# Γ-CONVERGENCE AND STOCHASTIC HOMOGENISATION OF PHASE-TRANSITION FUNCTIONALS

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Abstract. In this paper we study the asymptotics of singularly perturbed phase-transition functionals of the form

$$\mathscr{F}_k(u) = \frac{1}{\varepsilon_k}\int_A f_k(x,u,\varepsilon_k\nabla u)\,\mathrm{d}x\,,$$

where $u \in [0,1]$ is a phase-field variable, $\varepsilon_k > 0$ a singular-perturbation parameter*; i.e.*, $\varepsilon_k \to 0$, as $k \to +\infty$, and the integrands $f_k$ are such that, for every $x$ and every $k$, $f_k(x,\cdot,0)$ is a double well potential with zeros at 0 and 1. We prove that the functionals $\mathscr{F}_k$ Γ-converge (up to subsequences) to a surface functional of the form

$$\mathscr{F}_\infty(u) = \int_{S_u\cap A} f_\infty(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where $u \in BV(A;\{0,1\})$ and $f_\infty$ is characterised by the double limit of suitably scaled minimisation problems. Afterwards we extend our analysis to the setting of stochastic homogenisation and prove a Γ-convergence result for *stationary random* integrands.

**Keywords:** Singular perturbation, phase-field approximation, free discontinuity problems, Γ-convergence, deterministic and stochastic homogenisation.

**MSC 2010:** 49J45, 49Q20, 74Q05.

## 1. Introduction

The classical theory of phase transition is based on the assumption that at the equilibrium two immiscible fluids in a container will separate to create a sharp interface with minimal area. Alternatively Cahn and Hilliard in [19] (see also [35]) proposed a model which describes the transition as a continuous phenomenon concentrated on a thin layer, where a fine mixture of the two fluids is allowed. The connection between the classical theory of phase transition and the Cahn-Hilliard model was conjectured by Gurtin in [28] and rigorously derived, via Γ-convergence, by Modica in [29], by generalising a previous result together with Mortola [30] (see also [34], and [2]).

The model considered in [29, 30] is formalised as follows. Representing by $A \subset \mathbb{R}^n$, open bounded set with Lipschitz boundary, the container of the two fluids, and by $u\colon A \to [0,1]$ the density of the second fluid, the associated Modica-Mortola functionals are given by

$$\mathcal{M}_k(u) = \int_A \left(\frac{W(u)}{\varepsilon_k} + \varepsilon_k^{p-1}|\nabla u|^p\right)\mathrm{d}x\,. \tag{1.1}$$

Here $\varepsilon_k \searrow 0$, as $k \to +\infty$, is an elliptic regularisation parameter which corresponds to the characteristic length-scale of a phase transition. Moreover, $p > 1$ and $W\colon \mathbb{R} \to [0,+\infty)$ is a double well potential vanishing at 0 and 1. Clearly the functionals in (2.8) are finite in $W^{1,p}(A)$. Note that if $(u_k) \subset W^{1,p}(A)$, $0 \le u_k \le 1$, is a sequence satisfying $\sup_k \mathcal{M}_k(u_k) < +\infty$, then from the first term we may deduce that $u_k \to u$ in measure with $u(x) \in \{0,1\}$ a.e. $x \in A$ (which corresponds to a separation of the two phases). Actually the condition $0 \le u_k \le 1$ implies that $u_k$ converges to $u$ strongly in $L^1(A)$. On the other hand the second term penalizes spatial inhomogeneities of $u_k$ or equivalently the occurrence of too many transition regions. According to this, in [29] it was shown that the functionals in (1.1) Γ-converge to the functional

$$c_p\mathcal{H}^{n-1}(S_u), \tag{1.2}$$

where now $u \in BV(A;\{0,1\})$ (the space of functions with bounded variation taking values in $\{0,1\}$ a.e.). Here $S_u$ denotes the set of discontinuity points of $u$, while the constant $c_p > 0$ represents the optimal cost to make a transition between 0 and 1. The quantity in (1.2) can be interpreted as the surface tension between the two fluids. We stress also that this result is stable under the volume constraint $\int_A u\,\mathrm{d}x = m$ with $0 < m < \mathcal{L}^n(A)$ (that is if we fix the volume of the second fluid).

Starting from the result in [30] phase-field models have been extended in several directions including, among others, multi-phase models that describe the behaviour of an arbitrary number of immiscible fluids [10] (see also [34, 27]), models for heterogeneous fluids that can also undergo a temperature change [13], anisotropic models [11, 32], second order singular perturbations [20, 26, 21], and models that allow to analyse the interaction between singular perturbation and homogenisation [6, 12, 22, 23, 31]. These last variants are those which are closest in the spirit to the present work.

Drawing some inspiration from [9], in the present paper, we aim first at deriving Γ-convergence and integral representation results for a general class of phase-transition functionals that in some sense unify several of the variants mentioned above. Second, we will apply the abstract Γ-convergence result to obtain a stochastic homogenisation result for a sequence of phase-field functionals with random stationary integrands. Precisely, in the first part we perform a Γ-convergence analysis, as $k \to +\infty$, of a family of singularly perturbed phase-transition functionals of the form

$$\mathscr{F}_k(u) = \frac{1}{\varepsilon_k}\int_A f_k(x,u,\varepsilon_k\nabla u)\,\mathrm{d}x\,, \tag{1.3}$$

where $\varepsilon_k \searrow 0$ and $u\colon A \to [0,1]$. We assume that the integrands $f_k\colon \mathbb{R}^n \times [0,1] \times \mathbb{R}^n \to [0,+\infty)$ belong to a suitable class of functions denoted by $\mathcal{F}$ (see Section 2.2 for its definition). This in particular ensures that for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}^n$

$$c_1\left(W(u) + |\xi|^p\right) \le f_k(x,u,\xi) \le c_2\left(W(u) + |\xi|^p\right), \tag{1.4}$$

for every $u \in [0,1]$ and $\xi \in \mathbb{R}^n$, and for some $0 < c_1 \le c_2 < +\infty$. As a consequence $f_k(x,\cdot,0)$ is a double well potential that vanishes at 0 and 1. In addition the functional $\mathscr{F}_k$ is bounded from above and from below by the Modica-Mortola functional in (1.1).

Our first main result is contained in Theorem 3.1 which asserts the following: if the integrands $f_k$ belong to $\mathcal{F}$, then (up to subsequences) the functionals $\mathscr{F}_k$ Γ-converge to a surface-type functional of the form

$$\mathscr{F}_\infty(u) = \int_{S_u\cap A} f_\infty(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{1.5}$$

where $u \in BV(A;\{0,1\})$ and $\nu_u$ denotes an orientation for $S_u$. Moreover $f_\infty$ is characterised by a cell formula given by

$$f_\infty(x,\nu) = \limsup_{\rho\to 0}\liminf_{k\to+\infty}\frac{1}{\varepsilon_k}\int_{Q_\rho^\nu(x)} f_k(y,u,\varepsilon_k\nabla u)\,\mathrm{d}x, \tag{1.6}$$

where $Q_\rho^\nu(x)$ is a cube centred at $x$ with side-length $\rho$ with one face orthogonal to $\nu$, and the infimum is taken over all $u \in W^{1,p}(Q_\rho^\nu(x))$, $0 \le u \le 1$, such that $u = \bar{u}^\nu_{x,\varepsilon_k}$ near $\partial Q_\rho^\nu(x)$. Here

$\bar{u}^{\nu}_{x,\varepsilon_k}$ is a suitable regularisation (see Section 2.1 for the precise definition) of the jump function $u^{\nu}_{x}$ defined as

$$u^{\nu}_{x}(y) = \begin{cases} 1 & \text{if } (y-x)\cdot\nu \geq 0\,, \\ 0 & \text{if } (y-x)\cdot\nu < 0\,. \end{cases}$$

Notice that the presence of "regularised" boundary conditions is related to the fact that $u$ belongs to $W^{1,p}(A)$ while $u^{\nu}_{x}$ jumps on the hyperplane $\{(x-y)\cdot\nu = 0\}$.

Due to the generality of our model the proof of Theorem 3.1 relies on an abstract approach known as localisation method (cf. for instance [24]). This argument consists of the following main steps: we first localise the functionals $\mathscr{F}_k$ by introducing the dependence on the domain of integration; then, by means of a fundamental estimate, we show that, up to subsequence, the Γ-limit exists and admits a surface-integral representation. Eventually we identify the surface energy density with the cell formula in (1.6).

In the second part of the paper we are concerned with the case of stochastic homogenisation. We namely consider functionals of type (1.3) which depend also on a variable $\omega$ belonging to the set of events $\Omega$ of a probability space $(\Omega, \mathcal{T}, P)$. This dependence follows by choosing integrands $f_k$ of type

$$f_k(\omega, x, u, \xi) = f\left(\omega, \frac{x}{\varepsilon_k}, u, \xi\right). \tag{1.7}$$

Here $f$ is a stationary random variable such that $f(\omega,\cdot,\cdot,\cdot) \in \mathcal{F}$ and in addition is lower semicontinuous in the variable $\xi$ for every $\omega \in \Omega$ (see Section 7, $(F1)$-$(F3)$). Our second main result is contained in Theorem 7.4 which states that if $f_k$ are as in (1.7), then, almost surely, the functionals $\mathscr{F}_k$ Γ-converge to a surface functional of type (1.5) whose integrand is homogeneous in the space variable and is given by a cell formula.

The proof of Theorem 7.4 is achieved by resorting on the abstract Γ-convergence result Theorem 3.1 as follows. From Theorem 3.1 we first deduce a homogenisation result (cf. Theorem 3.3) in the deterministic setting without requiring any spatial periodicity of the integrands but instead assuming the existence and spatial homogeneity of the limit of a certain cell formula. Afterwards we show that in the above stochastic setting, almost surely this limit indeed exists and is homogeneous almost surely. This second step involves the combination of the subadditive ergodic Theorem (see e.g. [1, 25]) with a specific analysis for regularised surface functionals in the spirit of [3, 18].

Eventually to complete our analysis we show that if we restrict to the periodic setting, namely if $f_k$ are of the form $f(\frac{x}{\varepsilon_k}, u, \xi)$ with $f$ periodic in the space variable, then the corresponding homogenisation result Theorem 8.1 can be deduced without requiring any lower semicontinuity of $f$ in the variable $\xi$ (i.e., without assuming $(F3)$).

We notice that our stochastic/periodic homogenisation results (Theorem 7.4 and Theorem 8.1) cover in particular the cases considered in [31, 23] and the critical regime in [6]. As a final remark it would be interesting to fully characterise the Γ-limit for integrands of type (1.7) when the oscillation parameter $\varepsilon_k$ is replaced by a second scale parameter $\delta_k \searrow 0$. We expect a different behaviour for the two regimes $\varepsilon_k \ll \delta_k$ and $\varepsilon_k \gg \delta_k$. However this type of analysis goes beyond the purpose of the present paper. A similar analysis was performed in [6, 22, 23] in the deterministic setting (see also [7, 8] for the corresponding analysis for Ambrosio-Tortorelli type functionals).

**Outline of the paper.** This paper is organised as follows. In Section 2 we collect the notation adopted throughout the paper and introduce the functionals we will consider. In Section 3 we state the first main result of the paper, that is, the Γ-convergence and integral representation result (Theorem 3.1). We also give a homogenisation result without any periodicity assumptions (Theorem 3.3). Sections 4-6 contain the proof of Theorem 3.1). In particular, in Section 4 we implement the localisation method and prove a compactness and integral representation result for the Γ-limit of $\mathscr{F}_k$ (Theorem 4.2); in Section 5 we characterise the surface integrand (Proposition 5.1); eventually in Section 6 we prove that the limit surface integrand satisfies some properties (Proposition 6.2). In Section 7 we state and prove our second main result, namely a stochastic homogenisation result for stationary random integrands (Theorem 7.4). In Section 8 we prove a periodic homogenisation result (Theorem 8.1). Eventually in the Appendix we prove some technical lemmas which are used in Sections 7 8.

## 2. Setting of the problem and preliminaries

In this section we collect some notation and we introduce the family of functionals we will consider.

2.1. **Notation.** We start by listing the notation we will adopt throughout the paper.

(a) $n \geq 2$ is a fixed positive integer;
(b) $\mathbb{S}^{n-1} := \{\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n : \nu_1^2 + \dots + \nu_n^2 = 1\}$ and $\widehat{\mathbb{S}}^{n-1}_\pm := \{\nu \in \mathbb{S}^{n-1} : \pm \nu_{i(\nu)} > 0\}$, where $i(\nu) := \max\{i \in \{1, \dots, n\} : \nu_i \neq 0\}$;
(c) $\mathcal{L}^n$ and and $\mathcal{H}^{n-1}$ denote the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, respectively;
(d) for every $A \subset \mathbb{R}^n$ let $\chi_A$ denote the characteristic function of the set $A$;
(e) $\mathcal{A}$ denotes the collection of all open and bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary. If $A, B \in \mathcal{A}$ by $A \subset\subset B$ we mean that $A$ is relatively compact in $B$;
(f) $Q$ denotes the open unit cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at the origin; for $x \in \mathbb{R}^n$ and $r > 0$ we set $Q_r(x) := rQ + x$. Moreover, $Q'$ denotes the open unit cube in $\mathbb{R}^{n-1}$ with sides parallel to the coordinate axes, centred at the origin, for every $r > 0$ we set $Q'_r := rQ'$;
(g) for every $\nu \in \mathbb{S}^{n-1}$ let $R_\nu$ denote an orthogonal $(n \times n)$-matrix such that $R_\nu e_n = \nu$; we also assume that $R_{-\nu}Q = R_\nu Q$ for every $\nu \in \mathbb{S}^{n-1}$, $R_\nu \in \mathbb{Q}^{n\times n}$ if $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, and that the restrictions of the map $\nu \mapsto R_\nu$ to $\widehat{\mathbb{S}}^{n-1}_\pm$ are continuous. For an explicit example of a map $\nu \mapsto R_\nu$ satisfying all these properties we refer the reader, *e.g.*, to [17, Example A.1];
(h) for $x \in \mathbb{R}^n$, $r > 0$, and $\nu \in \mathbb{S}^{n-1}$, we define $Q^\nu_r(x) := R_\nu Q_r(x)$.
(i) for $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ we denote with $u^\nu_x$ the piecewise constant function taking values $0, 1$ and jumping across the hyperplane $\Pi^\nu(x) := \{y \in \mathbb{R}^n : (y - x) \cdot \nu = 0\}$; *i.e.*,
$$u^\nu_x(y) := \begin{cases} 1 & \text{if } (y-x)\cdot\nu \geq 0\,, \\ 0 & \text{if } (y-x)\cdot\nu < 0\,; \end{cases}$$
(j) let $\mathrm{u} \in C^1(\mathbb{R})$ with $0 \leq \mathrm{u} \leq 1$, be a one-dimensional function such that
$$\mathrm{u}(t) = \chi_{(0,+\infty)}(t) \quad \text{for} \quad |t| > 1;$$
(k) for $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ we set
$$\bar{u}^\nu_x(y) := \mathrm{u}((y-x)\cdot\nu)\,;$$
(l) for $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$ we set
$$\bar{u}^\nu_{x,\varepsilon}(y) := \mathrm{u}\big(\tfrac{1}{\varepsilon}(y-x)\cdot\nu\big)\,.$$
We notice that in particular, $\bar{u}^\nu_{x,\varepsilon}(y) = u^\nu_x$ in $\{y : |(y-x)\cdot\nu| > \varepsilon\}$, and $\bar{u}^\nu_{x,1} = \bar{u}^\nu_x$;
(m) for a given topological space $X$, $\mathcal{B}(X)$ denotes the Borel $\sigma$- algebra on $X$. If $X = \mathbb{R}^d$, with $d \in \mathbb{N}$, $d \geq 1$ we simply write $\mathcal{B}^d$ in place of $\mathcal{B}(\mathbb{R}^d)$. For $d = 1$ we write $\mathcal{B}$.

For $A \in \mathcal{A}$ we let $BV(A)$ be the space of functions with bounded variation (see for instance [5] for a detailed exposition of the subject). We recall that if $u \in BV(A)$, then its distributional derivative $Du$ is an $\mathbb{R}^n$- valued Radon measure on $A$. We denote by $|Du|(A)$ its total variation.

Moreover a set $E \subset \mathbb{R}^n$ is a set of finite perimeter in $A$, or a *Caccioppoli set*, if $\chi_E \in BV(A)$ and we let

$$\mathcal{P}_A(E) := |D\chi_E|(A)\,,$$

be the *perimeter* of $E$ in $A$. The family of sets with finite perimeter can be identified with the space $BV(A;\{0,1\})$, namely, the space of functions in $BV(A)$ taking values in $\{0,1\}$ almost everywhere. Indeed if $u \in BV(A;\{0,1\})$ then $u = \chi_E$ where $E = \{x \in A \colon u(x) = 1\}$ and

$$Du(B) = \int_{B\cap S_u} \nu_u \,\mathrm{d}\mathcal{H}^{n-1}\,,$$

for every $B \in \mathcal{B}^n$, where $S_u$ is the set of approximate discontinuity points of $u$ which, up to $\mathcal{H}^{n-1}$-negligible sets, coincides with the part of the reduced boundary of $E$ that lies in $A$, while $\nu_u$ is the external normal to $S_u$. Moreover

$$\mathcal{P}_A(E) = |Du|(A) = \mathcal{H}^{n-1}(S_u \cap A)\,.$$

Throughout the paper $C$ denotes a strictly positive constant which may vary from line to line and within the same expression.

