SOME CANONICAL METRICS *VIA* AUBIN'S LOCAL DEFORMATIONS

G. Catino[1], D. Dameno [2], P. Mastrolia [3].

Abstract. In this paper, using special metric deformations introduced by Aubin, we construct Riemannian metrics satisfying non-vanishing conditions concerning the Weyl tensor, on every closed manifolds. In particular, in dimension four, we show that there are no topological obstructions for the existence of metrics with non-vanishing Bach tensor.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. It is well-known that its Riemann curvature tensor, Riem_g, admits the decomposition

$$\mathrm{Riem}_g = \mathrm{W}_g + \frac{1}{n-2}\,\mathrm{Ric}_g \owedge g - \frac{R_g}{2(n-1)(n-2)} g \owedge g,$$

where W_g, Ric_g, R_g are the Weyl tensor, the Ricci tensor and the scalar curvature of (M, g), respectively, and $\owedge$ denotes the Kulkarni-Nomizu product.

If we require that the curvature of (M, g) satisfies certain condition, several obstructions to the validity of these properties may occur: indeed, the topology of M may not allow the existence of such metrics. Famous examples of this relation between curvature and topology are given, for instance, by metrics with positive scalar curvature ([11], [12], [15], [17]) or by locally conformally flat metrics, which, for $n \geq 4$, are the ones with vanishing Weyl tensor ([4], [6], [13], [14]).

On the contrary, there are curvature conditions which can be realized on every Riemannian manifold (and we say that they are "non-obstructed"): for instance, Aubin ([3]) showed that, if M is closed and $n \geq 3$, there always exists a Riemannian metric g such that $R_g \equiv -1$; he also proved that, if M is closed and $n \geq 4$, there always exists a Riemannian metric g such that the Weyl tensor W_g never vanishes ([2], [3]). The first author generalized these results showing that, given a Riemannian manifold (M, g), for every $t \in \mathbb{R}$, there exists a Riemannian metric $\widetilde{g}$ such that the *scalar-Weyl curvature* $R_g + t|\mathrm{W}_g|_g \equiv -1$ on M ([7]); on the other hand, the first and the third authors, together with D. D. Monticelli and F. Punzo,

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[1]Politecnico di Milano, Italy. Email: giovanni.catino@polimi.it.

[2]Università degli Studi di Milano, Italy. Email: davide.dameno@unimi.it.

[3]Università degli Studi di Milano, Italy. Email: paolo.mastrolia@unimi.it.

used Aubin's result concerning the Weyl tensor to show the existence of *weak harmonic-Weyl* metrics on every closed Riemannian four-manifold ([9]). More precisely, these metrics arise as minimizers of the functional

$$g \longmapsto \mathfrak{D}(g) := \mathrm{Vol}_g(M)^{\frac{1}{2}} \int_M |\delta_g W_g|_g^2 \, dV_g$$

in the conformal class with non-vanishing Weyl tensor constructed by Aubin.

Our main task in this paper is to investigate other curvature conditions which can be imposed without any topological obstruction: in particular, we focus on some properties involving geometric tensors related to W_g on closed manifolds of dimension $n \geq 4$.

First, for the sake of completeness, we provide a detailed proof of Aubin's result (see Theorem 3.1). Then, we focus on the case $n = 4$: it is well-known that, on an oriented four- dimensional Riemannian manifold (M, g), the Hodge operator $\star$ induces a splitting of the bundle of 2-forms into two subbundles $\Lambda = \Lambda_+ \oplus \Lambda_-$, where $\Lambda_\pm$ is the eigenspace of $\star$ corresponding to the eigenvalue ± 1. This leads to a decomposition of the Weyl tensor into a *self-dual* and an *anti-self-dual* part; namely,

$$\mathrm{W_g} = \mathrm{W}_g^+ + \mathrm{W}_g^- .$$

Exploiting Aubin's deformation method, we are able to prove the following

Theorem 1.1. *Let M be a closed smooth manifold, with* $\dim M = 4$. *Then, there exists a Riemannian metric $\bar{g}$ such that*

$$|\,\mathrm{W}_{\bar{g}}^+\,|_{\bar{g}}^2 \equiv 1 \quad \text{on } M.$$

The same result holds for the anti-self-dual component $\mathrm{W}_{\bar{g}}^-$.

As a consequence, using the metric g_0 constructed in Theorem 1.1 and following the same strategy as in [9], it is immediate to prove the

Corollary 1.2. *On every smooth, closed four-manifold M, there exists a Riemannian metric g_0 such that, in its conformal class $[g_0]$, there exist* weak half harmonic Weyl metrics, *i.e. minimizers of the quadratic curvature functional*

$$g \longmapsto \mathfrak{D}^\pm(g) := \mathrm{Vol}_g(M)^{\frac{1}{2}} \int_M |\delta_g W_g^\pm|_g^2 \, dV_g.$$

(see also Remark 4 in [9]).

Moreover, we generalize this statement, showing a "mixed-type" condition:

Theorem 1.3. *Let (M, g) be a closed Riemannian manifold, with* $\dim M = 4$. *Then, for every $t \in \mathbb{R}$, there exists a Riemannian metric $\bar{g}_t$ such that*

$$|\,\mathrm{W}_{\bar{g}_t}^+ + t\,\mathrm{W}_{\bar{g}_t}^-\,|^2 \equiv 1 \quad \text{on } M.$$

In the subsequent sections, we focus on two other relevant geometric tensors: the *Cotton tensor* and the *Bach tensor*, which we denote as C_g and B_g, respectively (see Subsection 2.1 for the definitions and the main properties of these tensors).

First, we obtain a "non-obstructed" condition for C_g on a closed Riemannian manifold of dimension $n \geq 4$:

Theorem 1.4. *Let M be a closed smooth manifold of dimension $n \geq 4$. Then, there exists a metric $\widetilde{g}$ such that the Cotton tensor $\mathrm{C}_{\widetilde{g}}$ of $(M, \widetilde{g})$ vanishes only at finitely many points $p_1, ..., p_k \in M$.*

The final section of the paper is dedicated to the tensor B_g, which has many applications, for instance, in General Relativity ([5]). This tensor is especially relevant when $n = 4$: indeed, in this case B_g is also divergence-free and conformally covariant, i.e., given a conformal change $\widetilde{g} = e^{2u} g$ of g, the Bach tensor transforms as

$$e^{4u} \widetilde{B}_{ij} = B_{ij},$$

which, in global notation, means

$$e^{2u}\, \mathrm{B}_{\widetilde{g}} = \mathrm{B}_g$$

When $\mathrm{B}_g \equiv 0$, we say that (M, g) is *Bach-flat*: these metrics are critical points of the *Weyl functional*

$$g \longmapsto \mathcal{W}(g) := \int_M |\mathrm{W}_g|_g^2 dV_g,$$

which is a conformally invariant functional, playing an important role in the study of Einstein four-manifolds: indeed, Bach-flatness is a necessary condition for a metric g to be *conformally Einstein* (i.e., there exists a metric $\widetilde{g}$ in the conformal class $[g]$ such that $(M, \widetilde{g})$ is an Einstein manifold). We point out that, in general, this condition is not sufficient (see [1]): however, Derdziński [10] showed that Bach-flatness is a sufficient condition for positive definite Kähler four-manifolds and recently LeBrun ([16]) classified Bach-flat compact Kähler complex surfaces.

Although the existence of topological obstructions for Bach-flat metrics on Riemannian four-manifolds is an open problem, in this paper we provide an answer to the "opposite" question, i.e. if the topology of the manifold plays a role in the existence of metrics with never vanishing Bach tensor. More precisely, we exploit Aubin's construction in the four-dimensional case to obtain the following:

Theorem 1.5. *Let M be a smooth manifold with $\dim M = 4$. Then, there exists a Riemannian metric $\bar{g}$ such that*

$$|\,\mathrm{B}_{\bar{g}}\,|_{\bar{g}}^2 \equiv 1 \quad \textit{on } M.$$

2. Aubin's deformation

2.1. **Preliminaries.** The $(1,3)$-Riemann curvature tensor of a smooth Riemannian manifold (M^n, g) is defined by

$$\mathrm{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,.$$

Throughout the article, the Einstein convention of summing over the repeated indices will be adopted. In a local coordinate system the components of the $(1,3)$-Riemann curvature tensor are given by $R^l_{ijk}\frac{\partial}{\partial x^l} = \mathrm{R}\big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\big)\frac{\partial}{\partial x^i}$ and we denote by Riem_g its $(0,4)$ version with components by $R_{ijkl} = g_{im}R^m_{jkl}$. The Ricci tensor is obtained by the contraction $R_{ik} = g^{jl}R_{ijkl}$ and $R = g^{ik}R_{ik}$ will denote the scalar curvature (g^{ij} are the coefficient of the inverse of the metric g). As recalled in the Introduction, the *Weyl tensor* W_g is defined by the decomposition formula, in dimension $n \geq 3$,

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}\right) \\ &\quad + \frac{R}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)\,. \end{aligned} \tag{2.1}$$

The Weyl tensor shares the algebraic symmetries of the curvature tensor. Moreover, as it can be easily seen by the formula above, all of its contractions with the metric are zero, i.e. W is totally trace-free. In dimension three, W is identically zero on every Riemannian manifold, whereas, when $n \geq 4$, the vanishing of the Weyl tensor is a relevant condition, since it is equivalent to the local conformal flatness of (M^n, g). We also recall that in dimension $n = 3$, local conformal flatness is equivalent to the vanishing of the *Cotton tensor* C_g, whose local components are

$$C_{ijk} = R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}\big(R_k g_{ij} - R_j g_{ik}\big) = A_{ij,k} - A_{ik,j}\,; \tag{2.2}$$

here $R_{ij,k} = \nabla_k R_{ij}$ and $R_k = \nabla_k R$ denote, respectively, the components of the covariant derivative of the Ricci tensor and of the differential of the scalar curvature, and $A_{ij,k}$ denote the components of the covariant derivative of the *Schouten tensor*

$$\mathrm{A}_g = \mathrm{Ric}_g - \frac{R_g}{2(n-1)}g;$$

hence, the Cotton tensor represents the obstruction for A_g to be a Codazzi tensor (i.e., $(\nabla_X \mathrm{A})Y = (\nabla_Y \mathrm{A})X$ for every pair of vector fields X, Y). By direct computation, we can see that C_g satisfies the symmetries

$$C_{ijk} = -C_{ikj}, \qquad C_{ijk} + C_{jki} + C_{kij} = 0\,, \tag{2.3}$$

moreover it is totally trace-free,

$$g^{ij}C_{ijk} = g^{ik}C_{ijk} = g^{jk}C_{ijk} = 0\,, \tag{2.4}$$

by its skew–symmetry and Schur lemma. We also recall that, for $n \geq 4$, the Cotton tensor can be defined as one of the possible divergences of the Weyl tensor:

$$C_{ijk} = \left(\frac{n-2}{n-3}\right) W_{tikj,t} = -\left(\frac{n-2}{n-3}\right) W_{tijk,t} = -\frac{n-2}{n-3}(\delta W)_{ijk}\,. \tag{2.5}$$

A computation shows that the two definitions coincide (see e.g. [8]).

The *Bach tensor* B_g of (M, g) is defined, in components, as

$$B_{ij} := \frac{1}{n-2}\Big(g^{ks} C_{jik,s} + g^{ks} g^{lt} R_{kl} W_{isjt}\Big). \tag{2.6}$$

It is immediate to show that B_g is a traceless tensor; moreover, since $(n-3)W_{jkil,lk} = (n-2)C_{ijk,k}$, exploiting the second covariant derivative commutation formulas, it can be shown that B_g is symmetric (see, for instance, [8, Lemma 2.8]). Also, recall that, if $n = 4$, the Bach tensor acquires two additional features: it is divergence-free and conformally covariant.

