

# RIGIDITY RESULTS FOR RIEMANNIAN TWISTOR SPACES UNDER VANISHING CURVATURE CONDITIONS

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ABSTRACT. In this paper we provide new rigidity results for four-dimensional Riemannian manifolds and their twistor spaces. In particular, using the moving frame method, we prove that  $\mathbb{C}\mathbb{P}^3$  is the only twistor space whose Bochner tensor is parallel; moreover, we classify Hermitian Ricci-parallel and locally symmetric twistor spaces and we show the nonexistence of conformally flat twistor spaces. We also generalize a result due to Atiyah, Hitchin and Singer concerning the self-duality of a Riemannian four-manifold.

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## 1. INTRODUCTION AND MAIN RESULTS

Let  $(M, g)$  be an oriented Riemannian manifold of dimension  $2n$ , with metric  $g$ . The *twistor space*  $Z$  associated to  $M$  is defined as the set of all the couples  $(p, J_p)$  such that  $p \in M$  and  $J_p$  is a complex structure on  $T_p M$  compatible with  $g$ , i.e. such that  $g_p(J_p(X), J_p(Y)) = g_p(X, Y)$  for every  $X, Y \in T_p M$ .<sup>4</sup>

Alternatively, we can define  $Z$  in an equivalent way as

$$Z = O(M) / U(n),$$

where  $O(M)$  denotes the orthonormal frame bundle over  $M$  and the unitary group  $U(n)$  is identified with a subgroup of  $SO(2n)$  (see [13] for further details).

These structures, introduced by Penrose ([24]) as an attempt to define an innovative framework for Physics, have been the subject of many investigations by the mathematical community, in virtue of the numerous geometrical and algebraic tools involved in the definition of their properties. In 1978, Atiyah, Hitchin and Singer ([1]) adapted Penrose's twistor theory to the Riemannian context, introducing the concept of twistor space associated to a Riemannian four-manifold and paving the way for many researches about this subject.

The orientation on  $M$  implies that  $O(M)$  has two connected components,  $O(M)_+$  and  $O(M)_-$ , and therefore we can define the two connected components of  $Z$

$$Z_{\pm} = O(M)_{\pm} / U(n) = SO(M) / U(n),$$

where  $SO(M)$  is the orthonormal oriented frame bundle over  $M$ . We choose the component  $Z_-$  to be the twistor space of  $(M, g)$  (see also [5] and [25]). It is possible to define a natural family of Riemannian metrics  $g_t$  on  $Z_-$ , where  $t > 0$  ([13] [21]); from now on, we systematically use the notation  $(Z, g_t)$  to denote the twistor space  $Z_-$  endowed with the Riemannian metric  $g_t$ .

In general, if  $(M, g)$  is a Riemannian manifold of dimension  $m$ , the Riemann curvature tensor  $\text{Riem}$  on  $M$  admits the well known decomposition

$$\text{Riem} = W + \frac{1}{m-2} \text{Ric} \otimes g - \frac{S}{2(m-1)(m-2)} g \otimes g,$$

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<sup>4</sup>In this work, we call *complex structure* an endomorphism  $J_V$  of a vector space  $V$  such that  $J_V^2 = -\text{Id}_V$ , while we call *almost complex structure* a  $(1, 1)$  tensor field  $J$  on a differentiable manifold  $M$  such that  $J$  assigns smoothly, to every point  $p$ , a complex structure  $J_p$  on  $T_p M$ .

where  $W$ ,  $\text{Ric}$  and  $S$  denote the *Weyl tensor*, the *Ricci tensor* and the *scalar curvature* of  $M$ , respectively, and  $\otimes$  is the *Kulkarni-Nomizu product*. Moreover, the Riemann curvature tensor defines a symmetric linear operator from the bundle of two-forms  $\Lambda^2$  to itself

$$\mathcal{R}: \Lambda^2 \longrightarrow \Lambda^2$$

$$\gamma \longmapsto \mathcal{R}(\gamma) = \frac{1}{4} R_{ijkl} \gamma_{kt} \theta^i \wedge \theta^j,$$

where  $\{\theta^i\}_{i=1,\dots,m}$  is a local orthonormal coframe on an open set  $U \subset M$ , with dual frame  $\{e_i\}_{i=1,\dots,m}$ ,  $\gamma_{kt} = \gamma(e_k, e_t)$  and  $R_{ijkl}$  are the components of the Riemann tensor with respect to the coframe  $\{\theta^i\}$ .

If  $m = 4$  and  $M$  is oriented,  $\Lambda^2$  splits, *via* the Hodge  $\star$  operator, into the direct sum of two subbundles  $\Lambda_+$  and  $\Lambda_-$ . This implies that the Riemann curvature operator  $\mathcal{R}$  assumes a block matrix form

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

where  $A$  (resp,  $C$ ) is a symmetric endomorphism of  $\Lambda_+$  (resp.,  $\Lambda_-$ ) and  $B$  is a symmetric linear map from  $\Lambda_+$  to  $\Lambda_-$  (see [1], [2] and [28]). Moreover,  $\text{tr } A = \text{tr } C = \frac{S}{4}$ . This corresponds to a decomposition of the Weyl tensor into a sum

$$W = W^+ + W^-,$$

where  $W^+$  (resp.,  $W^-$ ) is called the *self-dual* (resp., *anti-self-dual*) part of  $W$ . If  $W^+ = 0$  (resp.,  $W^- = 0$ ), we say that  $M$  is an *anti-self-dual* (resp., *self-dual*) manifold. If we consider the symmetric linear operators induced by  $W^+$  and  $W^-$ , we have that their representative matrices are  $A - \frac{S}{12} I_3$  and  $C - \frac{S}{12} I_3$ , respectively, with respect to any positively oriented local orthonormal coframe; thus,  $(M, g)$  is self-dual (resp., anti-self-dual) if and only if  $C = \frac{S}{12} I_3$  (resp.,  $A = \frac{S}{12} I_3$ ). Note that, if the coframe is negatively oriented,  $A$  and  $C$  need to be exchanged in the previous statements.

In this paper, starting from our previous work [5], we focus our attention on some rigidity results concerning twistor spaces satisfying vanishing conditions on relevant geometric tensors, such as the Weyl tensor, the Bochner tensor and the covariant derivatives of the Ricci tensor and the Riemann tensor. For instance, we are able to show the following results:

- nonexistence of locally conformally flat twistor spaces;
- a twistor space is Bochner-flat if and only if the underlying manifold is homotetically isometric to  $\mathbb{S}^4$ ;
- characterization of Ricci-parallel and locally symmetric twistor spaces;
- a generalization of Atiyah-Hitchin-Singer result, using the divergences of the Nijenhuis tensor(s).

The paper is organized as follows: in Section 2, we show that, given a Riemannian four-manifold  $(M, g)$ , its twistor space  $(Z, g_t)$  cannot be locally conformally flat for any  $t > 0$ .

Section 3 is devoted to the characterization of Bochner-flat twistor spaces: in particular, we show that the only Bochner-parallel twistor space is "essentially"  $\mathbb{C}\mathbb{P}^3$ , which is the one associated to the four-sphere  $\mathbb{S}^4$ .

In Section 4 we consider Ricci parallel and locally symmetric twistor spaces, providing rigidity results for twistor spaces whose Atiyah-Hitchin-Singer almost complex structure  $J^+ = J$  is integrable (see Appendix C for details).

In Section 5 we prove a general quadratic formula for  $|\nabla J|^2$ ; moreover, we generalize the necessary and sufficient condition for the integrability of  $J$ , first proven by Atiyah, Hitchin and Singer [1], through a vanishing condition on the divergences of the associated Nijenhuis tensor. We also prove a new result concerning the Nijenhuis tensor of the Eells-Salamon almost complex structure  $J^- = \mathbf{J}$  (see [14]).

To keep the paper self-contained as much as possible, we provide also five brief appendices devoted to technicalities and some heavy computations (for instance, the list of the local components of the Weyl tensor of a twistor space  $(Z, g_t)$ ).

## 2. LOCALLY CONFORMALLY FLAT TWISTOR SPACES

In this section, we want to show that the twistor space  $(Z, g_t)$  associated to a Riemannian four-manifold  $(M, g)$  is never locally conformally flat for any  $t > 0$ . By Weyl-Schouten Theorem, we know that a Riemannian manifold of dimension  $n \geq 4$  is locally conformally flat if and only if its Weyl tensor  $W$  vanishes identically (for a proof, see [19] or [22]).

Before we state the main result of this section, let us recall the transformation laws for the matrices  $A$  and  $B$  appearing in the decomposition of the Riemann curvature operator: we know that, given a local orthonormal frame  $e \in O(M)_-$ , if we choose another frame  $\tilde{e} \in O(M)_-$ , the change of frames is determined by a matrix  $a \in SO(4)$  and that the matrices  $A$  and  $B$  transform according to the equations

$$(2.1) \quad \tilde{A} = a_+^{-1} A a_+, \quad \tilde{B} = a_-^{-1} B a_+$$

where  $SO(3) \times SO(3) \ni (a_+, a_-) = \mu(a)$  and  $\mu$  is a surjective homomorphism from  $SO(4)$  to  $SO(3) \times SO(3)$  induced by the universal covers of  $SO(4)$  and  $SO(3)$  (see [2], [5] and [25] for a detailed description).

For the sake of simplicity, throughout the paper we adopt the following notation

$$(2.2) \quad \begin{aligned} \mathcal{Q}_{ab} &:= R_{12ab} + R_{34ab}; \\ \mathcal{Q}_{ab} &:= R_{13ab} + R_{42ab}; \\ \mathcal{Q}_{ab} &:= R_{14ab} + R_{23ab}. \end{aligned}$$

Now, we can state the following result, which is new, to the best of our knowledge:

**Theorem 2.1.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t)$  be its twistor space. Then,  $(Z, g_t)$  is not locally conformally flat for any  $t > 0$ .*

*Proof.* Let us suppose that  $(Z, g_t)$  is locally conformally flat, i.e., by Weyl-Schouten Theorem,  $\overline{W} \equiv 0$  on  $Z$ . By the vanishing of the coefficients  $\overline{W}_{ab56}$  in (B.4), we obtain the system

$$\begin{cases} \mathcal{Q}_{12} = \frac{t^2}{4}(\mathcal{Q}_{1c} \mathcal{Q}_{2c} - \mathcal{Q}_{2c} \mathcal{Q}_{1c}) \\ \mathcal{Q}_{34} = \frac{t^2}{4}(\mathcal{Q}_{3c} \mathcal{Q}_{4c} - \mathcal{Q}_{4c} \mathcal{Q}_{3c}) \end{cases} ;$$

expliciting the right-hand sides and then summing the equations, we derive the equality

$$2A_{11} + t^2(A_{23}^2 - A_{22}A_{33}) = 0.$$

Note that  $\overline{W} \equiv 0$  is a global condition: in particular, this means that the equation above must hold for every  $p \in M$  (it suffices to consider the pullback maps *via* any section of the twistor bundle). Moreover, since the locally conformally flatness is a frame-independent condition, the equation holds for every local negatively oriented orthonormal frame  $e \in O(M)_-$ . In particular, since  $A$  is a symmetric matrix, we have that the equality holds for every frame  $e$  such that  $A$  is diagonal; in this situation, we have that

$$2A_{11} - t^2 A_{22} A_{33} = 0,$$

for every frame with respect to which  $A$  is diagonal. By (2.1), we can exchange the diagonal entries of  $A$  with suitable changes of frames in order to obtain the additional equations

$$\begin{aligned} 0 &= 2\tilde{A}_{11} - t^2 \tilde{A}_{22} \tilde{A}_{33} = 2A_{22} - t^2 A_{11} A_{33} \\ 0 &= 2\hat{A}_{11} - t^2 \hat{A}_{22} \hat{A}_{33} = 2A_{33} - t^2 A_{11} A_{22}, \end{aligned}$$

where  $\tilde{A}_{ij}$  and  $\hat{A}_{ij}$  are the entries of the matrix  $A$  with respect to some frames  $\tilde{e}$  and  $\hat{e}$ , respectively. At a point  $p \in M$ , since  $t > 0$ , the system of these three equations admits three distinct solutions:

- (1)  $A_{11} = A_{22} = A_{33} = 0$ ;
- (2)  $A_{11} = A_{22} = A_{33} = \frac{2}{t^2}$ ;
- (3) two diagonal entries out of three are equal to  $-\frac{2}{t^2}$ , while the third is equal to  $\frac{2}{t^2}$ .

This means that, at  $p \in M$ , the scalar curvature  $S$  of  $(M, g)$  can attain the values  $0$ ,  $24/t^2$  or  $-8/t^2$ . Since the scalar curvature is a smooth function on  $M$  and, for every point of  $M$ , one of the three equations must hold, we can conclude that  $S$  is constant on  $M$ : indeed, the possible values for  $S$  are finitely many, therefore, if  $S(p) \neq S(p')$  for  $p, p' \in M$ ,  $S$  would not be a smooth function.

First, let us prove that the first two cases lead to a contradiction. Note that, in this situation,  $A$  is a scalar matrix for every point  $p \in M$  (and, by (2.1), for every frame), which

means that  $(M, g)$  is a self-dual manifold. By the vanishing of the components  $\overline{W}_{5ab5}$  and  $\overline{W}_{6ab6}$ , if  $a \neq b$ , we obtain

$$0 = \overline{W}_{5ab5} + \overline{W}_{6ab6} = \frac{1}{2}R_{ab} - \frac{t^2}{2}(\mathcal{Q}_{ac} \mathcal{Q}_{bc} + \mathcal{Q}_{ac} \mathcal{Q}_{bc});$$

in particular, for  $(a, b) = (1, 2)$  and  $(a, b) = (3, 4)$ , by the self-duality condition we can compute

$$\begin{aligned} \mathcal{Q}_{13} - \mathcal{Q}_{14} &= R_{12} = \frac{t^2 S}{6}(\mathcal{Q}_{13} - \mathcal{Q}_{14}) \\ \mathcal{Q}_{13} + \mathcal{Q}_{14} &= R_{34} = \frac{t^2 S}{6}(\mathcal{Q}_{13} + \mathcal{Q}_{14}), \end{aligned}$$

which imply immediately  $\mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{13} = \mathcal{Q}_{42} = 0$  on  $M$ . This is equivalent to say that the entries  $B_{23}$  and  $B_{32}$  of the matrix  $B$  vanish identically on  $M$ ; since this is a global condition, by suitable change of frames, equation (2.1) implies that the matrix  $B$  is the zero matrix, i.e.  $(M, g)$  is an Einstein manifold (see also [5] for a detailed proof). However, we have that

$$0 = \overline{W}_{5656} \implies |\mathcal{Q}_{ab}|^2 + |\mathcal{Q}_{ab}|^2 = \frac{2S}{3t^2} + \frac{8}{t^4};$$

the left-hand side of the second equation is equal to  $(S^2/18)$  for an Einstein, self-dual manifold, hence, for  $S$  equal to 0 or to  $24/t^2$ , we get a contradiction.

Thus, we can choose a frame  $e$  with respect to which  $A$  is diagonal and

$$A_{11} = -A_{22} = A_{33} = -\frac{2}{t^2}.$$

If we consider again the equations  $\overline{W}_{5ab5} = \overline{W}_{6ab6} = 0$ , we obtain

$$\begin{aligned} \frac{1}{2}(\mathcal{Q}_{13} - \mathcal{Q}_{14}) &= \frac{1}{2}R_{12} = \frac{t^2}{2}(\mathcal{Q}_{1c} \mathcal{Q}_{2c} + \mathcal{Q}_{1c} \mathcal{Q}_{2c}) = 2(\mathcal{Q}_{13} + \mathcal{Q}_{14}) \\ \frac{1}{2}(\mathcal{Q}_{13} + \mathcal{Q}_{14}) &= \frac{1}{2}R_{34} = \frac{t^2}{2}(\mathcal{Q}_{3c} \mathcal{Q}_{4c} + \mathcal{Q}_{3c} \mathcal{Q}_{4c}) = 2(\mathcal{Q}_{13} - \mathcal{Q}_{14}), \end{aligned}$$

which obviously imply  $\mathcal{Q}_{14} = \mathcal{Q}_{13} = 0$ , i.e.  $B_{32} = B_{23} = 0$  in the chosen frame  $e$ . In fact, we can say more: the equalities  $B_{32} = B_{23} = 0$  hold for every frame  $e'$  with respect to which the matrix  $A$  is in diagonal form with  $A_{11} = -A_{22} = A_{33} = -2/t^2$ . Note that we can choose suitable change of frames such that  $\tilde{A} = A$ , where  $\tilde{A}$  is the matrix associated to the transformed frame  $\tilde{e}$ : indeed, it suffices to choose  $a_+ = I_3$  in (2.1).

Therefore, with suitable choices of  $a_-$  and putting  $a_+ = I_3$  in (2.1), it is immediate to show that

$$B_{12} = B_{13} = B_{22} = B_{23} = B_{32} = B_{33} = 0$$

for a frame  $e$  with respect to which  $A$  is in diagonal form with  $A_{11} = -A_{22} = A_{33} = -2/t^2$ .

Finally, let us compute

$$0 = \overline{W}_{5656} \implies \frac{128}{t^4} = |\mathcal{Q}_{ab}|^2 + |\mathcal{Q}_{ab}|^2 = \frac{2S}{3t^2} + \frac{8}{t^4} = \frac{8}{3t^4},$$

which is obviously impossible. Thus,  $(Z, g_t)$  cannot be locally conformally flat.  $\square$

By a well-known result due to Glódek (see [16]), it is immediate to show the following

**Corollary 2.2.** *A twistor space  $(Z, g_t)$  is conformally symmetric, i.e.  $\nabla\overline{W} \equiv 0$ , if and only if it is locally symmetric, i.e.  $\nabla\overline{\text{Riem}} \equiv 0$ .*

### 3. BOCHNER-FLAT TWISTOR SPACES

Let  $(N, g, J)$  be a almost Hermitian manifold of dimension  $2n$ . We can define the *Bochner tensor*  $B$  of  $N$  as the  $(0, 4)$ -tensor whose components with respect to a local orthonormal frame are

$$(3.1) \quad \begin{aligned} B_{pqrs} = & R_{pqrs} + \frac{1}{2(n+2)} [\delta_{ps}R_{qr} - \delta_{pr}R_{qs} + \delta_{qr}R_{ps} - \delta_{qs}R_{pr} + \\ & + J_s^p J_r^t R_{qt} - J_r^p J_s^t R_{qt} - 2J_q^p J_s^t R_{rt} + \\ & + J_r^q J_s^t R_{pt} - J_s^q J_r^t R_{pt} - 2J_s^r J_q^t R_{pt}] + \\ & - \frac{S}{4(n+1)(n+2)} [\delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + J_s^p J_r^q - J_r^p J_s^q - 2J_q^p J_s^r]. \end{aligned}$$

This tensor was first introduced by Bochner as a "complex analogue" of the Weyl tensor [3]. It is important to note that some authors define the Bochner tensor as  $-B$ , because of a different convention for the sign of the Riemann tensor (see, for instance, [29] and [30]).