2.2. **Setting of the problem.** Let $p > 1$, let $c_1, c_2$ be given constants such that $0 < c_1 \leq c_2 < +\infty$. Let $W \colon \mathbb{R} \to [0,+\infty)$ be a double-well potential, that is a continuous function vanishing only at 0 and 1. We denote by $\mathcal{F} := \mathcal{F}(W,p,c_1,c_2)$ the collection of all functions $f \colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty)$, $f = f(x,u,\xi)$, satisfying the following conditions:

($f1$) (measurability) $f$ is Borel measurable on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$;

($f2$) (continuity in $u$) the function $f(x,\cdot,\xi)$ is continuous for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$;

($f3$) (lower bound) for every $x \in \mathbb{R}^n$, every $u \in \mathbb{R}$, and every $\xi \in \mathbb{R}^n$

$$c_1\big(W(u) + |\xi|^p\big) \leq f(x,u,\xi)\,;$$

($f4$) (upper bound) for every $x \in \mathbb{R}^n$, every $u \in \mathbb{R}$, and every $\xi \in \mathbb{R}^n$

$$f(x,u,\xi) \leq c_2(W(u) + |\xi|^p)\,.$$

For $k \in \mathbb{N}$ let $(f_k) \subset \mathcal{F}$ and let $(\varepsilon_k) \subset (0,1]$ be a decreasing sequence of real numbers converging to zero, as $k \to +\infty$. We consider the sequence of singularly perturbed phase-transitions functionals $\mathscr{F}_k \colon L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0,+\infty]$ defined by

$$\mathscr{F}_k(u,A) := \begin{cases} \dfrac{1}{\varepsilon_k}\displaystyle\int_A f_k(x,u,\varepsilon_k\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A)\,, 0 \leq u \leq 1\,,\\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.1}$$

It is convenient to introduce some further notation. For $A \in \mathcal{A}$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ we consider the following minimisation problem

$$\mathbf{m}_k(\bar{u}^\nu_{x,\varepsilon_k}, A) := \inf\{\mathscr{F}_k(u,A) \colon u \in \mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, A)\}\,, \tag{2.2}$$

where

$$\mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, A) := \big\{u \in W^{1,p}(A),\ 0 \leq u \leq 1 \colon\ u = \bar{u}^\nu_{x,\varepsilon_k} \text{ near } \partial A\ \big\}\,. \tag{2.3}$$

In (2.2) by "$u = \bar{u}^{\nu}_{x,\varepsilon_k}$ near $\partial A$" we mean that the boundary datum is attained in a neighbourhood of $\partial A$. Note that, if we choose $A = Q^{\nu}_{\rho}(x)$ for some $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $\rho > 0$ then (2.2) is well defined for $\rho > 2\varepsilon_k$. Therefore, for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ we can set

$$f'(x,\nu) := \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}} \liminf_{k\to+\infty} \mathbf{m}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, Q^{\nu}_{\rho}(x)) \,, \tag{2.4}$$

$$f''(x,\nu) := \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}} \limsup_{k\to+\infty} \mathbf{m}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, Q^{\nu}_{\rho}(x)) \,. \tag{2.5}$$

*Remark* 2.1. (*a*) From $(f3)$ and $(f4)$ together with the assumptions on $W$ we have that

$$f(x,u,\xi) = 0 \;\text{ if and only if }\; (u,\xi) \in \{(0,0),(1,0)\}\,, \tag{2.6}$$

for every $x \in \mathbb{R}^n$. As a consequence by $((f2))$ $f(x,\cdot,0)$ is a double-well potential vanishing at 0 and 1 for every $x \in \mathbb{R}^n$. Moreover the functionals

$$\frac{1}{\varepsilon_k}\int_A f(x,u,\varepsilon_k\nabla u)\,\mathrm{d}x\,,$$

decrease under the transformation $u \to \min\{\max\{u,0\},1\}$. Thus it is not restrictive to define $\mathscr{F}_k$ to be finite when $u \in W^{1,p}(A)$ satisfies the bounds $0 \le u \le 1$ and the $\Gamma$-convergence for functionals $\mathscr{F}_k$ that are finite when $u \in W^{1,p}(A)$ readily follows from Theorem 3.1.

(*b*) Assumptions $(f3)$–$(f4)$ imply that for every $A \in \mathcal{A}$ and every $u \in W^{1,p}(A)$ with $0 \le u \le 1$ it holds

$$c_1\mathcal{M}_k(u,A) \le \mathscr{F}_k(u,A) \le c_2\mathcal{M}_k(u,A)\,; \tag{2.7}$$

where

$$\mathcal{M}_k(u,A) := \int_A \left(\frac{W(u)}{\varepsilon_k} + \varepsilon_k^{p-1}|\nabla u|^p\right)\mathrm{d}x\,, \tag{2.8}$$

is the Modica-Mortola functional.

(*c*) From the assumption that $\mathscr{F}_k$ is finite when $u \in W^{1,p}(A)$, $0 \le u \le 1$, together with the bound in (2.7) and the result in [29] we easily deduce the following compactness property: Let $(u_k) \subset W^{1,p}(A)$ be a sequence satisfying $\sup_k \mathscr{F}_k(u_k,A) < +\infty$. Then $u_k$ converges strongly in $L^1(A)$ to some $u \in BV(A,\{0,1\})$.

If the function $W$ satisfies the following coercivity condition:

$$W(u) \ge \varphi(|u|) \quad \forall u \in \mathbb{R}\,,$$

with $\varphi\colon [0,+\infty) \to [0,+\infty)$ such that $\lim_{t\to+\infty}\varphi(t)/t = +\infty$, then the same compactness property holds if we define $\mathscr{F}_k$ to be finite when $u \in W^{1,p}(A)$ (cf. [29]).

(*d*) Let $A \in \mathcal{A}$ be such that $A = A' \times I$ with $A' \subset \mathbb{R}^{n-1}$ open and bounded and $I \subset \mathbb{R}$ open interval. Let $\nu \in \mathbb{S}^{n-1}$ and set $A_\nu := R_\nu A$, with $R_\nu$ as in (g). For every $k \in \mathbb{N}$ we have

$$\mathcal{M}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, A_\nu) \le \int_{A'}\int_{\mathbb{R}} \big(W(\mathrm{u}(t)) + |\mathrm{u}'(t)|^p\big)\,\mathrm{d}t\,\mathrm{d}y' = C_{\mathrm{u}}\mathcal{L}^{n-1}(A')\,, \tag{2.9}$$

where

$$C_{\mathrm{u}} := \int_{\mathbb{R}} \big(W(\mathrm{u}(t)) + |\mathrm{u}'(t)|^p\big)\,\mathrm{d}t = \int_{-1}^{1} \big(W(\mathrm{u}(t)) + |\mathrm{u}'(t)|^p\big)\,\mathrm{d}t < +\infty\,.$$

In particular from $(f4)$ and (2.9) we deduce

$$\mathbf{m}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, A_\nu) \le \mathscr{F}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, A_\nu) \le c_2 C_{\mathrm{u}}\mathcal{L}^{n-1}(A')\,. \tag{2.10}$$

## 3. Statements of the main results

In this section we collect some of the results of this paper. We start by stating our first main result (Theorem 3.1), that is a Γ-convergence and an integral representation result. Next we provide a homogenisation result without periodicity assumptions (Theorem 3.3). The latter, in particular, will be crucial to obtain our second main result in the stochastic setting (see Section 7).

### 3.1. Γ-convergence.

The following result shows that, up to subsequences, the functionals $\mathscr{F}_k$ Γ-converge to a surface integral functional. Furthermore, the surface energy density can be characterised as a double limit of a suitable minimisation problem.

**Theorem 3.1** (Γ-convergence)**.** *Let $(f_k) \subset \mathcal{F}$ and let $\mathscr{F}_k$ be the functionals as in* (2.1)*. Then there exists a subsequence $k_j$ such that for every $A \in \mathcal{A}$ the functionals $\mathscr{F}_{k_j}(\cdot\,, A)$ Γ-converge in $L^1(A)$ to $\mathscr{F}_\infty(\cdot\,, A)$ with $\mathscr{F}_\infty \colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0,+\infty]$ given by*

$$\mathscr{F}_\infty(u, A) := \begin{cases} \displaystyle\int_{S_u \cap A} f_\infty(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in BV(A; \{0,1\})\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

*where $f_\infty \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,+\infty)$ is a Borel function. Moreover for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ there hold*

$$f_\infty(x,\nu) = f'(x,\nu) = f''(x,\nu)\,,$$

*and*

$$c_1 c_p \leq f_\infty(x,\nu) \leq c_2 c_p\,,$$

*with $f'$, $f''$ defined as in* (2.4) *and* (2.5) *with $k$ replaced by $k_j$, and $c_p := p(p-1)^{\frac{1-p}{p}} \int_0^1 W(t)^{\frac{p-1}{p}}\, \mathrm{d}t$.*

For the reader's convenience we divide the proof of Theorem 3.1 in three parts which can be found in Sections 4, 5, and 6, respectively. Precisely, in Section 4 we show that there is a sequence $(k_j)$, with $k_j \to +\infty$ as $j \to +\infty$, such that for every $A \in \mathcal{A}$ the corresponding functionals $\mathscr{F}_{k_j}(\,\cdot\,, A)$ Γ-converge to a functional which is finite in $BV(A; \{0,1\})$ and is of the form

$$\int_{S_u \cap A} \hat{f}(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}\,,$$

for some Borel function $\hat{f}$ (see Theorem 4.2). In Section 5 we identify $\hat{f}$ by showing that it is equal both to $f'$ and $f''$ which in particular are shown to coincide along the Γ-convergent subsequence (see Proposition 5.1). Eventually in Section 6 we prove that the function $f'$ (respectively $f''$) is Borel measurable and satisfies $c_1 c_p \leq f'(x,\nu) \leq c_2 c_p$ (respectively $c_1 c_p \leq f''(x,\nu) \leq c_2 c_p$) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ (see Proposition 6.2).

By Theorem 3.1 and the Urysohn property of Γ-convergence [24, Proposition 8.3] we can deduce the following useful property.

**Corollary 3.2.** *Let $(f_k) \subset \mathcal{F}$ and let $\mathscr{F}_k$ be the functionals as in* (2.1)*. Let $f'$, $f''$ be as in* (2.4) *and* (2.5)*, respectively. Assume that*

$$f'(x,\nu) = f''(x,\nu) =: f_\infty(x,\nu), \quad \textit{for every } x \in \mathbb{R}^n \textit{ and every } \nu \in \mathbb{S}^{n-1},$$

*for some Borel function $f_\infty \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,+\infty)$. Then, for every $A \in \mathcal{A}$ the functionals $\mathscr{F}_k(\cdot\,, A)$ Γ-converge in $L^1(A)$ to $\mathscr{F}_\infty(\cdot\,, A)$ with $\mathscr{F}_\infty \colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0,+\infty]$ given by*

$$\mathscr{F}_\infty(u, A) := \begin{cases} \displaystyle\int_{S_u \cap A} f_\infty(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in BV(A; \{0,1\})\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

3.2. **Homogenisation.** In this subsection we prove a general homogenisation theorem without assuming any spatial periodicity of the integrands $f_k$. This theorem will be employed to prove the stochastic homogenisation result Theorem 7.4.

We need to introduce some further notation. We fix $f \in \mathcal{F}$, $A \in \mathcal{A}$, $u \in W^{1,p}(A)$, $z \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and let $\bar{u}_z^\nu$ be as in (k) (with $x = z$). Then we set

$$\mathscr{F}(u, A) := \int_A f(x, u, \nabla u)\, \mathrm{d}x\,; \tag{3.1}$$

and

$$\mathbf{m}(\bar{u}_z^\nu, A) := \inf\Big\{\mathscr{F}(u, A) \colon u \in \mathscr{A}(\bar{u}_z^\nu, A)\Big\}\,, \tag{3.2}$$

where $\mathscr{A}(\bar{u}_z^\nu, A)$ is as in (2.3), with $\bar{u}_{x,\varepsilon_k}^\nu$ replaced by $\bar{u}_z^\nu$ (that is, $x = z$ and $\varepsilon_k = 1$).

When the integrands $f_k$ in the definition of $\mathscr{F}_k$ are of type

$$f_k(x, u, \xi) := f\Big(\frac{x}{\varepsilon_k}, u, \xi\Big)\,, \tag{3.3}$$

we can deduce the following homogenisation result.

**Theorem 3.3** (Deterministic homogenisation)**.** *Let $f \in \mathcal{F}$ and let $\mathbf{m}(\bar{u}_{rx}^\nu, Q_r^\nu(rx))$ be as in* (3.2) *with $z = rx$ and $A = Q_r^\nu(rx)$. Assume that for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ the following limit*

$$\lim_{r\to+\infty} \frac{\mathbf{m}(\bar{u}_{rx}^\nu, Q_r^\nu(rx))}{r^{n-1}} =: f_{\mathrm{hom}}(\nu)\,, \tag{3.4}$$

*exists and is independent of $x$. Then, for every $A \in \mathcal{A}$ the functionals $\mathscr{F}_k(\cdot, A)$ defined in* (2.1) *with $f_k$ as in* (3.3) *$\Gamma$-converge in $L^1(A)$ to the functional $\mathscr{F}_{\mathrm{hom}}(\cdot, A)$, with $\mathscr{F}_{\mathrm{hom}} \colon L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ given by*

$$\mathscr{F}_{\mathrm{hom}}(u, A) := \begin{cases} \displaystyle\int_{S_u \cap A} f_{\mathrm{hom}}(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in BV(A; \{0,1\})\,, u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise}\,. \end{cases}$$

*Proof.* Let $f'$, $f''$ be as in (2.4), (2.5), respectively. By Corollary 3.2 it suffices to show that

$$f_{\mathrm{hom}}(\nu) = f'(x, \nu) = f''(x, \nu)\,, \tag{3.5}$$

for every $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$.

Let $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$ and $\rho > 2\varepsilon_k$ be fixed and let $u \in \mathscr{A}(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))$. Define $u_k \in W^{1,p}(Q^\nu_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k}))$ as $u_k(y) := u(\varepsilon_k y)$. Then clearly $u_k \in \mathscr{A}(\bar{u}^\nu_{\frac{x}{\varepsilon_k}}, Q^\nu_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k}))$ and a change of variables gives

$$\mathscr{F}_k(u, Q_\rho^\nu(x)) = \varepsilon_k^{n-1} \int_{Q^\nu_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k})} f(x, u_k, \nabla u_k)\, \mathrm{d}x\,.$$

Hence by setting $r_k := \frac{\rho}{\varepsilon_k}$ we get

$$\mathbf{m}_k(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) = \varepsilon_k^{n-1} \mathbf{m}\big(\bar{u}^\nu_{\frac{x}{\varepsilon_k}}, Q^\nu_{\frac{\rho}{\varepsilon_k}}\big(\tfrac{x}{\varepsilon_k}\big)\big) = \frac{\rho^{n-1}}{r_k^{n-1}} \mathbf{m}\big(\bar{u}^\nu_{r_k\frac{x}{\rho}}, Q_{r_k}\big(r_k \tfrac{x}{\rho}\big)\big)\,.$$

Finally using (3.4) with $x/\rho$ in place of $x$ we obtain

$$\lim_{k\to+\infty} \frac{\mathbf{m}_k(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))}{\rho^{n-1}} = \lim_{k\to+\infty} \frac{\mathbf{m}\big(\bar{u}^\nu_{r_k\frac{x}{\rho}}, Q^\nu_{r_k}\big(r_k\tfrac{x}{\rho}\big)\big)}{r_k^{n-1}} = f_{\mathrm{hom}}(\nu)\,,$$

and therefore $f'(x, \nu) = f''(x, \nu) = f_{\mathrm{hom}}(\nu)$, for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$. □

## 4. Γ-convergence and integral representation

In this section we prove that, up to subsequences, the functionals $\mathscr{F}_k$ Γ-converge to a surface integral functional. In order to do that we apply the so called localisation method of Γ-convergence. In particular we closely follow the theory described in [24, Chapters 14-18] and [16, Chapters 10, 11], for this reason we discuss here only the main adaptations to our case.

We start by proving a preliminary result, namely we show that the functionals $\mathscr{F}_k$ satisfy a fundamental estimate, uniformly in $k$ similarly to [6, Lemma 3.2].

**Proposition 4.1** (Fundamental estimate). *Let $\mathscr{F}_k$ be as in (2.1). Then there exist $K > 0$ depending only on $c_1, c_2, p$ and $W$ and functions $\omega_k \colon L^1_{\rm loc}(\mathbb{R}^n)^2 \times \mathcal{A}^3 \to [0, +\infty)$ (depending only on $k$) such that for every $A$, $A'$, $B \in \mathcal{A}$ with $A \subset\subset A'$ the following hold:*

(i) *For every $u \in W^{1,p}(A')$, $v \in W^{1,p}(B)$, $0 \le u, v \le 1$, there exists $w \in W^{1,p}(A \cup B)$ with $0 \le w \le 1$ such that*

$$w = u \quad \text{a.e. in} \quad A \quad \text{and} \quad w = v \quad \text{a.e. in} \quad B \setminus \overline{A}',$$

*and*

$$\mathscr{F}_k(w, A \cup B) \le (1 + K\varepsilon_k)(\mathscr{F}_k(u, A') + \mathscr{F}_k(v, B)) + \omega_k(u, v, A, A', B)\,;$$

(ii) *For every $u \in W^{1,p}(A')$, $v \in W^{1,p}(B)$ $0 \le u, v \le 1$ and every $\widetilde{S} \subset (A' \setminus \overline{A}) \cap B$ with*

$$u(x) = v(x) \in \{0, 1\} \quad \text{a.e. in} \quad \widetilde{S}\,;$$

*we have*

$$\omega_k(u, v, A, A', B) \le K\mathcal{L}^n((A' \setminus \overline{A}) \cap B \setminus \widetilde{S})\,;$$

(iii) *For every $(u_k), (v_k) \subset L^1_{\rm loc}(\mathbb{R}^n)$ having the same limit as $k \to \infty$ in $L^1((A' \setminus \overline{A}) \cap B)$ and satisfying*

$$\sup_{k\in\mathbb{N}} (\mathscr{F}_k(u_k, A') + \mathscr{F}_k(v_k, B)) < +\infty\,,$$

*it holds*

$$\lim_{k\to\infty} \omega_k(u_k, v_k, A, A', B) = 0\,. \tag{4.1}$$

*Proof.* Let $A$, $A'$, $B \in \mathcal{A}$ with $A \subset\subset A'$ be fixed.

