A characterization of BV and Sobolev functions via nonlocal functionals in metric spaces *

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July 5, 2022

Abstract

We study a characterization of BV and Sobolev functions via nonlocal functionals in metric spaces equipped with a doubling measure and supporting a Poincaré inequality. Compared with previous works, we consider more general functionals. We also give a counterexample in the case $p = 1$ demonstrating that unlike in Euclidean spaces, in metric measure spaces the limit of the nonlocal functions is only comparable, not necessarily equal, to the variation measure $\|Df\|(\Omega)$.

1 Introduction

Consider a sequence $\{\rho_i\}_{i=1}^{\infty}$ of nonnegative functions in $L^1(\mathbb{R}^n)$, $n \geq 1$, which are radial (i.e. only depend on $|x|$) and for which

$$\int_{\mathbb{R}^n} \rho_i(x)\,dx = 1 \quad \text{for all } i \in \mathbb{N} \quad \text{and} \quad \lim_{i\to\infty} \int_{|x|>\delta} \rho_i(x)\,dx = 0 \quad \text{for all } \delta > 0. \tag{1.1}$$

For an open set $\Omega \subset \mathbb{R}^n$ and function $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$, we define the Sobolev seminorm by

$$|f|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla f|^p \, dx \right)^{1/p},$$

and if $f \notin W^{1,p}_{\mathrm{loc}}(\Omega)$, then we let $|f|_{W^{1,p}(\Omega)} = \infty$. Bourgain, Brezis, and Mironescu [3, Theorem 3] showed that when $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain and $1 < p < \infty$, then for every $f \in L^p(\Omega)$ we have

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_i(|x-y|)\,dx\,dy = K_{p,n} |f|^p_{W^{1,p}(\Omega)};$$

here $K_{p,n}$ is a constant depending only on p, n. Dávila [14] generalized this result to functions of bounded variation (BV functions) f and their variation measures $\|Df\|$. He

*__2020 Mathematics Subject Classification__: 46E36, 26B30

Keywords: Sobolev function, function of bounded variation, metric measure space, nonlocal functional, Poincaré inequality

showed that when $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, then for every $f \in L^1(\Omega)$ we have

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \rho_i(|x-y|)\,dx\,dy = K_{1,n}\|Df\|(\Omega),$$

where we understand $\|Df\|(\Omega) = \infty$ if $f \notin \mathrm{BV}(\Omega)$. To unify the notation, we define the energy

$$E_{f,p}(\Omega) := \begin{cases} \|Df\|(\Omega) & \text{when } p = 1, \\ \int_\Omega |\nabla f|^p\,dx & \text{when } 1 < p < \infty. \end{cases}$$

Several different generalizations of these results in Euclidean spaces have been considered e.g. by Ponce [33], Leoni–Spector [25, 26], Brezis–Van Schaftingen–Yung [8, 9, 10], Nguyen–Pinamonti–Vecchi–Squassina [31, 32], Nguyen [30], Brezis–Nguyen [5, 6, 7], Garofalo–Tralli [16, 17] and Comi-Stefani [11, 12, 13].

Brezis [4, Remark 6] suggested generalizing the theory to more general metric measure spaces (X, d, μ). One generalization was given by Di Marino–Squassina [15], who assumed the measure μ to be doubling and the space to support a (p,p)–Poincaré inequality. Such spaces are often called PI spaces. We will give definitions in Section 2. They considered the mollifiers

$$\rho_s(x,y) := (1-s)\frac{1}{d(x,y)^{ps}\mu(B(y,d(x,y)))}, \quad x,y \in X, \quad 0 < s < 1,$$

and showed in [15, Theorem 1.4] that for a constant $C \geq 1$ and for every $f \in L^p(X)$, we have

$$\begin{aligned} C^{-1}E_{f,p}(X) &\leq \liminf_{s\nearrow 1}(1-s)\int_X\int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mu(B(y,d(x,y)))}\,d\mu(y)\,d\mu(x) \\ &\leq \limsup_{s\nearrow 1}(1-s)\int_X\int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mu(B(y,d(x,y)))}\,d\mu(y)\,d\mu(x) \leq CE_{f,p}(X). \end{aligned} \tag{1.2}$$

A similar result was proved previously in Ahlfors-regular spaces in [29]. Górny [18], resp. Han–Pinamonti [21], studied the problem in certain PI spaces that "locally look like" Euclidean spaces, resp. finite-dimensional Banach spaces or Carnot groups, and showed that for every $f \in N^{1,p}(X)$, with $1 < p < \infty$, we have

$$\lim_{r\to 0}\frac{1}{r^p}\int_X \fint_{B(y,r)} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y) = CE_{f,p}(X).$$

These results correspond to certain choices of the mollifiers ρ_i satisfying (1.1). In the current paper, our main goal is to study this problem for more general mollifiers ρ_i, of which the mollifiers considered in [15, 18, 29] are special cases. Moreover, we consider domains $\Omega \neq X$.

Our main result is the following.

Theorem 1.3. *Let $1 \le p < \infty$, and suppose μ is doubling and X supports a (p,p)–Poincaré inequality. Suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers satisfying conditions (2.8)–(2.11). Suppose $\Omega \subset X$ is a strong p-extension domain, and let $f \in L^p(\Omega)$. Then*

$$\begin{aligned} C_1 E_{f,p}(\Omega) &\le \liminf_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \\ &\le \limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \le C_2 E_{f,p}(\Omega) \end{aligned} \tag{1.4}$$

for some constants $C_1 \le C_2$ that depend only on p, the doubling constant of the measure, the constants in the Poincaré inequality, and the constant C_ρ associated with the mollifiers.

After giving definitions in Section 2 and some preliminary results in Section 3, we prove the two directions of (1.4) in Sections 4 and 5. In Section 6 we give corollaries to our main Theorem 1.3, showing that the mollifiers considered in [15] and [18], as well as other choices, can be handled as special cases. In Section 7 we give a counterexample demonstrating that we do not generally have $C_1 = C_2$ in (1.4).

Acknowledgement: The authors would like to thank Camillo Brena and Enrico Pasqualetto for some useful comments on a preliminary version of the paper.

2 Notation and definitions

Throughout this paper, we work in a complete and connected metric measure space (X, d, μ) equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \ge 1$ such that

$$0 < \mu(B(x, 2r)) \le C_d \mu(B(x,r)) < \infty$$

for every ball $B(x,r) := \{y \in X\colon\ d(y,x) < r\}$. We assume that $1 \le p < \infty$ and X consists of at least two points, that is, $\operatorname{diam} X > 0$.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into X. The length of a curve γ is denoted by ℓ_γ. We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [19, Theorem 3.2]). A nonnegative Borel function g on X is an upper gradient of a function $f\colon X \to [-\infty, \infty]$ if for all nonconstant curves $\gamma\colon [0, \ell_\gamma] \to X$, we have

$$|f(x) - f(y)| \le \int_\gamma g\, ds := \int_0^{\ell_\gamma} g(\gamma(s))\, ds, \tag{2.1}$$

where x and y are the end points of γ. We interpret $|f(x) - f(y)| = \infty$ whenever at least one of $|f(x)|$, $|f(y)|$ is infinite. Upper gradients were originally introduced in [23].

