RIGIDITY THEOREMS FOR BEST SOBOLEV INEQUALITIES

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Abstract. For $n \ge 2$, $p \in (1,n)$, the "best p-Sobolev inequality" on an open set $\Omega \subset \mathbb{R}^n$ is identified with a family Φ_Ω of variational problems with critical volume and trace constraints. When Ω is bounded we prove: (i) for every n and p, the existence of generalized minimizers that have at most one boundary concentration point, and: (ii) for $n > 2\,p$, the existence of (classical) minimizers. We then establish rigidity results for the comparison theorem "balls have the worst best Sobolev inequalities" by the first named author and Villani, thus giving the first affirmative answers to a question raised in [MV05].

Contents

1. Introduction

1.1. **Overview.** The goal of this paper is to answer some basic open questions concerning a "doubly critical" family $\{\Phi_\Omega(T)\}_{T\ge 0}$ of minimization problems on Sobolev functions, which, in a precise sense to be clarified below, can be interpreted as collectively defining the best Sobolev inequality on an open set $\Omega \subset \mathbb{R}^n$ with C^1-boundary. Given an integer $n \ge 2$ and $p \in (1,n)$, these problems are defined as

$$\Phi_\Omega(T) = \inf\Big\{\Big(\int_\Omega |\nabla u|^p\Big)^{1/p} : \int_\Omega |u|^{p^\star} = 1\,,\ \int_{\partial\Omega} |u|^{p^\#} = T^{p^\#}\Big\}\,, \tag{1.1}$$

and their minimizers, whenever they exist, satisfy the Euler–Lagrange equation

$$\begin{cases} -\Delta_p u = \lambda\, u^{p^\star - 1}\,, & \text{on } \Omega\,,\\ |\nabla u|^{p-2}\,\dfrac{\partial u}{\partial \nu_\Omega} = \sigma\, u^{p^\# - 1}\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

for suitable Lagrange multipliers $\lambda, \sigma \in \mathbb{R}$. We call $\int_\Omega |u|^{p^\star} = 1$ and $\int_{\partial\Omega} |u|^{p^\#} = T^{p^\#}$ the "volume" and "trace" constraints of $\Phi_\Omega(T)$. The critical Sobolev exponents associated to n and p, $p^\star$ and $p^\#$, are defined by

$$p^\star = \frac{np}{n-p}\,, \qquad p^\# = \frac{(n-1)p}{n-p} = \frac{n-1}{n}\,p^\star\,,$$

and their precise values guarantee the scale invariance of Φ_Ω, i.e.

$$\Phi_{x + r\,\Omega}(T) = \Phi_\Omega(T) \qquad \forall x \in \mathbb{R}^n\,, r > 0\,, T \ge 0\,.$$

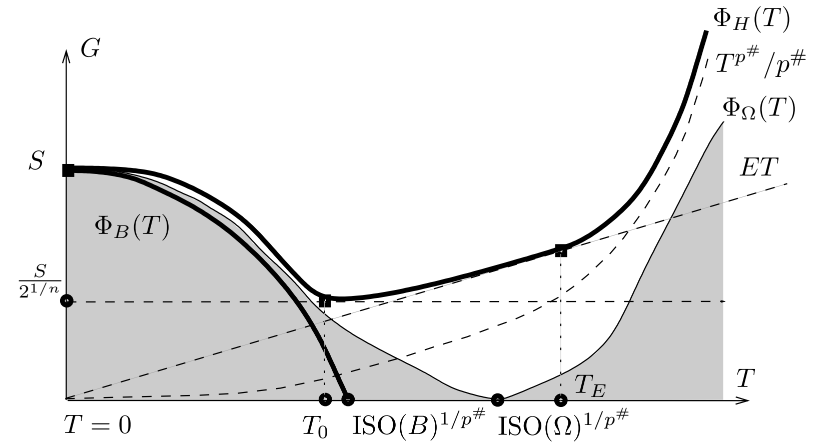

FIGURE 1.1. The state of the art about Φ_Ω. The "exclusion zone" for the values of $\|\nabla u\|_{L^p(\Omega)}$ (under the constraint $\|u\|_{L^{p^\star}(\Omega)}=1$) is depicted in gray. It always contains the subgraph of $\Phi_B(T)$ on $T\in[0,\mathrm{ISO}(B)^{1/p^\#}]$, see (1.12), but is always smaller than the one of a half-space H, see (1.19). The Sobolev inequality on $\mathbb{R}^n$ is equivalent to $\Phi_\Omega(0)=S(n,p)$, the Euclidean isoperimetric inequality to the fact that the only zero of Φ_Ω (i.e. $T=\mathrm{ISO}(\Omega)^{1/p^\#}$) is achieved to the right of the only zero of Φ_B, and the Escobar inequality (1.6) is equivalent to the linear bound $\Phi_H(T)\ge E\,T$. Both $\{(T,\Phi_B(T)):T\in[0,\mathrm{ISO}(B)^{1/p^\#}]\}$ and $\{(T,\Phi_H(T)):T\ge 0\}$ can be implicitly parametrized by looking at the explicit families of minimizers given in (1.11), (1.15), (1.16) and (1.17).

When Ω is bounded, the C^1-regularity of $\partial\Omega$ guarantees that every $u\in L^1_{\mathrm{loc}}(\Omega)$ with $\nabla u\in L^p(\Omega;\mathbb{R}^n)$ lies in the competition class of $\Phi_\Omega(T)$ for some $T\ge 0$. In particular,

$$\mathrm{Epi}(\Phi_\Omega)=\big\{(T,G)\in\mathbb{R}^2:T\ge 0\,,G\ge\Phi_\Omega(T)\big\}\,,$$

(the epigraph of Φ_Ω) collects the best possible information on the range of values achievable by $\|\nabla u\|_{L^p(\Omega)}$ when $\|u\|_{L^{p^\star}(\Omega)}$ is fixed: from this peculiar viewpoint, which is reminiscent of the one adopted in the study of Blaschke–Santaló diagrams, $\mathrm{Epi}(\Phi_\Omega)$ *is* "the best Sobolev inequality on Ω". The following list of results, summarized in Figure 1.1, aims to provide a hopefully complete state of the art on Φ_Ω, and illustrates the wealth of information stored in this family of variational problems. As a disclaimer: here we are definitely *not* attempting to exhaustively frame the study of Φ_Ω into the incredibly vast and layered context of the theory of Sobolev-type inequalities (see e.g. [Maz85]), as that would be a long and delicate exercise, lying well beyond the scope of this introduction.

(1) Sobolev inequality on $\mathbb{R}^n$: A scaling and localization argument shows that, for every open set Ω, one has

$$\Phi_\Omega(0)=S(n,p):=\inf\Big\{\Big(\int_{\mathbb{R}^n}|\nabla u|^p\Big)^{1/p}:\int_{\mathbb{R}^n}|u|^{p^\star}=1\Big\}\,, \tag{1.3}$$

that is, $\Phi_\Omega(0)$ is the best constant in the L^p-Sobolev inequality on $\mathbb{R}^n$. Minimizers of (1.3) are exactly given by the family $\{\tau_{x_0}[U_S^{(\alpha)}]\}_{x_0\in\mathbb{R}^n,\alpha>0}$ generated by

$$U_S(x)=\big(1+|x|^{p/(p-1)}\big)^{1-(n/p)}\,,\qquad x\in\mathbb{R}^n\,, \tag{1.4}$$

see [Aub76, Tal76, CENV04], we see that $\Phi_\Omega(0)$ is attained if and only if $\Omega=\mathbb{R}^n$. Here and in the following, we set

$$\tau_{x_0}[v](x)=v(x-x_0)\,,\qquad v^{(\alpha)}(x)=\alpha^{-n/p^\star}\,v(x/\alpha) \tag{1.5}$$

whenever $x_0\in\mathbb{R}^n$ and $\alpha>0$.

(2) Escobar inequality: The Escobar inequality ([Esc88, for $p = 2$], [Naz06, for $p \in (1,n)$]) states that if H is an (open) half-space in $\mathbb{R}^n$ with outer unit normal ν_H, then

$$\Big(\int_H |\nabla u|^p\Big)^{1/p} \geq E(n,p)\Big(\int_{\partial H} |u|^{p^\#}\Big)^{1/p^\#} \tag{1.6}$$

with equality if and only if[1] $u = \tau_{x_0}[U_E^{(\alpha)}]$ for some $x_0 \in \mathbb{R}^n \setminus \overline{H}$, and where

$$U_E(x) = |x|^{-(n-p)/(p-1)}\,, \qquad x \in \mathbb{R}^n \setminus \{0\}\,, \tag{1.7}$$

is a multiple of the fundamental solution of the p-Laplacian. The quantity

$$T_E = \|\tau_{x_0}[U_E^{(\alpha)}]\|_{L^{p^\#}(\partial H)} \Big/ \|\tau_{x_0}[U_E^{(\alpha)}]\|_{L^{p^\star}(H)} \tag{1.8}$$

is independent of $x_0 \in \mathbb{R}^n \setminus \overline{H}$, and is such that

$$\Phi_H(T_E) = E(n,p)\,. \tag{1.9}$$

The Escobar inequality (1.6) can be equivalently reformulated in the "Φ-setting" as a linear lower bound for Φ_H, i.e.

$$\Phi_H(T) \geq E(n,p)\,T\,, \qquad \forall T \geq 0\,.$$

This bound is sharp only if $T = T_E$, and is nearly optimal only if T is close to T_E; but it largely suboptimal away from T_E, see Figure 1.1.

(3) Euclidean isoperimetry: It is easily seen that $\Phi_\Omega(T) = 0$ for some $T > 0$ if and only if $|\Omega| < \infty$ and $T = \mathrm{ISO}(\Omega)^{1/p^\#}$, where $\mathrm{ISO}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)/|\Omega|^{(n-1)/n}$ stands for the isoperimetric ratio of Ω. Since the Euclidean isoperimetric inequality states that

$$\mathrm{ISO}(\Omega) \geq \mathrm{ISO}(B)\,, \tag{1.10}$$

(with equality if and only if Ω is a ball), in the Φ-setting, (1.10) is equivalent to saying that Φ_B has the left-most zero among all Φ_Ω.

(4) Balls have the worst best Sobolev inequalities: In [CL90, CL94, $p = 2$] (by symmetrization methods and conformal invariance) and in [MV05, $p \in (1,n)$] (via the mass transportation method pioneered in [Kno57, MS86, CENV04]) it is shown that if B is a ball, then for every $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$ there is a unique $\alpha > 0$ such that

$$\Phi_B(T) = \|\nabla U_S^{(\alpha)}\|_{L^p(B)} \Big/ \|U_S^{(\alpha)}\|_{L^{p^\star}(B)}\,, \tag{1.11}$$

with U_S defined in (1.4). Further elaborating on the proof of this partial characterization of Φ_B, again in [MV05] the comparison theorem that *balls have the worst best Sobolev inequalities*

$$\Phi_\Omega(T) \geq \Phi_B(T)\,, \qquad \forall T \in [0, \mathrm{ISO}(B)^{1/p^\#}] \tag{1.12}$$

is proved. This sharp lower bound, combined with (1.11), allows one to infer some sharp and more traditional-looking Sobolev-type inequalities, like the following sharp interpolation between (1.3) and (1.10)

$$\frac{\|\nabla u\|_{L^p(\Omega)}}{S(n,p)} + \frac{\|u\|_{L^{p^\#}(\partial\Omega)}}{\mathrm{ISO}(B)^{1/p^\#}} \geq \|u\|_{L^{p^\star}(\Omega)}\,, \tag{1.13}$$

and the following sharp Sobolev inequality, additive in the domain of the p-Dirichlet energy,

$$\frac{\|\nabla u\|^p_{L^p(\Omega)}}{S(n,p)^p} + \frac{\|u\|^p_{L^{p^\#}(\partial\Omega)}}{C(n,p)} \geq \|u\|^p_{L^{p^\star}(\Omega)}\,, \tag{1.14}$$

which was first conjectured by Brezis and Lieb in [BL85].

[1] The "only if" statement for $p \neq 2$ was left open in [Naz06], but was proven in [MN17, Theorem 2.3].

(5) Full characterization of Φ_H: In [CL90, CL94, for $p=2$] and, again by optimal mass transport arguments, in [MN17, for $p\in(1,n)$], it is proven that if H is half-space in $\mathbb{R}^n$, then: for each $T\in(0,T_E)$ (Sobolev range) there is a unique $t_T\in\mathbb{R}$ such that

$$\Phi_H(T)=\|\nabla(\tau_{t_T\,\nu_H}U_S)\|_{L^p(H)}\Big/\|\tau_{t_T\,\nu_H}U_S\|_{L^{p^\star}(H)}\,; \tag{1.15}$$

if $T=T_E$ (Escobar point), then

$$\Phi_H(T_E)=E(n,p)=\|\nabla(\tau_{\nu_H}U_E)\|_{L^p(H)}\Big/\|\tau_{\nu_H}U_E\|_{L^{p^\star}(H)}\,; \tag{1.16}$$

for each $T>T_E$ (beyond Escobar range), there is a unique $s_T>1$ s.t.

$$\Phi_H(T)=\|\nabla(\tau_{s_T\,\nu_H}U_{BE})\|_{L^p(H)}\Big/\|\tau_{s_T\,\nu_H}U_{BE}\|_{L^{p^\star}(H)}\,, \tag{1.17}$$

$$\text{where } U_{BE}(x)=(|x|^{p/(p-1)}-1)^{1-(n/p)}\,,\qquad |x|>1\,. \tag{1.18}$$

Up to the natural dilation and translation invariances, these functions are the unique minimizers of $\Phi_H(T)$. Moreover, again by [MN17]: **(a):** $\inf_{T\ge0}\Phi_H(T)$ is achieved at $T=T_0\in(0,T_E)$, where $t_{T_0}=0$ and $\Phi_H(T_0)=S(n,p)/2^{1/n}$; **(b):** by the divergence theorem, $\Phi_H(T)>T^{p^\#}/p^\#$ for every $T>0$, and this lower bound is sharp as $T\to\infty$; **(c):** finally, Φ_H has the *best* best Sobolev inequality, i.e.

$$\Phi_\Omega(T)\le\Phi_H(T)\,,\qquad \forall T\ge0\,. \tag{1.19}$$

1.2. **Statements of the main results.** With this summary on the state of the art for Φ_Ω in mind, there are two fundamental open questions that form the subject of our paper:

Question 1: When does $\Phi_\Omega(T)$ $(T>0)$ admit minimizers?

Question 2: Does rigidity hold in the comparison theorem (1.12)?

The main idea of this paper is attacking these two closely related questions by systematically exploiting the complete characterization of Φ_H obtained in [MN17].

Concerning Question 1, a classical concentration-compactness argument characterizes the limit behavior of minimizing sequences of $\Phi_\Omega(T)$ as the superposition of a standard weak limit plus at most countably many concentration points, located either in the interior of Ω, or on its boundary. By exploiting properties of Φ_H we are able to (i): exclude all interior concentrations and all but *at most one* boundary concentration, thus proving existence of minimizers for a suitable "relaxed problem" $\Phi^*_\Omega(T)$; and (ii): completely exclude concentrations, and thus establish the existence of minimizers of $\Phi_\Omega(T)$, as soon as $\partial\Omega$ is of class C^2 and $n>2p$. To give precise statements, it is convenient to let $\mathcal{X}_\Omega(T)$ denote the competition class of $\Phi_\Omega(T)$, and let $\mathcal{Y}_\Omega(T)$ denote the set of triples $(u,\mathsf{v},\mathsf{t})$ with either $u\in\mathcal{X}_\Omega(T)$ and $\mathsf{v}=\mathsf{t}=0$, or $u\in W^{1,p}(\Omega)$, $\mathsf{v}\in(0,1]$, $\mathsf{t}\in(0,T]$, and

$$\mathsf{v}^{p^\star}+\int_\Omega u^{p^\star}=1\,,\qquad \mathsf{t}^{p^\#}+\int_{\partial\Omega}u^{p^\#}=T^{p^\#}\,. \tag{1.20}$$

The relaxed problem associated to $\Phi_\Omega(T)$ is then given by

$$\Phi^*_\Omega(T)=\inf_{\mathcal{Y}_\Omega(T)}\mathcal{E}\,,\ \text{ where } \mathcal{E}(u,\mathsf{v},\mathsf{t})^p=\int_\Omega|\nabla u|^p+\mathsf{v}^p\,\Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p\,, \tag{1.21}$$

with the convention that $\mathsf{v}^p\Phi_H(\mathsf{t}/\mathsf{v})^p=0$ if $(\mathsf{v},\mathsf{t})=(0,0)$.

Theorem 1.1 (Existence of minimizers of Φ_Ω)**.** *If $n\ge2$, $p\in(1,n)$, and Ω is a bounded open set with C^1-boundary in $\mathbb{R}^n$, then:*

(i): *for every $T>0$, there is a minimizer $(u,\mathsf{v},\mathsf{t})$ of $\Phi^*_\Omega(T)$, and*

$$\Phi_\Omega(T)=\Phi^*_\Omega(T)\,; \tag{1.22}$$

moreover, if $\int_\Omega u^{p^\star}>0$, then $u/\|u\|_{L^{p^\star}(\Omega)}$ is a minimizer of $\Phi_\Omega(\|u\|_{L^{p^\#}(\partial\Omega)}/\|u\|_{L^{p^\star}(\Omega)})$;

(ii): *if Ω has boundary of class C^2, $n > 2p$, $T > 0$, and $(u, \mathsf{v}, \mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$, then $\mathsf{v} = \mathsf{t} = 0$, and thus u is a minimizer of $\Phi_\Omega(T)$.*

Remark 1.2. Minimizers of $\Phi_\Omega(T)$ solve the Euler–Lagrange equation (1.2). For the Euler–Lagrange equation satisfied by minimizers $(u, \mathsf{v}, \mathsf{t})$ of the relaxed problem $\Phi^*_\Omega(T)$, see Theorem 4.1 below.