*Step 1:* we show $(i)$. Let $d := {\rm dist}(A; \mathbb{R}^n \setminus A')$, $N_k := \lfloor 2\varepsilon_k^{-1} \rfloor$ (here $\lfloor a \rfloor$ denotes the integer part of $a$), and let $A_1, \dots, A_{N_k+1} \in \mathcal{A}$ with

$$A \subset\subset A_1 \subset\subset \dots \subset\subset A_{N_k+1} \subset\subset A'\,,$$

and

$${\rm dist}(A_i; \mathbb{R}^n \setminus A_{i+1}) \ge \frac{d}{N_k + 2} \quad i = 1, \dots, N_k\,.$$

For each $i = 1, \dots, N_k$ let $\varphi_i$ be a smooth cut-off function between $A_i$ and $A_{i+1}$ such that

$$M_k := \max_{1\le i\le N_k} \|\nabla\varphi_i\|_\infty \le \frac{2(N_k + 2)}{d}\,.$$

Let $u$ and $v$ be as in the statement. For $i = 1, \dots, N_k$ we define $w^i \in W^{1,p}(A \cup B)$ as follows

$$w^i := \varphi_i u + (1 - \varphi_i)v\,. \tag{4.2}$$

By setting $S_i := (A_{i+1} \setminus \overline{A}_i) \cap B$ we obtain

$$\begin{aligned} \mathscr{F}_k(w^i, A \cup B) &\le \mathscr{F}_k(u, A_i) + \mathscr{F}_k(v, B \setminus \overline{A}_{i+1}) + \mathscr{F}_k(w^i, S_i) \\ &\le \mathscr{F}_k(u, A') + \mathscr{F}_k(v, B) + \mathscr{F}_k(w^i, S_i)\,. \end{aligned} \tag{4.3}$$

Now from $(f4)$ we deduce

$$\mathscr{F}_k(w^i,S_i)\le c_2\int_{S_i\cap B}\bigg(\frac{W(w^i)}{\varepsilon_k}+\varepsilon_k^{p-1}|\nabla w^i|^p\bigg)\,\mathrm{d}x\,. \tag{4.4}$$

Moreover a convexity argument gives

$$\begin{aligned}|\nabla w_i|^p&\le 2^{2(p-1)}\left(|\nabla u|^p+|\nabla v|^p+|\nabla\varphi_i|^p|u-v|^p\right)\\&\le 2^{2(p-1)}\left(|\nabla u|^p+|\nabla v|^p+M_k^p|u-v|^p\right)\,.\end{aligned} \tag{4.5}$$

From (4.4), (4.5) and $(f3)$ we obtain

$$\begin{aligned}\mathscr{F}_k(w^i,S_i)\le{}&\frac{2^{2(p-1)}c_2}{c_1}\big(\mathscr{F}_k(u,S_i)+\mathscr{F}_k(v,S_i)\big)\\&+2^{2(p-1)}c_2M_k^p\varepsilon_k^{p-1}\int_{S_i}|u-v|^p\,\mathrm{d}x+c_2\int_{S_i}\frac{W(w^i)}{\varepsilon_k}\,\mathrm{d}x\,.\end{aligned} \tag{4.6}$$

Now summing up in (4.3) over all $i$, by averaging we find an index $i^*\in\{1,\dots,N_k\}$ such that

$$\mathscr{F}_k(w^{i^*},A\cup B)\le\frac{1}{N_k}\sum_{i=1}^{N_k}\mathscr{F}_k(w^i,A\cup B)\le\mathscr{F}_k(u,A')+\mathscr{F}_k(v,B)+\frac{1}{N_k}\sum_{i=1}^{N_k}\mathscr{F}_k(w^i,S_i)\,. \tag{4.7}$$

By (4.6) we have

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\mathscr{F}_k(w^i,S_i)\le\frac{\widehat{M}}{N_k}\left(\mathscr{F}_k(u,A')+\mathscr{F}_k(v,B)+M_k^p\varepsilon_k^{p-1}\int_S|u-v|^p\,\mathrm{d}x+\sum_{i=1}^{N_k}\int_{S_i}\frac{W(w^i)}{\varepsilon_k}\,\mathrm{d}x\right), \tag{4.8}$$

with $\widehat{M}:=\max\{\frac{2^{2(p-1)}c_2}{c_1},2^{2(p-1)}c_2,c_2\}$ and $S:=(A'\setminus\overline{A})\cap B$. Using that $\frac{2-\varepsilon_k}{\varepsilon_k}\le N_k\le\frac{2}{\varepsilon_k}$ we have

$$\frac{\widehat{M}}{N_k}\le\frac{\widehat{M}\varepsilon_k}{2-\varepsilon_k}\le\widehat{M}\varepsilon_k\,,\qquad\text{and}\qquad\frac{\widehat{M}}{N_k}M_k^p\varepsilon_k^{p-1}\le C\varepsilon_k^p\left(\frac{1}{\varepsilon_k}+1\right)^p\le C\,.$$

This together with (4.7) and (4.8) give

$$\mathscr{F}_k(w^{i^*},A\cup B)\le(1+\widehat{M}\varepsilon_k)\left(\mathscr{F}_k(u,A')+\mathscr{F}_k(v,B)\right)+\omega_k(u,v,A,A',B)$$

where

$$\omega_k(u,v,A,A',B):=C\left(\int_S|u-v|^p\,\mathrm{d}x+\sum_{i=1}^{N_k}\int_{S_i}W(w^i)\,\mathrm{d}x\right), \tag{4.9}$$

for some positive constant $C>0$ independent of $k$.

*Step 2:* we show $(ii)$. Assume that $0\le u,v\le 1$ are such that

$$u(x)=v(x)\in\{0,1\}\quad\text{a.e. in}\quad\widetilde{S}\subset S\,.$$

Then clearly $\int_{\widetilde{S}}|u-v|^p\,\mathrm{d}x=0$ and

$$W(w^i)=0\quad\text{a.e. in}\quad S_i\cap\widetilde{S}\,.$$

Moreover, since $0\le w^i\le 1$ for all $i=1,...,N_k$, by the continuity of $W$

$$W(w_i)\le M\quad\text{a.e. in}\quad S_i\setminus\widetilde{S}\,,\ \forall i=1,...,N_k\,.$$

Hence (4.9) readily implies

$$\omega_k(u,v,A,A',B)\le C(2^p+M)\mathcal{L}^n(S\setminus\widetilde{S})\,,$$

and by setting $K:=\max\{\widehat{M},C(2^p+M)\}$ we deduce $(ii)$.

*Step 3:* we show (*iii*). Let $(u_k),(v_k) \subset L^1_{\rm loc}(\mathbb{R}^n)$, $0 \le u_k, v_k \le 1$, be two sequences converging to $w$ in $L^1(S)$ and such that

$$\sup_{k\in\mathbb{N}} \big(\mathscr{F}_k(u_k, A') + \mathscr{F}_k(v_k, B)\big) < C\,.$$

Since $0 \le u_k, v_k \le 1$ we immediately deduce that $u_k$ and $v_k$ converge to $w$ also in $L^p(S)$.

Thus (4.1) follows if we show that

$$\lim_{k\to\infty} \sum_{i=1}^{N_k} \int_{S_i} W(w_k^i)\,\mathrm{d}x = 0\,,$$

with $w_k^i$ defined as in (4.2) with $u_k$ and $v_k$ in place of $u$ and $v$ respectively. Let

$$w_k := \begin{cases} w_k^i & \text{if } x \in S_i\,, i = 1, \dots N_k\,,\\ w & \text{otherwise}\,; \end{cases}$$

then clearly $w_k$ converges to $w$ in $L^p(S)$, moreover since $W$ is continuous

$$0 \le \lim_{k\to\infty} \sum_{i=1}^{N_k} \int_{S_i} W(w_k^i)\,\mathrm{d}x \le \lim_{k\to\infty} \int_S W(w_k)\,\mathrm{d}x = \lim_{k\to\infty} \int_S W(u_k)\,\mathrm{d}x \le \lim_{k\to\infty} \varepsilon_k C = 0\,.$$

□

With the help of Proposition 4.1, we can now prove the following Γ-convergence result.

**Theorem 4.2.** *Let $\mathscr{F}_k$ be as in (2.1). Then there exist a subsequence $(\mathscr{F}_{k_j})$ of $(\mathscr{F}_k)$ and a functional $\mathscr{F}_\infty\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0,+\infty]$ such that for every $A \in \mathcal{A}$ the functionals $\mathscr{F}_{k_j}(\cdot\,, A)$ Γ-converge in $L^1(A)$ to $\mathscr{F}_\infty(\cdot\,, A)$. Moreover, $\mathscr{F}_\infty$ is given by*

$$\mathscr{F}_\infty(u, A) := \begin{cases} \displaystyle\int_{S_u\cap A} \hat{f}(x, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \textit{if } u \in BV(A)\,,\ v \in \{0,1\} \textit{ a.e. in } A\,,\\ +\infty & \textit{otherwise}\,, \end{cases}$$

*with $\hat{f}\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,+\infty)$ defined as*

$$\hat{f}(x,\nu) := \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}} \widehat{\mathbf{m}}(u_x^\nu, Q_\rho^\nu(x))\,, \tag{4.10}$$

*for every $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, where for $A \in \mathcal{A}$ and $\bar{u} \in BV(A, \{0,1\})$*

$$\widehat{\mathbf{m}}(\bar{u}, A) := \inf\{\mathscr{F}_\infty(u, A)\colon u \in BV(A; \{0,1\})\,,\ u = \bar{u} \textit{ near } \partial A\}\,.$$

*Proof.* The proof is rather standard, thus here we only sketch it and refer to [24] for more details. By [24, Theorem 16.9] there exists a subsequence $(k_j)$, with $k_j \to +\infty$ as $j \to +\infty$, such that the inner regular envelopes of the Γ-lower limit and of the Γ-upper limit coincide. More precisely for every $u \in L^1_{\rm loc}(\mathbb{R}^n)$ and every $A \in \mathcal{A}$

$$\sup_{A'\subset\subset A,\, A'\in\mathcal{A}} \mathscr{F}'(u, A') \quad = \sup_{A'\subset\subset A,\, A'\in\mathcal{A}} \mathscr{F}''(u, A') =: \mathscr{F}_\infty(u, A)\,, \tag{4.11}$$

with $\mathscr{F}', \mathscr{F}''\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0,+\infty]$ defined as

$$\mathscr{F}'(\,\cdot\,, A) := \Gamma\text{-}\liminf_{k_j\to+\infty} \mathscr{F}_{k_j}(\,\cdot\,, A) \quad \text{and} \quad \mathscr{F}''(\,\cdot\,, A) := \Gamma\text{-}\limsup_{k_j\to+\infty} \mathscr{F}_{k_j}(\,\cdot\,, A)\,.$$

Now by recalling Remark 2.1 (*b*) and [30, Theorem 1] we can find a constant $C > 0$ such that

$$\frac{1}{C}\mathcal{H}^{n-1}(S_u \cap A) \le \mathscr{F}'(u, A) \le \mathscr{F}''(u, A) \le C\mathcal{H}^{n-1}(S_u \cap A)\,, \tag{4.12}$$

for every $A \in \mathcal{A}$ and every $u \in BV(A;\{0,1\})$ and

$$\mathscr{F}'(u,A) = \mathscr{F}''(u,A) = +\infty \quad \text{if } \ u \notin BV(A;\{0,1\})\,. \tag{4.13}$$

In particular, (4.12) together with [24, Proposition 18.6] and Proposition 4.1 imply that

$$\mathscr{F}_\infty(u,A) = \mathscr{F}'(u,A) = \mathscr{F}''(u,A) \quad \text{if} \quad u \in BV(A;\{0,1\})\,,$$

while (4.11) and (4.13) yield

$$\mathscr{F}_\infty(u,A) = \mathscr{F}'(u,A) = \mathscr{F}''(u,A) = +\infty \quad \text{if } \ u \notin BV(A;\{0,1\})\,.$$

Hence, $\mathscr{F}_\infty(\cdot,A)$ coincides with the $\Gamma$-limit of $\mathscr{F}_{k_j}(\cdot,A)$ on $L^1(\mathbb{R}^n)$, for every $A \in \mathcal{A}$.

We now observe that the functional $\mathscr{F}_\infty$ defined in (4.11) satisfies the following properties: the functional $\mathscr{F}_\infty(\cdot,A)$ is $L^1(\mathbb{R}^n)$ lower semicontinuous [24, Propositions 6.7–6.8 and Remark 15.10] and local [24, Remark 15.25 and Proposition 16.15], while the set function $\mathscr{F}_\infty(u,\cdot)$ is inner regular, increasing and superadditive [24, Propositions 6.7, 16.12 and Remark 15.10]. Finally by means of the fundamental estimate Proposition 4.1 we can appeal to [24, Proposition 18.4] to deduce that $\mathscr{F}_\infty(u,\cdot)$ is also a subadditive set function. This together with the measure-property criterion of De Giorgi and Letta (see *e.g.*, [24, Theorem 14.23]) imply that for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set function $\mathscr{F}_\infty(u,\cdot)$ is the restriction to $\mathcal{A}$ of a Borel measure.
Thus we can invoke [4, Theorem 3.1] (see also [14, Theorem 3]) and conclude that for every $A \in \mathcal{A}$ and $u \in BV(A;\{0,1\})$ the $\Gamma$-limit $\mathscr{F}_\infty$ can be represented in an integral form as

$$\mathscr{F}_\infty(u,A) = \int_{S_u \cap A} \hat{f}(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

with $\hat{f}$ given by (4.10). □

## 5. Identification of the surface integrand

In this section we identify the surface integrand $\hat{f}$. Namely, we show that $\hat{f}$ coincides with both $f'$ and $f''$, defined as in (2.4) and (2.5) with respect to the $\Gamma$-converging subsequence.

**Proposition 5.1.** *Let $(f_k) \subset \mathcal{F}$ ; let $(k_j)$ and $\hat{f}$ be as in Theorem 4.2. Then, it holds*

$$\hat{f}(x,\nu) = f'(x,\nu) = f''(x,\nu)\,,$$

*for every $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, where $f'$ and $f''$ are, respectively, as in* (2.4) *and* (2.5)*, with $k$ replaced by $k_j$.*

*Proof.* For notational simplicity, in what follows we still denote with $k$ the index of the subsequence provided by Theorem 4.2.

By definition we have $f' \leq f''$, hence to conclude we need to show that $\hat{f} \leq f'$ and $\hat{f} \geq f''$. We divide the proof into two steps.

*Step 1:* In this step we show that $\hat{f}(x,\nu) \leq f'(x,\nu)$ for every $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$.
Let $\rho > 0$ and $\eta > 0$ be fixed, and for every $k$ choose $u_k \in \mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, A)$ satisfying

$$\mathscr{F}_k(u_k, Q^\nu_\rho(x)) \leq \mathbf{m}_k(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) + \eta\rho^{n-1} \tag{5.1}$$

$$\leq (c_2 C_{\mathrm{u}} + \eta)\rho^{n-1}\,. \tag{5.2}$$

Here the last inequality follows from (2.10). Up to extracting a subsequence we may assume that

$$\lim_{k\to+\infty} \mathscr{F}_k(u_k, Q^\nu_\rho(x)) = \liminf_{k\to+\infty} \mathscr{F}_k(u_k, Q^\nu_\rho(x))\,.$$

Then extend $u_k$ to a $W^{1,p}_{\rm loc}(\mathbb{R}^n)$-function by setting

$$w_k := \begin{cases} u_k & \text{in } Q^\nu_\rho(x)\,, \\ \bar u^\nu_{x,\varepsilon_k} & \text{in } \mathbb{R}^n \setminus Q^\nu_\rho(x)\,. \end{cases}$$

From (5.2), (2.7) and [30, Lemma 4.1] there exists a subsequence (not relabelled) such that

$$w_k \to u \quad \text{in} \quad L^p_{\rm loc}(\mathbb{R}^n)\,,$$

for some $u \in L^p_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m) \cap BV(Q^\nu_{(1+\eta)\rho}(x);\{0,1\})$ such that $u = u^\nu_x$ in a neighbourhood of $\partial Q^\nu_{(1+\eta)\rho}(x)$. Hence by $\Gamma$-convergence, (2.6), (2.10) and (5.1) we obtain

$$\begin{aligned} \mathscr{F}_\infty(u, Q^\nu_{(1+\eta)\rho}(x)) &\le \liminf_{k\to+\infty} \mathscr{F}_{k_j}(w_k, Q^\nu_{(1+\eta)\rho}(x)) \\ &\le \lim_{k\to+\infty} \mathscr{F}_{k_j}(u_k, Q^\nu_\rho(x)) + c_2 C_{\rm u}\big((1+\eta)^{n-1} - 1\big)\rho^{n-1} \\ &\le \liminf_{k\to+\infty} \mathbf{m}_k(\bar u^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) + \eta\rho^{n-1} + c_2 C_{\rm u}\big((1+\eta)^{n-1} - 1\big)\rho^{n-1}\,. \end{aligned}$$

Combining the above inequality with

$$\widehat{\mathbf{m}}(u^\nu_x, Q^\nu_{(1+\eta)\rho}(x)) \le \mathscr{F}_\infty(u, Q^\nu_{(1+\eta)\rho}(x))\,,$$

dividing by $\rho^{n-1}$ and passing to the limsup as $\rho \to 0$ we infer

$$(1+\eta)^{n-1}\hat f(x,\nu) \le f'(x,\nu) + \eta + c_2 C_{\rm u}\big((1+\eta)^{n-1} - 1\big)\,.$$

Eventually we conclude by sending $\eta \to 0$.

*Step 2:* In this step we show that $\hat f(x,\nu) \ge f''(x,\nu)$ for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $\rho > 0$. We fix $\eta > 0$ and choose $u \in BV(Q^\nu_\rho(x);\{0,1\})$ with $u = u^\nu_x$ near $\partial Q^\nu_\rho(x)$ and

$$\mathscr{F}_\infty(u, Q^\nu_\rho(x)) \le \widehat{\mathbf{m}}(u^\nu_x, Q^\nu_\rho(x)) + \eta\,. \tag{5.3}$$

We extend $u$ to the whole $\mathbb{R}^n$ to the jump function $u^\nu_x$ in $\mathbb{R}^n \setminus Q^\nu_\rho(x)$. By $\Gamma$-convergence there exists a sequence $u_k$ converging to $u$ in $L^1_{\rm loc}(\mathbb{R}^n)$ such that

$$\lim_{k\to+\infty} \mathscr{F}_k(u_k, Q^\nu_\rho(x)) = \mathscr{F}_\infty(u, Q^\nu_\rho(x))\,. \tag{5.4}$$

We next properly modify the sequence $u_k$ in order to obtain a new sequence $\hat u_k \in \mathscr{A}(\bar u^\nu_{x,\varepsilon_k}, A)$. To this end let $0 < \rho'' < \rho' < \rho$ be such that $u = u^\nu_x$ on $Q^\nu_\rho(x) \setminus \overline{Q}^\nu_{\rho''}(x)$. Then we apply Proposition 4.1 with $A = Q^\nu_{\rho''}(x)$, $A' = Q^\nu_{\rho'}(x)$, $B = Q^\nu_\rho(x) \setminus \overline{Q}^\nu_{\rho''}(x)$ and $u = u_k$, $v = \bar u^\nu_{x,\varepsilon_k}$. In this way we get a new sequence $\hat u_k \subset W^{1,p}_{\rm loc}(\mathbb{R}^n)$ converging to $u$ in $L^p(Q^\nu_\rho(x))$ such that

$$\hat u_k = u_k \quad \text{in} \quad Q^\nu_{\rho''}(x), \quad \hat u_k = \bar u^\nu_{x,\varepsilon_k} \quad \text{in} \quad Q^\nu_\rho(x) \setminus \overline{Q}^\nu_{\rho'}(x),$$

and

$$\begin{aligned} \limsup_{k\to+\infty} \mathscr{F}_k(\hat u_k, Q^\nu_\rho(x)) &\le \limsup_{k\to+\infty} \Big( \mathscr{F}_k(u_k, Q^\nu_{\rho'}(x)) + \mathscr{F}_k(\bar u^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x) \setminus \overline{Q}^\nu_{\rho''}(x)) \Big) \\ &\le \mathscr{F}_\infty(u, Q^\nu_\rho(x)) + c_2 C_{\rm u} \mathcal{L}^{n-1}\big(Q'_\rho \setminus \overline{Q'_{\rho''}}\big)\,. \end{aligned} \tag{5.5}$$

Here the second inequality follows from (5.4) and (2.10) together with (2.6).