We always consider $1 \le p < \infty$. The p-modulus of a family of curves Γ is defined by

$$\operatorname{Mod}_p(\Gamma) := \inf \int_X \rho^p\, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho\, ds \ge 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.1) holds for p-almost every curve, we say that g is a p-weak upper gradient of f. By only considering curves γ in a set $A \subset X$, we can talk about a function g being a (p-weak) upper gradient of u in A.

We always let Ω denote an open subset of X. We define the Newton-Sobolev space $N^{1,p}(\Omega)$ to consist of those functions $f \in L^p(\Omega)$ for which there exists a p-weak upper gradient $g \in L^p(\Omega)$ of f in Ω. This space was first introduced in [35]. We write $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every $x \in \Omega$ there exists $r > 0$ such that $f \in N^{1,p}(B(x,r))$; other local function spaces are defined analogously. For every $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ there exists a minimal p-weak upper gradient of f in Ω, denoted by g_f, satisfying $g_f \le g$ μ-almost everywhere (a.e.) in Ω for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of f in Ω, see [2, Theorem 2.25].

Note that Newton-Sobolev functions are understood to be defined at every $x \in \Omega$, whereas the functionals that we consider are not affected by perturbations of f in a set of zero μ-measure. For this reason, we also define

$$\widehat{N}^{1,p}(\Omega) := \{f \colon f = h \ \mu\text{-a.e. in } \Omega \text{ for some } h \in N^{1,p}(\Omega)\}.$$

For every $f \in \widehat{N}^{1,p}(\Omega)$, we can also define $g_f := g_h$, where g_h is the minimal p-weak upper gradient of any h as above in Ω; this is well defined μ-a.e. in Ω by [2, Corollary 1.49, Proposition 1.59].

Next we define functions of bounded variation. Given an open set $\Omega \subset X$ and a function $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of f in Ω by

$$\|Df\|(\Omega) := \inf\left\{\liminf_{i\to\infty} \int_\Omega g_{f_i}\, d\mu :\ f_i \in N^{1,1}_{\mathrm{loc}}(\Omega),\ f_i \to f \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{f_i} is the minimal 1-weak upper gradient of f_i in Ω. We say that a function $f \in L^1(\Omega)$ is of bounded variation, and denote $f \in \mathrm{BV}(\Omega)$, if $\|Df\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Df\|(A) := \inf\{\|Df\|(W) :\ A \subset W,\, W \subset X \text{ is open}\}.$$

If $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, then $\|Df\|(\cdot)$ is a Radon measure on Ω by [28, Theorem 3.4].

Next we record Mazur's lemma and Fuglede's lemma, see e.g. [34, Theorem 3.12] and [2, Lemma 2.1].

Theorem 2.2. *Let $\{g_i\}_{i=1}^\infty$ be a sequence with $g_i \to g$ weakly in $L^p(\Omega)$. Then there exist convex combinations $\widehat{g}_i := \sum_{j=i}^{N_i} a_{i,j} g_j$, for some $N_i \in \mathbb{N}$, such that $\widehat{g}_i \to g$ in $L^p(\Omega)$.*

By convex combinations we mean that the numbers $a_{i,j}$ are nonnegative and that $\sum_{j=i}^{N_i} a_{i,j} = 1$ for every $i \in \mathbb{N}$.

Lemma 2.3. *Let $\{g_i\}_{i=1}^\infty$ be a sequence of functions with $g_i \to g$ in $L^p(\Omega)$. Then for p-a.e. curve γ in Ω, we have*

$$\int_\gamma g_i\, ds \to \int_\gamma g\, ds \quad \text{as } i \to \infty.$$

We say that X supports a (p,p)-Poincaré inequality, if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every $f \in L^p(X)$, and every p-weak upper gradient g of f, we have

$$\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \leq C_P r^p \int_{B(x,\lambda r)} g^p \, d\mu, \tag{2.4}$$

where

$$f_{B(x,r)} := \fint_{B(x,r)} f \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu.$$

In the case $p = 1$, the following BV version of the Poincaré inequality can be obtained by applying the $(1,1)$–Poincaré inequality to the approximating functions in the definition of the total variation: for every $f \in L^1(X)$, we have

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_P r \|Df\|(B(x,\lambda r)). \tag{2.5}$$

Suppose $f \in \mathrm{BV}(\Omega)$ if $p = 1$, and $f \in \widehat{N}^{1,p}(\Omega)$ if $1 < p < \infty$. In the latter case, denote the minimal p-weak upper gradient of f in Ω by g_f. For every Borel set $A \subset \Omega$, we denote the energy by

$$E_{f,p}(A) := \begin{cases} \|Df\|(A) & \text{when } p = 1 \\ \int_A g_f^p \, d\mu & \text{when } 1 < p < \infty. \end{cases}$$

Note that $E_{f,p}$ is then a Borel measure on Ω. If f is not in $\mathrm{BV}(\Omega)$ (in the case $p = 1$), respectively not in $\widehat{N}^{1,p}(\Omega)$ (in the case $1 < p < \infty$), then we let $E_{f,p}(\Omega) = \infty$. We can combine (2.4) and (2.5) to give: for every $1 \leq p < \infty$ and every $f \in L^p(X)$, we have

$$\int_{B(x,r)} |f - f_{B(x,r)}|^p \, d\mu \leq C_P r^p E_{f,p}(B(x,\lambda r)). \tag{2.6}$$

Definition 2.7. We say that an open set $\Omega \subset X$ is a strong p-extension domain if

- in the case $p = 1$, for every $f \in \mathrm{BV}(\Omega)$ there exists an extension $F \in \mathrm{BV}(X)$;
- in the case $1 < p < \infty$, for every $f \in \widehat{N}^{1,p}(\Omega)$ there exists an extension $F \in \widehat{N}^{1,p}(X)$;

and in both cases, $E_{F,p}(\partial\Omega) = 0$.

For example, in Euclidean spaces, a bounded domain with a Lipschitz boundary is a strong p-extension domain for all $1 \leq p < \infty$, see e.g. [1, Proposition 3.21].

Now we describe the mollifiers that we will use. We will consider a sequence of nonnegative $X \times X$-measurable functions $\{\rho_i(x,y)\}_{i=1}^{\infty}$, $x, y \in X$, and a fixed constant $1 \leq C_\rho < \infty$ satisfying the following conditions:

(1) For every $x, y \in X$ with $d(x,y) \leq 1$, we have for every $i \in \mathbb{N}$

$$\text{either} \quad \rho_i(x,y) \geq C_\rho^{-1} \frac{d(x,y)^p}{r_i^p} \frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))} \quad \text{or} \quad \rho_i(x,y) \geq d(x,y)^p \frac{\nu_i((d(x,y),\infty))}{\mu(B(y,d(x,y)))}, \tag{2.8}$$

where $r_i \searrow 0$ and each ν_i is a positive Radon measure on $[0, \infty)$ for which

$$\liminf_{i \to \infty} \int_0^\delta t^p \, d\nu_i \geq C_\rho^{-1} \text{ for all } \delta > 0. \tag{2.9}$$

Also for every $x, y \in X$ with $0 < d(x, y) \leq 1$, we have

$$\rho_i(x, y) \leq \sum_{j=1}^{\infty} d_{i,j} \frac{\chi_{B(y,2^{-j+1}) \setminus B(y,2^{-j})}(x)}{\mu(B(y, 2^{-j+1}))} \tag{2.10}$$

for numbers $d_{i,j} \geq 0$ for which $\sum_{j=1}^{\infty} d_{i,j} \leq C_\rho$.