We say that  $N$  is a *Bochner-flat manifold* if  $B$  vanishes identically, i.e. if  $B_{pqrs} = 0$  for every  $1 \leq p, q, r, s \leq 2n$ . It is known that, in general, the Bochner tensor does not satisfy the same symmetries as the Riemann tensor (see, for instance, [33]). However, if  $N$  is Bochner-flat, by (3.1) we obtain

$$(3.2) \quad \begin{aligned} R_{pqrs} = & - \frac{1}{2(n+2)} [\delta_{ps}R_{qr} - \delta_{pr}R_{qs} + \delta_{qr}R_{ps} - \delta_{qs}R_{pr} + \\ & + J_s^p J_r^t R_{qt} - J_r^p J_s^t R_{qt} - 2J_q^p J_s^t R_{rt} + \\ & + J_r^q J_s^t R_{pt} - J_s^q J_r^t R_{pt} - 2J_s^r J_q^t R_{pt}] + \\ & + \frac{S}{4(n+1)(n+2)} [\delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + J_s^p J_r^q - J_r^p J_s^q - 2J_q^p J_s^r], \end{aligned}$$

which means that the right-hand side of (3.2) satisfies the same symmetries as the Riemann tensor.

Now, let  $(M, g)$  be a four-dimensional Riemannian manifold and let  $(Z, g_t, J)$  be its twistor space, regarded as an almost Hermitian manifold. It is known that  $(Z, g_t, J)$  is a Kähler-Einstein manifold if and only if  $(M, g)$  is an Einstein, self-dual manifold with scalar curvature  $S = 12/t^2$  (see, for instance, [5], [10], [23] and Corollary 4.2 of this paper). Let us suppose that  $(Z, g_t, J)$  is a Kähler-Einstein manifold and let  $\overline{B}$  be its Bochner tensor. Under these

hypotheses, we can compute the components  $\overline{B}_{pqrs}$ :

$$(3.3) \quad \begin{aligned} \overline{B}_{pqrs} &= 0, \text{ if at least one of the indices is equal to 5 or 6;} \\ \overline{B}_{abcd} &= R_{abcd} - \frac{1}{t^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \end{aligned}$$

(note that, in this case, the Bochner tensor satisfies the same symmetries as the Riemann tensor. See also Remark 3.5). By direct inspection of these components and by recalling that  $\mathbb{S}^4$  is the only four-dimensional space form with positive sectional curvature, up to isometries, one can show the following

**Proposition 3.1.** *Let  $(M, g)$  be a Riemannian four-manifold such that its twistor space  $(Z, g_t, J)$  is Kähler-Einstein. Then  $(Z, g_t, J)$  is Bochner-flat if and only if  $(M, g)$  is isometric to  $\mathbb{S}^4$ , with its canonical Riemannian metric.*

It is natural to ask whether Proposition 3.1 can be generalized or not if there are no hypothesis on the almost complex structure  $J$ : more precisely, our goal is to characterize almost Hermitian, Bochner-flat twistor spaces. Rather surprisingly, it turns out that  $\mathbb{S}^4$  is the only Riemannian four-manifold whose twistor space is Bochner-flat.

First, let us define the covariant derivative  $\nabla B$  of the Bochner tensor  $B$  of an almost Hermitian manifold  $(N, g, J)$ , whose components with respect to a local orthonormal coframe are

$$(3.4) \quad \begin{aligned} B_{pqrs,u} &= R_{pqrs,u} + \frac{1}{2(n+2)} [\delta_{ps}R_{qr,u} - \delta_{pr}R_{qs,u} + \delta_{qr}R_{ps,u} - \delta_{qs}R_{pr,u} + \\ &\quad + R_{qt}(J_r^t J_s^p + J_s^p J_r^t) + J_s^p J_r^t R_{qt,u} + \\ &\quad - R_{qt}(J_s^t J_r^p + J_r^p J_s^t) - J_r^p J_s^t R_{qt,u} + \\ &\quad - 2R_{rt}(J_s^t J_q^p + J_q^p J_s^t) - 2J_q^p J_s^t R_{rt,u} + \\ &\quad + R_{pt}(J_s^t J_r^q + J_r^q J_s^t) + J_r^q J_s^t R_{pt,u} + \\ &\quad - R_{pt}(J_r^t J_s^q + J_s^q J_r^t) - J_s^q J_r^t R_{pt,u} + \\ &\quad - 2R_{pt}(J_q^t J_s^r + J_s^r J_q^t) - 2J_s^r J_q^t R_{pt,u}] + \\ &\quad - \frac{S_u}{4(n+1)(n+2)} [\delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + J_s^p J_r^q - J_r^p J_s^q - 2J_q^p J_s^r] \\ &\quad - \frac{S}{4(n+1)(n+2)} [J_r^q J_s^p + J_s^p J_r^q - J_s^q J_r^p - J_r^p J_s^q - 2J_s^r J_q^p - 2J_q^p J_s^r], \end{aligned}$$

and

$$\begin{aligned} \nabla \text{Ric} &= R_{pq,t} \theta^t \otimes \theta^p \otimes \theta^q \\ \nabla J &= J_{q,t}^p \theta^t \otimes \theta^q \otimes e_p \\ dS &= S_u \theta^u. \end{aligned}$$

We say that  $(N, g, J)$  is *Bochner-parallel* if  $\nabla B \equiv 0$ ; in this case, by (3.4) the components of  $\nabla \text{Riem}$  satisfy the equation

$$(3.5) \quad \begin{aligned} R_{pqrs,u} = & -\frac{1}{2(n+2)} [\delta_{ps}R_{qr,u} - \delta_{pr}R_{qs,u} + \delta_{qr}R_{ps,u} - \delta_{qs}R_{pr,u} + \\ & + R_{qt}(J_r^t J_{s,u}^p + J_s^p J_{r,u}^t) + J_s^p J_r^t R_{qt,u} + \\ & - R_{qt}(J_s^t J_{r,u}^p + J_r^p J_{s,u}^t) - J_r^p J_s^t R_{qt,u} + \\ & - 2R_{rt}(J_s^t J_{q,u}^p + J_q^p J_{s,u}^t) - 2J_q^p J_s^t R_{rt,u} + \\ & + R_{pt}(J_s^t J_{r,u}^q + J_r^q J_{s,u}^t) + J_r^q J_s^t R_{pt,u} + \\ & - R_{pt}(J_r^t J_{s,u}^q + J_s^q J_{r,u}^t) - J_s^q J_r^t R_{pt,u} + \\ & - 2R_{pt}(J_q^t J_{s,u}^r + J_s^r J_{q,u}^t) - 2J_s^r J_q^t R_{pt,u}] + \\ & + \frac{S_u}{4(n+1)(n+2)} [\delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + J_s^p J_r^q - J_r^p J_s^q - 2J_q^p J_s^r] \\ & + \frac{S}{4(n+1)(n+2)} [J_r^q J_{s,u}^p + J_s^p J_{r,u}^q - J_s^q J_{r,u}^p - J_r^p J_{s,u}^q - 2J_s^r J_{q,u}^p - 2J_q^p J_{s,u}^r]. \end{aligned}$$

Before we state and prove the main result of this section, we need the following

**Theorem 3.2.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. Then, the Ricci tensor  $\overline{\text{Ric}}$  of  $Z$  is complex linear, i.e.*

$$(3.6) \quad \overline{R}_{pt} J_q^t + \overline{R}_{qt} J_p^t = 0 \text{ on } Z \text{ for every } p, q = 1, \dots, 6,$$

if and only if  $(M, g)$  is an Einstein, self-dual manifold.

*Proof.* If  $(M, g)$  is an Einstein, self-dual manifold, the validity of (3.6) can be immediately shown by a direct inspection of the components listed in (B.2).

Thus, let us suppose that (3.6) holds on  $Z$ . First, note that

$$\begin{aligned} 0 &= \overline{R}_{1t} J_1^t + \overline{R}_{1t} J_1^t = 2\overline{R}_{1t} J_1^t = \overline{R}_{12} \\ 0 &= \overline{R}_{3t} J_3^t + \overline{R}_{3t} J_3^t = 2\overline{R}_{3t} J_3^t = \overline{R}_{34} \end{aligned}$$

on  $Z$ ; by (B.2), we can write

$$\begin{aligned} R_{12} &= \frac{t^2}{2} (\mathbb{Q}_{1c} \mathbb{Q}_{2c} + \mathbb{Q}_{1c} \mathbb{Q}_{2c}) \\ R_{34} &= \frac{t^2}{2} (\mathbb{Q}_{3c} \mathbb{Q}_{4c} + \mathbb{Q}_{3c} \mathbb{Q}_{4c}). \end{aligned}$$

Subtracting the first equation from the second, we obtain

$$B_{32} = t^2 (B_{22} B_{32} + B_{23} B_{33});$$

note that this is a global condition on the entries of the matrix  $B$ , which means that it holds for every choice of local orthonormal frame. By (2.1), we can choose

$$a_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a_+ = I_3,$$

where  $I_3$  is the  $3 \times 3$  identity matrix, to compute

$$\tilde{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ -B_{31} & -B_{32} & -B_{33} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

and to obtain

$$B_{22} = \tilde{B}_{32} = t^2(\tilde{B}_{22}\tilde{B}_{32} + \tilde{B}_{23}\tilde{B}_{33}) = -t^2(B_{32}B_{22} + B_{33}B_{23}) = -B_{32}.$$

Now, choosing

$$a_- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_+ = I_3,$$

we get

$$\tilde{B} = \begin{pmatrix} -B_{11} & -B_{12} & -B_{13} \\ -B_{21} & -B_{22} & -B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

and

$$B_{32} = \tilde{B}_{32} = t^2(\tilde{B}_{22}\tilde{B}_{32} + \tilde{B}_{23}\tilde{B}_{33}) = -t^2(B_{32}B_{22} + B_{33}B_{23}) = -B_{32};$$

therefore, we conclude that  $B_{22} = B_{32} = 0$ .

Now, we can choose the change of frames determined by

$$a_- = I_3, \quad a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad a_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

to compute

$$B_{33} = t^2 B_{23} B_{33} = -B_{23},$$

which obviously implies that  $B_{23} = B_{33} = 0$ . By an analogous computations, we can also obtain  $B_{21} = B_{31} = 0$ . Finally, choosing

$$a_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_+ = I_3$$

and repeating the argument above, we conclude that  $B_{11} = B_{12} = B_{13} = 0$ . Thus, for every  $p \in M$  there exists a local frame such that  $B = 0$ , i.e.  $(M, g)$  is an Einstein manifold.

Now, by (3.6), we obtain

$$0 = \bar{R}_{5t}J_6^t + \bar{R}_{6t}J_5^t \implies \bar{R}_{55} = \bar{R}_{66} \implies |\mathcal{Q}_{ab}|^2 = |\mathcal{Q}_{ab}|^2;$$

since  $(M, g)$  is an Einstein manifold, this equation can be rewritten as

$$(A_{12})^2 + (A_{22})^2 = (A_{13})^2 + (A_{33})^2,$$

which is another global condition and, therefore, does not depend on the choice of the local frame. In particular, for every frame with respect to which  $A$  is diagonal, we have that  $(A_{22})^2 = (A_{33})^2$  and, since we can exchange the diagonal entries of  $A$ , we can conclude

$$(A_{11})^2 = (A_{22})^2 = (A_{33})^2.$$

If  $A_{11} = A_{22} = A_{33}$ , then this holds for every local orthonormal frame by (2.1), hence  $(M, g)$  is self-dual and the claim is proven.

Thus, without loss of generality, we may suppose that, for instance,  $A_{22} \neq A_{33}$  at a point  $p \in M$  for some local frame  $e \in O(M)_-$  with respect to which  $A$  is diagonal. By the equation above, it is obvious that the diagonal entries satisfy

$$A_{11} = A_{22} = -A_{33} = x \neq 0;$$

if we choose the change of frames determined by

$$a_+ = \begin{pmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

we obtain

$$\tilde{A} = \begin{pmatrix} 0 & 0 & -\frac{x}{2} \\ 0 & x & 0 \\ -\frac{x}{2} & 0 & 0 \end{pmatrix}$$

and

$$(\tilde{A}_{12})^2 + (\tilde{A}_{22})^2 = (\tilde{A}_{13})^2 + (\tilde{A}_{33})^2 \implies x^2 = \frac{x^2}{4},$$

which is true if and only if  $x = 0$  and leads to a contradiction. Therefore, diagonalizing  $A$  we obtain a scalar matrix on  $M$ , which is equivalent to say that  $(M, g)$  is a self-dual manifold.  $\square$

We are now ready to state the following

**Theorem 3.3.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. Then,  $(Z, g_t, J)$  is Bochner-parallel if and only if  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$  with its canonical metric. In particular, the only Bochner-parallel twistor space is  $\mathbb{C}\mathbb{P}^3$  endowed with the Fubini-Study metric.*

An immediate consequence of Theorem 3.3 is the following

**Corollary 3.4.**  *$(Z, g_t, J)$  is Bochner-flat if and only if  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$  with its canonical metric.*

*Proof of Theorem 3.3.* First, note that, in order to prove the claim, it is sufficient to show that, if  $(Z, g_t, J)$  is Bochner-parallel, then it is a Kähler-Einstein manifold, i.e.  $(M, g)$  is an Einstein, self-dual manifold with scalar curvature  $S$  equal to  $12/t^2$ : indeed, by (3.3), it is immediate to show that, if  $(Z, g_t, J)$  is a Kähler-Einstein manifold, then it is Bochner-parallel if and only if it is Bochner-flat (it is sufficient to check the components  $\overline{B}_{pqr5,t}$ ); therefore, this ends the proof by Proposition 3.1.

Let us consider the local expression of  $\nabla B$  in (3.5): the right-hand side satisfies the same symmetries as the components of  $\nabla \text{Riem}$ . Thus, recalling that the Riemann tensor is skew-symmetric with respect to the last two indices, we obtain

$$0 = \overline{R}_{pqrs,u} + \overline{R}_{qprs,u} = \frac{1}{5} [J_{s,u}^r (\overline{R}_{pt} J_q^t + \overline{R}_{qt} J_p^t) + J_s^r (\overline{R}_{pt} J_{q,u}^t + \overline{R}_{qt} J_{p,u}^t + J_q^t \overline{R}_{pt,u} + J_p^t \overline{R}_{qt,u})]$$

for every  $p, q = 1, \dots, 6$ .

If we consider a pair of indices  $(r, s)$  such that  $J_s^r = 0$ , the equation becomes

$$J_{s,u}^r (\overline{R}_{pt} J_q^t + \overline{R}_{qt} J_p^t) = 0.$$

If  $J_{s,u}^r \neq 0$  for some  $r, s, u$  with respect to some local frame  $e \in O(M)$ , we conclude that  $\overline{R}_{pt} J_q^t + \overline{R}_{qt} J_p^t = 0$  for every  $p, q$ . On the other hand, if  $e'$  is a frame with respect to which  $J_{s,u}^r = 0$  for every  $r, s, u$ , we obtain

$$\begin{aligned} \mathcal{Q}_{13} = \mathcal{Q}_{42} = \mathcal{Q}_{14} = \mathcal{Q}_{23} &= \frac{1}{t^2} \\ \mathcal{Q}_{12} = \mathcal{Q}_{34} = \mathcal{Q}_{12} = \mathcal{Q}_{34} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{13} = \mathcal{Q}_{42} &= 0, \end{aligned}$$

which obviously imply that

$$\overline{R}_{12} = \overline{R}_{34} = 0 \text{ and } \overline{R}_{55} = \overline{R}_{66}$$

for the chosen frame. Thus, these equations hold on  $Z$  for every choice of frame  $e \in O(M)$ : therefore, we can repeat the argument exploited in the proof of Theorem 3.2 to conclude that  $(M, g)$  is an Einstein, self-dual manifold.

Now, by (3.5), (B.1) and the local expression of  $\overline{\nabla \text{Riem}}$ , it is easy to compute, for instance,

$$-\frac{2}{5} \overline{R}_{16,4} = \overline{R}_{1556,4} = \frac{t^3 S}{6912} \left( S^2 - \frac{36S}{t^2} + \frac{288}{t^4} \right);$$

since by (B.7)

$$\overline{R}_{16,4} = -\frac{t^3 S}{1728} \left( S^2 - \frac{18S}{t^2} + \frac{72}{t^4} \right),$$

we have the equality

$$\frac{S}{6912} \left( S - \frac{12}{t^2} \right) \left( S - \frac{24}{t^2} \right) = \frac{S}{4320} \left( S - \frac{12}{t^2} \right) \left( S - \frac{6}{t^2} \right)$$

and it follows immediately that this holds if and only if  $S \in \{0, 12/t^2\}$ .

If we suppose  $S = 0$ , by (B.1) and (3.5) it is easy to compute

$$0 = \overline{R}_{1334,6} = \frac{\overline{S}}{40} J_{3,6}^1 = \frac{1}{20t^2} J_{3,6}^1$$

which is impossible, since

$$J_{3,6}^1 = \frac{1}{2}t \left[ \frac{2}{t^2} - (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right] = \frac{1}{t} \neq 0.$$

Thus, by Corollary 4.2 we conclude that  $(Z, g_t, J)$  is a Kähler-Einstein manifold and this ends the proof.  $\square$

**Remark 3.5.** (1) It is worth to note that Theorem 3.2 is a generalization of a result due to Davidov, Grantcharov and Muškarov, who showed that the Riemann tensor  $\overline{\text{Riem}}$  of  $(Z, g_t, J)$  satisfies

$$(3.7) \quad \overline{R}_{pqrs} = \overline{R}_{tuvw} J_p^t J_q^u J_r^v J_s^w, \quad \forall 1 \leq p, q, r, s \leq 6$$

if and only if  $(M, g)$  is an Einstein, self-dual manifold (see [12]). Indeed, a straightforward computation shows that (3.7) implies (3.6). Almost Hermitian manifolds which satisfy (3.7) are sometimes called *RK-manifolds* see ([31] and [33]) and (3.7) is a condition satisfied by every nearly Kähler manifold (see [17]).

(2) We point out that one can directly prove Corollary 3.4 without exploiting Theorem 3.3; indeed, if we suppose that  $(Z, g_t, J)$  is a Bochner-flat manifold, by (3.2) we can show that  $\overline{\text{Riem}}$  is a *K-curvature-like tensor*, i.e. its components satisfy

$$(3.8) \quad \overline{R}_{pqts} J_r^t + \overline{R}_{pqrt} J_s^t = 0, \quad \forall 1 \leq p, q, r, s \leq 6.$$

The equality in (3.8) is sometimes referred to as *Kähler identity* and it is a deeply studied feature of almost Hermitian manifolds, aside from the twistor spaces context (we may refer the reader to [17], [26], [27], [32] and [33]).

By another result due to Davidov, Grantcharov and Muškarov [12],  $\overline{\text{Riem}}$  satisfies (3.8) if and only if  $(M, g)$  is an Einstein, self-dual manifold with  $S \in \{0, 12/t^2\}$ . If  $S = 0$ , by (3.1), (B.1), (B.2) and (B.3), we obtain

$$0 = \overline{B}_{5656} = \frac{S}{20} + \frac{3}{10t^2} - \frac{9t^2}{40} |\mathcal{Q}_{ab}|^2 = \frac{3}{10t^2},$$

which is a contradiction. Thus,  $(Z, g_t, J)$  is Kähler-Einstein and, by Proposition 3.1, the claim is proved. For detailed dissertations about Kähler, Bochner-flat manifolds, see, for instance, [4], [6], [7] and [8].

#### 4. RICCI PARALLEL AND LOCALLY SYMMETRIC TWISTOR SPACES

In this section, we discuss the case of a Riemannian four-manifold  $(M, g)$  whose twistor space  $(Z, g_t, J)$  is a Kähler-Einstein manifold.