Question 2 is motivated by the various rigidity statements associated to comparison theorems in Riemannian geometry (see, e.g. [CE08]). In that setting, a certain model space provides a universal bound on a certain global geometric quantity (comparison theorem), which is then shown to be saturated by the model space alone (rigidity statement). With this analogy in mind, we can reformulate more precisely Question 2 as follows:

Question 2, weak form: Does $\Phi_\Omega = \Phi_B$ on $(0, \mathrm{ISO}(B)^{1/p^\#})$ imply that Ω is a ball?

Question 2, strong form: Does $\Phi_\Omega(T) = \Phi_B(T)$ at just one value of $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$ imply that Ω is a ball?

Concerning the *weak form* of Question 2, through a careful use of the properties of Φ_H we answer affirmatively whenever Ω is bounded and connected. These conditions are optimal, as shown by unbounded or disconnected non-rigidity examples presented in [MV05]. In fact, the argument we propose gives rigidity under the mere assumption that $\Phi_\Omega = \Phi_B$ holds on an open neighborhood of $T = 0$. Concerning the *strong form* of Question 2, which was originally formulated in [MV05, Section 1.9], we can answer in the affirmative as a direct by-product of our existence result for minimizers of $\Phi_\Omega(T)$ (thus, when Ω has C^2-boundary and $n > 2p$) thanks to the following "conditional rigidity" statement, which is proved in [MV05] as a direct by-product of the proof of (1.12):

$$\begin{aligned}&\textit{if } \Omega \textit{ is connected (possibly unbounded)},\\ &\textit{if } \Phi_\Omega(T) = \Phi_B(T) \textit{ for a value of } T \in (0, \mathrm{ISO}(B)^{1/p^\#}),\\ &\textit{and if } \Phi_\Omega(T) \textit{ is known to admit minimizers (possibly just for that } T),\\ &\textit{then } \Omega \textit{ is a ball}.\end{aligned} \tag{1.23}$$

(We notice for future use an important consequence of (1.23), namely, we have

$$\Phi_B(T) < \Phi_H(T), \qquad \forall T \in \big(0, \mathrm{ISO}(B)^{1/p^\#}\big); \tag{1.24}$$

indeed, by [MN17], $\Phi_H(T)$ admits minimizers for every $T > 0$.) With these premises, we now state our main results concerning Question 2.

Theorem 1.3 (Rigidity of "Balls have the worst best Sobolev inequalities")**.** *Let $n \ge 2$, $p \in (1, n)$, Ω an open, bounded, connected set with C^1-boundary in $\mathbb{R}^n$, and assume that* **one** *of the following two conditions holds:*

(i): *there is $T_* > 0$ such that $\Phi_\Omega(T) = \Phi_B(T)$ for every $T \in (0, T_*)$; or*

(ii): *$n > 2p$, the boundary of Ω is of class C^2, and there is $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$ such that $\Phi_\Omega(T) = \Phi_B(T)$.*

Then, Ω is a ball.

1.3. **Strategy of proof.** Concentration-compactness arguments and the use of sharp Sobolev-type inequalities (like the Sobolev and Escobar inequalities (1.3) and (1.6)) are the standard tools of the trade in the analysis of variational problems with critical growth. As seen, if interpreted as assertions about Φ_H, (1.3) and (1.6) contain only very partial information (respectively, "$\Phi_H(0) = S(n,p)$" and "$\Phi_H(T) \ge E(n,p)\, T$ for every $T > 0$"). From this viewpoint, our arguments provide an interesting example of the potentialities of using, in the familiar context of concentration-compactness, the full characterization of

Φ_H obtained in [CL94, MN17]. We now explain how this characterization is used in this paper.

We have already mentioned how the mere knowledge of the existence of minimizers in $\Phi_H(T)$ for every $T>0$ allows one to reduce the analysis of concentrations to the simplest possible case of a *single* boundary concentration (thus leading to Theorem 1.1-(i)). Finer properties of Φ_H are exploited in the proof of Theorem 1.1-(ii), which goes as follows. We consider the existence of a minimizer $(u,\mathsf{v},\mathsf{t})$ of $\Phi^*_\Omega(T)$ with $\mathsf{v}>0$, and, keeping in mind that $\Phi_\Omega(T)=\Phi^*_\Omega(T)$, we aim to obtain a contradiction to $\mathsf{v}>0$ by constructing a competitor v of $\Phi_\Omega(T)$ with

$$\int_\Omega |\nabla v|^p < \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p .$$

We seek v in the form $v=u_\varepsilon$, for the *Ansatz* given by

$$u_\varepsilon(x) = (1-\varphi_\varepsilon(x))\,u(x) + \varphi_\varepsilon(x)\,\big(U^{(\varepsilon)}\circ g\big)(x)\,, \qquad x\in\Omega\,. \tag{1.25}$$

Here $x_0\in\partial\Omega$ is a boundary point of Ω with positive mean curvature[2], i.e. $\mathrm{H}_{\partial\Omega}(x_0)>0$; φ_ε is a cut-off function between $B_{\varepsilon^\beta}(x_0)$ and $B_{2\,\varepsilon^\beta}(x_0)$ for $\beta=\beta(n,p)\in(0,1)$ to be suitably chosen (the condition $n>2\,p$ enters in this choice); g is a boundary flattening diffeomorphism near x_0; and, finally, $U=U_\tau+b\,\varepsilon\,V_\tau$ for $\tau=\mathsf{t}/\mathsf{v}$, V_τ a standard perturbation of U_τ, and b a constant suitably chosen depending on n, p, $H_{\partial\Omega}(x_0)$ and τ. The energy, volume and trace expansions for u_ε as $\varepsilon\to 0^+$ are computed to be

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p \le \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p \qquad\qquad\qquad\qquad\qquad\qquad & \qquad (1.26)\\ -\Big\{\mathcal{L}(U_\tau) - \frac{(n-p)}{n}\,\lambda_H(\tau)\,\mathcal{M}(U_\tau)\Big\}\,\mathrm{H}_{\partial\Omega}(x_0)\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\,, &\end{aligned}$$

$$\int_\Omega u_\varepsilon^{p^\star} = 1+\mathrm{o}(\varepsilon)\,, \qquad \int_{\partial\Omega} u_\varepsilon^{p^\#} = T^{p^\#}+\mathrm{o}(\varepsilon)\,, \tag{1.27}$$

where $\lambda_H(T)$ is the volume Lagrange multiplier of U_T (see (3.7) below), and where $\mathcal{L}$ and $\mathcal{M}$ are functionals defined on $U:H\to\mathbb{R}$ by

$$\mathcal{L}(U) = \int_H x_n\,|\nabla U|^p - p\,x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,, \tag{1.28}$$

$$\mathcal{M}(U) = \int_H x_n\,U^{p^\star}\,. \tag{1.29}$$

Modulo $\mathrm{o}(\varepsilon)$-perturbations of u_ε aimed at correcting the volume and trace constraints to the exact values needed for inclusion in $\mathcal{X}_\Omega(T)$, we have constructed the required competitors, and proved Theorem 1.1-(ii), if we can show the existence of $c(n,p,T)>0$ such that

$$\mathcal{L}(U_T) - \frac{(n-p)}{n}\,\lambda_H(T)\,\mathcal{M}(U_T) \ge c(n,p,T)\,, \qquad \forall T>0\,. \tag{1.30}$$

Of course, the full characterization of Φ_H plays a crucial role in our proof of (1.30), see Lemma 3.1 below.

While Theorem 1.3-(ii) is immediate from Theorem 1.1-(ii) thanks to the rigidity criterion (1.23), the proof of Theorem 1.3-(i) requires an additional argument, which once more exploits several fine properties of Φ_B and Φ_H: these include (1.23), (1.24), and information on the signs of the Lagrange multipliers $\lambda_H(T)$ and $\sigma_H(T)$ for minimizers U_T of $\Phi_H(T)$ (see (3.12) and (3.11) below).

[2] Our convention is that the scalar mean curvature of $\partial\Omega$ is computed with respect to the outer unit normal to Ω, so that every bounded open set with C^2-boundary has at least one boundary point of positive mean curvature.

1.4. **Organization of the paper.** After collecting a few preliminary results in section 2, in section 3 we study in detail various properties of Φ_H and of its minimizers: in particular, we prove the key inequality (1.30) (see Lemma 3.1), and discuss in detail the *Ansatz* (1.25) (see Lemma 3.4). Sections 4 and 5 contain, respectively, the proofs of Theorem 1.1 and Theorem 1.3. Finally, we collect some auxiliary, routine proofs in an appendix.

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2. Notation and preparations

Some basic notation is presented in section 2.1. We then discuss, in separate subsections, four useful technical lemmas: a concentration-compactness lemma with boundary terms (Lemma 2.1); a second order expansion for the boundary flattening diffeomorphisms used in the *Ansatz* (1.25) (Lemma 2.3); some basic regularity information on minimizers of $\Phi_\Omega(T)$ (Lemma 2.5); and the basic technique of "volume/trace correcting variations" (Lemma 2.6). Some proofs are postponed to the appendix.

2.1. **Notation.** Throughout the paper we always assume that $n \geq 2$ and $p \in (1, n)$. We denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the Lebesgue measure and the k-dimensional Hausdorff measure of $\mathbb{R}^n$, although we simply set $|E|$ in place of $\mathcal{L}^n(E)$. We denote by $B_r(x)$ the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, and set $B_r = B_r(0)$, while B denotes a ball of unspecified center and radius.

Following a standard shorthand notation, by "$f(x) = g(x) + \mathrm{O}_{a,b}(|x|)$ for $|x| > R$" we mean that $|f(x) - g(x)| \leq C(a, b)\,|x|$ if $|x| > R$; by "$f(x) = g(x) + \mathrm{o}_{a,b}(|x|)$ as $|x| \to 0$" we mean that $\lim_{|x|\to 0} |f(x) - g(x)|/|x| = 0$ at a rate that is uniform with respect to the parameters a and b.

In general, we will use capital letters (e.g. U, V, Ψ) to denote functions defined on the half space H and lowercase letters (e.g. u, v, φ) to denote functions defined on an open bounded domain Ω.

2.2. **Concentration-compactness.** The following lemma is a version of Lions' celebrated concentration-compactness lemma and provides a natural starting point to study minimizing sequences of $\Phi_\Omega(T)$.

Lemma 2.1 (Concentration-compactness). *Let $n \geq 2$, $p \in (1, n)$, and let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. If $\{u_j\}_j$ is a sequence in $L^1_{\mathrm{loc}}(\Omega)$, $\{\nabla u_j\}_j$ is bounded in $L^p(\Omega; \mathbb{R}^n)$ and $u_j \rightharpoonup u$ as distributions in Ω, then the Radon measures on $\mathbb{R}^n$ defined by*

$$\mu_j = |\nabla u_j|^p \, \mathcal{L}^n \llcorner \Omega\,, \qquad \nu_j = |u_j|^{p^\star} \, \mathcal{L}^n \llcorner \Omega\,, \qquad \tau_j = |u_j|^{p^\#} \, \mathcal{H}^{n-1} \llcorner \partial\Omega\,, \tag{2.1}$$

have subsequential weak-star limits μ, ν and τ which satisfy

$$\nu = |u|^{p^\star} \mathcal{L}^n \llcorner \Omega + \sum_{i \in I} \mathsf{v}_i^{p^\star} \, \delta_{x_i}\,, \tag{2.2}$$

$$\tau = |u|^{p^\#} \mathcal{H}^{n-1} \llcorner \partial\Omega + \sum_{i \in I} \mathsf{t}_i^{p^\#} \, \delta_{x_i}\,, \tag{2.3}$$

$$\mu \geq |\nabla u|^p \mathcal{L}^n \llcorner \Omega + \sum_{i \in I} \mathsf{g}_i^p \, \delta_{x_i}, \tag{2.4}$$

where $\{x_i\}_{i \in I} \subset \overline{\Omega}$ is at most countable set, $\mathsf{v}_i > 0$ and $\mathsf{t}_i \geq 0$ for every $i \in I$, $\mathsf{t}_i > 0$ only if $x_i \in \partial\Omega$, and

$$\mathsf{g}_i \geq \mathsf{v}_i \, \Phi_H\Big(\frac{\mathsf{t}_i}{\mathsf{v}_i}\Big)\,, \qquad \forall i \in I\,. \tag{2.5}$$

In particular, $\mathsf{g}_i \geq S\,\mathsf{v}_i$ whenever $x_i \in \Omega$.

Proof. See appendix A. □

2.3. **Near-boundary coordinates.** In this section, we introduce two types of coordinates for a neighborhood of a boundary point of a domain Ω: one that requires minimal regularity of the boundary of Ω and will suffice in the proofs of Theorem 1.1**(i)** and Theorem 1.3**(i)**, and a second that requires C^2 regularity of the boundary of Ω and will be used in the proof of Theorem 1.1**(ii)** and Theorem 1.3**(ii)**.

Given an open set Ω with C^1-boundary, we denote by ν_Ω its outer unit normal and by $T_x(\partial\Omega)$ the tangent space to $x \in \partial\Omega$. When Ω has C^2-boundary, we denote by A_Ω and H_Ω the second fundamental form and the scalar mean curvature of $\partial\Omega$ defined by ν_Ω. To define coordinates near boundary points of Ω, for $x \in \mathbb{R}^n$ we set $\mathbf{p}(x) = x - x_n\, e_n$, $\mathbf{D}_r = \{x : x_n = 0\,, |\mathbf{p}x| < r\}$, and $\mathbf{C}_r = \big\{x : |x_n| < r\,, |\mathbf{p}(x)| < r\big\}$. In particular, if Ω is an open set with C^1-boundary such that

$$0 \in \partial\Omega\,, \qquad T_0\,(\partial\Omega) = \{x_n = 0\}\,, \qquad \nu_\Omega(0) = -e_n\,, \tag{2.6}$$

then we can find $r_0 > 0$ and $\ell : \mathbf{D}_{r_0} \to (-r_0, r_0)$ such that $\ell(0) = 0$, $\nabla\ell(0) = 0$, and

$$\begin{aligned} \Omega \cap \mathbf{C}_{r_0} &= \big\{x + t\,e_n : x \in \mathbf{D}_{r_0}\,, r_0 > t > \ell(x)\big\}\,, \\ (\partial\Omega) \cap \mathbf{C}_{r_0} &= \big\{x + \ell(x)\,e_n : x \in \mathbf{D}_{r_0}\big\}\,. \end{aligned}$$

We then define the maps $F : \mathbf{D}_{r_0} \to \partial\Omega$, $f : \mathbf{C}_{r_0} \to \mathbb{R}^n$ and $\hat f : \mathbf{C}_{r_0} \to \mathbb{R}^n$ by setting

$$F(x) = x + \ell(x)\,e_n\,, \qquad x \in \mathbf{D}_{r_0}\,, \tag{2.7}$$

$$\hat f(x) = F(\mathbf{p}x) + x_n e_n\,, \qquad x \in \mathbf{C}_{r_0}\,. \tag{2.8}$$

$$f(x) = F(\mathbf{p}x) - x_n\,\nu_\Omega(F(\mathbf{p}x))\,, \qquad x \in \mathbf{C}_{r_0}\,. \tag{2.9}$$

In this way, for every $y \in (\partial\Omega) \cap \mathbf{C}_{r_0}$, if we set $y = F(\mathbf{p}x)$, then

$$\nu_\Omega(y) = \frac{\nabla\ell(x) - e_n}{\sqrt{1 + |\nabla\ell(x)|^2}}\,, \qquad \mathrm{H}_\Omega(y) = \mathrm{div}\Big(\frac{\nabla\ell}{\sqrt{1+|\nabla\ell|^2}}\Big)(x)\,.$$

Notice that the map f need not be of class C^1 if the boundary of Ω is only of class C^1, while the map $\hat f$ will be as regular as the boundary of Ω. The following lemma summarizes basic properties about the map $\hat f$.

Lemma 2.2 (Near-boundary coordinates, one)**.** *If $H = \{x_n > 0\}$, Ω is an open set with C^1-boundary and* (2.6) *holds, then there exist r_0 and C_0 positive such that the map $\hat f$ in* (2.8) *defines a C^1-diffeomorphism from $\mathbf{C}_{r_0}$ to its image, taking $\mathbf{D}_{r_0}$ into $\partial\Omega$ and with*

$$\hat f(\mathbf{C}_{r/C_0} \cap H) \subset \Omega \cap B_r \subset \hat f(\mathbf{C}_{C_0\,r} \cap H) \qquad \forall r < r_0/C_0\,. \tag{2.10}$$

Moreover, letting $\hat g = \hat f^{-1}$ denote the inverse of $\hat f$, we have

$$\nabla\hat f = \mathrm{Id}_{\mathbb{R}^n} + \mathrm{o}(1)\,, \qquad (\nabla\hat g)^* \circ \hat f = \mathrm{Id}_{\mathbb{R}^n} + \mathrm{o}(1)\,, \tag{2.11}$$

$$J\hat f = 1 + \mathrm{o}(1)\,, \qquad 1 \le J^{\partial H}\hat f \le 1 + \mathrm{o}(1)\,, \qquad \text{for } x \in \mathbf{C}_{r_0}. \tag{2.12}$$

The orders in (2.11) *and* (2.12) *depend on $\partial\Omega$ and on $0 \in \partial\Omega$.*

Proof. See appendix B. □

The map f defined in (2.9) has the advantage that, when the boundary of Ω is at least of class C^2, curvature quantities appear in expansions of the metric coefficients and the volume form in these coordinates. These properties are the content of the following lemma.

