Interactions Between Homogenization and Phase-Transition Processes

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1 Introduction

The main topic of this note will be the application of homogenization theory to the study of phase transitions problems in composite media. The starting point of the research (see Section 2) is a result of Γ-convergence obtained by L. Modica and S. Mortola in 1977 ([MM77]), and later on rivisited by Modica [M87] in terms of Van der Waals - Cahn - Hilliard theory of phase transitions. In those two papers the authors study, with Γ-convergence techniques (see [DGF75], [DM93], [B]), the asymptotic behaviour of a family of functionals of the type

$$F_\varepsilon(u,\Omega)=\int_\Omega\Big[\frac{W(u)}{\varepsilon}+\varepsilon|Du|^2\Big]\,dx,\qquad u\in L^1(\Omega)$$

where $W:\mathbf{R}^+\to\mathbf{R}^+$ is equal to 0 in $u=\alpha$, $u=\beta$, and positive otherwise. In this model u may represent the concentration of a homogeneous isothermal fluid contained in the bounded region Ω and the determination of the stable configurations

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of the fluid corresponds to minimize the total energy

$$E(u, \Omega) = \int_\Omega W(u)\, dx$$

among all possible density distributions u with fixed total mass m

$$\int_\Omega u\, dx = m.$$

If $0 < m < |\Omega|$ the minimum problem

$$\min\Big\{E(u) : \int_\Omega u\, dx = m\Big\}$$

has in general infinitely many solutions, without restrictions on the shape and extension of the interface between the two sets $\{u = \alpha\}$, $\{u = \beta\}$. This is in contrast with the physically reasonable criterion that the interface have minimal area (Gurtin conjecture [G85]). The results obtained by Modica and Mortola show that the solutions u_ε of the problems

$$\min\Big\{F_\varepsilon(u) : \int_\Omega u\, dx = m\Big\}$$

converge, as $\varepsilon \to 0$, to a function u which takes only the values α and β (i. e., it is a phase transition) and whose interface set has minimal area.

Starting from these results we study the asymptotic behaviour of a family of functionals

$$F_{\varepsilon,\delta}(u, \Omega) = \int_\Omega \Big[\frac{W(u)}{\varepsilon} + \varepsilon f(\frac{x}{\delta}, Du)\Big]\, dx, \qquad u \in L^1(\Omega)$$

depending on two parameters $\varepsilon, \delta \to 0$, where W is a positive function equal to 0 only when $u = \alpha$ or $u = \beta$, while $f(y, \xi)$ satisfies suitable hypotheses of regularity and growth. In particular f is assumed to depend periodically on y, in order to describe by means of $F_{\varepsilon,\delta}$ the phase transitions of an etherogeneous fluid with periodic microstructure and period proportional to δ.

Aim of the research is to analyse the interactions between effects due to phase transitions ($\varepsilon \to 0$) and effects due to homogenization ($\delta \to 0$), by means of computing the Γ-limits of the family $F_{\varepsilon,\delta}$, under the assumption that δ is a function of ε.

The main result is that, as $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, $F_{\varepsilon,\delta}(u, \Omega)$ Γ-converges to a functional $F(u, \Omega)$ for every open set Ω and every function $u \in L^1_{loc}(\mathbb{R}^n)$. The limit has the form

$$F(u, \Omega) = \begin{cases} \displaystyle\int_{S_u \cap \Omega} \varphi(x, \nu_u)\, d\mathcal{H}^{n-1} & \text{if} \quad u \in BV(\Omega; \{0, 1\}) \\ +\infty & \text{otherwise,} \end{cases}$$

where the symbol S_u denotes the complement of the Lebesgue set of u, and the vector-valued measure Du can be expressed by means of a unit vector ν_u, defined $\mathcal{H}^{n-1}$-a.e. on S_u, as

$$Du(B) = \int_{S_u \cap B} \nu_u d\mathcal{H}^{n-1}$$

for every Borel set B contained in Ω. Note that, as u belongs to set $BV(\Omega; \{0,1\})$ of the functions of bounded variations defined on Ω which takes almost everywhere the values 0 or 1, then it coincides a.e. with the characteristic function of a set of finite perimeter E, S_u coincides with the reduced boundary $\partial^* E$ of E in Ω and ν_u is interpreted as the measure theoretical internal normal to E.

Moreover, we find different explicit formulas for the integrand φ in the cases $\delta \sim \varepsilon$, $\varepsilon = o(\delta)$, $\delta = o(\varepsilon)$. The proof of the formulas for φ will be given in Section 4, in the case $n = 1$, while the proofs in the general n-dimensional case can be found in the forthcoming paper [ABC].

2 Gradient theory for phase transitions

2.1 Starting point

Let us fix a bounded open set $\Omega \subset \mathbb{R}^n$ and a non-negative function $u : \Omega \to \mathbb{R}^+$, that may represent the concentration of a fluid contained in the region Ω. Assume that there exists a function $W_0 : \mathbb{R} \to \mathbb{R}^+$ such that $W_0(u)$ is the energy density of the fluid, and then

$$E_0(u) = \int_\Omega W_0(u(x))\, dx$$

is the total free energy and

$$m = \int_\Omega u\, dx$$

is the total mass of the fluid. We want to study the following problem.