Let us start proving a well-known result due to Friedrich and Grunewald (see [15]):

**Theorem 4.1.** *Let  $(M, g)$  a Riemannian four-manifold and  $(Z, g_t)$  be its twistor space. Then  $(Z, g_t)$  is Einstein if and only if  $(M, g)$  is Einstein, self-dual with scalar curvature  $S$  equal to  $6/t^2$  or to  $12/t^2$ .*

*Proof.* Let us suppose that  $M$  is Einstein, self-dual with  $S \in \{\frac{6}{t^2}, \frac{12}{t^2}\}$ . We obtain immediately that  $\bar{R}_{a5} = \bar{R}_{a6} = 0$ : indeed, since  $M$  is Einstein, then  $(M, g)$  has a harmonic curvature metric, i.e.  $\operatorname{div} \operatorname{Riem} = 0$ , which means that, in particular,  $\mathcal{Q}_{ac,c} = \mathcal{Q}_{ac,c} = 0$  for every  $a = 1, \dots, 4$ . We have that  $M$  is Einstein and self-dual if and only if

$$\begin{cases} \mathcal{Q}_{13} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{42} = 0; \\ \mathcal{Q}_{12} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{34} = 0; \\ \mathcal{Q}_{12} = \mathcal{Q}_{13} = \mathcal{Q}_{42} = \mathcal{Q}_{34} = 0; \\ \mathcal{Q}_{13} = \mathcal{Q}_{42} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{12} = \mathcal{Q}_{34} = \frac{S}{12}. \end{cases}$$

Therefore, for instance we have that

$$\bar{R}_{12} = -\frac{1}{2}t^2 \sum_{c=1}^4 (\mathcal{Q}_{1c} \mathcal{Q}_{2c} + \mathcal{Q}_{1c} \mathcal{Q}_{2c}) = 0;$$

with similar computations, we obtain that  $\bar{R}_{ab} = 0$  for every  $a \neq b$ . Obviously, the system also implies that  $\bar{R}_{56} = 0$ . Now, let us consider

$$\begin{aligned} \bar{R}_{11} &= \frac{S}{4} - \frac{1}{2}t^2 \sum_{c=1}^4 (\mathcal{Q}_{1c} \mathcal{Q}_{1c} + \mathcal{Q}_{1c} \mathcal{Q}_{1c}) = \\ &= \frac{S}{4} - \frac{1}{2}t^2 ((\mathcal{Q}_{13})^2 + (\mathcal{Q}_{14})^2) = \frac{S}{4} - t^2 \frac{S^2}{144}; \end{aligned}$$

we have that

$$\bar{S} = S + \frac{2}{t^2} - t^2 \frac{S^2}{72}$$

Thus, we have

$$\bar{R}_{11} = \frac{\bar{S}}{6} \Leftrightarrow t^2 S^2 - 18S + \frac{72}{t^2} = 0 \Leftrightarrow S \in \left\{ \frac{6}{t^2}, \frac{12}{t^2} \right\},$$

i.e.  $\bar{R}_{11} = \bar{S}/6$  by hypothesis. With analogous computations, we conclude that  $\bar{R}_{aa} = \bar{S}/6$  for every  $a$ . Finally,

$$\bar{R}_{55} = \bar{R}_{66} = \frac{1}{t^2} + t^2 \frac{S^2}{144}.$$

Again, the right-hand side is equal to  $\bar{S}/6$  if and only if  $S$  is equal to  $6/t^2$  or to  $12/t^2$ , which implies that  $Z$  is Einstein.

Conversely, let us suppose that  $(Z, g_t)$  is an Einstein manifold, i.e.  $\bar{R}_{pq} = (\bar{S}/6)\delta_{pq}$ . Recalling the expressions of  $\bar{\operatorname{Riem}}$ ,  $\bar{\operatorname{Ric}}$  and  $\bar{S}$  listed in (B.1), (B.2) and (B.3), respectively, we

easily obtain

$$\begin{aligned}\frac{S}{6} + \frac{1}{3t^2} - \frac{t^2}{24} \left( |\mathbb{Q}_{ab}|^2 + |\mathbb{Q}_{ab}|^2 \right) &= \frac{\bar{S}}{6} = \bar{R}_{55} = \frac{1}{t^2} + \frac{t^2}{4} |\mathbb{Q}_{ab}|^2; \\ \frac{S}{6} + \frac{1}{3t^2} - \frac{t^2}{24} \left( |\mathbb{Q}_{ab}|^2 + |\mathbb{Q}_{ab}|^2 \right) &= \frac{\bar{S}}{6} = \bar{R}_{66} = \frac{1}{t^2} + \frac{t^2}{4} |\mathbb{Q}_{ab}|^2.\end{aligned}$$

This implies immediately that

$$\begin{aligned}\frac{7t^2}{24} |\mathbb{Q}_{ab}|^2 &= \frac{S}{6} - \frac{2}{3t^2} - \frac{t^2}{24} |\mathbb{Q}_{ab}|^2; \\ \frac{7t^2}{24} |\mathbb{Q}_{ab}|^2 &= \frac{S}{6} - \frac{2}{3t^2} - \frac{t^2}{24} |\mathbb{Q}_{ab}|^2.\end{aligned}$$

Subtracting the two equations, it is easy to show that

$$|\mathbb{Q}_{ab}|^2 = |\mathbb{Q}_{ab}|^2 = \frac{S}{2t^2} - \frac{2}{t^4}.$$

Now, by the expression of the components in (B.2), we have that

$$R_{ab} - \frac{t^2}{2} \sum_{c=1}^4 (\mathbb{Q}_{ac} \mathbb{Q}_{bc} + \mathbb{Q}_{ac} \mathbb{Q}_{bc}) = \frac{\bar{S}}{6} \delta_{ab} = \left( \frac{S}{8} + \frac{1}{2t^2} \right) \delta_{ab};$$

a straightforward computation shows that

$$R_{11} + R_{22} - R_{33} - R_{44} = t^2 [(\mathbb{Q}_{12})^2 + (\mathbb{Q}_{12})^2 - (\mathbb{Q}_{34})^2 - (\mathbb{Q}_{34})^2],$$

that is,

$$B_{11} = t^2 (A_{12} B_{12} + A_{13} B_{13}),$$

for every local orthonormal frame  $e \in O(M)_-$ . Thus, we can choose a frame  $e$  such that the associated matrix  $A$  is diagonal, in order to obtain  $B_{11} = 0$  (note that this is not a global condition). By (2.1), if we use choose

$$a_+ = I_3, \quad a_- = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we easily obtain that

$$\tilde{B} = \begin{pmatrix} B_{21} & B_{22} & B_{23} \\ 0 & -B_{12} & -B_{13} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

which leads to

$$B_{21} = \tilde{B}_{11} = t^2 (\tilde{A}_{12} \tilde{B}_{12} + \tilde{A}_{13} \tilde{B}_{13}) = 0$$

(note that also  $\tilde{A}$  is a diagonal matrix). By analogous computations, we can conclude that  $B = 0$ , i.e.  $(M, g)$  is an Einstein manifold.

Now, by hypothesis we have that

$$0 = \tilde{R}_{56} = \frac{t^2}{4} \sum_{a,c=1}^4 \mathbb{Q}_{ac} \mathbb{Q}_{ac};$$

since  $(M, g)$  is Einstein, this equation assumes the form

$$A_{12}A_{13} + A_{23}(A_{22} + A_{33}) = 0.$$

Let us again choose  $e \in O(M)_-$  such that  $A$  is diagonal. Choosing suitable matrices  $a_+ \in SO(3)$ , it is easy to obtain that

$$A_{11} = A_{22} \vee A_{11} = -A_{22}.$$

If we suppose that  $A_{11} \neq A_{22}$ , analogous computations immediately lead to a contradiction. Thus,  $A_{11} = A_{22}$ . Exchanging  $A_{22}$  and  $A_{33}$ , we can repeat the same argument to obtain  $A_{11} = A_{33}$ . Thus,  $(M, g)$  is self-dual.

Finally, since  $(M, g)$  is Einstein and self-dual, we compute

$$\frac{S^2}{36} = |\mathbb{Q}_{ab}|^2 = \frac{S}{2t^2} - \frac{2}{t^4},$$

that is,

$$t^4 S^2 - 18t^2 S + 72 = 0.$$

Again, this equation holds if and only if

$$S \in \left\{ \frac{6}{t^2}, \frac{12}{t^2} \right\}$$

and this ends the proof.  $\square$

As a consequence, we can state another well known result (see [23] and [5]):

**Corollary 4.2.** *The twistor space  $(Z, g_t, J)$  associated to  $(M, g)$  is a Kähler-Einstein manifold if and only if  $(M, g)$  is an Einstein, self-dual manifold with scalar curvature equal to  $12/t^2$ .*

*Proof.* If  $(M, g)$  is an Einstein, self-dual manifold with  $S = 12/t^2$ , a direct inspection of the components in (B.2) and (C.3) shows that  $(Z, g_t)$  is Kähler-Einstein. Conversely, by Theorem 4.1 we know that  $(M, g)$  is Einstein, self-dual with  $S$  equal to  $6/t^2$  or  $12/t^2$ . If we suppose  $S = 6/t^2$ , we immediately obtain that

$$J_{3,6}^1 = -J_{4,5}^1 = \frac{1}{2t} \neq 0,$$

which contradicts the hypothesis that  $(Z, g_t, J)$  is a Kähler manifold and ends the proof.  $\square$

We recall that compact, Einstein, self-dual four-manifolds with positive scalar curvature have been classified: indeed, Hitchin showed that there are just two possibilities, up to conformal equivalences, which are  $S^4$  and  $\mathbb{C}\mathbb{P}^2$  with their canonical metrics [20].

Now, we want to provide an analogue of Proposition 3.1, in order to give another characterization of  $\mathbb{S}^4$ . Indeed, by direct inspection of the components listed in (B.6), one can immediately show the validity of

**Theorem 4.3.** *Let  $(Z, g_t, J)$  be a Kähler-Einstein twistor space. Then,  $(Z, g_t, J)$  is locally symmetric if and only if  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$  with its canonical metric.*

Note that, combining Theorem 4.3 with Theorem 3.3 and Corollary 2.2, we can state the following

**Theorem 4.4.** *Let  $(Z, g_t, J)$  be a Kähler-Einstein twistor space. Then the following conditions are equivalent:*

- (1)  $(Z, g_t, J)$  is a conformally symmetric manifold;
- (2)  $(Z, g_t, J)$  is a Bochner-parallel manifold;
- (3)  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$ , with its canonical metric.

Recall that the equivalence (2)  $\Leftrightarrow$  (3) holds without any *a priori* hypothesis on  $(Z, g_t, J)$ , by Theorem 3.3.

Moreover, the Einstein condition on  $g_t$  implies that  $(Z, g_t, J)$  is a harmonic curvature manifold, i.e.  $\operatorname{div} \overline{\operatorname{Riem}} \equiv 0$ . Then, by this consideration, equation (3.4) and Theorem 4.4 we can state the following

**Theorem 4.5.** *Let  $(Z, g_t, J)$  be a Kähler-Einstein twistor space. Then, either  $(Z, g_t, J)$  is Bochner-parallel (and, then,  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$ ) or  $\nabla \overline{\mathbf{B}} \neq 0$  and  $\operatorname{div} \overline{\mathbf{B}} \equiv 0$ .*

Though in Theorem 3.3 we characterized  $\mathbb{S}^4$  as the only four-dimensional manifold whose twistor space is Bochner-parallel, we cannot obtain an analogous characterization if we drop the hypothesis of  $g_t$  being Kähler-Einstein in Theorem 4.3. For instance, by (B.1), an easy computation of the local expression of  $\nabla \overline{\operatorname{Riem}}$  proves the following

**Proposition 4.6.** *Let  $(M, g)$  be a Ricci-flat, self-dual, locally symmetric four-manifold. Then, its twistor space  $(Z, g_t)$  is locally symmetric.*

Even though  $\mathbb{S}^4$  is not the only four-manifold whose twistor space satisfies  $\nabla \overline{\operatorname{Riem}} \equiv 0$ , if we consider Hermitian twistor spaces, i.e. the ones associated to self-dual manifolds (see [1] and [5]), we can state the following

**Theorem 4.7.** *Let  $(M, g)$  be a self-dual Riemannian manifold and let  $(Z, g_t)$  be its twistor space. Then,*

- (1)  $(Z, g_t)$  is Ricci parallel (i.e.,  $\nabla \overline{\operatorname{Ric}} \equiv 0$ ) if and only if  $(Z, g_t)$  is an Einstein manifold or  $(M, g)$  is Ricci-flat;
- (2)  $(Z, g_t)$  is locally symmetric if and only if  $(M, g)$  is homotetically isometric to  $\mathbb{S}^4$  with its canonical metric or  $(M, g)$  is Ricci-flat and locally symmetric.

*Proof.* (1) First, let us suppose that  $(M, g)$  is an Einstein, self-dual manifold with scalar curvature  $S$ . In particular,  $(M, g)$  is Ricci parallel. Moreover, since under these hypotheses

$$\mathcal{Q}_{de}\mathcal{Q}_{de} = \mathcal{Q}_{de}\mathcal{Q}_{de} = \mathcal{Q}_{de}\mathcal{Q}_{de} = \mathcal{Q}_{ab,c} = \mathcal{Q}_{ab,c} = \mathcal{Q}_{ab,c} = 0$$

for every  $a, b, c$ , we immediately obtain that the only components listed in (B.7) that may not vanish are  $\bar{R}_{ab,5}, \bar{R}_{ab,6}, \bar{R}_{a5,b}, \bar{R}_{a6,b}$  for some  $a, b$ . A straightforward computation shows that  $\bar{R}_{ab,5} = \bar{R}_{ab,6} = 0$ , regardless of the value of  $S$ .

Let us consider, for instance,

$$\bar{R}_{15,b} = -\frac{t^3 S}{1728} \left( S^2 - \frac{18S}{t^2} + \frac{72}{t^4} \right) \delta_{3b} = -\frac{t^3 S}{1728} \left( S - \frac{12}{t^2} \right) \left( S - \frac{6}{t^2} \right) \delta_{3b}.$$

Similar computations show that the other components of  $\nabla \bar{\text{Ric}}$  vanish for the same values of  $S$ . Thus, by Theorem 4.1, we conclude that, if  $(M, g)$  is an Einstein, self-dual manifold,  $(Z, g_t)$  is Ricci parallel if and only if it is an Einstein manifold or the scalar curvature of  $(M, g)$  vanishes.

Thus, in order to prove the statement, it is sufficient to show that any self-dual manifold whose twistor space is Ricci parallel must also be an Einstein manifold. Under these hypotheses, we can observe that

$$0 = \bar{R}_{55,6} = \bar{R}_{66,5} \implies 0 = \mathcal{Q}_{ab}\mathcal{Q}_{ab} = \mathcal{Q}_{ab}\mathcal{Q}_{ab};$$

by the self-duality condition, these equations become

$$\begin{cases} B_{13}B_{11} + B_{31}B_{33} + B_{21}B_{23} = 0 \\ B_{12}B_{11} + B_{21}B_{22} + B_{31}B_{32} = 0 \end{cases}.$$

Since the system holds for every  $e \in O(M)_-$ , by a suitable change of frames we obtain that

$$B_{12}B_{13} + B_{22}B_{23} + B_{32}B_{33} = 0.$$

The validity of these three equations is equivalent to the orthogonality of the columns of the matrix  $B$ ; this means that  $B^T B = D$ , where  $D$  is a diagonal matrix. Moreover, let us denote the  $i$ -th column of  $B$  as  $v_i$  and let us define

$$\|v_i\|^2 := \sum_{j=1}^3 B_{ji}B_{ji};$$

furthermore, let us suppose that  $v_i \neq 0$  for every  $i$ . Since the columns of  $B$  are the rows of  $B^T$ , if we replace the rows of  $B^T$  with  $v_i^T / \|v_i\|^2$ , we obtain that the new matrix, which we denote as  $B_{or}$ , is orthogonal. In particular, we may assume that  $B_{or} \in SO(3)$  (otherwise, we could replace it with  $-B_{or}$ ); thus, putting  $a_-^{-1} = B_{or}$  and  $a_+ = I_3$  in (2.1), we have that

$$a_-^{-1} B a_+ = B_{or} B = I_3.$$

By the expressions of the entries of the matrices  $A$  and  $B$ , it is easy to obtain

$$\begin{aligned}\bar{R}_{13,5} &= 2t\left(\frac{S}{3} + 2\right) - \frac{t^3 S}{12} = 0; \\ \bar{R}_{24,5} &= \frac{tS}{6} - \frac{t^3 S}{12}\left(\frac{S}{6} - 1\right) = 0.\end{aligned}$$

The two equations can hold simultaneously if and only if  $t = \sqrt{3 + \sqrt{17}}$ , which implies that

$$S = \frac{3(1 + \sqrt{17})}{2}.$$

However, since  $S$  is invariant under change of frames, we can choose

$$a_- = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_+ = I_3$$

in (2.1) in order to obtain

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

now, it is easy to compute

$$\bar{R}_{12,5} = -t + \frac{t^3 S}{12} = 0,$$

which holds if and only if

$$S = \frac{12}{t^2} = \frac{3(\sqrt{17} - 3)}{2} \neq \frac{3(1 + \sqrt{17})}{2}.$$

Therefore, we get a contradiction and we conclude that at least one of the columns of  $B$  must be made of zeros for every  $e \in O(M)_-$ , which obviously implies that  $(M, g)$  is an Einstein manifold.

- (2) We know that  $\mathbb{S}^4$  and every locally symmetric, self-dual, Ricci-flat manifold have locally symmetric twistor spaces, by Proposition 4.6. Conversely, let us suppose that  $(Z, g_t)$  is locally symmetric: in particular, it is Ricci parallel, therefore  $(Z, g_t)$  is an Einstein manifold or  $(M, g)$  is Ricci-flat.

If  $S \neq 0$ , by (B.1), it is easy to compute

$$\begin{aligned}\bar{R}_{5653,u}\theta^u &= \frac{t}{2} \left\{ \left[ -\frac{3}{2} \mathcal{Q}_{34} + \frac{t^2}{2} \mathcal{Q}_{23} \mathcal{Q}_{42} + \frac{t^2}{4} \mathcal{Q}_{13} \mathcal{Q}_{14} \right] \mathcal{Q}_{42} - \frac{t^3}{8} (\mathcal{Q}_{13})^2 \mathcal{Q}_{23} + \frac{1}{2t} \mathcal{Q}_{23} \right\} \theta^2 = \\ &= \frac{t^3 S}{6912} \left( S^2 - \frac{36S}{t^2} + \frac{288}{t^4} \right) = \frac{t^3 S}{6912} \left( S - \frac{12}{t^2} \right) \left( S - \frac{24}{t^2} \right) = 0,\end{aligned}$$

which holds if and only if  $S \in \{12/t^2, 24/t^2\}$ . Since  $(Z, g_t)$  must be an Einstein manifold, this implies that  $S \in \{6/t^2, 12/t^2\}$ : therefore,  $S = 12/t^2$  and, by Theorem 4.3, we conclude that  $(M, g)$  is a spherical space form.