Lemma 2.3 (Near-boundary coordinates, two)**.** *If $H = \{x_n > 0\}$, Ω is an open set with C^2-boundary and* (2.6) *holds, then there exist r_0 and C_0 positive such that the map f in* (2.9) *defines a C^1-diffeomorphism from $\mathbf{C}_{r_0}$ to its image, taking $\mathbf{D}_{r_0}$ into $\partial\Omega$ and with*

$$f(\mathbf{C}_{r/C_0} \cap H) \subset \Omega \cap B_r \subset f(\mathbf{C}_{C_0\,r} \cap H) \qquad \forall r < r_0/C_0\,. \tag{2.13}$$

Moreover, for $x \in \mathbf{C}_{r_0}$ and $x \in \mathbf{D}_{r_0}$ respectively, we have

$$Jf(x) = 1 - x_n\, \mathrm{H}_\Omega(0) + \mathrm{O}(|x|^2)\,, \qquad 1 \le J^{\partial H} f(x) \le 1 + \mathrm{O}(|x|^2)\,, \tag{2.14}$$

and if $\{e_i\}_{i=1}^{n-1}$ is an orthonormal basis of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ of eigenvectors of $\nabla^2\ell(0)$ and $\{\kappa_i\}_{i=1}^{n-1}$ denote the corresponding eigenvalues (so that, by (2.6), they are the principal curvatures of $\partial\Omega$ with respect to ν_Ω computed at $0 \in \partial\Omega$, and in particular $\mathrm{H}_\Omega(0) = \sum_{i=1}^{n-1} \kappa_i$), then, letting $g = f^{-1}$ denote the inverse of f, we have

$$(\nabla g)^* \circ f = \mathrm{Id}_{\mathbb{R}^n} + \big(\nabla\ell \otimes e_n - e_n \otimes \nabla\ell\big) + x_n \sum_{i=1}^{n-1} \kappa_i e_i \otimes e_i + \mathrm{O}(|x|^2)\,. \tag{2.15}$$

The orders in (2.14) and (2.15) depend on $\partial\Omega$ and on $0 \in \partial\Omega$.

Proof. See appendix B. □

Remark 2.4. Given $x_0 \in \partial\Omega$, we denote by π_{x_0} the rigid motion of $\mathbb{R}^n$ that maps x_0 to 0 such that (2.6) holds with $\pi_{x_0}(\Omega)$ in place of Ω. Then we set, for $\hat f$ and f defined as in (2.8) and (2.9) respectively but with $\pi_{x_0}(\Omega)$ in place of Ω,

$$\hat f_{x_0} = \pi_{x_0}^{-1} \circ \hat f\,, \qquad f_{x_0} = \pi_{x_0}^{-1} \circ f\,.$$

Clearly these maps are diffeomorphisms on $\mathbf{C}_{r_0}$, mapping $H \cap \mathbf{C}_{r_0}$ into a neighborhood of x_0 in Ω and $\mathbf{D}_{r_0} = (\partial H) \cap \mathbf{C}_{r_0}$ into a neighborhood of x_0 in $\partial\Omega$, and satisfies proper reformulations of the estimates in Lemmas 2.2 and 2.3. Here r_0 and C_0 depend also on the choice of x_0, and can of course be assumed uniform across $x_0 \in \partial\Omega$ if $\partial\Omega$ is bounded.

2.4. **Properties of minimizers.** The following lemma gathers some fundamental properties of minimizers of Φ_Ω that will be needed in the sequel.

Lemma 2.5. *If $n \ge 2$, $p \in (1, n)$, $T > 0$, Ω is a bounded open set with C^2-boundary, and u is a minimizer of $\Phi_\Omega(T)$, then u is bounded and Lipschitz continuous in Ω, there are λ and σ such that the Euler–Lagrange equation (1.2) holds in the weak sense, and the balance condition*

$$\lambda \int_\Omega u^{p^\star - 1} + \sigma \int_{\partial\Omega} u^{p^\# - 1} = 0\,, \tag{2.16}$$

holds.

Proof. By a standard argument, based on similar considerations to the one presented in Lemma 2.6 below, one sees that a minimizer u of $\Phi_\Omega(T)$ is a $W^{1,p}(\Omega)$-distributional solution of the Euler-Lagrange equation (1.2) for some $\lambda, \sigma \in \mathbb{R}$. As soon as Ω is bounded and has Lipschitz boundary, one can exploit (1.2) in conjunction with a Moser iteration argument to prove that $u \in L^\infty(\Omega)$ (see, e.g. [MW19, Theorem 3.1]; their result applies to (1.2) by taking, in the notation of their paper, $\mathcal{A}(x, u, \nabla u) = |\nabla u|^{p-2}\nabla u$, $\mathcal{B}(x, u, \nabla u) = \lambda u^{p^\star - 1}$, and $\mathcal{C}(x, u) = \sigma u^{p^\# - 1}$). On further assuming that $\partial\Omega$ is of class C^2, then the classical result [Lie92, Theorem 1.7] can be applied to deduce that $u \in C^{1,\beta}(\overline{\Omega})$ for a suitable $\beta = \beta(n, p) \in (0, 1)$ (for more details, see [MW19, Theorem 3.9]). In particular, u is bounded and Lipschitz continuous on Ω, as claimed. □

2.5. **Volume/trace correcting variations.** At various stages in our arguments we will need to slightly modify certain competitors so to restore the volume and trace constraints defining $\mathcal{X}_\Omega(T)$. The following lemma describes the basic mechanism used to this end.

Lemma 2.6 (Volume/trace correcting variations)**.** *If $n \ge 2$, $p \in (1, n)$, $M > 0$, Ω is an open set with C^1-boundary, $v \in L^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in L^p(\Omega; \mathbb{R}^n)$, and if $x_0 \in \mathbb{R}^n$ and $r > 0$ are such that*

$$\textstyle\int_{\Omega \setminus B_r(x_0)} v^{p^\star} \text{ and } \int_{(\partial\Omega) \setminus B_r(x_0)} v^{p^\#} \text{ are positive and finite}\,, \tag{2.17}$$

then there exist positive constants η and C, and functions $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ and $\psi \in C^\infty_c(\Omega\setminus\overline{B}_r(x_0))$, all depending on n, p, v and M only, and with the following property.

If $\{v_\varepsilon\}_{\varepsilon<\varepsilon_0} \subset L^1_{\rm loc}(\Omega)$ is such that, for every $\varepsilon < \varepsilon_0$,

$$v_\varepsilon = v \text{ on } \Omega \setminus B_r(x_0)\,, \qquad \int_\Omega |\nabla v_\varepsilon|^p \le M\,, \tag{2.18}$$

then for every (a,b) with $|a|,|b| < \eta/C$, we can find (s,t) with $|s|,|t| < \eta$ such that

$$w_\varepsilon = v_\varepsilon + s\,\varphi + t\,\psi$$

satisfies

$$\int_{\partial\Omega} |w_\varepsilon|^{p^\#} = a + \int_{\partial\Omega} |v_\varepsilon|^{p^\#}\,, \qquad \int_\Omega |w_\varepsilon|^{p^\star} = b + \int_\Omega |v_\varepsilon|^{p^\star}\,, \tag{2.19}$$

$$\Big|\int_\Omega |\nabla w_\varepsilon|^p - \int_\Omega |\nabla v_\varepsilon|^p\Big| \le C\,\big(|a| + |b|\big)\,. \tag{2.20}$$

Proof. By (2.17) there are $\xi \in C^\infty_c(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ and $\psi \in C^\infty_c(\Omega \setminus \overline{B}_r(x_0))$ such that

$$\int_{\partial\Omega} v^{p^\# - 1}\,\xi = 1\,, \qquad \int_\Omega v^{p^\star - 1}\,\psi = 1\,. \tag{2.21}$$

Setting $\varphi = \xi - (\int_\Omega v^{p^\star-1}\xi)\,\psi$, we have $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ with

$$\int_\Omega v^{p^\star - 1}\,\varphi = 0\,, \qquad \int_{\partial\Omega} v^{p^\# - 1}\,\varphi = 1\,. \tag{2.22}$$

We now define $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h_\varepsilon(s,t) = \Big(\int_{\partial\Omega} |v_\varepsilon + s\,\varphi + t\,\psi|^{p^\#} - \int_{\partial\Omega} |v_\varepsilon|^{p^\#},\ \int_\Omega |v_\varepsilon + s\,\varphi + t\,\psi|^{p^\star} - \int_\Omega |v_\varepsilon|^{p^\star}\Big)\,.$$

By (2.18) we have $h_\varepsilon \in C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$ for some $\alpha = \alpha(n,p) \in (0,1)$, with

$$\sup_{\varepsilon<\varepsilon_0} \|h_\varepsilon\|_{C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)} < \infty\,;$$

moreover, $h_\varepsilon(0,0) = 0$ and, by (2.21) and (2.22),

$$\nabla h_\varepsilon(0,0) = \begin{pmatrix} p^\# \int_{\partial\Omega} |v_\varepsilon|^{p^\#-1}\,\varphi & p^\# \int_{\partial\Omega} |v_\varepsilon|^{p^\#-1}\,\psi \\ p^\star \int_\Omega |v_\varepsilon|^{p^\star-1}\,\varphi & p^\star \int_\Omega |v_\varepsilon|^{p^\star-1}\,\psi \end{pmatrix} = \begin{pmatrix} p^\# & 0 \\ 0 & p^\star \end{pmatrix}.$$

We can thus apply the inverse function theorem *uniformly* in ε, to find positive constants η and C_1 depending on n, p, and v so that each h_ε is invertible on $E = \{(s,t) : |s|,|t| < \eta\}$, with $\{(a,b) : |a|,|b| < \eta/C_1\} \subset h_\varepsilon(E)$, and $\nabla h_\varepsilon^{-1}(a,b) \ge {\rm Id}_{\,2\times2}/C_1$ (in the sense of positive definite matrices) for every $(a,b) \in h_\varepsilon(E)$. In particular, if we let $(s,t) = h_\varepsilon^{-1}(a,b)$ for a pair (a,b) with $|a|,|b| < \eta/C_1$, then the function $w_\varepsilon = v_\varepsilon + s\,\varphi + t\,\psi$ satisfies (2.19) and $|(a,b)| = |h_\varepsilon^{-1}(s,t)| \ge |(s,t)|/C_1$. Moreover, by the elementary inequality

$$\big||X+Y|^p - |X|^p\big| \le p\,\max\big\{|X|,|Y|\big\}^{p-1}\,|Y| \qquad \forall X,Y \in \mathbb{R}^n\,,$$

we see that, setting $\gamma = \max\big\{|\nabla v_\varepsilon|, |\nabla\varphi|, |\nabla\psi|\big\}^p$,

$$\begin{aligned}\Big|\int_\Omega |\nabla w_\varepsilon|^p - \int_\Omega |\nabla v_\varepsilon|^p\Big| &\le p\int_\Omega \gamma^{(p-1)/p}\big(|s|\,|\nabla\varphi| + |t|\,|\nabla\psi|\big)\\ &\le C\Big(\int_\Omega \gamma\Big)^{(p-1)/p}\Big(\int_\Omega \big(|s|^p\,|\nabla\varphi|^p + |t|^p\,|\nabla\psi|^p\big)\Big)^{1/p}\\ &\le C\,|(s,t)| \le C_2\,|(a,b)|\,,\end{aligned} \tag{2.23}$$

for a constant C_2 depending on n, p, v, and M. Letting $C = \max\{C_1, C_2\}$ concludes the proof of the lemma. □

3. Boundary concentrations

3.1. **Properties of Φ_H-minimizers.** We recall some facts proved in [MN17] about Φ_H and its minimizers. Recall that we denote by T_0 the minimum point of Φ_H, so that

$$T_0 \in (0, T_E)\,, \qquad \Phi_H(T_0) = 2^{-1/n}\, S(n,p)\,, \tag{3.1}$$

where T_E is the "Escobar trace" defined in (1.8). If we set $H = \{x_n > 0\}$, the minimizers of U_T of $\Phi_H(T)$ for $T > 0$ are characterized (modulo the obvious scaling and translation invariance of Φ_H) as

$$U_T(x) = c_T \begin{cases} \tau_{t_T\, e_n} U_S(x) = (1 + |x - t_T\, e_n|^{p'})^{1-(n/p)}\,, & T \in (0, T_E)\,, \\ \tau_{e_n} U_E(x) = |x + e_n|^{(p-n)/(p-1)}\,, & T = T_E\,, \\ \tau_{s_T\, e_n} U_{BE}(x) = (|x - s_T\, e_n|^{p'} - 1)^{1-(n/p)}\,, & T > T_E\,, \end{cases} \tag{3.2}$$

where the constants c_T, t_T and s_T are chosen in such a way that

$$\int_H U_T^{p^\star} = 1\,, \qquad \int_{\partial H} U_T^{p^\sharp} = T^{p^\sharp}\,, \qquad \int_H |\nabla U_T|^p = \Phi_H(T)^p\,. \tag{3.3}$$

It is convenient to keep in mind that the various formulas for U_T listed in (3.2) all share the same decay behavior at infinity, that is (see (3.20) below), we have

$$U_T(x) \sim |x|^{(p-n)/(p-1)}, \qquad |\nabla U_T| \sim |x|^{(1-n)/(p-1)} \qquad \text{as } |x| \to \infty. \tag{3.4}$$

(where the rate depends on the specific value of T under consideration). The constants t_T and s_T have the following properties: $T \in (0, T_E) \mapsto t_T$ is continuous and strictly decreasing, with $t_T > 0$ if and only if $T \in (0, T_0)$, and

$$\lim_{T \to 0^+} t_T = +\infty\,, \qquad \lim_{t \to (T_E)^-} t_T = -\infty\,, \qquad t_{T_0} = 0\,, \tag{3.5}$$

while $T \in (T_E, \infty) \mapsto s_T$ is continuous, negative, strictly increasing, with

$$\lim_{T \to (T_E)^+} s_T = -\infty\,, \qquad \lim_{T \to +\infty} s_T = -1\,. \tag{3.6}$$

Denoting by $\Delta_p v = \mathrm{div}\,(|\nabla v|^{p-2} \nabla v)$ the p-Laplace operator, we have

$$\begin{cases} -\Delta_p U_T = \lambda_H(T)\, U_T^{p^\star - 1} & \text{on } H\,, \\ |\nabla U_T|^{p-2} \dfrac{\partial U_T}{\partial \nu_H} = \sigma_H(T)\, U_T^{p^\sharp - 1} & \text{on } \partial H\,, \end{cases} \qquad \forall T > 0\,, \tag{3.7}$$

where $\lambda_H, \sigma_H : (0, \infty) \to \mathbb{R}$ are continuous and satisfy the relations

$$\Phi_H(T)^p = \lambda_H(T) + \sigma_H(T)\, T^{p^\#}\,, \qquad \sigma_H(T) = \frac{\Phi_H(T)^{p-1}\, \Phi_H'(T)}{T^{p^\# - 1}}\,, \tag{3.8}$$

(see [MN17, Lemma 3.3][3]) as well as

$$\lim_{T \to 0^+} \sigma_H(T) = -\infty\,, \qquad \lim_{T \to +\infty} \sigma_H(T) = +\infty \tag{3.9}$$

$$\lim_{T \to 0^+} \lambda_H(T) > 0\,, \qquad \lim_{T \to +\infty} \lambda_H(T) = -\infty\,. \tag{3.10}$$

[3]Notice that in [MN17, (3.16)] it is incorrectly stated that $\sigma_H(T) = \Phi_H(T)^{p-1}\, \Phi_H'(T)/(p^\#\, T^{p^\# - 1})$, where the extra $1/p^\#$-factor is wrongly introduced in the penultimate displayed equation in the proof of Lemma 3.3, where $\tau'(0) = T^{1-p^\#}/p^\#$ should be replaced by $\tau'(0) = T^{1-p^\#}$. This error is inconsequential for the arguments in [MN17], since this expression for $\sigma_H(T)$ is only used in equation (3.22) and subsequent displayed equations, and since, in all these subsequent identities, a generic multiplicative factor $c(n,p)$ is used (in particular, two functions of (n,p) differing by a $1/p^\#$-factor are both $c(n,p)$). It will instead be important in the proof of (4.3) to work with correct expression for $\sigma_H(T)$.

The signs of σ_H and λ_H can be easily deduced from (3.2), and satisfy

$$(0,T_0)=\{\sigma_H<0\}\,,\qquad (T_0,\infty)=\{\sigma_H>0\}\,,\qquad \sigma_H(T_0)=0\,, \tag{3.11}$$

$$(0,T_E)=\{\lambda_H>0\}\,,\qquad (T_E,\infty)=\{\lambda_H<0\}\,,\qquad \lambda_H(T_E)=0\,. \tag{3.12}$$

3.2. **A key inequality and further properties of U_T.** In this section, we prove the key inequality (3.13) for the functions $\mathcal{L}$ and $\mathcal{M}$ introduced in (1.28) and (1.29), namely

$$\mathcal{L}(U)=\int_H x_n\,|\nabla U|^p-p\,x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,$$

$$\mathcal{M}(U)=\int_H x_n\,U^{p^\star}\,.$$

Whenever U satisfies the decay properties (3.4) (e.g., when U is a compactly supported perturbation of some U_T), we have that $\mathcal{M}(U)<\infty$; however, $\mathcal{L}(U)<\infty$ under (3.4) if and only if $n>2\,p-1$; see (3.24) and (3.25) below.