2.2. **Aubin's local deformations.** Let us introduce the following deformation of the metric g:

$$\widetilde{g} = g + d\phi \otimes d\phi, \tag{2.7}$$

where $\phi \in C^\infty(M)$. We denote the Weyl tensor of $(M, \widetilde{g})$ as $\mathrm{W}_{\widetilde{g}}$. If U is a local chart of M and $x_1, ..., x_n$ are local coordinates on U, the local components of the $(0, 4)$-version of $\mathrm{W}_{\widetilde{g}}$,

$\widetilde{W}_{ijkt}$, are given by the following expression (see also [8], Chapter 2):

(2.8)

$$\begin{aligned}
\widetilde{W}_{ijkt} = W_{ijkt} &+ \frac{1}{w}(\phi_{ik}\phi_{jt} - \phi_{it}\phi_{jk}) + \\
&+ \frac{1}{n-2}(R_{ik}\phi_j\phi_t - R_{it}\phi_j\phi_k + R_{jt}\phi_i\phi_k - R_{jk}\phi_i\phi_t) \\
&+ \frac{S}{(n-1)(n-2)}(g_{ik}\phi_j\phi_t - g_{it}\phi_j\phi_k + g_{jt}\phi_i\phi_k - g_{jk}\phi_i\phi_t) + \\
&+ \frac{\phi^p\phi^q}{w(n-2)}[R_{ipkq}(g_{jt} + \phi_j\phi_t) - R_{iptq}(g_{jk} + \phi_j\phi_k) + R_{jptq}(g_{ik} + \phi_i\phi_k) - R_{jpkq}(g_{it} - \phi_i\phi_t)] + \\
&- \frac{2R_{pq}\phi^p\phi^q}{w(n-1)(n-2)}[g_{ik}g_{jt} - g_{it}g_{jk} + g_{ik}\phi_j\phi_t - g_{it}\phi_j\phi_k + g_{jt}\phi_i\phi_k - g_{jk}\phi_i\phi_t] + \\
&- \frac{1}{w(n-2)}\{[(\Delta\phi)\phi_{ik} - \phi_{ip}\phi^p_k](g_{jt} + \phi_j\phi_t) - [(\Delta\phi)\phi_{it} - \phi_{ip}\phi^p_t](g_{jk} + \phi_j\phi_k)\} + \\
&- \frac{1}{w(n-2)}\{[(\Delta\phi)\phi_{jt} - \phi_{jp}\phi^p_t](g_{ik} + \phi_i\phi_k) - [(\Delta\phi)\phi_{jk} - \phi_{jp}\phi^p_k](g_{it} + \phi_i\phi_t)\} + \\
&+ \frac{1}{w(n-1)(n-2)}\Big[(\Delta\phi)^2 - |\mathrm{Hess}(\phi)|^2\Big][g_{ik}g_{jt} - g_{it}g_{jk} + g_{ik}\phi_j\phi_t - g_{it}\phi_j\phi_k + g_{jt}\phi_i\phi_k - g_{jk}\phi_i\phi_t] + \\
&+ \frac{\phi^p\phi^q}{w^2(n-2)}[(\phi_{ik}\phi_{pq} - \phi_{ip}\phi_{kq})(g_{jt} + \phi_j\phi_t) - (\phi_{it}\phi_{pq} - \phi_{ip}\phi_{tq})(g_{jk} + \phi_j\phi_k)] + \\
&+ \frac{\phi^p\phi^q}{w^2(n-2)}[(\phi_{jt}\phi_{pq} - \phi_{jp}\phi_{tq})(g_{ik} + \phi_i\phi_k) - (\phi_{jk}\phi_{pq} - \phi_{jp}\phi_{kq})(g_{it} + \phi_i\phi_t)] + \\
&- \frac{2}{w^2(n-1)(n-2)}[(\Delta\phi)\phi^p\phi^q\phi_{pq} - \phi^p\phi_{pq}\phi^{qr}\phi_r](g_{ik}g_{jt} - g_{it}g_{jk}) + \\
&- \frac{2}{w^2(n-1)(n-2)}[(\Delta\phi)\phi^p\phi^q\phi_{pq} - \phi^p\phi_{pq}\phi^{qr}\phi_r](g_{ik}\phi_j\phi_t - g_{it}\phi_j\phi_k + g_{jt}\phi_i\phi_k - g_{jk}\phi_i\phi_t),
\end{aligned}$$

where $w = 1 + |\nabla\phi|^2$ and

$$\begin{aligned}
\phi_i &= \partial_i\phi = \frac{\partial\phi}{\partial x_i}, \\
\phi^i &= g^{ip}\phi_p, \\
\phi_{ij} &= \partial_i\partial_j\phi - \Gamma^p_{ij}\phi_p, \\
\phi^i_j &= g^{ip}\phi_{pj} = \partial_j\phi^i + \phi^p\Gamma^i_{pj}, \\
\phi^{ij} &= g^{ip}\phi^j_p.
\end{aligned}$$

3. A detailed proof of Aubin's result

In this section we give a complete proof of Aubin's result (see [2] and [3]), i.e. we prove the following

Theorem 3.1 (Aubin ([2], [3]))**.** *On every smooth manifold of dimension at least 4 there exists a Riemannian metric g whose Weyl tensor never identically vanishes.*

Proof. Let g any Riemannian metric on M and consider the metric $\widetilde{g}$ given by (2.7). Let $p_0 \in M$ be such that W_g vanishes in p_0 and B_r an open ball of radius r and centered in p_0. Moreover, let us consider normal coordinates $x_1, ..., x_n$ on B_r such that $p_0 = (0, ..., 0)$. Thus, at p_0 we have

$$g_{ij} = g^{ij} = \delta_{ij}, \quad \phi_i = \phi^i, \quad \phi_{ij} = \partial_i \partial_j \phi = \phi^i_j = \phi^{ij}$$

From now on, we denote the local components of W_g ($\mathrm{W}_{\widetilde{g}}$, resp.) on B_r as W_{ijkl} ($\widetilde{W}_{ijkl}$, resp.).

We construct the function ϕ as follows: let $f \in C^\infty([0, +\infty))$ such that

$$\begin{cases} f(y) = 0, \text{ if } y \geq 1 \\ f'(y) > 0, f''(y) < 0, \text{ if } 0 \leq y < 1 \end{cases}.$$

For instance, we may choose

$$f(x) := \begin{cases} -e^{\left(\frac{b}{1-x}\right)} & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}, \tag{3.1}$$

where $b > 0$ is sufficiently large. Now, let $\lambda, \alpha_1, ..., \alpha_n$ be $n+1$ real numbers in the interval $[1, 2]$ and let

$$\phi = \frac{\lambda r^2}{2} f\left(\frac{\alpha_1 x_1^2 + ... + \alpha_n x_n^2}{r^2}\right). \tag{3.2}$$

By definition, $\phi \in C^\infty(B_r)$ and it vanishes outside B_r. Indeed, if $x_1, ..., x_n$ are such that $\alpha_1 x_1^2 + ... \alpha_n x_n^2 < r^2$, then, since $\alpha_i \in [1, 2]$ for every i,

$$\sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n \alpha_i x_i^2 < r^2,$$

i.e. $p = (x_1, ..., x_n) \in B_r$; in particular, this means that ϕ vanishes outside B_r.

The partial derivatives of ϕ satisfy

$$\phi_i = \lambda f' \cdot a_i x_i = O(r), \tag{3.3}$$

for a sufficiently small radius r. Since we chose a system of normal coordinates, for small radii the second partial derivatives of ϕ satisfy

$$\phi_{ij} = \lambda\Big(\alpha_i f' \delta_{ij} + 2\frac{\alpha_i \alpha_j}{r^2} x_i x_j f''\Big) = O(1). \tag{3.4}$$

Now, let us consider equation (2.8); for sufficiently small radii r, we can rewrite the expression as

$$
\begin{aligned}
\widetilde{W}_{ijkl} &= W_{ijkl} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + \\
&\quad - \frac{1}{n-2}\Delta\phi(\phi_{ik}\delta_{jl} - \phi_{il}\delta_{jk} + \phi_{jl}\delta_{ik} - \phi_{jk}\delta_{il}) + \\
&\quad + \frac{1}{n-2}(\phi_{ip}\phi_{pk}\delta_{jl} - \phi_{ip}\phi_{pl}\delta_{jk} + \phi_{jp}\phi_{pl}\delta_{ik} - \phi_{jp}\phi_{pk}\delta_{il}) \\
&\quad + \frac{1}{(n-1)(n-2)}\Big[(\Delta\phi)^2 - |\mathrm{Hess}(\phi)|^2\Big](\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + r^2\theta_{ijkl},
\end{aligned}
\tag{3.5}
$$

where $r^2\theta_{ijkl}$ contains all the terms in (2.8) whose order is the same as r or higher (i.e., all the terms involving the derivatives ϕ_i) . Thus, we informally distinguish a "principal part" and a "remainder" in the expression of the components $\widetilde{W}_{ijkl}$. Then, the key of the proof is to show that the principal parts of the components $\widetilde{W}_{ijkl}$ cannot be simultaneously zero on B_r.

Now, let $i \neq j \neq k \neq l$; inserting (3.3) and (3.4) into (3.5), we obtain

$$
\begin{aligned}
\widetilde{W}_{ijij} &= W_{ijij} + \lambda^2[a_{ij}(f')^2 + b_{ij}f'f''] + r^2\theta_{ijij}; \\
\widetilde{W}_{ijik} &= W_{ijik} + \lambda^2 a_{ijk}f'f''x_jx_k + r^2\theta_{ijik}; \\
\widetilde{W}_{ijkl} &= W_{ijkl} + r^2\theta_{ijkl},
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
a_{ij} &= \frac{1}{n-2}\left[(n-4)\alpha_i\alpha_j - (\alpha_i+\alpha_j)\sum_{k\neq i,j}\alpha_k + \frac{2}{n-1}\sum_{k<l}\alpha_k\alpha_l\right]; \\
b_{ij} &= \frac{2}{(n-2)r^2}\left[(n-4)(\alpha_i x_i^2 + \alpha_j x_j^2)\alpha_i\alpha_j - (\alpha_i^2x_i^2 + \alpha_j^2x_j^2)\sum_{k\neq i,j}\alpha_k + \right. \\
&\quad \left. -(\alpha_i+\alpha_j)\sum_{k\neq i,j}\alpha_k^2x_k^2 + \frac{2}{n-1}\sum_{k=1}^{n}\alpha_k\left(\sum_{l\neq k}\alpha_l^2x_l^2\right)\right]; \\
a_{ijk} &= \frac{2\alpha_j\alpha_k}{(n-2)r^2}\left[(n-3)\alpha_i - \sum_{l\neq i,j,k}\alpha_l\right].
\end{aligned}
\tag{3.7}
$$

It is important to note that there exist suitable choices for $\alpha_1, ..., \alpha_n$ such that, for every $i \neq j \neq k$, a_{ij} and a_{ijk} never vanish on B_r (observe that a_{ij} and a_{ijk} are scalars, while b_{ij} is a polynomial of degree 2 in the variables $x_1, ..., x_n$ for every $i \neq j \neq k$). For instance, we may define

$$
\begin{cases}
(\alpha_1, ..., \alpha_n) = (2, 2, 1, 1, ..., 1), & \text{if } n > 4; \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, \frac{5}{4}, \frac{3}{2}, 2\right), & \text{if } n = 4.
\end{cases}
$$

A direct inspection of (3.7) shows that, with this choice, $a_{ij}, a_{ijk} \neq 0$.