Finally, let us suppose that  $S = 0$ , i.e.  $(M, g)$  is Ricci-flat. Since

$$\overline{R}_{abcd,e} = R_{abcd,e}, \text{ for every } a, b, c, d, e = 1, \dots, 4,$$

it is apparent that  $(M, g)$  is locally symmetric by hypothesis and this ends the proof.  $\square$

## 5. A QUADRATIC FORMULA FOR $\nabla J$ AND HIGHER ORDER CONDITIONS

**5.1. General quadratic formula for the square norm of  $\nabla J$ .** We begin stating a general result:

**Theorem 5.1.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. Then, the equality*

$$(5.1) \quad |\nabla J|^2 = \frac{1}{3}|d\omega|^2 + \frac{1}{8}|N_J|^2$$

holds, where  $|d\omega|^2 = \sum_{p,q,t=1}^6 d\omega(e_p, e_q, e_t)d\omega(e_p, e_q, e_t)$  and  $|N_J|^2 = \sum_{p,q,t=1}^6 N_{pq}^t N_{pq}^t$ .

*Proof.* By direct computation, we obtain

$$\begin{aligned} |d\omega|^2 = 6t^2 & \left[ \mathcal{Q}_{12}^2 + \mathcal{Q}_{12}^2 + \mathcal{Q}_{13}^2 + \left( \mathcal{Q}_{13} - \frac{1}{t^2} \right)^2 + \left( \mathcal{Q}_{14} - \frac{1}{t^2} \right)^2 + \right. \\ & + \mathcal{Q}_{14}^2 + \left( \mathcal{Q}_{23} - \frac{1}{t^2} \right)^2 + \mathcal{Q}_{23}^2 + \mathcal{Q}_{42}^2 + \left( \mathcal{Q}_{42} - \frac{1}{t^2} \right)^2 + \\ & \left. + \mathcal{Q}_{34}^2 + \mathcal{Q}_{34}^2 \right]; \end{aligned}$$

$$|N_J|^2 = 8t^2 [(\mathcal{Q}_{13} + \mathcal{Q}_{42} - \mathcal{Q}_{14} - \mathcal{Q}_{23})^2 + (\mathcal{Q}_{14} + \mathcal{Q}_{23} + \mathcal{Q}_{13} + \mathcal{Q}_{42})^2].$$

Comparing these expressions with (C.4), it is easy to obtain (5.1).  $\square$

It is worth to point out that Theorem 5.1 allows to give alternate proofs of some well-known results due to Muškarov (see [23]), exploiting the quadratic relations among the invariants listed by Gray and Hervella (see [18]). For instance, we can reformulate the following

**Proposition 5.2.** *Let  $(M, g)$  be a Riemannian manifold and  $(Z, g_t, J)$  be its twistor space. Then,*

$$(Z, g_t, J) \in \mathcal{NK} \cup \mathcal{AK} \Leftrightarrow (Z, g_t, J) \in \mathcal{K},$$

where  $\mathcal{NK}$ ,  $\mathcal{AK}$  and  $\mathcal{K}$  denote the classes of nearly Kähler, almost Kähler and Kähler manifolds, respectively.

*Proof.* One direction is trivial. Let us suppose  $Z \in \mathcal{NK}$ . Then, by table IV in [18], we know that

$$|\nabla J|^2 = \frac{1}{9}|d\omega|^2;$$

inserting this equation in (5.1), it is easy to obtain

$$|d\omega|^2 = |N_J|^2 = 0 \Rightarrow |\nabla J|^2 = 0,$$

that is,  $Z \in K$ . Now, let us suppose  $Z \in \mathcal{AK}$ . By the same table, we have that

$$|\nabla J|^2 = \frac{1}{4}|N_J|^2 \text{ and } |d\omega|^2 = 0;$$

again, inserting these equations in (5.1), we obtain  $|\nabla J|^2 = 0$ , i.e.  $Z \in \mathcal{K}$ .  $\square$

In fact, by analogous calculations, we can prove more. Let us consider the sixteen classes of almost Hermitian manifolds listed in [18]. Then, by Theorem 5.1, we can obtain an alternate proof of the following statement, which was first proven by Muškarov (see [23]):

**Theorem 5.3.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. If  $(Z, g_t, J)$  belongs to one of the first fifteen classes of almost Hermitian manifolds, then  $(Z, g_t, J) \in \mathcal{H}$ ; consequently,  $(M, g)$  is self-dual.*

**5.2. Laplacian of the almost complex structures.** In this section, we consider the Laplacian  $\Delta_J J$  of the almost complex structure  $J$  (for the definition and the components, see D). We say that  $J$  is *harmonic* if  $\Delta_J J = 0$  (see also [35] and [36]). By a direct inspection of the components listed in D, we can provide an alternate proof to a well-known result, due to Davidov and Muškarov (see [11]):

**Theorem 5.4.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. Then,  $J$  is harmonic if and only if  $(M, g)$  is self-dual.*

*Proof.* One direction is trivial; indeed, since  $(M, g)$  is self-dual,

$$\begin{aligned} \Delta_J J_3^1 &= 2A_{12} + \frac{1}{2}t[N_{14}^5 A_{13} - N_{13}^5 A_{12}] = 0; \\ \Delta_J J_4^1 &= 2A_{13} + \frac{1}{2}t[N_{14}^5 A_{12} - N_{13}^5 A_{13}] = 0, \end{aligned}$$

where the  $N_{pq}^r$  are the components of the Nijenhuis tensor of  $J$  (see (E.1)). Moreover, as an immediate consequence of the self-duality condition and the second Bianchi identity, by (A.1) and (D.2) it is easy to show that

$$\Delta_J J_5^1 = \Delta_J J_6^1 = \Delta_J J_5^3 = \Delta_J J_6^3 = 0.$$

Conversely, let us suppose  $\Delta_J J = 0$ . By the explicit expression of  $\Delta_J J_v^u$  and  $N_{pq}^r$  listed in (D.1) and in (E.1), respectively, we obtain the global equations

$$(5.2) \quad \begin{aligned} 2A_{12} - 2t^2 A_{23} A_{13} - t^2 A_{12} (A_{33} - A_{22}) &= 0; \\ 2A_{13} - 2t^2 A_{23} A_{12} - t^2 A_{13} (A_{33} - A_{22}) &= 0. \end{aligned}$$

Again, let us choose a local orthonormal frame  $e \in O(M)_-$  such that  $A$  is a diagonal matrix. By the transformation law for the matrix  $A$  defined in (2.1), choosing the matrix

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and using the equations (5.2), we obtain

$$(A_{22} - A_{11}) \left[ \frac{2}{t^2} - A_{33} + \frac{1}{2}(A_{11} + A_{22}) \right] = 0.$$

Let us suppose that  $A_{11} \neq A_{22}$ ; this implies immediately that

$$A_{33} = \frac{1}{2}(A_{11} + A_{22}) + \frac{2}{t^2}.$$

With similar computations, it is easy to show that, if this equation holds, by (5.2) we must have

$$A_{11} - A_{22} = \frac{4}{t^2} \vee A_{11} - A_{22} = -\frac{4}{3t^2}.$$

In both cases, choosing the matrix

$$a_+ = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and applying the transformation law for  $A$ , the equations (5.2) lead to

$$\frac{\sqrt{3}}{t^2} = 0 \vee \frac{5}{3\sqrt{3}t^2} = 0,$$

which clearly are contradictions. Thus,  $A_{11} = A_{22}$ ; by analogous computations, we easily obtain  $A_{22} = A_{33}$ . Therefore, we can conclude that  $(M, g)$  is self-dual.  $\square$

**5.3. Nijenhuis tensors of  $J$  and  $\mathbf{J}$ .** Now, let us consider the Nijenhuis tensors  $N_J$  and  $N_{\mathbf{J}}$  associated to  $J$  and  $\mathbf{J}$  respectively. The expression of the components  $J_{q,t}^p$  and  $\mathbf{J}_{q,t}^p$  of the covariant derivatives of  $J$  and  $\mathbf{J}$  are listed in (C.3) and (C.5), while the components of the tensors  $N_J$  and  $N_{\mathbf{J}}$  are listed in (E.1) and (E.6), together with the components of their covariant derivatives and their divergences in (E.3) and (E.7). Thus, let us recall the local expression of the divergences of  $N_J$  with respect to a local orthonormal coframe and its dual frame:

$$(5.3) \quad \operatorname{div} N_J = N_{pq,t}^t \theta^p \otimes \theta^q \quad \overline{\operatorname{div}} N_J = N_{pt,t}^r \theta^p \otimes e_r,$$

where  $\nabla N_J = N_{tq,s}^p \theta^s \otimes \theta^t \otimes \theta^q \otimes e_p$ . The following Theorem is a generalization of a result due to Atiyah, Hitchin and Singer, which characterizes self-dual four-manifolds as the ones whose twistor space is Hermitian with respect to  $J$  (see [1] and [5]):

**Theorem 5.5.** *Let  $(M, g)$  be a Riemannian four-manifold and  $(Z, g_t, J)$  be its twistor space. Then,  $(M, g)$  is self-dual if and only if  $\operatorname{div} N_J \equiv 0 \vee \overline{\operatorname{div}} N_J \equiv 0$ .*

*Proof.* One direction is trivial: indeed, we know that, if  $(M, g)$  is self-dual,  $N_J \equiv 0$ . Therefore, it is easy to see that  $\operatorname{div} N_J = \overline{\operatorname{div}} N_J \equiv 0$ .

Conversely, let us suppose that  $\operatorname{div} N_J \equiv 0$ . By (E.1) and (E.4), we have

$$\begin{cases} A_{13} \left( A_{33} - A_{22} - \frac{4}{t^2} \right) + 2A_{23}A_{12} = 0 \\ A_{12} \left( A_{33} - A_{22} + \frac{4}{t^2} \right) - 2A_{23}A_{13} = 0 \end{cases}.$$

Let us choose a frame  $e \in O(M)_-$  such that  $A$  is diagonal. By (2.1), choosing

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for the transformed matrix we obtain

$$(A_{22} - A_{11}) \left[ \frac{1}{2}(A_{11} + A_{22}) - \frac{4}{t^2} - A_{33} \right] = 0,$$

that is,

$$A_{11} = A_{22} \vee A_{33} = \frac{1}{2}(A_{11} + A_{22}) - \frac{4}{t^2}.$$

Let us suppose  $A_{11} \neq A_{22}$ . By choosing

$$a_+ = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and applying again (2.1), it is easy to see that  $A_{11} = A_{22}$ , which is a contradiction. Thus, with respect to  $e$  we have that  $A_{11} = A_{22}$ . By a similar argument, one can easily show that  $A_{11} = A_{33}$ , i.e.  $(M, g)$  is self-dual.

Now, let us consider the case  $\overline{\operatorname{div}} N_J \equiv 0$  and let us choose again a frame  $e \in O(M)_-$  such that  $A$  is diagonal. Then, since by hypothesis

$$N_{1t,t}^1 + N_{2t,t}^2 = 0,$$

we can rewrite this equation as

$$(A_{33} - A_{22})^2 = 0,$$

which means that  $A_{22} = A_{33}$ . The equation above holds for every frame with respect to which the matrix  $A$  is diagonal; thus, similarly we can obtain  $A_{11} = A_{22}$ , i.e.  $(M, g)$  is self-dual.  $\square$

As a consequence, it is immediate to show the validity of

**Corollary 5.6.**  *$(M, g)$  is self-dual if and only if  $\nabla N_J \equiv 0$ .*

We point out that the equation  $\nabla N_{\mathbf{J}} \equiv 0$  has been studied in the wider context of almost Hermitian manifolds: for instance, Vezzoni showed that any almost Kähler manifold that satisfies this condition is, in fact, a Kähler manifold (see [34]).

Note that, on the contrary, a simple inspection of the coefficients listed in (E.7) shows that  $\nabla N_{\mathbf{J}}$  never vanishes (the fact that  $\mathbf{J}$  is never integrable was first proven by Eells and Salamon [14] and it is apparent by (E.6); we also mention that, if  $(M, g)$  is Einstein and self-dual,  $N_{\mathbf{J}}$  is parallel with respect to the Chern connection  $\tilde{\nabla}$  defined on  $(Z, g_t, \mathbf{J})$ , as shown by Davidov, Grantcharov and Muškarov [9]); thus, we cannot obtain an analogue of Corollary 5.6 for  $N_{\mathbf{J}}$ . However, we can consider the divergences  $\operatorname{div} N_{\mathbf{J}}$  and  $\overline{\operatorname{div}} N_{\mathbf{J}}$  of  $N_{\mathbf{J}}$  (the components are listed in (E.8) and (E.9)) and state the following

**Theorem 5.7.** *Let  $(M, g)$  be a four-dimensional, self-dual Riemannian manifold. If  $M$  is Ricci-flat, then  $\overline{\operatorname{div}} N_{\mathbf{J}} = 0$ . If the scalar curvature  $S$  of  $(M, g)$  is different from  $6/t^2$ , then the converse holds.*

*Proof.* Suppose that  $(M, g)$  is Ricci-flat, i.e.  $\operatorname{Ric} = 0$ . In particular,  $(M, g)$  is Einstein, self-dual with  $S = 0$ . This implies that

$$\begin{aligned} \mathcal{Q}_{12} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = \mathcal{Q}_{34} = \mathcal{Q}_{12} = \mathcal{Q}_{13} = \mathcal{Q}_{42} = \mathcal{Q}_{34} = 0; \\ \mathcal{Q}_{13} = \mathcal{Q}_{42} = \mathcal{Q}_{14} = \mathcal{Q}_{23} = 0. \end{aligned}$$

In particular, by the second equation we obtain

$$\Sigma = 0 \Rightarrow (\Sigma)_{,a} = 0, \quad \forall a = 1, \dots, 4.$$

Therefore,  $\overline{\operatorname{div}} N_{\mathbf{J}} = 0$ , by direct inspection.

Now, let us suppose that  $\overline{\operatorname{div}} N_{\mathbf{J}} = 0$  and  $S \neq 6/t^2$ . It is easily shown that the hypothesis on the scalar curvature leads to

$$\Sigma - \frac{2}{t^2} \neq 0, \quad \text{on } O(M)_-.$$

Indeed, if  $\Sigma - \frac{2}{t^2} = 0$ , the matrix  $A$  appearing in the decomposition of the Riemann curvature operator has the form

$$A = \begin{pmatrix} \frac{S}{12} & 0 & 0 \\ 0 & \frac{S}{12} & 0 \\ 0 & 0 & \frac{1}{t^2} - \frac{S}{12} \end{pmatrix}$$

for every local orthonormal frame, since  $M$  is self-dual. Then, we must have

$$\frac{S}{12} = \frac{1}{t^2} - \frac{S}{12} \Leftrightarrow S = \frac{6}{t^2},$$

which is a contradiction. Thus, by hypothesis, we must have

$$\begin{aligned} \mathcal{Q}_{13} + \mathcal{Q}_{14} = \mathcal{Q}_{13} + \mathcal{Q}_{23} = \mathcal{Q}_{42} + \mathcal{Q}_{23} = \mathcal{Q}_{42} + \mathcal{Q}_{14} = \mathcal{Q}_{13} + \mathcal{Q}_{42} = \mathcal{Q}_{14} + \mathcal{Q}_{23} = 0 \\ \mathcal{Q}_{12} = \mathcal{Q}_{34} = \mathcal{Q}_{12} = \mathcal{Q}_{34} = 0. \end{aligned}$$

These equations immediately imply that  $(M, g)$  is Einstein with  $S = 0$ , i.e.  $M$  is Ricci-flat.  $\square$

## APPENDIX A. DIVERGENCE OF THE SELF-DUAL PART OF THE WEYL TENSOR

We list the components of the divergence of  $W^+$  in a Riemannian four-manifold.

$$\begin{aligned}
W_{abcd,e}^+ \omega^e &= dW_{abcd}^+ - W_{ebcd}^+ \omega_a^e - W_{aecd}^+ \omega_b^e - W_{abed}^+ \omega_c^e - W_{abce}^+ \omega_d^e; \\
(\delta W^+)_{abc} &= \sum_{e=1}^4 W_{cbae}^+ = \sum_{e=1}^4 W_{eabc}^+; \quad (\delta W^+)_{abc} = -(\delta W^+)_{acb}.
\end{aligned}
\tag{A.1}$$

$$\begin{aligned}
(\delta W^+)_{121} &= \frac{1}{4}(\mathcal{Q}_{12,2} + \mathcal{Q}_{13,3} + \mathcal{Q}_{14,4}) - \frac{1}{24}S_2; & (\delta W^+)_{312} &= \frac{1}{4}(\mathcal{Q}_{13,1} + \mathcal{Q}_{23,2} - \mathcal{Q}_{34,4}) + \frac{1}{24}S_4; \\
(\delta W^+)_{131} &= \frac{1}{4}(\mathcal{Q}_{12,2} + \mathcal{Q}_{13,3} + \mathcal{Q}_{14,4}) - \frac{1}{24}S_3; & (\delta W^+)_{313} &= \frac{1}{4}(\mathcal{Q}_{13,1} + \mathcal{Q}_{23,2} - \mathcal{Q}_{34,4}) - \frac{1}{24}S_1; \\
(\delta W^+)_{141} &= \frac{1}{4}(\mathcal{Q}_{12,2} + \mathcal{Q}_{13,3} + \mathcal{Q}_{14,4}) - \frac{1}{24}S_4; & (\delta W^+)_{314} &= \frac{1}{4}(\mathcal{Q}_{13,1} + \mathcal{Q}_{23,2} - \mathcal{Q}_{34,4}) - \frac{1}{24}S_2; \\
(\delta W^+)_{212} &= \frac{1}{4}(\mathcal{Q}_{12,1} - \mathcal{Q}_{23,3} + \mathcal{Q}_{42,4}) - \frac{1}{24}S_1; & (\delta W^+)_{412} &= \frac{1}{4}(\mathcal{Q}_{14,1} - \mathcal{Q}_{42,2} + \mathcal{Q}_{34,3}) - \frac{1}{24}S_3; \\
(\delta W^+)_{213} &= \frac{1}{4}(\mathcal{Q}_{12,1} - \mathcal{Q}_{23,3} + \mathcal{Q}_{42,4}) - \frac{1}{24}S_4; & (\delta W^+)_{413} &= \frac{1}{4}(\mathcal{Q}_{14,1} - \mathcal{Q}_{42,2} + \mathcal{Q}_{34,3}) + \frac{1}{24}S_2; \\
(\delta W^+)_{214} &= \frac{1}{4}(\mathcal{Q}_{12,1} - \mathcal{Q}_{23,3} + \mathcal{Q}_{42,4}) + \frac{1}{24}S_3; & (\delta W^+)_{414} &= \frac{1}{4}(\mathcal{Q}_{14,1} - \mathcal{Q}_{42,2} + \mathcal{Q}_{34,3}) - \frac{1}{24}S_1.
\end{aligned}$$

## APPENDIX B. RIEMANN CURVATURE OF A TWISTOR SPACE

We recall here all the components of the Riemann tensor, the Ricci tensor and the scalar curvature of the twistor space for a Riemannian four-manifold (see also [21]).

