PRESCRIBING Q-CURVATURE ON EVEN-DIMENSIONAL MANIFOLDS WITH CONICAL SINGULARITIES

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ABSTRACT. On a $2m$-dimensional closed manifold we investigate the existence of prescribed Q-curvature metrics with conical singularities. We present here a general existence and multiplicity result in the supercritical regime. To this end, we first carry out a blow-up analysis of a $2m$th-order PDE associated to the problem and then apply a variational argument of min-max type. For $m > 1$, this seems to be the first existence result for supercritical conic manifolds different from the sphere.

Keywords: Q-curvature, conical singularities, blow-up analysis, variational methods

1. INTRODUCTION

In conformal geometry, one of the most fundamental problems is understanding the relationship between conformally covariant operators, their associated conformal invariants and the related PDEs.

As a first example, let us consider the Laplace-Beltrami operator in two dimensions on a closed surface (M, g) and the Gaussian curvature. Through a conformal change of metric $g_v = e^{2v}g$, we have the associated PDE

$$-\Delta_g v + K_g = K_{g_v} e^{2v}, \tag{1.1}$$

where Δ_g denotes the Laplace-Beltrami operator with respect to the background metric g, K_g and K_{g_v} are the Gaussian curvatures of the metric g and g_v, respectively. Observe that the latter equation yields in particular the conformal invariance of the total Gaussian curvature which is then tight to the topology of the surface via the Gauss-Bonnet formula

$$\int_M K_g \, dvol_g = \chi(M).$$

Here, $\chi(M)$ is the Euler characteristic of the surface.

A classical issue here is the prescribed Gaussian curvature problem or the Uniformization Theorem about the existence of a conformal metric in the conformal class of g with prescribed (possibly constant) curvature. This amounts to solve the PDE in (1.1) which has been systematically studied since the works of Berger [6], Kazdan-Warner [28] and Chang-Yang [13, 14].

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In higher dimensions, we have the so-called GJMS operators P_g^{2m} and the related Q-curvatures Q_g^{2m} which are the higher-order analogues of the Laplace-Beltrami operator and the Gaussian curvature for $2m$-dimensional closed manifolds, see [23, 24]. These are conformally covariant differential operators whose leading term is $(-\Delta_g)^m$. In particular, when $m = 1$ we recover the Laplace-Beltrami operator and the Gaussian curvature. Moreover, for $m = 2$, P_g^4 and Q_g^4 are related to the Paneitz operator and the standard Q-curvature:

$$\begin{aligned} P_g^4 f &= P_g f = \Delta_g^2 f + \mathrm{div}_g \left(\frac{2}{3} R_g g - 2\mathrm{Ric}_g \right) df, \\ Q_g^4 &= 2Q_g = -\frac{1}{6}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2), \end{aligned} \tag{1.2}$$

where Ric_g and R_g stands for the Ricci tensor and the scalar curvature of the manifold (M, g). See the original works of Paneitz [43, 42] and Branson [7] for more details.

The family of GJMS operators and the related Q-curvature functions play now an important role in modern differential geometry. As in the lower order case, under the conformal transformation $g_v = e^{2v} g$, P_g^{2m} and Q_g^{2m} satisfy the following law

$$P_{g_v}^{2m} = e^{-2mv} P_g^{2m}, \quad P_g^{2m} v + Q_g^{2m} = Q_{g_v}^{2m} e^{2mv}. \tag{1.3}$$

Again, the total Q-curvature $\int_M Q_g^{2m} \, dvol_g$ is seen to be conformally invariant for which the Chern-Gauss-Bonnet formula holds true. In analogy with the two-dimensional case, a core problem is the prescribed Q-curvature problem which is in turn related to the solvability of (1.3).

One can attack this problem variationally by looking at the critical points of the associated energy functional. A lot of work has been done in this direction, in particular for the four-dimensional case and the Paneitz operator (1.2). In this setting, assuming

$$P_g \geq 0, \quad \mathrm{Ker}\{P_g\} = \{constants\},$$

the problem has been first solved by Chang-Yang [15] for

$$\int_M Q_g^4 \, dvol_g = 2 \int_M Q_g \, dvol_g < 16\pi^2 = 2 \int_{\mathbb{S}^4} Q_{g_0} \, dvol_{g_0}.$$

Here, g_0 is the standard metric of the sphere. See also the related work of Gursky [25]. This is the so-called subcritical case in which the energy functional is coercive and bounded from below by means of the Adams-Trudinger-Moser inequality [1] and solutions corresponds to global minima using the direct methods of the calculus of variations. We refer to the discussion in the sequel for the precise definition of the subcritical, critical and supercritical case. The supercritical case $\int_M Q_g^4 \, dvol_g > 16\pi^2$, where the energy functional fails to be bounded from below, has been considered by Djadli-Malchiodi [20] via a new min-max method based on improved versions of the Adams-Trudinger-Moser inequality, solving the problem provided

$$\operatorname{Ker}\{P_g\} = \{constants\}, \quad \int_M Q_g^4 \, dvol_g \notin 16\pi^2\mathbb{N}.$$

Finally, some existence results for the critical case $\int_M Q_g^4 \, dvol_g \in 16\pi^2\mathbb{N}$ have been derived by Ndiaye [41] by making use of the critical point theory at infinity jointly with a blow-up analysis.

As far as the higher-dimensional case $2m > 4$ is concerned, the subcritical case has been solved in [8] via a geometric flow, while the Djadli-Malchiodi's argument has been generalized by Ndiaye [40] to treat the supercritical case.

In this paper we are interested in prescribing the Q-curvature on a general $2m$-dimensional closed manifold M with conical singularities. Let g be a smooth metric on M. We will say that a point $q \in M$ is a conical singularity of order $\alpha \in (-1, +\infty)$ for the new metric $g_v = e^{2v}g$ if

$$g_v(x) = f(x)|x|^{2\alpha}|dx|^2 \quad \text{locally around } q,$$

for some smooth function f. The set of conical singularities q_j of orders α_j is encoded in the divisor

$$D = \sum_{j=1}^{N} \alpha_j q_j,$$

while (M, D) will denote the related conical manifold. We define

$$\kappa_g = \int_M Q_g^{2m} \, dvol_g, \quad \kappa_{g_v} = \int_M Q_{g_v}^{2m} \, dvol_{g_v},$$

for which the following relation holds

$$\kappa_{g_v} = \kappa_g + \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j, \tag{1.4}$$

where $\Lambda_m = (2m-1)!|\mathbb{S}^{2m}|$. This can be regarded as a singular Chern-Gauss-Bonnet formula, see for example Theorem 2.3. The critical threshold of a singular manifold is essentially related to the singular Adams-Trudinger-Moser inequality stated in Theorem 2.4. In the spirit of Troyanov [45] we let

$$\tau(M, D) = \Lambda_m \Big(1 + \min_j \big\{\alpha_j, 0\big\}\Big)$$

and give the following classification.

Definition 1.1. *The singular manifold (M, D) is said to be:*

$$\begin{array}{lll} \textit{subcritical} & \textit{if} & \kappa_{g_v} < \tau(M, D) \\ \textit{critical} & \textit{if} & \kappa_{g_v} = \tau(M, D) \\ \textit{supercritical} & \textit{if} & \kappa_{g_v} > \tau(M, D). \end{array}$$

See also the recent work of Fang-Ma [22] for a similar discussion. We point out we have a slightly different notation for Λ_m with respect to the latter paper.