Note that, for $n = 4$, $\alpha_i \neq \alpha_j$ if $i \neq j$. For $n > 4$, observe that a_{ij} and a_{ijk} can be seen as homogeneous polynomials in the n variables $\alpha_1, ..., \alpha_n$, therefore, in particular, they are smooth functions of these variables: hence, since we found a n-tuple $(\alpha_1, ..., \alpha_n)$ such that $a_{ij}, a_{ijk} \neq 0$, we know that there exist sufficiently small $\epsilon_1 \neq ... \neq \epsilon_n$, with $\epsilon_i > 0$ for every i, such that $a_{ij}, a_{ijk} \neq 0$ for

$$(\alpha'_1, ..., \alpha'_n) := (2 - \epsilon_1, 2 - \epsilon_2, 1 + \epsilon_3, 1 + \epsilon_4, ..., 1 + \epsilon_n)$$

and $\alpha'_i \neq \alpha'_j$ for $i \neq j$. Therefore, without loss of generality, we may assume that $\alpha_i \neq \alpha_j$ whenever $i \neq j$.

Now, we show that, for every $p \in B_r$, the Weyl tensor $\mathrm{W}_{\widetilde{g}}$, whose local components are defined in (3.6), does not identically vanish. Consequently, since M is closed, we can repeat the argument finitely-many times on M and, therefore, the Theorem will be proven.

Let us distinguish three cases.

Case 1 ($p = p_0$). By hypothesis, W_g vanishes in p and, since $p_0 = (0, ..., 0)$ in our local coordinates, by (3.6) we obtain

$$\begin{aligned} \widetilde{W}_{ijij} &= \lambda^2 a_{ij} (f')^2 + r^2 \theta_{ijij}; \\ \widetilde{W}_{ijik} &= r^2 \theta_{ijik} \end{aligned}$$

as $r \to 0$, since $a_{ij}, f', \lambda \neq 0$, we have that

$$|\mathrm{W}_{\widetilde{g}}|^2_{\widetilde{g}} \geq 2 \sum_{i<j} \widetilde{W}^2_{ijij} = (\lambda f')^4 \sum_{i<j} (a_{ij})^2 > 0.$$

Case 2 ($p \in B_{r/2} \setminus \{p_0\}$). Since p lies in the open ball of radius $r/2$ and centered in p_0, by Taylor's Theorem we have that

$$|\mathrm{W}_g| \leq C \cdot r + o(r^2), \text{ as } r \to 0.$$

Let us suppose $\widetilde{W}_{ijij} = \widetilde{W}_{ijik} = 0$ for every $i \neq j \neq k$. By (3.6), we can write

$$\begin{aligned} a_{ij} (f')^2 + b_{ij} f' f'' + o(r) &= 0; \\ a_{ijk} x_j x_k + o(r) &= 0. \end{aligned}$$

Letting $r \to 0$, the previous equations become

$$\begin{cases} a_{ij} (f')^2 + b_{ij} f' f'' = 0; & \text{(3.8a)} \\ a_{ijk} x_j x_k = 0. & \text{(3.8b)} \end{cases}$$

Note that we obtained an overdetermined system in the variables $x_1, ... x_n$: indeed, since $i \neq j \neq k$ and the coefficients a_{ijk} are symmetric with respect to the indices j and k, we have $n(n-1)/2$ equations of the form (3.8b). Moreover, the polynomials $a_{ij}(f')^2 + b_{ij} f' f''$ are symmetric with respect to i and j and a straightforward computation shows that

$$\sum_{i \neq j} a_{ij} = \sum_{i \neq j} b_{ij} = 0, \text{ for every } j$$

(this can also be seen as a consequence of the fact that the Weyl tensor is traceless). Thus, we have

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

equations of the form (3.8a). Therefore, our system is made by

$$\frac{n(n-3)}{2} + \frac{n(n-1)}{2} = n(n-2)$$

independent equations, and $n(n-2) > n+1 > n$ for every $n \geq 4$.

Now, let us show that the system admits only the solution $x_1 = \cdots = x_n = 0$, which will lead to a contradiction, since $p \neq p_0$. Since $a_{ijk} \neq 0$, we obtain that $x_j x_k = 0$ for every $j \neq k$. This implies that at least $n-1$ coordinates of p must be zero; since $p \neq p_0$, there is exactly one coordinate x_i which is non-zero.

Let us consider $j \neq t \neq s \neq i$ (note that this is possible since $n \geq 4$): by $\widetilde{W}_{ijij} = \widetilde{W}_{itit} = \widetilde{W}_{isis} = 0$ we obtain

$$\begin{aligned}
0 &= \frac{1}{n-2}\left[(n-4)\alpha_i\alpha_j - (\alpha_i+\alpha_j)\sum_{k\neq i,j}\alpha_k + \frac{2}{n-1}\sum_{k<l}\alpha_k\alpha_l\right](f')^2 + \\
&\quad + \frac{2}{(n-2)r^2}\left[(n-4)\alpha_i^2\alpha_j x_i^2 - \alpha_i^2 x_i^2\sum_{k\neq i,j}\alpha_k + \frac{2}{n-1}\sum_{k=1}^{n}\alpha_k\left(\sum_{l\neq k}\alpha_l^2 x_l^2\right)\right] f'f'';\\
0 &= \frac{1}{n-2}\left[(n-4)\alpha_i\alpha_t - (\alpha_i+\alpha_t)\sum_{k\neq i,t}\alpha_k + \frac{2}{n-1}\sum_{k<l}\alpha_k\alpha_l\right](f')^2 + \\
&\quad + \frac{2}{(n-2)r^2}\left[(n-4)\alpha_i^2\alpha_t x_i^2 - \alpha_i^2 x_i^2\sum_{k\neq i,t}\alpha_k + \frac{2}{n-1}\sum_{k=1}^{n}\alpha_k\left(\sum_{l\neq k}\alpha_l^2 x_l^2\right)\right] f'f'';\\
0 &= \frac{1}{n-2}\left[(n-4)\alpha_i\alpha_s - (\alpha_i+\alpha_s)\sum_{k\neq i,s}\alpha_k + \frac{2}{n-1}\sum_{k<l}\alpha_k\alpha_l\right](f')^2 + \\
&\quad + \frac{2}{(n-2)r^2}\left[(n-4)\alpha_i^2\alpha_s x_i^2 - \alpha_i^2 x_i^2\sum_{k\neq i,s}\alpha_k + \frac{2}{n-1}\sum_{k=1}^{n}\alpha_k\left(\sum_{l\neq k}\alpha_l^2 x_l^2\right)\right] f'f'';
\end{aligned}$$

subtracting the second and the third equations from the first, since $\alpha_j \neq \alpha_t \neq \alpha_s$ and $f', f'' \neq 0$ on B_r, we get

$$\begin{aligned}
0 &= \left[(n-3)\alpha_i - \sum_{k\neq i,j,t}\alpha_k\right] f' + \frac{2}{r^2}(n-3)\alpha_i^2 x_i^2 f'',\\
0 &= \left[(n-3)\alpha_i - \sum_{k\neq i,j,s}\alpha_k\right] f' + \frac{2}{r^2}(n-3)\alpha_i^2 x_i^2 f''.
\end{aligned}$$

It is immediate to observe that these two equations hold simultaneously if and only if

$$\sum_{k\neq i,j,t} \alpha_k = \sum_{k\neq i,j,s} \alpha_k \quad \Leftrightarrow \quad \alpha_s = \alpha_t,$$

which is impossible. Thus, not all the components of $\mathrm{W}_{\widetilde{g}}$ vanish at p.

Case 3 ($p \in B_r \setminus B_{\frac{r}{2}}$). Let us suppose again that $\widetilde{W}_{ijij} = \widetilde{W}_{ijik} = 0$ for every $i \neq j \neq k$. As $r \to 0$, the first two equations in (3.6) become

$$\begin{cases} W_{ijij} + \lambda^2(a_{ij}(f')^2 + b_{ij} f' f'') = 0; & \text{(3.9a)} \\ W_{ijik} + \lambda^2 a_{ijk} x_j x_k f' f'' = 0. & \text{(3.9b)} \end{cases}$$

If $W_{ijij} = W_{ijik} = 0$ at p, we get a contradiction by the conclusions of Case 2. Thus, let us suppose that $|W_g|_g^2 > 0$ at p: for instance, let $W_{ijik} \neq 0$ for some i, j, k. The equation $\widetilde{W}_{ijik} = 0$ allows us to compute λ:

$$\lambda^2 = -\frac{W_{ijik}}{a_{ijk} x_j x_k}.$$

This equation holds for every point whose coordinates are solutions of the system above; however, $\lambda \in [1, 2]$ appears as a free parameter in (3.2), therefore it is sufficient to choose $\lambda_1 \in [1, 2]$ such that $\lambda_1^2 \neq \lambda^2$ and repeat the argument of the proof to obtain a contradiction. Thus, $W_{ijik} = 0$. If, for instance, λ_1 is such that the equation

$$W_{i'j'i'k'} + \lambda_1^2 a_{i'j'k'} x_{j'} x_{k'} f' f'' = 0$$

holds for some $i' \neq j' \neq k'$, it is sufficient to choose $\lambda_2 \in [1, 2]$ such that $\lambda_2^2 \neq \lambda_1^2$ to get the same contradiction. Note that we can repeat the procedure for every equation of the system above.

Therefore, eventually choosing λ in (3.2) out of a finite set $\{\lambda_1, ..., \lambda_k\}$, we can conclude that the system holds if and only if $W_{ijij} = W_{ijik} = 0$ at p: however, by the argument of Case 2, this leads to a contradiction.

Hence, not all the components of the Weyl tensor $\mathrm{W}_{\widetilde{g}}$ vanish at p and this ends the proof. □

Remark 3.2. If $\left|\mathrm{W}_{\widetilde{g}}\right|_{\widetilde{g}} > 0$ for every point of M, then, operating the conformal change

$$\overline{g} := \left|\mathrm{W}_{\widetilde{g}}\right| \widetilde{g},$$

we obtain that the metric $\overline{g}$ is such that its Weyl tensor $\mathrm{W}_{\overline{g}}$ satisfies

$$|\mathrm{W}_{\overline{g}}|^2_{\overline{g}} \equiv 1 \text{ on } M.$$

4. Proof of Theorems 1.1 and 1.3

In this section we extend Aubin's result in dimension four to the self-dual and anti-self dual components of the Weyl tensor in order to prove Theorem 1.1.

Proof of Theorem 1.1. First, note that, by Remark 3.2, it is sufficient to show that there exists a Riemannian metric whose self-dual Weyl tensor never vanishes on M.

Let g any Riemannian metric on M and let again $p \in B_r$, where B_r is an open ball in M of radius r and centered in a point $p_0 \in M$. Moreover, let x_1, x_2, x_3, x_4 be normal coordinates on B_r such that $p_0 = (0,0,0,0)$. Let $\widetilde{g}$ be the metric defined in (2.7), with the same ϕ introduced in (3.2).