Problem 2.1 *Determine stable configurations of the fluid by minimizing the total energy $E_0(u)$ among all distributions u of prescribed total mass m:*

$$\min\Big\{ E_0(u) : \int_\Omega u\, dx = m \Big\}. \tag{2.1}$$

In the above problem, we can change $W_0(u)$ into $W(u) = W_0(u) - (au + b)$ and $E_0(u)$ into $E(u) = E_0(u) - (am + b|\Omega|)$. In particular we may assume that $W \geq 0$ and that $W(u) = 0$ if and only if $u = \alpha$ or $u = \beta$. Anyway, problem 2.1 has the *same solutions* as the following problem.

Problem 2.2 *Determine stable configurations of the fluid by minimizing the total energy $E(u)$ among all distributions u of prescribed total mass m;*

$$\min\{E(u) : \int_\Omega u\,dx = m\}. \tag{2.2}$$

¿From now on we shall study Problem 2.2. We begin by remarking that if u is an absolute minimizer of Problem 2.2 then

$$\int_\Omega u\,dx = m \qquad \text{and} \qquad E(u) = 0,$$

from which we deduce that $W(u(x)) \equiv 0$, and hence there exists a set A such that

$$u(x) = \begin{cases} \alpha & \text{if } x \in A \\ \beta & \text{if } x \in \Omega \setminus A \end{cases}$$

i.e., u is a *phase transition.* The values α and β are called *phases.* A is the region occupied by phase α.

2.2 Existence of solutions

These phase-transition solutions may exist under a *compatibility condition* of the type

$$\alpha|\Omega| \leq m \leq \beta|\Omega|.$$

We remark that under the compatibility condition Problem 2.2 has infinitely many solutions, with *no restriction on the interface* between the sets $\{x \in \Omega : u(x) = \alpha\}$ and $\{x \in \Omega : u(x) = \beta\}$. This is in constrast with the physically reasonable criterion that the interface should have minimal area (Gurtin conjecture [G85]).

2.3 Van der Waals - Cahn - Hilliard gradient theory

In order to recover this *minimal interface property* it is customary to take into account a different energy functional of the form

$$E_\varepsilon(u) = \int_\Omega \Big(W(u(x)) + \varepsilon |Du|^2 \Big)\,dx$$

where $\varepsilon > 0$ and to study the following problem.

Problem 2.3 *Study the asymptotic behaviour as $\varepsilon \to 0+$ of the solutions u_ε of*

$$\min\Big\{ E_\varepsilon(u) : \int_\Omega u\,dx = m \Big\}. \tag{2.3}$$

Concerning Problem 2.3 we sinthetically report the results obtained by L. Modica and S. Mortola (see [MM77] and [M87]) in the following theorem.

Theorem 2.4 *Under some technical conditions (for instance, if (u_ε) is bounded in L^∞ or if W has polynomial growth at ∞) then*

1. *upon extracting a subsequence u_ε converge to some u in $L^1(\Omega)$;*
2. *$u(x) \in \{\alpha, \beta\}$ for a.a. $x \in \Omega$;*
3. *the interface between $\{x \in \Omega : u(x) = \alpha\}$ and $\{x \in \Omega : u(x) = \beta\}$ has minimal area.*

We remark that the notion of area used here is the well-known variational notion of perimeter (see e.g. [DG54], [DG55], [F68], [G84], [MM84] [AFP]). To follow the ideas underlying the preceding results we point out that the main tools to perfom the asymptotic analysis of Problem 2.3 are given by Γ-convergence theory. For a better comprehension of what follows, we shall briefly recall first the definition of perimeter and its main properties, and then the notion of Γ-convergence together with some useful related results.

2.4 Sets with finite perimeter

Definition 2.5 *We say that a function u is of* bounded variation ($u \in \mathrm{BV}(\Omega)$) *if $u \in L^1(\Omega)$ and the total variation of the vector-valued measure Du given by the distributional gradient of u is finite, i.e.*

$$\int_\Omega |Du| = \sup\Big\{ \int_\Omega u \mathrm{div}\, \varphi \, dx : \varphi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n), |\varphi| \leq 1 \Big\} < +\infty.$$

Definition 2.6 *A measurable set $E \subseteq \mathbb{R}^n$ is said to have* finite perimeter *in Ω if its characteristic function χ_E belongs to* $\mathrm{BV}(\Omega)$. *Moreover we say that the total variation of $D\chi_E$*

$$\int_\Omega |D\chi_E| = \mathcal{P}_\Omega(E)$$

is the perimeter of E in Ω.

Remark 2.7 If u is the characteristic function of some measurable set E having finite perimeter in Ω, i.e., $u \in \mathrm{BV}(\Omega, \{0,1\})$ then the vector-valued measure Du can be represented as

$$Du(B) = \int_{B \cap S_u} \nu_u \, d\mathcal{H}^{n-1}$$

for every borel set $B \subseteq \Omega$, where S_u denotes the complement of the Lebesgue set of u, $\nu_u \in \mathbb{R}^n$ is a unit vector which is $\mathcal{H}^{n-1}$-a.e. defined in S_u and $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Haudorff measure of $\mathbb{R}^n$. Moreover one can prove that

$$\mathcal{P}_\Omega(E) = \int_\Omega |Du| = \mathcal{H}^{n-1}(S_u \cap \Omega) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega).$$

The last inequality can be strict, if E is not regular enough (see [G84]).