Due to the singular behavior of the conformal factor v around a conical point, prescribing the Q-curvature on a manifold with conical singularities at $q_j \in M$ of order $\alpha_j \in (-1, +\infty)$ is related to the solvability of the following singular PDE

$$P_g^{2m} v + Q_g^{2m} = Q_{g_v}^{2m} e^{2mv} - \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j \delta_{q_j}, \tag{1.5}$$

where δ_{q_j} stands for the Dirac measure located at the point $q_j \in M$. One may desingularize the behavior of v around the conical points by considering

$$u = v - \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j G(x, q_j),$$

where $G(x, p)$ is the Green function of P_g^{2m}, see for example Lemma 2.2. Then u satisfies

$$P_g^{2m} u + Q_g^{2m} + \frac{\Lambda_m}{2|M|} \sum_{j=1}^{N} \alpha_j = \widetilde{Q} e^{2mu}, \tag{1.6}$$

where

$$\widetilde{Q} = Q_{g_v}^{2m} e^{-m\Lambda_m \sum_{j=1}^{N} \alpha_j G(x, q_j)}, \tag{1.7}$$

which is now singular at the points q_j.

The singular equation (1.6) has been studied mainly in the two-dimensional case, that is in relation to the prescribed Gaussian curvature problem. After the initial work of Troyanov [45], there have been contributions by many authors, as for example [16, 17, 18, 31, 37]. This problem has received a lot of attention also in recent years, see [3, 4, 5, 12, 33]. See also [21, 36, 38, 39] for further developments in this direction.

In the higher-dimensional case $m > 1$ there are very few results available. The subcritical regime have been just recently solved by Fang-Ma [22], where the four-dimensional case is considered. The authors point out their method could be applied for higher dimensions too. In any case, the existence here follows by direct methods of the calculus of variations once the singular Adams-Trudinger-Moser inequality in Theorem 2.4 is derived. See also [26] for a related result on the sphere via a fixed point argument. For a blow-up analysis in dimension four we refer instead to [2]. Concerning the existence problem in the supercritical case, the only result we are aware of is [27] where the authors consider a slightly supercritical problem on the sphere, again with a fixed point argument in the spirit of [26].

The goal of this paper is to give a first general existence result for $2m$-dimensional conic manifolds in the supercritical regime. We define a critical set of values Γ as follows:

$$\Gamma = \left\{ n\Lambda_m + \Lambda_m \sum_{i\in J}(1+\alpha_i) \mid n \in \mathbb{N}\cup\{0\} \quad \text{and} \quad J \subset \{1,\dots,N\} \right\}. \tag{1.8}$$

Observe that if $\alpha_j \in \mathbb{N}$ for all j, then we simply have $\Gamma = \Lambda_m\mathbb{N}$. Recall now the definition of the total singular curvature κ_{g_v} given in (1.4). Let $M^{\mathrm{R}} \subset M$ be a closed n-dimensional submanifold, $n \in [1, 2m]$, such that $\alpha_j \notin M^{\mathrm{R}}$ for all $j = 1, \dots, N$. Then, we have:

Theorem 1.1. *Let (M, D) be a supercritical singular $2m$-dimensional closed manifold with $\alpha_j > 0$ for $j = 1, \dots, N$ such that there exists a retraction $R : M \to M^R$. Let Q be a smooth positive function on M. If*

$$Ker\{P_g^{2m}\} = \{constants\}, \quad \kappa_{g_v} \notin \Gamma,$$

then there exists a conformal metric on (M, D) with Q^{2m}-curvature equal to Q.

Remark 1.1. *We point out that a retraction $R : M \to M^R$ as above exists for a wide class of manifolds. For example we can consider manifolds of the type $M^n \times M^{2m-n}$, where we denote by M^l any l-dimensional closed manifold. Indeed, it is easy to see that we can define a retraction $R : M^n \times M^{2m-n} \to M^n \times \{p\}$ for some $p \in M^{2m-n}$ with the desired properties. Observe that the torus $\mathbb{T}^{2m}$ belongs to this class of manifolds. One could also consider the connected sum $(M^n \times M^{2m-n})\# N^{2m}$, modifying the above retraction so that it is constant on N^{2m}.*

We can even deduce the following multiplicity result. Here, M_k^{R} are the formal barycenters of M^{R} according to (4.3) and $\widetilde{H}_q(M_k^{\mathrm{R}})$ denotes its reduced q-th homology group.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let $\kappa_{g_v} \in (k\Lambda_m, (k+1)\Lambda_m)$. Then, if $\mathcal{E}$ in* (4.1) *is a Morse functional,*

$$\#\{\textit{solutions of } (1.6)\} \geq \sum_{q\geq 0} \dim \widetilde{H}_q(M_k^R).$$

Remark 1.2. *Consider for example the class of manifolds $M^n \times M^{2m-n}$ in Remark 1.1. We will get an explicit lower bound on the number of solutions as far as we can explicitly estimate the homology groups of M_k^n. One can find such computations in* [19] *for general manifolds M^n, focusing on the cases $n = 2$ and $n = 4$. For some simple manifolds we can easily compute the homology groups. For example, if M^n is a* 2*-dimensional G-torus (connected sum of G tori), then we have at least $\frac{(N+G-1)!}{N!(G-1)!}$ solutions, see* [3].

The argument of the proof of the existence result is in the spirit of the celebrated min-max scheme of [20], extended to high dimensions by [40], jointly with some ideas of [3] to treat the singularities. Roughly speaking, the strategy is based on the study of the sublevels of the energy functional, in particular by showing the low sublevels are non-contractible. This is done by using improved versions of the singular Adams-Trudinger-Moser inequality. We will then overcome the complexity due to the singularities by retracting the manifold onto M^{R}, not containing the singular points. This leads us to study the low sublevels just by looking at functions concentrating on such submanifold which is enough to gain some non-trivial homology.

To conclude the min-max argument we would need some compactness property as the Palais-Smale conditions is not available in this setting. We thus use Struwe's monotonicity trick [44], which is by now a standard tool in this class of problems, to deduce the existence of a sequence of solutions u_k satisfying (1.6). We will then conclude by showing the following compactness result which actually holds for any $2m$-dimensional manifold and $\alpha_j > -1$.

Theorem 1.3. *Let u_k be a sequence of solutions of* (1.6) *with $\widetilde{Q} > 0$ and $\alpha_j > -1$ for $j = 1, \dots, N$. If*

$$Ker\{P_g^{2m}\} = \{constants\}, \quad \kappa_{g_v} \notin \Gamma,$$

then there exists a constant C independent of k such that

$$\|u_k\|_{L^\infty(M)} \leq C.$$

The latter result is a consequence of a quantization phenomenon of blowing-up solutions which is derived via Pohozaev-type inequalities in the spirit of [2, 5, 30].

The above analysis, together with Morse inequalities, allows us to deduce also the multiplicity result of Theorem 1.2.

Remark 1.3. *We conclude the introduction with the following observations.*

1. *The existence result is derived for the case $\alpha_j > 0$ for all j. In principle, the same strategy can be carried out for the case $\alpha_j \in (-1, 0)$. However, in this scenario we get a worse Adams-Trudinger-Moser inequality in Theorem 2.4 and this in turn affects the topology of the low sublevels in a non-trivial way, see for instance* [12]*. We postpone this study to a future paper.*

2. *The same analysis should work in the odd-dimensional case with some further technical difficulties, as explained in Section 5 of* [40]*. We will not discuss this case in the present paper.*

This paper is organized as follows: in Section 2 we collect some useful preliminary results, Section 3 is devoted to blow-up analyis and the proof of Theorem 1.3 and in Section 4 we carry out the min-max method to derive the existence and multiplicity results of Theorems 1.1 and 1.2. A Pohozaev-type identity is provided in the last section.