Lemma 3.1 (Key inequality)**.** *If $n\geq 2$, $p\in(1,n)$, $n>2p-1$, and $T>0$, then there is a positive constant $c(n,p,T)$ such that*

$$\mathcal{L}(U_T)-\frac{n-p}{n}\,\lambda_H(T)\,\mathcal{M}(U_T)\geq c(n,p,T)\,. \tag{3.13}$$

The following lemma will be useful in proving Lemma 3.1.

Lemma 3.2. *If $H=\{x_n>0\}$, $U:H\to\mathbb{R}$ is radially symmetric with respect to $t\,e_n$ for some $t\in\mathbb{R}$, and $\int_H x_n|\nabla U|^p$ is finite, then*

$$\int_H x_n\,|\nabla U|^{p-2}\big\{(\partial_n U)^2-(\partial_1 U)^2\big\}>0\,.$$

Proof of Lemma 3.2. We have $U(x)=\eta(|x-t\,e_n|)$, $y=x-t\,e_n$, $r=|y|$, and $\hat y=y/|y|$, so that

$$x_n\,|\nabla U|^{p-2}\big\{(\partial_n U)^2-(\partial_1 U)^2\big\}=(y_n+t)\,|\eta'(r)|^p\big\{(\hat y_n)^2-(\hat y_1)^2\big\}\,,$$

and, setting $y=r\,z$,

$$\begin{aligned}\int_H x_n\,|\nabla U|^{p-2}\big\{(\partial_n U)^2-(\partial_1 U)^2\big\}&=\int_{\{y_n>-t\}}(y_n+t)\,|\eta'(r)|^p\big\{(\hat y_n)^2-(\hat y_1)^2\big\}\\ =\int_0^\infty|\eta'(r)|^p\,dr&\int_{\{y_n>-t\}\cap\partial B_r}(y_n+t)\,\big\{(\hat y_n)^2-(\hat y_1)^2\big\}\,d\mathcal{H}^{n-1}_y\\ =\int_0^\infty r^n\,|\eta'(r)|^p\,dr&\int_{\{z_n>-t/r\}\cap\partial B_1}\Big(z_n+\frac{t}{r}\Big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^{n-1}_z\,.\end{aligned}$$

We conclude the proof by showing that

$$\int_{\{z_n>-s\}\cap\partial B_1}\big(z_n+s\big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^{n-1}_z>0\qquad\text{for all } s\in(-1,1), \tag{3.14}$$

noting that for each $s\in[1,\infty)$, this integral vanishes by symmetry while for $s\in(-\infty,-1]$ the domain of integration is empty. To see (3.14), let $\mathbf{p}:\mathbb{R}^n\to\mathbb{R}^{n-2}$ denote the projection map $\mathbf{p}(x)=(x_2,...,x_{n-1})$ (if $n=2$ there is no need to introduce $\mathbf{p}$). The tangential coarea factor of $\mathbf{p}$ along ∂B_1 defines a positive function $K:\partial B_1\to(0,\infty]$ which is $\mathcal{H}^{n-1}$-a.e. finite on ∂B_1, and which is *independent* of the variables (x_1,x_n), i.e. $K(x_1,w,x_n)=K(w)$ for every $(x_1,w,x_n)\in\partial B_1$. Therefore, setting for brevity $M_s=\{z_n>-s\}\cap\partial B_1$,

$$\int_{M_s}\big(z_n+s\big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^{n-1}_z=\int_{\mathbf{p}(M_s)}\frac{d\mathcal{L}^{n-2}_w}{K(w)}\int_{M_s\cap\mathbf{p}^{-1}(w)}\big(z_n+s\big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^1_{(z_1,z_n)}\,,$$

where

$$M_s \cap \mathbf{p}^{-1}(w) = \Big\{(z_1, z_n) : z_1^2 + z_n^2 = 1 - |w|^2 \,, z_n > -s\Big\}\,, \qquad s \in (-1, 1)\,.$$

As before, the inner integral above vanishes when $1 - |w|^2 \le s$ by symmetry and the domain of integration is empty when $-s \ge 1 - |w|^2$. We are thus left to prove that

$$\int_{-\alpha}^{\pi+\alpha} \big(\sin\theta + \sin\alpha\big)\,\big\{\sin^2\theta - \cos^2\theta\big\}\,d\theta > 0$$

for $\alpha \in (0, \pi/2)$ (corresponding to the case when $s \ge 0$) and

$$\int_{\alpha}^{\pi-\alpha} \big(\sin\theta - \sin\alpha\big)\,\big\{\sin^2\theta - \cos^2\theta\big\}\,d\theta > 0$$

for $\alpha \in (0, \pi/2)$ (corresponding to the case when $s < 0$). Direct computation shows that both of these integrals are equal to the positive quantity $(2/3)\,(\cos\alpha)^3$. ☐

We now prove Lemma 3.1.

Proof of Lemma 3.1. Testing (3.7) with $x_n\,U_T$ we find

$$\lambda_H(T)\int_H x_n\,U_T^{p^\star} = -\int_H x_n\,U_T\,\Delta_p U_T = \int_H |\nabla U_T|^{p-2}\,\nabla U_T\cdot\nabla(x_n\,U_T)\,.$$

Here the integration by parts is justified since $x_n = 0$ on ∂H and since by (3.4),

$$\Big|\int_{H\cap\partial B_R} x_n\,U_T\,|\nabla U_T|^{p-2}(\nu_{B_R}\cdot\nabla U_T)\Big| \le C\,\frac{R^{n-1}\,R}{R^{(n-p)/(p-1)}}\,\Big(\frac{1}{R^{(n-1)/(p-1)}}\Big)^{p-1} \to 0$$

like $R^{-(n+1)/(p-1)}$ as $R\to\infty$. We thus find that

$$\begin{aligned}\mathcal{L}(U_T) - \frac{n-p}{n}\,\lambda_H(T)\,\mathcal{M}(U_T) &= \int_H x_n\,|\nabla U_T|^p - p\,x_n\,(\partial_1 U_T)^2\,|\nabla U_T|^{p-2}\\ &\qquad -\frac{n-p}{n}\Big\{\int_H x_n\,|\nabla U_T|^p + \int_H U_T\,|\nabla U_T|^{p-2}\,\partial_n U_T\Big\}\,.\end{aligned}$$

Now, since $|\nabla U_T|$ is symmetric by reflection with respect to the hyperplanes $\{x_i = 0\}$, $i = 1, ..., n-1$, we see that

$$\begin{aligned}\int_H x_n\,|\nabla U_T|^p &= \sum_{i=1}^{n-1}\int_H x_n\,(\partial_i U_T)^2|\nabla U_T|^{p-2} + \int_H x_n\,(\partial_n U_T)^2|\nabla U_T|^{p-2}\\ &= (n-1)\int_H x_n\,(\partial_1 U_T)^2|\nabla U_T|^{p-2} + \int_H x_n\,(\partial_n U_T)^2|\nabla U_T|^{p-2}\end{aligned}$$

so that continuing from above we have

$$\begin{aligned}&\mathcal{L}(U_T) - \frac{n-p}{n}\,\lambda_H(T)\,\mathcal{M}(U_T)\\ &= \frac{p}{n}\int_H x_n\,|\nabla U_T|^p - p\int_H x_n\,(\partial_1 U_T)^2\,|\nabla U_T|^{p-2} - \frac{n-p}{n}\int_H U_T\,|\nabla U_T|^{p-2}\,\partial_n U_T\\ &= \frac{p}{n}\int_H x_n\,|\nabla U_T|^{p-2}\Big\{(\partial_n U_T)^2 - (\partial_1 U_T)^2\Big\} + \frac{n-p}{n}\int_H U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big)\,.\end{aligned}$$

In particular, the lemma is proved by showing that

$$\int_H x_n\,|\nabla U_T|^{p-2}\big\{(\partial_n U_T)^2 - (\partial_1 U_T)^2\big\} > 0\,, \tag{3.15}$$

$$\int_H U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big) > 0\,, \tag{3.16}$$

where the first inequality, (3.15), is immediate from Lemma 3.2 (recall that $n > 2p-1$ and U_T is radially symmetric with respect to te_n for some $t\in\mathbb{R}$).

We are thus left to prove (3.16). This is immediate in the case when $T \ge T_0$, because in that case, by (3.5) and (3.6), U_T has center of symmetry at $t\,e_n$ for some $t \le 0$, and thus $\partial_n U_T < 0$ on H. By (3.5), if $T \in (0, T_0)$, then U_T has center of symmetry at $t\,e_n$ for some $t > 0$. Correspondingly, $U_T\,\partial_n U_T$ is odd with respect to $\{x_n = t\}$, with $U_T\,\partial_n U_T < 0$ on $\{x_n > t\}$ and $U_T\,\partial_n U_T > 0$ on $\{0 < x_n < t\}$: in particular, if p_t denotes the reflection with respect to $\{x_n = t\}$, then

$$
\begin{aligned}
\int_{2t > x_n > t} x_n\, U_T\,(-\partial_n U_T) &= \int_{t > x_n > 0} (p_t(x)\cdot e_n)\ [U_T\,(-\partial_n U_T)](p_t(x))\,dx \\
&= \int_{t > x_n > 0} (p_t(x)\cdot e_n)\ [U_T\,\partial_n U_T)](x)\,dx \ge \int_{t > x_n > 0} x_n\ U_T\,\partial_n U_T\,,
\end{aligned}
$$

so that

$$
\int_H U_T\,|\nabla U_T|^{p-2}\big(-\partial_n U_T\big) \ge \int_{\{x_n > 2t\}} U_T\,|\nabla U_T|^{p-2}\big(-\partial_n U_T\big)\,,
$$

and the latter integral is positive because $\partial_n U_T < 0$ on $\{x_n > t\}$. □

3.3. **Standard variations of Φ_H-minimizers.** We now introduce a "class of standard variations" of minimizers of Φ_H. With $H = \{x_n > 0\}$, we define $\zeta = \zeta(r, T) : [0,\infty) \times (0,\infty) \to [0,\infty)$, so that, setting $V_T(x) = \zeta(|x - e_n|, T)$ for $x \in \mathbb{R}^n$, we have

$$
V_T \in C^\infty_c(H;[0,\infty))\,, \qquad \int_H U_T^{p^\star - 1}\,V_T = 1\,, \qquad \int_{\partial H} U_T^{p^\# - 1}\,V_T = 0\,. \tag{3.17}
$$

Given $T > 0$ we denote by

$$
\mathcal{U}_T \tag{3.18}
$$

the family of functions $U : H \to \mathbb{R}$ of the form

$$
U = U_T + t\,V_T\,, \qquad |t| \le 1\,.
$$

The following lemma contains some basic properties of functions in $\mathcal{U}_T$. We notice that

$$
\begin{gathered}
\text{every } U \in \mathcal{U}_T \text{ is symmetric by reflection} \\
\text{with respect to the coordinates } x_1,\, ...,\, x_{n-1}\,.
\end{gathered} \tag{3.19}
$$

Lemma 3.3 (Standard variations of U_T)**.** *If $n \ge 2$, $p \in (1, n)$, and $T > 0$, then there are positive constants R_0 and C_0 depending on n, p, T, and V_T such that the following properties hold:*

(i): *if $U \in \mathcal{U}_T$, then for every $|x| > R_0$ we have*

$$
\begin{aligned}
\frac{1}{C_0|x|^{(n-p)/(p-1)}} \ &\le U(x) \le \frac{C_0}{|x|^{(n-p)/(p-1)}}\,, \\
\frac{1}{C_0|x|^{(n-1)/(p-1)}} &\le |\nabla U(x)| \le \frac{C_0}{|x|^{(n-1)/(p-1)}}\,,
\end{aligned} \tag{3.20}
$$

and for every $R > R_0$,

$$
\int_{H\setminus B_R} U^{p^\star} \le \frac{C_0}{R^{n/(p-1)}}\,, \qquad \int_{H\setminus B_R} U^{p^\star - 1} \le \frac{C_0}{R^{p/(p-1)}}\,, \tag{3.21}
$$

$$
\int_{H\cap(B_{2R}\setminus B_R)} U^p \le \frac{C_0}{R^{(n-p^2)/(p-1)}}\,, \qquad \int_{H\setminus B_R} |\nabla U|^p \le \frac{C_0}{R^{(n-p)/(p-1)}}\,, \tag{3.22}
$$

$$
\int_{(\partial H)\setminus B_R} U^{p^\#} \le \frac{C_0}{R^{(n-1)/(p-1)}}\,, \qquad \int_{(\partial H)\setminus B_R} U^{p^\# - 1} \le \frac{C_0}{R}\,, \tag{3.23}
$$

$$
\int_{H\setminus B_R} |x|\,U^{p^\star} \le \frac{C_0}{R^{[1+n-p]/(p-1)}}\,, \tag{3.24}
$$

$$
\int_{H\setminus B_R} |x|\,|\nabla U|^p \le \frac{C_0}{R^{(n+1-2p)/(p-1)}}\,, \qquad \textit{if } n > 2\,p - 1\,. \tag{3.25}
$$

(ii): *for every $U \in \mathcal{U}_T$ we have*

$$\int_H |\nabla U|^p = \Phi_H(T)^p + p\,\lambda_H(T)\,t + \mathrm{o}(t)\,, \tag{3.26}$$

$$\int_H U^{p^\star} = 1 + p^\star\,t + \mathrm{o}(t)\,, \tag{3.27}$$

$$\int_{\partial H} U^{p^\#} = T^{p^\#}\,. \tag{3.28}$$

Proof of Lemma 3.3. Since V_T is assumed to be compactly supported, statement (i) follows immediately from the corresponding properties for U_T. More specifically, we deduce (3.20) from (3.2), and (3.21)–(3.25) from (3.20). Statement (ii) follows from (3.7). □

3.4. **The** *Ansatz* **for boundary concentrations.** We next use the standard variations of minimizers of Φ_H described in Lemma 3.3 to define certain competitors for Φ_Ω that provide us with a notion of "standard boundary concentration." Recall the notation $U^{(\varepsilon)}$ for dilations introduced in (1.5).

Lemma 3.4. *Fix $n \ge 2$, $p \in (1,n)$, $T > 0$, $U \in \mathcal{U}_T$. Let Ω be an open set with C^1-boundary, $x_0 \in \partial\Omega$, and let $\hat f = \hat f_{x_0}$, $\hat g = \hat f^{-1}$, $f = f_{x_0}$ and $g = f^{-1}$ be determined as in Remark 2.4 starting from Ω and x_0. Then the following statements hold:*

(i): *If $v \in W^{1,p}(\Omega)$, $\beta \in (0,1)$, $r_1 = \varepsilon^\beta$, $r_2 = 2\,\varepsilon^\beta$, and φ_ε is a cut-off function between $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$ with $|\nabla\varphi_\varepsilon| \le C\varepsilon^{-\beta}$, then*

$$v_\varepsilon(x) = (1-\varphi_\varepsilon(x))\,v(x) + \varphi_\varepsilon(x)\,\big(U^{(\varepsilon)} \circ \hat g\big)(x)\,, \qquad x \in \Omega\,, \tag{3.29}$$

satisfies

$$\lim_{\varepsilon\to0^+}\int_\Omega |\nabla v_\varepsilon|^p = \int_H |\nabla U|^p + \int_\Omega |\nabla v|^p\,, \tag{3.30}$$

$$\lim_{\varepsilon\to0^+}\int_\Omega v_\varepsilon^{p^\star} = \int_H U^{p^\star} + \int_\Omega v^{p^\star}\,, \tag{3.31}$$

$$\lim_{\varepsilon\to0^+}\int_{\partial\Omega} v_\varepsilon^{p^\#} = \int_{\partial H} U^{p^\#} + \int_{\partial\Omega} v^{p^\#}\,. \tag{3.32}$$

(ii): *If $n > 2p$, $v \in \mathrm{Lip}(\Omega)$ and Ω has C^2-boundary, then there exists a choice of $\beta = \beta(n,p) \in (0,1)$ (used in the definition of $r_1 = \varepsilon^\beta$ and $r_2 = 2\,\varepsilon^\beta$), such that the function*

$$v_\varepsilon(x) = (1-\varphi_\varepsilon(x))\,v(x) + \varphi_\varepsilon(x)\,\big(U^{(\varepsilon)} \circ g\big)(x)\,, \qquad x \in \Omega\,, \tag{3.33}$$

satisfies (3.30), (3.31), *and* (3.32) *in the more precise form*

$$\int_\Omega |\nabla v_\varepsilon|^p = \int_H |\nabla U|^p + \int_\Omega |\nabla v|^p - \mathrm{H}_{\partial\Omega}(x_0)\,\mathcal{L}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,, \tag{3.34}$$

$$\int_\Omega v_\varepsilon^{p^\star} = \int_H U^{p^\star} + \int_\Omega v^{p^\star} - \mathrm{H}_{\partial\Omega}(x_0)\,\mathcal{M}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,, \tag{3.35}$$

$$\int_{\partial\Omega} v_\varepsilon^{p^\#} = \int_{\partial H} U^{p^\#} + \int_{\partial\Omega} v^{p^\#} + \mathrm{o}(\varepsilon)\,, \tag{3.36}$$

as $\varepsilon \to 0^+$. Here $\mathcal{L}(U)$ and $\mathcal{M}(U)$ are defined in (1.28) *and* (1.29) *and the orders in* (3.34), (3.35), *and* (3.36) *depend on n, p, T, and v.*

Proof. Without loss of generality we assume that $x_0 = 0 \in \partial\Omega$, $T_0\,(\partial\Omega) = \{x_n = 0\}$ and $\nu_\Omega(0) = -e_n$. We carry out the proof in several steps.

