilon}{\varepsilon^4}-\gamma_\varepsilon-\frac{2}{\varepsilon^2}\Big)=0\,. \tag{C.7}$$

Finally, by (C.1), (C.4), (C.6), (C.7), and using that $W^s_{0,\varepsilon}\in C^4(A_{\varepsilon,R}(0))$, we deduce that $W^s_{0,\varepsilon}\in\mathscr{B}_{\varepsilon,R}$.

Now we prove that $W^s_{0,\varepsilon}$ is the minimizer of $\mathcal{J}^s_{0,\varepsilon}$ in $\mathscr{B}_{\varepsilon,R}$. In view of (3.21), for every $\phi\in\mathscr{B}_{\varepsilon,R}$, $W^s_{0,\varepsilon}$ must satisfy

$$\begin{aligned}
0=&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon}+t\phi)\\
=&\frac{1+\nu}{E}\Big(\int_{A_{\varepsilon,R}(0)}\nabla^2W^s_{0,\varepsilon}:\nabla^2\phi\,\mathrm{d}x-\nu\int_{A_{\varepsilon,R}(0)}\Delta W^s_{0,\varepsilon}\,\Delta\phi\,\mathrm{d}x\Big)+\frac{s}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}\phi\,\mathrm{d}\mathcal{H}^1\\
=&\frac{1-\nu^2}{E}\int_{A_{\varepsilon,R}(0)}\Delta^2W^s_{0,\varepsilon}\,\phi\,\mathrm{d}x\\
&+\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(0)}\partial_n\Delta W^s_{0,\varepsilon}\phi\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\int_{\partial B_\varepsilon(0)}\big(\nu\Delta W^s_{0,\varepsilon}-(\nabla^2W^s_{0,\varepsilon})_{nn}\big)\partial_n\phi\,\mathrm{d}\mathcal{H}^1\\
&+\int_{\partial B_\varepsilon(0)}\frac{s}{2\pi\varepsilon}\partial_{x_1}\phi\,\mathrm{d}\mathcal{H}^1\,,
\end{aligned}$$

where we have used that $\phi=\partial_n\phi=0$ on $\partial B_R(0)$ and integration by parts to get

$$\begin{aligned}
\int_{A_{\varepsilon,R}(0)}\nabla^2W^s_{0,\varepsilon}:\nabla^2\phi\,\mathrm{d}x\;=\;&\int_{A_{\varepsilon,R}(0)}\Delta^2W^s_{0,\varepsilon}\,\phi\,\mathrm{d}x+\int_{\partial A_{\varepsilon,R}(0)}\langle\nabla^2W^s_{0,\varepsilon}n,\nabla\phi\rangle\,\mathrm{d}\mathcal{H}^1\\
&-\int_{\partial A_{\varepsilon,R}(0)}\partial_n(\Delta W^s_{0,\varepsilon})\phi\,\mathrm{d}\mathcal{H}^1\\
=\;&\int_{A_{\varepsilon,R}(0)}\Delta^2W^s_{0,\varepsilon}\,\phi\,\mathrm{d}x+\int_{\partial B_\varepsilon(0)}\big(\phi\,\partial_n(\Delta W^s_{0,\varepsilon})-\langle\nabla^2W^s_{0,\varepsilon}n,\nabla\phi\rangle\big)\,\mathrm{d}\mathcal{H}^1\,,\\
\int_{A_{\varepsilon,R}(0)}\Delta W^s_{0,\varepsilon}\,\Delta\phi\,\mathrm{d}x\;=\;&\int_{A_{\varepsilon,R}(0)}\Delta^2W^s_{0,\varepsilon}\,\phi\,\mathrm{d}x+\int_{\partial A_{\varepsilon,R}(0)}\Delta W^s_{0,\varepsilon}\,\partial_n\phi\,\mathrm{d}\mathcal{H}^1\\
&-\int_{\partial A_{\varepsilon,R}(0)}\phi\,\partial_n(\Delta W^s_{0,\varepsilon})\,\mathrm{d}\mathcal{H}^1\\
=\;&\int_{A_{\varepsilon,R}(0)}\Delta^2W^s_{0,\varepsilon}\,\phi\,\mathrm{d}x+\int_{\partial B_\varepsilon(0)}\big(\phi\,\partial_n(\Delta W^s_{0,\varepsilon})-\Delta W^s_{0,\varepsilon}\,\partial_n\phi\big)\,\mathrm{d}\mathcal{H}^1\,.
\end{aligned}$$

Therefore, proving the minimality of $W^s_{0,\varepsilon}$ is equivalent to showing that

$$\Delta^2W^s_{0,\varepsilon}=0\qquad\text{in }A_{\varepsilon,R}(0)\,, \tag{C.8}$$

$$\begin{aligned}
&\frac{1-\nu^2}{E}\int_{\partial B_\varepsilon(0)}\partial_n\Delta W^s_{0,\varepsilon}\phi\,\mathrm{d}\mathcal{H}^1+\frac{1+\nu}{E}\int_{\partial B_\varepsilon(0)}\big(\nu\Delta W^s_{0,\varepsilon}-(\nabla^2W^s_{0,\varepsilon})_{nn}\big)\partial_n\phi\,\mathrm{d}\mathcal{H}^1\\
&\qquad+\int_{\partial B_\varepsilon(0)}\frac{s}{2\pi\varepsilon}\partial_{x_1}\phi\,\mathrm{d}\mathcal{H}^1=0\qquad\text{for every }\phi\in\mathscr{B}_{\varepsilon,R}\,.
\end{aligned} \tag{C.9}$$

By (C.3) and the very definition of $\widehat{W}_{0,\varepsilon}$ in (C.2), the biharmonicity in (C.8) follows by a direct computation, so that we are left with proving (C.9). To this purpose, we notice that

$$\begin{aligned}
\Delta W^s_{0,\varepsilon}&=\frac{s}{16\pi}\frac{E}{1-\nu^2}\Delta\widehat{W}_{0,\varepsilon}=\frac{s}{2\pi}\frac{E}{1-\nu^2}x_1\Big(\gamma_\varepsilon+\frac{1}{|x|^2}\Big)\,,\\
\partial_n\Delta W^s_{0,\varepsilon}&=\frac{s}{2\pi}\frac{E}{1-\nu^2}\frac{x_1}{|x|}\Big(\gamma_\varepsilon-\frac{1}{|x|^2}\Big)\,,\\
(\nabla^2W^s_{0,\varepsilon})_{nn}&=\frac{s}{8\pi}\frac{E}{1-\nu^2}x_1\Big(\frac{\beta_\varepsilon}{|x|^4}+3\gamma_\varepsilon+\frac{2}{|x|^2}\Big)\,,
\end{aligned}$$

whence we deduce, recalling (3.27),

$$\Delta W^s_{0,\varepsilon}\Big|_{\partial B_\varepsilon(0)} = \frac{s}{2\pi}\frac{E}{1-\nu^2}x_1\Big(\gamma_\varepsilon+\frac{1}{\varepsilon^2}\Big) = \frac{s}{2\pi}\frac{E}{1-\nu^2}\frac{R^2-\varepsilon^2}{\varepsilon^2(R^2+\varepsilon^2)}x_1\,, \tag{C.10}$$

$$\partial_n\Delta W^s_{0,\varepsilon}\Big|_{\partial B_\varepsilon(0)} = \frac{s}{2\pi}\frac{E}{1-\nu^2}\frac{x_1}{\varepsilon}\Big(\gamma_\varepsilon-\frac{1}{\varepsilon^2}\Big) = -\frac{s}{2\pi}\frac{E}{1-\nu^2}\frac{R^2+3\varepsilon^2}{\varepsilon^2(R^2+\varepsilon^2)}\frac{x_1}{\varepsilon}\,, \tag{C.11}$$

$$(\nabla^2 W^s_{0,\varepsilon})_{nn}\Big|_{\partial B_\varepsilon(0)} = \frac{s}{8\pi}\frac{E}{1-\nu^2}x_1\Big(\frac{\beta_\varepsilon}{\varepsilon^4}+3\gamma_\varepsilon+\frac{2}{\varepsilon^2}\Big) = \frac{s}{2\pi}\frac{E}{1-\nu^2}\frac{R^2-\varepsilon^2}{\varepsilon^2(R^2+\varepsilon^2)}x_1\,. \tag{C.12}$$