(2) For all $\delta > 0$, we have

$$\lim_{i \to \infty} \left(\sup_{y \in \Omega} \int_{\Omega \setminus B(y,\delta)} \frac{\rho_i(x, y)}{d(x, y)^p} \, d\mu(x) + \sup_{x \in \Omega} \int_{\Omega \setminus B(x,\delta)} \frac{\rho_i(x, y)}{d(x, y)^p} \, d\mu(y) \right) = 0. \tag{2.11}$$

Remark 2.12. Conditions (2.10) and (2.11) will be used to prove the upper bound of our main Theorem 1.3, and they are quite close to the Euclidean assumptions (1.1). Since we do not have as many tools at our disposal as in Euclidean spaces, we additionally impose the somewhat stronger conditions (2.8) and (2.9); these will be used to prove the lower bound. In Section 6 we will see that these two conditions are also very natural.

Throughout the paper, we assume that μ is doubling, but we do not always assume that X satisfies a Poincaré inequality. Nonetheless, for convenience we assume that X is connected, from which it follows that $\mu(\{x\}) = 0$ for every $x \in X$, see e.g. [2, Corollary 3.9]. Thus, integrating over the set where $x = y$ in the functionals that we consider does not cause any problems.

3 Preliminary results

First we note the following basic fact: for every $f \in L^p(X)$ and every ball $B(z, r)$, using the estimate

$$|f(x) - f(y)|^p \leq 2^{p-1}(|f(x) - f_{B(z,r)}|^p + |f(y) - f_{B(z,r)}|^p), \quad x, y \in B(z, r),$$

we get

$$\int_{B(z,r)} \int_{B(z,r)} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \leq 2^p \mu(B(z, r)) \int_{B(z,r)} |f - f_{B(z,r)}|^p \, d\mu. \tag{3.1}$$

The next lemma is similar to [15, Lemma 3.1(ii)].

Lemma 3.2. *For any $h(x, y) \geq 0$ that is $\mu \times \mu$-measurable in $X \times X$ and satisfies $h(x, y) = 0$ for all $x, y \in X$ with $d(x, y) \geq \delta > 0$, we have*

$$\int_X \int_X h(x, y) \, d\mu(x) \, d\mu(y) \leq C_d \int_X \frac{1}{\mu(B(z, \delta))} \iint_{B(z,2\delta) \times B(z,2\delta)} h(x, y) \, d\mu(x) \, d\mu(y) \, d\mu(z).$$

Proof. For all $x, y \in X$ with $h(x, y) \neq 0$, we have $d(x, y) < \delta$, and then

$$B(x, \delta) \subset B(x, 2\delta) \cap B(y, 2\delta). \tag{3.3}$$

Note also that $\chi_{B(z,2\delta)}(x)$ and $\chi_{B(z,2\delta)}(y)$ are lower semicontinuous functions in the product space $X \times X \times X$, and so

$$(x, y, z) \mapsto \frac{1}{\mu(B(z, \delta))} \chi_{B(z,2\delta)}(x) \chi_{B(z,2\delta)}(y) h(x, y)$$

is $\mu \times \mu \times \mu$-measurable, and we can apply Fubini's theorem. We estimate

$$\begin{aligned}
&\int_X \frac{1}{\mu(B(z, \delta))} \iint_{B(z,2\delta)\times B(z,2\delta)} h(x, y)\, d\mu(x)\, d\mu(y)\, d\mu(z) \\
&\quad = \int_X \int_X \int_X \frac{1}{\mu(B(z, \delta))} \chi_{B(z,2\delta)}(x) \chi_{B(z,2\delta)}(y) h(x, y)\, d\mu(x)\, d\mu(y)\, d\mu(z) \\
&\quad = \int_X \int_X \int_{B(x,2\delta)\cap B(y,2\delta)} \frac{1}{\mu(B(z, \delta))}\, d\mu(z)\, h(x, y)\, d\mu(x)\, d\mu(y) \quad \text{by Fubini} \\
&\quad \geq \int_X \int_X \int_{B(x,\delta)} \frac{1}{\mu(B(z, \delta))}\, d\mu(z)\, h(x, y)\, d\mu(x)\, d\mu(y) \quad \text{by (3.3)} \\
&\quad \geq \frac{1}{C_d} \int_X \int_X \int_{B(x,\delta)} \frac{1}{\mu(B(x, \delta))}\, d\mu(z)\, h(x, y)\, d\mu(x)\, d\mu(y) \quad \text{since } B(z, \delta) \subset B(x, 2\delta) \\
&\quad = \frac{1}{C_d} \int_X \int_X h(x, y)\, d\mu(x)\, d\mu(y).
\end{aligned}$$

□

For an open set $U \subset X$ and $\delta > 0$, denote

$$U_\delta := \{x \in U : d(x, X \setminus U) > \delta\} \quad \text{and} \quad U(\delta) := \{x \in X : d(x, U) < \delta\}. \tag{3.4}$$

Lemma 3.5. *For any function $h(x, y) \geq 0$ that is $\mu \times \mu$-measurable on X and satisfies $h(x, y) = 0$ for all $x, y \in X$ with $d(x, y) \geq \delta > 0$, and for an open set $U \subset X$, we have*

$$\begin{aligned}
&\int_U \int_X h(x, y)\, d\mu(x)\, d\mu(y) \\
&\qquad \leq C_d \int_{U(2\delta)} \frac{1}{\mu(B(z, \delta))} \iint_{B(z,2\delta)\times B(z,2\delta)} h(x, y)\, d\mu(x)\, d\mu(y)\, d\mu(z).
\end{aligned}$$

Proof. Apply Lemma 3.2 with the function h replaced by $h(x, y)\chi_U(y)$. □

4 Upper bound of Theorem 1.3

Recall that we always denote by Ω an open subset of X, and that $1 \leq p < \infty$.

In order to prove the upper bound of our main Theorem 1.3, we first prove the following result. Recall the notation $U(R)$ from (3.4).

Proposition 4.1. *Suppose X supports the (p,p)-Poincaré inequality (2.4). Let $f \in L^p(X)$ and $0 < R \le 1$, and suppose $U \subset X$ is open. Suppose $\{\rho_i\}_{i=1}^\infty$ is a sequence of mollifiers that satisfy (2.10). Then*

$$\int_U \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \le C E_{f,p}(U(8\lambda R)) \tag{4.2}$$

for every $i \in \mathbb{N}$ and for a constant $C = C(C_d, C_P, \lambda, C_\rho)$.