Step one: We start by noticing the following estimates for the energy, volume and trace of v_ε in transition region for the cut-off function φ_ε. The estimates in this step hold in identical form with the same proofs for v_ε defined from $\hat f$ as in (3.29) and for v_ε defined from f as in (3.33); we write the proof for (3.33). First, with $v\in W^{1,p}(\Omega)$,

$$\lim_{\varepsilon\to 0}\max\Big\{\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|\nabla v_\varepsilon|^p,\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\star},\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\#}\Big\}=0\,,\tag{3.37}$$

and, second, under the additional assumption that $v\in\mathrm{Lip}(\Omega)$,

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|\nabla v_\varepsilon|^p\le C\,\max\Big\{\varepsilon^{(1-\beta)\,(n-p)/(p-1)},\varepsilon^{\beta(n-p)}\Big\}\,,\tag{3.38}$$

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\star}\le C\,\max\Big\{\varepsilon^{(1-\beta)\,n/(p-1)},\varepsilon^{\beta\,n}\Big\}\tag{3.39}$$

$$\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\#}\le C\,\max\Big\{\varepsilon^{(1-\beta)\,(n-1)/(p-1)},\varepsilon^{\beta(n-1)}\Big\}\,.\tag{3.40}$$

Indeed, we have $\nabla v_\varepsilon=a_\varepsilon+b_\varepsilon$ for

$$a_\varepsilon=\varphi_\varepsilon\,(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]+(U^{(\varepsilon)}\circ g)\,\nabla\varphi_\varepsilon\,,\qquad b_\varepsilon=(1-\varphi_\varepsilon)\,\nabla v-v\,\nabla\varphi_\varepsilon\,.$$

By (2.13), and thanks to $|\nabla g|,Jf\le 2$ on $\mathbf{C}_{r_0}$, we find

$$\begin{aligned}\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|a_\varepsilon|^p&\le C\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p+\frac{U^{(\varepsilon)}(g)^p}{\varepsilon^{\beta\,p}}\\ &\le\ C\int_{H\cap(B_{Cr_2}\setminus B_{r_1/C})}|\nabla U^{(\varepsilon)}|^p+\frac{(U^{(\varepsilon)})^p}{\varepsilon^{\beta\,p}}\\ &=\ C\int_{H\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}|\nabla U|^p+\varepsilon^{np/p^\star}\,\frac{U^p}{\varepsilon^{\beta\,p}}\,\varepsilon^n\\ &\le C\Big\{\varepsilon^{(1-\beta)(n-p)/(p-1)}+\frac{\varepsilon^p\,\varepsilon^{(\beta-1)(p^2-n)/(p-1)}}{\varepsilon^{\beta\,p}}\Big\}=C\,\varepsilon^{(1-\beta)(n-p)/(p-1)}\,,\end{aligned}$$

where in the last inequality we have used (3.22). Concerning b_ε, we notice that if we only know that $v\in W^{1,p}(\Omega)$ then by $v\in L^{p^\star}(\Omega)$ and $\nabla v\in L^p(\Omega)$ we find that

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|b_\varepsilon|^p\le\int_{\Omega\cap B_{r_2}}|\nabla v|^p+\int_{\Omega\cap B_{r_2}}\frac{|v|^p}{\varepsilon^{\beta\,p}}\le\int_{\Omega\cap B_{r_2}}|\nabla v|^p+\Big(\int_{\Omega\cap B_{r_2}}|v|^{p^\star}\Big)^{p/p^\star}$$

where the latter quantity converges to 0 at an non-quantified rate as $\varepsilon\to 0^+$ (as stated in (3.37)); while, if $v\in\mathrm{Lip}(\Omega)$, then

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|b_\varepsilon|^p\le C\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|v|^p\,|\nabla\varphi_\varepsilon|^p+|\nabla v|^p\le C\,r_2^n\,\mathrm{Lip}(\varphi_\varepsilon)^p\le C\varepsilon^{\beta\,(n-p)}\,,$$

and (3.38) is proved. The other two limits in (3.37) follow similarly (with non-quantified rates), while if $v\in\mathrm{Lip}(\Omega)$, then (3.39) and (3.40) follow from (3.21), (3.23), and

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}u_\varepsilon^{p^\star}\le C\varepsilon^{\beta n}+C\int_{H\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}U^{p^\star}\le C\varepsilon^{\beta n}+C\,\varepsilon^{(1-\beta)n/(p-1)}$$

$$\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})}u_\varepsilon^{p^\#}\le C\varepsilon^{\beta(n-1)}+C\int_{(\partial H)\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}U^{p^\#}\le C\varepsilon^{\beta(n-1)}+C\,\varepsilon^{(1-\beta)\,\frac{(n-1)}{(p-1)}}\,.$$

Step two: We prove statement (i). By (3.37),

$$\int_\Omega|\nabla v_\varepsilon|^p=\int_{\Omega\cap B_{r_1}}|(\nabla\hat g)^*[(\nabla U^{(\varepsilon)})\circ\hat g]|^p+\int_{\Omega\setminus B_{r_2}}|\nabla v|^p+\mathrm{o}(1)\,,$$

and, similarly,

$$\int_\Omega |\nabla v|^p \geq \int_{\Omega\setminus B_{r_2}} |\nabla v|^p \geq \int_\Omega |\nabla v|^p + \mathrm{o}(1)\,. \tag{3.41}$$

Moreover, if we set $E_\varepsilon = g(B_{r_1}) \subset H$ and $\tilde{E}_\varepsilon = E_\varepsilon/\varepsilon \subset H$, then keeping in mind (2.13), (2.11), (2.12), and (3.22), we have

$$\begin{aligned}\int_{\Omega\cap B_{r_1}} |(\nabla\hat{g})^*[(\nabla U^{(\varepsilon)})\circ\hat{g}]|^p &= \int_{E_\varepsilon} |((\nabla\hat{g})\circ\hat{f})^*[\nabla U^{(\varepsilon)}]|^p\, J\hat{f} = (1+\mathrm{o}(1))\int_{E_\varepsilon} |\nabla U^{(\varepsilon)}|^p \\ &= (1+\mathrm{o}(1))\Big\{\int_H |\nabla U|^p - \int_{H\setminus\tilde{E}_\varepsilon} |\nabla U|^p\Big\} = (1+\mathrm{o}(1))\int_H |\nabla U|^p\,.\end{aligned}$$

This proves (3.30). Entirely analogous arguments prove (3.31) and (3.32).

Step three: We now start the proof of statement (ii); in particular, from now on, Ω has C^2-boundary, $n > 2p$, and v_ε is defined as in (3.33); moreover, for the sake of brevity, we set $h = \mathrm{H}_{\partial\Omega}(0)$. In this step, we discuss the choice of $\beta = \beta(n,p) \in (0,1)$, which is determined by the rates in (3.38), (3.39) and (3.40), and by the fact that in (3.34), (3.35) and (3.36) we want errors of size $\mathrm{o}(\varepsilon)$: therefore, by

$$\begin{aligned}\beta\,\min\Big\{n, n-1, n-p\Big\} > 1 \qquad &\text{iff} \qquad \beta > \frac{1}{n-p}\,,\\ (1-\beta)\,\min\Big\{\frac{n-p}{p-1}, \frac{n-1}{p-1}, \frac{n}{p-1}\Big\} > 1 \qquad &\text{iff} \qquad \beta < \frac{n+1-2\,p}{n-p}\,,\end{aligned}$$

we are led to choose

$$\beta \in \Big(\frac{1}{n-p}, \min\Big\{1, \frac{n+1-2\,p}{n-p}\Big\}\Big)\,, \tag{3.42}$$

(where the interval appearing in (3.42) is non-empty thanks to $n > 2p$). With this choice of β, we have $\min\{\beta\,(n-p), (1-\beta)\,\frac{n-p}{p-1}\} > 1$, and thus deduce from (3.38), (3.39) and (3.40) that

$$\max\Big\{\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} |\nabla v_\varepsilon|^p, \int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\star}, \int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\#}\Big\} = \mathrm{o}(\varepsilon)\,. \tag{3.43}$$

Step four: We prove (3.34). We first notice that by (3.43) and (3.41) we have

$$\int_\Omega |\nabla v_\varepsilon|^p = \int_{\Omega\cap B_{r_1}} |(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p + \int_\Omega |\nabla v|^p + \mathrm{o}(\varepsilon)\,. \tag{3.44}$$

Now, by (2.15) we have

$$\begin{aligned}&\nabla(U^{(\varepsilon)}\circ g)\circ f = [(\nabla g)\circ f]^*\,\nabla U^{(\varepsilon)}\\ =\;& \nabla U^{(\varepsilon)} + \partial_n U^{(\varepsilon)}\,\nabla\ell - (\nabla\ell\cdot\nabla U^{(\varepsilon)})\,e_n + x_n\sum_{i=1}^{n-1}\kappa_i\,\partial_i U^{(\varepsilon)}\,e_i + \mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|\,,\end{aligned}$$

so that, recalling that $|\nabla\ell| = \mathrm{O}(|x|)$,

$$\begin{aligned}\Big|\nabla(U^{(\varepsilon)}\circ g)\circ f\Big|^2 &= |\nabla U^{(\varepsilon)}|^2 + 2\,\big((\nabla\ell\cdot\nabla U^{(\varepsilon)})\,e_n - \partial_n U^{(\varepsilon)}\,\nabla\ell\big)\cdot\nabla U^{(\varepsilon)}\\ &\quad +2\,x_n\sum_{i=1}^{n-1}\kappa_i\,(\partial_i U^{(\varepsilon)})^2 + \mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^2\\ &= |\nabla U^{(\varepsilon)}|^2 + 2\,x_n\sum_{i=1}^{n-1}\kappa_i\,(\partial_i U^{(\varepsilon)})^2 + \mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^2\,.\end{aligned}$$

Now set $a = |\nabla U^{(\varepsilon)}|$ and $b = [2\sum_{i=1}^{n-1}\kappa_i\,(\partial_i U^{(\varepsilon)})^2]^{1/2}$, so that $0 \le b \le C\,a$ for a constant depending on $|A_{\partial\Omega}(x_0)|$. Since $z \mapsto (1+z)^{p/2}$ is smooth in a neighborhood of $z = 0$, we see that if $|x| < 1/C$ for a constant C depending on $|A_{\partial\Omega}(x_0)|$, then

$$
\begin{aligned}
&\big(a^2 + x_n\,b^2 + \mathrm{O}(|x|^2)\,a^2\big)^{p/2} = a^p\,\big(1 + (b/a)^2\,x_n + \mathrm{O}(|x|^2)\big)^{p/2}\\
&= a^p\,\Big(1 + \frac{p}{2}(b/a)^2\,x_n + \mathrm{O}(|x|^2)\Big) = a^p + \frac{p}{2}\,a^{p-2}\,b^2\,x_n + \mathrm{O}(|x|^2)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Big|\nabla(U^{(\varepsilon)}\circ g)\circ f\Big|^p\,Jf &= |\nabla U^{(\varepsilon)}|^p\Big(1 + p\,x_n\sum_{i=1}^{n-1}\kappa_i\frac{(\partial_i U^{(\varepsilon)})^2}{|\nabla U^{(\varepsilon)}|^2} + \mathrm{O}(|x|^2)\Big)\Big(1 - x_n\,h + \mathrm{O}(|x|^2)\Big)\\
&= |\nabla U^{(\varepsilon)}|^p - x_n\Big(h - p\sum_{i=1}^{n-1}\kappa_i\,\frac{(\partial_i U^{(\varepsilon)})^2}{|\nabla U^{(\varepsilon)}|^2}\Big)\,|\nabla U^{(\varepsilon)}|^p + \mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^p\,.
\end{aligned}
$$

Then, by (2.13),

$$
\begin{aligned}
&\int_{\Omega\cap B_{r_1}}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p \le \int_{H\cap B_{C\,r_1}}\big|[(\nabla g)\circ f]^*\,(\nabla U^{(\varepsilon)})\big|^p\,Jf\\
&\le \int_{H\cap B_{C\,r_1}}|\nabla U^{(\varepsilon)}|^p - h\int_{H\cap B_{C\,r_1}}x_n\,|\nabla U^{(\varepsilon)}|^p\\
&+p\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{C\,r_1}}x_n\,(\partial_i U^{(\varepsilon)})^2\,|\nabla U^{(\varepsilon)}|^{p-2} + C\int_{H\cap B_{C\,r_1}}|x|^2\,|\nabla U^{(\varepsilon)}|^p\\
&= \int_{H\cap B_{Cr_1/\varepsilon}}|\nabla U|^p - h\,\varepsilon\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,|\nabla U|^p\\
&+p\,\varepsilon\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2} + C\,\varepsilon^2\int_{H\cap B_{Cr_1/\varepsilon}}|x|^2\,|\nabla U|^p\,.
\end{aligned}
\tag{3.45}
$$

Now, by the reflection symmetries of U with respect to $\{x_i = 0\}$, $i = 1, ..., n-1$ (recall (3.19)), we have

$$
\int_{H\cap B_R}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2} = \int_{H\cap B_R}x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,\qquad \forall i = 1, ..., n-1\,, \forall R > 0\,,
$$

and therefore

$$
\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2} = h\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,.
$$

Setting

$$
\mathcal{L}(U, B_R) = \int_{H\cap B_R}x_n\,|\nabla U|^p - p\,x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,
\tag{3.46}
$$

we can thus rewrite (3.45) as

$$
\int_{\Omega\cap B_{r_1}}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p \le \int_H|\nabla U|^p - h\,\varepsilon\,\mathcal{L}(U, B_{C\,r_1/\varepsilon}) + C\,\varepsilon^2\int_{H\cap B_{C\,r_1/\varepsilon}}|x|^2\,|\nabla U|^p\,.
$$

At the same time, $\varepsilon^2\,|x|^2 \le C\,\varepsilon\,r_1\,|x|$ for any $x \in B_{C\,r_1/\varepsilon}$, so

$$
\begin{aligned}
&\varepsilon^2\int_{H\cap B_{Cr_1/\varepsilon}}|x|^2\,|\nabla U|^p \le C\,\varepsilon\,r_1\int_{H\cap B_{Cr_1/\varepsilon}}|x|\,|\nabla U|^p\\
&\le C\,\varepsilon^{1+\beta}\int_H|x|\,|\nabla U_T|^p + C\,\varepsilon^{1+\beta}\int_H|x|\,|\nabla V_T|^p \le C(n,p,T)\,\varepsilon^{1+\beta},
\end{aligned}
$$

where we have used the facts that V_T is compactly supported and that $n > 2p-1$ to guarantee the convergence of the integrals in the final line. Hence,

$$\begin{aligned}\int_{\Omega\cap B_{r_1}}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p &= \int_H |\nabla U|^p - h\,\mathcal{L}(U,B_{C\,r_1/\varepsilon})\,\varepsilon + \mathrm{o}(\varepsilon)\\ &= \int_H |\nabla U|^p - h\,\mathcal{L}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,,\end{aligned}$$

where the $\mathrm{o}(\varepsilon)$ term depends on n, p, and T, and in the second line we have applied (3.25). By (3.44) we deduce (3.34).

Step five: We prove (3.35). We first notice that by (3.43), $v \in \mathrm{Lip}(\Omega)$, $r_2^n = \varepsilon^{\beta\,n} = \mathrm{o}(\varepsilon)$ and our choice of β we have

$$\int_\Omega v_\varepsilon^{p^\star} = \int_{\Omega\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\star} + \int_\Omega v^{p^\star} + \mathrm{o}(\varepsilon)\,. \tag{3.47}$$

Let $E_\varepsilon = g(B_{r_1}\cap\Omega) \subset H$ and $\tilde E_\varepsilon = E_\varepsilon/\varepsilon \subset E$ as in step two. Then keeping in mind (2.13) and (2.14),

$$\begin{aligned}\int_{\Omega\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\star} &= \int_{E_\varepsilon}(U^{(\varepsilon)})^{p^\star} - h\int_{E_\varepsilon} x_n\,(U^{(\varepsilon)})^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\\ &= \int_{\tilde E_\varepsilon} U^{p^\star} - h\,\varepsilon\int_{\tilde E_\varepsilon} x_n\,U^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\\ &= \int_H U^{p^\star} - h\,\varepsilon\mathcal{M}(U) + \Big\{-\int_{H\setminus\tilde E_\varepsilon}U^{p^\star} + h\varepsilon\int_{H\setminus\tilde E_\varepsilon}x_nU^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\Big\}\,.\end{aligned}$$

By (3.21) and (3.24), along with our choice of β, we see that

$$-\int_{H\setminus\tilde E_\varepsilon}U^{p^\star} = o(\varepsilon), \qquad h\,\varepsilon\int_{H\setminus\tilde E_\varepsilon}x_nU^{p^\star} = o(\varepsilon)\,.$$

Moreover, since $U = U_T + tV_T$ with V_T compactly supported in H and $|t|\le 1$, we have

$$\begin{aligned}\int_{E_\varepsilon}|x|^2\,(U^{(\varepsilon)})^{p^\star} &\le C\,r_1\,\varepsilon\int_{\tilde E_\varepsilon}|x|\,U^{p^\star} \le \varepsilon^{1+\beta}\int_H |x|\,U^{p^\star}\\ &\le C\varepsilon^{1+\beta}\int_H |x|\,U_T^{p^\star} + C\varepsilon^{1+\beta}\int_H |x|V^{p^\star} = \mathrm{o}(\varepsilon),\end{aligned}$$

with $\mathrm{o}(\varepsilon)$ depending on n, p, and T. So, the entire term in brackets above can be written as $\mathrm{o}(\varepsilon)$. Combining this estimate with (3.47), we deduce (3.35).