Furthermore, every $\phi\in\mathscr{B}_{\varepsilon,R}$ satisfies $\phi(x)=a_\phi+b_\phi x_1+c_\phi x_2$ for every $x\in\partial B_\varepsilon(0)$, for some $a_\phi,b_\phi,c_\phi\in\mathbb{R}$, so that the equation in (C.9) can be rewritten as

$$\begin{cases} \dfrac{1-\nu^2}{E}\displaystyle\int_{\partial B_\varepsilon(0)}\partial_n\Delta W^s_{0,\varepsilon}\,\mathrm{d}\mathcal{H}^1=0\\ \displaystyle\int_{\partial B_\varepsilon(0)}x_1\Big(\frac{1-\nu^2}{E}\partial_n\Delta W^s_{0,\varepsilon}+\frac{1+\nu}{E}\frac{1}{\varepsilon}\Big(\nu\Delta W^s_{0,\varepsilon}-\big(\nabla^2W^s_{0,\varepsilon}\big)_{nn}\Big)\Big)\,\mathrm{d}\mathcal{H}^1=-s\\ \displaystyle\int_{\partial B_\varepsilon(0)}x_2\Big(\frac{1-\nu^2}{E}\partial_n\Delta W^s_{0,\varepsilon}+\frac{1+\nu}{E}\frac{1}{\varepsilon}\Big(\nu\Delta W^s_{0,\varepsilon}-\big(\nabla^2W^s_{0,\varepsilon}\big)_{nn}\Big)\Big)\,\mathrm{d}\mathcal{H}^1=0\,, \end{cases} \tag{C.13}$$

which follow by straightforward computations from (C.10), (C.11), and (C.12).

Now we compute $\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})$. As for the second summand in the right-hand side of (3.20), we have

$$\frac{s}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}W^s_{0,\varepsilon}\,\mathrm{d}\mathcal{H}^1=-\frac{s^2}{4\pi}\frac{E}{1-\nu^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)\,. \tag{C.14}$$

Moreover,

$$\begin{aligned}\int_{A_{\varepsilon,R}(0)}|\Delta W^s_{0,\varepsilon}|^2\,\mathrm{d}x&=\frac{s^2}{4\pi^2}\frac{E^2}{(1-\nu^2)^2}\int_{A_{\varepsilon,R}(0)}x_1^2\Big(\gamma_\varepsilon+\frac{1}{|x|^2}\Big)^2\mathrm{d}x=\frac{s^2}{4\pi}\frac{E^2}{(1-\nu^2)^2}\int_\varepsilon^R\rho^3\Big(\gamma_\varepsilon+\frac{1}{\rho^2}\Big)^2\mathrm{d}\rho\\ &=\frac{s^2}{4\pi}\frac{E^2}{(1-\nu^2)^2}\Big(\gamma_\varepsilon^2\frac{R^4-\varepsilon^4}{4}+\gamma_\varepsilon(R^2-\varepsilon^2)+\log\frac{R}{\varepsilon}\Big)\\ &=\frac{s^2}{4\pi}\frac{E^2}{(1-\nu^2)^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)\end{aligned}$$

and

$$\begin{aligned}|\nabla^2\widehat{W}_{0,\varepsilon}|^2&=|\Delta\widehat{W}_{0,\varepsilon}|^2-2\partial^2_{x_1^2}\widehat{W}_{0,\varepsilon}\partial^2_{x_2^2}\widehat{W}_{0,\varepsilon}+2|\partial^2_{x_1x_2}\widehat{W}_{0,\varepsilon}|^2\\ &=64x_1^2\Big(\gamma_\varepsilon+\frac{1}{|x|^2}\Big)^2+\frac{8}{|x|^6}\beta_\varepsilon^2-16\gamma_\varepsilon^2x_1^2-\frac{32}{|x|^6}\beta_\varepsilon x_2^2-\frac{32}{|x|^2}\gamma_\varepsilon x_1^2\\ &\quad+8(x_1^2-x_2^2)\Big(-\gamma_\varepsilon^2+2\frac{\beta_\varepsilon\gamma_\varepsilon}{|x|^4}-\frac{4}{|x|^4}-\frac{4}{|x|^2}\gamma_\varepsilon\Big)\,,\end{aligned}$$

so that, recalling (3.27),

$$\int_{A_{\varepsilon,R}(0)}|\nabla^2W^s_{0,\varepsilon}|^2\,\mathrm{d}x=\frac{s^2}{4\pi}\frac{E^2}{(1-\nu^2)^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)\,.$$

It follows that

$$\mathcal{G}(W^s_{0,\varepsilon};A_{\varepsilon,R}(0))=\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)\,, \tag{C.15}$$

and hence, in view of (C.14),

$$\mathcal{J}^s_{0,\varepsilon}(W^s_{0,\varepsilon})=-\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\Big(\log\frac{R}{\varepsilon}-\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}\Big)$$

i.e., (3.28). □

The next result follows from the proof of Lemma 3.6 by straightforward computations.

Corollary C.1. *Let $s\in\mathbb{R}\setminus\{0\}$, $0<\varepsilon<R$ and let $w^s_{0,\varepsilon}$ be the function defined in* (3.26)*. Then, for every $\varepsilon<r\le R$,*

$$\begin{aligned}\mathcal{G}(W^s_{0,\varepsilon};A_{\varepsilon,r}(0)) =&\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\log\frac{r}{\varepsilon}+\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\frac{r^2-\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{r^2+\varepsilon^2}{R^2+\varepsilon^2}-2\Big)\\&+\frac{s^2}{32\pi}\frac{E}{(1-\nu)^2(1+\nu)}\frac{r^2-\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{R^2}{r^2}-1\Big)\Big(\frac{r^2+\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{R^2}{r^2}+1\Big)-2\Big),\end{aligned}$$

and hence

$$\mathcal{G}(W^s_{0,\varepsilon};A_{\varepsilon,r}(0))+\frac{s}{2\pi\varepsilon}\int_{\partial B_\varepsilon(0)}\partial_{x_1}W^s_{0,\varepsilon}\,\mathrm{d}\mathcal{H}^1=-\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\log\frac{1}{\varepsilon}+\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\log r+f_\varepsilon(r,R;s),$$

where

$$\begin{aligned}f_\varepsilon(r,R;s):=&\frac{s^2}{8\pi}\frac{E}{1-\nu^2}\Big(2\frac{R^2-\varepsilon^2}{R^2+\varepsilon^2}+\frac{r^2-\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{r^2+\varepsilon^2}{R^2+\varepsilon^2}-2\Big)-2\log R\Big)\\&+\frac{s^2}{32\pi}\frac{E}{(1-\nu)^2(1+\nu)}\frac{r^2-\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{R^2}{r^2}-1\Big)\Big(\frac{r^2+\varepsilon^2}{R^2+\varepsilon^2}\Big(\frac{R^2}{r^2}+1\Big)-2\Big).\end{aligned}\tag{C.16}$$

Appendix D. ε-independent integral inequalities for H^2 functions

Let $\alpha=\sum_{j=1}^J b^j\boldsymbol{\delta}_{x^j}\in\mathscr{E}\mathscr{D}(\Omega)$. Here we prove a Poincaré-type inequality for functions in $H^2(\Omega_\varepsilon(\alpha))$ where the Poincaré constant is shown to be independent of ε. The proof is obtained by combining the "ε-independent Poincaré inequality" contained in [20, Proposition A.1] and the generalized Poincaré inequality contained in [25, Theorem 6.1-8(b)], which we recall here.