Proof. We can assume that $E_p(f, U(8\lambda R)) < \infty$. Recall the condition (2.10). Note that on the left-hand side of (4.2) we require $d(x,y) < R$, but we also know that $\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)$ can be nonzero only when $2^{-j} < d(x,y)$. Then necessarily

$$2^{-j} \le R. \tag{4.3}$$

For every $j \in \mathbb{Z}$ satisfying (4.3), we estimate

$$\begin{aligned} &\int_U \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y) \\ &\quad \le \int_U \int_X \frac{|f(x)-f(y)|^p}{d(x,y)^p} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y) \\ &\quad \le C_d \int_{U(2^{-j+2})} \frac{1}{\mu(B(z,2^{-j+1}))} \iint_{[B(z,2^{-j+2})]^2} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \\ &\qquad \times \frac{\chi_{X\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y)\, d\mu(z) \quad \text{by Lemma 3.5.} \end{aligned} \tag{4.4}$$

We estimate further

$$\begin{aligned} &\frac{1}{\mu(B(z,2^{-j+1}))} \iint_{[B(z,2^{-j+2})]^2} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \frac{\chi_{X\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y) \\ &\quad \le 2^{jp} \frac{C_d^3}{\mu(B(z,2^{-j+2}))^2} \iint_{[B(z,2^{-j+2})]^2} |f(x)-f(y)|^p\, d\mu(x)\, d\mu(y) \\ &\quad \le 2^{(j+1)p} \frac{C_d^3}{\mu(B(z,2^{-j+2}))} \int_{B(z,2^{-j+2})} |f(x) - f_{B(z,2^{-j+2})}|^p\, d\mu(x) \quad \text{by (3.1)} \\ &\quad \le 8^p C_P C_d^3 \frac{E_{f,p}(B(z,2^{-j+2}\lambda))}{\mu(B(z,2^{-j+2}))} \quad \text{by the Poincaré inequality (2.6).} \end{aligned} \tag{4.5}$$

Denote the smallest integer at least $a \in \mathbb{R}$ by $\lceil a \rceil$. Combining the above with (4.4), we get

$$\begin{aligned}
&\int_U \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y) \\
&\quad \le 8^p C_P C_d^4 \int_{U(2^{-j+2})} \frac{E_{f,p}(B(z,2^{-j+2}\lambda))}{\mu(B(z,2^{-j+2}))}\, d\mu(z) \\
&\quad = 8^p C_P C_d^4 \int_X \int_X \chi_{U(2^{-j+2})}(z) \frac{\chi_{B(z,2^{-j+2}\lambda)}(w)}{\mu(B(z,2^{-j+2}))}\, dE_{f,p}(w)\, d\mu(z) \\
&\quad \le 8^p C_P C_d^4 \int_X \int_X \chi_{U(2^{-j+3}\lambda)}(w) \frac{\chi_{B(w,2^{-j+2}\lambda)}(z)}{\mu(B(z,2^{-j+2}))}\, dE_{f,p}(w)\, d\mu(z) \\
&\quad = 8^p C_P C_d^2 \int_X \int_X \chi_{U(2^{-j+3}\lambda)}(w) \frac{\chi_{B(w,2^{-j+2}\lambda)}(z)}{\mu(B(z,2^{-j+2}))}\, d\mu(z)\, dE_{f,p}(w) \quad \text{by Fubini} \\
&\quad \le 8^p C_P C_d^{3+\lceil \log_2 \lambda \rceil} \int_X \chi_{U(2^{-j+3}\lambda)}(w)\, dE_{f,p}(w) \\
&\quad = 8^p C_P C_d^{3+\lceil \log_2 \lambda \rceil} E_{f,p}(U(2^{-j+3}\lambda)).
\end{aligned} \tag{4.6}$$

Recalling (2.10), we get

$$\begin{aligned}
&\int_U \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \\
&\quad \le \sum_{j=1}^{\infty} d_{i,j} \int_U \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}\, d\mu(x)\, d\mu(y) \\
&\quad \le 8^p C_P C_d^{3+\lceil \log_2 \lambda \rceil} \sum_{j=1}^{\infty} d_{i,j} E_{f,p}(U(2^{-j+3}\lambda)) \quad \text{by (4.6)} \\
&\quad \le 8^p C_P C_d^{3+\lceil \log_2 \lambda \rceil} \sum_{j=1}^{\infty} d_{i,j} E_{f,p}(U(8\lambda R)) \quad \text{by (4.3)} \\
&\quad \le 8^p C_P C_d^{3+\lceil \log_2 \lambda \rceil} C_\rho E_{f,p}(U(8\lambda R))
\end{aligned} \tag{4.7}$$

by the assumption $\sum_{j=1}^{\infty} d_{i,j} \le C_\rho$. □

Now we can prove one direction of our main theorem. Recall the definition of a strong p-extension domain from Definition 2.7.

Theorem 4.8. *Suppose X supports a (p,p)–Poincaré inequality. Suppose $\Omega \subset X$ is a strong p-extension domain, and let $f \in \mathrm{BV}(\Omega)$ if $p = 1$, and $f \in \widehat{N}^{1,p}(\Omega)$ if $1 < p < \infty$. Suppose $\{\rho_i\}_{i=1}^{\infty}$ is a sequence of mollifiers that satisfy (2.10) and (2.11). Then*

$$\limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \le C E_{f,p}(\Omega)$$

for a constant $C = C(C_d, C_P, \lambda, C_\rho)$.

Proof. Consider $0 < R \le 1$. Recalling the notation $\Omega_{8\lambda R}$ from (3.4), we have

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad = \int_\Omega \int_{\Omega\setminus B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\qquad + \int_{\Omega_{8\lambda R}} \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\qquad\quad + \int_{\Omega\setminus\Omega_{8\lambda R}} \int_{\Omega\cap B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y).
\end{aligned}$$

For the first term, we estimate

$$\begin{aligned}
&\int_\Omega \int_{\Omega\setminus B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad \le 2^p \int_\Omega \int_{\Omega\setminus B(y,R)} \frac{|f(x)|^p+|f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad \le 2^p \int_\Omega |f(y)|^p \int_{\Omega\setminus B(y,R)} \frac{\rho_i(x,y)}{d(x,y)^p}\,d\mu(x)\,d\mu(y)\\
&\qquad + 2^p \int_\Omega |f(x)|^p \int_{\Omega\setminus B(x,R)} \frac{\rho_i(x,y)}{d(x,y)^p}\,d\mu(y)\,d\mu(x) \quad \text{by Fubini}\\
&\quad \le 2^p \int_\Omega |f|^p\,d\mu \left(\sup_{y\in\Omega} \int_{\Omega\setminus B(y,R)} \frac{\rho_i(x,y)}{d(x,y)^p}\,d\mu(x) + \sup_{x\in\Omega} \int_{\Omega\setminus B(x,R)} \frac{\rho_i(x,y)}{d(x,y)^p}\,d\mu(y) \right)\\
&\quad \to 0
\end{aligned}$$

as $i \to \infty$ by (2.11).

For the second term, we get from Proposition 4.1 that

$$\begin{aligned}
\int_{\Omega_{8\lambda R}} \int_{B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y) &\le C E_{f,p}(\Omega_{8\lambda R}(8\lambda R))\\
&\le C E_{f,p}(\Omega).
\end{aligned}$$

Then we estimate the third term. Since Ω is a strong p-extension domain, we find an extension $F \in \mathrm{BV}(X)$ in the case $p = 1$, and $F \in \widehat{N}^{1,p}(X)$ in the case $1 < p < \infty$, and in both cases $E_{F,p}(\partial\Omega) = 0$. Write $U := \Omega \setminus \Omega_{8\lambda R}$, and note that $U(8\lambda R) \subset \Omega(8\lambda R) \setminus \Omega_{16\lambda R}$. Thus we can estimate the third term by

$$\begin{aligned}
&\limsup_{i\to\infty} \int_{\Omega\setminus\Omega_{8\lambda R}} \int_{\Omega\cap B(y,R)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad \le \limsup_{i\to\infty} \int_U \int_{B(y,R)} \frac{|F(x)-F(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad \le C E_{F,p}(\Omega(8\lambda R) \setminus \Omega_{16\lambda R})
\end{aligned}$$

by Proposition 4.1. This goes to zero as $R \to 0$, since $E_{F,p}(\partial\Omega) = 0$. Combining the three terms, we get

$$\limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \le C E_{f,p}(\Omega).$$