2.5 Γ-convergence

The notion of Γ-convergence was introduced by E. De Giorgi and T. Franzoni in [DGF75] in a very general setting. We recall here this definition only in the special case of functionals defined on the space $L^1(\Omega)$, considered with the strong topology.

Definition 2.8 *Let $F_\varepsilon,\ F : L^1(\Omega) \to [0,+\infty],\ \varepsilon > 0$. We say that $(F_\varepsilon)_\varepsilon$ Γ-converges to F as $\varepsilon \to 0+$, if the following two conditions are satisfied:*

1. *for every sequence $\varepsilon_h \to 0+$, for every $u \in L^1(\Omega)$, and for every $u_h \to u$ strongly in $L^1(\Omega)$, then $F(u) \leq \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h)$*

2. *for every sequence $\varepsilon_h \to 0+$ and for every $u \in L^1(\Omega)$, there exists $u_h \to u$ strongly in $L^1(\Omega)$ such that $F(u) = \lim_{h\to+\infty} F_{\varepsilon_h}(u_h)$.*

For what concerns our interests, we point out that Γ-convergence has at least two nice properties, reported in the following two theorems, concerning the existence of convergent subsequences and the convergence of minimum values.

Theorem 2.9 *(Compactness for Γ-convergence) For every family of functionals F_ε and every sequence $\varepsilon_h \to 0+$ there exists a functional F and a subsequence ε_{h_k} such that $(F_{\varepsilon_{h_k}})_k$ Γ-converges to F.*

Theorem 2.10 *(Convergence of minima) If $(F_\varepsilon)_\varepsilon$ Γ-converges to F as $\varepsilon \to 0+$, and $F_\varepsilon(u_\varepsilon) = \min\{F_\varepsilon(v) : v \in L^1(\Omega)\}$, with $u_\varepsilon \to u$ strongly in $L^1(\Omega)$, then $F(u) = \min\{F(v) : v \in L^1(\Omega)\}$ and $F_\varepsilon(u_\varepsilon) \to F(u)$.*

The proofs of the preceding results con be found, for instance, in [DM91], [B], [BDF].

2.6 The minimal interface property

We want to return back to the content of Theorem 2.4, in order to give a precise meaning to result *3*.

Theorem 2.11 *The set $A = \{x \in \Omega : u(x) = \alpha\}$, where u is given by Theorem* 2.4*, is a solution of the variational problem*

$$\mathcal{P}_\Omega(A) = \min\left\{\mathcal{P}_\Omega(A') : A' \subseteq \Omega, |A'| = \frac{\beta|\Omega| - m}{\beta - \alpha}\right\} \tag{2.4}$$

i.e., A minimizes the perimeter $\mathcal{P}_\Omega(A')$ among all subsets A' of Ω with prescribed n-dimensional Lebesgue measure $|A'| = \frac{\beta|\Omega|-m}{\beta-\alpha}$.

To prove the result (see [MM77], [M87]), one can compute the Γ-limit of the rescaled functional $F_\varepsilon = \frac{E_\varepsilon}{\sqrt{\varepsilon}}$ or, more precisely

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\frac{W(u)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}|Du|^2\Big)\,dx & \text{if } \int_\Omega u\,dx = m \\ +\infty & \text{otherwise.} \end{cases}$$

It turns out that F_ε Γ-converges to F, with

$$F(u) = \begin{cases} \displaystyle c_0 \int_\Omega |Du| & \text{if } \int_\Omega u\,dx = m \text{ and } u \in \mathrm{BV}(\Omega; \{\alpha, \beta\}) \\ +\infty & \text{otherwise} \end{cases}$$

and $c_0 = 2\int_\alpha^\beta \sqrt{W(s)}\,ds$. Since by Theorem 2.10, Γ-convergence implies the convergence of minima, and since $u \in \{\alpha, \beta\}$ implies

$$\int_\Omega |Du| = \int_{\Omega \cap S_u} |\beta - \alpha|\,d\mathcal{H}^{n-1} = |\beta - \alpha|\,\mathcal{P}_\Omega(A)$$

then property (2.4) holds.

3 Phase transitions in composite media

In this section we consider the following problem.

Problem 3.1 *Compute the* Γ*-limit of the family of functionals*

$$F_{\varepsilon,\delta}(u) = \int_\Omega \Big[\frac{W(u)}{\varepsilon} + \varepsilon f(\frac{x}{\delta}, Du)\Big]\,dx. \tag{3.5}$$

Here, W is as in Section 2 (but we assume for simplicity $\alpha = 0$ and $\beta = 1$), $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ is a Borel function satisfying the following conditions:

$$f(\cdot, \xi) \qquad \text{is } Y\text{-periodic for a.e. } \xi \in \mathbb{R}^n, \tag{3.6}$$

$$f(y, \cdot) \qquad \text{positively homogeneous of degree 2 for a.e. } y \in \mathbb{R}^n, \tag{3.7}$$

$$c_1|\xi|^2 \le f(y, \xi) \le c_2|\xi|^2 \qquad \text{for a.e. } y \in \mathbb{R}^n, \qquad \text{for every } \xi \in \mathbb{R}^n, \tag{3.8}$$

with $Y = (0,1)^n$, $0 < c_1 \le c_2$, and $\delta = \delta(\varepsilon) \to 0+$ as $\varepsilon \to 0+$. The physical model we have in mind is the study of phase transitions in composite media having a δ-periodic microstructure. More precisely, we want to analyse the interplay between phase-transition effect ($\varepsilon \to 0$) and homogenization effect ($\delta \to 0$), by computing the Γ-limit of $F_{\varepsilon,\delta}$ for any choice of the function $\delta = \delta(\varepsilon)$. In the following we describe sinthetically the main results we can prove; i.e.,