Proposition D.1 ([25, Theorem 6.1-8(b)])**.** *Let $\Omega'\subset\Omega$ and let $\Gamma_0\subseteq\partial\Omega'$ be a portion of the boundary with $\mathcal{H}^1(\Gamma_0)>0$. Then there exists a constant $C(\Omega')>0$ depending only on Ω' such that for every function $u\in H^1(\Omega)$ it holds*

$$\int_{\Omega'}|u(x)|^2\,\mathrm{d}x\le C(\Omega')\Big(\int_{\Omega'}|\nabla u(x)|^2\,\mathrm{d}x+\Big|\int_{\Gamma_0}u(x)\,\mathrm{d}\mathcal{H}^1(x)\Big|^2\Big).\tag{D.1}$$

Take now $\Omega'\subset\Omega$ such that $\partial\Omega'\supset\partial\Omega$ and $\Gamma_0=\partial\Omega$ in Proposition D.1. Moreover, let f be a function which is smooth in a neighborhood of $\partial\Omega$. Then for every $u\in H^2(\Omega)$ with $u=f$ and $\partial_n u=\partial_n f$ on $\partial\Omega$, thanks to Jensen's inequality, formula (D.1) reads

$$\int_{\Omega'}|u(x)|^2\,\mathrm{d}x\le C(\Omega')\int_{\Omega'}|\nabla u(x)|^2\,\mathrm{d}x+C(\Omega',\partial\Omega)\int_{\partial\Omega}|f(x)|^2\,\mathrm{d}\mathcal{H}^1(x);\tag{D.2}$$

analogously, noticing that $\nabla(u-f)=0$ on $\partial\Omega$, by applying (D.1) to $\partial_{x_1}u$ and $\partial_{x_2}u$, we obtain

$$\int_{\Omega'}|\nabla u(x)|^2\,\mathrm{d}x\le 2C(\Omega')\int_{\Omega'}|\nabla^2 u(x)|^2\,\mathrm{d}x+C(\Omega',\partial\Omega)\int_{\partial\Omega}|\nabla f(x)|^2\,\mathrm{d}\mathcal{H}^1(x).\tag{D.3}$$

Proposition D.2 (ε-independent Poincaré inequality)**.** *Let $\alpha\in\mathscr{E}\mathscr{D}(\Omega)$ and let $0<\varepsilon<\frac{D}{2}$ with D defined in* (4.2)*. Then there exists a constant $C_1(\Omega,\alpha)>0$ depending only on Ω and on $\operatorname{spt}\alpha$, and independent of ε, such that the following holds true. For every function f which is smooth in a neighborhood of $\partial\Omega$ and for every $u\in H^2(\Omega)$ with $u=f$ and $\partial_n u=\partial_n f$ on $\partial\Omega$*

$$\int_{\Omega_\varepsilon(\alpha)}|u(x)|^2\,\mathrm{d}x+\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x\le C_1(\Omega,\alpha)\Big(\int_{\Omega_\varepsilon(\alpha)}|\nabla^2 u(x)|^2\,\mathrm{d}x+\|f\|^2_{C^\infty(\partial\Omega)}\Big).\tag{D.4}$$

Proof. The proof follows [20, Proposition A.1]. We recall here the main lines of the proof for the reader's convenience.

Let $j=1,\dots,J$ be fixed and let the pair $(r;\vartheta)$ denote the polar coordinates centered at x^j. Let $\varepsilon\le s\le\rho<D$ (with D defined in (4.2)) and let $\vartheta\in[0,2\pi]$. By the Fundamental Theorem of Calculus, we can write

$$u(s,\vartheta)=u(\rho,\vartheta)-\int_s^\rho\frac{\partial u}{\partial r}(r,\vartheta)\,\mathrm{d}r,$$

so that (by recalling that $(a-b)^2 \le 2a^2 + 2b^2$)

$$|u(s,\vartheta)|^2 \le 2|u(\rho,\vartheta)|^2 + 2\bigg|\int_s^\rho \frac{\partial u}{\partial r}(r,\vartheta)\,\mathrm{d}r\bigg|^2$$

and in turn, by Jensen's inequality,

$$|u(s,\vartheta)|^2 \le 2|u(\rho,\vartheta)|^2 + 2(\rho-s)\int_s^\rho \bigg|\frac{\partial u}{\partial r}(r,\vartheta)\bigg|^2\,\mathrm{d}r \le 2|u(\rho,\vartheta)|^2 + 2D\int_s^D \bigg|\frac{\partial u}{\partial r}(r,\vartheta)\bigg|^2\,\mathrm{d}r\,.$$

We now multiply by s and integrate with respect to ϑ to obtain

$$\begin{aligned}\int_0^{2\pi} s|u(s,\vartheta)|^2\,\mathrm{d}\vartheta &\le 2s\int_0^{2\pi}|u(\rho,\vartheta)|^2\,\mathrm{d}\vartheta + 2D\int_0^{2\pi}\int_s^D \bigg|\frac{\partial u}{\partial r}(r,\vartheta)\bigg|^2 s\,\mathrm{d}r\,\mathrm{d}\vartheta\\ &\le 2\int_0^{2\pi}|u(\rho,\vartheta)|^2\,\rho\,\mathrm{d}\vartheta + 2D\int_0^{2\pi}\int_\varepsilon^D |\nabla u(r,\vartheta)|^2\,r\,\mathrm{d}r\,\mathrm{d}\vartheta\\ &= 2\int_0^{2\pi}|u(\rho,\vartheta)|^2\,\rho\,\mathrm{d}\vartheta + 2D\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\,.\end{aligned} \tag{D.5}$$

We now integrate with respect to s in $\big[\varepsilon,\frac{D}{2}\big]$ (notice that the right-hand side does not depend on s) to get

$$\int_\varepsilon^{\frac{D}{2}}\int_0^{2\pi}|u(s,\vartheta)|^2\,s\,\mathrm{d}s\,\mathrm{d}\vartheta \le 2\Big(\frac{D}{2}-\varepsilon\Big)\bigg(\int_0^{2\pi}|u(\rho,\vartheta)|^2\,\rho\mathrm{d}\vartheta + D\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\bigg)\,,$$

whence

$$\int_{A_{\varepsilon,\frac{D}{2}}(x^j)}|u(x)|^2\,\mathrm{d}x \le D\int_0^{2\pi}|u(\rho,\vartheta)|^2\,\rho\,\mathrm{d}\vartheta + D^2\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\,;$$

an integration with respect to ρ in $\big[\frac{D}{2},D\big]$ now yields

$$\begin{aligned}\int_{A_{\varepsilon,\frac{D}{2}}(x^j)}|u(x)|^2\,\mathrm{d}x &\le 2\int_{A_{\frac{D}{2},D}(x^j)}|u(x)|^2\,\mathrm{d}x + D^2\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\\ &\le 2\int_{\Omega_{\frac{D}{2}}(\alpha)}|u(x)|^2\,\mathrm{d}x + D^2\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\\ &\le 2C\big(\Omega_{\frac{D}{2}}(\alpha)\big)\int_{\Omega_{\frac{D}{2}}(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x + 2C\big(\Omega_{\frac{D}{2}}(\alpha)\big)\bigg|\int_{\partial\Omega} f(x)\,\mathrm{d}\mathcal{H}^1(x)\bigg|^2\\ &\quad + D^2\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x\,,\end{aligned}$$