□

5 Lower bound of Theorem 1.3

In this section we prove the lower bound of Theorem 1.3. Note that in this section we do not need to assume a Poincaré inequality. As usual, $\Omega \subset X$ is an open set. Given a ball $B = B(x,r)$ with a specific center $x \in X$ and radius $r > 0$, we denote $2B := B(x, 2r)$. The distance between two sets $A, D \subset X$ is denoted by

$$\operatorname{dist}(A, D) := \inf\{d(x,y) \colon x \in A,\ y \in D\}.$$

Lemma 5.1. *Consider an open set $U \subset \Omega$ with $\operatorname{dist}(U, X \setminus \Omega) > 0$, and a scale $0 < R < \operatorname{dist}(U, X\setminus\Omega)/10$. Then we can choose an at most countable covering $\{B_j = B(x_j, R)\}_j$ of $U(5R)$ such that $x_j \in U(5R)$, each ball $5B_j$ is contained in Ω, and the balls $\{5B_j\}_{j=1}^\infty$ can be divided into at most C_d^8 collections of pairwise disjoint balls.*

Proof. Consider a covering $\{B(x, R/5)\}_{x\in U(5R)}$. By the 5-covering theorem, see e.g. [24, p. 60], we can choose a countable collection of disjoint balls $\{B(x_j, R/5)\}_j$ such that the balls $B_j = B(x_j, R)$ cover $U(5R)$. Consider a ball B_j and denote by I_j those $k \in \mathbb{N}$ for which $5B_k \cap 5B_j \neq \emptyset$. Then

$$\begin{aligned}\sum_{k\in I_j} \mu(B_k) \le C_d^3 \sum_{k\in I_j} \mu(\tfrac{1}{5}B_k) \le C_d^3 \mu(11B_j) \le C_d^3\mu(21B_l) \quad &\text{for any } l \in I_j\\ \le C_d^8 \mu(B_l),&\end{aligned}$$

and so I_j has cardinality at most C_d^8. We can recursively choose maximal collections of pairwise disjoint balls $5B_j$. After at most C_d^8 steps, we have exhausted all of the balls $5B_j$. □

Let $C_0 := 3C_d^8$. Given such a covering of $U(5R)$, we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to the covering, such that $0 \le \phi_j \le 1$,

$$\text{each } \phi_j \text{ is a } C_0/R\text{-Lipschitz function,} \tag{5.2}$$

and $\operatorname{spt}(\phi_j) \subset 2B_j$ for each $j \in \mathbb{N}$; see e.g. [24, p. 104]. Finally, we can define a *discrete convolution* h of any $f \in L^1(\Omega)$ with respect to the covering by

$$h := \sum_j f_{B_j} \phi_j.$$

Clearly $h \in \operatorname{Lip}_{\mathrm{loc}}(U)$.

Theorem 5.3. *Suppose ρ_i is a sequence of mollifiers satisfying (2.8). Suppose $f \in L^p(\Omega)$. Then*

$$C_1 E_{f,p}(\Omega) \le \liminf_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \tag{5.4}$$

for some constant C_1 depending only on C_ρ and on the doubling constant of the measure.

Note that here we do not impose any conditions on the open set $\Omega \subset X$.

Proof. We can assume that

$$\liminf_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) =: M < \infty.$$

Fix $0 < \varepsilon < 1$. Passing to a subsequence (not relabeled), we can assume that

$$\liminf_{i\to\infty} \int_\Omega \int_\Omega |f(x)-f(y)|^p \rho_i(x,y)\, d\mu(x)\, d\mu(y) \le M + \varepsilon \quad \text{for all } i \in \mathbb{N}.$$

Assuming the second option of (2.8), we get

$$\int_\Omega \int_\Omega |f(x)-f(y)|^p \frac{\nu_i((d(x,y),\infty))}{\mu(B(y,d(x,y)))} \chi_{B(y,1)}(x)\, d\mu(x)\, d\mu(y) \le M + \varepsilon.$$

It follows that

$$\begin{aligned} M + \varepsilon &\ge \int_\Omega \int_\Omega \int_{d(x,y)}^\infty |f(x)-f(y)|^p \frac{1}{\mu(B(y,d(x,y)))} \chi_{B(y,1)}(x)\, d\nu_i(t)\, d\mu(x)\, d\mu(y) \\ &= \int_0^\infty \iint_{\{x,y\in\Omega:\, d(x,y)<t\}} \frac{|f(x)-f(y)|^p}{t^p} \frac{\chi_{B(y,1)}(x)}{\mu(B(y,d(x,y)))}\, d\mu(x)\, d\mu(y)\, t^p\, d\nu_i(t) \end{aligned}$$

by Fubini's theorem. By (2.9), given an arbitrarily small $0 < \delta < 1$, we have

$$\liminf_{i\to\infty} \int_0^\delta t^p\, d\nu_i \ge C_\rho^{-1},$$

and so $\int_0^\delta t^p\, d\nu_i \ge (1-\varepsilon) C_\rho^{-1}$ for all sufficiently large $i \in \mathbb{N}$. Then there necessarily exists $0 < t \le \delta$ such that

$$\iint_{\{x,y\in\Omega:\, d(x,y)<t\}} \frac{|f(x)-f(y)|^p}{t^p} \frac{\chi_{B(y,1)}(x)}{\mu(B(y,d(x,y)))}\, d\mu(x)\, d\mu(y) \le \frac{M+\varepsilon}{(1-\varepsilon)C_\rho^{-1}}.$$

In other words, we find arbitarily small $t > 0$ such that

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{t^p} \frac{\chi_{B(y,t)\cap\Omega}(x)}{\mu(B(y,t))}\, d\mu(x)\, d\mu(y) \le \frac{(M+\varepsilon)C_\rho}{1-\varepsilon}. \tag{5.5}$$

We obviously obtain this also if the first option of (2.8) holds. Fix a small $t > 0$. Let $U \subset \Omega$ with $\operatorname{dist}(U, X \setminus \Omega) > t$, and let $R := t/10$. Consider a covering $\{B_j\}_{j=1}^\infty$ of $U(5R)$ at scale $R > 0$, as described in Lemma 5.1. Then consider the discrete convolution

$$h := \sum_j f_{B_j} \phi_j.$$

We define the pointwise asymptotic Lipschitz number by

$$\mathrm{Lip}_h(x) := \limsup_{r\to 0} \frac{\sup_{y\in B(x,r)} |h(y)-h(x)|}{r}, \quad x\in U.$$