By definition

$$W_{ijkl} = W^+_{ijkl} + W^-_{ijkl};$$

moreover, it is not hard to show that, for every $i,j,k,l = 1,...,4$ such that $i \neq j$ and $k \neq l$, there exist indices k' and l' such that

$$W^\pm_{ijkl} = \pm W^\pm_{ijk'l'}.$$

This implies immediately that

$$W^\pm_{ijkl} = \frac{1}{2}(W_{ijkl} \pm W_{ijk'l'}).$$

Let us now focus on W^+_g. By (3.6) and (3.7), for $i \neq j$ one can easily obtain

$$\begin{aligned} \widetilde{W}^+_{ijij} &= \frac{1}{2}(\widetilde{W}_{ijij} + \widetilde{W}_{iji'j'}) = \\ &= \frac{1}{2}[W_{ijij} + W_{iji'j'} + \lambda^2(a_{ij}(f')^2 + b_{ij}f'f'') + r^2\theta_{ijij}] = \\ &= W^+_{ijij} + \frac{\lambda^2}{2}(a_{ij}(f')^2 + b_{ij}f'f'') + r^2\theta_{iji'j'} \end{aligned} \tag{4.1}$$

(note that $(i',j') = (k,l)$ are such that $i \neq j \neq k \neq l$). Analogously, for $i \neq j \neq k$, we obtain

$$\begin{aligned} \widetilde{W}^+_{ijik} &= \frac{1}{2}(\widetilde{W}_{ijik} + \widetilde{W}_{iji'k'}) = \\ &= \frac{1}{2}[W_{ijik} \pm W_{jijl} + \lambda^2(a_{ijk}x_jx_k \pm a_{jil}x_ix_l)f'f'' + r^2(\theta_{ijik} \pm \theta_{jijl})] = \\ &= W^+_{ijik} + \frac{\lambda^2}{2}(a_{ijk}x_jx_k \pm a_{jil}x_ix_l)f'f'' + r^2\theta_{ijik}. \end{aligned} \tag{4.2}$$

Here, $\pm$ appears in the equations since we may have $(i',k') = (l,j)$ or $(i',k') = (j,l)$.

Now, we are ready to prove the statement. Let us choose

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1, \frac{5}{4}, \frac{3}{2}, 2\right);$$

thus, an easy computation shows that

$$\begin{cases} a_{12} = \frac{5}{48} = a_{34} \\ a_{13} = -\frac{1}{48} = a_{24} \\ a_{14} = -\frac{1}{12} = a_{23} \end{cases} \quad \text{and} \quad \begin{cases} a_{123} = -\frac{15}{8r^2}, \ a_{214} = -\frac{1}{2r^2} \\ a_{124} = -\frac{5}{4r^2}, \ a_{213} = -\frac{9}{8r^2} \\ a_{134} = -\frac{3}{4r^2}, \ a_{312} = -\frac{5}{8r^2} \end{cases} . \tag{4.3}$$

We recall that

$$\sum_{i \neq j} a_{ij} = 0 \text{ for every } j \text{ and } \sum_{i \neq j,k} a_{ijk} = 0 \text{ for every } j \neq k.$$

As before, we distinguish three cases.

Case 1 ($p = p_0$). As we did for Aubin's result, since $a_{ij} \neq 0$ for every $i \neq j$, by (4.1) and (3.7) we have

$$\left|\mathrm{W}^+{}_{\tilde{g}}\right|^2_{\tilde{g}} \geq 2 \sum_{i<j} (\widetilde{W}^+_{ijij})^2 = (\lambda f')^4 \sum_{i<j} (a_{ij})^2 > 0.$$

Case 2 ($p \in B_{r/2} \setminus \{p_0\}$). We can apply again Taylor's Theorem to conclude that

$$\left|\mathrm{W}^+_g\right| \leq C \cdot r + o(r^2), \ \text{ as } r \to 0.$$

Let us suppose $\widetilde{W}^+_{ijij} = \widetilde{W}^+_{ijik} = 0$ for every $i \neq j \neq k$. By (4.2), letting $r \to 0$ we have

$$a_{ijk} x_j x_k \pm a_{jil} x_i x_l = 0.$$

More explicitly, we obtain the system

$$\begin{cases} a_{123} x_2 x_3 + a_{214} x_1 x_4 = 0 \\ a_{124} x_2 x_4 - a_{213} x_1 x_3 = 0 \\ a_{134} x_3 x_4 + a_{312} x_1 x_2 = 0 \end{cases} ;$$

by (4.3), the system becomes

$$\begin{cases} 4 x_1 x_4 = -15 x_2 x_3 \\ 9 x_1 x_3 = 10 x_2 x_4 \\ 5 x_1 x_2 = -6 x_3 x_4 \end{cases} .$$

If $x_i \neq 0$ for every $i = 1, 2, 3, 4$, a straightforward computation shows that the system does not admit any real solution: therefore, the components $\widetilde{W}^+_{ijik}$ cannot simultaneously vanish. Thus, without loss of generality, we may suppose $x_4 = 0$. This implies immediately that two out of the three remaining variables must be zero. Let us suppose that $x_2 = x_3 = x_4 = 0$ and $x_1 \neq 0$ (the other cases are analogous). By $\widetilde{W}^+_{ijij} = 0$, letting $r \to 0$, (4.1) implies that

$$a_{ij} (f')^2 + b_{ij} f' f'' = 0.$$

However, since using (4.3) and (3.7) one has

$$a_{13} (f')^2 + b_{13} f' f'' = 0 \quad \Longrightarrow \quad x_1^2 = \frac{r^2}{4} \cdot \frac{f'}{f''},$$

we get a contradiction, since, by definition of f, the ratio f'/f'' is negative on B_r.

Case 3 ($p \in B_r \setminus B_{\frac{r}{2}}$). As before, let us suppose that $\widetilde{W}^+_{ijij} = \widetilde{W}^+_{ijik} = 0$ for every $i \neq j \neq k$. As $r \to 0$, by (4.1) and (4.2) we obtain the system

$$\begin{cases} W^+_{ijij} + \frac{\lambda^2}{2}(a_{ij}(f')^2 + b_{ij}f'f'') = 0 \\ W^+_{ijik} + \frac{\lambda^2}{2}(a_{ijk}x_jx_k \pm a_{jil}x_ix_l)f'f'' = 0 \end{cases}.$$

As in the proof of Theorem 3.1, if we suppose that W^+ does not identically vanish at p, eventually choosing λ outside of a finite set of values, we obtain a contradiction: therefore, $\mathrm{W}^+ = 0$ at p, which is impossible for the conclusions of Case 2.

Thus,

$$\left|\mathrm{W}^+_{\widetilde{g}}\right|^2_{\widetilde{g}} > 0$$

on B_r: since M is closed, we can repeat the argument finitely-many times to prove the claim.

Note that the proof is analogous if we consider $\mathrm{W}^-_{\widetilde{g}}$. ☐

Now, we prove the general condition defined in Theorem 1.3

Proof of Theorem 1.3. First, note that, if $t = 1$, there is nothing to show: indeed $\mathrm{W} = \mathrm{W}^+ + \mathrm{W}^-$, therefore Aubin's Theorem guarantees that the claim is true. If $t = 0$, we obtain Theorem 1.1.

Now, let us suppose $t = -1$. A straightforward computation shows that

$$\begin{aligned} W^+_{ijij} - W^-_{ijij} &= W_{iji'j'} \\ W^+_{ijik} - W^-_{ijik} &= \pm W_{iji'k'} \\ W^+_{ijkl} - W^-_{ijkl} &= \pm W_{ijij}; \end{aligned}$$

hence, we can apply again Theorem 3.1 to show the claim.

Therefore, let $t \neq -1, 0, 1$. We consider again the deformed metric $\widetilde{g}_t$ defined by (2.7), with ϕ as in (3.2). It is easy to obtain the system

$$\begin{cases} \widetilde{W}^+_{ijij} + t\widetilde{W}^-_{ijij} = W^+_{ijij} + tW^-_{ijij} + \dfrac{\lambda^2}{2}(1+t)[a_{ij}(f')^2 + b_{ij}f'f''] + r^2\theta_{ijij} & (4.4a) \\ \widetilde{W}^+_{ijik} + t\widetilde{W}^-_{ijik} = W^+_{ijik} + tW^-_{ijik} + \dfrac{\lambda^2}{2}[(1+t)a_{ijk}x_jx_k \pm (1-t)a_{jil}x_ix_l]f'f'' + r^2\theta_{ijik} & (4.4b) \\ \widetilde{W}^+_{ijkl} + t\widetilde{W}^-_{ijkl} = W^+_{ijkl} + tW^-_{ijkl} \pm \dfrac{\lambda^2}{2}(1-t)[a_{ij}(f')^2 + b_{ij}f'f''] + r^2\theta_{ijkl} & (4.4c) \end{cases}$$

where $i \neq j \neq k \neq l$. As we did for the proof of Aubin's Theorem, let $p_0 \in M$ be a point such that $\mathrm{W}^+_g + t\,\mathrm{W}^-_g\,|_{p_0} = 0$ and let B_r be an open ball of radius r and centered in p_0; moreover, let us define normal coordinates $x_1, \ldots x_4$ such that $p_0 = (0,0,0,0)$ and let $p \in B_r$. Finally, we choose the coefficients $(\alpha_1, \ldots, \alpha_4)$ such that $a_{ij}, a_{ijk} \neq 0$ for every i, j, k: note that the coefficients can be chosen in such a way that the numbers a_{ijk} have the same sign. By (4.3), it is easy to see that $\underline{\alpha} = (1, 5/4, 3/2, 2)$ is a suitable choice.

Case 1 ($p = p_0$). As usual, since $a_{ij} \neq 0$, we have that

$$\widetilde{W}^+_{ijij} + t\widetilde{W}^-_{ijij} = \frac{\lambda^2}{2}(1+t)a_{ij}(f')^2 \neq 0, \qquad \widetilde{W}^+_{ijkl} + t\widetilde{W}^-_{ijkl} = \frac{\lambda^2}{2}(1-t)a_{ij}(f')^2 \neq 0$$

at p_0; therefore $\mathrm{W}^+_{\widetilde{g}_t} + t\,\mathrm{W}^-_{\widetilde{g}_t} \not\equiv 0$ at p_0.

Case 2 ($p \in B_{r/2} \setminus \{p_0\}$). For a sufficiently small radius r, we again have that

$$\left|\mathrm{W}^+_g + t\,\mathrm{W}^-_g\right| \leq C\dot{r} + o(r^2), \text{ as } r \to 0.$$

Let us suppose that $\widetilde{W}^+_{ijkl} + t\widetilde{W}^-_{ijkl} = 0$ at p: therefore, as $r \to 0$ the subsystem consisting of the equations of the form (4.4b) becomes

$$\begin{cases} (1+t)a_{123}x_2x_3 + (1-t)a_{214}x_1x_4 &= 0 \\ (1+t)a_{124}x_2x_4 - (1-t)a_{213}x_1x_3 &= 0 \\ (1+t)a_{134}x_3x_4 + (1-t)a_{312}x_1x_2 &= 0 \end{cases}.$$

Let us suppose that $x_1, ..., x_4 \neq 0$: hence, we have

$$\frac{1-t}{1+t} = \frac{a_{124}}{a_{213}} \cdot \frac{x_2x_4}{x_1x_3} = -\frac{a_{123}}{a_{214}} \cdot \frac{x_2x_3}{x_1x_4} \Rightarrow \frac{a_{124}}{a_{213}} \cdot \frac{x_4^2}{x_3^2} = -\frac{a_{123}}{a_{214}},$$

which is impossible, since, by hypothesis, the coefficients a_{ijk} all have the same sign. Thus, at least one coordinate x_i must vanish and, by the system above, this implies that there is just one coordinate of p different from zero. Without loss of generality, we may suppose that $x_1 \neq 0$. However, by choosing the coefficients $\alpha_1, ...\alpha_4$ in such a way that a_{ij} and the coefficient of x_1^2 in b_{ij} have opposite signs for some $i \neq j$, we get a contradiction, since $(f')^2$ and $f'f''$ have opposite signs on B_r: for instance, if $\underline{\alpha} = (1, 5/4, 3/2, 2)$, by (3.7) we have

$$a_{12} = \frac{5}{16} \quad \text{and} \quad b_{12} = -\frac{1}{3r^2}x_1^2.$$

Thus, the only solution of the system is $x_1 = ... = x_4 = 0$, which is impossible, since $p \neq p_0$: hence, we conclude that $\mathrm{W}^+_{\widetilde{g}_t} + t\,\mathrm{W}^-_{\widetilde{g}_t}$ does not identically vanish at p.