where we have used (D.2) in the last inequality. Therefore, by using (D.2) again, we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon(\alpha)} |u(x)|^2\,\mathrm{d}x &= \sum_{j=1}^J \int_{A_{\varepsilon,\frac{D}{2}}(x^j)} |u(x)|^2\,\mathrm{d}x + \int_{\Omega_{\frac{D}{2}}(\alpha)} |u(x)|^2\,\mathrm{d}x \\
&\le (2J+1)C\big(\Omega_{\frac{D}{2}}(\alpha)\big)\int_{\Omega_{\frac{D}{2}}(\alpha)} |\nabla u(x)|^2\,\mathrm{d}x + D^2\sum_{j=1}^J\int_{A_{\varepsilon,D}(x^j)}|\nabla u(x)|^2\,\mathrm{d}x \\
&\quad + 2(J+1)C\big(\Omega_{\frac{D}{2}}(\alpha),\partial\Omega\big)\int_{\partial\Omega}|f(x)|^2\,\mathrm{d}\mathcal{H}^1(x) \\
&\le (2J+1)C_{\Omega_{\frac{D}{2}}(\alpha)}\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x + JD^2\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x \\
&\quad + 2(J+1)C\big(\Omega_{\frac{D}{2}}(\alpha),\partial\Omega\big)\int_{\partial\Omega}|f(x)|^2\,\mathrm{d}\mathcal{H}^1(x) \\
&\le \max\big\{(2J+1)C\big(\Omega_{\frac{D}{2}}(\alpha)\big), JD^2\big\}\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\mathrm{d}x \\
&\quad + 2(J+1)C\big(\Omega_{\frac{D}{2}}(\alpha),\partial\Omega\big)\int_{\partial\Omega}|f(x)|^2\,\mathrm{d}\mathcal{H}^1(x) \\
&\le \widetilde{C}_1(\Omega,\alpha)\Big(\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x + \|f\|^2_{L^\infty(\partial\Omega)}\Big),
\end{aligned}
\tag{D.6}
$$

where we have set $\widetilde{C}_1(\Omega,\alpha) \coloneqq \max\big\{(2J+1)C\big(\Omega_{\frac{D}{2}}(\alpha)\big), JD^2\big\}+2(J+1)C\big(\Omega_{\frac{D}{2}}(\alpha)\big)$. By repeating the same reasoning for $\partial_{x_1}u$ and $\partial_{x_2}u$ and by using (D.3) in place of (D.2), we obtain

$$
\int_{\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}x \le 2\widetilde{C}_1^2(\Omega,\alpha)\Big(\int_{\Omega_\varepsilon(\alpha)}|\nabla^2 u(x)|^2\,\mathrm{d}x + \|\nabla f\|^2_{L^\infty(\partial\Omega)}\Big),
$$

and the proposition is proved with $C_1(\Omega,\alpha) \coloneqq 3\widetilde{C}_1^2(\Omega,\alpha)$. □

Proposition D.3 (ε-independent trace inequality)**.** *Let $\alpha \in \mathscr{E}\mathscr{D}(\Omega)$ and let $\varepsilon > 0$ satisfy (4.2). Then, there exists a constant $C_2(\Omega,\alpha) > 0$ depending only on Ω and on $\operatorname{spt}\alpha$, and independent of ε, such that, for every function f which is smooth in a neighborhood of $\partial\Omega$ and for every $u \in H^2(\Omega)$ with $u = f$ and $\partial_n u = \partial_n f$ on $\partial\Omega$, the following fact holds true:*

$$
\int_{\partial\Omega_\varepsilon(\alpha)}|u(x)|^2\,\mathrm{d}\mathcal{H}^1(x) + \int_{\partial\Omega_\varepsilon(\alpha)}|\nabla u(x)|^2\,\mathrm{d}\mathcal{H}^1(x) \le C_2(\Omega,\alpha)\Big(\int_{\Omega_\varepsilon(\alpha)}|\nabla^2 u(x)|^2\,\mathrm{d}x + \|f\|^2_{C^\infty(\partial\Omega)}\Big).
$$

Proof. By [20, Proposition A.6], there exists a constant $C(\Omega,\alpha)$ depending only on Ω and on $\operatorname{spt}\alpha$ such that, for any function $v \in H^1(\Omega)$, there holds

$$
\int_{\partial\Omega_\varepsilon(\alpha)}|v|^2\,\mathrm{d}x \le C(\Omega,\alpha)\Big(\int_{\Omega_\varepsilon(\alpha)}|v|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon(\alpha)}|\nabla v|^2\,\mathrm{d}x\Big).
\tag{D.7}
$$

We conclude by applying (D.7) with $v = u$, $v = \partial_{x_1}u$, and $v = \partial_{x_2}u$ and using (D.4). □

Proposition D.4. *Let $\alpha = \sum_{j=1}^J b^j\boldsymbol{\delta}_{x^j} \in \mathscr{E}\mathscr{D}(\Omega)$ and let $\varepsilon > 0$ satisfy (4.2). For every $j = 1,\dots,J$ let f^j and a^j_ε be two functions with $f^j \in C^\infty(B_{\frac{D}{2}}(x^j))$ and a^j_ε affine. Moreover, let f be a function which is smooth in a neighborhood of $\partial\Omega$ and $u \in H^2(\Omega_\varepsilon(\alpha))$ be such that $u = f$ and $\partial_n u = \partial_n f$ on $\partial\Omega$ and $u = a^j_\varepsilon + f^j$ and $\partial_n u = \partial_n a^j_\varepsilon + \partial_n f^j$ on $\partial B_\varepsilon(x^j)$ for every $j = 1,\dots,J$. Then the function $\widehat{u}\colon \Omega \to \mathbb{R}$ defined by*

$$
\widehat{u}(x) \coloneqq \begin{cases} u(x) & \text{if } x \in \Omega_\varepsilon(\alpha) \\ a^j_\varepsilon(x) + f^j & \text{if } x \in B_\varepsilon(x^j) \end{cases}
$$

is in $H^2(\Omega)$ and satisfies

$$
\|\widehat{u}\|_{H^2(\Omega)} \le C\Big(\|\nabla^2 u\|_{L^2(\Omega_\varepsilon(\alpha);\mathbb{R}^{2\times 2})} + \|f\|_{C^\infty(\partial\Omega)} + \sum_{j=1}^J\|f^j\|_{C^\infty(B_{\frac{D}{2}}(x^j))}\Big),
$$

for some constant C independent of u and of ε.

Proof. By assumption and by Proposition D.3, we have

$$\sum_{j=1}^{J}\|a_\varepsilon^j+f^j\|^2_{H^1(\partial B_\varepsilon(x^j))}\le JC_2(\Omega,\alpha)\Big(\|\nabla^2u\|^2_{L^2(\Omega_\varepsilon(\alpha);\mathbb{R}^{2\times2})}+\|f\|^2_{C^\infty(\partial\Omega)}\Big),$$

which implies, in particular,

$$\sum_{j=1}^{J}\|a_\varepsilon^j\|^2_{H^1(\partial B_\varepsilon(x^j))}\le JC_2(\Omega,\alpha)\Big(\|\nabla^2u\|^2_{L^2(\Omega_\varepsilon(\alpha);\mathbb{R}^{2\times2})}+\|f\|^2_{C^\infty(\partial\Omega)}+\sum_{j=1}^{J}\|f^j\|^2_{H^1(\partial B_\varepsilon(x^j))}\Big).$$