1. general compactness and integral representation theorem for the Γ-limit
2. different explicit formulas for the integrand in the Γ-limit, in the cases $\delta \sim \varepsilon$, $\varepsilon = o(\delta)$, $\delta = o(\varepsilon)$.

3.1 The Γ-limit in the general case

We start by *localizing* the functionals $F_{\varepsilon,\delta}$; i. e., we consider

$$F_{\varepsilon,\delta}(u,A)=\int_A\Big[\frac{W(u)}{\varepsilon}+\varepsilon f(\frac{x}{\delta},Du)\Big]\,dx$$

as a function of the pair (u,A), for all $u\in L^1_{\rm loc}(\mathbb{R}^n)$ and for every $A\in\mathcal{A}$, where $\mathcal{A}$ denotes the family of all bounded open subsets of $\mathbb{R}^n$. This procedure is usually required when one has to deal with Γ-convergence of integral functionals and the Γ-limit itself is expected to be an integral (see, for instance, [DM91], [BDF]).

Theorem 3.2 *(Compactness result) For every sequence $\varepsilon_j\to 0$ there exists a subsequence ε_{j_k} and a functional $F: L^1_{\rm loc}(\mathbb{R}^n)\times\mathcal{A}\to[0,+\infty]$, such that $F(u,\cdot)$ is a Borel measure, for every $u\in L^1_{\rm loc}(\mathbb{R}^n)$, and the sequence $F_{\varepsilon_{j_k},\delta_{j_k}}(\cdot,A)$ Γ-converges to $F(\cdot,A)$, for every $A\in\mathcal{A}$.*

Theorem 3.3 *(Integral representation result) There exists a function $\varphi:\mathbb{R}^n\to[0,+\infty[$ which is Borel measurable, convex, and positively homogeneous of degree one, such that*

$$F(u,A)=\begin{cases}\displaystyle\int_{S_u\cap A}\varphi(\nu_u)d\mathcal{H}^{n-1} & \textit{if } u\in BV(A;\{0,1\})\\ +\infty & \textit{otherwise.}\end{cases}\tag{3.9}$$

Moreover

$$c_0\sqrt{c_1}\le\varphi(\nu)\le c_0\sqrt{c_1}\qquad\text{if}\quad |\nu|=1\tag{3.10}$$

where $c_0=2\int_0^1\sqrt{W(s)}\,ds$.

Theorem 3.4 *(Derivation formula for φ) For all unit vectors $\nu\in\mathbb{R}^n$ and $\rho>0$*

$$\varphi(\nu)=\rho^{1-n}\inf\{F(u,\overline{Q^\nu_\rho}):u=u^\nu\quad\text{on}\quad\mathbb{R}^n\setminus Q^\nu_\rho\}$$

where Q^ν_ρ is the cube of $\mathbb{R}^n$ centered at 0, with side-lenght ρ and a face orthogonal to ν and

$$u^\nu(x)=\begin{cases}1 & \textit{if } \langle x,\nu\rangle>0\\ 0 & \textit{if } \langle x,\nu\rangle\le 0.\end{cases}\tag{3.11}$$

Remark 3.5 We remark that all the preceding results (compactness, integral representation form, derivation formula for φ) hold true in a more general form for functionals of the type

$$F_\varepsilon(u,A)=\int_A\Big[\frac{W(u)}{\varepsilon}+\varepsilon f_\varepsilon(x,Du)\Big]\,dx,$$

without periodicity assumptions on f_ε. It is enough to assume that $f_\varepsilon(x,\xi)$ is a Borel-measurable function, 2-positively homogeneous with respect to ξ and such that

$$c_1|\xi|^2\le f_\varepsilon(x,\xi)\le c_2|\xi|^2$$

on $\mathbb{R}^n \times \mathbb{R}^n$, with c_1 and c_2 independent of ε. In this case the integrand φ may depend both on x and ν and may be non convex in ν. Moreover, for every $x \in \mathbb{R}^n$ and for every unit vector $\nu \in \mathbb{R}^n$, φ is given by the following derivation formula

$$\varphi(x,\nu) = \limsup_{\rho\to 0+} \rho^{1-n} \inf\{F(u, \overline{Q^\nu_\rho(x)}) : u = u^{\nu,x} \text{ on } \mathbb{R}^n \setminus Q^\nu_\rho(x)\},$$

where $Q^\nu_\rho(x)$ is the cube of $\mathbb{R}^n$ centered at x, with side-lenght ρ and a face orthogonal to ν and

$$u^{\nu,x}(y) = \begin{cases} 1 & \text{if } \langle y-x,\nu\rangle > 0 \\ 0 & \text{if } \langle y-x,\nu\rangle \le 0. \end{cases} \tag{3.12}$$

The proofs of these results are based on standard Γ-convergence techniques that can be found in the papers [BC95], [BC96].