Case 3 ($p \in B_r \setminus B_{r/2}$). If we suppose that $\mathrm{W}^+_{\widetilde{g}_t} + t\,\mathrm{W}^-_{\widetilde{g}_t}$ identically vanish at p, as $r \to 0$ the system consisting of the equations (4.4a), (4.4b) and (4.4c) becomes

$$\begin{cases} 0 &= W^+_{ijij} + tW^-_{ijij} + \frac{\lambda^2}{2}(1+t)[a_{ij}(f')^2 + b_{ij}f'f''] \\ 0 &= W^+_{ijik} + tW^-_{ijik} + \frac{\lambda^2}{2}[(1+t)a_{ijk}x_jx_k \pm (1-t)a_{jil}x_ix_l]f'f'' \\ 0 &= W^+_{ijkl} + tW^-_{ijkl} \pm \frac{\lambda^2}{2}(1-t)[a_{ij}(f')^2 + b_{ij}f'f''] \end{cases}.$$

However, if we suppose that $\mathrm{W}^+_g + t\,\mathrm{W}^-_g$ does not identically vanish at p, as we did in the proofs of Theorem 3.1 and Theorem (1.1), by eventually choosing λ out of a finite set of values, we get a contradiction. Therefore, $\mathrm{W}^+_g + t\,\mathrm{W}^-_g$ must vanish at p, which is impossible.

By the hypothesis of compactness on M, the claim is proven. □

5. Proof of Theorem 1.4

In this section we prove Theorem 1.4. If we use again Aubin's deformation of g as described in (2.7), we can write the components of the Cotton tensor with respect to the deformed metric $\widetilde{g}$ as

(5.1)

$$
\begin{aligned}
\widetilde{C}_{ijk} &= C_{ijk} - \frac{1}{w}[(\phi^t_k\phi^s + \phi^s_k\phi^t)R_{itjs} - (\phi^t_j\phi^s + \phi^s_j\phi^t)R_{itks}] + \\
&- \frac{\phi^p}{w}\phi_{ik}\left\{R_{jp} - \frac{1}{w}\left[\phi^t\phi^s(R_{ptjs} + \phi_{jp}\phi_{ts} - \phi_{pt}\phi_{js}) - (\Delta\phi)\phi_{jp} + \phi_{pt}\phi^t_j\right]\right\} + \\
&+ \frac{\phi^p}{w}\phi_{ij}\left\{R_{kp} - \frac{1}{w}\left[\phi^t\phi^s(R_{ptks} + \phi_{kp}\phi_{ts} - \phi_{pt}\phi_{ks}) - (\Delta\phi)\phi_{kp} + \phi_{pt}\phi^t_k\right]\right\} + \\
&+ \frac{1}{w}[(\Delta\phi)_k\phi_{ij} - (\Delta\phi)_j\phi_{ik} + (\Delta\phi)\phi^s R_{sijk} - \phi^t_i\phi^s R_{stjk} + \phi^t_k\phi_{itj} - \phi^t_j\phi_{itk} + \phi^t\phi^s(R_{itjs,k} - R_{itks,j})] + \\
&+ \frac{2\phi^p}{w^2}[\phi^t\phi^s(\phi_{kp}R_{itjs} - \phi_{jp}R_{itks}) + \phi_{jp}((\Delta\phi)\phi_{ik} - \phi_{it}\phi^t_k) - \phi_{kp}((\Delta\phi)\phi_{ij} - \phi_{it}\phi^t_j)] + \\
&- \frac{1}{w^2}\left\{\phi^p[\phi^s_k(\phi_{ij}\phi_{sp} - \phi_{is}\phi_{jp}) - \phi^t_j(\phi_{ik}\phi_{pt} - \phi_{it}\phi_{kp})]\right\} + \\
&- \frac{1}{w^2}\left\{\phi^p[\phi^s_k(\phi_{ij}\phi_{ps} - \phi_{ip}\phi_{js}) - \phi^t_j(\phi_{ik}\phi_{pt} - \phi_{ip}\phi_{kt})]\right\} + \\
&- \frac{1}{w^2}\left\{\phi^s\phi^t(\phi^r(R_{rijk}\phi_{ts} - R_{rsjk}\phi_{it}) + \phi_{tsk}\phi_{ij} - \phi_{tsj}\phi_{ik} - \phi_{itk}\phi_{js} + \phi_{itj}\phi_{ks})\right\} + \\
&- \frac{4\phi^p}{w^3}\phi^t\phi^s[\phi_{kp}(\phi_{ij}\phi_{ts} - \phi_{it}\phi_{js}) - \phi_{jp}(\phi_{ik}\phi_{ts} - \phi_{it}\phi_{ks})] + \\
&- \frac{1}{2w(n-1)}[\phi^p\phi^q R_{pq,k} + 2R_{pq}\phi^p\phi^q_k + 2(\Delta\phi)(\Delta\phi)_k - 2\phi^{pq}\phi_{pqk}](g_{ij} + \phi_i\phi_j) + \\
&+ \frac{1}{2w(n-1)}[\phi^p\phi^q R_{pq,j} + 2R_{pq}\phi^p\phi^q_j + 2(\Delta\phi)(\Delta\phi)_j - 2\phi^{pq}\phi_{pqj}](g_{ik} + \phi_i\phi_k) + \\
&- \frac{1}{2w^2(n-1)}\left\{2\phi^p\phi_{pk}\left[2R_{st}\phi^s\phi^t - (\Delta\phi)^2 + \phi_{st}\phi^{st} + \frac{4}{w}((\Delta\phi)\phi^s\phi^t\phi_{st} - \phi^r\phi_{rs}\phi^{st}\phi_t)\right] + \right. \\
&\left. + (\Delta\phi)_k\phi^p\phi^q\phi_{pq} + (\Delta\phi)\phi^p\phi^q\phi_{pqk} + 2(\Delta\phi)\phi^p\phi^q_k\phi_{pq} - 2\phi^p\phi^q\phi^s_p\phi_{sqk} - 2\phi^p\phi_{pq}\phi^{qs}\phi_{sk}\right\}(g_{ij} + \phi_i\phi_j) + \\
&+ \frac{1}{2w^2(n-1)}\left\{2\phi^p\phi_{pj}\left[2R_{st}\phi^s\phi^t - (\Delta\phi)^2 + \phi_{st}\phi^{st} + \frac{4}{w}((\Delta\phi)\phi^s\phi^t\phi_{st} - \phi^r\phi_{rs}\phi^{st}\phi_t)\right] + \right. \\
&\left. + (\Delta\phi)_j\phi^p\phi^q\phi_{pq} + (\Delta\phi)\phi^p\phi^q\phi_{pqj} + 2(\Delta\phi)\phi^p\phi^q_j\phi_{pq} - 2\phi^p\phi^q\phi^s_p\phi_{sqj} - 2\phi^p\phi_{pq}\phi^{qs}\phi_{sj}\right\}(g_{ik} + \phi_i\phi_k) + \\
&- \frac{2}{n-1}(S_k\phi_i\phi_j - S_j\phi_i\phi_k).
\end{aligned}
$$

Proof. Let g any Riemannian metric on M and consider the deformed metric $\widetilde{g}$ defined in (2.7), where ϕ is chosen as in (3.2), with $\alpha_1, ..., \alpha_n \in [1, 2]$ and such that the derivatives of f satisfies the following inequalities

$$f' > 0, \qquad f'' < 0, \qquad f''' > 0 \qquad \text{on } [0, 1)$$

(for instance, we can choose (3.1) with a sufficiently large b). Let us choose a point $p_0 \in M$ where the Cotton tensor C of (M, g) vanishes and let us consider again normal coordinates in p_0. Note that, in addition to (3.3) and (3.4), for sufficiently small radii r we have

$$\phi_{ijk} = \frac{2\lambda}{r^2}\alpha_i\Big[(\alpha_j x_i \delta_{jk} + \alpha_j x_j \delta_{ik} + \alpha_k x_k \delta_{ij})f'' + \frac{2\alpha_j\alpha_k}{r^2}x_i x_j x_k f'''\Big] = O\Big(\frac{1}{r}\Big). \tag{5.2}$$

By (3.4) and (5.2), we obtain

$$\Delta\phi = \lambda\left(f'\sum_{p=1}^{n}\alpha_p + \frac{2}{r^2}f''\sum_{p=1}^{n}\alpha_p^2 x_p^2\right) \tag{5.3}$$

$$(\Delta\phi)_k = \frac{2\lambda}{r^2}\left[\left(2\alpha_k^2 x_k + \alpha_k x_k\sum_{p=1}^{n}\alpha_p\right)f'' + \frac{2\alpha_k}{r^2}f'''\left(\sum_{p=1}^{n}\alpha_p^2 x_p^2\right)x_k\right] \tag{5.4}$$

As we did for $\widetilde{\mathrm{W}}$ in (3.5), for sufficiently small radii we can consider the principal part of the transformed Cotton tensor:

$$\begin{aligned}\widetilde{C}_{ijk} = {} & C_{ijk} + (\Delta\phi)_k\phi_{ij} - (\Delta\phi)_j\phi_{ik} + \phi_{tk}\phi_{itj} - \phi_{tj}\phi_{itk} + \\ & -\frac{1}{n-1}[((\Delta\phi)(\Delta\phi)_k - \phi_{pq}\phi_{pqk})g_{ij} - ((\Delta\phi)(\Delta\phi)_j - \phi_{pq}\phi_{pqj})g_{ik}] + r\theta_{ijk},\end{aligned} \tag{5.5}$$

where the expression $r\theta_{ijk}$ contains all the terms in (5.5) whose order is the same as r or higher. By inserting (5.2), (5.3) and (5.4) into (5.5), we obtain

$$\begin{aligned}\widetilde{C}_{iji} &= C_{iji} + \lambda^2\big\{a_{ij}f'f'' + b_{ij}\big[f'f''' + (f'')^2\big]\big\}x_j + r\theta_{ij} \\ \widetilde{C}_{ijk} &= C_{ijk} + \lambda^2 a_{ijk}x_i x_j x_k[(f'')^2 + f'f'''] + r\theta_{ijk},\end{aligned} \tag{5.6}$$

where $i \neq j \neq k$ and

$$\begin{aligned} a_{ij} &= \frac{2\alpha_j}{r^2}\left[-4\alpha_i\alpha_j - \alpha_i\sum_{k\neq i,j}\alpha_k + \frac{2}{n-1}\left(\alpha_j\sum_{k\neq j}\alpha_k + \sum_{k<l}\alpha_k\alpha_l\right)\right]; \\ b_{ij} &= \frac{4\alpha_j}{r^4}\left[-\alpha_i\left(\alpha_i\alpha_j x_i^2 + \sum_{k\neq i}\alpha_k^2 x_k^2\right) + \frac{1}{n-1}\sum_k \alpha_k\left(\sum_{l\neq k}\alpha_l^2 x_l^2\right)\right]; \\ a_{ijk} &= \frac{4\alpha_i\alpha_j\alpha_k}{r^4}(\alpha_k - \alpha_j). \end{aligned} \tag{5.7}$$

Note that it is sufficient to choose $\alpha_1, ..., \alpha_n$ such that $\alpha_i \neq \alpha_j$ for every $i \neq j$ to obtain $a_{ijk} \neq 0$ for every $i \neq j \neq k$.

It is immediate to observe that, by (5.6), the deformed cotton tensor $\mathrm{C}_{\widetilde{g}}$ vanish at p_0. Thus, we want to show that $\mathrm{C}_{\widetilde{g}}$ does not identically vanish on $B_r \setminus \{p_0\}$: by the compactness of M, since we can cover M with a finite open cover $\{B^i_{r_i}\}_{i=1}^k$, we will conclude that the Cotton tensor $\mathrm{C}_{\widetilde{g}}$ does not identically vanish on $M \setminus \{p_0 = p_0^1, ..., p_0^k\} =: M \setminus \{p_1, ..., p_k\}$.