Recalling that $\hat u_k$ is a test function for the definition of $\mathbf{m}_k(\bar u^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x))$, combining (5.3) and (5.5) we get

$$\limsup_{k\to+\infty} \mathbf{m}_k(\bar u^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) \le \mathbf{m}(u^\nu_x, Q^\nu_\rho(x)) + \eta + C(\rho^{n-1} - (\rho'')^{n-1})\,.$$

We divide the above inequality by $\rho^{n-1}$ and conclude by passing to the limit first as $\rho'' \to \rho$, second as $\eta \to 0$ and eventually to the limsup as $\rho \to 0$. $\square$

## 6. Properties of $f', f''$

This section is devoted to prove some properties satisfied by the functions $f'$, and $f''$ defined in (2.4) and (2.5) respectively. To this purpose is convenient to characterise them in an alternative way. Let us fix first some notation. For $\rho, \delta, \varepsilon_k > 0$ with $\rho > \delta > 2\varepsilon_k$ we set

$$\mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) := \inf\{\mathscr{F}_k(u, Q_\rho^\nu(x)) \colon u \in \mathscr{A}^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))\},$$

where

$$\mathscr{A}^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) := \{u \in W^{1,p}(Q_\rho^\nu(x)),\ 0 \le u \le 1 \colon u = \bar{u}_{x,\varepsilon_k}^\nu \text{ in } Q_\rho^\nu(x) \setminus \overline{Q}_{\rho-\delta}^\nu(x)\}\,.$$

Moreover, let $f'_\rho, f''_\rho : \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0, +\infty]$ be the functions defined as

$$\begin{aligned} f'_\rho(x,\nu) &:= \inf_{\delta>0} \liminf_{k\to+\infty} \mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) = \lim_{\delta\to0} \liminf_{k\to+\infty} \mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))\,, \\ f''_\rho(x,\nu) &:= \inf_{\delta>0} \limsup_{k\to+\infty} \mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) = \lim_{\delta\to0} \limsup_{k\to+\infty} \mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))\,. \end{aligned} \tag{6.1}$$

Next we prove the following technical Lemma.

**Lemma 6.1.** *Let $f'_\rho, f''_\rho$ be as in* (6.1). *Then the following hold:*

- $(i)$ *The restrictions of $f'_\rho$, $f''_\rho$ to the sets $\mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$ and $\mathbb{R}^n \times \widehat{\mathbb{S}}_-^{n-1}$ are upper semicontinuous;*
- $(ii)$ *The functions $\rho \to (f'_\rho(x,\nu) - c_2 C_{\mathrm{u}} \rho^{n-1})$ and $\rho \to (f''_\rho(x,\nu) - c_2 C_{\mathrm{u}} \rho^{n-1})$ are nonincreasing on $(0, +\infty)$;*
- $(iii)$ *For every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ there hold*

$$f'(x,\nu) = \limsup_{\rho\to0} \frac{1}{\rho^{n-1}} f'_\rho(x,\nu) \quad \textit{and} \quad f''(x,\nu) = \limsup_{\rho\to0} \frac{1}{\rho^{n-1}} f''_\rho(x,\nu)\,,$$

*with $f'$ and $f''$ as in* (2.4) *and* (2.5) *respectively.*

*Proof.* We prove the statement only for $f'_\rho$ and we show $(i)$ only for its restriction to the set $\mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$, the other cases can be treated arguing similarly.

*Step 1:* we show the validity of $(i)$. Let $\rho > 0$, $x \in \mathbb{R}^n$, and $\nu \in \widehat{\mathbb{S}}_+^{n-1}$ be fixed. Let $(x_j, \nu_j) \subset \mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$ be a sequence converging to $(x, \nu)$. We want to show that

$$\limsup_{j\to+\infty} f'_\rho(x_j, \nu_j) \le f'_\rho(x,\nu)\,. \tag{6.2}$$

To this aim we notice that (6.1) implies the following: for fixed $\eta > 0$ there exists $\delta_\eta > 0$ such that

$$\liminf_{k\to+\infty} \mathbf{m}_k^\delta(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x)) \le f'_\rho(x,\nu) + \eta, \tag{6.3}$$

for every $\delta \in (0, \delta_\eta)$. Hence we can choose $\delta_0$ such that $3\delta_0 \in (0, \delta_\eta)$ and $\rho - 5\delta_0 > 0$, in particular (6.3) holds true with $\delta = 3\delta_0$. For $2\varepsilon_k \le \rho - 5\delta_0$ let also $u_k \in \mathscr{A}^{3\delta_0}(\bar{u}_{x,\varepsilon_k}^\nu, Q_\rho^\nu(x))$ that satisfies

$$\mathscr{F}_k(u_k, Q_\rho^\nu(x)) \le \mathbf{m}_k^{3\delta_0}(u_x^\nu, Q_\rho^\nu(x)) + \eta\,. \tag{6.4}$$

In what follows we modify $u_k$ in order to obtain a new sequence $\widetilde{u}_k \in \mathscr{A}^\delta(\bar{u}_{x_j,\varepsilon_k}^{\nu_j}, Q_\rho^{\nu_j}(x_j))$ for $j$ large enough and any $\delta \in (0, \delta_0)$ without essentially increasing the energy. We define

$$h_{k,j} := \varepsilon_k + |x - x_j| + \frac{\sqrt{n}}{2}\rho|\nu - \nu_j|$$

and

$$R_{k,j} := R_\nu\Big(\big(Q'_{\rho-2\delta_0} \setminus \overline{Q}'_{\rho-3\delta_0}\big) \times \big(-h_{k,j}, h_{k,j}\big)\Big) + x\,.$$

Since the map $\nu \mapsto R_\nu$ is continuous on $\hat{\mathbb{S}}^{n-1}_+$, there exists $\hat{\jmath} = \hat{\jmath}(\delta_0) \in \mathbb{N}$ such that

$$Q^{\nu_j}_{\rho-5\delta_0}(x_j) \subset Q^{\nu}_{\rho-4\delta_0}(x) \subset Q^{\nu}_{\rho-2\delta_0}(x) \subset Q^{\nu_j}_{\rho-\delta_0}(x_j)\,, \tag{6.5}$$

and

$$R_{k,j} \subset Q^{\nu}_{\rho-2\delta_0}(x) \setminus \overline{Q}^{\nu}_{\rho-3\delta_0}(x),$$

for every $j \geq \hat{\jmath}$. For later convenience we observe that

$$u_k(y) = \bar{u}^{\nu}_{x,\varepsilon_k}(y) = \bar{u}^{\nu_j}_{x_j,\varepsilon_k}(y) \in \{0,1\} \ \text{ for a.e. } y \in \Big(Q^{\nu}_{\rho-2\delta_0}(x) \setminus \overline{Q}^{\nu}_{\rho-3\delta_0}(x)\Big) \setminus \overline{R}_{k,j}\,. \tag{6.6}$$

Indeed we have

$$\bar{u}^{\nu}_{x,\varepsilon_k} = u^{\nu}_x = u^{\nu_j}_{x_j} = \bar{u}^{\nu_j}_{x_j,\varepsilon_k} \ \text{ in } \ \{(y-x)\cdot\nu \geq \varepsilon_k\} \cap \{(y-x_j)\cdot\nu_j \geq \varepsilon_k\}\,. \tag{6.7}$$

On the other hand for $y \in \Big(Q^{\nu}_{\rho-3\delta_0+2\varepsilon_k}(x) \setminus \overline{Q}^{\nu}_{\rho-3\delta_0}(x)\Big) \setminus \overline{R}_{k,j}$ it holds

$$|(y-x)\cdot\nu| = |R^T_\nu(y-x)\cdot e_n| > h_{k,j} > \varepsilon_k\,,$$

and by applying the triangular inequality twice we have

$$\begin{aligned}|(y-x_j)\cdot\nu_j| &> h_{k,j} - |x-x_j| - |y-x||\nu-\nu_j| \\ &\geq \varepsilon_k + \Big(\frac{\sqrt{n}}{2}\rho - |y-x|\Big)|\nu-\nu_j| \geq \varepsilon_k\,,\end{aligned}$$

where the last inequality is a consequence of $Q^\nu_\rho(x) \subset B_{\frac{\sqrt{n}}{2}\rho}(x)$.
Next we apply Proposition 4.1 with

$$A := Q^{\nu}_{\rho-3\delta_0}(x)\,, \quad A' := Q^{\nu}_{\rho-2\delta_0}(x)\,, \quad B := Q^{\nu_j}_{\rho}(x_j) \setminus \overline{Q}^{\nu_j}_{\rho-5\delta_0}(x_j)\,,$$

and $u := u_k$, $v := \bar{u}^{\nu_j}_{x_j,\varepsilon_k}$. Note that thanks to (6.5) the following hold

$$A \cup B = Q^{\nu_j}_\rho(x_j)\,, \quad B \setminus \overline{A}' = Q^{\nu_j}_\rho(x_j) \setminus \overline{Q}^{\nu}_{\rho-2\delta_0}(x)\,,$$

and

$$(A' \setminus \overline{A}) \cap B = A' \setminus \overline{A} = Q^{\nu}_{\rho-2\delta_0}(x) \setminus \overline{Q}^{\nu}_{\rho-3\delta_0}(x)\,. \tag{6.8}$$

In particular Proposition 4.1 $(i)$ provides us with $\widetilde{u}_k \in W^{1,p}(Q^{\nu_j}_\rho(x_j))$, $0 \leq \widetilde{u}_k \leq 1$ with

$$\widetilde{u}_k = u_k \quad \text{ a.e. in } Q^{\nu}_{\rho-3\delta_0}(x)\,, \quad \widetilde{u}_k = \bar{u}^{\nu_j}_{x_j,\varepsilon_k} \quad \text{ a.e. in } Q^{\nu_j}_\rho(x_j) \setminus \overline{Q}^{\nu}_{\rho-2\delta_0}(x)\,, \tag{6.9}$$

and such that

$$\begin{aligned}\mathscr{F}_k(\widetilde{u}_k, Q^{\nu_j}_\rho(x_j)) \leq (1+K\varepsilon_k)\Big(\mathscr{F}_k(u_k, Q^{\nu}_\rho(x)) + \mathscr{F}_k(\bar{u}^{\nu_j}_{x_j,\varepsilon_k}, Q^{\nu_j}_\rho(x_j) \setminus \overline{Q}^{\nu_j}_{\rho-5\delta_0}(x_j))\Big) \\ + \omega_k(u_k, u^{\nu_j}_{x_j,\varepsilon_k}, A, A', B)\,.\end{aligned} \tag{6.10}$$

Moreover from (6.8), property (6.6) becomes

$$u_k(y) = u^{\nu_j}_{x_j,\varepsilon_k}(y) \in \{0,1\} \quad \text{ for a.e. } y \text{ in } ((A' \setminus \overline{A}) \cap B) \setminus R_{k,j}\,.$$

Thus, applying Proposition 4.1 $(ii)$ with $\widetilde{S} = ((A' \setminus \overline{A}) \cap B) \setminus R_{k,j}$ yields

$$\omega_k(u_k, u^{\nu_j}_{x_j,\varepsilon_k}, A, A', B) \leq K\mathcal{L}^n(R_{k,j})\,. \tag{6.11}$$

Now (2.10) together with (2.6) give

$$\mathscr{F}_k(\bar{u}^{\nu_j}_{x_j,\varepsilon_k}, Q^{\nu_j}_\rho(x_j) \setminus \overline{Q}^{\nu_j}_{\rho-5\delta_0}(x_j)) \leq c_2 C_{\mathrm{u}} \mathcal{L}^{n-1}\Big(Q'_\rho \setminus \overline{Q'_{\rho-5\delta_0}}\Big) \leq C\delta_0\rho^{n-2}\,, \tag{6.12}$$

while

$$\mathcal{L}^n(R_{k,j}) \leq C\delta_0\rho^{n-2}h_{k,j}\,. \tag{6.13}$$

By (6.9) we have $\widetilde{u}_k \in \mathscr{A}^\delta(\bar{u}^{\nu_j}_{x_j,\varepsilon_k}, Q^{\nu_j}_{\rho}(x_j))$ for any $j \geq \hat{\jmath}$ and any $\delta \in (0, \delta_0)$. Hence gathering (6.4), (6.10), (6.11),(6.12), and (6.13) we get

$$\mathbf{m}^\delta_k(\bar{u}^{\nu_j}_{x_j,\varepsilon_k}, Q^{\nu_j}_{\rho}(x_j)) \leq (1 + C\varepsilon_k)(\mathbf{m}^{3\delta_0}_k(\bar{u}^{\nu}_{x,\varepsilon_k}, Q^{\nu}_{\rho}(x)) + \eta) + C\rho^{n-2}\delta_0(1 + h_{k,j})\,. \tag{6.14}$$

Using (6.3) with $\delta = 3\delta_0$ and passing to the limit in (6.14) first as $k \to +\infty$, then as $\delta \to 0$, and finally as $j \to +\infty$ gives

$$\limsup_{j\to+\infty} f'_\rho(x_j, \nu_j) \leq f'_\rho(x, \nu) + 2\eta + C\delta_0\rho^{n-2}\,,$$

since $\lim_j \lim_k h_{k,j} = 0$. By letting $\delta_0 \to 0$ (6.2) and then $\eta \to 0$ we finally conclude.

*Step 2:* we show $(ii)$. Let $0 < \rho \leq \rho'$, $x \in \mathbb{R}^n$, $\nu \in S^{n-1}$ be fixed. We show that

$$f'_{\rho'}(x, \nu) - c_2C_{\mathrm{u}}(\rho')^{n-1} \leq f'_\rho(x, \nu) - c_2C_{\mathrm{u}}(\rho)^{n-1}\,. \tag{6.15}$$

Let $\eta > 0$ be fixed and let $\delta_\eta > 0$ be such that (6.3) holds true for every $\delta \in (0, \delta_\eta)$. For fixed $\delta \in (0, \delta_\eta)$ let also $u_k \in \mathscr{A}^\delta(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_{\rho'}(x))$ that satisfies

$$\mathscr{F}_k(u_k, Q^\nu_\rho(x)) \leq \mathbf{m}^\delta_k(u^\nu_x, Q^\nu_\rho(x)) + \eta\,. \tag{6.16}$$

we now extend $u_k$, without relabelling it, to $\bar{u}^\nu_{x,\varepsilon_k}$ in $Q^\nu_{\rho'}(x) \setminus \overline{Q}^\nu_\rho(x)$ so that it belongs to the class $\mathscr{A}^\delta(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_{\rho'}(x))$. Then (2.6) and (2.10) yield

$$\begin{aligned}\mathscr{F}_k(u_k, Q^\nu_{\rho'}(x)) &\leq \mathscr{F}_k(u_k, Q^\nu_\rho(x)) + \mathscr{F}_k\big(\bar{u}^\nu_{x,\varepsilon_k}, (Q^\nu_{\rho'}(x) \setminus \overline{Q}^\nu_\rho(x)\big)\\ &\leq \mathscr{F}_k(u_k, Q^\nu_\rho(x)) + c_2C_{\mathrm{u}}((\rho')^{n-1} - (\rho)^{n-1})\,.\end{aligned} \tag{6.17}$$

Finally letting first $k \to \infty$ and then $\delta \to 0$ from (6.3) and (6.16) we get

$$f'_{\rho'}(x, \nu) \leq f'_\rho(x, \nu) + c_2C_{\mathrm{u}}((\rho')^{n-1} - (\rho)^{n-1}) + 2\eta\,.$$

Eventually by sending $\eta \to 0$ we deduce (6.15).

*Step 3:* we show $(iii)$. One inequality follows immediately. Indeed as $\mathscr{A}^\delta(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) \subset \mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x))$ for every $\delta \in (0, \rho)$, we have

$$f'(x, \nu) \leq \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}} f'_\rho(x, \nu)\,.$$

It remains to prove the opposite inequality. For fixed $\rho > 0$, $x \in \mathbb{R}^n$, and $\nu \in \mathbb{S}^{n-1}$ and for every $k \in \mathbb{N}$ such that $\varepsilon_k \in (0, \frac{\rho}{2})$ let $u_k \in \mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x))$ satisfy

$$\mathscr{F}_k(u_k, Q^\nu_\rho(x)) \leq \mathbf{m}_k(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) + \rho^n\,, \tag{6.18}$$

Fix $\alpha > 0$, let $\rho_\alpha := (1+\alpha)\rho$, and extend $u_k$, without relabelling it, to $\bar{u}^\nu_{x,\varepsilon_k}$ in $Q^\nu_{\rho_\alpha}(x) \setminus \overline{Q}^\nu_\rho(x)$. In this way $u_k \in \mathscr{A}^\delta(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_{\rho_\alpha}(x))$ for any $\delta \in (0, \alpha\rho)$ and similarly to (6.17) it holds

$$\begin{aligned}\mathscr{F}_k(u_k, Q^\nu_{\rho_\alpha}(x)) &\leq \mathscr{F}_k(u_k, Q^\nu_\rho(x)) + \mathscr{F}_k\big(\bar{u}^\nu_{x,\varepsilon_k}, (Q^\nu_{\rho_\alpha}(x) \setminus \overline{Q}^\nu_\rho(x)\big)\\ &\leq \mathscr{F}_k(u_k, Q^\nu_\rho(x)) + c_2C_{\mathrm{u}}((1+\alpha)^{n-1} - 1))\rho^{n-1}\,.\end{aligned} \tag{6.19}$$

Now (6.18) and (6.19) give

$$\inf_{\delta>0} \liminf_{k\to+\infty} \mathbf{m}^\delta_k(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_{(1+\alpha)\rho}(x)) \leq \liminf_{k\to+\infty} \mathbf{m}_k(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) + \rho^n + c_2C_{\mathrm{u}}((1+\alpha)^{n-1} - 1))\rho^{n-1}\,.$$

Rescaling the above inequality by $((1+\alpha)\rho)^{n-1}$ and passing to the limsup as $\rho \to 0$ we infer

$$(1+\alpha)^{n-1} \limsup_{\rho\to 0} \frac{1}{\rho^{n-1}} f'_\rho(x, \nu) \leq f'(x, \nu) + c_2C_{\mathrm{u}}((1+\alpha)^{n-1} - 1))\,.$$

We finally conclude by the arbitrariness of $\alpha > 0$. □

We now state the main result of this section:

**Proposition 6.2.** *Let $(f_k) \subset \mathcal{F}$; then the functions $f'$ and $f''$ defined, respectively, as in (2.4) and (2.5) are Borel measurable and satisfy the following: for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$ it holds*

$$c_1 c_p \le f'(x,\nu) \le c_2 c_p, \quad c_1 c_p \le f''(x,\nu) \le c_2 c_p, \tag{6.20}$$

*with*

$$c_p := p(p-1)^{\frac{1-p}{p}} \int_0^1 W(t)^{\frac{p-1}{p}}\, \mathrm{d}t\,.$$

*Proof.* We prove the statement only for $f'$, as the proof for $f''$ can be achieved in a similar way.