Notations:

$B_r^M(p)$	the ball centered at $p \in M$ with geodesic radius r on the manifold M.
$B_r(p)$	the ball centered at p with radius r in $\mathbb{R}^{2m}$.

2. Preliminary facts

In this section we recall briefly some known results which can be easily derived from the existing literature. Let p be a point in M and $B_r^M(p)$ be a neighborhood of p such that $B_r^M(p)$ is mapped by $\exp_p^{-1}$ diffeomorphically onto a neighborhood of $0 \in T_p(M)$, where $T_p(M)$ refers to the tangent space of p which can be identified with $\mathbb{R}^{2m}$. The local coordinates defined by the chart $(\exp_p^{-1}, B_r^M(p))$ are called normal coordinates with center p. In such coordinates, the Riemannian metric at the point p satisfies

$$g_{ij} = \delta_{ij},\ g_{ij,k} = 0,\ \Gamma^i_{jk} = 0, \quad \text{for all } i, j, k \in \{1, \cdots, 2m\}, \tag{2.1}$$

where Γ^i_{jk} stands for the Christoffel symbols. With the above preparation, we have

Lemma 2.1. *Let $-\Delta_g$ be the Laplace-Beltrami operator and p be any point of M. In normal coordinates at p, we have*

$$(-\Delta_g)^m u = (-\Delta)^m u + \mathcal{D}^{2m} u + \mathcal{D}^{2m-1} u, \tag{2.2}$$

where $\mathcal{D}^{2m}$ is a linear differential operator of order $2m$ whose the coefficients are $O(|x - p|^2)$ as x tends to p, while $\mathcal{D}^{2m-1}$ is a linear differential operator of order at most $2m-1$, and whose coefficients belong to $C^l_{\text{loc}}(\mathbb{R}^{2m})$ for all $l \geq 0$.

Proof. By the definition of Laplace-Beltrami operator, we have

$$-\Delta_g u = -\frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j u), \tag{2.3}$$

where g^{ij} is the inverse of g_{ij}. Using (2.1) we can write

$$-\Delta_g u = -g^{ij} \partial_{ij} u - \frac{\partial_i(\sqrt{\det g} g^{ij})}{\sqrt{\det g}} \partial_j u. = -g^{ij}\partial_{ij} u - \vartheta_j \partial_j u, \tag{2.4}$$

where ϑ_j is a smooth function. Based on (2.4), it is easy to see that in the final expression of $(-\Delta)_g^m$ the leading differential order is

$$g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_m j_m} \partial_{i_1 j_1 i_2 j_2 \cdots i_m j_m} \cdot,$$

where $i_a, j_b \in \{1, \cdots, 2m\}$, $\forall a, b \in \{1, \cdots, m\}$. While the left terms are order at $2m-1$ and the coefficients are smooth due to the exponential map is differentiable with arbitrary order. Consider the leading term, using (2.1), we see that

$$g^{ij}(x) = \delta_{ij}(x) + O(r^2), \quad \text{if } x \in B_r^M(p).$$

Therefore we can write

$$\prod_{a=1}^m g^{i_a j_a}(x) = \prod_{a=1}^m \delta_{i_a j_a} + O(r^2), \quad \text{if } x \in B_r^M(p).$$

As a consequence, we can write

$$g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_m j_m} \partial_{i_1 j_1 i_2 j_2 \cdots i_m j_m} \cdot = (-\Delta)^m \cdot + \mathcal{D}^{2m} \cdot, \tag{2.5}$$

with $\mathcal{D}^{2m}$ satisfies the property stated in the lemma. Then we finish the proof. $\square$

Remark 2.1. *Throughout the paper, when performing local computations, we may consider conformal normal coordinates, see* [11] *or* [29], *if needed. These are normal coordinates at a point x_0 for a metric $g_w = e^{2w}g$ with $\det(g_w) = 1$ in a small neighborhood of x_0 and other useful properties, for which we refer the interested reader to* [46]. *Observe that the differential operator P_g^{2m}, after this change of the metric, can be still expanded as the right hand side of* (2.2). *Indeed, $w(x) = O(d^2_{g_w}(x, x_0))$ and it is smooth in a small neighborhood of x_0. Moreover, by* (1.3) *we can write* (1.6) *as*

$$P^{2m}_{g_w} u = e^{-2mw} P_g u = e^{-2mw}(-Q^{2m}_g + \widetilde{Q} e^{2mu}),$$

which is equivalent to

$$e^{2mw} P^{2m}_{g_w} u = -Q^{2m}_g + \widetilde{Q} e^{2mu}.$$

Concerning the differential operator $e^{2mw}P^{2m}_{g_w}$, it is known that the leading order operator is $e^{2mw}(-\Delta_{g_w})^m$. Then, by using the above Lemma 2.1 and the asymptotic behavior of $w(x)$, we can write

$$e^{2mw}(-\Delta_{g_w})^m u = (-\Delta)^m u + \mathcal{P}^{2m} u + \mathcal{P}^{2m-1} u,$$

with $\mathcal{P}^{2m}$ and $\mathcal{P}^{2m-1}$ satisfying the same properties as $\mathcal{D}^{2m}$ and $\mathcal{D}^{2m-1}$ in Lemma 2.1.

In what follows, recall $\Lambda_m = (2m-1)!|\mathbb{S}^{2m}|$. We will need the following structural result on the Green functional of the operators under consideration, see Lemma 2.1 in [40].

Lemma 2.2. *Suppose $Ker\{P^{2m}_g\} = \{constants\}$. Then the Green function $G(x, y)$ of P^{2m}_g exists and has the following properties:*

(1) *For all $u \in C^{2m}(M)$ we have for $x \neq y \in M$*

$$u(x) - \bar{u} = \int_M G(x, y) P^{2m}_g u(y) dV_g(y), \quad \int_M G(x, y) P^{2m}_g dV_g(y) = 0,$$

$$P^{2m}_g G(x, p) = \delta_p - \frac{1}{|M|},$$

where $\bar{u}$ is the average of u.

(2) *The function*

$$G(x, y) = H(x, y) + K(x, y)$$

is smooth on $M \times M$, away from the diagonal. The function K extends to a $C^{2,\alpha}$ function on $M \times M$ and H satisfies

$$H(x, y) = \frac{2}{\Lambda_m} \log\left(\frac{1}{r}\right) f(r),$$

where r is the geodesic distance from x to y and f is a smooth positive, decreasing function such that $f(r) = 1$ in a neighborhood of $r = 0$ and $f(r) = 0$ for $r \geq inj_g(M)$.

As mentioned in the introduction, the total Q-curvature is a conformal invariant for which the following singular Chern-Gauss-Bonnet formula holds true.

Theorem 2.3. *Consider the divisor $D = \sum_{i=1}^{N} p_i\alpha_i$ where $p_i \in M$ and $\alpha_i > -1$. Let g be a smooth metric on M and $g_v = e^{-2mv}g$ be the conical metric representing the divisor as explained before* (1.4)*. Then, it holds*

$$\int_M Q^{2m}_{g_v}\, dvol_{g_v} = \int_M Q^{2m}_g\, dvol_g + \frac{\Lambda_m}{2}\sum_{i=1}^{N}\alpha_i. \tag{2.6}$$

Proof. The proof is a standard argument (see e.g. [22] for $m = 2$), using Lemmata 2.1 and 2.2. See also [10] for a more general result which implies this statement as a particular case. ☐

Finally, we state the general singular Adams-Trudinger-Moser inequality suitable to treat our problem. We focus here for simplicity on the case $P^{2m}_g \geq 0$ and refer to the discussion in [40] for the general case.