Step six: We finally prove (3.36). Notice that, by (3.43), $v \in \mathrm{Lip}(\Omega)$, $r_2^{n-1} = \varepsilon^{\beta\,(n-1)} = \mathrm{o}(\varepsilon)$ (by the choice of β), we have

$$\int_{\partial\Omega} v_\varepsilon^{p^\#} = \int_{(\partial\Omega)\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\#} + \int_{\partial\Omega} v^{p^\#} + \mathrm{o}(\varepsilon)\,. \tag{3.48}$$

Now, by $J^{\partial H}f \ge 1$, (3.23) and our choice of β we have

$$\int_{(\partial\Omega)\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\#} \ge \int_{(\partial H)\cap B_{r_1/C}}(U^{(\varepsilon)})^{p^\#} = \int_{\partial H}U^{p^\#} - \int_{(\partial H)\setminus B_{r_1/C\varepsilon}}U^{p^\#} \ge \int_{\partial H}U^{p^\#} + \mathrm{o}(\varepsilon)\,.$$

At the same time, by $J^{\partial H}f \le 1 + C\,|x|^2$, we have

$$\begin{aligned}\int_{(\partial\Omega)\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\#} &\le \int_{(\partial H)\cap B_{C\,r_1}}(1+C\,|x|^2)\,(U^{(\varepsilon)})^{p^\#}\\ &\le \int_{\partial H}U^{p^\#} + C\,\varepsilon^2\int_{(\partial H)\cap B_{C\,r_1/\varepsilon}}|x|^2\,U^{p^\#}\end{aligned}$$

where

$$\varepsilon^2\int_{(\partial H)\cap B_{C\,r_1/\varepsilon}}|x|^2\,U^{p^\#}\le C\,\varepsilon^2\int_0^{C\,r_1/\varepsilon}\frac{r^2\,r^{n-2}\,dr}{(r^{(n-p)/(p-1)})^{p^\#}}$$
$$\le C\,\varepsilon^2\,\big(r_1/\varepsilon\big)^{1+n-(n-1)\,[p/(p-1)]}$$
$$\le C\,\varepsilon^2\,\big(r_1/\varepsilon\big)^{(2p-n-1)/(p-1)}\le C\,\varepsilon^2\,\varepsilon^{(1-\beta)\,(n+1-2p)/(p-1)}\le C\,\varepsilon^2=\mathrm{o}(\varepsilon)\,,$$

thanks to $n>2p-1$. This completes the proof. □

4. Existence of minimizers

We first establish the existence of generalized minimizers. Recall that $\Phi^*_\Omega(T)$ was defined in (1.21).

Theorem 4.1. *Let $n\ge 2$, $p\in(1,n)$, and let Ω be a bounded open set with C^1-boundary in $\mathbb{R}^n$. Then:*

(i): *for every $T>0$, $\Phi_\Omega(T)=\Phi^*_\Omega(T)$;*

(ii): *there is a minimizer $(u,\mathsf{v},\mathsf{t})$ of $\Phi^*_\Omega(T)$;*

(iii): *if $(u,\mathsf{v},\mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$ with $\int_\Omega u^{p^\star}>0$, then $\int_{\partial\Omega}u^{p^\#}>0$,*

$$u\Big/\|u\|_{L^{p^\star}(\Omega)}\ \text{is a minimizer of}\ \Phi_\Omega\Big(\|u\|_{L^{p^\#}(\partial\Omega)}\Big/\|u\|_{L^{p^\star}(\Omega)}\Big)\,,\tag{4.1}$$

and there exists $\lambda,\sigma\in\mathbb{R}$ such that

$$\begin{cases}-\Delta_p u=\lambda\,u^{p^\star-1}\,, & \text{on }\Omega\,,\\[4pt] |\nabla u|^{p-2}\,\dfrac{\partial u}{\partial\nu_\Omega}=\sigma\,u^{p^\#-1}\,, & \text{on }\partial\Omega\,,\end{cases}\tag{4.2}$$

In particular, $u\in\mathrm{Lip}(\Omega)$. If, in addition, $\mathsf{v}>0$, then λ and σ are given by

$$\lambda=\mathsf{v}^{p-p^\star}\,\lambda_H(\mathsf{t}/\mathsf{v})\,,\qquad \sigma=\mathsf{v}^{p-p^\#}\,\sigma_H(\mathsf{t}/\mathsf{v})\,,\tag{4.3}$$

and, in particular,

$$\frac{\mathsf{t}}{\mathsf{v}}\in(0,T_0)\cup(T_E,\infty)\,.\tag{4.4}$$

Proof. Step one: Since $(u,0,0)\in\mathcal{Y}_\Omega(T)$ if $u\in\mathcal{X}_\Omega(T)$, we have $\Phi^*_\Omega(T)\le\Phi_\Omega(T)$. To prove the converse inequality it is enough to show that for every $(u,\mathsf{v},\mathsf{t})\in\mathcal{Y}_\Omega(T)$,

$$\exists\,u_j\in\mathcal{X}_\Omega(T)\ \text{s.t.}\ \lim_{j\to\infty}\int_\Omega|\nabla u_j|^p=\int_\Omega|\nabla u|^p+\mathsf{v}^p\,\Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p\,.\tag{4.5}$$

Looking back at the definition of $\mathcal{Y}_\Omega(T)$ in the paragraph preceding the statement of Theorem 1.1, we can assume without loss of generality that $\mathsf{v}>0$ and $\mathsf{t}>0$. Moreover, given $(u,\mathsf{v},\mathsf{t})\in\mathcal{Y}_\Omega(T)$ with v and t positive we can easily find $(u_j,\mathsf{v}_j,\mathsf{t}_j)\in\mathcal{Y}_\Omega(T)$ with v_j, t_j, $\int_\Omega u_j^{p^\star}$, and $\int_{\partial\Omega}u_j^{p^\#}$ positive and such that $\mathcal{E}(u_j,\mathsf{v}_j,\mathsf{t}_j)\to\mathcal{E}(u,\mathsf{v},\mathsf{t})$. By a diagonal argument, it is thus sufficient proving (4.5) under the assumption that $\int_\Omega u^{p^\star}$ and $\int_{\partial\Omega}u^{p^\#}$ are positive. This said, we apply Lemma 3.4**(i)** with

$$v=\frac{u}{\mathsf{v}}\,,\qquad U=U_{\mathsf{t}/\mathsf{v}}\,,$$

to find functions v_j with $v_j = v$ on $\Omega \setminus B_{2\varepsilon_j}(x_0)$ for some $x_0 \in \partial\Omega$ and $\varepsilon_j \to 0^+$, and with

$$\begin{aligned}\int_\Omega |\nabla v_j|^p &= \frac{1}{\mathsf{v}^p}\int_\Omega |\nabla u|^p + \Phi_H(\mathsf{t}/\mathsf{v})^p + G_j\,,\\ \int_\Omega v_j^{p^\star} &= \frac{1}{\mathsf{v}^{p^\star}}\int_\Omega u^{p^\star} + 1 + V_j\\ \int_{\partial\Omega} v_j^{p^\#} &= \frac{1}{\mathsf{v}^{p^\#}}\int_{\partial\Omega} u^{p^\#} + (\mathsf{t}/\mathsf{v})^{p^\#} + T_j\,,\end{aligned}$$

where G_j, V_j, $T_j \to 0$ as $j \to \infty$ at a rate depending on n, p, Ω, t/v and u only. By Lemma 2.6, there exist η and C depending on n, p, Ω, t/v and u, but independent from j, such that for any (a_j, b_j) with $|a_j| + |b_j| < \eta$, we have functions w_j such that

$$\int_{\partial\Omega} |w_j|^{p^\#} = a_j + \int_{\partial\Omega} |v_j|^{p^\#}\,, \qquad \int_\Omega |w_j|^{p^\star} = b_j + \int_\Omega |v_j|^{p^\star}\,, \tag{4.6}$$

$$\Big|\int_\Omega |\nabla w_j|^p - \int_\Omega |\nabla v_j|^p\Big| \le C\left(|a_j| + |b_j|\right). \tag{4.7}$$

For j large enough we can apply this statement with $a_j = -T_j$ and $b_j = -V_j$ to find a sequence $\{w_j\}_j$ with

$$\int_{\partial\Omega} |w_j|^{p^\#} = \frac{1}{\mathsf{v}^{p^\#}}\int_{\partial\Omega} u^{p^\#} + (\mathsf{t}/\mathsf{v})^{p^\#} = \frac{T^{p^\#}}{\mathsf{v}^{p^\#}}\,, \qquad \int_\Omega |w_j|^{p^\star} = \frac{1}{\mathsf{v}^{p^\star}}\int_\Omega u^{p^\star} + 1 = \frac{1}{\mathsf{v}^{p^\star}}\,, \tag{4.8}$$

$$\Big|\int_\Omega |\nabla w_j|^p - \frac{1}{\mathsf{v}^p}\int_\Omega |\nabla u|^p - \Phi_H(\mathsf{t}/\mathsf{v})^p - G_j\Big| \le C\left(|T_j| + |V_j|\right). \tag{4.9}$$

Setting $u_j = \mathsf{v}\, w_j$, we obtain a sequence in $\mathcal{X}_\Omega(T)$ that satisfies (4.5).

Step two: We prove that there is a minimizer for the generalized problem $\Phi^*_\Omega(T)$. By the argument in step one we can find a sequence $\{u_j\}_j$ in $\mathcal{X}_\Omega(T)$ such that $\int_\Omega |\nabla u_j|^p \to \Phi^*_\Omega(T)^p$. By Lemma 2.1, the measures μ_j, ν_j and τ_j defined in (2.1) have subsequential weak-star limits μ, ν and τ satisfying (2.2), (2.3) and (2.4) and (2.5). In particular, there is an at most countable set $\{x_i\}_{i\in I} \subset \overline{\Omega}$ and corresponding $\mathsf{v}_i > 0$ and $\mathsf{t}_i \ge 0$ for every $i \in I$, such that

$$\Phi^*_\Omega(T)^p = \lim_{j\to\infty}\int_\Omega |\nabla u_j|^p \ge \int_\Omega |\nabla u|^p + S^p \sum_{i\in I\setminus I_{\rm bd}} \mathsf{v}_i^p + \sum_{i\in I_{\rm bd}} \mathsf{v}_i^p\, \Phi_H(\mathsf{t}_i/\mathsf{v}_i)^p\,, \tag{4.10}$$

where u is the subsequential weak limit of u_j, and

$$1 = \int_\Omega u^{p^\star} + \sum_{i\in I} \mathsf{v}_i^{p^\star}\,, \qquad T^{p^\#} = \int_{\partial\Omega} u^{p^\#} + \sum_{i\in I_{\rm bd}} \mathsf{t}_i^{p^\#}\,. \tag{4.11}$$

Now set

$$\mathsf{v}_c^{p^\star} = \sum_{i\in I} \mathsf{v}_i^{p^\star}\,, \qquad \mathsf{t}_c^{p^\#} = \sum_{i\in I_{\rm bd}} \mathsf{t}_i^{p^\#}\,.$$

By an immediate adaptation of the proof of Lemma 3.4 we can easily construct a sequence $\{W_j\}_j$ in $\mathcal{X}_H(\mathsf{t}_c/\mathsf{v}_c)$ with the property that

$$\int_H |\nabla W_j|^p \to \sum_{i\in I}\left(\frac{\mathsf{v}_i}{\mathsf{v}_c}\right)^p \Phi_H(\mathsf{t}_i/\mathsf{v}_i)^p\,.$$

Since $W_j \in \mathcal{X}_H(\mathsf{t}_c/\mathsf{v}_c)$ implies $\int_H |\nabla W_j|^p \ge \Phi_H(\mathsf{t}_c/\mathsf{v}_c)^p$, we deduce from (4.10) that

$$\Phi^*_\Omega(T)^p \ge \int_\Omega |\nabla u|^p + \mathsf{v}_c^p\, \Phi_H(\mathsf{t}_c/\mathsf{v}_c)^p\,,$$

while (4.11) gives $(u, \mathsf{v}_c, \mathsf{t}_c) \in \mathcal{Y}_\Omega(T)$. This proves that $(u, \mathsf{v}_c, \mathsf{t}_c)$ is a minimizer of $\Phi^*_\Omega(T)$.

Step three: We finally prove statement (iii). If $(u,\mathsf{v},\mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$ with $\int_\Omega u^{p^\star}>0$, it is immediate to deduce (4.1), and since $\Phi_\Omega(0)$ does not admit minimizers, it must also be $\int_{\partial\Omega}u^{p^\#}>0$. By Lemma 2.5, the Euler-Lagrange equation (4.2) for u holds for some $\lambda,\sigma\in\mathbb{R}$ and $u\in\mathrm{Lip}(\Omega)$. Assuming now that $\mathsf{v}>0$, we can prove (4.3) by noticing that, given $\varphi\in C^\infty_c(\mathbb{R}^n)$, if we define

$$\alpha(\delta)=\Big(1-\int_\Omega(u+\delta\,\varphi)^{p^\star}\Big)^{1/p^\star}-\mathsf{v}\,,\qquad \beta(\delta)=\Big(T^{p^\#}-\int_{\partial\Omega}(u+\delta\,\varphi)^{p^\#}\Big)^{1/p^\#}-\mathsf{t}\,,$$

then there is $\delta_0>0$ such that $(u+\delta\,\varphi,\mathsf{v}+\alpha(\delta),\mathsf{t}+\beta(\delta))\in\mathcal{Y}_\Omega(T)$ for every $|\delta|<\delta_0$. In particular,

$$0=\frac{d}{d\delta}\Big|_{\delta=0}\int_\Omega|\nabla(u+\delta\varphi)|^p+(\mathsf{v}+\alpha(\delta))^p\,\Phi_H\Big(\frac{\mathsf{t}+\beta(\delta)}{\mathsf{v}+\alpha(\delta)}\Big)^p\,,$$

and exploiting (4.2) as well as

$$\alpha'(0)=-\mathsf{v}^{1-p^\star}\int_\Omega u^{p^\star-1}\varphi\,,\qquad \beta'(0)=-\mathsf{t}^{1-p^\#}\int_{\partial\Omega}u^{p^\#-1}\varphi\,,$$

and (3.8) (i.e. $\Phi_H(T)=\lambda_H(T)+\sigma_H(T)\,T^{p^\#}$ and $\sigma_H(T)\,T^{p^\#-1}=\Phi_H(T)^{p-1}\Phi_H'(T)$ for every $T>0$), we see that

$$\begin{aligned}
0\;&=\;\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi+\mathsf{v}^{p-1}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\,\beta'(0)\\
&\qquad\qquad+\Big\{\mathsf{v}^{p-1}\,\Phi_H(\mathsf{t}/\mathsf{v})^p-\mathsf{t}\,\mathsf{v}^{p-2}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\Big\}\alpha'(0)\\
&=\;\lambda\int_\Omega u^{p^\star-1}\,\varphi+\sigma\int_{\partial\Omega}u^{p^\#-1}\varphi-\mathsf{v}^{p-1}\,\mathsf{t}^{1-p^\#}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\int_{\partial\Omega}u^{p^\#-1}\varphi\\
&\quad-\mathsf{v}^{p-p^\star}\Big\{\Phi_H(\mathsf{t}/\mathsf{v})^p-(\mathsf{t}/\mathsf{v})\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\Big\}\int_\Omega u^{p^\star-1}\varphi\\
&=\;\big(\sigma-\mathsf{v}^{p-p^\#}\sigma_H(\mathsf{t}/\mathsf{v})\big)\int_{\partial\Omega}u^{p^\#-1}\varphi+\big(\lambda-\mathsf{v}^{p-p^\star}\lambda_H(\mathsf{t}/\mathsf{v})\big)\int_\Omega u^{p^\star-1}\varphi\,.
\end{aligned}$$