Now, let $p \in B_r$ and let us consider $\mathrm{C}_{\widetilde{g}}$ at p.

Case 1 ($p \in B_{r/2} \setminus \{p_0\}$). As usual, we have that

$$|\mathrm{C}_g| \leq D \cdot r + o(r^2), \text{ as } r \to 0;$$

if we suppose that $\widetilde{C}_{iji} = \widetilde{C}_{ijk} = 0$ for every $i \neq j \neq k$, we have that

$$\begin{cases} a_{ij} f' f'' + b_{ij} [f' f''' + (f'')^2] x_j & = 0 \\ a_{ijk} x_i x_j x_k [(f'')^2 + f' f'''] & = 0 \end{cases}$$

as $r \to 0$. By the properties of f and our choice of $\alpha_1, ..., \alpha_n$, we have that $x_i x_j x_k = 0$ for every $i \neq j \neq k$, which implies that at most two coordinates of p are not zero.

Therefore, let us suppose that $x_i, x_j \neq 0$. By hypothesis, $\widetilde{C}_{iji} = \widetilde{C}_{jij} = 0$: hence, by (5.6) and (5.7) we obtain the following equations

$$\begin{aligned} 0 = & \left[-4\alpha_i\alpha_j - \alpha_i \sum_{k \neq i,j} \alpha_k + \frac{2}{n-1} \left(\alpha_j \sum_{k \neq j} \alpha_k + \sum_{k<l} \alpha_k \alpha_l \right) \right] f' f'' + \\ & + \frac{2}{r^2} \left[-\alpha_i \big(\alpha_i \alpha_j x_i^2 + \alpha_j^2 x_j^2 \big) + \frac{1}{n-1} \sum_k \alpha_k \left(\sum_{l \neq k} \alpha_l^2 x_l^2 \right) \right] [(f'')^2 + f' f''']; \\ 0 = & \left[-4\alpha_i\alpha_j - \alpha_j \sum_{k \neq i,j} \alpha_k + \frac{2}{n-1} \left(\alpha_i \sum_{k \neq i} \alpha_k + \sum_{k<l} \alpha_k \alpha_l \right) \right] f' f'' + \\ & + \frac{2}{r^2} \left[-\alpha_j \big(\alpha_i \alpha_j x_j^2 + \alpha_i^2 x_i^2 \big) + \frac{1}{n-1} \sum_k \alpha_k \left(\sum_{l \neq k} \alpha_l^2 x_l^2 \right) \right] [(f'')^2 + f' f''']; \end{aligned}$$

subtracting the second equation from the first, it is easy to obtain

$$(\alpha_j - \alpha_i) \sum_{k \neq i,j} \alpha_k + \frac{2}{n-1} \left(\alpha_j \sum_{k \neq j} \alpha_k - \alpha_i \sum_{k \neq i} \alpha_k \right) = 0 \Leftrightarrow \frac{n-3}{n-1} (\alpha_j - \alpha_i) \sum_{k \neq i,j} \alpha_k = 0,$$

which is impossible, since $\alpha_i \neq \alpha_j$ by hypothesis. This implies that exactly one coordinate of p is different from zero (say, x_j). Since $n \geq 4$, if $i \neq t \neq j$, by $\widetilde{C}_{iji} = \widetilde{C}_{tjt} = 0$ we obtain

$$0 = \left[-4\alpha_i\alpha_j - \alpha_i \sum_{k\neq i,j} \alpha_k + \frac{2}{n-1} \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) \right] f'f'' +$$

$$+ \frac{2}{r^2}\alpha_j^2 x_j^2 \left[-\alpha_i + \frac{1}{n-1} \sum_{k\neq j} \alpha_k \right] [(f'')^2 + f'f'''];$$

$$0 = \left[-4\alpha_t\alpha_j - \alpha_t \sum_{k\neq t,j} \alpha_k + \frac{2}{n-1} \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) \right] f'f'' +$$

$$+ \frac{2}{r^2}\alpha_j^2 x_j^2 \left[-\alpha_t + \frac{1}{n-1} \sum_{k\neq j} \alpha_k \right] [(f'')^2 + f'f'''].$$

It is not hard to see that, for a suitable choice of $\alpha_1 \neq ... \neq \alpha_n$, the coefficients of $[(f'')^2 + f'f''']$ in the equations do not vanish: this allows us to compute x_j^2 as

$$x_j^2 = \frac{r^2 \left[4\alpha_i\alpha_j + \alpha_i \sum_{k\neq i,j} \alpha_k - \frac{2}{n-1} \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) \right] f'f''}{2\alpha_j^2 \left[-\alpha_i + \frac{1}{n-1} \sum_{k\neq j} \alpha_k \right] [(f'')^2 + f'f''']}.$$

However, inserting this into the other equation, we obtain

$$\left[4\alpha_t\alpha_j + \alpha_t \sum_{k\neq t,j} \alpha_k - \frac{2}{n-1} \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) \right] \left[-\alpha_i + \frac{1}{n-1} \sum_{k\neq j} \alpha_k \right] =$$

$$= \left[4\alpha_i\alpha_j + \alpha_i \sum_{k\neq i,j} \alpha_k - \frac{2}{n-1} \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) \right] \left[-\alpha_t + \frac{1}{n-1} \sum_{k\neq j} \alpha_k \right],$$

which implies

$$\frac{4}{n-1}(\alpha_t - \alpha_i) \sum_{k\neq j} \alpha_k + \alpha_i\alpha_t(\alpha_t - \alpha_i) +$$

$$+ \frac{1}{n-1}(\alpha_t - \alpha_i) \left(\sum_{k\neq i,j,t} \alpha_k \right) \left(\sum_{l\neq j} \alpha_l \right) + \frac{2}{(n-1)^2}(\alpha_t - \alpha_i) \left(\alpha_j \sum_{k\neq j} \alpha_k + \sum_{k<l} \alpha_k\alpha_l \right) = 0$$

and this is clearly impossible. Since $p \neq p_0$, we have that the Cotton tensor $\mathrm{C}_{\widetilde{g}}$ cannot identically vanish at p.

Case 2 ($p \in B_r \setminus B_{r/2}$). As usual, let us suppose that $\mathrm{C}_{\widetilde{g}}$ identically vanishes at p. If C does not vanish at p, we can exploit the argument of Theorem 3.1 to conclude that, if we eventually choose λ out of a finite set of values, this is impossible. Therefore, $\mathrm{C} \equiv 0$ at p, which is a contradiction, by the proof of Case 1; hence, $\mathrm{C}_{\widetilde{g}}$ does not vanish at p.

By the hypothesis of compactness on M, we repeat the argument of the proof finitely-many times to conclude that

$$\left|\mathrm{C}_{\widetilde{g}}\right|^2_{\widetilde{g}} > 0 \text{ on } M \setminus \{p_1, ..., p_k\}.$$

□

Remark 5.1. We point out that Aubin's method in the proof of Theorem 1.4 does not lead to a sharp conclusion: indeed, one can prove the existence of left-invariant, non-Einstein metrics on the standard sphere whose Cotton tensor never vanishes for every $n \geq 3$ (we would like to thank Professor A. Derdziński for the useful suggestion). Moreover, if $n = 3$, the method used in the proof does not work, due to the lack of independent equations in the case $p \in B_{r/2} \setminus \{p_0\}$.

6. Proof of Theorem 1.5

In this section, we focus on four-dimensional manifolds and we prove Theorem 1.5. If $n = 4$, the Bach tensor acquires two additional properties: it is conformally invariant and divergence-free (see [8], Section 1.4 and Section 2.2.2).

Proof. As we did in the proof of Theorem 3.1, let g any Riemannian metric on M and let $p_0 \in M$ such that B_g vanishes and let B_r an open ball of radius r and centered in p_0. Let us choose normal coordinates $x_1, ..., x_4$ such that $p_0 = (0, 0, 0, 0)$ and let us define the function ϕ as in (3.2), with f defined as in (3.1). We know that $f \in C^\infty([0, +\infty))$: therefore, $\phi \in C^\infty(B_r)$ and it vanishes outside B_r. Moreover, for a sufficiently large b, the function f satisfies the following inequalities

$$f' > 0, \quad f'' < 0, \quad f''' > 0, \quad f^{IV} < 0 \quad \text{on } [0, 1).$$

By (5.2) and (5.3), we obtain the following additional expressions:

(6.1)
$$\begin{aligned}\phi_{ijkt} = \frac{2}{r^2}\lambda\alpha_i \bigg\{ &\frac{4}{r^4}\alpha_j\alpha_k\alpha_t x_i x_j x_k x_t f^{IV} + (\alpha_k\delta_{kt}\delta_{ij} + \alpha_j\delta_{jt}\delta_{ik} + \alpha_j\delta_{it}\delta_{jk})f'' + \\ &+ \frac{2}{r^2}[\alpha_j\alpha_k(\delta_{it}x_j x_k + \delta_{jt}x_i x_k + \delta_{kt}x_i x_j) + \alpha_t x_t(\delta_{ij}\alpha_k x_k + \delta_{ik}\alpha_j x_j + \delta_{jk}\alpha_j x_i)]f''' \bigg\}.\end{aligned}$$

(6.2)
$$\begin{aligned}(\Delta\phi)_{jk} = \frac{2\lambda\alpha_j}{r^2} \Bigg\{ &\left(2\alpha_j + \sum_p \alpha_p\right) f''\delta_{jk} + \\ &+ \frac{2}{r^2}\left[\alpha_k\left(2\alpha_j + \sum_p \alpha_p\right)x_j x_k + 2\alpha_k^2 x_j x_k + \sum_p \alpha_p^2 x_p^2 \delta_{jk}\right] f''' + \frac{4\alpha_k}{r^4}\left(\sum_p \alpha_p^2 x_p^2\right) x_j x_k f^{IV} \Bigg\}.\end{aligned}$$

$$(6.3)\quad (\Delta\phi)_{kk}=\frac{2\lambda}{r^2}\Bigg[\Bigg(2\sum_p\alpha_p^2+\Big(\sum_q\alpha_q\Big)^2\Bigg)f''+\frac{4}{r^2}\Big(2\sum_p\alpha_p^3x_p^2+\sum_p\alpha_p\sum_q\alpha_q^2x_q^2\Big)f'''+ \\ +\frac{4}{r^4}\Big(\sum_p\alpha_p^2x_p^2\Big)^2f^{IV}\Bigg].$$

Note that, for a sufficiently small radius r,

$$\phi_{ijkt}=O\Big(\frac{1}{r^2}\Big).$$

We consider the principal part of the transformed Bach tensor: by (5.1), (5.5) and the definition of the Bach tensor, we obtain

$$(6.4)\qquad \widetilde{B}_{ij}=B_{ij}+(\Delta\phi)_{kk}\phi_{ij}+(\Delta\phi)_k\phi_{ijk}-(\Delta\phi)_{jk}\phi_{ik}-(\Delta\phi)_j\phi_{ikk}+ \\ +\phi_{tkk}\phi_{itj}+\phi_{tk}\phi_{itjk}-\phi_{tjk}\phi_{itk}-\phi_{tj}\phi_{itkk}+ \\ -\frac{1}{n-1}[((\Delta\phi)_k(\Delta\phi)_k+(\Delta\phi)(\Delta\phi)_{kk}-\phi_{pqk}\phi_{pqk}-\phi_{pq}\phi_{pqkk})\delta_{ij}+ \\ -((\Delta\phi)_i(\Delta\phi)_j+(\Delta\phi)(\Delta\phi)_{ji}-\phi_{pqi}\phi_{pqj}-\phi_{pq}\phi_{pqji})]+\theta_{ij},$$

where θ_{ij} is the usual"remainder" term. Note that, as $r\to 0$, the terms given by $\widetilde{R}_{kl}\widetilde{W}_{ijkl}$ in the definition of the Bach tensor (2.6) do not appear in (6.4), since their order is lower than the order of $\widetilde{C}_{ijk,k}$; however, as we did for the Cotton tensor, we make explicit the coefficients of B_g, since they do not depend on f (and, therefore, they do not *a priori* vanish as the argument of f goes to 1).