Since a_ε^j is affine, this implies that, for every $j=1,\dots,J$,

$$\|a_\varepsilon^j\|^2_{H^1(B_\varepsilon(x^j))}\le JC_2(\Omega,\alpha)\varepsilon\Big(\|\nabla^2u\|^2_{L^2(\Omega_\varepsilon(\alpha);\mathbb{R}^{2\times2})}+\|f\|^2_{C^\infty(\partial\Omega)}+\sum_{j=1}^{J}\|f^j\|^2_{H^1(\partial B_\varepsilon(x^j))}\Big),$$

which immediately provides the claim. □

Appendix E. A density result for traction-free H^2 functions

In this appendix we prove that, given $\alpha=\sum_{j=1}^J b^j\boldsymbol{\delta}_{x^j}\in\mathscr{E}\mathscr{D}(\Omega)$, any function $w\in\widetilde{\mathscr{B}}^\alpha_{0,\Omega}$ (see (4.13)) can be approximated in the strong H^2 norm by a sequence of functions $w_\varepsilon\in\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ (see (4.5)). The rough idea is (up to modifying the boundary datum) to replace $w+W_0^\alpha$ (see Remark 4.2), with its first-order Taylor expansion in $B_\varepsilon(x^j)$ $(j=1,\dots,J)$. We highlight that W_0^α is not even in $H^2(\Omega)$ but, in view of Remark 4.2, it is the strong H^2_{loc} limit of $W_\varepsilon^\alpha:=\sum_{j=1}^J W_\varepsilon^j$, where W_ε^j is affine in $B_\varepsilon(x^j)$ and smooth in $B_\varepsilon(x^i)$ with $i\ne j$. This allows us to approximate W_0^α in the desired manner. Then we approximate w by a sequence $\{v_k\}_k$ of smooth functions and we apply Taylor's formula with Lagrange remainder to further approximate each v_k by a sequence $\{v_{k,\varepsilon}\}_\varepsilon$ that is affine in $\bigcup_{j=1}^J B_\varepsilon(x^j)$. Finally, the claim is obtained by summing $v_{k,\varepsilon}$ to the contribution approximating W_0^α, and by using a diagonal argument.

Proposition E.1. *Let $\alpha\in\mathscr{E}\mathscr{D}(\Omega)$. For every $w\in\widetilde{\mathscr{B}}^\alpha_{0,\Omega}$ there exists a sequence $\{w_\varepsilon\}_\varepsilon\subset H^2(\Omega)$ with $w_\varepsilon\in\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ for $\varepsilon>0$ small enough, such that $w_\varepsilon\to w$ strongly in $H^2(\Omega)$ as $\varepsilon\to0$.*

Proof. By standard density arguments, there exists a sequence $\{v_k\}_{k\in\mathbb{N}}\subset C^\infty(\Omega)$ such that $v_k\to w$ strongly in $H^2(\Omega)$ as $k\to\infty$. Furthermore, we can assume that there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}}$ with $\delta_k\to0$ as $k\to\infty$, such that $v_k\equiv -W_0^\alpha$ in a δ_k-neighborhood of $\partial\Omega$.

Let $\operatorname{spt}\alpha=\{x^1,\dots,x^J\}$. For every $j=1,\dots,J$ and for every $k\in\mathbb{N}$, we set

$$\hat v_k^j(x):=v_k(x^j)+\langle\nabla v_k(x^j),x-x^j\rangle\qquad\text{for every }x\in\mathbb{R}^2;$$

moreover, we consider a C^2 function $\gamma\colon[1,2]\to[0,1]$ with $\gamma(1)=0$, $\gamma(2)=1$, $\gamma'\ge0$ in $(1,2)$, $\gamma'_+(1)=0=\gamma'_-(2)$ and $\gamma''_+(1)=0=\gamma''_-(2)$. For every $0<\varepsilon<\frac12\min\{D,\delta_k\}$ (with D defined in (4.2)) we define the function $v_{k,\varepsilon}\colon\Omega\to\mathbb{R}$ as

$$v_{k,\varepsilon}(x):=\begin{cases}\hat v_k^j(x) & \text{if } x\in B_\varepsilon(x^j)\\ \Big(1-\gamma\Big(\frac{|x-x^j|}{\varepsilon}\Big)\Big)\hat v_k^j(x)+\gamma\Big(\frac{|x-x^j|}{\varepsilon}\Big)v_k(x) & \text{in } A_{\varepsilon,2\varepsilon}(x^j)\\ v_k(x) & \text{if } x\in\Omega_{2\varepsilon}(\alpha).\end{cases}$$

Notice that, since $\varepsilon<\frac{\delta_k}{2}$, we have that $v_{k,\varepsilon}$ coincide with $-W_0^\alpha$ in a $\frac{\delta_k}{2}$-neighborhood of $\partial\Omega$. We claim that, for every $k\in\mathbb{N}$,

$$\|v_{k,\varepsilon}-v_k\|_{H^2(\Omega)}=\sum_{j=1}^J\|v_{k,\varepsilon}-v_k\|_{H^2(B_{2\varepsilon}(x^j))}\to0\qquad\text{as }\varepsilon\to0.\tag{E.1}$$

To this end, we prove that for every $j=1,\dots,J$

$$\|v_{k,\varepsilon}-v_k\|_{H^2(B_{2\varepsilon}(x^j))}\le C\varepsilon\|\nabla^2v_k\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})},\tag{E.2}$$

for some universal constant C independent of k and ε. Indeed, fix $j=1,\dots,J$. By the Taylor expansion formula with Lagrange remainder, we have that

$$\begin{aligned}
\|\hat v_k^j - v_k\|^2_{L^2(B_{2\varepsilon}(x^j))} &= \frac14\int_{B_{2\varepsilon}(x^j)} |\langle \nabla^2 v_k(\xi_x^j)(x-x^j), x-x^j\rangle|^2\,\mathrm{d}x \\
&\le C\varepsilon^6\|\nabla^2 v_k\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times 2})}\,,
\end{aligned}\tag{E.3}$$

$$\begin{aligned}
\|\nabla\hat v_k^j - \nabla v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^2)} &= \int_{B_{2\varepsilon}(x^j)} |\nabla v_k(x^j)-\nabla v_k(x)|^2\,\mathrm{d}x \\
&\le C\varepsilon^4\|\nabla^2 v_k\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times 2})}\,,
\end{aligned}\tag{E.4}$$

$$\begin{aligned}
\|\nabla^2\hat v_k^j - \nabla^2 v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^{2\times2})} &= \|\nabla^2 v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^{2\times 2})} \\
&\le C\varepsilon^2\|\nabla^2 v_k\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times 2})}\,,
\end{aligned}\tag{E.5}$$

where in (E.3) ξ_x^j is a point in the segment joining x^j and x. Furthermore, since

$$\Big\|\nabla\gamma\Big(\frac{|\cdot|}{\varepsilon}\Big)\Big\|_{L^\infty(A_{\varepsilon,2\varepsilon}(0);\mathbb{R}^2)}\le\frac{C}{\varepsilon}\,,\qquad \Big\|\nabla^2\gamma\Big(\frac{|\cdot|}{\varepsilon}\Big)\Big\|_{L^\infty(A_{\varepsilon,2\varepsilon}(0);\mathbb{R}^{2\times2})}\le\frac{C}{\varepsilon^2}\,,$$

by (E.3), (E.4), and (E.5), we deduce that

$$\begin{aligned}
\|v_{k,\varepsilon}-v_k\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x^j))} &\le \|\hat v_k^j - v_k\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x^j))} \\
&\le C\varepsilon^6\|\nabla^2 v_k\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\,,\\
\|\nabla v_{k,\varepsilon}-\nabla v_k\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x^j);\mathbb{R}^2)} &\le \frac{C}{\varepsilon^2}\|\hat v_k^j - v_k\|^2_{L^2(B_{2\varepsilon}(x^j))} \\
&\quad + C\|\nabla\hat v_k^j-\nabla v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^2)} \\
&\le C\varepsilon^4\|\nabla^2 v_k\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\,,\\
\|\nabla^2 v_{k,\varepsilon}-\nabla^2 v_k\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x^j);\mathbb{R}^{2\times2})} &\le \frac{C}{\varepsilon^4}\|\hat v_k^j - v_k\|^2_{L^2(B_{2\varepsilon}(x^j))} \\
&\quad + \frac{C}{\varepsilon^2}\|\nabla\hat v_k^j-\nabla v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^2)} \\
&\quad + C\|\nabla^2\hat v_k^j-\nabla^2 v_k\|^2_{L^2(B_{2\varepsilon}(x^j);\mathbb{R}^{2\times2})} \\
&\le C\varepsilon^2\|\nabla^2 v_k\|^2_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\,,
\end{aligned}$$

whence we get that

$$\|v_{k,\varepsilon}-v_k\|_{H^2(A_{\varepsilon,2\varepsilon}(x^j))}\le C\varepsilon\|\nabla^2 v_k\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\,;$$

this fact, together with (E.2), implies (E.1). Moreover, up to using a cut-off function, in view of Remark 4.2, we can assume that $v_{k,\varepsilon}\equiv -W_\varepsilon^\alpha$ in an ε-neighborhood of $\partial\Omega$, so that the boundary condition in the definition of $\widetilde{\mathscr{B}}^\alpha_{\varepsilon,\Omega}$ in (4.5) is satisfied.