**Components of the Riemann curvature tensor**  $\overline{\text{Riem}}$  on  $(Z, g_t)$ :

$$\begin{aligned}
\text{(B.1)} \quad \overline{R}_{abcd} &= R_{abcd} - \frac{1}{4}t^2 [(\mathcal{Q}_{ac} \mathcal{Q}_{bd} - \mathcal{Q}_{ad} \mathcal{Q}_{bc}) \\
&\quad + (\mathcal{Q}_{ac} \mathcal{Q}_{bd} - \mathcal{Q}_{ad} \mathcal{Q}_{bc})] \\
&\quad - \frac{1}{2}t^2 (\mathcal{Q}_{ab} \mathcal{Q}_{cd} + \mathcal{Q}_{ab} \mathcal{Q}_{cd}); \\
\overline{R}_{ab56} &= \mathcal{Q}_{ab} - \frac{1}{4}t^2 \sum_{c=1}^4 (\mathcal{Q}_{ac} \mathcal{Q}_{bc} - \mathcal{Q}_{bc} \mathcal{Q}_{ac}); \\
\overline{R}_{abc5} &= -\frac{1}{2}t(\mathcal{Q}_{ab})_c; \quad \overline{R}_{abc6} = -\frac{1}{2}t(\mathcal{Q}_{ab})_c; \\
\overline{R}_{5656} &= \frac{1}{t^2}; \quad \overline{R}_{565b} = \overline{R}_{56a6} = 0; \\
\overline{R}_{5ab5} &= -\frac{1}{4}t^2 \sum_{c=1}^4 \mathcal{Q}_{ac} \mathcal{Q}_{bc}; \\
\overline{R}_{5ab6} &= -\frac{1}{2} \mathcal{Q}_{ab} - \frac{1}{4}t^2 \sum_{c=1}^4 \mathcal{Q}_{bc} \mathcal{Q}_{ac}; \\
\overline{R}_{6ab6} &= -\frac{1}{4}t^2 \sum_{c=1}^4 \mathcal{Q}_{ac} \mathcal{Q}_{bc}; \\
\overline{R}_{6ab5} &= \frac{1}{2} \mathcal{Q}_{ab} - \frac{1}{4}t^2 \sum_{c=1}^4 \mathcal{Q}_{bc} \mathcal{Q}_{ac}.
\end{aligned}$$

**Components of the Ricci tensor and  $\overline{\text{Ric}}$  and of the scalar curvature  $\overline{S}$ :**

$$\begin{aligned}
\text{(B.2)} \quad \overline{R}_{ab} &= R_{ab} - \frac{1}{2}t^2 \sum_{c=1}^4 (\mathcal{Q}_{ac} \mathcal{Q}_{bc} + \mathcal{Q}_{ac} \mathcal{Q}_{bc}); \\
\overline{R}_{a5} &= \frac{1}{2}t \sum_{c=1}^4 (\mathcal{Q}_{ac})_c; \\
\overline{R}_{a6} &= \frac{1}{2}t \sum_{c=1}^4 (\mathcal{Q}_{ac})_c;
\end{aligned}$$

$$\begin{aligned}\bar{R}_{55} &= \frac{1}{t^2} + \frac{t^2}{4} |\mathcal{Q}_{ab}|^2; \\ \bar{R}_{56} &= \frac{1}{4} t^2 \sum_{a,c=1}^4 \mathcal{Q}_{ac} \mathcal{Q}_{ac}; \\ \bar{R}_{66} &= \frac{1}{t^2} + \frac{t^2}{4} |\mathcal{Q}_{ab}|^2,\end{aligned}$$

$$(B.3) \quad \bar{S} = S + \frac{2}{t^2} - \frac{1}{4} t^2 (|\mathcal{Q}_{ab}|^2 + |\mathcal{Q}_{ab}|^2)$$

(here:  $|\mathcal{Q}_{ab}|^2 = \sum_{a,b=1}^4 (\mathcal{Q}_{ab})^2$ , and similarly for  $\mathcal{Q}_{ab}$ ).

### Components of the Weyl tensor $\bar{W}$ :

(B.4)

$$\begin{aligned}\bar{W}_{abcd} &= R_{abcd} - \frac{1}{4} t^2 [(\mathcal{Q}_{ac} \mathcal{Q}_{bd} - \mathcal{Q}_{ad} \mathcal{Q}_{bc}) + (\mathcal{Q}_{ac} \mathcal{Q}_{bd} - \mathcal{Q}_{ad} \mathcal{Q}_{bc})] - \frac{1}{2} t^2 (\mathcal{Q}_{ab} \mathcal{Q}_{cd} + \mathcal{Q}_{ab} \mathcal{Q}_{cd}) \\ &\quad + -\frac{1}{4} \left\{ \left[ R_{ac} - \frac{1}{2} t^2 (\mathcal{Q}_{ae} \mathcal{Q}_{ce} + \mathcal{Q}_{ae} \mathcal{Q}_{ce}) \right] \delta_{bd} - \left[ R_{bc} - \frac{1}{2} t^2 (\mathcal{Q}_{be} \mathcal{Q}_{ce} + \mathcal{Q}_{be} \mathcal{Q}_{ce}) \right] \delta_{ad} + \right. \\ &\quad \left. + \left[ R_{bd} - \frac{1}{2} t^2 (\mathcal{Q}_{be} \mathcal{Q}_{de} + \mathcal{Q}_{be} \mathcal{Q}_{de}) \right] \delta_{ac} - \left[ R_{ad} - \frac{1}{2} t^2 (\mathcal{Q}_{ae} \mathcal{Q}_{de} + \mathcal{Q}_{ae} \mathcal{Q}_{de}) \right] \delta_{bc} \right\} + \frac{\bar{S}}{20} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}); \\ \bar{W}_{abc5} &= -\frac{1}{2} t \left( \mathcal{Q}_{ab,c} + \frac{1}{4} \mathcal{Q}_{bd,d} \delta_{ac} - \frac{1}{4} \mathcal{Q}_{ad,d} \delta_{bc} \right); \\ \bar{W}_{abc6} &= -\frac{1}{2} t \left( \mathcal{Q}_{ab,c} + \frac{1}{4} \mathcal{Q}_{bd,d} \delta_{ac} - \frac{1}{4} \mathcal{Q}_{ad,d} \delta_{bc} \right); \\ \bar{W}_{ab56} &= \bar{R}_{ab56} = \mathcal{Q}_{ab} - \frac{1}{4} t^2 (\mathcal{Q}_{ac} \mathcal{Q}_{bc} - \mathcal{Q}_{bc} \mathcal{Q}_{ac}); \\ \bar{W}_{5ab6} &= -\frac{1}{2} \mathcal{Q}_{ab} - \frac{1}{4} t^2 \mathcal{Q}_{ac} \mathcal{Q}_{bc} + \frac{1}{16} t^2 \sum_{c,d=1}^4 \mathcal{Q}_{cd} \mathcal{Q}_{cd} \delta_{ab}; \\ \bar{W}_{5ab5} &= \frac{1}{4} R_{ab} - \frac{3}{8} t^2 \mathcal{Q}_{ac} \mathcal{Q}_{bc} - \frac{1}{8} t^2 \mathcal{Q}_{ac} \mathcal{Q}_{bc} + \left( \frac{3}{20t^2} + \frac{3}{40} t^2 |\mathcal{Q}_{cd}|^2 + \frac{1}{80} t^2 |\mathcal{Q}_{cd}|^2 - \frac{S}{20} \right) \delta_{ab}; \\ \bar{W}_{6ab6} &= \frac{1}{4} R_{ab} - \frac{3}{8} t^2 \mathcal{Q}_{ac} \mathcal{Q}_{bc} - \frac{1}{8} t^2 \mathcal{Q}_{ac} \mathcal{Q}_{bc} + \left( \frac{3}{20t^2} + \frac{3}{40} t^2 |\mathcal{Q}_{cd}|^2 + \frac{1}{80} t^2 |\mathcal{Q}_{cd}|^2 - \frac{S}{20} \right) \delta_{ab}; \\ \bar{W}_{56a5} &= \frac{1}{8} \mathcal{Q}_{ab,b}; \quad \bar{W}_{56a6} = -\frac{1}{8} \mathcal{Q}_{ab,b}; \\ \bar{W}_{5656} &= \frac{3}{5t^2} + \frac{S}{20} - \frac{3}{40} t^2 (|\mathcal{Q}_{ab}|^2 + |\mathcal{Q}_{ab}|^2).\end{aligned}$$

Let  $(Z, g_t, J)$  be a Kähler-Einstein manifold. By (B.1), the components of the Riemann tensor  $\bar{\text{Riem}}$  of  $Z$  are

(B.5)

$$\begin{aligned}\bar{R}_{5656} &= \frac{1}{t^2}, \quad \bar{R}_{56a6} = \bar{R}_{565b} = 0; \\ \bar{R}_{5ab5} &= \bar{R}_{6ab6} = -\frac{1}{4t^2} \delta_{ab}; \\ \bar{R}_{5126} &= \bar{R}_{5346} = -\bar{R}_{5216} = -\bar{R}_{5436} = -\frac{1}{4t^2}; \\ \bar{R}_{5ab6} &= 0 \quad \text{for } (a, b) \neq (1, 2), (3, 4); \\ \bar{R}_{abc5} &= \bar{R}_{abc6} = 0; \\ \bar{R}_{1256} &= \bar{R}_{3456} = \frac{1}{2t^2}, \\ \bar{R}_{ab56} &= 0 \quad \text{for } (a, b) \neq (1, 2), (3, 4);\end{aligned}$$

$$\begin{aligned}
\bar{R}_{1212} &= R_{1212}, & \bar{R}_{3434} &= R_{3434}; \\
\bar{R}_{1313} &= R_{1313} - \frac{3}{4t^2}, & \bar{R}_{4242} &= R_{4242} - \frac{3}{4t^2}; \\
\bar{R}_{1414} &= R_{1414} - \frac{3}{4t^2}, & \bar{R}_{2323} &= R_{2323} - \frac{3}{4t^2}; \\
\bar{R}_{1234} &= R_{1234} + \frac{1}{2t^2}; \\
\bar{R}_{1342} &= R_{1342} - \frac{1}{4t^2}; \\
\bar{R}_{1423} &= R_{1423} - \frac{1}{4t^2}; \\
\bar{R}_{abcd} &= R_{abcd} \quad \text{for the other components.}
\end{aligned}$$

The components of  $\nabla \overline{\text{Riem}}$  are

(B.6)

$$\begin{aligned}
\bar{R}_{5656,t} &= \bar{R}_{565b,t} = \bar{R}_{56a6,t} = \bar{R}_{5ab5,t} = \bar{R}_{6ab6,t} = \bar{R}_{5ab6,t} = \bar{R}_{ab56,t} = 0; \\
\bar{R}_{abcd,e} &= R_{abcd,e}, \quad \bar{R}_{abcd,5} = \bar{R}_{abcd,6} = \bar{R}_{abc5,5} = \bar{R}_{abc5,6} = \bar{R}_{abc6,5} = \bar{R}_{abc6,6} = 0;
\end{aligned}$$

$$\begin{aligned}
&\begin{cases} 2t\bar{R}_{12c5,1} = -R_{12c3}; \\ 2t\bar{R}_{12c5,2} = R_{12c4}; \\ 2t\bar{R}_{12c5,3} = \frac{1}{t^2}\delta_{2c} - R_{121c}; \\ 2t\bar{R}_{12c5,4} = \frac{1}{t^2}\delta_{1c} - R_{12c2}; \end{cases} & \begin{cases} 2t\bar{R}_{13c5,1} = \frac{1}{t^2}\delta_{1c} - R_{13c3}; \\ 2t\bar{R}_{13c5,2} = R_{13c4}; \\ 2t\bar{R}_{13c5,3} = \frac{1}{t^2}\delta_{3c} - R_{131c}; \\ 2t\bar{R}_{13c5,4} = -R_{13c2}; \end{cases} & \begin{cases} 2t\bar{R}_{14c5,1} = -R_{14c3}; \\ 2t\bar{R}_{14c5,2} = R_{14c4} - \frac{1}{t^2}\delta_{1c}; \\ 2t\bar{R}_{14c5,3} = \frac{1}{t^2}\delta_{4c} - R_{141c}; \\ 2t\bar{R}_{14c5,4} = -R_{14c2}; \end{cases} \\
&\begin{cases} 2t\bar{R}_{23c5,1} = \frac{1}{t^2}\delta_{2c} - R_{23c3}; \\ 2t\bar{R}_{23c5,2} = R_{23c4}; \\ 2t\bar{R}_{23c5,3} = R_{23c1}; \\ 2t\bar{R}_{23c5,4} = R_{232c} - \frac{1}{t^2}\delta_{3c}; \end{cases} & \begin{cases} 2t\bar{R}_{24c5,1} = R_{42c3}; \\ 2t\bar{R}_{24c5,2} = R_{424c} - \frac{1}{t^2}\delta_{2c}; \\ 2t\bar{R}_{24c5,3} = -R_{42c1}; \\ 2t\bar{R}_{24c5,4} = R_{42c2} - \frac{1}{t^2}\delta_{4c}; \end{cases} & \begin{cases} 2t\bar{R}_{34c5,1} = R_{343c} - \frac{1}{t^2}\delta_{4c}; \\ 2t\bar{R}_{34c5,2} = R_{34c4} - \frac{1}{t^2}\delta_{3c}; \\ 2t\bar{R}_{34c5,3} = R_{34c1}; \\ 2t\bar{R}_{34c5,4} = -R_{34c2} \end{cases} \\
&\begin{cases} 2t\bar{R}_{12c6,1} = -R_{12c4}; \\ 2t\bar{R}_{12c6,2} = -R_{12c3}; \\ 2t\bar{R}_{12c6,3} = R_{12c2} - \frac{1}{t^2}\delta_{1c}; \\ 2t\bar{R}_{12c6,4} = \frac{1}{t^2}\delta_{2c} - R_{121c}; \end{cases} & \begin{cases} 2t\bar{R}_{13c6,1} = -R_{13c4}; \\ 2t\bar{R}_{13c6,2} = \frac{1}{t^2}\delta_{1c} - R_{13c3}; \\ 2t\bar{R}_{13c6,3} = R_{13c2}; \\ 2t\bar{R}_{13c6,4} = \frac{1}{t^2}\delta_{3c} - R_{131c}; \end{cases} & \begin{cases} 2t\bar{R}_{14c6,1} = \frac{1}{t^2}\delta_{1c} - R_{14c4}; \\ 2t\bar{R}_{14c6,2} = -R_{14c3}; \\ 2t\bar{R}_{14c6,3} = R_{14c2}; \\ 2t\bar{R}_{14c6,4} = \frac{1}{t^2}\delta_{4c} - R_{141c}; \end{cases} \\
&\begin{cases} 2t\bar{R}_{23c6,1} = -R_{23c4}; \\ 2t\bar{R}_{23c6,2} = \frac{1}{t^2}\delta_{2c} - R_{23c3}; \\ 2t\bar{R}_{23c6,3} = \frac{1}{t^2}\delta_{3c} - R_{232c}; \\ 2t\bar{R}_{23c6,4} = -R_{231c}; \end{cases} & \begin{cases} 2t\bar{R}_{24c6,1} = \frac{1}{t^2}\delta_{2c} - R_{424c}; \\ 2t\bar{R}_{24c6,2} = R_{42c3}; \\ 2t\bar{R}_{24c6,3} = \frac{1}{t^2}\delta_{4c} - R_{42c2}; \\ 2t\bar{R}_{24c6,4} = R_{421c}; \end{cases} & \begin{cases} 2t\bar{R}_{34c6,1} = \frac{1}{t^2}\delta_{3c} - R_{34c4}; \\ 2t\bar{R}_{34c6,2} = R_{343c} - \frac{1}{t^2}\delta_{4c}; \\ 2t\bar{R}_{34c6,3} = R_{34c2}; \\ 2t\bar{R}_{34c6,4} = -R_{341c} \end{cases}
\end{aligned}$$

The general expressions of the components of  $\nabla \overline{\text{Ric}}$  are

(B.7)

$$\begin{aligned}
\bar{R}_{ab,c} &= R_{ab,c} - \frac{t^2}{2} [\mathcal{Q}_{ad,c} \mathcal{Q}_{bd} + \mathcal{Q}_{bd,c} \mathcal{Q}_{ad} + \mathcal{Q}_{ad,c} \mathcal{Q}_{bd} + \mathcal{Q}_{bd,c} \mathcal{Q}_{ad}] \\
&\quad - \frac{t^2}{4} [\mathcal{Q}_{ad,d} \mathcal{Q}_{bc} + \mathcal{Q}_{bd,d} \mathcal{Q}_{ac} + \mathcal{Q}_{ad,d} \mathcal{Q}_{bc} + \mathcal{Q}_{bd,d} \mathcal{Q}_{ac}]; \\
\bar{R}_{ab,5} &= \frac{t}{2} (\mathcal{Q}_{ac} \mathcal{Q}_{bc} + \mathcal{Q}_{bc} \mathcal{Q}_{ac}) - \frac{t}{2} (R_{ac} \mathcal{Q}_{bc} + R_{bc} \mathcal{Q}_{ac}) + \\
&\quad + \frac{t^3}{4} [\mathcal{Q}_{ac} (\mathcal{Q}_{cd} \mathcal{Q}_{bd} + \mathcal{Q}_{cd} \mathcal{Q}_{bd}) + \mathcal{Q}_{bc} (\mathcal{Q}_{cd} \mathcal{Q}_{ad} + \mathcal{Q}_{cd} \mathcal{Q}_{ad})]; \\
\bar{R}_{ab,6} &= \frac{t}{2} (\mathcal{Q}_{ac} \mathcal{Q}_{bc} + \mathcal{Q}_{bc} \mathcal{Q}_{ac}) - \frac{t}{2} (R_{ac} \mathcal{Q}_{bc} + R_{bc} \mathcal{Q}_{ac}) + \\
&\quad + \frac{t^3}{4} [\mathcal{Q}_{ac} (\mathcal{Q}_{cd} \mathcal{Q}_{bd} + \mathcal{Q}_{cd} \mathcal{Q}_{bd}) + \mathcal{Q}_{bc} (\mathcal{Q}_{cd} \mathcal{Q}_{ad} + \mathcal{Q}_{cd} \mathcal{Q}_{ad})]; \\
\bar{R}_{a5,b} &= \frac{t}{2} \mathcal{Q}_{ad,db} - \frac{t}{2} \left[ \mathcal{Q}_{ab} \left( \frac{t^2}{4} |\mathcal{Q}_{cd}|^2 + \frac{1}{t^2} \right) + \frac{t^2}{4} \mathcal{Q}_{ab} \mathcal{Q}_{dc} \mathcal{Q}_{dc} \right] + \\
&\quad + \frac{t}{2} \mathcal{Q}_{db} \left[ R_{ad} - \frac{t^2}{2} (\mathcal{Q}_{ac} \mathcal{Q}_{dc} + \mathcal{Q}_{ac} \mathcal{Q}_{dc}) \right]; \\
\bar{R}_{a5,5} &= -\frac{t^2}{4} \mathcal{Q}_{cd,d} \mathcal{Q}_{ac}, \quad \bar{R}_{a5,6} = \frac{1}{2} \mathcal{Q}_{ac,c} - \frac{t^2}{4} \mathcal{Q}_{cd,d} \mathcal{Q}_{ac};
\end{aligned}$$

$$\begin{aligned}
\bar{R}_{a6,b} &= \frac{t}{2} \mathcal{Q}_{ad,db} - \frac{t}{2} \left[ \mathcal{Q}_{ab} \left( \frac{t^2}{4} |\mathcal{Q}_{cd}|^2 + \frac{1}{t^2} \right) + \frac{t^2}{4} \mathcal{Q}_{ab} \mathcal{Q}_{dc} \mathcal{Q}_{dc} \right] + \\
&\quad + \frac{t}{2} \mathcal{Q}_{db} \left[ R_{ad} - \frac{t^2}{2} (\mathcal{Q}_{ac} \mathcal{Q}_{dc} + \mathcal{Q}_{ac} \mathcal{Q}_{dc}) \right]; \\
\bar{R}_{a6,5} &= -\frac{1}{2} \mathcal{Q}_{ac,c} - \frac{t^2}{4} \mathcal{Q}_{cd,d} \mathcal{Q}_{ac}, \quad \bar{R}_{a6,6} = -\frac{t^2}{4} \mathcal{Q}_{cd,d} \mathcal{Q}_{ac}; \\
\bar{R}_{55,a} &= \frac{t^2}{2} (\mathcal{Q}_{bc} \mathcal{Q}_{bc,a} + \mathcal{Q}_{bc,c} \mathcal{Q}_{ba}), \quad \bar{R}_{55,5} = 0, \quad \bar{R}_{55,6} = \frac{t}{2} \mathcal{Q}_{dc} \mathcal{Q}_{dc}; \\
\bar{R}_{66,a} &= \frac{t^2}{2} (\mathcal{Q}_{bc} \mathcal{Q}_{bc,a} + \mathcal{Q}_{bc,c} \mathcal{Q}_{ba}), \quad \bar{R}_{66,5} = -\frac{t}{2} \mathcal{Q}_{dc} \mathcal{Q}_{dc}, \quad \bar{R}_{66,6} = 0; \\
\bar{R}_{56,a} &= \frac{t^2}{4} [\mathcal{Q}_{bc} \mathcal{Q}_{bc,a} + \mathcal{Q}_{bc} \mathcal{Q}_{bc,a} + \mathcal{Q}_{ba} \mathcal{Q}_{bc,c} + \mathcal{Q}_{ba} \mathcal{Q}_{bc,c}]; \\
\bar{R}_{56,5} &= -\frac{t}{4} \mathcal{Q}_{ab} \mathcal{Q}_{ab}, \quad \bar{R}_{56,6} = \frac{t}{4} \mathcal{Q}_{ab} \mathcal{Q}_{ab}.
\end{aligned}$$