Inserting (3.4), (5.3), (5.2), (6.1) and (6.2) into (6.4), for a sufficiently small radius r we obtain the following expression for the Bach tensor:

(6.5)

$$\widetilde{B}_{ij}=B_{ij}+\frac{2\lambda^2}{3r^2}\Bigg\{\alpha_i\Bigg[8\sum_p\alpha_p^2+4\sum_q\alpha_q\Big(\sum_t\alpha_t-\alpha_i\Big)-8\alpha_i^2\Bigg]-\sum_p\alpha_p\Bigg[\sum_q\alpha_q^2+\Big(\sum_t\alpha_t\Big)^2\Bigg]+2\sum_p\alpha_p^3\Bigg\}f'f''\delta_{ij}+ \\ +\frac{4\lambda^2}{3r^4}\Bigg\{4\sum_p\alpha_p^4x_p^2+\Big(14\alpha_i-3\sum_p\alpha_p\Big)\sum_q\alpha_q^3x_q^2+ \\ +\sum_p\alpha_p^2x_p^2\Bigg[\alpha_i\Big(7\sum_q\alpha_q-6\alpha_i\Big)+\sum_t\alpha_t^2-2\Big(\sum_r\alpha_r\Big)^2\Bigg]\Bigg\}\Big[f'f'''+(f'')^2\Big]\delta_{ij}+ \\ +\frac{8\lambda^2}{3r^6}\sum_p\alpha_p^2x_p^2\Bigg[\sum_q\alpha_q^3x_q^2+\sum_q\alpha_q^2x_q^2\Big(3\alpha_i-\sum_t\alpha_t\Big)\Bigg]\Big(f'f^{IV}+3f''f'''\Big)\delta_{ij}+ \\ +\frac{4\lambda^2\alpha_i\alpha_j}{3r^4}x_ix_j\Bigg[2\sum_p\alpha_p^2+\Big(\sum_q\alpha_q\Big)^2-2(\alpha_i^2+\alpha_j^2+6\alpha_i\alpha_j)-(\alpha_i+\alpha_j)\sum_t\alpha_t\Bigg]\Big[f'f'''+(f'')^2\Big]+ \\ +\frac{8\lambda^2\alpha_i\alpha_j}{3r^6}x_ix_j\Bigg[2\sum_p\alpha_p^3x_p^2-\Big(3\alpha_i+3\alpha_j-\sum_q\alpha_q\Big)\sum_t\alpha_t^2x_t^2\Bigg]\Big(f'f^{IV}+3f''f'''\Big)+\theta_{ij}.$$

Let

$$A:=f'f'',\qquad B:=f'f'''+(f'')^2,\qquad C:=f'f^{IV}+3f''f'''$$

and let us choose $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, \frac{5}{4}, \frac{3}{2}, 2)$. For $i \neq j$, we obtain the following equations

$$
\begin{aligned}
\widetilde{B}_{12} &= B_{12} + \frac{5\lambda^2}{3r^4}\left[\frac{2}{r^2}\left(x_1^2 + \frac{75}{32}x_2^2 + \frac{9}{2}x_3^2 + 12x_4^2\right)C + \frac{141}{8}B\right]x_1x_2 \\
\widetilde{B}_{13} &= B_{13} + \frac{2\lambda^2}{r^4}\left[\frac{2}{r^2}\left(\frac{1}{4}x_1^2 + \frac{75}{64}x_2^2 + \frac{45}{16}x_3^2 + 9x_4^2\right)C + \frac{189}{16}B\right]x_1x_3 \\
\widetilde{B}_{14} &= B_{14} + \frac{8\lambda^2}{3r^4}\left[-\frac{2}{r^2}\left(\frac{5}{4}x_1^2 + \frac{75}{64}x_2^2 + \frac{9}{16}x_3^2 - 3x_4^2\right)C - \frac{9}{16}B\right]x_1x_4 \\
\widetilde{B}_{23} &= B_{23} + \frac{5\lambda^2}{2r^4}\left[\frac{2}{r^2}\left(-\frac{1}{2}x_1^2 + \frac{9}{8}x_3^2 + 6x_4^2\right)C + \frac{19}{4}B\right]x_2x_3 \\
\widetilde{B}_{24} &= B_{24} + \frac{10\lambda^2}{3r^4}\left[-\frac{2}{r^2}\left(x_1^2 + \frac{75}{32}x_2^2 + \frac{9}{4}x_3^2\right)C - \frac{73}{8}B\right]x_2x_4 \\
\widetilde{B}_{34} &= B_{34} + \frac{4\lambda^2}{r^4}\left[-\frac{2}{r^2}\left(\frac{11}{4}x_1^2 + \frac{225}{64}x_2^2 + \frac{63}{16}x_3^2 + 3x_4^2\right)C - \frac{287}{16}B\right]x_3x_4;
\end{aligned} \tag{6.6}
$$

for $i = j$, we have the additional expressions

$$
\begin{aligned}
\widetilde{B}_{11} &= B_{11} - \frac{323\lambda^2}{12r^2}A + \frac{\lambda^2}{3r^4}\left(\frac{7}{2}x_1^2 - \frac{4175}{32}x_2^2 - \frac{2727}{16}x_3^2 - 217x_4^2\right)B + \\
&+ \frac{8\lambda^2}{3r^6}\left[\left(x_1^2 + \frac{25}{16}x_2^2 + \frac{9}{4}x_3^2 + 4x_4^2\right)\left(-\frac{7}{4}x_1^2 - \frac{75}{32}x_2^2 - \frac{45}{16}x_3^2 - 3x_4^2\right) + \right. \\
&\left. + x_1^2\left(\frac{7}{4}x_1^2 + \frac{225}{64}x_2^2 + \frac{99}{16}x_3^2 + 15x_4^2\right)\right] C \\
\widetilde{B}_{22} &= B_{22} - \frac{41\lambda^2}{6r^2}A + \frac{\lambda^2}{r^4}\left(-\frac{97}{6}x_1^2 + \frac{75}{24}x_2^2 - 21x_3^2 + \frac{2}{3}x_4^2\right)B + \\
&+ \frac{\lambda^2}{3r^6}\left[\left(8x_1^2 + \frac{25}{2}x_2^2 + 18x_3^2 + 32x_4^2\right)\left(-x_1^2 - \frac{75}{64}x_2^2 - \frac{9}{8}x_3^2\right) + \right. \\
&\left. + x_2^2\left(\frac{25}{8}x_1^2 + \frac{1875}{128}x_2^2 + \frac{1125}{32}x_3^2 + \frac{225}{2}x_4^2\right)\right] C \\
\widetilde{B}_{33} &= B_{33} + \frac{53\lambda^2}{6r^2}A + \frac{\lambda^2}{r^4}\left(-\frac{43}{12}x_1^2 + \frac{25}{24}x_2^2 + \frac{39}{8}x_3^2 + \frac{209}{3}x_4^2\right)B + \\
&+ \frac{2\lambda^2}{r^6}\left[\left(\frac{4}{3}x_1^2 + \frac{25}{12}x_2^2 + 3x_3^2 + \frac{16}{3}x_4^2\right)\left(-\frac{1}{4}x_1^2 + \frac{9}{16}x_3^2 + 3x_4^2\right) + \right. \\
&\left. + x_3^2\left(-\frac{15}{4}x_1^2 - \frac{225}{64}x_2^2 - \frac{27}{16}x_3^2 + 9x_4^2\right)\right] C \\
\widetilde{B}_{44} &= B_{44} + \frac{299\lambda^2}{12r^2}A + \frac{\lambda^2}{r^4}\left(\frac{223}{12}x_1^2 + \frac{3775}{96}x_2^2 + \frac{1167}{16}x_3^2 + 2x_4^2\right)B + \\
&+ \frac{8\lambda^2}{3r^6}\left[\left(x_1^2 + \frac{25}{16}x_2^2 + \frac{9}{4}x_3^2 + 4x_4^2\right)\left(\frac{5}{4}x_1^2 + \frac{75}{32}x_2^2 + \frac{63}{16}x_3^2 + 9x_4^2\right) + \right. \\
&\left. - x_4^2\left(17x_1^2 + \frac{375}{16}x_2^2 + \frac{117}{4}x_3^2 + 36x_4^2\right)\right] C
\end{aligned} \tag{6.7}
$$

Of course, the equations in (6.7) cannot be all independent, since the Bach tensor is traceless.

As we did for Theorem 3.1, we consider three cases.

Case 1 ($p = p_0$). In our local coordinates, $p_0 = (0,0,0,0)$; therefore, since $B_g = 0$ in p_0 and $A < 0$ on B_r, by (6.6) and (6.7) we obtain

$$|\mathrm{B}_{\widetilde{g}}|^2_{\widetilde{g}} = 2\sum_{i=1}^{4} \widetilde{B}^2_{ii} = CA^2 > 0,$$

where $C = \frac{105845\lambda^4}{36r^4}$.

Case 2 ($p \in B_{r/2} \setminus \{p_0\}$). In this case, we have again that

$$|B_g| \leq C \cdot r + o(r^2), \text{ as } r \to 0.$$

Thus, we may consider just the principal parts in the system defined by (6.6) and (6.7).

Let us suppose that $\widetilde{B}_{ij} = 0$ for every i, j at $p = (x_1, x_2, x_3, x_4)$. We want to show that the only solution of the system is given by $x_i = 0$ for every i, which leads to a contradiction for the previous argument.

If we suppose that $x_i \neq 0$ for every i, we have that, for instance,

$$B = -\frac{16}{141r^2}\left(x_1^2 + \frac{75}{32}x_2^2 + \frac{9}{2}x_3^2 + 12x_4^2\right)C$$

by the first equation in (6.6). Since $B > 0$ and $C < 0$ in B_r and $x_1, ..., x_4 \neq 0$, inserting this into the other equations in (6.6), we obtain a system of five equations in the variables $x_1, ..., x_4$: a straightforward computation shows that this system admits only the trivial solution and, therefore, one of the variables $x_1, ..., x_4$ must be zero.

Now, let us suppose that $x_i \neq 0$ for at least two indices i. If $x_i \neq 0$ for one index i, by (6.6) and (6.7) we obtain a system of 5 independent equations in x_j, x_k, x_l, where $j, k, l \neq i$: by an analogous argument, we can show that the system admits no solutions, which implies that at least two variables x_i and x_j must be zero. In this case, expressing B in terms of C as before, by (6.7) we can express A in terms of C as well and, therefore, obtain two independent equations in x_k, x_l; however, by our choice of the coefficients $\alpha_1, ..., \alpha_4$, the system is once again inconsistent.

Therefore, as in the proof of Theorem 3.1, we obtain that exactly one variable x_i is different from zero. Let us suppose that, for instance, $x_1 \neq 0$. By (6.7), we have that

$$\widetilde{B}_{11} = -\frac{323\lambda^2}{12r^2}A + \frac{7\lambda^2}{6r^4}x_1^2 B > 0,$$

since $A < 0$ and $B > 0$ on B_r. Thus, the system admits no solution. The other cases can be shown in an analogous way. Hence, we conclude that $|\mathrm{B}_{\widetilde{g}}|^2_{\widetilde{g}}$ must be strictly positive at p.