*Step 1:* we show that $f'$ is Borel measurable. Let $\rho > 0$ and let $f'_\rho$ be the function defined in (6.1). By Lemma 6.1 $(ii)$ we have that the function $\rho \to f'_\rho(x,\nu) - c_2 C_{\mathrm{u}} \rho^{n-1}$ is nonincreasing on $(0,+\infty)$, hence in particular

$$\lim_{\rho' \to \rho^-} f'_{\rho'}(x,\nu) \ge f'_\rho(x,\nu) \ge \lim_{\rho' \to \rho^+} f'_{\rho'}(x,\nu),$$

for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and every $\rho > 0$. This together with Lemma 6.1 $(iii)$ imply that

$$f'(x,\nu) = \limsup_{\rho \to 0} \frac{1}{\rho^{n-1}} f'_\rho(x,\nu) = \limsup_{\substack{\rho \to 0 \\ \rho \in D}} \frac{1}{\rho^{n-1}} f'_\rho(x,\nu)$$

where $D$ is any countable dense subset of $(0,+\infty)$. Since by Lemma 6.1 $(i)$ the function $(x,\nu) \mapsto f'_\rho(x,\nu)$ is Borel measurable for every $\rho > 0$, and the limit on a countable set of Borel functions is Borel we conclude.

*Step 2:* we show that $f'$ is bounded. For every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $\rho > 0$ we define the minimisation problem

$$\mathbf{m}_{k,\rho}(x,\nu) := \min\Big\{\mathcal{M}_k(u, Q^\nu_\rho(x))\colon u \in \mathscr{A}(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x))\Big\},$$

where $\mathcal{M}_k$ is defined as in (2.8). From Remark 2.1 $(b)$ we have

$$c_1 \mathbf{m}_{k,\rho}(x,\nu) \le \mathbf{m}_k(\bar{u}^\nu_{x,\varepsilon_k}, Q^\nu_\rho(x)) \le c_2 \mathbf{m}_{k,\rho}(x,\nu).$$

Therefore to prove that $f'$ satisfies the bounds in (6.20) it is enough to show that

$$\lim_{k \to +\infty} \mathbf{m}_{k,\rho}(x,\nu) = c_p \rho^{n-1}. \tag{6.21}$$

By the homogeneity and rotation invariance of the Modica-Mortola functional there holds

$$\mathbf{m}_{k,\rho}(x,\nu) = \mathbf{m}_{k,\rho}(0,e_n).$$

Let $\eta > 0$ be arbitrary fixed. By the $\Gamma$-convergence result in [30] we can find a sequence $(u_k) \subset W^{1,p}(Q_\rho(0))$, $0 \le u_k \le 1$ such that $u_k$ converges to $u_0^{e_n}$ in $L^p(Q_\rho(0))$ and

$$\limsup_{k \to +\infty} \mathcal{M}_k(u_k, Q_\rho(0)) = (c_p + \eta)\rho^{n-1}\,.$$

Up to suitably modifying $u_k$ (for example by using Proposition 4.1), we can assume that $u_k = u_0^{e_n}$ in $\{|y_n| > \varepsilon_k\}$ so that in particular $(u_k) \subset \mathscr{A}(\bar{u}^{e_n}_{0,\varepsilon_k}, Q_\rho(0))$. In particular, by the arbitrariness of $\eta > 0$ we deduce that

$$\limsup_{k \to +\infty} \mathbf{m}_{k,\rho}(x,\nu) = \limsup_{k \to +\infty} \mathbf{m}_{k,\rho}(0,e_n) \le c_p \rho^{n-1}\,. \tag{6.22}$$

On the other hand by Fubini's Theorem we have

$$\begin{aligned}\mathcal{M}_k(u,Q_\rho(0)) &= \int_{Q'_\rho}\int_{-\frac{\rho}{2}}^{\frac{\rho}{2}}\Big(\frac{W(u(x',x_n))}{\varepsilon_k}+\varepsilon_k^{p-1}|\nabla u(x',x_n)|^p\Big)\,\mathrm{d}x_n\,\mathrm{d}x' \\ &\geq \int_{Q'_\rho}\int_{-\frac{\rho}{2}}^{\frac{\rho}{2}}\Big(\frac{W(u(x',x_n))}{\varepsilon_k}+\varepsilon_k^{p-1}\Big|\frac{\partial u(x',x_n)}{\partial x_n}\Big|^p\Big)\,\mathrm{d}x_n\,\mathrm{d}x'\,,\end{aligned} \tag{6.23}$$

and by Young's inequality

$$\begin{aligned}&\int_{-\frac{\rho}{2}}^{\frac{\rho}{2}}\Bigg(\frac{W(u(x',x_n))}{\varepsilon_k}+\varepsilon_k^{p-1}\Big|\frac{\partial u(x',x_n)}{\partial x_n}\Big|^p\Bigg)\,\mathrm{d}x_n \\ &\qquad\geq\Big(\frac{p}{p-1}\Big)^{\frac{p-1}{p}}p^{\frac{1}{p}}\int_{-\frac{\rho}{2}}^{\frac{\rho}{2}}\Bigg((W(u(x',x_n)))^{\frac{p-1}{p}}\Big|\frac{\partial u(x',x_n)}{\partial x_n}\Big|\Bigg)\,\mathrm{d}x_n\,,\end{aligned} \tag{6.24}$$

for $\mathcal{L}^{n-1}$-a.e. $x'\in Q'_\rho$. Now if $u\in\mathscr{A}((\bar{u}^{e_n}_{0,\varepsilon_k},Q_\rho(0))$, then it coincides with $u_0^{e_n}$ in a neighbourhood of

$$\partial^{\pm}Q_\rho(0):=\Big\{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\colon x'\in\overline{Q'_\rho},\ x_n=\pm\frac{\rho}{2}\Big\}\,.$$

Therefore, for $\mathcal{L}^{n-1}$-a.e. $x'\in Q'_\rho$, the function $u_{x'}(t):=u(x',t)$ belongs to $W^{1,p}(-\frac{\rho}{2},\frac{\rho}{2})$ and satisfies $u_{x'}(-\frac{\rho}{2})=0$, and $u_{x'}(\frac{\rho}{2})=1$, and a change of variable in (6.24) yields

$$\int_{-\frac{\rho}{2}}^{\frac{\rho}{2}}\Big(\frac{W(u(x',x_n))}{\varepsilon_k}+\varepsilon_k^{p-1}\Big|\frac{\partial u(x',x_n)}{\partial x_n}\Big|^p\Big)\,\mathrm{d}x_n\geq\Big(\frac{p}{p-1}\Big)^{\frac{p-1}{p}}p^{\frac{1}{p}}\int_0^1 W(t)^{\frac{p-1}{p}}\,\mathrm{d}t=c_p\,. \tag{6.25}$$

Thus, combining (6.23) with (6.25) we obtain

$$\mathcal{M}_k(u,Q_\rho(0))\geq c_p\rho^{n-1},$$

for every $u\in\mathscr{A}((\bar{u}^{e_n}_{0,\varepsilon_k},Q_\rho(0))$. Passing to the infimum on $u$ and to the liminf as $k\to+\infty$ we get

$$\liminf_{k\to+\infty}\mathbf{m}_{k,\rho}(x,\nu)=\liminf_{k\to+\infty}\mathbf{m}_{k,\rho}(0,e_n)\geq c_p\rho^{n-1}, \tag{6.26}$$

for every $\rho>0$. Eventually, gathering (6.22) and (6.26) we get (6.21), and hence the thesis. □

We now have all the ingredients to prove the main result of this paper, namely, Theorem 3.1.

*Proof of Theorem 3.1.* The proof follows by combining Theorem 4.2, Proposition 5.1 and Proposition 6.2. □

## 7. Stochastic homogenisation

In this section we derive a Γ-convergence result for functionals of type $\mathscr{F}_k$ when $f_k$ are random integrands of the form

$$f_k(\omega,x,u,\xi)=f\Big(\omega,\frac{x}{\varepsilon_k},u,\xi\Big),$$

where $\omega$ belongs to the sample space $\Omega$ of a complete probability space $(\Omega,\mathcal{T},P)$.

In order to do that we need to recall some useful definitions.

**Definition 7.1** (Group of $P$-preserving transformations)**.** Let $d\in\mathbb{N}$, $d\geq 1$. A group of $P$-preserving transformations on $(\Omega,\mathcal{T},P)$ is a family $(\tau_z)_{z\in\mathbb{Z}^d}$ of mappings $\tau_z\colon\Omega\to\Omega$ satisfying the following properties:

(1) (measurability) $\tau_z$ is $\mathcal{T}$-measurable for every $z\in\mathbb{Z}^d$;
(2) (invariance) $P(\tau_z(E))=P(E)$, for every $E\in\mathcal{T}$ and every $z\in\mathbb{Z}^d$;
(3) (group property) $\tau_0=\mathrm{id}_\Omega$ and $\tau_{z+z'}=\tau_z\circ\tau_{z'}$ for every $z,z'\in\mathbb{Z}^d$.

If, in addition, every $(\tau_z)_{z\in\mathbb{Z}^d}$-invariant set (*i.e.*, every $E \in \mathcal{T}$ with $\tau_z(E) = E$ for every $z \in \mathbb{Z}^d$) has probability 0 or 1, then $(\tau_z)_{z\in\mathbb{Z}^d}$ is called ergodic.

Let $a := (a_1, \dots, a_d)$, $b := (b_1, \dots, b_d) \in \mathbb{Z}^d$ with $a_i < b_i$ for all $i \in \{1, \dots, d\}$; we define the $d$-dimensional interval

$$[a, b) := \{x \in \mathbb{Z}^d \colon a_i \le x_i < b_i \text{ for } i = 1, \dots, d\}$$

and we set

$$\mathcal{I}_d := \{[a, b) \colon a, b \in \mathbb{Z}^d \,,\ a_i < b_i \text{ for } i = 1, \dots, d\} \,.$$

**Definition 7.2** (Subadditive process)**.** A discrete subadditive process with respect to a group $(\tau_z)_{z\in\mathbb{Z}^d}$ of $P$-preserving transformations on $(\Omega, \mathcal{T}, P)$ is a function $\mu \colon \Omega \times \mathcal{I}_d \to \mathbb{R}$ satisfying the following properties:

(1) (measurability) for every $A \in \mathcal{I}_d$ the function $\omega \mapsto \mu(\omega, A)$ is $\mathcal{T}$-measurable;
(2) (covariance) for every $\omega \in \Omega$, $A \in \mathcal{I}_d$, and $z \in \mathbb{Z}^d$ we have $\mu(\omega, A + z) = \mu(\tau_z(\omega), A)$;
(3) (subadditivity) for every $A \in \mathcal{I}_d$ and for every finite family $(A_i)_{i\in I} \subset \mathcal{I}_d$ of pairwise disjoint sets such that $A = \cup_{i\in I} A_i$, we have

$$\mu(\omega, A) \le \sum_{i\in I} \mu(\omega, A_i) \quad \text{for every } \omega \in \Omega \,;$$

(4) (boundedness) there exists $c > 0$ such that $0 \le \mu(\omega, A) \le c\mathcal{L}^d(A)$ for every $\omega \in \Omega$ and $A \in \mathcal{I}_d$.

**Definition 7.3** (Stationarity)**.** Let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of $P$-preserving transformations on $(\Omega, \mathcal{T}, P)$. We say that $f \colon \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ is stationary with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$ if

$$f(\omega, x + z, u, \xi) = f(\tau_z(\omega), x, u, \xi)$$

for every $\omega \in \Omega$, $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$, $u \in [0, 1]$ and $\xi \in \mathbb{R}^n$.

We consider random integrands $f \colon \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ satisfying the following assumptions:

$(F1)$ $f$ is $(\mathcal{T} \otimes \mathcal{B}^n \otimes \mathcal{B} \otimes \mathcal{B}^n)$-measurable;
$(F2)$ $f(\omega, \cdot\,, \cdot\,, \cdot\,) \in \mathcal{F}$ for every $\omega \in \Omega$;
$(F3)$ For every $\omega \in \Omega$, and every $x \in \mathbb{R}^n$ the map $\xi \mapsto f(\omega, x, u, \xi)$ is lower semicontinuous.

For any $f$ as above we define the sequence of random phase-field functionals $\mathscr{F}_k(\omega) \colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ given by

$$\mathscr{F}_k(\omega)(u, A) := \frac{1}{\varepsilon_k} \int_A f\left(\omega, \frac{x}{\varepsilon_k}, u, \varepsilon_k \nabla u\right) \mathrm{d}x \,, \tag{7.1}$$

if $u \in W^{1,p}(A)$, $0 \le u \le 1$ and extended to $+\infty$ otherwise. For $\omega \in \Omega$, $A \in \mathcal{A}$ we define

$$\mathscr{F}(\omega)(u, A) := \int_A f(\omega, x, u, \nabla u) \,\mathrm{d}x \,,$$

and

$$\mathbf{m}_\omega(\bar{u}^\nu_z, A) := \inf\Big\{\mathscr{F}(\omega)(u, A) \colon u \in \mathscr{A}(\bar{u}^\nu_z, A)\Big\} \,, \tag{7.2}$$

where $\mathscr{A}(\bar{u}^\nu_z, A)$ is as in (2.3), with $\bar{u}^\nu_{x,\varepsilon_k}$ replaced by $\bar{u}^\nu_z$; *i.e.*, with $\varepsilon_k = 1$. Moreover for every $A \subset \mathbb{R}^n$ with $\mathrm{int\,A} \in \mathcal{A}$ we set

$$\mathbf{m}_\omega(\bar{u}^\nu_x, A) := \mathbf{m}_\omega(\bar{u}^\nu_x, \mathrm{int\,A}), \quad \text{and} \quad \mathscr{A}(\bar{\mathrm{u}}^\nu_0, \mathrm{A}) := \mathscr{A}(\bar{\mathrm{u}}^\nu_0, \mathrm{int\,A}).$$

Finally we are able to state the main result of this section.

**Theorem 7.4** (Stochastic homogenisation)**.** *Let $f$ be a random integrand satisfying $(F1)$-$(F3)$. Assume also that $f$ is stationary with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of $P$-preserving transformations on $(\Omega,\mathcal{T},P)$. For every $\omega\in\Omega$ let $\mathscr{F}_k(\omega)$ be as in (7.1) and $\mathbf{m}_\omega$ be as in (7.2). Then there exists $\Omega'\in\mathcal{T}$, with $P(\Omega')=1$ such that for every $\omega\in\Omega'$, $x\in\mathbb{R}^n$, $\nu\in\mathbb{S}^{n-1}$ the limit*

$$\lim_{r\to+\infty}\frac{\mathbf{m}_\omega(\bar u^\nu_{rx},Q^\nu_r(rx))}{r^{n-1}}=\lim_{r\to+\infty}\frac{\mathbf{m}_\omega(\bar u^\nu_0,Q^\nu_r(0))}{r^{n-1}}=:f_{\rm hom}(\omega,\nu) \tag{7.3}$$

*exists and is independent of $x$. The function $f_{\rm hom}\colon\Omega\times\mathbb{S}^{n-1}\to[0,+\infty)$ is $(\mathcal{T}\otimes\mathcal{B}(\mathbb{S}^{n-1}))$-measurable.*

*Moreover, for every $\omega\in\Omega'$ and for every $A\in\mathcal{A}$ the functionals $\mathscr{F}_k(\omega)(\cdot,A)$ $\Gamma$-converge in $L^1_{\rm loc}(\mathbb{R}^n)$ to the functional $\mathscr{F}_{\rm hom}(\omega)(\cdot,A)$ with $\mathscr{F}_{\rm hom}(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n)\times\mathcal{A}\longrightarrow[0,+\infty]$ given by*

$$\mathscr{F}_{\rm hom}(\omega)(u,A):=\begin{cases}\displaystyle\int_{S_u\cap A}f_{\rm hom}(\omega,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } u\in BV(A;\{0,1\})\\ +\infty & \text{otherwise}\,.\end{cases}$$

*If, in addition, $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic, then $f_{\rm hom}$ is independent of $\omega$ and*

$$f_{\rm hom}(\nu)=\lim_{r\to+\infty}\frac{1}{r^{n-1}}\int_\Omega\mathbf{m}_\omega(\bar u^\nu_0,Q^\nu_r(0))\,\mathrm{d}P(\omega)\,,$$

*thus, $\mathscr{F}_{\rm hom}$ is deterministic.*

*Remark* 7.5. The notion of stationarity given in Definition 7.3 generalises the notion of spatial periodicity. Therefore Theorem 7.4 applies also to the case of deterministic periodic homogenisation. However we will show later (see Section 8) that condition $(F3)$ is not needed if we restrict to the periodic setting.

The rest of the section is dedicated to prove Theorem 7.4.

7.1. **Existence of the limit.** The almost sure $\Gamma$-convergence result in Theorem 7.4 readily follows by Theorem 3.3 if there exists a $\mathcal{T}$-measurable set $\Omega'\subset\Omega$, with $P(\Omega')=1$, such that for every $\omega\in\Omega'$ the limit in (7.3) exists and is independent of $x$, and if $f_{\rm hom}$ is $(\mathcal{T}\otimes\mathcal{B}(\mathbb{S}^{n-1}))$-measurable. We will prove that this is actually the case when the integrand $f$ satisfies $(F1)$-$(F3)$ and is stationary in the sense of Definition 7.3.

Since the boundary conditions given in the definition of $\mathbf{m}_\omega(\bar u^\nu_{rx},Q^\nu_r(rx))$ depend on $x$, the proof of the existence and $x$-homogeneity of the limit in (7.3) does not follow by a direct application of the Subadditive Ergodic Theorem [1, Theorem 2.4]. For this reason we need to follow a more technical argument, in the same way as in [3, 18, 15], which can be divided into three main steps. In the first step we prove that when $x=0$ the minimisation problem (7.2) defines a subadditive process on $\Omega\times\mathcal{I}_{n-1}$ (see Proposition 7.6). In the second step we prove the almost sure existence of the limit in (7.3) when $x=0$ (see Proposition 7.7). Eventually in the third one we show that the same holds for an arbitrary $x\in\mathbb{R}^n$ (see Proposition 7.8).

For $\nu\in\mathbb{S}^{n-1}$ we let $R_\nu$ be an orthogonal matrix as in (g). Note that $\{R_\nu e_i\colon i=1,\dots,n-1\}$ is an orthonormal basis for $\Pi^\nu$, and $R_\nu\in\mathbb{Q}^{n\times n}$, if $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$. Let $M_\nu>2$ be an integer such that $M_\nu R_\nu\in\mathbb{Z}^{n\times n}$; therefore $M_\nu R_\nu(z',0)\in\Pi^\nu\cap\mathbb{Z}^n$ for every $z'\in\mathbb{Z}^{n-1}$.

Let $I\in\mathcal{I}_{n-1}$*; i.e.*, $I=[a,b)$ with $a,b\in\mathbb{Z}^{n-1}$. For every $\nu\in\mathbb{S}^{n-1}$ and every $I\in\mathcal{I}_{n-1}$ we define the $n$-dimensional interval $I_\nu$ as

$$I_\nu:=M_\nu R_\nu\big(I\times[-c,c)\big)\quad\text{where}\quad c:=\frac12\max_{i=1,\dots,n-1}(b_i-a_i)\,. \tag{7.4}$$

Eventually, for fixed $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ we consider the function $\mu_\nu \colon \Omega \times \mathcal{I}_{n-1} \to \mathbb{R}$ given by

$$\mu_\nu(\omega, I) := \frac{1}{M_\nu^{n-1}} \mathbf{m}_\omega(\bar{u}_0^\nu, I_\nu)\,, \tag{7.5}$$

where $\mathbf{m}_\omega(\bar{u}_0^\nu, I_\nu)$ is as in (7.2) with $z = 0$ and $A = I_\nu$. Then the following result holds true.