Remark 3.6 At this stage (both in the more general case and in the homogenization case) the functions F and φ *may depend* on the subsequence ε_{j_k}.

3.2 Explicit formulas for φ

In the following we shall deal only with the periodic case; i.e.,

$$f_\varepsilon(x,\xi) = f\Big(\frac{x}{\delta(\varepsilon)},\xi\Big), \qquad f(\cdot,\xi)\ Y\text{−periodic}$$

with $\delta(\varepsilon)\to 0$ as $\varepsilon \to 0$. In this case we can prove explicit formulas for φ and, as a corollary, we can obtain that F and φ do not depend on ε_{j_k} and the whole family $(F_{\varepsilon,\delta(\varepsilon)})_\varepsilon$ Γ- converges to F. We distinguish three different cases

$$\delta \sim \varepsilon, \quad \varepsilon = o(\delta), \quad \delta = o(\varepsilon).$$

Case 1: $\varepsilon \sim \delta$ (Oscillations with the same lenght-scale as the transition layer) We assume that $\frac{\varepsilon}{\delta} \to c \in]0,+\infty[$. Then, for every unit vector $\nu \in \mathbb{R}^n$

$$\varphi(\nu) = \lim_{T\to+\infty} T^{1-n} \inf\Big\{\int_{TQ^\nu} \Big(W(u) + f(cx, Du)\Big)\,dx : u = v^\nu \text{ on } \partial(TQ^\nu)\Big\}, \tag{3.13}$$

where $v^\nu(x) = w(\langle x,\nu\rangle)$, $w \in H^1_{\rm loc}(\mathbb{R})$, $\lim_{t\to-\infty} w(t) = 0$, $\lim_{t\to+\infty} w(t) = 1$,

$$\int_{\mathbb{R}} \Big(W(w(t)) + |w'(t)|^2\Big)\,dt < +\infty,$$

and Q^ν is the unit cube of $\mathbb{R}^n$, centered at 0, with a face orthogonal to ν.

We notice that, in this case, the effects of phase-transition and of homogenization are 'mixed' in the explicit formula. Moreover, if we are in the simple case where $f(x,\xi) = |\xi|^2$, then we recover the Modica-Mortola result (see Theorems 2.4 and 2.11). In this case in fact formula (3.13) simplifies into

$$\varphi(\nu) = \lim_{T\to+\infty} T^{1-n} \inf\{\int_{TQ^\nu} \Big(W(u) + |Du|^2\Big)\, dx : u = v^\nu \text{ on } \partial(TQ^\nu)\},$$

and we obtain $\varphi(\nu) = c_0$ by Fubini's theorem, once we remark (see e.g. [B98] Remark 3.11) that

$$c_0 = 2\int_0^1 \sqrt{W(s)}\, ds = \min \int_{-\infty}^{+\infty} \Big(W(w(t)) + |w'(t)|^2\Big)\, dt,$$

where the minimum is taken over all functions $w \in H^1_{\rm loc}(\mathbb{R};[0,+\infty))$ such that $\lim\limits_{t\to-\infty} w(t) = 0$, $\lim\limits_{t\to+\infty} w(t) = 1$.

Case 2: $\varepsilon = o(\delta)$ **(Oscillations a lenght-scale lower than the transition layer)** We assume that $\frac{\varepsilon}{\delta} \to 0$. Then, for every unit vector $\nu \in \mathbb{R}^n$

$$\varphi(\nu) = c_0\, \psi_{\rm hom}(\nu), \tag{3.14}$$

where $c_0 = 2\int_0^1 \sqrt{W(s)}\, ds$, $\psi(x,\xi) = \sqrt{f(x,\xi)}$ and

$$\begin{aligned}\psi_{\rm hom}(\nu) \quad = \quad & \lim_{T\to+\infty} T^{1-n} \inf\Big\{\int_{\overline{TQ^\nu}\cap S_u} \psi(x,\nu_u) d\mathcal{H}^{n-1} \\ & \qquad : u \in BV(\Omega;\{0,1\}),\ u = u^\nu \text{ on}\, \mathbb{R}^n \setminus TQ^\nu\Big\}\end{aligned} \tag{3.15}$$

and u^ν, Q^ν are defined as before.

We remark that in this case the effects of phase-transition and of homogenization separate in the explicit formula. Essentially, it happens that

$$\Gamma\text{-}\lim_{\varepsilon\to0} F_{\varepsilon,\delta(\varepsilon)} = \Gamma\text{-}\lim_{\delta\to0}\Big(\Gamma\text{-}\lim_{\varepsilon\to0} F_{\varepsilon,\delta(\varepsilon)}\Big).$$