To recover the traction-free condition on each $\partial B_\varepsilon(x^j)$, we notice that each function W_ε^j (defined in (4.3)) is affine in $\overline{B}_\varepsilon(x^j)$, whereas it is smooth in $\bigcup_{i\neq j}B_\varepsilon(x^i)$. Therefore, for every $j=1,\dots,J$ we define the function $\widehat{W}_\varepsilon^{\neq j}\colon B_D(x^j)\to\mathbb{R}$ as the affine contribution of all of the W_ε^i for $i\neq j$, *i.e.*,

$$\widehat{W}_\varepsilon^{\neq j}(x)\coloneqq\sum_{i\neq j}\Big(W_\varepsilon^i(x^j)+\langle\nabla W_\varepsilon^i(x^j),x-x^j\rangle\Big)\,.$$

Now, we define the function $\overline{W}_\varepsilon^\alpha\colon\Omega\to\mathbb{R}$ as

$$\overline{W}_\varepsilon^\alpha(x)\coloneqq\begin{cases}\widehat{W}_\varepsilon^{\neq j}(x)+W_\varepsilon^j(x)-W_\varepsilon^\alpha(x) & \text{if } x\in B_\varepsilon(x^j)\\[2mm] \Big(1-\gamma\Big(\dfrac{|x-x^j|}{\varepsilon}\Big)\Big)\big(\widehat{W}_\varepsilon^{\neq j}(x)+W_\varepsilon^j(x)-W_\varepsilon^\alpha(x)\big) & \text{if } x\in A_{\varepsilon,2\varepsilon}(x^j)\\[2mm] 0 & \text{if } x\in\Omega_{2\varepsilon}(\alpha)\,.\end{cases}$$

By the very definition of W_ε^α (see (4.3) again) it is easy to check that

$$\|\overline{W}_\varepsilon^\alpha\|_{H^2(\Omega)}\to 0\qquad\text{as }\varepsilon\to0\,.$$

For every k and ε as above, we define $w_{k,\varepsilon}\colon \Omega \to \mathbb{R}$ as $w_{k,\varepsilon} := v_{k,\varepsilon} + \overline{W}^{\alpha}_{\varepsilon}$, and we notice that it belongs to $\widetilde{\mathscr{B}}^{\alpha}_{\varepsilon,\Omega}$ by construction. Therefore, by a standard diagonal argument, there exists a sequence $\{w_\varepsilon\}_\varepsilon$ with $w_\varepsilon = w_{k(\varepsilon),\varepsilon}$ satisfying the desired properties. □

References

[1] E. Abe, Y. Kawamura, K. Hayashi, and A. Inoue. Long-period ordered structure in a high-strength nanocrystalline Mg-1 at% Zn-2 at% Y alloy studied by atomic-resolution Z-contrast STEM. *Acta Materialia*, 50(15):3845–3857, 2002.

[2] A. Acharya. On Weingarten-Volterra Defects. *Journal of Elasticity*, 134(1):79–101, 2019.

[3] A. Acharya and C. Fressengeas. Continuum mechanics of the interaction of phase boundaries and dislocations in solids. In Gui-Qiang G. Chen, Michael Grinfeld, and R. J. Knops, editors, *Differential Geometry and Continuum Mechanics*, pages 123–165, Cham, 2015. Springer International Publishing.

[4] R. Alicandro, L. De Luca, A. Garroni, and M. Ponsiglione. Metastability and dynamics of discrete topological singularities in two dimensions: a Γ-convergence approach. *Archive for Rational Mechanics and Analysis*, 214(1):269–330, 2014.

[5] R. Alicandro, L. De Luca, A. Garroni, and M. Ponsiglione. Dynamics of discrete screw dislocations on glide directions. *J. Mech. Phys. Solids*, 92:87–104, 2016.

[6] R. Alicandro, L. De Luca, A. Garroni, and M. Ponsiglione. Minimising movements for the motion of discrete screw dislocations along glide directions. *Calc. Var. Partial Differ. Equ.*, 56:no. 5, 2017.

[7] R. Alicandro, L. De Luca, G. Lazzaroni, M. Palombaro, and M. Ponsiglione. Coarse-graining of a discrete model for edge dislocations in the triangular lattice. *Preprint*, 2022.

[8] K. Anthony, U. Essmann, A. Seeger, and H. Träuble. *Disclinations and the Cosserat-Continuum with Incompatible Rotations*, volume Mechanics of Generalized Continua, Proceedings of the IUTAM-Symposium on The Generalized Cosserat Continuum and the Continuum Theory of Dislocations with Applications, Freudenstadt and Stuttgart (Germany) 1967, pages 355–358. Springer-Verlag Berlin Heidelberg, 1968.

[9] M. P. Ariza and M. Ortiz. Discrete crystal elasticity and discrete dislocations in crystals. *Archive for Rational Mechanics and Analysis*, 178(2):149–226, 2005.

[10] X. Balandraud, D. Delpueyo, M. Grédiac, and G. Zanzotto. Almost compatible microstructures in shape memory alloys. *Acta Materialia*, 58(14):4559–4577, 2010.

[11] X. Balandraud and G. Zanzotto. Stressed microstructures in thermally induced M9R–M18R martensites. *Journal of the Mechanics and Physics of Solids*, 55(1):194–224, 2007.

[12] J. Ball, P. Cesana, and B. Hambly. A probabilistic model for martensitic avalanches. *MATEC Web of Conferences*, 33, 2015.

[13] J. Ball and R. James. Fine Phase Mixtures as Minimizers of Energies. *Arch. Ration. Mech. Anal.*, 100:13–52, 1987.

[14] F. Banhart, J. Kotakoski, and A. V. Krasheninnikov. Structural defects in graphene. *ACS Nano*, 5(1):26–41, 2011. PMID: 21090760.

[15] K. Bhattacharya. *Microstructure of martensite: why it forms and how it gives rise to the shape-memory effect*. Oxford University Press, 2003.

[16] T. Blass, I. Fonseca, G. Leoni, and M. Morandotti. Dynamics for systems of screw dislocations. *SIAM Journal on Applied Mathematics*, 75(2):393–419, 2015.

[17] T. Blass and M. Morandotti. Renormalized energy and Peach-Köhler forces for screw dislocations with antiplane shear. *Journal of Convex Analysis*, 24(2):547–570, 2017.

[18] A. Braides. *Gamma-convergence for Beginners*, volume 22. Oxford University Press, Oxford, 2002.

[19] P. Cermelli and M. E. Gurtin. The motion of screw dislocations in crystalline materials undergoing antiplane shear: glide, cross-slip, fine cross-slip. *Archive for Rational Mechanics and Analysis*, 148(1):3–52, 1999.

[20] P. Cermelli and G. Leoni. Renormalized energy and forces on dislocations. *SIAM Journal on Mathematical Analysis*, 37(4):1131–1160, 2005.

[21] P. Cesana, F. Della Porta, A. Rüland, C. Zillinger, and B. Zwicknagl. Exact constructions in the (non-linear) planar theory of elasticity: From elastic crystals to nematic elastomers. *Archive for Rational Mechanics and Analysis*, 237(1):383–445, 2020.