We also point out that the same system was solved *via* technical computing through Wolfram Mathematica (see Appendix A for the code). Also note that the system in the Appendix is more general than the one we are considering in this proof: indeed, we showed that the system (6.6)+(6.7), with $B_{ij} = 0$, would admit no real solutions even if A, B and C were free real parameters satisfying $A, B, C \neq 0$.

Case 3 ($p \in B_r \setminus B_{r/2}$). In this case, we need to consider the components of the Bach tensor B_g in (6.6) and (6.7).

If $\mathrm{B}_g \equiv 0$ at p, we can immediately conclude that $|\mathrm{B}_{\widetilde{g}}|^2_{\widetilde{g}} > 0$ at p, by the proof of Case 2. Thus, let us suppose that $\widetilde{B}_{ij} = 0$ at p for every i, j and that $|\mathrm{B}_g|^2_g > 0$ at p. In particular, we may suppose that $B_{12} \neq 0$ at p. By the first equation in (6.6), we obtain that

$$\lambda^2 = -\frac{3r^4 B_{12}}{5\left[\frac{2}{r^2}\left(x_1^2 + \frac{75}{32}x_2^2 + \frac{9}{2}x_3^2 + 12x_4^2\right)C + \frac{141}{8}B\right]x_1 x_2}$$

at p. However, we may choose $\lambda_1 \in \mathbb{R}$ such that $\lambda_1^2 \neq \lambda^2$ in (3.2), since λ is a free parameter: if we repeat the argument of the proof with λ_1 instead of λ, we get a contradiction and, therefore, we conclude that $B_{12} = 0$ at p.

Now, if $B_{13} \neq 0$ at p, the second equation in (6.6) implies that

$$\lambda_1^2 = -\frac{r^4 B_{13}}{2\left[\frac{2}{r^2}\left(\frac{1}{4}x_1^2 + \frac{75}{64}x_2^2 + \frac{45}{16}x_3^2 + 9x_4^2\right)C + \frac{189}{16}B\right]x_1 x_3};$$

again, possibly choosing λ_2 such that $\lambda_2^2 \neq \lambda_1^2$, we obtain that $B_{13} = 0$ at p. Iterating this argument for every component B_{ij}, we conclude that, possibly choosing $\overline{\lambda}$ outside a finite set $\{\lambda, ..., \lambda_k\}$, the components B_{ij} must all vanish at p. Therefore, we repeat the argument of Case 2 to conclude that

$$|\mathrm{B}_{\widetilde{g}}|^2_{\widetilde{g}} > 0 \text{ at } p.$$

Now, as in the proof of Theorem 3.1, since M is compact, we can deform the metric g on a finite cover of M: using the argument of Remark 3.2, the claim is proven. □

Remark 6.1. As we recalled in the Introduction, when $\dim M = 4$, Bach-flatness is a necessary condition for (M, g) to be an Einstein manifold; therefore, an immediate consequence of Theorem 1.5 is that, given a smooth manifold M of dimension four, one can always choose a conformal class $[g]$ of Riemannian metrics which contains no Einstein metrics. In fact, we can say more: since we found a quadruple $\alpha_1, ..., \alpha_4$ such that the system of equations (6.6)+(6.7) admits no solutions, there exists an open neighborhood $U_{\vec{\alpha}}$ of $\alpha = (\alpha_1, ..., \alpha_4)$ in $Q := [1, 2] \times [1, 2] \times [1, 2] \times [1, 2]$ such that, for every $\alpha' \in U_{\vec{\alpha}}$, the system admits no solutions on M. Therefore, there exist infinitely-many conformal classes of Riemannian metrics on M which contain no Einstein metrics.

Although we did not prove it in this paper, we expect that, given any Riemannian metric g on M, the subset

$$Q' := \left\{\alpha \in Q : |\mathrm{B}_{g_\alpha}|^2_{g_\alpha} \equiv 1, \text{ where } g_\alpha = g + d\phi_\alpha \otimes d\phi_\alpha \text{ and } \phi_\alpha \text{ is defined as in (3.2)} \right\}$$

is such that $Q \setminus Q'$ has Lebesgue measure zero in Q.

Appendix A. Solutions of the systems (6.6) and (6.7) in the homogeneous case

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In[1]:= B12 := (2 / r ^ 2 (x1 ^ 2 + 75 / 32 x2 ^ 2 + 9 / 2 x3 ^ 2 + 12 x4 ^ 2) C + 141 / 8 B) x1 * x2;
B13 := (2 / r ^ 2 × (1 / 4 x1 ^ 2 + 75 / 64 x2 ^ 2 + 45 / 16 x3 ^ 2 + 9 x4 ^ 2) C + 189 / 16 B) x1 * x3;
B14 := (-2 / r ^ 2 × (5 / 4 x1 ^ 2 + 75 / 64 x2 ^ 2 + 9 / 16 x3 ^ 2 - 3 x4 ^ 2) C - 9 / 16 B) x1 * x4;
B23 := (2 / r ^ 2 × (-1 / 2 x1 ^ 2 + 9 / 8 x3 ^ 2 + 6 x4 ^ 2) C + 19 / 4 B) x2 * x3;
B24 := (-2 / r ^ 2 (x1 ^ 2 + 75 / 32 x2 ^ 2 + 9 / 4 x3 ^ 2) C - 73 / 8 B) x2 * x4;
B34 := (-2 / r ^ 2 × (11 / 4 x1 ^ 2 + 225 / 64 x2 ^ 2 + 63 / 16 x3 ^ 2 + 3 x4 ^ 2) C - 287 / 16 B) x3 * x4;
B11 := -323 / 12 / r ^ 2 A + 1 / 3 / r ^ 4 × (7 / 2 x1 ^ 2 - 4175 / 32 x2 ^ 2 - 2727 / 16 x3 ^ 2 - 217 x4 ^ 2) B +
   8 / 3 / r ^ 6 ((x1 ^ 2 + 25 / 16 x2 ^ 2 + 9 / 4 x3 ^ 2 + 4 x4 ^ 2) (-7 / 4 x1 ^ 2 - 75 / 32 x2 ^ 2 - 45 / 16 x3 ^ 2 - 3 x4 ^ 2) +
      x1 ^ 2 (7 / 4 x1 ^ 2 + 225 / 64 x2 ^ 2 + 99 / 16 x3 ^ 2 + 15 x4 ^ 2)) C;
B22 := -41 / 6 / r ^ 2 A + 1 / r ^ 4 × (-97 / 6 x1 ^ 2 + 75 / 24 x2 ^ 2 - 21 x3 ^ 2 + 2 / 3 x4 ^ 2) B +
  1 / 3 / r ^ 6 ((8 x1 ^ 2 + 25 / 2 x2 ^ 2 + 18 x3 ^ 2 + 32 x4 ^ 2) (-x1 ^ 2 - 75 / 64 x2 ^ 2 - 9 / 8 x3 ^ 2) +
     x2 ^ 2 (25 / 8 x1 ^ 2 + 1875 / 128 x2 ^ 2 + 1125 / 32 x3 ^ 2 + 225 / 2 x4 ^ 2)) C;
B33 := 53 / 6 / r ^ 2 A + 1 / r ^ 4 × (-43 / 12 x1 ^ 2 + 25 / 24 x2 ^ 2 + 39 / 8 x3 ^ 2 + 209 / 3 x4 ^ 2) B +
  2 / r ^ 6 ((4 / 3 x1 ^ 2 + 25 / 12 x2 ^ 2 + 3 x3 ^ 2 + 16 / 3 x4 ^ 2) × (-1 / 4 x1 ^ 2 + 9 / 16 x3 ^ 2 + 3 x4 ^ 2) +
     x3 ^ 2 (-15 / 4 x1 ^ 2 - 225 / 64 x2 ^ 2 - 27 / 16 x3 ^ 2 + 9 x4 ^ 2)) C;

In[9]:= Solve[{B12 == 0, B13 == 0, B14 == 0, B23 == 0, B24 == 0, B34 == 0, B11 == 0, B22 == 0, B33 == 0},
   {x1, x2, x3, x4, A, B, C}]

Out[9]= {{A → 0, B → 0, C → 0}, {x2 → 0, A → 0, B → 0, C → 0}, {x1 → 0, A → 0, B → 0, C → 0}, {x3 → 0, A → 0, B → 0, C → 0},
 {x4 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x2 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x3 → 0, A → 0, B → 0, C → 0},
 {x1 → 0, x3 → -(4 i x4)/√3, A → 0, B → 0, C → 0}, {x1 → 0, x2 → -(8 √2 x4)/5, x3 → -(4 i x4)/√3, A → 0, B → 0},
 {x1 → 0, x2 → (8 √2 x4)/5, x3 → -(4 i x4)/√3, A → 0, B → 0}, {x1 → 0, x3 → (4 i x4)/√3, A → 0, B → 0, C → 0},
 {x1 → 0, x2 → -(8 √2 x4)/5, x3 → (4 i x4)/√3, A → 0, B → 0}, {x1 → 0, x2 → (8 √2 x4)/5, x3 → (4 i x4)/√3, A → 0, B → 0},
 {x1 → 0, x2 → 0, x3 → 0, x4 → 0, A → 0}, {x2 → 0, x3 → 0, A → 0, B → 0, C → 0},
 {x2 → 0, x3 → -2/3 i √2 x1, A → 0, B → 0, C → 0}, {x2 → 0, x3 → 2/3 i √2 x1, A → 0, B → 0, C → 0},
 {x2 → 0, x4 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x4 → 0, A → 0, B → 0, C → 0},
 {x1 → 0, x2 → 0, x3 → 0, x4 → 0, A → 0}, {x3 → 0, x4 → 0, A → 0, B → 0, C → 0},
 {x3 → -(2 x1)/3, x4 → 0, A → 0, B → 0, C → 0}, {x2 → -4/5 i √2 x1, x3 → -(2 x1)/3, x4 → 0, A → 0, B → 0},
 {x2 → 4/5 i √2 x1, x3 → -(2 x1)/3, x4 → 0, A → 0, B → 0}, {x3 → (2 x1)/3, x4 → 0, A → 0, B → 0, C → 0},
 {x2 → -4/5 i √2 x1, x3 → (2 x1)/3, x4 → 0, A → 0, B → 0}, {x2 → 4/5 i √2 x1, x3 → (2 x1)/3, x4 → 0, A → 0, B → 0},
 {x2 → 0, x3 → -2/3 i √2 x1, x4 → -x1/2, A → 0, B → 0}, {x2 → 0, x3 → 2/3 i √2 x1, x4 → -x1/2, A → 0, B → 0},
 {x2 → 0, x3 → -2/3 i √2 x1, x4 → x1/2, A → 0, B → 0}, {x2 → 0, x3 → 2/3 i √2 x1, x4 → x1/2, A → 0, B → 0},
 {x1 → 0, x2 → 0, x3 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x2 → 0, x4 → 0, A → 0, B → 0, C → 0},
 {x1 → 0, x2 → 0, x4 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x3 → 0, x4 → 0, A → 0, B → 0, C → 0},
 {x1 → 0, x2 → 0, x3 → 0, x4 → 0, A → 0, B → 0}, {x1 → 0, x2 → 0, x4 → 0, A → 0, B → 0, C → 0},
 {x1 → 0, x3 → 0, x4 → 0, A → 0, B → 0, C → 0}, {x1 → 0, x2 → 0, x3 → 0, x4 → 0, A → 0, B → 0},
 {x2 → 0, x3 → 0, x4 → 0, A → 0, B → 0, C → 0}, {x2 → 0, x3 → 0, x4 → 0, A → 0, B → 0, C → 0}}
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