To check this, assume for simplicity that

$$f(y,\xi) = a(y)|\xi|^2, \qquad 0 < c_1 \le a(y) \le c_2 < +\infty$$

and think of $y = \frac{x}{\delta}$ as a *parameter*. Then, if $u \in {\rm BV}(\Omega;\{0,1\})$,

$$\begin{aligned} & \Gamma\text{-}\lim_{\varepsilon\to0} \int_\Omega \Big(\frac{W(u(x))}{\varepsilon} + \varepsilon a(y)|Du(x)|^2\Big)\, dx \\ = \quad & \Gamma\text{-}\lim_{\varepsilon\to0} a(y) \int_\Omega \Big(\frac{W(u(x))}{a(y)\varepsilon} + \varepsilon|Du(x)|^2\Big)\, dx \\ = \quad & a(y) 2\int_0^1 \sqrt{\frac{W(s)}{a(y)}}\, ds \cdot \int_{\Omega\cap S_u} |\nu_u| d\mathcal{H}^{n-1}, \end{aligned}$$

where the last equality is due to Theorems 2.4 and 2.11. Hence the integrand is

$$c_0\sqrt{a(y)}|\nu| = c_0\sqrt{f(y,\nu)} = c_0\psi(y,\nu).$$

Now, replacing $y = \frac{x}{\delta}$ and computing

$$\Gamma\text{-}\lim_{\delta\to 0}\int_{\Omega\cap S_u}\sqrt{f(\frac{x}{\delta},\nu_u)}d\mathcal{H}^{n-1}$$

with $u \in \mathrm{BV}(\Omega;\{0,1\})$, by [AB] one gets finally

$$\varphi(\nu) = c_0\psi_{\mathrm{hom}}(\nu).$$

The formula for ψ_{hom} is obtained in [BC95]. Another homogenization formula for integrands $\psi(x,\xi)$ having linear growth in ξ, is due to Bouchitté (see [B86]).

Case 3: $\delta = o(\varepsilon)$ **(Oscillations with a lenght-scale faster than the transition layer)** We assume that $\frac{\varepsilon}{\delta} \to +\infty$. Then, for every unit vector $\nu \in \mathbb{R}^n$

$$\varphi(\nu) = c_0\sqrt{f_{\mathrm{hom}}(\nu)}, \tag{3.16}$$

where $c_0 = 2\int_0^1 \sqrt{W(s)}\,ds$,

$$f_{\mathrm{hom}}(\nu) = \inf\Big\{\int_{(0,1)^n} f(x, Du(x)+\nu_u)\,dx : u \in H^1_{per}((0,1)^n)\Big\} \tag{3.17}$$

and $H^1_{per}((0,1)^n) = \{u \in H^1((0,1)^n) : u$ has the same traces on opposite faces of $(0,1)^n\}$.

We remark that also in this case the effects of phase-transition and of homogenization separate in the explicit formula. Essentially, it happens that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)} = \Gamma\text{-}\lim_{\varepsilon\to 0}\Big(\Gamma\text{-}\lim_{\delta\to 0} F_{\varepsilon,\delta(\varepsilon)}\Big).$$

In fact, if we first let $\delta \to 0$ and note that the term

$$\int_\Omega \frac{W(u)}{\varepsilon}\,dx$$

is a continuous perturbation that does not affect the $\Gamma\text{-}\lim_{\delta\to 0}$, we find

$$\Gamma\text{-}\lim_{\delta\to 0}\int_\Omega\Big[\frac{W(u)}{\varepsilon} + \varepsilon f(\frac{x}{\delta}, Du)\Big]\,dx = \int_\Omega\Big[\frac{W(u)}{\varepsilon} + \varepsilon f_{\mathrm{hom}}(Du)\Big]\,dx$$

for all $u \in H^1(\Omega)$ (see e.g. [BDF]). Then we apply Modica-Mortola's technique, noticing that f_{hom} is positively homogeneous of degree 2, and we prove for all $u \in \mathrm{BV}(\Omega;\{0,1\})$

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\int_\Omega\Big[\frac{W(u)}{\varepsilon} + \varepsilon f_{\mathrm{hom}}(\nu)|Du|^2\Big]\,dx = c_0\int_{\Omega\cap S_u}\sqrt{f_{\mathrm{hom}}(\nu)}d\mathcal{H}^{n-1}.$$

4 A proof in the one-dimensional case

We now prove the results stated in the previous section in a one-dimensional linear setting; i.e., with energies of the form

$$F_{\varepsilon,\delta}(u)=\int_{(a,b)}\Big(\frac{W(u)}{\varepsilon}+\varepsilon g^2\Big(\frac{t}{\delta}\Big)|u'|^2\Big)\,dt \tag{4.18}$$

defined on $u\in W^{1,2}(a,b)$.

In order to avoid some technical difficulties we assume that $W,g:\mathbb{R}\to[0,+\infty)$ are continuous, W vanishes only at 0 and 1, g is 1-periodic and

$$0<m=\min g\le\max g=M.$$

We set as usual $c_0=2\int_0^1\sqrt{W(s)}\,ds$. Note that, by the inequalities

$$\int_{(a,b)}\Big(\frac{W(u)}{\varepsilon}+\varepsilon m^2|u'|^2\Big)\,dt\le F_{\varepsilon,\delta}(u)\le\int_{(a,b)}\Big(\frac{W(u)}{\varepsilon}+\varepsilon M^2|u'|^2\Big)\,dt$$

we immediately obtain that, however we choose $\delta=\delta(\varepsilon)$ the Γ-limit F_0 (if it exists) of $F_{\varepsilon,\delta(\varepsilon)}$ is finite only on functions u piecewise constant on (a,b) with $u\in\{0,1\}$ a.e. Moreover, for such an u we have the estimate

$$mc_0\,\#(S_u)\le F_0(u)\le Mc_0\,\#(S_u).$$

The exact value of F_0 depends on the behaviour of $\delta(\varepsilon)$ with respect to ε.