The covariant derivative of the scalar curvature  $\bar{S}$  has components

$$\begin{aligned}
\bar{S}_a &= S_a - \frac{t^2}{2} [\mathcal{Q}_{bc} \mathcal{Q}_{bc,a} + \mathcal{Q}_{bc} \mathcal{Q}_{bc,a}]; \\
\bar{S}_5 &= \frac{t}{2} \mathcal{Q}_{bc} \mathcal{Q}_{bc}, \quad \bar{S}_6 = -\frac{t}{2} \mathcal{Q}_{bc} \mathcal{Q}_{bc}.
\end{aligned}$$

APPENDIX C. COVARIANT DERIVATIVE OF THE ALMOST COMPLEX STRUCTURES AND  
DIFFERENTIAL OF THE KÄHLER FORMS

In this appendix we list all the components of the covariant derivative of the almost complex structures  $J^\pm$  on  $Z$ . Recall that

$$(C.1) \quad \begin{aligned} J^\pm &= \sum_{k=1}^3 \left( \theta^{2k-1} \otimes e_{2k} - \theta^{2k} \otimes e_{2k-1} \right) \\ &= \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3 \pm \theta^5 \otimes e_6 \mp \theta^6 \otimes e_5; \end{aligned}$$

using the same notation of the article, we write  $J^+ = J$ ,  $J^- = \mathbf{J}$ .

**Computation of  $\nabla J$  on  $(Z, g_t, J)$ :**

The covariant derivative  $\nabla J$  of an almost complex structure is defined as

$$(C.2) \quad \nabla J = J_{q,t}^p \theta^t \otimes \theta^q \otimes e_p, \text{ where } J_{q,t}^p = dJ_q^p - J_r^p \omega_q^r + J_q^r \omega_r^p, \quad J_{p,t}^q = -J_{q,t}^p,$$

with respect to a local orthonormal frame  $\{\theta_p\}$  and its dual frame  $\{e_q\}$ . Using (C.2), a long but straightforward computation shows that:

$$(C.3) \quad \begin{array}{ll} J_{2,t}^1 = J_{4,t}^3 = J_{6,t}^5 = 0; & J_{6,5}^1 = J_{6,6}^1 = 0; \\ J_{3,a}^1 = 0; & J_{3,t}^2 = J_{4,t}^1; \\ J_{3,5}^1 = -\frac{1}{2}t(\mathcal{Q}_{14} + \mathcal{Q}_{23}); & J_{4,t}^2 = -J_{3,t}^1 \\ J_{3,6}^1 = \frac{1}{2}t \left[ \frac{2}{t^2} - (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right]; & J_{5,t}^2 = J_{6,t}^1 \\ J_{4,a}^1 = 0; & J_{6,t}^2 = -J_{5,t}^1 \\ J_{4,5}^1 = -\frac{1}{2}t \left[ \frac{2}{t^2} - (\mathcal{Q}_{13} + \mathcal{Q}_{42}) \right]; & J_{5,a}^3 = -\frac{1}{2}t(\mathcal{Q}_{4a} + \mathcal{Q}_{3a}); \\ J_{4,6}^1 = \frac{1}{2}t(\mathcal{Q}_{13} + \mathcal{Q}_{42}); & J_{5,5}^3 = J_{5,6}^3 = 0; \\ J_{5,a}^1 = -\frac{1}{2}t(\mathcal{Q}_{2a} + \mathcal{Q}_{1a}); & J_{6,a}^3 = \frac{1}{2}t(\mathcal{Q}_{3a} - \mathcal{Q}_{4a}); \\ J_{5,5}^1 = J_{5,6}^1 = 0; & J_{6,5}^3 = J_{6,6}^3 = 0; \\ J_{6,a}^1 = \frac{1}{2}t(\mathcal{Q}_{1a} - \mathcal{Q}_{2a}); & J_{5,t}^4 = J_{6,t}^3 \\ & J_{6,t}^4 = -J_{5,t}^3. \end{array}$$

The square norm  $|\nabla J|^2 = \sum_{p,q,t=1}^6 J_{p,q}^t J_{p,q}^t$  is given by

$$(C.4) \quad \frac{1}{t^2} |\nabla J|^2 = [(\mathcal{Q}_{14} + \mathcal{Q}_{23})^2 + (\mathcal{Q}_{14} + \mathcal{Q}_{23})^2 + (\mathcal{Q}_{13} + \mathcal{Q}_{42})^2 + (\mathcal{Q}_{13} + \mathcal{Q}_{42})^2 + (\mathcal{Q}_{13} - \mathcal{Q}_{14})^2 + (\mathcal{Q}_{42} - \mathcal{Q}_{14})^2 + (\mathcal{Q}_{13} - \mathcal{Q}_{23})^2 + (\mathcal{Q}_{14} + \mathcal{Q}_{42})^2 + (\mathcal{Q}_{14} + \mathcal{Q}_{13})^2 + (\mathcal{Q}_{23} - \mathcal{Q}_{42})^2 + (\mathcal{Q}_{14} - \mathcal{Q}_{13})^2 + (\mathcal{Q}_{23} + \mathcal{Q}_{42})^2] + 2[\mathcal{Q}_{12}^2 + \mathcal{Q}_{12}^2 + \mathcal{Q}_{34}^2 + \mathcal{Q}_{34}^2] - \frac{4}{t^2}[(\mathcal{Q}_{13} + \mathcal{Q}_{42}) + (\mathcal{Q}_{14} + \mathcal{Q}_{23})] + \frac{8}{t^4}.$$

**Computation of  $\nabla \mathbf{J}$  on  $(Z, g_t, J)$ :**

Again, using (C.2), we have:

$$(C.5) \quad \begin{array}{ll} \mathbf{J}_{2,t}^1 = \mathbf{J}_{4,t}^3 = \mathbf{J}_{6,t}^5 = 0; & \mathbf{J}_{6,5}^1 = \mathbf{J}_{6,6}^1 = 0; \\ \mathbf{J}_{3,a}^1 = 0; & \mathbf{J}_{3,t}^2 = \mathbf{J}_{4,t}^1; \\ \mathbf{J}_{3,5}^1 = -\frac{1}{2}t(\mathcal{Q}_{14} + \mathcal{Q}_{23}); & \mathbf{J}_{4,t}^2 = -\mathbf{J}_{3,t}^1 \\ \mathbf{J}_{3,6}^1 = \frac{1}{2}t \left[ \frac{2}{t^2} - (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right]; & \mathbf{J}_{5,t}^2 = -\mathbf{J}_{6,t}^1 \\ \mathbf{J}_{4,a}^1 = 0; & \mathbf{J}_{6,t}^2 = \mathbf{J}_{5,t}^1 \\ \mathbf{J}_{4,5}^1 = -\frac{1}{2}t \left[ \frac{2}{t^2} - (\mathcal{Q}_{13} + \mathcal{Q}_{42}) \right]; & \mathbf{J}_{5,a}^3 = -\frac{1}{2}t(\mathcal{Q}_{4a} - \mathcal{Q}_{3a}); \\ \mathbf{J}_{4,6}^1 = \frac{1}{2}t(\mathcal{Q}_{13} + \mathcal{Q}_{42}); & \mathbf{J}_{5,5}^3 = \mathbf{J}_{5,6}^3 = 0; \\ \mathbf{J}_{5,a}^1 = -\frac{1}{2}t(\mathcal{Q}_{2a} - \mathcal{Q}_{1a}); & \mathbf{J}_{6,a}^3 = -\frac{1}{2}t(\mathcal{Q}_{3a} + \mathcal{Q}_{4a}); \\ \mathbf{J}_{5,5}^1 = \mathbf{J}_{5,6}^1 = 0; & \mathbf{J}_{6,5}^3 = \mathbf{J}_{6,6}^3 = 0; \\ \mathbf{J}_{6,a}^1 = -\frac{1}{2}t(\mathcal{Q}_{1a} + \mathcal{Q}_{2a}); & \mathbf{J}_{5,t}^4 = -\mathbf{J}_{6,t}^3 \\ & \mathbf{J}_{6,t}^4 = \mathbf{J}_{5,t}^3. \end{array}$$

**Kähler forms of  $J$  and  $\mathbf{J}$ :**

denoting by  $\omega_+$  and  $\omega_-$  the Kähler forms of  $J$  and  $\mathbf{J}$ , respectively, we have:

$$(C.6) \quad \begin{aligned} d\omega_+ = & -t \mathcal{Q}_{12} \theta^1 \wedge \theta^2 \wedge \theta^5 + t \mathcal{Q}_{12} \theta^1 \wedge \theta^2 \wedge \theta^6 - t \mathcal{Q}_{13} \theta^1 \wedge \theta^3 \wedge \theta^5 + \\ & + \left( t \mathcal{Q}_{13} - \frac{1}{t} \right) \theta^1 \wedge \theta^3 \wedge \theta^6 + \left( \frac{1}{t} - t \mathcal{Q}_{14} \right) \theta^1 \wedge \theta^4 \wedge \theta^5 + \\ & + t \mathcal{Q}_{14} \theta^1 \wedge \theta^4 \wedge \theta^6 + \left( \frac{1}{t} - t \mathcal{Q}_{23} \right) \theta^2 \wedge \theta^3 \wedge \theta^5 + t \mathcal{Q}_{23} \theta^2 \wedge \theta^3 \wedge \theta^6 + \\ & - t \mathcal{Q}_{42} \theta^4 \wedge \theta^2 \wedge \theta^5 + \left( t \mathcal{Q}_{42} - \frac{1}{t} \right) \theta^4 \wedge \theta^2 \wedge \theta^6 - t \mathcal{Q}_{34} \theta^3 \wedge \theta^4 \wedge \theta^5 + \\ & + t \mathcal{Q}_{34} \theta^3 \wedge \theta^4 \wedge \theta^6. \end{aligned}$$

$$\begin{aligned}
(C.7) \quad d\omega_- &= t \mathbb{Q}_{12} \theta^1 \wedge \theta^2 \wedge \theta^5 - t \mathbb{Q}_{12} \theta^1 \wedge \theta^2 \wedge \theta^6 + t \mathbb{Q}_{13} \theta^1 \wedge \theta^3 \wedge \theta^5 + \\
&\quad - \left( t \mathbb{Q}_{13} + \frac{1}{t} \right) \theta^1 \wedge \theta^3 \wedge \theta^6 + \left( \frac{1}{t} + t \mathbb{Q}_{14} \right) \theta^1 \wedge \theta^4 \wedge \theta^5 + \\
&\quad - t \mathbb{Q}_{14} \theta^1 \wedge \theta^4 \wedge \theta^6 + \left( \frac{1}{t} + t \mathbb{Q}_{23} \right) \theta^2 \wedge \theta^3 \wedge \theta^5 - t \mathbb{Q}_{23} \theta^2 \wedge \theta^3 \wedge \theta^6 + \\
&\quad + t \mathbb{Q}_{42} \theta^4 \wedge \theta^2 \wedge \theta^5 - \left( t \mathbb{Q}_{42} + \frac{1}{t} \right) \theta^4 \wedge \theta^2 \wedge \theta^6 + t \mathbb{Q}_{34} \theta^3 \wedge \theta^4 \wedge \theta^5 + \\
&\quad - t \mathbb{Q}_{34} \theta^3 \wedge \theta^4 \wedge \theta^6.
\end{aligned}$$

As far as the codifferentials of  $\omega_+$  and  $\omega_-$  are concerned, we have:

$$(C.8) \quad \delta\omega_+ = \delta\omega_- = t(\mathbb{Q}_{12} + \mathbb{Q}_{34})\theta^5 + t(\mathbb{Q}_{12} + \mathbb{Q}_{34})\theta^6.$$

#### APPENDIX D. HESSIAN AND LAPLACIAN OF $J$ AND $\mathbf{J}$

By definition, we have

$$\nabla^2 J = J_{q,rt}^p \theta^t \otimes \theta^r \otimes \theta^q \otimes e_p,$$

and

$$J_{q,rt}^p \theta^t = dJ_{q,r}^p - J_{t,r}^p \theta_q^t - J_{q,t}^p \theta_r^t + J_{q,r}^t \theta_t^p; \quad J_{q,rt}^p = -J_{p,rt}^q.$$

$$\begin{aligned}
J_{2,rt}^1 \theta^t &= -\frac{1}{2} t [J_{5,r}^1 (\mathbb{Q}_{2a} + \mathbb{Q}_{1a}) + J_{6,r}^1 (\mathbb{Q}_{2a} - \mathbb{Q}_{1a})] \theta^a \\
&\quad - \frac{1}{2} t \left\{ J_{3,r}^1 (\mathbb{Q}_{14} + \mathbb{Q}_{23}) + J_{4,r}^1 \left[ \frac{2}{t^2} - (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \right] \right\} \theta^5 \\
&\quad + \frac{1}{2} t \left\{ J_{3,r}^1 \left[ \frac{2}{t^2} - (\mathbb{Q}_{14} + \mathbb{Q}_{23}) \right] + J_{4,r}^1 (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \right\} \theta^6; \\
J_{4,rt}^3 \theta^t &= \frac{1}{2} t [J_{6,r}^3 (\mathbb{Q}_{3a} - \mathbb{Q}_{4a}) - J_{5,r}^3 (\mathbb{Q}_{4a} + \mathbb{Q}_{3a})] \theta^a \\
&\quad - \frac{1}{2} t \left\{ J_{3,r}^1 (\mathbb{Q}_{14} + \mathbb{Q}_{23}) + J_{4,r}^1 \left[ \frac{2}{t^2} - (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \right] \right\} \theta^5 \\
&\quad + \frac{1}{2} t \left\{ J_{3,r}^1 \left[ \frac{2}{t^2} - (\mathbb{Q}_{14} + \mathbb{Q}_{23}) \right] + J_{4,r}^1 (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \right\} \theta^6; \\
J_{6,rt}^5 \theta^t &= \frac{1}{2} t [J_{6,r}^1 (\mathbb{Q}_{1a} - \mathbb{Q}_{2a}) + J_{6,r}^2 (\mathbb{Q}_{2a} + \mathbb{Q}_{1a}) + J_{6,r}^3 (\mathbb{Q}_{3a} - \mathbb{Q}_{4a}) + J_{6,r}^4 (\mathbb{Q}_{4a} + \mathbb{Q}_{3a})] \theta^a; \\
J_{3,at}^1 \theta^t &= \frac{1}{4} t^2 \{ (\mathbb{Q}_{3a} - \mathbb{Q}_{4a}) \mathbb{Q}_{1b} - (\mathbb{Q}_{4a} + \mathbb{Q}_{3a}) \mathbb{Q}_{1b} + (\mathbb{Q}_{2a} + \mathbb{Q}_{1a}) \mathbb{Q}_{3b} \} \theta^b \\
&\quad + \frac{1}{4} t^2 \left\{ -(\mathbb{Q}_{1a} - \mathbb{Q}_{2a}) \mathbb{Q}_{3b} + (\mathbb{Q}_{14} + \mathbb{Q}_{23}) \mathbb{Q}_{ab} - \left[ \frac{2}{t^2} - (\mathbb{Q}_{14} + \mathbb{Q}_{23}) \right] \mathbb{Q}_{ab} \right\} \theta^b; \\
J_{3,5t}^1 \theta^t &= -\frac{1}{2} t (\mathbb{Q}_{14} + \mathbb{Q}_{23})_a \theta^a + \frac{1}{4} t^2 (\mathbb{Q}_{12} + \mathbb{Q}_{34}) \left[ \frac{4}{t^2} - (\mathbb{Q}_{q3} + \mathbb{Q}_{42}) \right] \theta^5 \\
&\quad - \frac{1}{4} t^2 [(\mathbb{Q}_{12} + \mathbb{Q}_{34})(\mathbb{Q}_{13} + \mathbb{Q}_{42})] \theta^6; \\
J_{3,6t}^1 \theta^t &= -\frac{1}{2} t (\mathbb{Q}_{14} + \mathbb{Q}_{23})_a \theta^a + \frac{1}{4} t^2 \left[ \frac{4}{t^2} (\mathbb{Q}_{12} + \mathbb{Q}_{34}) - (\mathbb{Q}_{13} + \mathbb{Q}_{42})(\mathbb{Q}_{12} + \mathbb{Q}_{34}) \right] \theta^5 \\
&\quad - \frac{1}{4} t^2 (\mathbb{Q}_{13} + \mathbb{Q}_{42})(\mathbb{Q}_{12} + \mathbb{Q}_{34}) \theta^6; \\
J_{4,at}^1 \theta^t &= \frac{1}{4} t^2 \{ (\mathbb{Q}_{3a} - \mathbb{Q}_{4a}) \mathbb{Q}_{1b} + (\mathbb{Q}_{4a} + \mathbb{Q}_{3a}) \mathbb{Q}_{1b} + (\mathbb{Q}_{2a} + \mathbb{Q}_{1a}) \mathbb{Q}_{4b} \} \theta^b \\
&= +\frac{1}{4} t^2 \left\{ -(\mathbb{Q}_{1a} - \mathbb{Q}_{2a}) \mathbb{Q}_{4b} - (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \mathbb{Q}_{ab} + \left[ \frac{2}{t^2} - (\mathbb{Q}_{13} + \mathbb{Q}_{42}) \right] \mathbb{Q}_{ab} \right\} \theta^b;
\end{aligned}$$