Testing with $\varphi\in C^\infty_c(\Omega)$ we find $\lambda=\mathsf{v}^{p-p^\star}\lambda_H(\mathsf{t}/\mathsf{v})$, and testing with $\varphi=1$ on $\overline{\Omega}$ then gives $\sigma=\mathsf{v}^{p-p^\#}\sigma_H(\mathsf{t}/\mathsf{v})$. We finally prove (4.4), i.e. $\mathsf{t}/\mathsf{v}\notin[T_0,T_E]$. Indeed, combining the balance condition (2.16) with (4.3) we find

$$\mathsf{v}^{p-p^\star}\,\lambda_H(\mathsf{t}/\mathsf{v})\int_\Omega u^{p^\star-1}+\mathsf{v}^{p-p^\#}\,\sigma_H(\mathsf{t}/\mathsf{v})\int_{\partial\Omega}u^{p^\#-1}=0\,,\qquad(4.12)$$

where by (3.11) and (3.12) we have $\lambda_H>0$ in $[T_0,T_E)$ and $\sigma_H>0$ on $(T_0,T_E]$ (with $\lambda_H(T_E)=\sigma_H(T_0)=0$ by continuity). If $\mathsf{t}/\mathsf{v}\in[T_0,T_E)$, then (4.12) implies $\int_\Omega u^{p^\star}=0$, a contradiction; if $\mathsf{t}/\mathsf{v}=T_E$, then (4.12) gives $u=0$ on $\partial\Omega$, so that, by (4.1), u is a minimizer of $\Phi_\Omega(0)$, and thus u is optimal in the Sobolev inequality on $\mathbb{R}^n$, so that $\Omega=\mathbb{R}^n$, contradicting the fact that Ω is bounded. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Statement (i) is an immediate consequence of Theorem 4.1. We thus focus on statement (ii), and assume that Ω is of class C^2 and that $n>2p$. We want to prove that if $(u,\mathsf{v},\mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$, then $\mathsf{v}=\mathsf{t}=0$. We assume by way of contradiction that either $\mathsf{v}>0$ *or* $\mathsf{t}>0$; recalling the definition of $\mathcal{Y}_\Omega(T)$, this implies that $\mathsf{v}>0$ *and* $\mathsf{t}>0$. We apply Lemma 3.4 **(ii)** with the choice $(v,T)=(u/\mathsf{v},\tau)$ at a point $x_0\in\partial\Omega$ of positive mean curvature, noting that if $\mathsf{v}=1$ then $v\equiv0$ and that v is

Lipschitz continuous if $\mathsf{v} \in (0,1)$ thanks to (4.1) and Lemma 2.5. Then, for every $U \in \mathcal{U}_\tau$, we have

$$\int_\Omega |\nabla v_\varepsilon|^p \le \int_H |\nabla U|^p + \int_\Omega |\nabla v|^p - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{L}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,, \tag{4.13}$$

$$\int_\Omega v_\varepsilon^{p^\star} = \int_H U^{p^\star} + \int_\Omega v^{p^\star} - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,, \tag{4.14}$$

$$\int_{\partial\Omega} v_\varepsilon^{p^\#} = \int_{\partial H} U^{p^\#} + \int_{\partial\Omega} v^{p^\#} + \mathrm{o}(\varepsilon)\,, \tag{4.15}$$

where $\mathcal{L}(U)$ and $\mathcal{M}(U)$ are defined in (3.46) and (1.29). We apply this with $U \in \mathcal{U}_\tau$ given by

$$U = U_\tau + b\,\varepsilon\,V_\tau\,, \qquad |\varepsilon| < \frac{1}{|b|}\,, \qquad b = \frac{\mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U_\tau)}{p^\star}\,,$$

The reason for the choice of b will become apparent in a moment. Indeed, thanks to (3.26), (3.27) and (3.28), we have

$$\int_H |\nabla U|^p = \Phi_H(\tau)^p + p\,b\,\lambda_H(\tau)\varepsilon + \mathrm{o}(\varepsilon)\,,$$

$$\int_H U^{p^\star} = 1 + p^\star\,b\,\varepsilon + \mathrm{o}(\varepsilon)\,, \qquad \int_{\partial H} U^{p^\#} = \tau^{p^\#}\,,$$

which, combined with (4.13), (4.14), (4.15) and

$$\frac{\Phi_\Omega(T)^p}{\mathsf{v}^p} = \Phi_H(\tau)^p + \int_\Omega |\nabla v|^p, \qquad \frac{1}{\mathsf{v}^{p^\star}} = 1 + \int_\Omega v^{p^\star}\,, \qquad (T/\mathsf{v})^{p^\#} = \tau^{p^\#} + \int_{\partial\Omega} v^{p^\#}\,,$$

implies that $w_\varepsilon = \mathsf{v}\,v_\varepsilon$ satisfies

$$\int_\Omega |\nabla w_\varepsilon|^p \le \Phi_\Omega(T)^p + \Big\{p\,b\,\lambda_H(\tau) - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{L}(U)\Big\}\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\,, \tag{4.16}$$

$$\int_\Omega w_\varepsilon^{p^\star} = 1 + \big\{p^\star\,b - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\big\}\,\mathsf{v}^{p^\star}\varepsilon + \mathrm{o}(\varepsilon) = 1 + \mathrm{o}(\varepsilon) \tag{4.17}$$

$$\int_{\partial\Omega} w_\varepsilon^{p^\#} = T^{p^\#} + \mathrm{o}(\varepsilon)\,, \tag{4.18}$$

where in (4.16) we have used the choice of b to deduce

$$\mathcal{M}(U) = \mathcal{M}(U_\tau) + p^\star \int_H x_n\,U_\tau^{p^\star-1}\,V_\tau\,b\,\varepsilon + \mathrm{o}(\varepsilon)\,, \quad \big\{p^\star\,b - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\big\}\,\varepsilon = \mathrm{o}(\varepsilon)\,.$$

In the same spirit, by

$$\lim_{\varepsilon\to 0^+} |\mathcal{L}(U_\tau + b\,\varepsilon\,V_\tau) - \mathcal{L}(U_\tau)| = 0\,,$$

we deduce from (4.16) that

$$\begin{aligned}\int_\Omega |\nabla w_\varepsilon|^p &\le \Phi_\Omega(T)^p - \Big\{\mathcal{L}(U_\tau) - \frac{(n-p)}{n}\,\mathcal{M}(U_\tau)\,\lambda_H(\tau)\Big\}\,\mathrm{H}_{\partial\Omega}(0)\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\\ &\le \Phi_\Omega(T)^p - C(n,p,\tau)\,\mathrm{H}_{\partial\Omega}(0)\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\,,\end{aligned} \tag{4.19}$$

where in the second line we apply Lemma 3.1. We thus conclude that

$$\int_\Omega |\nabla w_\varepsilon|^p \le \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H(\mathsf{t}/\mathsf{v})^p - M\,\mathsf{v}^p\,\varepsilon + \mathrm{o}(\varepsilon)\,. \tag{4.20}$$

It remains to modify the functions w_ε to obtain $w_\varepsilon^* \in \mathcal{X}_\Omega(T)$ also satisfying (4.20), allowing us to conclude the proof of the theorem by choosing ε sufficiently small. We will distinguish between two cases, applying Lemma 2.6 in different ways in the two cases.

Case one: Suppose first that $\mathsf{v} < 1$ and thus $\int_\Omega u^{p^\star} > 0$. This also implies that $\int_{\partial\Omega} u^{p^\#} > 0$ by Theorem 4.1-(iii). Taking into account (4.17) and (4.18), we can thus apply Lemma 2.6

in an analogous way to step one of the proof of Theorem 4.1 in order to slightly modify w_ε into $w_\varepsilon^* \in \mathcal{X}_\Omega(T)$ with

$$\begin{aligned}\Phi_\Omega(T)^p &\le \int_\Omega |\nabla w_\varepsilon^*|^p = \int_\Omega |\nabla w_\varepsilon|^p + (\mathsf{v}^{p^\#} + \mathsf{v}^{p^\star})\,\mathrm{o}(\varepsilon)\\ &\le \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H(\mathsf{t}/\mathsf{v})^p - \frac{M}{2}\,\mathsf{v}^p\,\varepsilon = \Phi_\Omega(T)^p - \frac{M}{2}\,\mathsf{v}^p\,\varepsilon < \Phi_\Omega(T)^p\,,\end{aligned}$$

thus reaching a contradiction.

Case two: Next, suppose that $\mathsf{v} = 1$. So, $\Phi_\Omega(T) = \Phi_H(T)$, $u = v \equiv 0$ and $v_\varepsilon = \varphi_\varepsilon\,(U^{(\varepsilon)} \circ g)$. In this case, we will pull the relevant quantities back to the half space H and apply Lemma 2.6 there to correct the volume and trace constraints. More specifically, for $\varepsilon < \varepsilon_0$, the support of φ_ε is entirely contained in the domain of the diffeomorphism f_{x_0}, and so we can define $\Psi_\varepsilon : H \to \mathbb{R}$ by $\Psi_\varepsilon(y) = \varphi_\varepsilon \circ f(\varepsilon\,y)$. In this way, we can rewrite

$$w_\varepsilon = v_\varepsilon = (\Psi_\varepsilon\,U)^{(\varepsilon)} \circ g.$$

Thanks to (2.13), Ψ_ε is identically equal to one in $B_{\varepsilon^{\beta-1}/C} \cap H$ and vanishes outside of $B_{C\,\varepsilon^{\beta-1}} \cap H$. Using the area formula, we rewrite (4.17), (4.18), and (4.20) as

$$\begin{aligned}\int_H (\Psi_\varepsilon\,U)^{p^\star}\,m_\varepsilon &= \int_\Omega w_\varepsilon^{p^\star}\,dx = 1 + o(\varepsilon)\,,\\ \int_{\partial H} (\Psi_\varepsilon\,U)^{p^\sharp}\,\hat{m}_\varepsilon &= \int_{\partial\Omega} w_\varepsilon^{p^\sharp} = T^{p^\sharp} + o(\varepsilon).\end{aligned}$$

$$\int_H |A_\varepsilon\,[\nabla(\Psi_\varepsilon\,U)]|^p\,m_\varepsilon = \int_\Omega |\nabla w_\varepsilon|^p \le \Phi_H(T) - M\varepsilon + o(\varepsilon),$$

where we have set

$$m_\varepsilon(x) = Jf(\varepsilon\,x)\,, \qquad \hat{m}_\varepsilon(x) = J^{\partial H} f(\varepsilon\,x)\,, \qquad A_\varepsilon(x) = (\nabla g \circ f(\varepsilon\,x))^*\,.$$

We now repeat the argument used in the proof of Lemma 2.6: exploiting the fact that

$$\Psi_\varepsilon\,U = U_T \qquad \text{on } (H \cap B_R) \setminus (\mathrm{spt} V_T)\,, \qquad R = \frac{\varepsilon_0^{\beta-1}}{C}\,, \tag{4.21}$$

as well as that both $\int_H (\Psi_\varepsilon\,U)^{p^\star}\,m_\varepsilon$ and $\int_{\partial H} (\Psi_\varepsilon\,U)^{p^\sharp}\,\hat{m}_\varepsilon$ are positive and finite, we can easily find $\psi \in C_c^\infty((H \cap B_R) \setminus (\mathrm{spt} V_T))$ and $\varphi \in C_c^\infty((\mathbb{R}^n \cap B_R) \setminus (\mathrm{spt} V_T))$ such that

$$\begin{aligned}\int_H (\Psi_\varepsilon\,U_T)^{p^\star-1}\,m_\varepsilon\,\psi &= \int_{\partial H} (\Psi_\varepsilon\,U_T)^{p^\#-1}\,\hat{m}_\varepsilon \varphi = 1\,,\\ \int_H (\Psi_\varepsilon\,U_T)^{p^\star-1}\,m_\varepsilon\,\varphi &= \int_{\partial H} (\Psi_\varepsilon\,U_T)^{p^\#-1}\,\hat{m}_\varepsilon\,\psi = 0\,.\end{aligned} \tag{4.22}$$

Correspondingly, we consider the maps $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h_\varepsilon(s,t) = \Big(\int_{\partial H} \big(|v_\varepsilon + s\,\varphi + t\,\psi|^{p^\#} - |v_\varepsilon|^{p^\#}\big)\,\hat{m}_\varepsilon,\ \int_H \big(|v_\varepsilon + s\,\varphi + t\,\psi|^{p^\star} - |v_\varepsilon|^{p^\star}\big)\,m_\varepsilon \Big)\,.$$

By (4.21), $h_\varepsilon \in C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$ for some $\alpha = \alpha(n,p) \in (0,1)$, with

$$\sup_{\varepsilon<\varepsilon_0} \|h_\varepsilon\|_{C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)} < \infty\,;$$

moreover, $h_\varepsilon(0,0) = 0$ and, by (4.22),

$$\nabla h_\varepsilon(0,0) = \begin{pmatrix} p^\# & 0 \\ 0 & p^\star \end{pmatrix}.$$

We can thus apply the inverse function theorem uniformly in ε, to obtain functions $W_\varepsilon^* : H \to \mathbb{R}$ with support in $B_{C\varepsilon^{\beta-1}}$ such that

$$\begin{gathered}\int_H (W_\varepsilon^*)^{p^\star} m_\varepsilon = 1\,, \qquad \int_{\partial H} (W_\varepsilon^*)^{p^\sharp} \hat{m}_\varepsilon = T^{p^\sharp}\,, \\ \left| \int_H |A_\varepsilon[\nabla(\Psi_\varepsilon U)]|^p m_\varepsilon \; - \; \int_H |A_\varepsilon[\nabla W_\varepsilon^*]|^p m_\varepsilon \right| = o(\varepsilon).\end{gathered} \tag{4.23}$$

Finally, for $\varepsilon < \varepsilon_0$, define $w_\varepsilon^* : \Omega \to \mathbb{R}$ by $w_\varepsilon^* = (W_\varepsilon^*)^{(\varepsilon)} \circ g$. Changing variables once again, (4.23) tells us that $w_\varepsilon \in \mathcal{X}_\Omega(T)$ and that

$$\begin{aligned}\Phi_\Omega(T)^p &\le \int_\Omega |\nabla w_\varepsilon^*|^p = \int_\Omega |\nabla w_\varepsilon|^p + o(\varepsilon) \\ &\le \Phi_H(T)^p - M\varepsilon + \mathrm{o}(\varepsilon) \le \Phi_H(T)^p - \frac{M\varepsilon}{2} < \Phi_H(T) = \Phi_\Omega(T),\end{aligned}$$

giving us a contradiction in this case as well. This completes the proof of the theorem. □

5. Rigidity theorems for best Sobolev inequalities

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. Rigidity under assumption (ii) is immediate by combining Theorem 1.1-(ii) with (1.23) and (1.24). Let us now consider assumption (i), namely, there is $T_* > 0$ such that

$$\Phi_\Omega(T) = \Phi_B(T) \qquad \forall T \in (0, T_*)\,. \tag{5.1}$$

Without loss of generality we can assume that $T_* < \mathrm{ISO}(B)^{1/p^\#}$. We argue by contradiction and assume that Ω is not a ball.

By Theorem 4.1, for every $T > 0$ there is $(u_T, \mathsf{v}_T, \mathsf{t}_T)$ a minimizer of $\Phi^*_\Omega(T)$. The basic idea of the proof will be to show that the trace-to-volume ratio of u_T must be, on one hand, uniformly positive and, on the other hand, tending to zero as $T \to 0$, giving us a contradiction. Since Ω is connected and is not a ball by assumption, the rigidity criterion (1.23) together with (1.24) and (5.1) tell us that a classical minimizer for $\Phi_\Omega(T)$ cannot exist for $T \in (0, T_*)$, and so we immediately deduce that $\mathsf{v}_T < 1$ for all such T. In other words, if we set

$$\nu_T = \big(1 - \mathsf{v}_T^{p^\star}\big)^{1/p^\star} = \|u_T\|_{L^{p^\star}(\Omega)}\,, \qquad \tau_T = (T^{p^\#} - \mathsf{t}^{p^\#})^{1/p^\#} = \|u\|_{L^{p^\#}(\partial\Omega)}\,,$$

then $\nu_T > 0$ for all $T \in (0, T_*)$. So, Theorem 4.1-(iii) implies that

$$u_T/\nu_T \text{ is a minimizer of } \Phi_\Omega(\tau_T/\nu_T)\,. \tag{5.2}$$

In particular, this means that

$$\frac{\tau_T}{\nu_T} \ge T_* \qquad \text{for all } T \in (0, T_*)\,, \tag{5.3}$$

since as we noted above, no minimizer of $\Phi_\Omega(\tilde{T})$ can exist for $\tilde{T} = \tau_T/\nu_T < T_*$. Since $T \ge \tau_T$, (5.3) tells us that $T/\nu_T \ge T_*$; rearranging this inequality gives us the following lower bound on v_T:

$$\mathsf{v}_T^{p^\star} \ge 1 - (T/T_*)^{p^\star} \qquad \forall T \in (0, T_*)\,. \tag{5.4}$$

From this, an upper bound on the ratio $\mathsf{t}_T/\mathsf{v}_T$ follows immediately:

$$\frac{\mathsf{t}_T}{\mathsf{v}_T} \le \big(1 - (T/T_*)^{p^\star}\big)^{-1/p^\star}\, T \qquad \forall T \in (0, T_*)\,. \tag{5.5}$$

In particular $\mathsf{t}_T/\mathsf{v}_T \to 0$ as $T \to 0^+$.