Theorem 2.4. *Consider the divisor $D = \sum_{i=1}^{N} p_i\alpha_i$ where $p_i \in M$ and $\alpha_i > -1$. Let $\widetilde{Q} > 0$ be as in* (1.7)*. Assume $P^{2m}_g \geq 0$ and $Ker\{P^{2m}_g\} = \{constants\}$. Then, there exists a constant $C = C(\alpha, M)$ such that for any $u \in H^m(M)$ we have*

$$\Lambda_m\Big(1 + \min_j\{\alpha_j, 0\}\Big)\log\int_M \widetilde{Q}e^{2m(u-\bar{u})}\, dvol_g \leq m\int_M uP^{2m}_g u\, dvol_g + C,$$

where $\bar{u}$ is the average of u.

Proof. The case without singularities is Proposition 2.2 in [40]. The conic case follows by the same approach as in [22]. ☐

3. Compactness property

In this section we shall prove the compactness result of Theorem 1.3. For simplicity of notation, there is no loss of generality to consider a blow-up sequence u_k to

$$P^{2m}_g u_k + Q^{2m}_g = \widetilde{Q}e^{2mu_k}, \tag{3.1}$$

where

$$\widetilde{Q} = Q^{2m}_{g_v}e^{-m\Lambda_m\sum_{j=1}^{N}\alpha_j G(x,q_j)} > 0, \quad \Lambda_m = (2m-1)!|\mathbb{S}^{2m}|.$$

Theorem 1.3 will follow by showing a concentration phenomenon:

$$\widetilde{Q}e^{2mu_k} \rightharpoonup \sum_{p\in\mathcal{B}}(1+\alpha_p)\Lambda_m\delta_p \quad \text{as} \quad k \to +\infty,$$

weakly in the sense of measures, where $\mathcal{B}$ is the blow up set of u_k,

$$\alpha_p = \begin{cases} 0, & \text{if} \quad p \notin \{q_1, \cdots, q_N\}, \\ \alpha_j, & \text{if} \quad p = q_j. \end{cases}$$

It follows that when blow-up occurs, then necessarily

$$\int_M \widetilde{Q}e^{2mu_k}\, dvol_g \to \sigma \in \Gamma \quad \text{as} \quad k \to +\infty,$$

where Γ is given in (1.8).

First, we establish the following lemma.

Lemma 3.1. *Let $\{u_k\}$ be a sequence of functions on (M,g) satisfying* (3.1). *Then for $i = 1, \cdots, 2m-1$ we have*

$$\int_{B_r^M(x)} |\nabla^i u_k|^l \, dy \le C(n) r^{2m-il}, \quad 1 \le l < \frac{2m}{i}, \, \forall x \in M, \, 0 < r < r_{inj}. \tag{3.2}$$

where r_{inj} is the injectivity radius of (M,g).

Proof. Set $f_k := \widetilde{Q} e^{2mu_k} - Q_g^{2m}$, which is bounded in $L^1(M)$. By Green's representation formula we have

$$u_k(x) = \fint_M u_k \, dvol_g + \int_M G(x,y) f_k(y) \, dvol_g(y). \tag{3.3}$$

For $x, y \in M$, $x \neq y$, we have (see [40, Lemma2.1])

$$|\nabla_y^i G(x,y)| \le \frac{C}{d_g(x,y)^i}, \quad 1 \le i \le 2m-1. \tag{3.4}$$

Then differentiating (3.3) and using (3.4) and Jensen's inequality, we get

$$\begin{aligned} |\nabla^i u_k(x)|^l &\le C \left(\int_M \frac{1}{d_g(x,y)^i} |f_k(y)| \, dvol_g \right)^l \\ &\le C \int_M \left(\frac{\|f_k\|_{L^1(M)}}{d_g(x,y)^i} \right)^l \frac{|f_k(y)|}{\|f_k\|_{L^1(M)}} \, dvol_g. \end{aligned}$$

From Fubini's theorem we conclude that

$$\int_{B_r^M(x)} |\nabla^i u_k(x)|^l \, dvol_g \le C \sup_{y \in M} \int_{B_r^M(x)} \frac{\|f_k\|_{L^1(M)}^l}{d_g(x,z)^{il}} \, dvol_g(z) \le C r^{2m-il}. \tag{3.5}$$

It proves the lemma. ∎

Next, we shall give the minimal local mass around a blow-up point.

Lemma 3.2. *Let the sequence u_k satisfy* (3.1) *and blowing-up at q_j. Suppose that*

$$\widetilde{Q} e^{2mu_k} \rightharpoonup \mathfrak{m}, \text{ weakly in the sense of measure in } M,$$

then

$$\mathfrak{m}(q_j) \ge \frac{1}{2} \min\{\Lambda_m(1+\alpha_j), \Lambda_m\}.$$

Proof. To show the thesis, it suffices to prove that if the following inequality holds

$$\int_{B^M(q_j,2r)} \widetilde{Q} e^{2mu_k} \, dvol_g < \frac{1}{2} \min\{\Lambda_m(1+\alpha_j), \Lambda_m\}, \quad r < \frac{r_{inj}}{2}, \tag{3.6}$$

then

$$u_k \le C \text{ in } B^M(q_j, r). \tag{3.7}$$

We study equation (3.1) in terms of the local normal coordinates at q_j (see for example Lemma 2.1 and Remark 2.1). By the exponential map we define the pre-image of $B^M(q_j, r)$ by $B_r(0)$ and we use the same notation to denote $x \in M$ and its pre-image. We decompose u_k as $u_k = u_{1k} + u_{2k}$, where u_{1k} is the solution of

$$\begin{cases} (-\Delta)^m u_{1k} = \widetilde{Q} e^{2mu_k} \Xi_r(x), & \text{in } B_{2r}(0), \\ u_{1k} = \Delta u_{1k} = \cdots = (-\Delta)^{m-1} u_{1k} = 0, & \text{on } \partial B_{2r}(0). \end{cases} \tag{3.8}$$

where $\Xi_r(x) = dvol_g(x)/dx = 1 + O(r^2)$ due to the metric tensor $g_{ij}(x) = \delta_{ij} + O(r^2)$. By [34, Theorem 7], we have

$$e^{2m\ell|u_{1k}|} \in L^1(B_{2r}(0)) \quad \text{for } \ell \in \left(0, \frac{\Lambda_m}{2\|\widetilde{Q} e^{2mu_k} \Xi_r(x)\|_{L^1(B_{2r}(0))}}\right) \tag{3.9}$$

and

$$\int_{B_{2r}(0)} e^{2m\ell|u_{1k}|} dx \le C(p) r^{2m}. \tag{3.10}$$

Let $G_r(x, y)$ be the Green's function of $(-\Delta)^m$ on $B_{2r}(0)$ satisfying the Navier boundary condition, i.e.,

$$\begin{cases} (-\Delta)^m G_r(x, y) = \delta_x(y), & \text{in } B_{2r}(0), \\ G_r(x, y) = \cdots = \Delta^{m-1} G_r(x, y) = 0, & \text{on } \partial B_{2r}(0). \end{cases}$$

$G_r(x, y)$ can be decomposed as

$$G_r(x, y) = -\frac{2}{\Lambda_m} \log|x - y| + R_r(x, y)$$

with $R_r(x, y)$ a smooth function for $x, y \in B_{2r}$. By the Green's representation formula we have

$$u_{1k}(x) = -\frac{2}{\Lambda_m} \int_{B_{2r}(0)} \log|x - y| \widetilde{Q} e^{2mu_k} \Xi_r(y) dy + O(1), \quad x \in B_{3r/2}(0). \tag{3.11}$$