[22] P. Cesana and B. Hambly. A probabilistic model for interfaces in a martensitic phase transition. *Journal of Applied Probability*, 2022.

[23] P. Cesana, M. Porta, and T. Lookman. Asymptotic analysis of hierarchical martensitic microstructure. *Journal of the Mechanics and Physics of Solids*, 72, 2014.

[24] P. Cesana and P. van Meurs. Discrete-to-continuum limits of planar disclinations. *ESAIM: COCV*, 27:23, 2021.

[25] P. G. Ciarlet. *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, volume 20 of *Studies in Mathematics and Its Applications*. Elsevier, North-Holland Publishing Co., Amsterdam, 1988.

[26] P. G. Ciarlet. *Mathematical Elasticity, Volume II: Theory of Plates*, volume 27 of *Studies in Mathematics and Its Applications*. Elsevier, North-Holland Publishing Co., Amsterdam, 1997.

[27] P. G. Ciarlet and P. Ciarlet. Another approach to linearized elasticity and a new proof of Korn's inequality. *Mathematical Models and Methods in Applied Sciences*, 15(02):259–271, 2005.

[28] S. Conti, A. Garroni, and S. Müller. Singular kernels, multiscale decomposition of microstructure, and dislocation models. *Archive for Rational Mechanics and Analysis*, 199(3):779–819, 2011.
[29] S. Conti, A. Garroni, and S. Müller. Derivation of strain-gradient plasticity from a generalized Peierls-Nabarro model. *J. Eur. Math. Soc.*, 2022.
[30] S. Conti, A. Garroni, and M. Ortiz. The line-tension approximation as the dilute limit of linear-elastic dislocations. *Archive for Rational Mechanics and Analysis*, 218(2):699–755, 2015.
[31] P. Cordier, S. Demouchy, B. Beausir, V. Taupin, F. Barou, and C. Fressengeas. Disclinations provide the missing mechanism for deforming olivine-rich rocks in the mantle. *Nature*, 507(7490):51–56, 2014.
[32] G. Dal Maso. *An Introduction to Γ-Convergence*. Birkhäuser Verlag, Boston, 1993.
[33] L. De Luca. Γ-convergence analysis for discrete topological singularities: the anisotropic triangular lattice and the long range interaction energy. *Asymptot. Anal.*, 96:185–221, 2016.
[34] L. De Luca, A. Garroni, and M. Ponsiglione. Γ-convergence analysis of systems of edge dislocations: the self energy regime. *Archive for Rational Mechanics and Analysis*, 206(3):885–910, 2012.
[35] R. de Wit. Linear theory of static disclinations. In *J. A. Simmons, R. de Wit, and R. Bullough (eds.) Fundamental Aspects of Dislocation Theory*, volume I, pages 651–673. Nat. Bur. Stand. (US), Spec. Publ. 317, 1970.
[36] R. de Wit. Partial disclinations. *Journal of Physics C: Solid State Physics*, 5(5):529–534, mar 1972.
[37] R. de Wit. Theory of disclinations: II. continuous and discrete disclinations in anisotropic elasticity. *Journal of Research of the National Bureau of Standards - A, Physics and Chemistry*, 77A(1):49–100, 1973.
[38] R. de Wit. Theory of disclinations: III. continuous and discrete disclinations in isotropic elasticity. *Journal of Research of the National Bureau of Standards - A, Physics and Chemistry*, 73A(3), 1973.
[39] R. de Wit. Theory of disclinations: IV. straight disclinations. *Journal of Research of the National Bureau of Standards - A Physics and Chemistry*, 77A(5), 1973.
[40] J. D. Eshelby. A simple derivation of the elastic field of an edge dislocation. *British Journal of Applied Physics*, 17(9):1131–1135, sep 1966.
[41] D. Essmann and H. Träuble. The Direct Observation of Individual Flux Lines in Type II Superconductors. *Phys. Letters*, 24A(526), 1967.
[42] N. A. Fleck and J. W. Hutchinson. A phenomenological theory for strain gradient effects in plasticity. *J. Mech. Phys. Solids*, 41:1825–1857, 1993.
[43] C. Fressengeas, V. Taupin, and L. Capolungo. An elasto-plastic theory of dislocation and disclination fields. *International Journal of Solids and Structures*, 48(25):3499–3509, 2011.
[44] A. Garroni, G. Leoni, and M. Ponsiglione. Gradient theory for plasticity via homogenization of discrete dislocations. *Journal European Mathematical Society*, 12(5):1231–1266, 2010.
[45] A. Garroni, R. Marziani, and R. Scala. Derivation of a line-tension model for dislocations from a nonlinear three-dimensional energy: The case of quadratic growth. *SIAM J. Math. Anal.*, 53:4252–4302, 2021.
[46] A. Garroni and S. Müller. Γ-limit of a phase-field model of dislocations. *SIAM Journal on Mathematical Analysis*, 36(6):1943–1964, 2005.
[47] A. Garroni and S. Müller. A variational model for dislocations in the line tension limit. *Archive for Rational Mechanics and Analysis*, 181(3):535–578, 2006.
[48] F. Gazzola, H. C. Grunau, and G. Sweers. *Polyharmonic boundary value problems: positivity preserving and nonlinear higher order elliptic equations in bounded domains*. Springer Science & Business Media, 2010.
[49] V. Yu. Gertsman, A. A. Nazarov, A. E. Romanov, R. Z. Valiev, and V. I. Vladimirov. Disclination-structural unit model of grain boundaries. *Philosophical Magazine A*, 59(5):1113–1118, 1989.
[50] G. Geymonat and F. Krasucki. Hodge decomposition for symmetric matrix fields and the elasticity complex in Lipschitz domains. *Communications on Pure and Applied Analysis*, 8(1):295–309, 2009.
[51] J. Ginster. Plasticity as the Γ-Limit of a Two-dimensional Dislocation Energy: The Critical Regime Without the Assumption of Well-Separateness. *Arch. Rational Mech. Anal.*, 233:1253–1288, 2019.
[52] J. Ginster. Strain-gradient plasticity as the Γ-limit of a nonlinear dislocation energy with mixed growth. *SIAM J. Math. Anal.*, 51:3424–3464, 2019.
[53] I. Groma. Link between the microscopic and mesoscopic length-scale description of the collective behavior of dislocations. *Physical Review B*, 56(10):5807–5813, 1997.
[54] I. Groma, G. Györgyi, and P. D. Ispánovity. Variational approach in dislocation theory. *Phil. Magazine*, 90(27):3679–3695, 2010.
[55] M. E. Gurtin and L. Anand. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. i. small deformations. *J. Mech. Phys. Solids*, 53(7):1624–1649, 2005.
[56] M. Y. Gutkin. 12 - Elastic and plastic deformation in nanocrystalline metals. In Sung H. Whang, editor, *Nanostructured Metals and Alloys*, Woodhead Publishing Series in Metals and Surface Engineering, pages 329–374. Woodhead Publishing, 2011.
[57] K. Hagihara, T. Mayama, M. Honnami, M. Yamasaki, H. Izuno, T. Okamoto, T. Ohashi, T. Nakano, and Y. Kawamura. Orientation dependence of the deformation kink band formation behavior in Zn single crystal. *International Journal of Plasticity*, 77:174–191, 2016.
[58] K. Hagihara, T. Okamoto, H. Izuno, M. Yamasaki, M. Matsushita, T. Nakano, and Y. Kawamura. Plastic deformation behavior of 10H-type synchronized LPSO phase in a Mg–Zn–Y system. *Acta Materialia*, 109:90–102, 2016.