**Proposition 7.6.** *Let $f$ satisfy (F1)-(F3) and assume that it is stationary with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of $P$-preserving transformations on $(\Omega, \mathcal{T}, P)$. Let $\nu \in \mathbb{S}^{n-1}\cap\mathbb{Q}^n$ and let $\mu_\nu \colon \Omega\times\mathcal{I}_{n-1} \mapsto \mathbb{R}$ be as in (7.5). Then there exists a group of $P$-preserving transformations $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$ on $(\Omega, \mathcal{T}, P)$ such that $\mu_\nu$ is a subadditive process on $(\Omega, \mathcal{T}, P)$ with respect to $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$. Moreover, it holds*

$$0 \le \mu_\nu(\omega, I) \le c_2 C_{\rm u} \mathcal{L}^{n-1}(I), \tag{7.6}$$

*for $P$-a.e. $\omega \in \Omega$ and for every $I \in \mathcal{I}_{n-1}$.*

*Proof.* Let $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ be fixed; below we show that there exists a group of $P$-preserving transformations $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$ for which $\mu_\nu$ satisfies conditions (1)–(4) of Definition 7.2.

*Step 1: measurability.* The measurabilty can be achieved by suitably adapting the proof of [33, Lemma C.1.], for this reason here we only sketch it. Let $I \in \mathcal{I}_{n-1}$ and let $I_\nu \subset \mathbb{R}^n$ be as in (7.4). For every $n \in \mathbb{N}$ we let $f^n$ be the Moreau-Yosida-regularisation of $f$ with respect to the last two variables, that is

$$f^n(\omega, x, u, \xi) := \inf_{(v,\zeta)\in\mathbb{R}\times\mathbb{R}^n} \{f(\omega, x, v, \zeta) + n|(v, \zeta) - (u, \xi)|\}\,.$$

It is known that $f^n$ is $n$-Lipschitz in $(u, \xi)$. Then arguing as in the proof of [33, Lemma C.1.] it can be shown that the map $\mathscr{F}^n(\omega) \colon W^{1,p}(I_\nu) \to [0, +\infty)$ given by

$$\mathscr{F}^n(\omega)(u) := \int_{I_\nu} f^n(\omega, x, u, \nabla u)\,\mathrm{d}x\,,$$

is well defined for every $\omega \in \Omega$ and that $(\omega, u) \mapsto \mathscr{F}^n(\omega)(u)$ is $\mathcal{T} \otimes \mathcal{B}(W^{1,p}(I_\nu))$-measurable. By $(F3)$ $f^n \nearrow f$ pointwise so that in particular $\mathscr{F}^n(\omega)(u)$ converges to $\mathscr{F}(\omega)(u, I_\nu)$ pointwise. This in turn implies that $(\omega, u) \mapsto \mathscr{F}(\omega)(u, I_\nu)$ is $\mathcal{T} \otimes \mathcal{B}(W^{1,p}(I_\nu))$-measurable as well. Now recall that $(\Omega, \mathcal{T}, P)$ is a complete probability space, while the set

$$\mathscr{A}(\bar{u}_0^\nu, I_\nu) = \{u \in W^{1,p}(I_\nu)\,,\ 0 \le u \le 1\,,\ u = \bar{u}_0^\nu \ \text{ near } \ \partial I_\nu\} \subset W^{1,p}(I_\nu)$$

defines a separable metric space, when endowed with the distance induced by the $W^{1,p}(I_\nu)$-norm. Moreover from $(F3)$ the map $u \mapsto \mathscr{F}(\omega)(u, I_\nu)$ is lower semicontinuous and not constantly $+\infty$ since from (2.10) we have $\mathscr{F}(\omega)(\bar{u}_0^\nu, I_\nu) < +\infty$. Hence we can apply [33, Lemma C.2.] to deduce the $\mathcal{T}$-measurability of $\omega \mapsto \mathbf{m}_\omega(\bar{u}_0^\nu, I_\nu)$ and in particular of $\omega \mapsto \mu_\nu(\omega, I)$.

*Step 2: covariance.* Let $z' \in \mathbb{Z}^{n-1}$ be fixed and let $I \in \mathcal{I}_{n-1}$. Note that (7.4) implies the following equality

$$(I + z')_\nu = I_\nu + M_\nu R_\nu(z', 0) = I_\nu + z'_\nu\,,$$

where $z'_\nu := M_\nu R_\nu(z', 0) \in \mathbb{Z}^n \cap \Pi^\nu$. Therefore we have

$$\mu_\nu(\omega, I + z') = \frac{1}{M_\nu^{n-1}} \mathbf{m}_\omega(\bar{u}_0^\nu, I_\nu + z'_\nu)\,. \tag{7.7}$$

Let $u \in \mathscr{A}(\bar{u}_0^\nu, I_\nu + z'_\nu)$ and define $\widetilde{u}(x) := u(x + z'_\nu)$. Clearly $\widetilde{u} \in \mathscr{A}(\bar{u}_0^\nu, I_\nu)$. Indeed since $z'_\nu \in \Pi^\nu$, we have $\widetilde{u} = \bar{u}_0^\nu(\cdot + z'_\nu) = \bar{u}_0^\nu$ near $\partial I_\nu$. Further by a change of variables and using the stationarity of $f$ we obtain the following

$$\begin{aligned}\mathscr{F}(\omega)(u,\operatorname{int}(I_\nu+z'_\nu)) &= \int_{I_\nu+z'_\nu} f(\omega,x,u,\nabla u)\,\mathrm{d}x = \int_{I_\nu} f(\omega,x+z'_\nu,\widetilde{u},\nabla\widetilde{u})\,\mathrm{d}x\\ &= \int_{I_\nu} f(\tau_{z'_\nu}(\omega),x,\widetilde{u},\nabla\widetilde{u})\,\mathrm{d}x = \mathscr{F}(\tau_{z'_\nu}(\omega))(\widetilde{u},\operatorname{int} I_\nu)\,.\end{aligned}$$

Let $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}} := (\tau_{z'_\nu})_{z'\in\mathbb{Z}^{n-1}}$; then $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$ is well defined since $z'_\nu \in \mathbb{Z}^n$ and it defines a group of $P$-preserving transformations on $(\Omega,\mathcal{T},P)$. In particular the equality above becomes

$$\mathscr{F}(\omega)(u,\operatorname{int}(I_\nu+z'_\nu)) = \mathscr{F}(\tau^\nu_{z'}(\omega))(\widetilde{u},\operatorname{int} I_\nu)\,. \tag{7.8}$$

Eventually, gathering (7.7) and (7.8), by the arbitrariness of $u$ we infer

$$\mu_\nu(\omega,I+z') = \mu_\nu(\tau^\nu_{z'}(\omega),I)\,,$$

and the covariance of $\mu_\nu$ with respect to $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$ is shown.

*Step 3: subadditivity.* Let $\omega\in\Omega$, $I\in\mathcal{I}_{n-1}$, and let $\{I_1,\dots,I_N\}\subset\mathcal{I}_{n-1}$ be a finite family of pairwise disjoint sets such that $I=\bigcup_i I_i$. For $\eta>0$ fixed and for every $i=1,\dots,N$ we let $u_i\in W^{1,p}((I_i)_\nu)$ be admissible for $\mathbf{m}_\omega(\bar{u}^\nu_0,(I_i)_\nu)$ such that

$$\mathscr{F}(\omega)(u_i,\operatorname{int}(I_i)_\nu) \le \mathbf{m}_\omega(\bar{u}^\nu_0,(I_i)_\nu)+\eta\,. \tag{7.9}$$

We let

$$u := \begin{cases} u_i & \text{in } (I_i)_\nu\,,\ i=1\,,\dots,N\,,\\ \bar{u}^\nu_0 & \text{in } I_\nu\setminus\bigcup_i (I_i)_\nu\,,\end{cases}$$

in this way $u\in W^{1,p}(I_\nu;\mathbb{R}^m)$ and $u=\bar{u}^\nu_0$ near $\partial I_\nu$. Hence in particular $u\in\mathscr{A}(\bar{u}^\nu_0,I_\nu)$ and

$$\mathscr{F}(\omega)(u,\operatorname{int} I_\nu) = \sum_{i=1}^N \mathscr{F}(\omega)(u_i,\operatorname{int}(I_i)_\nu) + \mathscr{F}(\omega)\big(\bar{u}^\nu_0,\operatorname{int}(I_\nu\setminus\bigcup_{i=1}^N (I_i)_\nu)\big)\,. \tag{7.10}$$

We observe that

$$\mathscr{F}(\omega)\big(\bar{u}^\nu_0,\operatorname{int}(I_\nu\setminus\bigcup_{i=1}^N (I_i)_\nu)\big) = 0\,. \tag{7.11}$$

Indeed, being $M_\nu>2$ and $c\ge\frac{1}{2}$ in (7.4) we have that $\{y\in I_\nu\colon |y\cdot\nu|\le 1\}\subset\bigcup_i (I_i)_\nu$ and thus $\bar{u}^\nu_0\equiv u^\nu_0$ in $I_\nu\setminus\bigcup_i (I_i)_\nu$. Now recalling (2.6) we derive (7.11). Putting together (7.9)–(7.11) we obtain

$$\mathbf{m}_\omega(\bar{u}^\nu_0,I_\nu) \le \mathscr{F}(\omega)(u,\operatorname{int} I_\nu) = \sum_{i=1}^N \mathscr{F}(\omega)(u_i,\operatorname{int}(I_i)_\nu) \le \sum_{i=1}^N \mathbf{m}_\omega(\bar{u}^\nu_0,(I_i)_\nu) + N\eta\,,$$

and the subadditivity of $\mu_\nu$ follows by letting $\eta\to 0$-

*Step 4: boundedness.* Let $\omega\in\Omega$ and $I\in\mathcal{I}_{n-1}$. From (7.4) and (2.10) we easily deduce

$$0\le\mu_\nu(\omega,I) = \frac{1}{M_\nu^{n-1}}\mathbf{m}_\omega(\bar{u}^\nu_0,I_\nu) \le c_2 C_{\mathrm{u}}\mathcal{L}^{n-1}(I).$$

□

Proposition 7.6 is the key ingredient to prove that almost surely the limit defining $f_{\mathrm{hom}}$ exists when $x=0$.

**Proposition 7.7** (Homogenised surface integrand for $x = 0$)**.** *Let $f$ satisfy (F1)-(F3) and assume that it is stationary with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of $P$-preserving transformations on $(\Omega, \mathcal{T}, P)$. For $\omega \in \Omega$ let $\mathbf{m}_\omega$ be as in (7.2). Then there exist $\widetilde{\Omega} \in \mathcal{T}$ with $P(\widetilde{\Omega}) = 1$ and a $(\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1}))$-measurable function $f_{\rm hom} : \Omega \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that*

$$\lim_{r\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}} = f_{\rm hom}(\omega, \nu) \tag{7.12}$$

*for every $\omega \in \widetilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$. Moreover, $\widetilde{\Omega}$ and $f_{\rm hom}$ are $(\tau_z)_{z\in\mathbb{Z}^n}$-translation invariant; i.e., $\tau_z(\widetilde{\Omega}) = \widetilde{\Omega}$ for every $z \in \mathbb{Z}^n$ and*

$$f_{\rm hom}(\tau_z(\omega), \nu) = f_{\rm hom}(\omega, \nu), \tag{7.13}$$

*for every $z \in \mathbb{Z}^n$, for every $\omega \in \widetilde{\Omega}$, and every $\nu \in \mathbb{S}^{n-1}$. Eventually, if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic then $f_{\rm hom}$ is independent of $\omega$ and given by*

$$f_{\rm hom}(\nu) = \lim_{r\to+\infty} \frac{1}{r^{n-1}} \int_\Omega \mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0))\, \mathrm{d}P(\omega). \tag{7.14}$$

*Proof.* We divide the proof into three steps.

*Step 1: existence of the limit for $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.* In this step we show that there exists $\widetilde{\Omega} \in \mathcal{T}$ with $P(\widetilde{\Omega}) = 1$ such that the following holds: for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ there is a $\mathcal{T}$-measurable function $f_\nu \colon \Omega \to [0, +\infty)$ such that

$$f_\nu(\omega) = \lim_{r\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}, \quad \forall \omega \in \widetilde{\Omega}.$$

We fix $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. By Proposition 7.6 we know that the map $\mu_\nu$ given in (7.5) is a subadditive process on $(\Omega, \mathcal{T}, P)$ with respect to some group of $P$-preserving transformations. This allows us to apply the Subadditive Ergodic Theorem [1, Theorem 2.4] and deduce the following: let $I = [-1, 1)^{n-1}$, so that $I_\nu = 2M_\nu Q^\nu(0)$; then there exists a set $\Omega_\nu \in \mathcal{T}$, with $P(\Omega_\nu) = 1$, and a $\mathcal{T}$-measurable function $f_\nu \colon \Omega \to [0, +\infty)$ such that

$$f_\nu(\omega) = \lim_{j\to+\infty} \frac{\mu_\nu(\omega, jI)}{(2j)^{n-1}} = \lim_{j\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, j2M_\nu Q^\nu(0))}{(j2M_\nu)^{n-1}}. \tag{7.15}$$

for every $\omega \in \Omega_\nu$.

Let now $(r_j)$ be a sequence of strictly positive real numbers with $r_j \to +\infty$, as $j \to +\infty$ and define

$$r_j^- := 2M_\nu\Big(\Big\lfloor \frac{r_j}{2M_\nu} \Big\rfloor - 1\Big) \quad \text{and} \quad r_j^+ := 2M_\nu\Big(\Big\lfloor \frac{r_j}{2M_\nu} \Big\rfloor + 2\Big).$$

For $j$ sufficiently large $r_j > 4(1 + M_\nu)$, and thus $r_j^- > 4$, moreover

$$Q^\nu_{r_j^-+2}(0) \subset\subset Q^\nu_{r_j}(0) \subset\subset Q^\nu_{r_j+2}(0) \subset\subset Q^\nu_{r_j^+}(0).$$

We then can apply Lemma A.1 twice: the first time with $x = \widetilde{x} = 0$, $r = r_j^-$, and $\widetilde{r} = r_j$ and the second one with $x = \widetilde{x} = 0$, $r = r_j^+$, and $\widetilde{r} = r_j$ and get the two following estimates

$$\frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q^\nu_{r_j}(0))}{r_j^{n-1}} \le \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q^\nu_{r_j^-}(0))}{(r_j^-)^{n-1}} + \frac{L(r_j - r_j^- + 1)}{r_j}, \tag{7.16}$$

$$\frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q^\nu_{r_j}(0))}{r_j^{n-1}} \ge \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q^\nu_{r_j^+}(0))}{(r_j^+)^{n-1}} - \frac{L(r_j^+ - r_j + 1)}{r_j}. \tag{7.17}$$

Using that $r_j^+ - r_j \leq 4M_\nu$ and $r_j - r_j^- \leq 4M_\nu$, thus thanks to (7.15), passing to the limsup in (7.16) and to the liminf in (7.17) as $j \to +\infty$ yield

$$\limsup_{j\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} \leq f_\nu(\omega) \quad \forall \omega \in \Omega_\nu, \tag{7.18}$$

and

$$\liminf_{j\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} \geq f_\nu(\omega) \quad \forall \omega \in \Omega_\nu. \tag{7.19}$$

From (7.18) and (7.19) we have that for every $\omega \in \Omega_\nu$ the limit along $(r_j)$ exists and satisfies

$$\lim_{j\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} = f_\nu(\omega).$$

Eventually to conclude we define

$$\widetilde{\Omega} := \bigcap_{\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n} \Omega_\nu,$$

which satisfies the desired properties.

*Step 2: existence of the limit for $\nu \in \mathbb{S}^{n-1} \setminus \mathbb{Q}^n$.* In this step we prove that there is a $(\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1}))$-measurable function $f_{\rm hom} : \Omega \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that (7.12) holds for every $\omega \in \widetilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$.

Let $\underline{f}, \overline{f} \colon \widetilde{\Omega} \times \mathbb{S}^{n-1} \to [0, +\infty]$ be given by

$$\underline{f}(\omega,\nu) := \liminf_{r\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}, \quad \overline{f}(\omega,\nu) := \limsup_{r\to+\infty} \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}.$$

We first observe that $\widehat{\mathbb{S}}_\pm^{n-1} \cap \mathbb{Q}^n$ is dense in $\widehat{\mathbb{S}}_\pm^{n-1}$. Moreover in Step 1 we showed that $\underline{f}(\omega,\nu) = \overline{f}(\omega,\nu) = f_\nu(\omega)$, for every $\omega \in \widetilde{\Omega}$ and for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Therefore, in order to obtain the same equality for every for every $\omega \in \widetilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$, it is enough to show that the restrictions of the functions $\nu \mapsto \underline{f}(\omega,\nu)$ and $\nu \mapsto \overline{f}(\omega,\nu)$ to the sets $\widehat{\mathbb{S}}_\pm^{n-1}$ are continuous. Further once we have that we can also deduce the following:

$$\omega \mapsto \overline{f}(\omega,\nu) \text{ is } \mathcal{T}\text{-measurable in } \widetilde{\Omega}, \text{ for every } \nu \in \mathbb{S}^{n-1}$$

together with

$$\nu \mapsto \overline{f}(\omega,\nu) \text{ is continuous in } \widehat{\mathbb{S}}_\pm^{n-1}, \text{ for every } \omega \in \widetilde{\Omega},$$

readily imply that the restriction of $\overline{f}$ to $\widetilde{\Omega} \times \widehat{\mathbb{S}}_\pm^{n-1}$ is measurable with respect to the $\sigma$-algebra induced in $\widetilde{\Omega} \times \widehat{\mathbb{S}}_\pm^{n-1}$ by $\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1})$. Hence the claim follows by setting

$$f_{\rm hom}(\omega,\nu) := \begin{cases} \overline{f}(\omega,\nu) & \text{if } \omega \in \widetilde{\Omega}, \\ c_2 c_p & \text{if } \omega \in \Omega \setminus \widetilde{\Omega}. \end{cases} \tag{7.20}$$

We now prove that $\overline{f}(\omega,\cdot)$ is continuous in $\widehat{\mathbb{S}}_+^{n-1}$. The other proofs are analogous and therefore are left to the readers. Let $\nu \in \widehat{\mathbb{S}}_+^{n-1}$, $(\nu_j) \subset \widehat{\mathbb{S}}_+^{n-1}$ be such that $\nu_j \to \nu$, as $j \to +\infty$. Then, for every $\alpha \in (0, \frac{1}{2})$ we find $j_\alpha \in \mathbb{N}$ such that (A.7) holds true for every $j \geq j_\alpha$. Hence we can apply Lemma A.2 with $x = 0$ and $\widetilde{\nu} = \nu_j$ and get

$$\mathbf{m}_\omega(\bar{u}_0^{\nu_j}, Q_{(1+\alpha)r}^{\nu_j}(0)) - c_\alpha r^{n-1} \leq \mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0)) \leq \mathbf{m}_\omega(\bar{u}_0^{\nu_j}, Q_{(1-\alpha)r}^{\nu_j}(0)) + c_\alpha r^{n-1},$$

where $c_\alpha \to 0$, as $\alpha \to 0$. Now dividing the above inequality by $r^{n-1}$ and passing to the limsup as $r \to +\infty$ we get

$$(1+\alpha)^{n-1}\,\overline{f}(\omega,\nu_j) \le \overline{f}(\omega,\nu) + c_\alpha\,, \tag{7.21}$$

$$(1-\alpha)^{n-1}\,\overline{f}(\omega,\nu_j) \ge \overline{f}(\omega,\nu) - c_\alpha\,. \tag{7.22}$$

Eventually passing to the limsup as $j \to +\infty$ in (7.21) and to the liminf as $j \to +\infty$ in (7.22), and letting $\alpha \to 0$ we have

$$\limsup_{j\to+\infty} \overline{f}(\omega,\nu_j) \le \overline{f}(\omega,\nu) \le \liminf_{j\to+\infty} \overline{f}(\omega,\nu_j)\,,$$

and the proof of step 2 is achieved.