Suppose $x \in U$. Then $x \in B_j$ for some $j \in \mathbb{N}$. Consider any other point $y \in B_j$. Denote by I_j those $k \in \mathbb{N}$ for which $2B_k \cap 2B_j \neq \emptyset$. We estimate

$$\begin{aligned}
|h(x)-h(y)| &= \left|\sum_{k\in I_j} f_{B_k}(\phi_k(x)-\phi_k(y))\right| \\
&= \left|\sum_{k\in I_j} (f_{B_k}-f_{B_j})(\phi_k(x)-\phi_k(y))\right| \\
&\le \frac{C_0 d(x,y)}{R}\left(\sum_{k\in I_j} \fint_{B_k} |f-f_{5B_j}|\,d\mu + \sum_{k\in I_j}\fint_{B_j}|f-f_{5B_j}|\,d\mu\right) \quad \text{by (5.2)} \\
&\le \frac{2C_0C_d^3 d(x,y)}{R}\sum_{k\in I_j}\fint_{5B_j}|f-f_{5B_j}|\,d\mu \\
&\le \frac{2C_0^2C_d^3 d(x,y)}{R}\fint_{5B_j}\fint_{5B_j}|f(z)-f(w)|\,d\mu(z)\,d\mu(w),
\end{aligned} \tag{5.6}$$

since by Lemma 5.1 we know that I_j has cardinality at most C_0. Letting $y \to x$, we obtain an estimate for Lip_h in the ball B_j. In total, we conclude (we track the constants for a while in order to make the estimates more explicit)

$$\mathrm{Lip}_h \le \frac{2C_0^2C_d^3}{R}\sum_j \chi_{B_j}\fint_{5B_j}\fint_{5B_j}|f(x)-f(y)|\,d\mu(x)\,d\mu(y).$$

Since the balls $\{B_j\}_j$ can be divided into at most C_0 collections of pairwise disjoint balls, we get

$$\begin{aligned}
(\mathrm{Lip}_h)^p &\le \frac{(2C_0^2C_d^3)^pC_0^p}{R^p}\sum_j \chi_{B_j}\left(\fint_{5B_j}\fint_{5B_j}|f(x)-f(y)|^p\,d\mu(x)\,d\mu(y)\right)^p \\
&\le \frac{(2C_0^2C_d^3)^pC_0^p}{R^p}\sum_j \chi_{B_j}\fint_{5B_j}\fint_{5B_j}|f(x)-f(y)|^p\,d\mu(x)\,d\mu(y) \quad \text{by Hölder} \\
&\le \frac{(2C_0^2C_d^3)^p(10C_0)^pC_d^2}{(10R)^p}\sum_j \chi_{B_j}\fint_{5B_j}\int_{5B_j}|f(x)-f(y)|^p\frac{\chi_{B(y,10R)}(x)}{\mu(B(y,10R))}\,d\mu(x)\,d\mu(y).
\end{aligned} \tag{5.7}$$

Thus

$$\begin{aligned}\int_U (\mathrm{Lip}_h)^p\,d\mu &\le \frac{(2C_0^2C_d^3)^p(10C_0)^pC_d^2}{(10R)^p}\sum_j \int_{5B_j}\int_{5B_j}|f(x)-f(y)|^p\frac{\chi_{B(y,10R)}(x)}{\mu(B(y,10R))}\,d\mu(x)\,d\mu(y)\\ &\le \frac{(2C_0^2C_d^3)^p(10C_0)^pC_d^2C_0}{(10R)^p}\int_\Omega\int_\Omega |f(x)-f(y)|^p\frac{\chi_{B(y,10R)\cap\Omega}(x)}{\mu(B(y,10R))}\,d\mu(x)\,d\mu(y)\\ &\le C\frac{(M+\varepsilon)C_\rho}{(1-\varepsilon)}\end{aligned} \tag{5.8}$$

by (5.5), with $C := (2C_0^2C_d^3)^p(10C_0)^pC_d^2C_0$. We know that the minimal p-weak upper gradient g_h of h in U satisfies $g_h \le \mathrm{Lip}_h$ μ-a.e. in U, see e.g. [2, Proposition 1.14].

Recall that we can do the above for arbitrarily small $t>0$ and thus arbitrarily small $R>0$. From now on, we can consider any open $U\subset\Omega$ with $\mathrm{dist}(U, X\setminus\Omega)>0$. We get a sequence of discrete convolutions $\{h_i\}_{i=1}^\infty$ corresponding to scales $R_i \searrow 0$, such that $\{g_{h_i}\}_{i=1}^\infty$ is a bounded sequence in $L^p(U)$. From the properties of discrete convolutions, see e.g. [22, Lemma 5.3], we know that $h_i\to f$ in $L^p(U)$. Passing to a subsequence (not relabeled), we also have $h_i(x)\to f(x)$ for μ-a.e. $x\in U$. When $p=1$, we get

$$\|Df\|(U) \le \liminf_{i\to\infty}\int_U g_{h_i}\,d\mu \le \liminf_{i\to\infty}\int_U \mathrm{Lip}_{h_i}\,d\mu \le C\frac{(M+\varepsilon)C_\rho}{(1-\varepsilon)},$$

and so $f\in\mathrm{BV}(U)$. In the case $1<p<\infty$, by reflexivity of the space $L^p(U)$, we find a subsequence of $\{h_i\}_{i=1}^\infty$ (not relabeled) and $g\in L^p(U)$ such that $g_{h_i}\to g$ weakly in $L^p(U)$ (see e.g. [24, Section 2]). By Mazur's lemma (Theorem 2.2), for suitable convex combinations we get the strong convergence $\sum_{l=i}^{N_i} a_{i,l}g_{h_l}\to g$ in $L^p(U)$. We still have $\sum_{l=i}^{N_i} a_{i,l}h_l(x)\to f(x)$ for μ-a.e. $x\in U$. Define

$$\widetilde{h}(x) := \limsup_{i\to\infty}\sum_{l=i}^{N_i} a_{i,l}h_l(x), \quad x\in U.$$

Then $\widetilde{h}=f$ μ-a.e. in U. Denote $N:=\{x\in U\colon |\widetilde{h}(x)|<\infty\}$, so that $\mu(N)=0$. For p-a.e. curve γ in U, denoting the end points by x,y, we have that either $x\notin N$ or $y\notin N$; see [2, Corollary 1.51]. For such γ, we obtain

$$|\widetilde{h}(x)-\widetilde{h}(y)| \le \limsup_{i\to\infty}\left|\sum_{l=i}^{N_i} a_{i,l}h_l(x) - \sum_{l=i}^{N_i} a_{i,l}h_l(y)\right| \le \limsup_{i\to\infty}\int_\gamma \sum_{l=i}^{N_i} a_{i,l}g_{h_l}\,ds = \int_\gamma g\,ds$$

by Fuglede's lemma (Lemma 2.3), exclusing another curve family of zero p-modulus. Hence g is a p-weak upper gradient of $\widetilde{h}$ in U, and so for the minimal p-weak upper gradient we have

$$\int_U g_{\widetilde{h}}^p\,d\mu \le \int_U g^p\,d\mu \le \limsup_{i\to\infty}\int_U g_{h_i}^p\,d\mu \le C\frac{(M+\varepsilon)C_\rho}{(1-\varepsilon)}$$

by (5.8). Since $f=\widetilde{h}$ μ-a.e. in U, we have $f\in \widehat{N}^{1,p}(U)$. Note that now $E_{f,p}$ is a Radon measure on Ω. Exhausting Ω by sets U, in both cases we obtain

$$\begin{aligned}E_{f,p}(\Omega)&\le C\frac{(M+\varepsilon)C_\rho}{(1-\varepsilon)}\\&=\frac{CC_\rho}{1-\varepsilon}\liminf_{i\to\infty}\left[\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_i(x,y)\,d\mu(x)\,d\mu(y)+\varepsilon\right].\end{aligned}$$

Letting $\varepsilon\to 0$, this proves (5.4). $\square$

6 Corollaries

The conditions (2.8)–(2.11) that we impose on the mollifiers ρ_i are quite flexible, and so we can obtain various existing results in the literature as special cases of our main Theorem 1.3. The following is essentially [15, Theorem 1.4], except that we consider an open set Ω instead of the whole space X.