Observe that

$$\widetilde{Q} = d_g(x, q_j)^{2m\alpha_j} \widehat{Q}$$

where

$$\widehat{Q} = Q_{g_v}^{2m} e^{-m\Lambda_m \alpha_j R(x, q_j) - m\Lambda_m \sum_{i \neq j}^N \alpha_i G(x, q_i)} \text{ is a smooth function in } B_{2r}(0). \tag{3.12}$$

On the other hand, by using $G(x, y)$ and the Green's representation formula we get that

$$u_k(x) = u_{1k}(x) + u_{2k}(x) = \overline{u}_k + \int_M G(x, y) \widetilde{Q} e^{2mu_k} \, dvol_g, \tag{3.13}$$

where $\overline{u}_k$ is the average of u_k. Since it is known that the leading term of G and G_{2r} carry the same singular potential, we get from Jensen's inequality that

$$u_{2k} = \overline{u}_k + O(1) = \overline{v}_k + O(1) \le \log \fint_M e^{v_k} + O(1) = O(1), \tag{3.14}$$

where we used the average of Green function on M is zero. Therefore, we conclude that u_{2k} is bounded uniformly from above. Next, we shall prove that u_{1k} is bounded and our discussion is separated into two cases:

Case 1. $\alpha_j > 0$, then (3.6) is equivalent to

$$\int_{B^M_{2r}(q_j)} \widetilde{Q} e^{2mu_k}\, dvol_g < \frac{1}{2}\Lambda_m.$$

Since u_{2k} is bounded from above in $B_{3r/2}(0)$, we see from (3.10) that there exists some $\ell > 1$ such that $\widetilde{Q}e^{2mu_k} \in L^\ell(B^M_{2r}(q_1))$. It is easy to see that $u_k \in L^1(M)$. Together with (3.11) we can easily see that $u_{1k}, u_{2k} \in L^1(B_{3r/2}(0))$. By the interior regularity results in [9, Theorem 1] we get that

$$\|u_{1k}\|_{W^{2m,\ell}(B_r(0))} \le \|\widetilde{Q}e^{2mu_k}\|_{L^\ell(B_{3r/2}(0))} + \|u_{1k}\|_{L^1(B_r(0))} \le C. \tag{3.15}$$

Thus by the classical Sobolev inequality we get that $u_{1k} \in L^\infty(B_r(0))$.

Case 2. If $\alpha_j \in (-1, 0)$ then (3.6) is equivalent to

$$\int_{B^M_{2r}(q_j)} \widetilde{Q} e^{2mu_k}\, dvol_g < \frac{1}{2}\Lambda_m(1+\alpha_j).$$

It is not difficult to see $|x|^{2m\alpha_j} \in L^\ell(B_{2r})$ for any $\ell \in [1, -\frac{1}{\alpha_j})$ and $e^{2mu_{1k}} \in L^p(B_{2r})$ for $p \in [1, \frac{1}{1+\alpha_1} + \epsilon)$ for some small strictly positive number ϵ by (3.9). As a consequence, we get that $|x|^{2m\alpha_1}e^{2mu_{1k}} \in L^\ell(B_{2r}(0))$ for some $\ell > 1$ by Hölder inequality. Repeating the arguments as in Case 1, we obtain that u_{1k} is bounded uniformly in $B_r(0)$.

After establishing that u_{1k} is bounded in $B_r(0)$, combining with (3.14) we derive that u_k is bounded above in $B^M_r(q_j)$. Then we finish the proof of this lemma. □

We shall derive now the quantization result and the concentration property of the bubbling solution.

Proposition 3.3. *Let $\{u_k\}$ be a sequence of solution to* (3.1) *and $\mathcal{B}$ be its blow-up set, then we have the following convergence in the sense of measures*

$$\widetilde{Q}e^{2mu_k} \rightharpoonup \sum_{p\in\mathcal{B}} \Lambda_m(1+\alpha_p)\delta_p, \quad \text{as } k \to +\infty, \tag{3.16}$$

where

$$\alpha_p = \begin{cases} \alpha_i, & \text{if} \quad p = q_i \in \{q_1, \cdots, q_N\}, \\ 0, & \text{if} \quad p \in \mathcal{B} \setminus \{q_1, \cdots, q_N\}. \end{cases}$$

In particular, $u_k \to -\infty$ uniformly on any compact subset of $M \setminus \mathcal{B}$.

Proof. For any compact set $K \subset M \setminus \mathcal{B}$, we can use the Green's representation formula

$$u_k(x) - u_k(y) = \int_M (G(x,z) - G(y,z)) \left(\widetilde{Q}e^{2mu_k} - Q^{2m}_g\right) dz \tag{3.17}$$

together with the estimate (3.4) to derive that

$$|\nabla^i u_k(x)| \leq C(K) \text{ for } x \in K, \quad 1 \leq i \leq 2m-1. \tag{3.18}$$

Then from equation (3.1) and classical elliptic estimates we get the $2m$ order derivatives of u_k

$$|\nabla^{2m} u_k(x)| \leq C(K) \text{ for } x \in K. \tag{3.19}$$

To proceed with our discussion we introduce the following quantity

$$\sigma_p = \lim_{r \to 0} \lim_{k \to +\infty} \int_{B^M(p,r)} \widetilde{Q} e^{2mu_k} \, dvol_g. \tag{3.20}$$

It has been shown in Lemma 3.2 that σ_p has a positive lower bound at the blow-up point. From the fact that $\int_M Q_g^{2m} \, dvol_g$ is finite, we conclude that the blow-up points are finite. At a regular blow-up point p it has been already shown in [35, Theorem 2] that

$$\widetilde{Q} e^{2mu_k} \rightharpoonup \Lambda_m \delta_p \quad \text{in} \quad B^M_{r_p}(p),$$

where r_p is chosen such that $B^M(p, r_p) \cap (\mathcal{B} \setminus \{p\}) = \emptyset$. In the following discussion we will focus on the singular blow-up point. Without loss of generality, we shall consider u_k in $B^M_{2r}(q_1)$, where r is chosen such that $B^M_{2r}(q_1)$ only contains q_1 from $\mathcal{B}$. We first claim that

$$u_k \to -\infty \quad \text{for} \quad x \in B^M_{2r}(q_1) \setminus \{q_1\}. \tag{3.21}$$

We prove it by contradiction. Suppose that u_k is uniformly bounded below at some point away from q_1. Then by (3.18) we derive that

$$u_k \to u_0 \quad \text{in} \quad C^{2m-1,\sigma}_{\text{loc}}(B^M_{2r}(q_1) \setminus \{q_1\}), \quad \sigma \in (0,1), \tag{3.22}$$

with the limit function verifying

$$P^{2m}_g u_0 + Q^{2m}_g = d_g(x, q_1)^{2m\alpha_1} \widehat{Q} e^{2mu_0} \quad \text{in} \quad B^M_{2r}(q_1) \setminus \{q_1\}. \tag{3.23}$$

where $\widehat{Q}$ is a smooth function around q_1 defined analogously as in (3.12) and $d_g(x, q_1)$ denotes the geodesic distance between x and q_1 with respect to the metric g. According to the definition σ_p (see (3.20)) we see that u_0 satisfies

$$P^{2m}_g u_0 + Q^{2m}_g = d_g(x, q_1)^{2m\alpha_1} \widehat{Q} e^{2mu_0} + \sigma_{q_1} \delta_{q_1} \quad \text{in} \quad B^M_{2r}(q_1).$$