On the other hand, we will now use the Euler-Lagrange equation for u_T to show that

$$\lim_{T\to 0^+}\frac{\tau_T}{\nu_T}=0\,,\tag{5.6}$$

which is a clear contradiction to (5.3). Indeed, by Theorem 4.1-(iii) and (5.2), u_T satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta_p u_T=\mathsf{v}_T^{p-p^\star}\,\lambda_H(\mathsf{t}_T/\mathsf{v}_T)\,u_T^{p^\star-1} & \text{on }\Omega\,,\\ |\nabla u_T|^{p-2}\,\dfrac{\partial u}{\partial\nu_\Omega}=\mathsf{v}_T^{p-p^\#}\,\sigma_H(\mathsf{t}_T/\mathsf{v}_T)\,u_T^{p^\#-1} & \text{on }\partial\Omega\,;\end{cases}\tag{5.7}$$

see (4.2), (4.3) and (3.7) (for the definition of$\lambda_H(T)$ and $\sigma_H(T)$). Testing (5.7) with u_T, we find that

$$\begin{aligned}\int_\Omega|\nabla u_T|^p &= \mathsf{v}_T^{p-p^\star}\,\lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\int_\Omega u_T^{p^\star}+\mathsf{v}_T^{p-p^\#}\,\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\int_{\partial\Omega}u_T^{p^\#}\\ &= \mathsf{v}_T^{p-p^\star}\,\lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\,\nu_T^{p^\star}+\mathsf{v}_T^{p-p^\#}\,\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\,\tau_T^{p^\#}\,.\end{aligned}$$

After rearranging and multiplying through by $\mathsf{v}_T^{p^\sharp-p}\nu_T^{-p^\sharp}>0$, we arrive at the inequality

$$-\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\Big(\frac{\tau_T}{\nu_T}\Big)^{p^\#}\le\lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\Big(\frac{\nu_T}{\mathsf{v}_T}\Big)^{p^\star-p^\#}\,.\tag{5.8}$$

By (3.12) and the continuity of $T\mapsto\lambda_H(T)$, there are $C>0$ and $T_{**}>0$ such that $\lambda_H(T)\in(1/C,C)$ for every $T\in(0,T_{**})$. Moreover, thanks to (5.4), we can ask that $\mathsf{v}_T\ge 1/C$ for $T\in(0,T_{**})$. In particular, by (5.5) and up to further increasing C, if $T<1/C$, then $\mathsf{t}_T/\mathsf{v}_T<T_{**}$ and thus (5.8), $\nu_T\le 1$ and $\mathsf{v}_T\ge 1/C$ give

$$-\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\Big(\frac{\tau_T}{\nu_T}\Big)^{p^\#}\le C\,.\tag{5.9}$$

By (3.11) and the fact that $\mathsf{t}_T/\mathsf{v}_T\to 0$ as $T\to 0$, we see that $\sigma_H(T)\to-\infty$ as $T\to 0^+$, so that (5.9) implies (5.6). We reach a contradiction to (5.3), completing the proof. $\square$

Appendix A. Proof of Lemma 2.1

We will use the *Brezis–Lieb lemma* (if (X,μ) is a measure space, $q\ge 1$, and $\{f_j\}_j$ is bounded in $L^q(X)$, then

$$\int_X|f|^q\,d\mu=\lim_{j\to\infty}\int_X|f_j|^q\,d\mu-\int_X|f_j-f|^q\,d\mu\,,\tag{A.1}$$

provided f is a μ-a.e. limit of $\{f_j\}_j$ on X), and the two Sobolev-type inequalities

$$\|u\|_{L^{p^\star}(\Omega)}\le C_1\Big(\|\nabla u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)}\Big)\,,\tag{A.2}$$

$$\|u\|_{L^{p^\#}(\partial\Omega)}\le C_2\Big(\|\nabla u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)}\Big)\,,\tag{A.3}$$

which are valid, with constants C_1 and C_2 depending on n, p and Ω only, as soon as Ω is bounded and has Lipschitz boundary.

Proof of Lemma 2.1. Step one: Since Ω is a bounded open set with Lipschitz boundary, $u_j\rightharpoonup u$ as distributions in Ω, and $\{\nabla u_j\}_j$ is bounded in $L^p(\Omega)$, standard considerations show that $u\in W^{1,p}(\Omega)$, $\{u_j\}_j$ is bounded in $L^{p^\star}(\Omega)$ and in $L^{p^\#}(\partial\Omega)$, and, up to extracting subsequences, $u_j\to u$ in $L^q(\Omega)$ for every $q\in[1,p^\star)$, $u_j\to u$ in $L^r(\partial\Omega)$ for every $r<p^\#$, $u_j\to u$ pointwise $\mathcal{L}^n$-a.e. on Ω and $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and that the sequences of Radon measures $\{\nu_j\}_j$, $\{\tau_j\}_j$ and $\{\mu_j\}_j$ defined in (2.1) admits weak-$*$ limits ν, τ and μ, with $\operatorname{spt}\nu$ and $\operatorname{spt}\mu$ contained in $\overline{\Omega}$, and $\operatorname{spt}\tau$ contained in $\partial\Omega$.

Step two: We let $\tilde\mu$ denote the weak-$*$ limit of $|\nabla(u_j-u)|^p\,\mathcal{L}^n\llcorner\Omega$ (which exists up to possibly extracting a further subsequence), and claim that $\tilde\mu(\{x\})=\mu(\{x\})$ for every $x\in\mathbb{R}^n$. Indeed, by the elementary inequality

$$\big||v+w|^p-|w|^p\big|\le\varepsilon\,|v|^p+C(p,\varepsilon)\,|w|^p\,,\qquad v,w\in\mathbb{R}^n\,,\varepsilon>0\,,$$

given $x\in\mathbb{R}^n$ and $r>0$ we see that

$$\Big|\int_{B_r(x)}|\nabla u_j|^p-\int_{B_r(x)}|\nabla(u_j-u)|^p\Big|\le\varepsilon\int_{B_r(x)}|\nabla(u_j-u)|^p+C(p,\varepsilon)\int_{B_r(x)}|\nabla u|^p\,,$$

so that letting first $j\to\infty$ (for $r>0$ such that $\mu(\partial B_r(x))=\tilde\mu(\partial B_r(x))=0$) and then $r\to0^+$ (along a generic sequence of radii), we find indeed $|\mu(\{x\})-\tilde\mu(\{x\})|\le\varepsilon\,\tilde\mu(\{x\})$ for every $\varepsilon>0$.

Step three: If now pick $\eta\in C^\infty_c(\mathbb{R}^n)$ and set $\tilde\nu=\nu-|u|^{p^\star}\,\mathcal{L}^n\llcorner\Omega$ (notice that, by lower semicontinuity, $\tilde\nu$ is a Radon measure), then, by exploiting, in order, $\nu_j\overset{*}{\rightharpoonup}\nu$, (A.1), (A.2), and $u_j\to u$ in $L^p(\Omega)$, we find

$$\begin{aligned}&\int_{\mathbb{R}^n}|\eta|^{p^\star}\,d\tilde\nu\le\liminf_{j\to\infty}\int_\Omega|\eta|^{p^\star}\,(|u_j|^{p^\star}-|u|^{p^\star})\;=\lim_{j\to\infty}\int_\Omega|\eta\,(u_j-u)|^{p^\star}\\&\le C_1^{p^\star}\,\lim_{j\to\infty}\Big(\|\nabla(\eta(u_j-u))\|_{L^p(\Omega)}+\|\eta(u_j-u)\|_{L^p(\Omega)}\Big)^{p^\star}\\&=C_1^{p^\star}\,\lim_{j\to\infty}\|\nabla(\eta(u_j-u))\|^{p^\star}_{L^p(\Omega)}=C_1^{p^\star}\,\lim_{j\to\infty}\|\eta\,\nabla(u_j-u))\|^{p^\star}_{L^p(\Omega)}\\&\le C_1^{p^\star}\,\Big(\int_{\mathbb{R}^n}|\eta|^p\,d\tilde\mu\Big)^{p^\star/p}\,.\end{aligned}$$

In particular, for every $x\in\mathbb{R}^n$ one has

$$\tilde\nu(B_r(x))\le C_1^{p^\star}\,\tilde\mu(B_r(x))^{p^\star/p}\qquad\text{for a.e. }r>0\,.\tag{A.4}$$

By (A.4), $\tilde\nu$ is absolutely continuous with respect to $\tilde\mu$, so that $\tilde\nu=f\,\tilde\mu$ where f is such that, for $\tilde\mu$-a.e. $x\in\mathbb{R}^n$,

$$f(x)=\lim_{r\to0^+}\frac{\tilde\nu(B_r(x))}{\tilde\mu(B_r(x))}\,;$$

in particular, again by (A.4), for $\tilde\mu$-a.e. $x\in\mathbb{R}^n$ we have

$$f(x)\le C_1^{p^\star}\,\lim_{r\to0^+}\tilde\mu(B_r(x))^{(p^\star/p)-1}=C_1^{p^\star}\,\mu(\{x\})^{(p^\star/p)-1}\,.$$

In particular, as $p^\star>p$, if $X=\{x\in\mathbb{R}^n:\mu(\{x\})>0\}=\{x_i\}_{i\in I}\subset\overline{\Omega}$ (I at most countable) denotes the set of atoms of μ, then $f(x)=0$ μ-a.e. on $\mathbb{R}^n\setminus X$, and we have proved that

$$\mathrm{spt}\tilde\nu\subset\{x_i\}_{i\in I}\,.\tag{A.5}$$

An entirely analogous argument, this time based on (A.3) rather than on (A.2), shows that, if $\tilde\tau=\tau-|u|^{p^\#}\,\mathcal{H}^{n-1}\llcorner\partial\Omega$, then

$$\mathrm{spt}\tilde\tau\subset\{x_i\}_{i\in I}\cap\partial\Omega\,.\tag{A.6}$$

We have thus proved the validity of (2.2), (2.3) and (2.4) for suitable $\mathsf{v}_i,\mathsf{t}_i\ge0$ and $\mathsf{g}_i>0$: and of course we can discard possible points x_i with $\mathsf{v}_i=0$ from these decompositions, and directly assume that $\mathsf{v}_i>0$ for every i. The fact that $\mathsf{g}_i\ge S\,\mathsf{v}_i$ if $x_i\in\Omega$ is immediate by repeating the above argument with arbitrary $\eta\in C^\infty_c(\Omega)$ (in which case C_1 can be replaced by $1/S$). An analogous argument, this time using Lemma 2.2, shows that $\mathsf{g}_i\ge\mathsf{v}_i\,\Phi_H(\mathsf{t}_i/\mathsf{v}_i)$ if $x_i\in\partial\Omega$. □

Appendix B. Proof of Lemmas 2.2 and 2.3

This section is dedicated to the proof of Lemmas 2.2 and 2.3. We recall the standard Taylor expansions for the inverse and determinant of a matrix that is a perturbation of the identity:

$$(\mathrm{Id}_{\mathbb{R}^n} + tA)^{-1} = \mathrm{Id}_{\mathbb{R}^n} - tA + O(t^2)\,, \tag{B.1}$$

$$\det(\mathrm{Id}_{\mathbb{R}^n} + tA) = 1 + t\,\mathrm{trace}A + O(t^2); \tag{B.2}$$

see for instance [Mag12, Lemma 17.4].

Proof of Lemma 2.2. Note that $\hat{f}$ can equivalently be written as $\hat{f}(x) = x + \ell(\mathbf{p}(x))e_n$. We directly see that $\hat{f}$ maps C^1-diffeomorphically onto its image with inverse $\hat{g}(y) = y - \ell(\mathbf{p}(y))x_n$ and (2.10) holds because $\ell(0) = |\nabla\ell(0)| = 0$ and ℓ is C^1. We compute, in the standard basis for $\mathbb{R}^n$, that

$$\nabla\hat{f}(\mathbf{p}(x), x_n) = \begin{pmatrix} Id & 0 \\ \nabla\ell(\mathbf{p}(x)) & 1 \end{pmatrix} = Id + \begin{pmatrix} 0 & 0 \\ \nabla\ell(\mathbf{p}(x)) & 0 \end{pmatrix}.$$

Since ℓ is C^1 with $\nabla\ell(0) = 0$, this gives the first estimate in (2.11). The second estimate in (2.11) follows in the same way using the explicit form of g above. The first estimate in (2.12) follows from (B.2) and the expression for $\nabla\hat{f}$ above. The second estimate in (2.12) follows because $\hat{f} = F$ on $\mathbf{D}_{r_0}$ (compare with (2.7)), and so

$$J^{\partial H}\hat{f} = J^{\partial H}F = \sqrt{1 + |\nabla\ell|^2}\,.$$

This completes the proof of the lemma. □

Proof of Lemma 2.3. Step one: We compute some geometric quantities for $\partial\Omega$ using the graphical coordinates defined by the map F given in (2.7). We start with first order quantities. For $x \in \mathbf{D}_{r_0}$ and $i = 1, \dots, n-1$, we set $\tau_i := dF_x(e_i)$, i.e.,

$$\tau_i = \partial_i F(x) = e_i + \partial_i\ell(x)e_n, \tag{B.3}$$

so that $\{\tau_1, \dots, \tau_{n-1}\}$ forms a basis for $T_{F(x)}\partial\Omega$. Since ℓ is C^2 and $\nabla\ell(x) = 0$, we have

$$g_{ij} = \langle\tau_i, \tau_j\rangle_{\mathbb{R}^n} = \delta_{ij} + \partial_i\ell\,\partial_j\ell = \delta_{ij} + \mathrm{O}(|x|^2)\,. \tag{B.4}$$

In other words, the metric coefficients in graphical coordinates are Euclidean up to second order in $|x|$. The volume measure of $\partial\Omega$ is given by

$$J^{\partial H}F = \sqrt{\det g_{ij}} = \sqrt{1 + |\nabla\ell|^2}\,;$$

this immediately gives the second estimate in (2.14) since $f = F$ for $x \in \mathbf{D}_{r_0}$. The inverse metric coefficients are

$$g^{ij} = \delta^{ij} + \frac{\delta^{ia}\,\delta^{ib}\,\partial_a\ell\,\partial_b\ell}{1 + |\nabla\ell|^2}\delta^{ij} + O(|x|^2)\,. \tag{B.5}$$

Recall that, without loss of generality, we have chosen an orthonormal basis for $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that diagonalizes the Hessian of ℓ at $x = 0$. Let $\{\kappa_1, \dots, \kappa_{n-1}\}$ denote the eigenvalues. The second fundamental form of $\partial\Omega$ at x is defined by $A_x(v, w) = -\langle d\nu_\Omega(v), w\rangle$ for tangent vectors $v, w \in T_x(\partial\Omega)$. Using the shorthand $\nu := \nu_\Omega \circ F : \mathbf{D}_{r_0} \to S^{n-1}$, the coefficients of the second fundamental form in the coordinates defined by F are given by $A_{ij} = \langle\partial_{ij}F, \nu\rangle_{\mathbb{R}^n}$. Differentiating (B.3) above, we have $\partial_{ij}F = \partial_j\tau_i = \partial_{ij}\ell\,e_n$, and thus for $i, j \in \{1, \dots, n-1\}$ we have

$$A_{ij} = \langle\partial_{ij}F, \nu\rangle = \partial_{ij}\ell\,\langle e_n, \nu\rangle = \frac{-\partial_{ij}\ell}{\sqrt{1 + |\nabla\ell|^2}} = -\partial_{ij}\ell + \mathrm{O}(|x|^2)\,. \tag{B.6}$$

We have

$$\partial_i\nu = -A_i^j\tau_j\,, \tag{B.7}$$

and from (B.5) and (B.6) we directly compute that $A_i^j = g^{ik} A_{kj}$ is given by

$$A_i^j = \frac{-\partial_{ij}\ell}{\sqrt{1+|\nabla\ell|^2}} + O(|x|^2) = -\partial_{ij}\ell + O(|x|^2)$$
$$= -\partial_{ij}\ell(0) + O(|x|^2) = -\kappa_j \delta_i^j + O(|x|^2).$$

Step 2: Next, we use the previous step to compute geometric quantities associated to the coordinates defined by f. For $x \in \mathbf{C}_{r_0}$ and $i = 1, \ldots, n-1$, we note that $\partial_i \mathbf{p}(x) = e_i$ and thus from (B.7), In what follows we will suppress the composition with $\mathbf{p}$ in our notation, writing for instance τ_i in place of $\tau_i \circ \mathbf{p}(x)$. For $i = 1, \ldots, n-1$, we have

$$\partial_i f(x) = \tau_i - x_n \partial_i \nu(\mathbf{p}(x)) \tag{B.8}$$
$$= e_i + \partial_i \ell\, e_n + x_n A_i^j \tau_j = e_i + \partial_i \ell\, e_n - x_n \kappa_i e_i + O(|x|^2),$$

and $$\partial_n f(x) = -\nu = \frac{e_n - \nabla\ell}{\sqrt{1+|\nabla\ell|^2}} = e_n - \nabla\ell + O(|x|^2). \tag{B.9}$$

Together (B.8) and (B.9) can be expressed in consolidated form as

$$\nabla f = \sum_{i=1}^n \partial_i f \otimes e_i = \sum_{i=1}^n e_i \otimes e_i + e_n \otimes \nabla\ell - \nabla\ell \otimes e_n - x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2)$$
$$= \mathrm{Id}_{\mathbb{R}^n} + e_n \otimes \nabla\ell - \nabla\ell \otimes e_n - x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2).$$

In particular, from (B.2) we see that the volume form is given by

$$Jf = \sqrt{\det g_{ij}} = 1 - x_n H_\Omega(0) + O(|x|^2)\,,$$

giving us the first estimate in (2.14). We see directly from the definition that f is a C^1 map, and since we see from the expression for ∇f above that $\nabla f(0) = \mathrm{Id}_{\mathbb{R}^n}$, we may apply the inverse function theorem to see that, up to decreasing r_0, f defines a C^1 diffeomorphism onto its image. Letting $g = f^{-1}$ and using the expansion of the inverse (B.1), we find

$$(\nabla g) \circ f = (\nabla f)^{-1} = \mathrm{Id}_{\mathbb{R}^n} - e_n \otimes \nabla\ell + \nabla\ell \otimes e_n + x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2).$$

Finally, (2.13) follows from these expressions for ∇f and ∇g, along with the assumptions that $\nabla\ell(0) = 0$. This completes the proof of the lemma. □

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