Corollary 6.1. *Suppose X supports a (p,p)–Poincaré inequality. Let $\Omega\subset X$ be a strong p-extension domain, and let $f\in L^p(\Omega)$. Then*

$$\begin{aligned}C^{-1}E_{f,p}(\Omega)&\le\liminf_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mu(B(y,d(x,y)))}\,d\mu(x)\,d\mu(y)\\&\le\limsup_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mu(B(y,d(x,y)))}\,d\mu(x)\,d\mu(y)\le CE_{f,p}(\Omega)\end{aligned}\tag{6.2}$$

for some constant $C\ge 1$ depending only on p, the doubling constant of the measure, and the constants in the Poincaré inequality.

Proof. This is obtained from Theorem 1.3 with the choice of mollifiers

$$\rho_i(x,y):=(1-s_i)\frac{1}{d(x,y)^{p(s_i-1)}\mu(B(y,d(x,y)))},\quad x,y\in X,$$

where $s_i\nearrow 1$ as $i\to\infty$. We only need to check that conditions (2.8)–(2.11) are satisfied. We have

$$\rho_i(x,y)=d(x,y)^p\frac{\nu_i((d(x,y),\infty))}{\mu(B(y,d(x,y)))}\quad\text{with}\quad d\nu_i(t):=ps_i(1-s_i)t^{-ps_i-1}\,dt,$$

and so

$$\liminf_{i\to\infty}\int_0^\delta t^p\,d\nu_i=p\liminf_{i\to\infty}s_i(1-s_i)\int_0^\delta t^{-p(s_i-1)-1}\,dt=1\quad\text{for all }\delta>0,$$

satisfying the second option of (2.8), and (2.9).

For every $x,y\in X$ with $0<d(x,y)\le 1$, we have

$$(1-s_i)\frac{1}{d(x,y)^{p(s_i-1)}\mu(B(y,d(x,y)))}\le\sum_{j=1}^\infty d_{i,j}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}$$

with $d_{i,j} = C_d(1-s_i)2^{(-j+1)p(1-s_i)}$. Here

$$\sum_{j=1}^{\infty} 2^{(-j+1)p(1-s_i)} \le 2\int_0^2 t^{p(1-s_i)-1}\,dt = \frac{2^{1+p(1-s_i)}}{p(1-s_i)} \le \frac{2^{1+p}}{p(1-s_i)},$$

and so

$$\sum_{j=0}^{\infty} d_{i,j} \le \frac{2^{1+p}C_d}{p},$$

satisfying (2.10). Finally, we estimate

$$\begin{aligned}\frac{\rho_i(x,y)}{d(x,y)^p} &= (1-s_i)\frac{1}{d(x,y)^{ps_i}\mu(B(y,d(x,y)))}\\ &\le C_d(1-s_i)\sum_{j\in\mathbb{Z}} 2^{jps_i}\frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))},\end{aligned}$$

and so (the notation $\sum_{j\le -\log_2\delta}$ means that we sum over integers j at most $-\log_2\delta$)

$$\begin{aligned}\int_{X\setminus B(y,\delta)} \frac{\rho_i(x,y)}{d(x,y)^p}\,d\mu(x) &\le C_d(1-s_i)\sum_{j\le -\log_2\delta} 2^{jps_i}\\ &\le C_d(1-s_i)\frac{\delta^{-ps_i}}{1-2^{-ps_i}}\\ &\to 0 \quad \text{as } i\to\infty,\end{aligned}$$

and so (2.11) holds. □

In particular, in the Euclidean setting, the functional considered in the Corollary 6.1 reduces to the fractional Sobolev seminorm

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.$$

The functional appearing in the following corollary was previously considered by Marola–Miranda–Shanmugalingam [27] as well as Górny [18] and Han–Pinamonti [21]

Corollary 6.3. *Suppose X supports a (p,p)–Poincaré inequality, and let $f \in L^p(X)$. Then*

$$\begin{aligned}C^{-1}E_{f,p}(X) &\le \liminf_{r\searrow 0}\frac{1}{r^p}\int_X \fint_{B(y,r)} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y)\\ &\le \limsup_{r\searrow 0}\frac{1}{r^p}\int_X \fint_{B(y,r)} |f(x)-f(y)|^p\,d\mu(x)\,d\mu(y) \le CE_{f,p}(X)\end{aligned} \tag{6.4}$$

for some constant C depending only on p, the doubling constant of the measure, and the constants in the Poincaré inequality.

Proof. This is obtained from Theorem 1.3 with the choice

$$\rho_i(x,y) = r_i^{-p} d(x,y)^p \frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))},$$

where $r_i \searrow 0$ as $i \to \infty$. Now the first option of (2.8) holds. For every $x, y \in X$ with $0 < d(x,y) \le 1$, we have

$$\rho_i(x,y) \le \sum_{j=1}^{\infty} d_{i,j} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))}$$

with $d_{i,j} = r_i^{-p} 2^{(-j+1)p} \mu(B(y,2^{-j+1}))\mu(B(y,r_i))^{-1}$ for $j \ge -\log_2 r_i$, and $d_{i,j} = 0$ otherwise. Now

$$\begin{aligned}\sum_{j=1}^{\infty} d_{i,j} &= r_i^{-p} \sum_{j\ge -\log_2 r_i} 2^{(-j+1)p} \mu(B(y,2^{-j+1}))\mu(B(y,r_i))^{-1} \\ &\le C_d r_i^{-p} \sum_{j\ge -\log_2 r_i} 2^{(-j+1)p} \\ &\le 2^p C_d,\end{aligned}$$

and thus (2.10) is satisfied. The condition (2.11) obviously holds. □

The following simple choice of mollifiers, considered in the Euclidean setting e.g. by Brezis [4, Eq. (45)], is also natural. This will be used also in a counterexample in the last section.

Corollary 6.5. *Suppose X supports a (p,p)–Poincaré inequality. Let $f \in L^p(X)$. Then*

$$\begin{aligned}C^{-1} E_{f,p}(X) &\le \liminf_{r\searrow 0} \int_X \fint_{B(y,r)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\, d\mu(x)\, d\mu(y) \\ &\le \limsup_{r\searrow 0} \int_X \fint_{B(y,r)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\, d\mu(x)\, d\mu(y) \le C E_{f,p}(X).\end{aligned} \tag{6.6}$$

for some constant C depending only on p, the doubling constant of the measure, and the constants in the Poincaré inequality.

Proof. This is obtained from Theorem 1.3 with the choice

$$\rho_i(x,y) = \frac{\chi_{B(y,r_i)}(x)}{\mu(B(y,r_i))},$$

where $r_i \searrow 0$ as $i \to \infty$. Again the first option of (2.8) holds. We can assume that $r_i < \min\{1, \operatorname{diam} X/4\}$ for all $i \in \mathbb{N}$. For every $x, y \in X$ with $0 < d(x,y) \le 1$, we have

$$\rho_i(x,y) \le \sum_{j=1}^{\infty} d_{i,j} \frac{\chi_{B(y,2^{-j+1})\setminus B(y,2^{-j})}(x)}{\mu(B(y,2^{-j+1}))},$$

where $d_{i,j} = \mu(B(y, 2^{-j+1}))\mu(B(y, r_i))^{-1}$ for $j \geq -\log_2 r_i$ and $d_{i,j} = 0$ otherwise.