Using the Green's representation formula for u_0, we have

$$u_0(x) = \sigma_{q_1} G(x, q_1) + v_0(x), \tag{3.24}$$

where

$$\begin{aligned} v_0(x) = & \int_M \frac{2}{\Lambda_m} \log d_g(x,y) \left(d_g(y, q_1)^{2m\alpha_1} \widehat{Q} e^{2mu_0} \right) dvol_g \\ & + \int_M R(x,y) \left(d_g(y, q_1)^{2m\alpha_1} \widehat{Q} e^{2mu_0} \right) dvol_g. \end{aligned} \tag{3.25}$$

Denoting the two terms on right hand side by $\hat{v}_1$ and $\hat{v}_2$ respectively, it is not difficult to see that $\hat{v}_2$ is smooth. In the following we shall prove that $\hat{v}_1(x)$ is bounded in $x \in B_r^M(q_1)$. In fact, for $x \in B_r^M(q_1)$ we have

$$\begin{aligned} v_0(x) &= \int_{B_r^M(q_1)} G(x,y) d_g(y,q_1)^{2m\alpha_1} \widehat{Q} e^{2mu_0}\, dvol_g + O(1), \\ &\geq \frac{2}{\Lambda_m} \log \frac{1}{r} \| \widehat{Q} |x|^{2m\alpha_1} e^{2mu_0} \|_{L^1(B_{2r}^M(q_1))} + O(1). \end{aligned} \tag{3.26}$$

This provides a lower bound for $v_0(x)$. On the other hand, we have

$$d_g(x,q_1)^{2m\alpha_1} e^{2mu_0} \geq C d_g(x,q_1)^{2m(\alpha_1 - \frac{2\sigma_{q_1}}{\Lambda_m})}. \tag{3.27}$$

Using the fact that the left hand side of (3.27) is integrable we get

$$\alpha_1 - \frac{2\sigma_{q_1}}{\Lambda_m} > -1. \tag{3.28}$$

When $\alpha_1 < 0$, we see that the above inequality (3.28) implies that $\sigma_{q_1} < \Lambda_m(1+\alpha_1)$. Then u_k can not blow-up at q_1 by Lemma 3.2 and we get a contradiction. This implies (3.21) for $\alpha_1 < 0$. For $\alpha_1 > 0$ we have

$$C d_g(x,q_1)^{2m\alpha_1 - \frac{4m\sigma_{q_1}}{\Lambda_m}} e^{v_0(x)} \geq d_g(x,q_1)^{2m\alpha_1} e^{2mu_0} \geq C d_g(x,q_1)^{2m\alpha_1 - \frac{4m\sigma_{q_1}}{\Lambda_m}}. \tag{3.29}$$

In order to show that $\hat{v}_1(x)$ is bounded in $B_r^M(q_1)$, we study $\hat{v}_1(x)$ in terms of local coordinates at q_1. Then $d_g(x,q_1)$ can be regarded as $|x^p - 0|$, where x^p denotes its pre-image of x under the exponential map at q_1. By a little abuse of notation, we still denote x^p by x. Then we notice that $\hat{v}_1(x)$ satisfies

$$(-\Delta)^m \hat{v}_1(x) = |x|^{2m\alpha_1} \widehat{Q} e^{2mu_0} \Xi(x) \quad \text{in} \quad B_{2r}(0), \tag{3.30}$$

where $\Xi(x) = \frac{dvol_g(x)}{dx}$ is bounded above and below in $B_{2r}(0)$ since the metric tensor is comparable to the standard Euclidean metric. Since we only consider the local behavior of $\hat{v}_1(x)$ in $B_r(0)$, by multiplying a cut-off function $\chi(x)$ with $\chi(x) = 1$ for $|x| \leq r$ and $\chi(x) = 0$ for $|x| \geq 2r$, we have $\tilde{v}_1(x) := \chi(x)\hat{v}_1(x)$ verifies

$$\begin{cases} (-\Delta)^m \tilde{v}_1(x) = |x|^{2m\alpha_1} \widehat{Q} e^{2mu_0} \Xi(x) + \Xi_0(x) & \text{in} \quad B_{2r}(0), \\ \\ \tilde{v}_1(x) = \Delta \tilde{v}_1(x) = \cdots = \Delta^{m-1} \tilde{v}_1(x) = 0 & \text{on} \quad \partial B_{2r}(0), \end{cases} \tag{3.31}$$

where $\Xi_0(x)$ is smooth in $B_{2r}(0)$. It is not difficult to see that

$$|x|^{2m\alpha_1} \widehat{Q} e^{2mu_0} \Xi(x) + \Xi_0(x) \in L^1(B_2 r(0)).$$

We decompose $|x|^{2m\alpha_1} \widehat{Q} e^{2mu_0} \Xi(x) + \Xi_0(x)$ into $P_1(x)$ and $P_2(x)$ with

$$\|P_1\|_{L^1(B_{2r}(0))} \leq \varepsilon \quad \text{and} \quad P_2 \in L^\infty(B_{2r}(0)). \tag{3.32}$$

Correspondingly we decompose $\tilde{v}_1$ into $\tilde{v}_{11}$ and $\tilde{v}_{12}$, where $\tilde{v}_{1i},\ i=1,2$ solve

$$\begin{cases} (-\Delta)^m \tilde{v}_{1i}(x) = P_i(x) & \text{in} \quad B_{2r}(0), \\ \\ \tilde{v}_{1i}(x) = \Delta\tilde{v}_{1i}(x) = \cdots = \Delta^{m-1}\tilde{v}_{1i}(x) = 0 & \text{on} \quad \partial B_{2r}(0). \end{cases} \tag{3.33}$$

For $\tilde{v}_{11}$, by [34, Theorem 7] we get that $e^{\frac{\Lambda_m}{2\varepsilon}\tilde{v}_{11}} \in L^1(B_{2r}(0))$. While for $\tilde{v}_{12}$, using the classical elliptic regularity theory we derive that $\tilde{v}_{12} \in L^\infty(B_{2r}(0))$. Together with (3.29) we can select ε sufficiently small such that

$$|x|^{2m\alpha_1}\widehat{Q}e^{2mu_0}\Xi(x) \in L^l(B_{2r}(0)) \quad \text{for some} \quad l>1.$$

Returning to equation (3.31) we apply the regularity theory to deduce that $\hat{v}_1 \in W^{2m,l}(B_{2r}(0))$. As a consequence, we have $v_0 \in W^{2m,l}(B_{2r}(0))$ and it implies that $|v_0| \le C$ for some constant C in $B_r(0)$ by the classical Sobolev inequality. Thus we have proved v_0 is bounded in $B_r^M(q_1)$. Together with (3.29) we get that

$$d_g(x,q_1)^{2m\alpha_1}\widehat{Q}e^{2mu_0} \sim d_g(x,q_1)^{2m(\alpha_1 - \frac{2\sigma_{q_1}}{\Lambda_m})} \quad \text{if} \quad \alpha_1 > 0. \tag{3.34}$$