$$\begin{aligned}
J_{4,5t}^1 \theta^t &= \frac{1}{2} t (\mathcal{Q}_{13} + \mathcal{Q}_{42})_a \theta^a - \frac{1}{4} t^2 (\mathcal{Q}_{12} + \mathcal{Q}_{34}) (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \theta^5 \\
&\quad + \frac{1}{4} t^2 \left[ \frac{4}{t^2} (\mathcal{Q}_{12} + \mathcal{Q}_{34}) - (\mathcal{Q}_{14} + \mathcal{Q}_{23}) (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \right] \theta^6; \\
J_{4,6t}^1 \theta^t &= \frac{1}{2} t (\mathcal{Q}_{13} + \mathcal{Q}_{42})_a \theta^a - \frac{1}{4} t^2 (\mathcal{Q}_{12} + \mathcal{Q}_{34}) (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \theta^5 \\
&\quad + \frac{1}{4} t^2 (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \left[ \frac{4}{t^2} - (\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right] \theta^6; \\
J_{5,at}^1 \theta^t &= -\frac{1}{2} t (\mathcal{Q}_{2a} + \mathcal{Q}_{1a})_b \theta^b + \frac{1}{2} (\mathcal{Q}_{1a} - \mathcal{Q}_{4a}) \theta^5 + \frac{1}{2} (\mathcal{Q}_{3a} - \mathcal{Q}_{2a}) \theta^6 \\
&\quad + \frac{1}{4} t^2 [(\mathcal{Q}_{2c} + \mathcal{Q}_{1c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{12} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{13} - (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{14}] \theta^5 \\
&\quad + \frac{1}{4} t^2 [(\mathcal{Q}_{2c} + \mathcal{Q}_{1c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{12} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{13} - (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{14}] \theta^6; \\
J_{5,5t}^1 \theta^t &= \frac{1}{2} t (J_{c,5}^1 + J_{5,c}^1) \mathcal{Q}_{cb} \theta^b; & J_{5,6t}^1 \theta^t &= \frac{1}{2} t (J_{c,6}^1 \mathcal{Q}_{cb} + J_{5,c}^1 \mathcal{Q}_{cb}) \theta^b; \\
J_{6,at}^1 \theta^t &= \frac{1}{2} t (\mathcal{Q}_{1a} - \mathcal{Q}_{2a})_b \theta^b + \frac{1}{2} (\mathcal{Q}_{2a} - \mathcal{Q}_{4a}) \theta^5 + \frac{1}{2} (\mathcal{Q}_{1a} + \mathcal{Q}_{3a}) \theta^6 \\
&\quad - \frac{1}{4} t^2 [(\mathcal{Q}_{1c} - \mathcal{Q}_{2c}) \mathcal{Q}_{ac} + (\mathcal{Q}_{2a} + \mathcal{Q}_{1a}) \mathcal{Q}_{12} + (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{13} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{14}] \theta^5 \\
&\quad - \frac{1}{4} t^2 [(\mathcal{Q}_{1c} - \mathcal{Q}_{2c}) \mathcal{Q}_{ac} + (\mathcal{Q}_{3a} + \mathcal{Q}_{1a}) \mathcal{Q}_{12} + (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{13} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{14}] \theta^6; \\
J_{6,5t}^1 \theta^t &= \frac{1}{2} t (J_{6,c}^1 \mathcal{Q}_{cb} + J_{c,5}^1 \mathcal{Q}_{cb}) \theta^b; \\
J_{6,6t}^1 \theta^t &= \frac{1}{2} t (J_{c,6}^1 + J_{6,c}^1) \mathcal{Q}_{cb} \theta^b; \\
J_{5,at}^3 \theta^t &= -\frac{1}{2} t (\mathcal{Q}_{4a} + \mathcal{Q}_{3a})_b \theta^b + \frac{1}{2} (\mathcal{Q}_{3a} + \mathcal{Q}_{2a}) \theta^5 - \frac{1}{2} (\mathcal{Q}_{4a} + \mathcal{Q}_{1a}) \theta^6 \\
&\quad + \frac{1}{4} t^2 [(\mathcal{Q}_{4c} + \mathcal{Q}_{3c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{2a} + \mathcal{Q}_{1a}) \mathcal{Q}_{13} + (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{23} - (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{34}] \theta^5 \\
&\quad + \frac{1}{4} t^2 [(\mathcal{Q}_{4c} + \mathcal{Q}_{3c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{2a} + \mathcal{Q}_{1a}) \mathcal{Q}_{13} + (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{23} - (\mathcal{Q}_{3a} - \mathcal{Q}_{4a}) \mathcal{Q}_{34}] \theta^6; \\
J_{5,5t}^3 \theta^t &= \frac{1}{2} t (J_{c,5}^3 + J_{5,c}^3) \mathcal{Q}_{cb} \theta^b; \\
J_{5,6t}^3 \theta^t &= \frac{1}{2} t (J_{c,6}^3 \mathcal{Q}_{cb} + J_{5,c}^3 \mathcal{Q}_{cb}) \theta^b; \\
J_{6,at}^3 \theta^t &= \frac{1}{2} t (\mathcal{Q}_{3a} - \mathcal{Q}_{4a})_b \theta^b + \frac{1}{2} (\mathcal{Q}_{4a} + \mathcal{Q}_{2a}) \theta^5 + \frac{1}{2} (\mathcal{Q}_{3a} - \mathcal{Q}_{1a}) \theta^6 \\
&\quad - \frac{1}{4} t^2 [(\mathcal{Q}_{3c} - \mathcal{Q}_{4c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{13} - (\mathcal{Q}_{2a} + \mathcal{Q}_{1a}) \mathcal{Q}_{23} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{34}] \theta^5 \\
&\quad - \frac{1}{4} t^2 [(\mathcal{Q}_{3c} - \mathcal{Q}_{4c}) \mathcal{Q}_{ac} - (\mathcal{Q}_{1a} - \mathcal{Q}_{2a}) \mathcal{Q}_{13} - (\mathcal{Q}_{2a} + \mathcal{Q}_{1a}) \mathcal{Q}_{23} + (\mathcal{Q}_{4a} + \mathcal{Q}_{3a}) \mathcal{Q}_{34}] \theta^6; \\
J_{6,5t}^3 \theta^t &= \frac{1}{2} t (J_{6,c}^3 \mathcal{Q}_{cb} + J_{c,5}^3 \mathcal{Q}_{cb}) \theta^b; \\
J_{6,6t}^3 \theta^t &= \frac{1}{2} t (J_{c,6}^3 + J_{6,c}^3) \mathcal{Q}_{cb} \theta^b; \\
J_{5,rt}^4 &= J_{6,rt}^3; & J_{6,rt}^4 &= -J_{5,rt}^3; & J_{3,rt}^2 &= J_{4,rt}^1; & J_{4,rt}^2 &= -J_{3,rt}^1; & J_{5,rt}^2 &= J_{6,rt}^1; & J_{6,rt}^2 &= -J_{5,rt}^1.
\end{aligned}$$

$$(D.1) \quad \Delta_J J_v^u := (\Delta J - J \nabla_p J \nabla_p J)_v^u = \Delta J_v^u - J_t^u J_{q,p}^t J_{v,p}^q.$$

$$\begin{aligned}
\Delta_J J_2^1 &= \Delta_J J_4^3 = \Delta_J J_6^5 \equiv 0; \\
\Delta_J J_3^1 &= (\mathcal{Q}_{12} + \mathcal{Q}_{34}) + \frac{1}{4}t[N_{14}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) - N_{13}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34})] = -\Delta_J J_4^2 \\
\Delta_J J_4^1 &= (\mathcal{Q}_{12} + \mathcal{Q}_{34}) + \frac{1}{4}t[N_{14}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) - N_{13}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34})] = \Delta_J J_3^2 \\
\Delta_J J_5^1 &= -\frac{1}{2}t \sum_{a=1}^4 (\mathcal{Q}_{2a} + \mathcal{Q}_{1a})_a = -\Delta_J J_6^2; \\
\Delta_J J_6^1 &= \frac{1}{2}t \sum_{a=1}^4 (\mathcal{Q}_{1a} - \mathcal{Q}_{2a})_a = \Delta_J J_5^2; \\
\Delta_J J_5^3 &= -\frac{1}{2}t \sum_{a=1}^4 (\mathcal{Q}_{4a} + \mathcal{Q}_{3a})_a = -\Delta_J J_6^4; \\
\Delta_J J_6^3 &= \frac{1}{2}t \sum_{a=1}^4 (\mathcal{Q}_{3a} - \mathcal{Q}_{4a})_a = \Delta_J J_5^4.
\end{aligned}$$

Note that

$$\begin{aligned}
\text{(D.2)} \quad \Delta_J J_5^1 &= 2t[(\delta W^+)_{213} - (\delta W^+)_{141}]; \\
\Delta_J J_6^1 &= 2t[(\delta W^+)_{214} + (\delta W^+)_{131}]; \\
\Delta_J J_5^3 &= 2t[(\delta W^+)_{413} + (\delta W^+)_{314}]; \\
\Delta_J J_6^3 &= 2t[(\delta W^+)_{414} - (\delta W^+)_{313}].
\end{aligned}$$

#### APPENDIX E. COMPUTATION OF $N_J$ , $N_{\mathbf{J}}$

Here we consider the Nijenhuis tensors  $N_J$  and  $N_{\mathbf{J}}$  of  $J = J^+$  and  $\mathbf{J} = J^-$ , respectively. We have

$$\begin{aligned}
N_J &= N_{tq}^p \theta^t \otimes \theta^q \otimes e_p, & N_{tq}^p &= -N_{qt}^p, \\
N_{\mathbf{J}} &= \mathbf{N}_{tq}^p \theta^t \otimes \theta^q \otimes e_p, & \mathbf{N}_{tq}^p &= -\mathbf{N}_{qt}^p,
\end{aligned}$$

where  $N_{tq}^p = J_t^r J_{r,q}^p - J_q^r J_{r,t}^p + J_q^s J_{t,s}^p - J_t^s J_{q,s}^p$  and analogously for  $\mathbf{N}_{tq}^p$ .

Using the definition, (C.3) and (C.5), we have

$$\begin{aligned}
\text{(E.1)} \quad N_{pq}^a &= 0 = N_{12}^5 = N_{34}^5 = N_{12}^6 = N_{34}^6; & N_{13}^5 &= -N_{24}^5 = -N_{14}^6 = -N_{23}^6; \\
N_{13}^5 &= -t(\mathcal{Q}_{13} + \mathcal{Q}_{42} - \mathcal{Q}_{14} - \mathcal{Q}_{23}) = 2t(A_{33} - A_{22}); & N_{14}^5 &= N_{23}^5 = N_{13}^6 = -N_{24}^6; \\
N_{14}^5 &= -t(\mathcal{Q}_{14} + \mathcal{Q}_{23} + \mathcal{Q}_{13} + \mathcal{Q}_{42}) \\
&= -2t(\mathcal{Q}_{14} + \mathcal{Q}_{23}) = -2t(\mathcal{Q}_{13} + \mathcal{Q}_{42}) = -4tA_{23};
\end{aligned}$$

Components of  $\nabla N_J$ : by

$$\text{(E.2)} \quad \nabla N_J = N_{tq,s}^p \theta^s \otimes \theta^t \otimes \theta^q \otimes e_p, \quad N_{tq,s}^p = -N_{qt,s}^p;$$

$$\begin{aligned}
\text{(E.3)} \quad N_{pq,5}^a &= N_{pq,6}^a = 0; \\
N_{pq,b}^a &= -\frac{1}{2}(N_{pq}^5 \mathcal{Q}_{ab} + N_{pq}^6 \mathcal{Q}_{ab}).
\end{aligned}$$

$$\begin{aligned}
N_{12,a}^5 &= 0; & N_{36,a}^5 &= -\frac{1}{2}t(N_{13}^5 \mathcal{Q}_{1a} + N_{14}^5 \mathcal{Q}_{2a}); \\
N_{12,5}^5 &= -N_{14}^5 \left[ \frac{1}{t} - \frac{1}{2}t(\mathcal{Q}_{13} + \mathcal{Q}_{42}) \right] + \frac{1}{4}N_{13}^5 N_{14}^5; & N_{36,5}^5 &= N_{36,6}^5 = 0; \\
N_{12,6}^5 &= N_{13}^5 \left[ \frac{1}{t} - \frac{1}{2}t(\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right] - \frac{1}{4}(N_{14}^5)^2; & N_{45,a}^5 &= -\frac{1}{2}t(N_{14}^5 \mathcal{Q}_{1a} - N_{13}^5 \mathcal{Q}_{2a}); \\
N_{13,a}^5 &= -t[(\mathcal{Q}_{13} + \mathcal{Q}_{42})_a - (\mathcal{Q}_{14} + \mathcal{Q}_{23})_a]; & N_{45,5}^5 &= N_{45,6}^5 = 0; \\
N_{13,5}^5 &= -\frac{1}{2}tN_{14}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) - 2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{46,a}^5 &= -\frac{1}{2}t(N_{14}^5 \mathcal{Q}_{1a} - N_{13}^5 \mathcal{Q}_{2a}); \\
N_{13,6}^5 &= -\frac{1}{2}tN_{14}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) - 2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{46,5}^5 &= N_{46,6}^5 = 0; \\
N_{14,a}^5 &= -2t(\mathcal{Q}_{14} + \mathcal{Q}_{23})_a = -2t(\mathcal{Q}_{13} + \mathcal{Q}_{42})_a; & N_{56,s}^5 &= 0; \\
N_{14,5}^5 &= \frac{1}{2}tN_{13}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) + 2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{12,a}^6 &= 0; \\
N_{14,6}^5 &= \frac{1}{2}tN_{13}^5(\mathcal{Q}_{12} + \mathcal{Q}_{34}) - 2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{12,5}^6 &= N_{13}^5 \left[ \frac{1}{t} - \frac{1}{2}t(\mathcal{Q}_{13} + \mathcal{Q}_{42}) \right] + \frac{1}{4}(N_{14}^5)^2; \\
N_{15,a}^5 &= \frac{1}{2}t(N_{13}^5 \mathcal{Q}_{3a} + N_{14}^5 \mathcal{Q}_{4a}); & N_{12,6}^6 &= N_{14}^5 \left[ \frac{1}{t} - \frac{1}{2}t(\mathcal{Q}_{14} + \mathcal{Q}_{23}) \right] + \frac{1}{4}N_{13}^5 N_{14}^5; \\
N_{15,5}^5 &= N_{15,6}^5 = 0; & N_{13,s}^6 &= N_{14,s}^5; \\
N_{16,a}^5 &= \frac{1}{2}t(N_{13}^5 \mathcal{Q}_{3a} + N_{14}^5 \mathcal{Q}_{4a}); & N_{14,s}^6 &= -N_{13,s}^5; \\
N_{16,5}^5 &= N_{16,6}^5 = 0; & N_{15,s}^6 &= N_{25,s}^5; \\
N_{23,s}^5 &= N_{14,s}^5; & N_{16,s}^6 &= N_{26,s}^5; \\
N_{24,s}^5 &= -N_{13,s}^5; & N_{23,s}^6 &= -N_{13,s}^5; \\
N_{25,a}^5 &= \frac{1}{2}t(N_{14}^5 \mathcal{Q}_{3a} - N_{13}^5 \mathcal{Q}_{4a}); & N_{24,s}^6 &= -N_{14,s}^5; \\
N_{25,5}^5 &= N_{25,6}^5 = 0; & N_{25,s}^6 &= -N_{15,s}^5; \\
N_{26,a}^5 &= \frac{1}{2}t(N_{14}^5 \mathcal{Q}_{3a} - N_{13}^5 \mathcal{Q}_{4a}); & N_{26,s}^6 &= -N_{16,s}^5; \\
N_{26,5}^5 &= N_{26,6}^5 = 0; & N_{34,s}^6 &= N_{12,s}^6; \\
N_{34,s}^5 &= N_{12,s}^5; & N_{35,s}^6 &= N_{46,s}^5; \\
N_{35,a}^5 &= -\frac{1}{2}t(N_{13}^5 \mathcal{Q}_{1a} + N_{14}^5 \mathcal{Q}_{2a}); & N_{36,s}^6 &= N_{46,s}^5; \\
N_{35,5}^5 &= N_{35,6}^5 = 0; & N_{45,s}^6 &= -N_{35,s}^5; \\
& & N_{46,s}^6 &= -N_{36,s}^5; \\
& & N_{56,s}^6 &= 0.
\end{aligned}$$

Thus, we can now compute the components of the two divergences

$$\operatorname{div} N_J = N_{pq,t}^t \theta^p \otimes \theta^q, \quad \overline{\operatorname{div}} N_J = N_{pt,t}^r \theta^p \otimes e_r.$$

For the sake of simplicity, let

$$\Gamma := \mathcal{Q}_{13} + \mathcal{Q}_{42} - \mathcal{Q}_{14} - \mathcal{Q}_{23} = 2(A_{22} + A_{33}).$$

Then, we have

$$(E.4) \quad \begin{aligned} N_{12,t}^t &= N_{34,t}^t = 0; \\ N_{13,t}^t &= -N_{24,t}^t = \frac{t}{2} \left[ (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \left( N_{13}^5 - \frac{8}{t} \right) - N_{14}^5 (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \right]; \\ N_{14,t}^t &= N_{23,t}^t = \frac{t}{2} \left[ (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \left( N_{13}^5 + \frac{8}{t} \right) + N_{14}^5 (\mathcal{Q}_{12} + \mathcal{Q}_{34}) \right]; \\ N_{a5,t}^t &= N_{a6,t}^t = 0. \end{aligned}$$

$$(E.5) \quad \begin{aligned} N_{1t,t}^1 &= -\frac{1}{2} [N_{13}^5 (\mathcal{Q}_{13} - \mathcal{Q}_{14}) + N_{14}^5 (\mathcal{Q}_{14} + \mathcal{Q}_{13})]; & N_{3t,t}^3 &= -\frac{1}{2} [N_{13}^5 (\mathcal{Q}_{13} - \mathcal{Q}_{23}) + N_{14}^5 (\mathcal{Q}_{23} + \mathcal{Q}_{13})]; \\ N_{2t,t}^1 &= -\frac{1}{2} [N_{14}^5 (\mathcal{Q}_{13} - \mathcal{Q}_{14}) - N_{13}^5 (\mathcal{Q}_{14} + \mathcal{Q}_{13})]; & N_{4t,t}^3 &= -\frac{1}{2} [N_{14}^5 (\mathcal{Q}_{13} - \mathcal{Q}_{23}) - N_{13}^5 (\mathcal{Q}_{23} + \mathcal{Q}_{13})]; \\ N_{3t,t}^1 &= -N_{4t,t}^2 = \frac{1}{2} [N_{14}^5 \mathcal{Q}_{12} - N_{13}^5 \mathcal{Q}_{12}]; & N_{3t,t}^4 &= \frac{1}{2} [N_{14}^5 (\mathcal{Q}_{42} - \mathcal{Q}_{14}) - N_{13}^5 (\mathcal{Q}_{14} + \mathcal{Q}_{42})]; \\ N_{4t,t}^1 &= N_{3t,t}^2 = -\frac{1}{2} [N_{13}^5 \mathcal{Q}_{12} + N_{14}^5 \mathcal{Q}_{12}]; & N_{4t,t}^4 &= -\frac{1}{2} [N_{13}^5 (\mathcal{Q}_{42} - \mathcal{Q}_{14}) + N_{14}^5 (\mathcal{Q}_{14} + \mathcal{Q}_{42})]; \\ N_{1t,t}^2 &= \frac{1}{2} [N_{14}^5 (\mathcal{Q}_{42} - \mathcal{Q}_{23}) - N_{13}^5 (\mathcal{Q}_{23} + \mathcal{Q}_{42})]; & N_{1t,t}^5 &= -N_{2t,t}^6 = -t(\Gamma)_3 - 2t(\mathcal{Q}_{14} + \mathcal{Q}_{23})_4; \\ N_{2t,t}^2 &= -\frac{1}{2} [N_{13}^5 (\mathcal{Q}_{42} - \mathcal{Q}_{23}) + N_{14}^5 (\mathcal{Q}_{23} + \mathcal{Q}_{42})]; & N_{2t,t}^5 &= N_{1t,t}^6 = t(\Gamma)_4 - 2t(\mathcal{Q}_{14} + \mathcal{Q}_{23})_3; \\ N_{1t,t}^3 &= -N_{2t,t}^4 = \frac{1}{2} (N_{13}^5 \mathcal{Q}_{34} - N_{14}^5 \mathcal{Q}_{34}); & N_{3t,t}^5 &= -N_{4t,t}^6 = t(\Gamma)_1 + 2t(\mathcal{Q}_{14} + \mathcal{Q}_{23})_2; \\ N_{2t,t}^3 &= N_{1t,t}^4 = \frac{1}{2} (N_{13}^5 \mathcal{Q}_{34} + N_{14}^5 \mathcal{Q}_{34}); & N_{4t,t}^5 &= N_{3t,t}^6 = -t(\Gamma)_2 - 2t(\mathcal{Q}_{14} + \mathcal{Q}_{23})_1; \\ & & N_{5t,t}^p &= N_{6t,t}^p = 0. \end{aligned}$$