[59] K. Hagihara, N. Yokotani, and Y. Umakoshi. Plastic deformation behavior of Mg12YZn with 18R long-period stacking ordered structure. *Intermetallics*, 18(2):267–276, 2010.
[60] W. F. Harris. Disclinations. *Scientific American*, 237(6):130–145, 1977.
[61] J. P. Hirth, G. Hirth, and J. Wang. Disclinations and disconnections in minerals and metals. *Proceedings of the National Academy of Sciences*, 117(1):196–204, 2020.
[62] J. P. Hirth and J. Lothe. *Theory of Dislocations*. John Wiley & Sons, New York, 1982.
[63] D. Hull and D. J. Bacon. *Introduction to Dislocations*. Butterworth Heinemann, Oxford, 2001.
[64] T. Inamura. Geometry of kink microstructure analysed by rank-1 connection. *Acta Materialia*, 173:270–280, 2019.
[65] T. Inamura, H. Hosoda, and S. Miyazaki. Incompatibility and preferred morphology in the self-accommodation microstructure of β-titanium shape memory alloy. *Philosophical Magazine*, 93(6):618–634, 2013.
[66] T. Inamura, M. Li, M. Tahara, and H. Hosoda. Formation process of the incompatible martensite microstructure in a beta-titanium shape memory alloy. *Acta Materialia*, 124:351–359, 2017.
[67] T. Inamura and T. Shinohara. Rank-1 connection of kink bands formed by non-parallel shears. *Materials Transactions*, 61:870–874, 2020.
[68] Y. Kawamura, K. Hayashi, A. Inoue, and T. Masumoto. Rapidly Solidified Powder Metallurgy $Mg_{97}Zn_1Y_2$ Alloys with Excellent Tensile Yield Strength above 600 MPa. *Materials Transactions*, 42(7):1172–1176, 2001.
[69] Y. Kitano and K. Kifune. HREM study of disclinations in MgCd ordered alloy. *Ultramicroscopy*, 39:279–286, 1991.
[70] M. Lazar. Wedge disclination in the field theory of elastoplasticity. *Physics Letters A*, 311:416–425, 2003.
[71] X. W. Lei and A. Nakatani. A deformation mechanism for ridge-shaped kink structure in layered solids. *Journal of Applied Mechanics*, 82:071016, 1–6, 2015.
[72] J. C. M. Li. Disclination model of high angle grain boundaries. *Surface Science*, 31:12–26, 1972.
[73] S. Limkumnerd and E. Van der Giessen. Statistical approach to dislocation dynamics: From dislocation correlations to a multiple-slip continuum theory of plasticity. *Physical Review B*, 77(18), 2008.
[74] C. Manolikas and S. Amelinckx. Phase transitions in ferroelastic lead orthovanadate as observed by means of electron microscopy and electron diffraction. *Phys. Stat. Sol.*, 60(2):607–617, 1980.
[75] C. Manolikas and S. Amelinckx. Phase transitions in ferroelastic lead orthovanadate as observed by means of electron microscopy and electron diffraction. II. Dynamic Observations. *physica status solidi (a)*, 61(1):179–188, 1980.
[76] V. V. Meleshko. Selected topics in the history of the two-dimensional biharmonic problem. *Applied Mechanics Reviews*, 56(1):33–85, 01 2003.
[77] S. Müller, L. Scardia, and C. I. Zeppieri. Geometric rigidity for incompatible fields, and an application to strain-gradient plasticity. *Indiana Univ. Math. J.*, 63:1365–1396, 2014.
[78] F. R. N. Nabarro. *Theory of crystal dislocations*. International Series of Monographs on Physics. Oxford: Clarendon Press, 1967.
[79] F. R. N. Nabarro and W. F. Harris. Presence and function of disclinations in surface coats of unicellular organisms. *Nature*, 232(5310):423–423, 1971.
[80] A. A. Nazarov. Disclinations in bulk nanostructured materials: their origin, relaxation and role in material properties. *Advances in Natural Sciences: Nanoscience and Nanotechnology*, 4(3):033002, jun 2013.
[81] J. Nečas. *Direct Methods in the Theory of Elliptic Equations*. Springer Science and Business Media, 2011.
[82] E. Orowan. Zur Kristallplastizität. III. *Z. Physik*, 89:634–659, 1934.
[83] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47:397–462, 1999.
[84] M. Polanyi. Über eine Art Gitterstörung, die einen Kristall plastisch machen könnte. *Z. Physik*, 89:660–664, 1934.
[85] M. Porta and T. Lookman. Heterogeneity and phase transformation in materials: Energy minimization, iterative methods and geometric nonlinearity. *Acta Materialia*, 61(14):5311–5340, 2013.
[86] A. E. Romanov. Disclinations in crystalline solids. *Dislocations in solids*, 9, 1992.
[87] A. E. Romanov and A. L. Kolesnikova. Application of disclination concept to solid structures. *Progress in Materials Science*, 54(6):740–769, 2009. "The Nabarro Legacy – Perspectives for Advances Materials in the 21st Century" A collection of invited lectures at the MRS Spring Meeting 2008.
[88] A. J. C. B. Saint-Venant. Mémoire sur la torsion des prismes. *Mem. Divers Savants*, 14:233–560, 1855.
[89] L. Scardia and C. I. Zeppieri. Line-tension model for plasticity as the Γ-limit of a nonlinear dislocation energy. *SIAM J. Math. Anal.*, 44:2372–2400, 2012.
[90] L. Schwartz. *Théorie des distributions*. Hermann, Paris, 1966.
[91] H. S. Seung and D. R. Nelson. Defects in flexible membranes with crystalline order. *Physical Review A*, 38:1005–1018, 1988.
[92] V. Taupin, L. Capolungo, C. Fressengeas, M. Upadhyay, and B. Beausir. A mesoscopic theory of dislocation and disclination fields for grain boundary-mediated crystal plasticity. *International Journal of Solids and Structures*, 71(Complete):277–290, 2015.
[93] G. I. Taylor. The Mechanism of Plastic Deformation of Crystals. Part I. Theoretical. *Proceedings of the Royal Society of London. Series A*, 145:362–387, 1934.

[94] H. Träuble and D. Essmann. Fehler im Flussliniengitter von Supraleitern zweiter Art. *Phys. Stat. Sol.*, 25(373), 1968.

[95] V. Volterra. Sur l'équilibre des corps élastiques multiplement connexes. *Annales scientifiques de l'École Normale Supérieure*, 24:401–517, 1907.

[96] K. Yamasaki and T. Hasebe. Duality of the incompatibility tensor. *Materials Transactions*, 61(5):875–877, 2020.

[97] G. Yang, L. Li, W. B. Lee, and M. C. Ng. Structure of graphene and its disorders: a review. *Science and Technology of Advanced Materials*, 19(1):613–648, 2018. PMID: 30181789.

[98] A. Yavari and A. Goriely. Riemann–Cartan Geometry of Nonlinear Dislocation Mechanics. *Archive for Rational Mechanics and Analysis*, 205(1):59–118, 2012.

[99] A. Yavari and A. Goriely. Riemann–Cartan geometry of nonlinear disclination mechanics. *Mathematics and Mechanics of Solids*, 18(1):91–102, 2013.

[100] O. V. Yazyev and S. G. Louie. Topological defects in graphene: Dislocations and grain boundaries. *Phys. Rev. B*, 81:195420, May 2010.

[101] C. Zhang and A. Acharya. On the relevance of generalized disclinations in defect mechanics. *Journal of the Mechanics and Physics of Solids*, 119:188–223, 2018.

[102] C. Zhang, A. Acharya, and S. Puri. Finite element approximation of the fields of bulk and interfacial line defects. *Journal of the Mechanics and Physics of Solids*, 114:258–302, 2018.

[103] T. Zhang, X. Li, and H. Gao. Defects controlled wrinkling and topological design in graphene. *Journal of the Mechanics and Physics of Solids*, 67:2–13, 2014.

[104] L. M. Zubov. *Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies*. Lecture Notes in Physics Monographs. Springer, 1997.

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