Next we shall derive a contradiction by making use of the Pohzozaev identity. It is known that equation (3.1) can be written as

$$(-\Delta_g)^m u_k(x) + \mathcal{A}u_k + Q_g^{2m} = \widetilde{Q}e^{2mu_k} \quad \text{in} \quad B_r^M(q_1), \tag{3.35}$$

where $\mathcal{A}$ is a linear differential operator of order at most $2m-1$, moreover the coefficients of A belong to $C^l_{\text{loc}}(M)$ for all $l \ge 0$. Using the local normal coordinate, by Lemma 2.1 (see also Remark 2.1) we could write (3.35) as

$$(-\Delta)^m u_k + \mathcal{D}^{2m}u_k + \mathcal{C}u_k + Q_g^{2m} = \widetilde{Q}e^{2mu_k} \quad \text{in} \quad B_r(0), \tag{3.36}$$

where $\mathcal{D}^{2m}$ is a linear differential operator of order $2m$ and the coefficients are of order $O(|x|^2)$ with its derivative of arbitrary order smooth, while $\mathcal{C}$ is a linear differential operator of order at most $2m-1$, and the coefficients of B_k belong to $C^l_{\text{loc}}(\mathbb{R}^{2m})$ for all $l \ge 0$. Multiplying $x \cdot \nabla u_k$ on both sides, concerning the right hand side, we have

$$\begin{aligned} \text{R.H.S. of (3.36)} &= \frac{1}{2m}\int_{B_r(0)} x\cdot\nabla(|x|^{2m\alpha_1}\widehat{Q}e^{2mu_k})\,dx - \alpha_1\int_{B_r(0)}|x|^{2\alpha_1}\widehat{Q}e^{2mu_k}\,dx \\ &\quad - \frac{1}{2m}\int_{B_r(0)}(x\cdot\nabla\widehat{Q})|x|^{2m\alpha_1}e^{2mu_k}\,dx \\ &= \int_{\partial B_r(0)}|x|^{2\alpha_1+1}\widehat{Q}e^{2mu_k}\,ds - (\alpha_1+1)\int_{B_r(0)}|x|^{2\alpha_1}\widehat{Q}e^{2mu_k}\,dx \\ &\quad - \frac{1}{2m}\int_{B_r(0)}(x\cdot\nabla\widehat{Q})|x|^{2m\alpha_1}e^{2mu_k}\,dx \\ &\to -(1+\alpha_1)\sigma_{q_1} + o_r(1) \quad \text{as} \quad k\to+\infty. \end{aligned} \tag{3.37}$$

Next, we consider the left hand side of (3.36). At first, for the fourth term, we have

$$\left|\int_{B_r(0)} \langle x, \nabla u_k \rangle Q_g^{2m} dx\right| \le r \int_{B_r(0)} |\nabla u_k| dx \le Cr^{2m}, \tag{3.38}$$

where we used Lemma 3.1. Therefore

$$\lim_{r\to 0} \lim_{k\to+\infty} \left|\int_{B_r(0)} \langle x, \nabla u_k \rangle Q_g^{2m} dx\right| = 0. \tag{3.39}$$

For the third term, we have

$$\begin{aligned}\left|\int_{B_r(0)} \langle x, \nabla u_k \rangle \mathcal{C} u_k\right| &\le C \sum_{i=0}^{2m-1} \int_{B_r(0)} |x||\nabla^i u_k||\nabla u_k| dx \\ &\le \sum_{i=1}^{2m-1} \left(\int_{B_r(0)} |x|^{s_i} |\nabla u_k|^{s_i} dx\right)^{\frac{1}{s_i}} \left(\int_{B_r(0)} |\nabla^i u_k|^{t_i} dx\right)^{\frac{1}{t_i}} \\ &\quad + C \int_{B_r(0)} |\nabla u_k| dx,\end{aligned} \tag{3.40}$$

where we used $|x||u_k| \le C$ in $B_r(0)$,

$$t_i = \frac{2m}{i} - \delta \quad \text{and} \quad s_i = \frac{2m - \delta i}{2m - i - \delta i}, \qquad \delta \in \left(0, \frac{1}{2(2m-1)}\right).$$

From (3.24), (3.25) and since $\hat{v}_1$, $\hat{v}_2$ are bounded from above, we can see that $|x \cdot \nabla u_0| \le C$ in $B_r(0)$. Together with Lemma 3.1 we have

$$\int_{B_r(0)} |x|^{s_i} |\nabla u_0|^{s_i} dx \le Cr^{2m}, \quad \text{and} \quad \int_{B_r(0)} |\nabla^i u_0|^{t_i} dx \le Cr^{i\delta}. \tag{3.41}$$

As a consequence of (3.40) and (3.41), we see that

$$\lim_{r\to 0} \lim_{k\to+\infty} \left|\int_{B_r(0)} \langle x, \nabla u_k \rangle \mathcal{C} u_k\right| dx = 0. \tag{3.42}$$

For the second term, we have already seen that $v_0 \in W^{2m,l}(B_r(0))$ for some $l > 1$, then using (3.24), we see that

$$\int_{B_r(0)} x \cdot \nabla u_0 \mathcal{D}^{2m} u_0 \, dx \le C \int_{B_r(0)} \frac{1}{|x|^{m-2}} \, dx + Cr^2 \|v_0\|_{W^{2m,1}(B_r(0))} \le Cr^2.$$

It leads to

$$\lim_{r\to 0} \lim_{k\to+\infty} \left|\int_{B_r(0)} \langle x, \nabla u_k \rangle \mathcal{D}^{2m} u_k \, dx\right| = 0. \tag{3.43}$$

Therefore, from (3.39), (3.42) and (3.43) we get that except from the first on the left hand side of (3.36), the other terms vanish in the limit. It remains to study the term $\int_{B_r(0)} (-\Delta)^m u_k x \cdot \nabla u_k dx$. We shall only consider the case when m is even, the argument for the case m is odd goes almost the same. We set $m = 2m_0$. Using the Pohozaev identity(5.1), replacing f by $\frac{2\sigma_{q_1}}{\Lambda_m}\log|x|$ plus a smooth function, after direct computations we get that

$$\begin{aligned}
&\int_{B_r(0)}(-\Delta)^m u_0\langle x,\nabla u_0\rangle dx\\
&=\sum_{i=2}^{m_0}\int_{\partial B_r(0)}2^{2m}(m-1)!(m-1)!\left(1-\frac{i-1}{m-i}\right)\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}ds\\
&\quad+\sum_{i=1}^{m_0}\int_{\partial B_r(0)}2^{2m}(m-1)!(m-1)!\left(\frac{i-1}{m-i}-1\right)\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}ds\\
&\quad-\int_{\partial B_r(0)}2^{2m-1}(m-1)!(m-1)!\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}ds+o_r(1)\\
&\to -2^{2m-1}(m-1)!(m-1)!\frac{\sigma_{q_1}^2}{\Lambda_m^2}|\mathbb{S}^{2m-1}|\quad\text{as}\quad r\to 0.
\end{aligned}\tag{3.44}$$

It is known that we can write

$$\Lambda_m=(2m-1)!|\mathbb{S}^{2m}|=2^{2m-1}(m-1)!(m-1)!|\mathbb{S}^{2m-1}|.$$

Together with (3.37) and (3.44) we derive that

$$(1+\alpha_1)\sigma_{q_1}=\frac{\sigma_{q_1}^2}{\Lambda_m}.\tag{3.45}$$

Recalling also Lemma 3.2 we get

$$\sigma_{q_1}=(1+\alpha_1)\Lambda_m.\tag{3.46}$$

Returning to equation (3.34) we see that

$$d_g(x,q_1)^{2m\alpha_1}\widehat{Q}e^{2mu_0}\sim d_g(x,q_1)^{-2m-2m(1+\alpha_1)}\geq d_g(x,q_1)^{-2m},\tag{3.47}$$

which contradicts $d_g(x,q_1)^{2m\alpha_1}\widehat{Q}e^{2mu_0}\in L^1(B_r(0))$. Therefore, $u_k\to-\infty$ uniformly on any compact subset of $B_{2r}^M(q_1)\setminus\{q_1\}$.