The components of  $N_{\mathbf{J}}$  are

$$(E.6) \quad \begin{aligned} N_{13}^5 &= -t(\mathcal{Q}_{13} + \mathcal{Q}_{42} + \mathcal{Q}_{14} + \mathcal{Q}_{23}) = -2t(A_{22} + A_{33}); \\ N_{35}^1 &= -\frac{2}{t}; \\ N_{13}^5 &= -N_{24}^5 = N_{14}^6 = N_{23}^6; \\ N_{35}^1 &= -N_{15}^3 = -N_{16}^4 = N_{25}^4 = -N_{26}^3 = N_{36}^2 = -N_{45}^2 = N_{46}^1; \end{aligned}$$

For the sake of simplicity, let

$$\Sigma := \mathcal{Q}_{13} + \mathcal{Q}_{42} + \mathcal{Q}_{14} + \mathcal{Q}_{23} = 2(A_{22} + A_{33}).$$

Therefore, the components of  $\nabla N_{\mathbf{J}}$  are:

$$(E.7) \quad \begin{aligned} N_{13,a}^5 &= -t(\Sigma)_{,a}; & N_{26,s}^3 &= N_{36,s}^2 = N_{45,s}^2 = N_{46,s}^1 = 0; N_{12,s}^1 = 0; \\ N_{13,5}^5 &= 2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{13,a}^1 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{1a}; \\ N_{13,6}^5 &= -2(\mathcal{Q}_{12} + \mathcal{Q}_{34}); & N_{13,5}^1 &= N_{13,6}^1 = 0; \\ N_{13,s}^5 &= -N_{24,s}^5 = N_{14,s}^6 = N_{23,s}^6 = 0; & N_{14,a}^1 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{1a}; \\ N_{13,5}^1 &= N_{15,s}^3 = N_{16,s}^4 = N_{25,s}^4 = 0; & & \end{aligned}$$

$$\begin{aligned}
\mathbb{N}_{14,5}^1 &= \mathbb{N}_{14,6}^1 = 0; & \mathbb{N}_{24,5}^2 &= \mathbb{N}_{24,6}^2 = 0; \\
\mathbb{N}_{15,s}^1 &= \mathbb{N}_{16,s}^1 = 0; & \mathbb{N}_{25,s}^2 &= \mathbb{N}_{26,s}^2 = 0; \\
\mathbb{N}_{23,a}^1 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{1a} - \frac{2}{t^2} \mathcal{Q}_{2a} \right); & \mathbb{N}_{34,a}^2 &= \mathcal{Q}_{3a} + \mathcal{Q}_{4a}; \\
\mathbb{N}_{23,5}^1 &= \mathbb{N}_{23,6}^1 = 0; & \mathbb{N}_{34,5}^2 &= \mathbb{N}_{34,6}^2 = 0; \\
\mathbb{N}_{24,a}^1 &= -\frac{t^2}{2} \left( \Sigma \mathcal{Q}_{1a} + \frac{2}{t^2} \mathcal{Q}_{2a} \right); & \mathbb{N}_{35,s}^2 &= \mathbb{N}_{46,s}^2 = -\mathbb{N}_{36,s}^1; \\
\mathbb{N}_{24,5}^1 &= \mathbb{N}_{24,6}^1 = 0; & \mathbb{N}_{56,s}^2 &= -\mathbb{N}_{34,s}^2; \\
\mathbb{N}_{25,a}^1 &= 0; & \mathbb{N}_{12,a}^3 &= \mathcal{Q}_{1a} - \mathcal{Q}_{2a}; \\
\mathbb{N}_{25,5}^1 &= \mathcal{Q}_{14} + \mathcal{Q}_{23}; & \mathbb{N}_{12,5}^3 &= \mathbb{N}_{12,6}^3 = 0; \\
\mathbb{N}_{25,6}^1 &= (\mathcal{Q}_{14} + \mathcal{Q}_{23}) - \frac{2}{t^2}; & \mathbb{N}_{13,a}^3 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{3a}; \\
\mathbb{N}_{26,a}^1 &= 0; & \mathbb{N}_{13,5}^3 &= \mathbb{N}_{13,6}^3 = 0; \\
\mathbb{N}_{26,5}^1 &= \frac{2}{t^2} - (\mathcal{Q}_{13} + \mathcal{Q}_{42}); & \mathbb{N}_{14,a}^3 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{3a} - \frac{2}{t^2} \mathcal{Q}_{4a} \right); \\
\mathbb{N}_{26,6}^1 &= -(\mathcal{Q}_{13} + \mathcal{Q}_{42}) = -\mathbb{N}_{25,5}^1; & \mathbb{N}_{14,5}^3 &= \mathbb{N}_{14,6}^3 = 0; \\
\mathbb{N}_{34,a}^1 &= \mathcal{Q}_{4a} - \mathcal{Q}_{3a}; & \mathbb{N}_{16,s}^3 &= -\mathbb{N}_{36,s}^1; \\
\mathbb{N}_{34,5}^1 &= \mathbb{N}_{34,6}^1 = 0; & \mathbb{N}_{23,a}^3 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{3a}; \\
\mathbb{N}_{36,a}^1 &= 0; & \mathbb{N}_{23,5}^3 &= \mathbb{N}_{23,6}^3 = 0; \\
\mathbb{N}_{36,5}^1 &= \mathcal{Q}_{12} + \mathcal{Q}_{34}; & \mathbb{N}_{24,a}^3 &= -\frac{t^2}{2} \left( \Sigma \mathcal{Q}_{3a} + \frac{2}{t^2} \mathcal{Q}_{4a} \right); \\
\mathbb{N}_{36,6}^1 &= \mathcal{Q}_{12} + \mathcal{Q}_{34}; & \mathbb{N}_{24,5}^3 &= \mathbb{N}_{24,6}^3 = 0; \\
\mathbb{N}_{45,s}^1 &= -\mathbb{N}_{36,s}^1; & \mathbb{N}_{25,s}^3 &= \mathbb{N}_{36,s}^1; \\
\mathbb{N}_{46,s}^1 &= 0; & \mathbb{N}_{34,s}^3 &= 0; \\
\mathbb{N}_{56,s}^1 &= -\mathbb{N}_{34,s}^1; & \mathbb{N}_{35,s}^3 &= \mathbb{N}_{36,s}^3 = 0; \\
\mathbb{N}_{12,s}^2 &= 0; & \mathbb{N}_{45,s}^3 &= \mathbb{N}_{25,s}^1; \\
\mathbb{N}_{13,a}^2 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{2a} - \frac{2}{t^2} \mathcal{Q}_{1a} \right); & \mathbb{N}_{46,s}^3 &= \mathbb{N}_{26,s}^1; \\
\mathbb{N}_{13,5}^2 &= \mathbb{N}_{13,6}^2 = 0; & \mathbb{N}_{56,s}^3 &= -\mathbb{N}_{12,s}^3; \\
\mathbb{N}_{14,a}^2 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{2a} + \frac{2}{t^2} \mathcal{Q}_{1a} \right); & \mathbb{N}_{12,a}^4 &= -(\mathcal{Q}_{1a} + \mathcal{Q}_{2a}); \\
\mathbb{N}_{14,5}^2 &= \mathbb{N}_{14,6}^2 = 0; & \mathbb{N}_{12,5}^4 &= \mathbb{N}_{12,6}^4 = 0; \\
\mathbb{N}_{15,s}^2 &= -\mathbb{N}_{25,s}^1; & \mathbb{N}_{13,a}^4 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{4a} - \frac{2}{t^2} \mathcal{Q}_{3a} \right); \\
\mathbb{N}_{16,s}^2 &= -\mathbb{N}_{26,s}^1; & \mathbb{N}_{13,5}^4 &= \mathbb{N}_{13,6}^4 = 0; \\
\mathbb{N}_{23,a}^2 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{2a}; & \mathbb{N}_{14,a}^4 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{4a}; \\
\mathbb{N}_{23,5}^2 &= \mathbb{N}_{23,6}^2 = 0; & \mathbb{N}_{14,5}^4 &= \mathbb{N}_{14,6}^4 = 0; \\
\mathbb{N}_{24,a}^2 &= \frac{t^2}{2} \left( \frac{2}{t^2} - \Sigma \right) \mathcal{Q}_{2a}; & & 
\end{aligned}$$

$$\begin{aligned}
\mathbb{N}_{15,s}^4 &= \mathbb{N}_{36,s}^1; & \mathbb{N}_{25,s}^5 &= -\mathbb{N}_{24,s}^4; \\
\mathbb{N}_{23,a}^4 &= \frac{t^2}{2} \left( \Sigma \mathcal{Q}_{4a} + \frac{2}{t^2} \mathcal{Q}_{3a} \right); & \mathbb{N}_{26,s}^5 &= -\mathbb{N}_{23,s}^4; \\
\mathbb{N}_{23,5}^4 &= \mathbb{N}_{23,6}^4 = 0; & \mathbb{N}_{34,s}^5 &= \mathbb{N}_{12,s}^5; \\
\mathbb{N}_{24,a}^4 &= \frac{t^2}{2} \left( \frac{2}{t^2} - \Sigma \right) \mathcal{Q}_{4a}; & \mathbb{N}_{35,s}^5 &= \mathbb{N}_{13,s}^1; \\
\mathbb{N}_{24,5}^4 &= \mathbb{N}_{24,6}^4 = 0; & \mathbb{N}_{36,s}^5 &= \mathbb{N}_{23,s}^1; \\
\mathbb{N}_{26,s}^4 &= \mathbb{N}_{36,s}^1; & \mathbb{N}_{45,s}^5 &= \mathbb{N}_{24,s}^2; \\
\mathbb{N}_{34,s}^4 &= 0; & \mathbb{N}_{46,s}^5 &= -\mathbb{N}_{14,s}^2; \\
\mathbb{N}_{35,s}^4 &= -\mathbb{N}_{25,s}^1; & \mathbb{N}_{56,s}^5 &= 0; \\
\mathbb{N}_{36,s}^4 &= -\mathbb{N}_{26,s}^1; & \mathbb{N}_{12,a}^6 &= 0; \\
\mathbb{N}_{45,s}^4 &= \mathbb{N}_{46,s}^4 = 0; & \mathbb{N}_{12,5}^6 &= \frac{t^2}{2} \Sigma \left[ \frac{2}{t^2} - (\mathcal{Q}_{13} + \mathcal{Q}_{42}) \right]; \\
\mathbb{N}_{56,s}^4 &= -\mathbb{N}_{12,s}^4; & \mathbb{N}_{12,6}^6 &= -\frac{t^2}{2} \Sigma (\mathcal{Q}_{13} + \mathcal{Q}_{42}) = -\mathbb{N}_{12,5}^5; \\
\mathbb{N}_{12,a}^5 &= 0; & \mathbb{N}_{13,s}^6 &= -\mathbb{N}_{14,s}^5; \\
\mathbb{N}_{12,5}^5 &= \frac{t^2}{2} \Sigma (\mathcal{Q}_{14} + \mathcal{Q}_{23}); & \mathbb{N}_{15,s}^6 &= -\mathbb{N}_{13,s}^4; \\
\mathbb{N}_{12,6}^5 &= \frac{t^2}{2} \Sigma \left[ (\mathcal{Q}_{14} + \mathcal{Q}_{23}) - \frac{2}{t^2} \right]; & \mathbb{N}_{16,s}^6 &= -\mathbb{N}_{14,s}^4; \\
\mathbb{N}_{14,a}^5 &= 0; & \mathbb{N}_{24,s}^6 &= \mathbb{N}_{14,s}^5; \\
\mathbb{N}_{14,5}^5 &= -\frac{t^2}{2} \Sigma (\mathcal{Q}_{12} + \mathcal{Q}_{34}); & \mathbb{N}_{25,s}^6 &= \mathbb{N}_{24,s}^3; \\
\mathbb{N}_{14,6}^5 &= -\frac{t^2}{2} \Sigma (\mathcal{Q}_{12} + \mathcal{Q}_{34}); & \mathbb{N}_{26,s}^6 &= -\mathbb{N}_{23,s}^3; \\
\mathbb{N}_{15,s}^5 &= -\mathbb{N}_{13,s}^3; & \mathbb{N}_{34,s}^6 &= \mathbb{N}_{12,s}^6; \\
\mathbb{N}_{16,s}^5 &= -\mathbb{N}_{14,s}^3; & \mathbb{N}_{35,s}^6 &= \mathbb{N}_{13,s}^2; \\
\mathbb{N}_{23,s}^5 &= \mathbb{N}_{14,s}^5; & \mathbb{N}_{36,s}^6 &= \mathbb{N}_{23,s}^2; \\
& & \mathbb{N}_{45,s}^6 &= -\mathbb{N}_{24,s}^1; \\
& & \mathbb{N}_{46,s}^6 &= \mathbb{N}_{14,s}^1; \\
& & \mathbb{N}_{56,s}^6 &= 0.
\end{aligned}$$

As far as the two divergences

$$\operatorname{div} N_{\mathbf{J}} = \mathbb{N}_{pq,t}^t \theta^p \otimes \theta^q, \quad \overline{\operatorname{div}} N_{\mathbf{J}} = \mathbb{N}_{pt,t}^r \theta^p \otimes e_r.$$

are concerned, we have, for the first one,

$$\begin{aligned}
\text{(E.8)} \quad \mathbb{N}_{13,t}^t &= \frac{t^2}{2} \left( \Sigma + \frac{2}{t^2} \right) (\mathcal{Q}_{12} + \mathcal{Q}_{34}) = -\mathbb{N}_{24,t}^t; \\
\mathbb{N}_{14,t}^t &= -\frac{t^2}{2} \left( \Sigma + \frac{2}{t^2} \right) (\mathcal{Q}_{12} + \mathcal{Q}_{34}) = \mathbb{N}_{23,t}^t;
\end{aligned}$$

all other components are zero. For the second one, we obtain

$$\begin{aligned}
(E.9) \quad \mathbb{N}_{1t,t}^1 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{13} + \mathcal{Q}_{14}); & \mathbb{N}_{1t,t}^4 &= \mathbb{N}_{2t,t}^3; \\
\mathbb{N}_{2t,t}^1 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{13} - \mathcal{Q}_{14}); & \mathbb{N}_{2t,t}^4 &= -\mathbb{N}_{1t,t}^3; \\
\mathbb{N}_{3t,t}^1 &= \frac{t^2}{2} \left( \frac{2}{t^2} - \Sigma \right) \mathcal{Q}_{12}; & \mathbb{N}_{3t,t}^4 &= \mathbb{N}_{4t,t}^3; \\
\mathbb{N}_{4t,t}^1 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{12}; & \mathbb{N}_{4t,t}^4 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{42} + \mathcal{Q}_{14}); \\
\mathbb{N}_{5t,t}^1 &= \mathbb{N}_{6t,t}^1 = 0; & \mathbb{N}_{5t,t}^4 &= \mathbb{N}_{6t,t}^4 = 0; \\
\mathbb{N}_{1t,t}^2 &= \mathbb{N}_{2t,t}^1; & \mathbb{N}_{1t,t}^5 &= -t(\Sigma)_{,3}; \\
\mathbb{N}_{2t,t}^2 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{42} + \mathcal{Q}_{23}); & \mathbb{N}_{2t,t}^5 &= t(\Sigma)_{,4}; \\
\mathbb{N}_{3t,t}^2 &= \mathbb{N}_{4t,t}^1; & \mathbb{N}_{3t,t}^5 &= t(\Sigma)_{,1}; \\
\mathbb{N}_{4t,t}^2 &= -\mathbb{N}_{3t,t}^1; & \mathbb{N}_{4t,t}^5 &= -t(\Sigma)_{,2}; \\
\mathbb{N}_{5t,t}^2 &= \mathbb{N}_{6t,t}^2 = 0; & \mathbb{N}_{5t,t}^5 &= t^2 \left( \frac{2}{t^2} - \Sigma \right) (\mathcal{Q}_{13} + \mathcal{Q}_{42}); \\
\mathbb{N}_{1t,t}^3 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) \mathcal{Q}_{34}; & \mathbb{N}_{6t,t}^5 &= t^2 \left( \frac{2}{t^2} - \Sigma \right) (\mathcal{Q}_{14} + \mathcal{Q}_{23}); \\
\mathbb{N}_{2t,t}^3 &= \frac{t^2}{2} \left( \frac{2}{t^2} - \Sigma \right) \mathcal{Q}_{34}; & \mathbb{N}_{1t,t}^6 &= -\mathbb{N}_{2t,t}^5; \\
\mathbb{N}_{3t,t}^3 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{13} + \mathcal{Q}_{23}); & \mathbb{N}_{2t,t}^6 &= \mathbb{N}_{1t,t}^5; \\
\mathbb{N}_{4t,t}^3 &= \frac{t^2}{2} \left( \Sigma - \frac{2}{t^2} \right) (\mathcal{Q}_{13} - \mathcal{Q}_{23}); & \mathbb{N}_{3t,t}^6 &= -\mathbb{N}_{4t,t}^5; \\
\mathbb{N}_{5t,t}^3 &= \mathbb{N}_{6t,t}^3 = 0; & \mathbb{N}_{4t,t}^6 &= \mathbb{N}_{3t,t}^5; \\
& & \mathbb{N}_{5t,t}^6 &= \mathbb{N}_{6t,t}^5; \\
& & \mathbb{N}_{6t,t}^6 &= t^2 \left( \frac{2}{t^2} - \Sigma \right) (\mathcal{Q}_{14} + \mathcal{Q}_{23}).
\end{aligned}$$

#### APPENDIX F. ACKNOWLEDGEMENTS

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