*Step 3: $(\tau_z)_{z\in\mathbb{Z}^n}$-translation invariance.* In this step we show that $\widetilde{\Omega}$ and $f_{\rm hom}$ are $(\tau_z)_{z\in\mathbb{Z}^n}$-translation invariant.

Let $z \in \mathbb{Z}^n$, $\omega \in \widetilde{\Omega}$, and $\nu \in \mathbb{S}^{n-1}$ be fixed. Let $r > 4$ and $u \in \mathscr{A}(\bar{u}_0^\nu, Q_r^\nu(0))$ satisfying

$$\mathscr{F}(\omega)(u, Q_r^\nu(0)) \le \mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0)) + 1\,. \tag{7.23}$$

Setting $\widetilde{u}(y) := u(y+z)$, then since $f$ is stationary there holds

$$\mathscr{F}(\omega)(u, Q_r^\nu(0)) = \mathscr{F}(\tau_z(\omega))(\widetilde{u}, Q_r^\nu(-z)).$$

This together with (7.23) and the fact that $\widetilde{u} \in \mathscr{A}(\bar{u}_{-z}^\nu, Q_r^\nu(-z))$ yield

$$\mathbf{m}_{\tau_z(\omega)}(\bar{u}_{-z}^\nu, Q_r^\nu(-z)) \le \mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0)) + 1\,. \tag{7.24}$$

We choose $r, \widetilde{r}$ such that $\widetilde{r} > r$ and

$$Q_{r+2}^\nu(-z) \subset\subset Q_{\widetilde{r}}^\nu(0) \quad \text{and} \quad \operatorname{dist}(0, \Pi^\nu(-z)) \le \frac{r}{4}.$$

We next apply Lemma A.1 twice: once with $x = -z$ and $\widetilde{x} = 0$ to the minimisation problem $\mathbf{m}_{\tau_z(\omega)}$ and once with $x = z$ and $\widetilde{x} = 0$ to the minimisation problem $\mathbf{m}_\omega$ and get

$$\mathbf{m}_{\tau_z(\omega)}(\bar{u}_0^\nu, Q_{\widetilde{r}}^\nu(0)) \le \mathbf{m}_{\tau_z(\omega)}(\bar{u}_{-z}^\nu, Q_r^\nu(-z)) + L\big(|z| + |r-\widetilde{r}| + 1\big)(\widetilde{r})^{n-2}\,, \tag{7.25}$$

and

$$\mathbf{m}_\omega(\bar{u}_0^\nu, Q_{\widetilde{r}}^\nu(0)) \le \mathbf{m}_\omega(\bar{u}_z^\nu, Q_r^\nu(z)) + L\big(|z| + |r-\widetilde{r}| + 1\big)(\widetilde{r})^{n-2}\,. \tag{7.26}$$

Hence, combining (7.24) with (7.25) and (7.26) we have

$$\frac{\mathbf{m}_{\tau_z(\omega)}(\bar{u}_0^\nu, Q_{\widetilde{r}}^\nu(0))}{\widetilde{r}^{n-1}} \le \frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_r^\nu(0)) + 1}{r^{n-1}} + \frac{L\big(|z| + |r-\widetilde{r}| + 1\big)}{\widetilde{r}}\,, \tag{7.27}$$

and

$$\frac{\mathbf{m}_\omega(\bar{u}_0^\nu, Q_{\widetilde{r}}^\nu(0))}{\widetilde{r}^{n-1}} \le \frac{\mathbf{m}_{\tau_z(\omega)}(\bar{u}_0^\nu, Q_r^\nu(0)) + 1}{r^{n-1}} + \frac{L\big(|z| + |r-\widetilde{r}| + 1\big)}{\widetilde{r}}\,. \tag{7.28}$$

We now take in (7.27) the limsup as $\widetilde{r} \to +\infty$ and the limit as $r \to +\infty$ and find

$$\limsup_{\widetilde{r}\to+\infty} \frac{\mathbf{m}_{\tau_z(\omega)}(\bar{u}_0^\nu, Q_{\widetilde{r}}^\nu(0))}{\widetilde{r}^{n-1}} \le f_{\rm hom}(\omega,\nu). \tag{7.29}$$

Similarly we take in (7.28) the limit as $\widetilde{r} \to +\infty$ and the liminf as $r \to +\infty$ and obtain

$$f_{\rm hom}(\omega,\nu) \le \liminf_{r\to+\infty} \frac{\mathbf{m}_{\tau_z(\omega)}(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}. \tag{7.30}$$

Gathering (7.29) and (7.30) we deduce that $\tau_z(\omega) \in \widetilde{\Omega}$ and that

$$f_{\rm hom}(\tau_z(\omega),\nu) = f_{\rm hom}(\omega,\nu)\,,$$

for every $z \in \mathbb{Z}$, $\omega \in \widetilde{\Omega}$, and $\nu \in \mathbb{S}^{n-1}$. Notice also that thanks to the group properties of $(\tau_z)_{z\in\mathbb{Z}^n}$ we also have that $\omega \in \tau_z(\widetilde{\Omega})$, for every $z \in \mathbb{Z}^n$.

We conclude by observing that if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic, then the independence of $\omega$ of the function of $f_{\rm hom}$ follows by (7.13) (cf. [18, Corollary 6.3]) and (7.14) follows by integrating (7.12) on $\Omega$ and by using the Dominated Convergence Theorem, thanks to (7.6) (see also (7.5)). $\square$

We conclude this section by establishing the last crucial result which extends Proposition 7.7 to the case of an arbitrary $x\in\mathbb{R}^n$. More precisely, Proposition 7.8 below states that the limit in (7.12) exists when $x=0$ is replaced by any $x\in\mathbb{R}^n$ and that it is $x$-independent, and hence it coincides with (7.12).

The proof of the following proposition can be obtained arguing exactly as in [18, Theorem 6.1] (see also [3, Theorem 5.5]), now appealing to Proposition 7.7, Lemma A.1, and Lemma A.2. For this reason we skip its proof here.

**Proposition 7.8** (Homogenised surface integrand)**.** *Let $f$ satisfy $(F1)$-$(F3)$ and assume that it is stationary with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of $P$-preserving transformations on $(\Omega,\mathcal{T},P)$. For $\omega\in\Omega$ let $\mathbf{m}_\omega$ be as in* (7.2)*. Then there exists $\Omega'\in\mathcal{T}$ with $P(\Omega')=1$ such that*

$$\lim_{r\to+\infty}\frac{\mathbf{m}_\omega(\bar u^\nu_{rx},Q^\nu_r(rx))}{r^{n-1}}=f_{\rm hom}(\omega,\nu) \tag{7.31}$$

*for every $\omega\in\Omega'$, every $x\in\mathbb{R}^n$, and every $\nu\in\mathbb{S}^{n-1}$, where $f_{\rm hom}$ is given by* (7.12)*. In particular, the limit in* (7.31) *is independent of $x$. Moreover, if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic, then $f_{\rm hom}$ is independent of $\omega$ and given by* (7.14)*.*

We conclude this section with the proof of Theorem 7.4.

*Proof of Theorem 7.4.* The proof follows by Theorem 3.3 now invoking Proposition 7.8. $\square$

## 8. Periodic Homogenisation

In this last section we prove a periodic homogenisation result without requiring any lower semicontinuity on the integrand $f$ (i.e., without assuming $(F3)$).

A careful inspection of the proof of Theorem 7.4 shows that condition $(F3)$ is used only once in the proof of Proposition 7.6 to establish the $\mathcal{T}$-measurability of the map $\mu_\nu$ defined in (7.5). This suggests that in the periodic setting Theorem 7.4 should still be true if we drop condition $(F3)$.

Let $f\in\mathcal{F}$ be such that for every $u\in\mathbb{R}$ and every $\xi\in\mathbb{R}^n$ $f(\cdot,u,\xi)$ is $Q$-periodic. Let $f_k$ and $\mathscr{F}_k$ be defined as in (3.3) and (2.1) accordingly. We now state and prove the following result.

**Theorem 8.1** (Periodic homogenisation)**.** *Let $f\in\mathcal{F}$ be such that for every $u\in\mathbb{R}$ and every $\xi\in\mathbb{R}^n$ $f(\cdot,u,\xi)$ is $Q$-periodic. Let $\mathbf{m}$ be as in* (3.2)*. Then the limit*

$$\lim_{r\to+\infty}\frac{\mathbf{m}(\bar u^\nu_{rx},Q^\nu_r(rx))}{r^{n-1}}=\lim_{r\to+\infty}\frac{\mathbf{m}(\bar u^\nu_{0},Q^\nu_r(0))}{r^{n-1}}=:f_{\rm hom}(\nu) \tag{8.1}$$

*exists and is independent of $x$. The function $f_{\rm hom}\colon\mathbb{S}^{n-1}\to[0,+\infty)$ is $\mathcal{B}(\mathbb{S}^{n-1})$-measurable.*

*Moreover, for every $A\in\mathcal{A}$, the functionals $\mathscr{F}_k(\cdot,A)$, defined in* (2.1) *with $f_k$ as in* (3.3)*, $\Gamma$-converge in $L^1_{\rm loc}(\mathbb{R}^n)$ to the functional $\mathscr{F}_{\rm hom}(\cdot,A)$ with $\mathscr{F}_{\rm hom}\colon L^1_{\rm loc}(\mathbb{R}^n)\times\mathcal{A}\longrightarrow[0,+\infty]$ given by*

$$\mathscr{F}_{\rm hom}(u,A):=\begin{cases}\displaystyle\int_{S_u\cap A}f_{\rm hom}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } u\in BV(A;\{0,1\})\\ +\infty & \text{otherwise}.\end{cases}$$

*Proof.* From Theorem 3.3 it is sufficient to show that the limit in (8.1) exists and is independent of $x$ and that $f_{\rm hom}$ is $\mathcal{B}(\mathbb{S}^{n-1})$-measurable. We divide the proof into a nuber of steps.

*Step 1: existence of the limit for $x = 0$.* In this step we show that for $x = 0$ and for every $\nu \in \mathbb{S}^{n-1}$ the following limit in (8.1) exists. Let $\nu \in \mathbb{S}^{n-1}$ and let $s > r > 0$ be fixed. By reasoning as in step 2 of the proof of Proposition 7.7 we can assume without loss of generality that $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Next we choose $u_r \in \mathscr{A}(\bar{u}_0^\nu, Q_r^\nu(0))$ such that

$$\mathscr{F}(u_r, Q_r^\nu(0)) \leq \mathbf{m}(\bar{u}_0^\nu, Q_r^\nu(0)) + 1\,. \tag{8.2}$$

In what follows we extend $u_r$ to $Q_s^\nu(0)$ to get a new function $u_s \in \mathscr{A}(\bar{u}_0^\nu, Q_s^\nu(0))$ without essentially increasing the energy. For every $z \in \mathbb{Z}^{n-1} \times \{0\}$ we set

$$z_r^\nu := \left(\left\lfloor \frac{r}{M_\nu} \right\rfloor + 1\right) M_\nu R_\nu z\,, \quad \widetilde{Q}_z := Q_r^\nu(0) + z_r^\nu\,,$$

with $R_\nu$ as in (g) and $M_\nu$ defined as in section 7.1, and let

$$I_s := \Big\{ z \in \mathbb{Z}^{n-1} \times \{0\} \colon \widetilde{Q}_z \subset Q_s^\nu(0) \Big\}\,.$$

Note that by definition of $M_\nu$ and $R_\nu$ we have $z_r^\nu \in \mathbb{Z}^n \cap \Pi_\nu$ for every $z \in \mathbb{Z}^{n-1} \times \{0\}$. Moreover a direct computation shows that

$$s^{n-1}\left(\frac{1}{r+1} - \frac{1}{s}\right)^{n-1} \leq \#(I_s) \leq \frac{s^{n-1}}{r^{n-1}}\,. \tag{8.3}$$

Let us define

$$u_s(y) := \begin{cases} u_r(y - z_r^\nu) & \text{if } y \in \widetilde{Q}_z,\ z \in I_s\,, \\ \bar{u}_0^\nu(y) & \text{otherwise in } Q_s^\nu(0)\,. \end{cases}$$

Clearly $u_s \in \mathscr{A}(\bar{u}_0^\nu, Q_s^\nu(0))$. Furthermore a change of variable together with the the $Q$-periodicity of $f$ yield

$$\mathscr{F}(u_s, Q_s^\nu(0)) \leq \#(I_s)\mathscr{F}(u_r, Q_r^\nu(0)) + \mathscr{F}(\bar{u}_0^\nu, Q_s^\nu(0) \setminus \cup_{z\in I_s} \widetilde{Q}_z)\,. \tag{8.4}$$

Moreover by (2.10) we may deduce that

$$\mathscr{F}(\bar{u}_0^\nu, Q_s^\nu(0) \setminus \cup_{z\in I_s} \widetilde{Q}_z) \leq c_2 C_{\rm u}(s^{n-1} - \#(I_s) r^{n-1})\,. \tag{8.5}$$

Gathering together (8.3)-(8.5) we obtain

$$\frac{\mathbf{m}(\bar{u}_0^\nu, Q_s^\nu(0))}{s^{n-1}} \leq \frac{\mathscr{F}(u_s, Q_s^\nu(0))}{s^{n-1}} \leq \frac{\mathscr{F}(u_r, Q_r^\nu(0))}{r^{n-1}} + c_2 C_{\rm u}\left(1 - r^{n-1}\Big(\frac{1}{r+1} - \frac{1}{s}\Big)^{n-1}\right)\,. \tag{8.6}$$

Finally combining (8.6) with (8.2) and passing first to the limsup as $s \to +\infty$ and then to the liminf as $r \to +\infty$ we get

$$\limsup_{s\to+\infty} \frac{\mathbf{m}(\bar{u}_0^\nu, Q_s^\nu(0))}{s^{n-1}} \leq \liminf_{r\to+\infty} \frac{\mathbf{m}(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}\,,$$

and, being the converse inequality trivial, the proof of Step 1 is achieved.

*Step 2: existence of the limit for every $x$.* In this step we show that for every $x \in \mathbb{R}^n$ the limit in (8.1) exists and is $x$-independent. Let $x \neq 0$, and let $r \geq 4\sqrt{n}$ be fixed. Let $u_r \in \mathscr{A}(\bar{u}_0^\nu, Q_r^\nu(0))$ be such that (8.2) holds. Define $x_r := \lfloor (r+3)x \rfloor$ (i.e., the vector whose componets are the integer parts of the components of $(r+3)x$) so that $|(r+3)x - x_r| \leq \sqrt{n}$ and moreover

$$Q_{r+2}^\nu(x_r) \subset\subset Q_{r+3}^\nu((r+3)x)\,, \quad \text{and} \quad \operatorname{dist}((r+3)x, \Pi^\nu(x_r)) \leq \frac{r}{4}\,.$$

The function $\hat{u}_r(y) := u_r(y - x_r)$ belongs to $\mathscr{A}(\bar{u}^\nu_{x_r}, Q^\nu_r(x_r))$. Moreover by $Q$-periodicity of $f$ and (8.2) we have

$$\mathbf{m}(\bar{u}^\nu_{x_r}, Q^\nu_r(x_r)) \leq \mathscr{F}(\hat{u}_r, Q^\nu_r(x_r)) = \mathscr{F}(u_r, Q^\nu_r(0)) \leq \mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0)) + 1\,. \tag{8.7}$$

Next we invoke Lemma A.1 and get

$$\mathbf{m}(\bar{u}^\nu_{(r+3)x}, Q^\nu_{r+3}((r+3)x)) \leq \mathbf{m}(\bar{u}^\nu_{x_r}, Q^\nu_r(x_r)) + L\big(\sqrt{n}+3+1\big)(r+3)^{n-2}\,. \tag{8.8}$$

Combining (8.7) with (8.8) and rescaling by $(r+3)^{n-1}$ we have

$$\frac{\mathbf{m}(\bar{u}^\nu_{(r+3)x}, Q^\nu_{r+3}((r+3)x))}{(r+3)^{n-1}} \leq \frac{r^{n-1}}{(r+3)^{n-1}}\frac{\mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0))}{r^{n-1}} + \frac{C}{r+3}\,,$$

hence, up to replacing $r+3$ with $r$, passing to the limsup as $r \to +\infty$ we find

$$\limsup_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_{rx}, Q^\nu_r(rx))}{r^{n-1}} \leq \limsup_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0))}{r^{n-1}} = \lim_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0))}{r^{n-1}}\,.$$

In a similar way one can show that also

$$\lim_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0))}{r^{n-1}} = \liminf_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_0, Q^\nu_r(0))}{r^{n-1}} \leq \liminf_{r\to+\infty}\frac{\mathbf{m}(\bar{u}^\nu_{rx}, Q^\nu_r(rx))}{r^{n-1}}\,.$$

The above two inequalities conclude step 2.

*Conclusions.* From step 2 we deduce (8.1). Moreover arguing as in the step 2 of the proof of Proposition 7.7 we may also deduce that $f_{\rm hom}$ is $\mathcal{B}(\mathbb{S}^{n-1})$-measurable.

□

*Remark* 8.2. We conclude this section by observing that our analysis in the stochastic/periodic setting (and the corresponding results Theorem 7.4 and Theorem 8.1) covers, in particular, the case of functionals considered in [31, 23] and the critical regime of [6].

## Acknowledgments

The author wishes to thank Caterina Ida Zeppieri for having drawn her attention to this problem and for many helpful discussions and suggestions, and Annika Bach for useful comments on a preliminary version. The present work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) project 3160408400 and under the Germany Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure.

## Appendix

In this last section we state and prove two technical lemmas which are used in Subsection 7.1.

For $A \in \mathcal{A}$, $x \in \mathbb{R}^n$, and $\nu \in \mathbb{S}^{n-1}$, in what follows $\mathbf{m}(\bar{u}^\nu_x, A)$ denotes the infimum value given by (3.2).