Case 1: oscillations on the same scale of the transition layer This is the case when

$$\lim_{\varepsilon\to0}\frac{\varepsilon}{\delta(\varepsilon)}=K\in(0,+\infty).$$

We can reason as in the case when g is a constant (following for example the proof of [B98] Theorem 3.10), showing that $F_0(u)=c_{W,K}\,\#(S_u)$, where

$$c_{W,K}=\min\Big\{\int_{-\infty}^{+\infty}(W(v)+g(Ks)|v'|^2)\,ds:\ v(-\infty)=0,\ v(+\infty)=1\Big\}.$$

The interaction of the two limit processes results in a contribution of K and g to the definition of the constant $c_{W,K}$.

Case 2: oscillations on a finer scale than the transition layer This is the case when

$$\lim_{\varepsilon\to0}\frac{\delta(\varepsilon)}{\varepsilon}=0.$$

The 'guess' for the limit functional is that

$$F_0 = \Gamma\text{-}\lim_{\varepsilon\to 0}\Big(\Gamma\text{-}\lim_{\delta\to 0} F_{\varepsilon,\delta}\Big);$$

i.e., first we consider δ as independent of ε and we let it tend to 0, obtaining

$$\Gamma\text{-}\lim_{\delta\to 0} F_{\varepsilon,\delta} = \int_{(a,b)} \Big(\frac{W(u)}{\varepsilon} + \varepsilon c_g^2 |u'|^2\Big)\, dt,$$

where

$$c_g = \Big(\int_0^1 \frac{1}{g^2(s)}\, ds\Big)^{-1/2}.$$

Note in fact that the first term is continuous with respect to the L^1 convergence, while we can apply the homogenization theorem (see e.g. [BDF]) to the second one. By letting $\varepsilon \to 0$ we then have

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\Big(\Gamma\text{-}\lim_{\delta\to 0} F_{\varepsilon,\delta}(u)\Big) = c_g c_0 \#(S_u)$$

for u piecewise constant on (a,b) with $u \in \{0,1\}$ a.e., which gives the form of F_0.

In order to prove that this *ansatz* is correct, we have to check the two Γ-limit inequalities for u piecewise constant on (a,b) with $u \in \{0,1\}$ a.e. We begin with the liminf inequality. Let $u_\varepsilon \to u$ in $L^1(a,b)$ be such that $\liminf_{\varepsilon\to 0+} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon) < +\infty$. Upon extracting a subsequence we can suppose this liminf is a limit. With fixed $\eta \in (0,1/2)$, let $(a_\varepsilon, b_\varepsilon)$ be an interval such that $\{u_\varepsilon(a_\varepsilon), u_\varepsilon(b_\varepsilon)\} = \{\eta, 1-\eta\}$ and $u_\varepsilon(t) \in (\eta, 1-\eta)$ for $t \in (a_\varepsilon, b_\varepsilon)$. We then have

$$(b_\varepsilon - a_\varepsilon) \le \varepsilon \frac{1}{\min_{[\eta,1-\eta]} W} \int_{(a_\varepsilon,b_\varepsilon)} \frac{W(u_\varepsilon)}{\varepsilon}\, dt \le \varepsilon c,$$

with $c = c(W,\eta)$. It is not restrictive to suppose indeed that $b_\varepsilon - a_\varepsilon = \varepsilon T$ for some $T > 0$. We then have, upon an affine change of variables,

$$\begin{aligned} &\int_{(a_\varepsilon,b_\varepsilon)} \Big(\frac{W(u_\varepsilon)}{\varepsilon} + \varepsilon g^2\Big(\frac{t}{\delta(\varepsilon)}\Big)|u_\varepsilon'|^2\Big)\, dt \\ \ge\ & \min\Big\{\int_0^T \Big(W(v) + g^2\Big(\frac{t}{\delta(\varepsilon)/\varepsilon}\Big)|v'|^2\Big)\, dt :\ v(0) = \eta, v(T) = 1-\eta\Big\}. \end{aligned}$$

Note that the number of intervals $(a_\varepsilon, b_\varepsilon)$ as above is definitely not larger than $\#(S_u)$, and that, by homogenization, the minimum problems above converge, as $\varepsilon \to 0$ (so that $\delta(\varepsilon)/\varepsilon \to 0$) to

$$\min\Big\{\int_0^T \Big(W(v) + c_g^2 |v'|^2\Big)\, dt :\ v(0) = \eta, v(T) = 1-\eta\Big\} \ge 2c_g \int_\eta^{1-\eta} \sqrt{W(s)}\, ds.$$

Taking all these facts into account, we obtain the desired inequality by the arbitrariness of η.