Since X is connected, there exists $z \in \partial B(y, \frac{3}{2} \cdot 2^{-j})$ for all $j \geq -\log_2 r_i$, and so

$$B(z, 2^{-j-1}) \subset B(y, 2^{-j+1}) \setminus B(y, 2^{-j}) \quad \text{and} \quad B(y, 2^{-j+1}) \subset B(z, 2^{-j+2}).$$

It follows that

$$\mu(B(y, 2^{-j+1}) \setminus B(y, 2^{-j})) \geq C_d^{-3}\mu(B(y, 2^{-j+1})),$$

and so

$$\begin{aligned}
\sum_{j=1}^{\infty} d_{i,j} &= \mu(B(y, r_i))^{-1} \sum_{j \geq -\log_2 r_i} \mu(B(y, 2^{-j+1})) \\
&\leq C_d^3 \mu(B(y, r_i))^{-1} \sum_{j \geq -\log_2 r_i} \mu(B(y, 2^{-j+1}) \setminus B(y, 2^{-j})) \\
&\leq C_d^3 \mu(B(y, r_i))^{-1} \mu(B(y, 2r_i)) \\
&\leq C_d^4,
\end{aligned}$$

satisfying (2.10). The condition (2.11) obviously holds. □

7 A counterexample

Recall that the conclusion of our main Theorem 1.3 has the form

$$\begin{aligned}
C_1 E_{f,p}(\Omega) &\leq \liminf_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \\
&\leq \limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_i(x,y)\, d\mu(x)\, d\mu(y) \leq C_2 E_{f,p}(\Omega).
\end{aligned} \tag{7.1}$$

One natural question to ask is whether $C_1 = C_2$ might hold. Górny [18] shows that with a suitable choice of the mollifiers ρ_i, this holds in the case $1 < p < \infty$ if we additionally assume that at μ-a.e. point $x \in X$, the tangent space is Euclidean with fixed dimension. On the other hand, he gives an example where the dimension of the tangent space takes two different values in two different parts of the space, and then it is necessary to choose $C_1 < C_2$. In the example below, inspired by [20, Example 4.8], it is easy to check that the tangent space of X is Euclidean with dimension 1 at μ-a.e. $x \in X$ (see definitions in [18]), but nonetheless we show in the case $p = 1$ that $C_1 < C_2$.

First consider the real line equipped with the Euclidean metric and the one-dimensional Lebesgue measure $\mathcal{L}^1$. Similarly to Corollary 6.5, consider the sequence of mollifiers

$$\rho_i(x,y) := \frac{\chi_{[-1/i,1/i]}(|x-y|)}{2/i}, \quad x, y \in \mathbb{R}, \quad i \in \mathbb{N}.$$

From the Euclidean theory, see Dávila [14, Theorem 1.1], we know that for every $f \in L^1(\mathbb{R})$, we have

$$\lim_{i\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|}{|x-y|} \rho_i(x,y)\, d\mathcal{L}^1(x)\, d\mathcal{L}^1(y) = \|Df\|(\mathbb{R}). \tag{7.2}$$

Example 7.3. Consider the space $X = [0,1]$, equipped with the Euclidean metric and a weighted measure μ that we will next define. First we construct a fat Cantor set A as follows. Let $A_0 := [0,1]$. Then in each step $i \in \mathbb{N}$, we remove from A_{i-1} the set D_i, which consists of 2^{i-1} open intervals of length 2^{-2i}, centered at the middle points of the intervals that make up A_{i-1}. We denote $L_i := \mathcal{L}^1(A_i)$, and we let $A = \bigcap_{i=1}^{\infty} A_i$. Then we have

$$L := \mathcal{L}^1(A) = \lim_{i\to\infty} L_i = 1/2.$$

Then define the weight

$$w := \begin{cases} 2 & \text{in } A, \\ 1 & \text{in } X \setminus A, \end{cases}$$

and equip the space X with the weighted Lebesgue measure $d\mu := w\, d\mathcal{L}^1$. Obviously the measure is doubling, and X supports a $(1,1)$–Poincaré inequality.

Let

$$g := 2\chi_A \quad \text{and} \quad g_i = \frac{1}{L_{i-1} - L_i}\chi_{D_i}, \quad i \in \mathbb{N}.$$

Then

$$\int_0^1 g(s)\, ds = \int_0^1 g_i(s)\, ds = 1 \quad \text{for all } i \in \mathbb{N}.$$

Next define the function

$$f(x) = \int_0^x g(s)\, ds, \quad x \in [0,1].$$

Now $f \in \mathrm{Lip}(X)$, since g is bounded. Approximate f with the functions

$$f_i(x) = \int_0^x g_i(s)\, ds, \quad x \in [0,1], \quad i \in \mathbb{N}.$$

Now also $f_i \in \mathrm{Lip}(X)$, and $f_i \to f$ uniformly. This can be seen as follows. Given $i \in \mathbb{N}$, the set A_i consists of 2^i intervals of length $L_i/2^i$. If I is one of these intervals, we have

$$2^{-i} = \int_I g(s)\, ds = \int_I g_{i+1}(s)\, ds,$$

and also

$$\int_{X\setminus A_i} g\, d\mathcal{L}^1 = 0 = \int_{X\setminus A_i} g_{i+1}\, d\mathcal{L}^1.$$

Hence $f_{i+1} = f$ in $X \setminus A_i$, and elsewhere $|f_{i+1} - f|$ is at most 2^{-i}. In particular, $f_i \to f$ in $L^1(X)$ and so

$$\|Df\|(X) \le \lim_{i\to\infty} \int_0^1 g_i\, d\mu = \lim_{i\to\infty} \int_0^1 g_i\, d\mathcal{L}^1 = 1.$$

For a.e. $x \in A$, f is differentiable at x and so we have

$$\lim_{i\to\infty} \int_X \frac{|f(x) - f(y)|}{|x-y|} \rho_i(x,y)\, d\mathcal{L}^1(x) = |f'(y)|.$$

Thus

$$\begin{aligned}
&\liminf_{i\to\infty}\int_X\int_X \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\quad\geq 2\liminf_{i\to\infty}\int_A\int_X \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x,y)\,d\mathcal{L}^1(x)\,d\mathcal{L}^1(y)\\
&\quad\geq 2\int_A\liminf_{i\to\infty}\int_X \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x,y)\,d\mathcal{L}^1(x)\,d\mathcal{L}^1(y)\quad\text{by Fatou}\\
&\quad= 2\int_X |f'(y)|\,d\mathcal{L}^1(y)\\
&\quad= 2.
\end{aligned}$$

We conclude that

$$\liminf_{i\to\infty}\int_X\int_X \frac{|f(x)-f(y)|}{|x-y|}\rho_i(x,y)\,d\mu(y)\,d\mu(x)\geq 2\|Df\|(X). \tag{7.4}$$

On the other hand, consider any nonzero Lipschitz function f_0 supported in $(3/8, 5/8)$. For such a function, from (7.2) we have

$$\lim_{i\to\infty}\int_X\int_X \frac{|f_0(x)-f_0(y)|}{|x-y|}\rho_i(x,y)\,d\mu(x)\,d\mu(y)=\|Df_0\|(X),$$

since both sides are equal to the classical quantities, that is, the quantities obtained when the measure μ is $\mathcal{L}^1$. This combined with (7.4) shows that we cannot have $C_1 = C_2$ in (7.1).

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