**Lemma A.1.** *Let $f \in \mathcal{F}$; let $\nu \in \mathbb{S}^{n-1}$, $x, \widetilde{x} \in \mathbb{R}^n$, and $\widetilde{r} > r > 4$ be such that*

$$\text{(i)}\ \ Q^\nu_{r+2}(x) \subset\subset Q^\nu_{\widetilde{r}}(\widetilde{x})\,, \qquad \text{(ii)}\ \ \mathrm{dist}(\widetilde{x}, \Pi^\nu(x)) \leq \frac{r}{4}\,.$$

*Then there exists a constant $L > 0$ (independent of $\nu, x, \widetilde{x}, r, \widetilde{r}$) such that*

$$\mathbf{m}(\bar{u}^\nu_{\widetilde{x}}, Q^\nu_{\widetilde{r}}(\widetilde{x})) \leq \mathbf{m}(\bar{u}^\nu_x, Q^\nu_r(x)) + L\big(|x-\widetilde{x}| + |r-\widetilde{r}| + 1\big)\widetilde{r}^{n-2}\,. \tag{A.1}$$

*Proof.* Let $\nu \in S^{n-1}$, $\eta > 0$ be fixed and let $u \in \mathscr{A}(\bar{u}_x^\nu, Q_r^\nu(x))$ with

$$\mathscr{F}(u, Q_r^\nu(x)) \leq \mathbf{m}(\bar{u}_x^\nu, Q_r^\nu(x)) + \eta\,. \tag{A.2}$$

Then $u = \bar{u}_x^\nu$ a.e. in $U$, where $U$ is a neighbourhood of $\partial Q_r^\nu(x)$. Let moreover $\beta \in (0,1)$ be such that $Q_r^\nu(x) \setminus \overline{Q}^\nu_{r-\beta}(x) \subset U$. Let $\widetilde{x} \in \mathbb{R}^n$ and $\widetilde{r} > r > 4$ satisfy (i)-(ii). We set

$$R := R_\nu\bigg(\big(Q'_r \setminus \overline{Q'}_{r-\beta}\big) \times \Big(-1 - \frac{|(x-\widetilde{x})\cdot\nu|}{2}, 1 + \frac{|(x-\widetilde{x})\cdot\nu|}{2}\Big)\bigg) + x + \frac{(\widetilde{x}-x)\cdot\nu}{2}\nu\,,$$

where $R_\nu$ be as in (g). Let $\varphi$ be a smooth cutoff between $Q^\nu_{r-\beta}(x)$ and $Q^\nu_{\widetilde{r}}(\widetilde{x})$ and define

$$\widetilde{u} := \varphi u + (1-\varphi)\bar{u}^\nu_{\widetilde{x}}\,,$$

which clearly belongs to $\mathscr{A}(\bar{u}^\nu_{\widetilde{x}}, Q^\nu_{\widetilde{r}}(\widetilde{x}))$ and moreover it satisfies

$$\mathscr{F}(\widetilde{u}, Q^\nu_{\widetilde{r}}(\widetilde{x})) \leq \mathscr{F}(u, Q^\nu_r(x)) + \mathscr{F}(\widetilde{u}, Q^\nu_{\widetilde{r}}(\widetilde{x}) \setminus \overline{Q}^\nu_{r-\beta}(\widetilde{x}))\,. \tag{A.3}$$

Notice that

$$\widetilde{u} = \varphi\bar{u}^\nu_x + (1-\varphi)\bar{u}^\nu_{\widetilde{x}} \quad \text{ in } \quad Q^\nu_{\widetilde{r}}(\widetilde{x}) \setminus \overline{Q}^\nu_{r-\beta}(x)\,,$$

and in particular

$$\widetilde{u} = u^\nu_x = u^\nu_{\widetilde{x}} \in \{0,1\} \quad \text{ a.e. in } \quad (Q^\nu_{\widetilde{r}}(\widetilde{x}) \setminus \overline{Q}^\nu_{r-\beta}(x)) \setminus \overline{R}\,.$$

This together with (2.6) and $(f4)$ imply

$$\begin{aligned}
\mathscr{F}(\widetilde{u}, Q^\nu_{\widetilde{r}}(\widetilde{x}) \setminus \overline{Q}^\nu_{r-\beta}(\widetilde{x})) &\leq c_2 \int_R (W(\widetilde{u}) + |\nabla\widetilde{u}|^p)\,\mathrm{d}y \\
&\leq C\mathcal{L}^n(R) + C\int_R (|\nabla\bar{u}^\nu_x|^p + |\nabla\bar{u}^\nu_{\widetilde{x}}|^p + |\nabla\varphi|^p|\bar{u}^\nu_{\widetilde{x}} - \bar{u}^\nu_x|^p)\,\mathrm{d}y \\
&\leq C\mathcal{L}^n(R) + C\mathcal{L}^{n-1}(Q'_r \setminus \overline{Q}'_{r-\beta})\,,
\end{aligned} \tag{A.4}$$

where the last two inequality follow by recalling that $W$ is bounded on compact sets and by (2.10). Now we observe that

$$\mathcal{L}^{n-1}\big(Q'_{\widetilde{r}} \setminus \overline{Q}'_{r-\beta}\big) \leq C|r-\widetilde{r}|\widetilde{r}^{n-2}\,, \tag{A.5}$$

while

$$\mathcal{L}^n(R) \leq C\beta r^{n-2}(|x-\widetilde{x}|+1)\,. \tag{A.6}$$

Finally gathering (A.2), (A.3), (A.4), (A.5) and (A.6) we obtain

$$\mathbf{m}(\bar{u}^\nu_{\widetilde{x}}, Q^\nu_{\widetilde{r}}(\widetilde{x})) \leq \mathscr{F}\big(\widetilde{u}, Q^\nu_{\widetilde{r}}(\widetilde{x})\big) \leq \mathbf{m}(\bar{u}^\nu_x, Q^\nu_r(x)) + L\big(|x-\widetilde{x}| + |r-\widetilde{r}| + 1\big)\widetilde{r}^{n-2} + \eta\,,$$

for some $L > 0$ independent of $x, \widetilde{x}, r, \widetilde{r}, \nu$, thus (A.1) follows by the arbitrariness of $\eta > 0$. □

**Lemma A.2.** *Let $f \in \mathcal{F}$; let $\alpha \in (0, \frac{1}{2})$ and $\nu, \widetilde{\nu} \in \mathbb{S}^{n-1}$ be such that*

$$\max_{1\leq i\leq n-1} |R_\nu e_i - R_{\widetilde{\nu}} e_i| + |\nu - \widetilde{\nu}| < \frac{\alpha}{\sqrt{n}}, \tag{A.7}$$

*where $R_\nu$ and $R_{\widetilde{\nu}}$ are orthogonal $(n\times n)$-matrices as in (g). Then there exists a constant $c_\alpha > 0$ (independent of $\nu, \widetilde{\nu}$), with $c_\alpha \to 0$ as $\alpha \to 0$, such that for every $x \in \mathbb{R}^n$ and every $r > 2$ we have*

$$\begin{aligned}
\mathbf{m}\big(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{(1+\alpha)r}(rx)\big) - c_\alpha r^{n-1} &\leq \mathbf{m}\big(\bar{u}^\nu_{rx}, Q^\nu_r(rx)\big) \\
&\leq \mathbf{m}\big(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{(1-\alpha)r}(rx)\big) + c_\alpha r^{n-1}\,.
\end{aligned} \tag{A.8}$$

*Proof.* We show only that

$$\mathbf{m}\big(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{(1+\alpha)r}(rx)\big) - c_\alpha r^{n-1} \le \mathbf{m}\big(\bar{u}^{\nu}_{rx}, Q^{\nu}_{r}(rx)\big), \tag{A.9}$$

for some $c_\alpha > 0$, with $c_\alpha \to 0$, as $\alpha \to 0$; as the proof of the other inequality is analogous. Let $x \in \mathbb{R}^n$, $r > 2$, and set $r^{\pm}_{\alpha} := (1 \pm \alpha)r$. From (A.7) we might deduce that

$$Q^{\widetilde{\nu}}_{r^-_\alpha}(rx) \subset\subset Q^{\nu}_{r}(rx) \subset\subset Q^{\widetilde{\nu}}_{r^+_\alpha}(rx).$$

Let $\eta > 0$ be fixed and let $u \in W^{1,p}(Q^\nu_r(rx))$ be a test function for $\mathbf{m}(\bar{u}^\nu_{rx}, Q^\nu_r(rx))$ satisfying

$$\mathscr{F}(u, Q^\nu_r(rx)) \le \mathbf{m}(\bar{u}^\nu_{rx}, Q^\nu_r(rx)) + \eta. \tag{A.10}$$

Choose $\beta \in (0,1)$ such that $Q^{\widetilde{\nu}}_{r^-_\alpha}(rx) \subset\subset Q^\nu_{r-\beta}(rx)$ and $u = \bar{u}^\nu_{rx}$ a.e. in $Q^\nu_r(rx) \setminus \overline{Q}^\nu_{r-\beta}(rx)$ and set

$$R := R_\nu\Big(Q'_r \setminus \overline{Q'}_{r-\beta} \times (-1-\alpha r, 1+\alpha r)\Big) + rx\,,$$

where $R_\nu$ is as in (g). Next define

$$\widetilde{u} := \varphi u + (1-\varphi)\bar{u}^{\widetilde{\nu}}_{rx}\,,$$

where $\varphi$ is a smooth cutoff between $Q^\nu_{r-\beta}(rx)$ and $Q^\nu_r(rx)$. Then $\widetilde{u} \in \mathscr{A}(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{r^+_\alpha}(rx))$ and it also satisfies

$$\mathscr{F}(\widetilde{u}, Q^{\widetilde{\nu}}_{r^+_\alpha}(rx)) \le \mathscr{F}(u, Q^\nu_r(rx)) + \mathscr{F}(\widetilde{u}, Q^\nu_r(rx) \setminus \overline{Q}^\nu_{r-\beta}(rx)) + \mathscr{F}(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{r^+_\alpha}(rx) \setminus \overline{Q}^\nu_{r^-_\alpha}(rx))\,. \tag{A.11}$$

By definition we have

$$\widetilde{u} = \varphi \bar{u}^\nu_{rx} + (1-\varphi)\bar{u}^{\widetilde{\nu}}_{rx} \quad \text{in} \quad Q^\nu_r(rx) \setminus \overline{Q}^\nu_{r-\beta}(rx)\,,$$

and in particular

$$\widetilde{u} = u^\nu_{rx} = u^{\widetilde{\nu}}_{rx} \in \{0,1\} \quad \text{a.e. in} \quad (Q^\nu_r(rx) \setminus \overline{Q}^\nu_{r-\beta}(rx)) \setminus \overline{R}\,.$$

Therefore from (2.6), ($(f4)$) and (2.10) we deduce that

$$\begin{aligned}\mathscr{F}(\widetilde{u}, Q^\nu_r(rx) \setminus \overline{Q}^\nu_{r-\beta}(rx)) &\le c_2 \int_R (W(\widetilde{u}) + |\nabla \widetilde{u}|^p)\,\mathrm{d}y \\ &\le C\mathcal{L}^n(R) + C\int_R \Big(|\nabla \bar{u}^\nu_{rx}|^p + |\nabla \bar{u}^{\widetilde{\nu}}_{rx}|^p + |\nabla\varphi|^p|\bar{u}^\nu_{rx} - \bar{u}^{\widetilde{\nu}}_{rx}|^p\Big)\,\mathrm{d}y \\ &\le C\mathcal{L}^n(R) + C\mathcal{L}^{n-1}(Q'_r \setminus \overline{Q}'_{r-\beta})\,.\end{aligned} \tag{A.12}$$

On the other hand from (2.10) and (2.6) we infer

$$\mathscr{F}(\bar{u}^{\widetilde{\nu}}_{rx}(y), Q^{\widetilde{\nu}}_{r^+_\alpha}(rx) \setminus Q^{\widetilde{\nu}}_{r^-_\alpha}(rx)) \le c_2 C_{\mathrm{u}}((1+\alpha)^{n-1} - (1-\alpha)^{n-1})r^{n-1}\,. \tag{A.13}$$

Using that

$$\mathcal{L}^{n-1}(Q'_r \setminus \overline{Q}'_{r-\beta}) \le Cr^{n-2}\,, \quad \text{and} \quad \mathcal{L}^n(R) \le C\beta r^{n-2}\alpha r \le C\alpha r^{n-1}\,.$$

together with (A.11), (A.12), and (A.13) we obtain

$$\mathscr{F}(\widetilde{u}, Q^{\widetilde{\nu}}_{r^+_\alpha}(rx)) \le \mathscr{F}(u, Q^\nu_r(rx)) + \mathscr{F}(\bar{u}^{\widetilde{\nu}}_{rx}(y), Q^{\widetilde{\nu}}_{r^+_\alpha}(rx) \setminus Q^{\widetilde{\nu}}_{r^-_\alpha}(rx) + c_\alpha r^{n-1}\,,$$

where $c_\alpha := c_4 C_{\mathrm{u}}\big((1+\alpha)^{n-1} - (1-\alpha)^{n-1}\big) + C\alpha$. Eventually as the above inequality together with (A.10) imply

$$\mathbf{m}(\bar{u}^{\widetilde{\nu}}_{rx}, Q^{\widetilde{\nu}}_{r^+_\alpha}(rx)) \le \mathbf{m}(\bar{u}^\nu_{rx}, Q^\nu_r(rx)) + c_\alpha r^{n-1} + \eta\,,$$

we may deduce (A.9) by the arbitrariness of $\eta > 0$. □

## References


[1] M. A. Akcoglu and U. Krengel, Ergodic theorems for superadditive processes, *J. Reine Angew. Math.* **323** (1981), 53–67.

[2] G. Alberti, Variational models for phase transitions, an approach via Γ-convergence, in L. Ambrosio and N. Dancer, *Calculus of Variations and Partial Differential Equations.* Springer Verlag, Berlin (2000), 95–114.

[3] R. Alicandro, M. Cicalese, and M. Ruf, Domain formation in magnetic polymer composites: an approach via stochastic homogenization, *Arch. Ration. Mech. Anal.* **218** (2015), 945–984.

[4] L. Ambrosio and A. Braides, *Functionals defined on partitions of sets of finite perimeter, I: integral representation and* Γ*-convergence*, *J. Math. Pures. Appl.* **69** (1990), 285–306.

[5] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[6] N. Ansini, A. Braides, and V. Chiadó Piat, Gradient theory of phase transitions in composite media, *Proc. R. Soc. Edind.* **133A** (2003), 265–296.

[7] A. Bach, T. Esposito, R. Marziani, and C. I. Zeppieri, Gradient damage models for heterogeneous materials, *Arxiv preprint: 2205.13966* (2022).

[8] A. Bach, T. Esposito, R. Marziani, and C. I. Zeppieri, Interaction between oscillations and singular perturbations in a one-dimensional phase-field model , *Accepted Paper: AWM Volume - Research in the Mathematics of Materials Science* (2022).

[9] A. Bach, R. Marziani, and C. I. Zeppieri, Γ-Convergence and stochastic homogenisation of singularly perturbed elliptic functionals, *arXiv:2102.09872* (2021).

[10] S. Baldo, Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **7** (1990), 67–90.

[11] A. C. Barroso and I. Fonseca, Anisotropic singular perturbations - the vectorial case, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 527–571.

[12] A. Braides and C. I. Zeppieri, Multiscale analysis of a prototypical model for the interaction between microstructure and surface energy, *Interfaces Free Bound.* **11** (2009), 61–118.

[13] G. Bouchittè, Singular perturbations of variational problems arising from a two-phase transition model, *Appl. Math. and Opt.* **21** (1990), 289–314.

[14] G. Bouchittè, I. Fonseca, G. Leoni, and L. Mascarenhas, A global method for relaxation in $W^{1,p}$ and $SBV_p$, *Arch. Ration. Mech. Anal.* **165** (2002), 187–242.

[15] A. Braides, M. Cicalese, and M. Ruf, Continuum limit and stochastic homogenization of discrete ferromagnetic thin films, *Anal. PDE* **11** (2018), 499–553.

[16] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, New York, 1998.

[17] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri, Γ-convergence of free-discontinuity problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **36** (2019), 1035–1079.

[18] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri, Stochastic homogenisation of free discontinuity problems, *Arch. Ration. Mech. Anal.* **233** (2019), 935–974.

[19] J. W. Cahn, and J. E. Hilliard, Free energy of a nonuniform system. i. interfacial free energy, *J. Chem. Ph.m* **28** (1958), 258–267.

[20] M. Chermisi, G. Dal Maso, I. Fonseca, and G. Leoni, Singular perturbation models in phase transitions for second-order materials, *Indiana Univ. Math. J.* **60** (2011), 367–409.

[21] M. Cicalese, E. Spadaro, and C. I. Zeppieri, Asymptotic analysis of a second-order singular perturbation model for phase transitions, *Calc. Var. Partial Differential Equations* **41** (2011), 127–150.

[22] R. Cristoferi, I. Fonseca,and L. Ganedi Homogenization and phase separation with space dependent wells -the subcritical case., *Arxiv preprint: 2205.12893* (2022).

[23] R. Cristoferi, I. Fonseca, A. Hagerty, and C. Popovici A homogenization result in the gradient theory of phase transitions, *Interfaces and Free Boundaries* **21** (2019), 367–408.

[24] G. Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkhäuser Boston Inc., Boston, MA, 1993.

[25] G. Dal Maso and L. Modica, Nonlinear stochastic homogenization and ergodic theory, *J. Reine Angew. Math.* **368** (1986), 28–42.

[26] I. Fonseca and C. Mantegazza, Second order singular perturbation models for phase transitions, *SIAM J. Math. Anal.* **31** (2000), no. 5, 1121–1143.

[27] I. Fonseca and L. Tartar, The gradient theory of phase transitions for systems with two potential wells, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), 89–102.

[28] M. E. Gurtin, Some results and conjectures in the gradient theory of phase transitions, in Metastability and incompletely posed problems (Minneapolis, Minn., 1985), *volume 3 of IMA Vol. Math.* (1987), 135–146.

[29] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Ration. Mech. Anal.* **98** (1987), 123–142.

[30] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital.* **14-B** (1977), 285–299.

[31] P. S. Morfe, Surface Tension and Γ-Convergence of Van der Waals-Cahn-Hilliard Phase Transitions in Stationary Ergodic Media, *J. Stat. Phys.* **181** (2020), no. 6, 2225–2256.

[32] N. C. Owen and P. Sternberg, Nonconvex variational problems with anisotropic perturbations, *Nonlinear Anal.* **16** (1991), 705–719.

[33] M. Ruf and T. Ruf, Stochastic homogenization of degenerate integral functionals and their Euler-Lagrange equations *Arxiv preprint: 2109.13013* (2021).

[34] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems, *Arch. Ration. Mech. Anal.* **101** (1988), 209–260.

[35] J. D. Van der Waals, The thermodynamics theory of capillarity under the hypothesis of a continuous variation of density, *Verhaendel kronik. Akad. Weten. Amsterdam*, **1** (1893), 386–398.

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