To check the converse inequality, it will suffice to consider the case of $u = \chi_{(0,+\infty)}$. In order to exhibit a recovery sequence, let $\eta > 0$, $T > 0$ be such that

$$\min\Big\{\int_{-T}^{T}\Big(W(v) + c_g^2|v'|^2\Big)\,dt: \ v(0) = 0, v(T) = 1\Big\} \le c_g c_0 + \eta.$$

Again, note that by homogenization this minimum is the limit of

$$\min\Big\{\int_{-T}^{T}\Big(W(v) + g^2\Big(\frac{t}{\delta(\varepsilon)/\varepsilon}\Big)|v'|^2\Big)\,dt: \ v(0) = 0, v(T) = 1\Big\}$$

as $\varepsilon \to 0$. Let v_ε be functions realizing the corresponding minimum, and let

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } t < -\varepsilon T \\ v_\varepsilon(t/\varepsilon) & \text{if } -\varepsilon T \le t \le \varepsilon T \\ 1 & \text{if } t > \varepsilon T. \end{cases}$$

Then $u_\varepsilon \to u$ and $\limsup_{\varepsilon\to 0+} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon) \le c_g c_0 + \eta$. By the arbitrariness of η we conclude the proof.

Case 3: oscillations on a slower scale than the transition layer This is the case when

$$\lim_{\varepsilon\to 0}\frac{\delta(\varepsilon)}{\varepsilon} = +\infty.$$

Now, the 'guess' for the limit functional is that

$$F_0 = \Gamma\text{-}\lim_{\delta\to 0}\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta}\Big);$$

i.e., first we consider δ as fixed, and let ε tend to 0, obtaining

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta} = c_0 \sum_{t\in S_u} g\Big(\frac{t}{\delta}\Big).$$

By letting $\delta \to 0$ we then have

$$\Gamma\text{-}\lim_{\delta\to 0}\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta}(u)\Big) = m\, c_0 \#(S_u)$$

for u piecewise constant on (a, b) with $u \in \{0, 1\}$ a.e., which gives the form of F_0.

To check the validity of this guess it will suffices to exhibit a recovery sequence for $u = \chi_{(0,+\infty)}$, the liminf inequality being already proven by comparison. With fixed $\eta > 0$ let $T > 0$ and $v \in H^1(-T, T)$ be such that $v(-T) = 0$, $v(T) = 1$ and

$$\int_{-T}^{T}\Big(W(v) + m^2|v'|^2\Big)\,dt \le mc_0 + \eta.$$

Let $t_m \in [0,1]$ be such that $g(t_m) = \min g$. Define

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } t < \delta(\varepsilon)t_m - \varepsilon T \\ v((t - \delta(\varepsilon)t_m)/\varepsilon) & \text{if } \delta(\varepsilon)t_m - \varepsilon T \leq t \leq \delta(\varepsilon)t_m + \varepsilon T \\ 1 & \text{if } t > \delta(\varepsilon)t_m + \varepsilon T. \end{cases}$$

We then have $u_\varepsilon \to u$ and

$$F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon) = \int_{-T}^{T} \Big(W(v) + m^2|v'|^2\Big)\,dt + \int_{-T}^{T} \Big(g^2\Big(t_m + \frac{t\varepsilon}{\delta(\varepsilon)}\Big) - g^2(t_m)\Big)|v'|^2\,dt.$$

As the last term tends to 0 as $\varepsilon \to 0$, the proof is concluded.

References

[AB] L. Ambrosio, A. Braides, Functionals defined on partitions in sets of finite perimeter, I: integral representation and Γ-convergence, J. Math. Pures Appl. 69 (1990) 285-305.

[AFP] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[ABC] N. Ansini, A. Braides, V. Chiadò Piat, Gradient theory of phase transitions in composite media, to appear.

[B86] G. Bouchitté, Convergence et relaxation de fonctionelles du calcul des variations a croissance linéaire. Application a l'homogénéisation en plasticité. *Ann. Fac. Sci. Uni. Toulouse math.* **VIII** (1986/87), 7-36.

[B98] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1998.

[B] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, to appear.

[BC95] A. Braides, V. Chiadò Piat, A derivation formula for convex integral functionals defined on BV(Ω), *J. Convex Anal.* **2** (1995), 69-85.

[BC96] A. Braides, V. Chiadò Piat, Integral representation results for functionals defined on $SBV(\Omega, \mathbb{R}^m)$, *J. Math. Pures Appl.* **75** (1996), 595-626.

[BDF] A. Braides, A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[DG54] E. De Giorgi, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio a r dimensioni, *Ann. Mat. Pura Appl.* **36** (1954), 191-213.

[DG55] E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni, *Ricerche Mat.* **4** (1955), 95-113.

[DGF75] E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei, *Rend. Cl. Sci. Mat. Fis. Natur.* **58** (1975), 842-850.

[DM93] G. Dal Maso, *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.

[F68] H. Federer, *Geometric Measure Theory*, Springer Verlag, Berlin, 1968.

[G84] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[G85] M.E. Gurtin, Some results and conjectures in the gradient theory of phase transitions, in "Metastability and incompletely posed problems, Proc. Workshop, Minneapolis/Minn. 1984/85", *IMA Vol. Math. Appl.* **3** (1987), 135-146.

[MM84] U. Massari, M. Miranda, *Minimal Surfaces of Codimension One*, North-Holland Math. Studies **91**, North-Holland 1984.

[MM77] L. Modica, S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital.* **14-B** (1977), 285-299.

[M87] L.Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. Anal.* **98** (1987), 123-142.