It remains to show the quantization σ_{q_1} is exactly $\Lambda_m(1+\alpha_1)$. We consider the function $\hat{u}_k(x)=u_k(x)-c_k$, with $c_k=\fint_{\partial B_r(0)}u_k(x)\to-\infty$. As before, we can show that $\hat{u}_k(x)$ converges to some function $\hat{u}_0(x)$ in $C^{2m}_{\mathrm{loc}}(B_{2r}\setminus\{0\})$ and we can write

$$\hat{u}_0(x)=-\frac{2\sigma_{q_1}}{\Lambda_m}\log d_g(x,q_1)+\hat{v}(x).$$

Repeating the previous argument, again by the Pohozaev identity we derive that $\sigma_{q_1}=\Lambda(1+\alpha_1)$ and we finish the proof. $\square$

4. Existence result

In this section we are going to prove the existence and multiplicity results of Theorems 1.1 and 1.2. To make the exposition more transparent we assume hereafter for simplicity $P_g^{2m} \geq 0$. The general case can be treated by suitable adaptations, see Remark 4.1.

Solutions of (1.6) are critical points of the functional

$$\mathcal{E}(u) = 2m \int_M u P_g^{2m} u \, dvol_g + 4m \int_M \Big(Q_g^{2m} + \frac{\Lambda_m}{2|M|} \sum_{j=1}^N \alpha_j \Big) u \, dvol_g - 2\kappa_{g_v} \log \int_M \widetilde{Q} e^{2mu} \, dvol_g \tag{4.1}$$

with $u \in H^m(M)$, where we recall

$$\widetilde{Q} = Q_{g_v}^{2m} e^{-m\Lambda_m \sum_{j=1}^N \alpha_j G(x,q_j)} > 0, \quad \Lambda_m = (2m-1)! |\mathbb{S}^{2m}|$$

and

$$\kappa_{g_v} = \int_M \widetilde{Q} e^{2mu} \, dvol_g = \int_M Q_g^{2m} \, dvol_g + \frac{\Lambda_m}{2} \sum_{j=1}^N \alpha_j.$$

We point out we consider here

$$\alpha_j > 0 \quad \forall j = 1, \dots, N.$$

Then, in particular, the Adams-Trudinger-Moser inequality in Theorem 2.4 implies the functional $\mathcal{E}$ is coercive and bounded from below provided $\kappa_{g_v} < \Lambda_m$. Thus, existence of solutions in this subcritical case follows by direct method of calculus of variations.

In the supercritical case $\kappa_{g_v} > \Lambda_m$ the functional is unbounded from below and we need to apply a min-max method based on the topology of the sublevels of the functional

$$\mathcal{E}^a = \big\{ u \in H^m(M) : \mathcal{E}(u) \leq a \big\}.$$

The rough idea is that the low sublevels carry some non-trivial topology while the high sublevels are contractible and such change of topology jointly with the compactness property of Theorem 1.3 (provided $\kappa_{g_v} \notin \Gamma$) detects a critical point. The main step is the study of low sublevels which is done by improved version of the Adams-Trudinger-Moser inequality and suitable test functions.

Let us start by pointing out that once the Adams-Trudinger-Moser inequality is available (Theorem 2.4), then a standard argument yields improved versions of it under a spreading of the conformal volume $\widetilde{Q} e^{2mu}$ in, say, l disjoint subsets as expressed in (4.2). Somehow, it is possible to sum up localized versions of the inequality which are in turn based on cut-off functions and improve the Adams-Trudinger-Moser constant to roughly $l\Lambda_m$. We refer the interested readers for example to Lemma 4.1 in [40] and references therein.

Lemma 4.1. *Let $\delta, \theta > 0$, $l \in \mathbb{N}$ and $\Omega_1, \dots, \Omega_l \subset M$ be such that $d(\Omega_i, \Omega_j) > \delta$ for any $i \neq j$. Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \theta, L, M)$ such that for any $u \in H^m(M)$ such that*

$$\int_{\Omega_i} \frac{\widetilde{Q} e^{2mu}\, dvol_g}{\int_M \widetilde{Q}\, e^{2mu}\, dvol_g} \geq \theta \qquad \forall i \in \{1, \dots, l\} \tag{4.2}$$

it holds

$$l\Lambda_m \log \int_M \widetilde{Q} e^{2m(u-\bar{u})}\, dvol_g \leq (1+\varepsilon) m \int_M u P_g^{2m} u \, dvol_g + C,$$

where $\bar{u}$ is the average of u.

Improved inequalities then yield improved lower bounds on the functional $\mathcal{E}$. As a consequence, in the low sublevels $\widetilde{Q} e^{2mu}$ has to be concentrated in not too many different subsets, as shown in the following result.

Lemma 4.2. *Suppose $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, $\forall \varepsilon, r > 0$ there exists $L = L(\varepsilon, r) \gg 1$ such that $\forall u \in \mathcal{E}^{-L}$ there exist k points $\{p_1, \dots, p_k\} \subset M$ such that*

$$\int_{\cup_{i=1}^k B_r^M(p_i)} \frac{\widetilde{Q} e^{2mu}\, dvol_g}{\int_M \widetilde{Q}\, e^{2mu}\, dvol_g} \geq 1 - \varepsilon.$$

Proof. We sketch here the proof. Suppose the thesis is false. Then, using a standard covering argument as in Lemma 2.3 of [20] there exist $k+1$ disjoint subsets $\Omega_1, \dots, \Omega_{k+1} \subset M$ in which $\widetilde{Q} e^{2mu}$ is spread in the sense of (4.2). Therefore, applying the improved Adams-Trudinger-Moser inequality of Lemma 4.1 we would get a lower bound of the functional

$$\mathcal{E}(u) \geq 2m \left(1 - \frac{\kappa_{g_v}}{(k+1)\Lambda_m}(1+\varepsilon)\right) \int_M u P_g^{2m} u\, dvol_g + l.o.t.$$

By assumption $\kappa_{g_v} < (k+1)\Lambda_m$ and hence we can take a sufficiently small $\varepsilon > 0$ such that

$$1 - \frac{\kappa_{g_v}}{(k+1)\Lambda_m}(1+\varepsilon) \geq 0$$

which yields $\mathcal{E}(u) \geq -L$ for some $L \gg 1$. This is not possible since we were considering $u \in \mathcal{E}^{-L}$. $\square$

It is then convenient to describe the low sublevels by means of formal barycenters of M of order k, that is unit measures supported in at most k points on M, defined by

$$M_k = \left\{ \sum_{i=1}^k t_i \delta_{p_i} \, : \, \sum_{i=1}^k t_i = 1,\ t_i \geq 0,\ p_i \in M,\ \forall i = 1, \dots, k \right\}. \tag{4.3}$$

The idea is to use a projection within unit measures such that

$$\frac{\widetilde{Q} e^{2mu}\, dvol_g}{\int_M \widetilde{Q}\, e^{2mu}\, dvol_g} \mapsto \sigma \in M_k.$$

This is done exactly as in Proposition 3.1 of [20] by using Lemma 4.2 to get the following.

Proposition 4.3. *Suppose $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, there exists $L \gg 1$ and a projection $\Psi : \mathcal{E}^{-L} \to M_k$.*

Recall now that we are assuming there exists a retraction $R : M \to M^{\mathrm{R}}$ with $M^{\mathrm{R}} \subset M$ a closed n-dimensional submanifold, $n \in [1, 2m]$, such that $\alpha_j \notin M^{\mathrm{R}}$ for all $j = 1, \dots, N$. Let M_k^{R} be the set of formal barycenters of M^{R}. We can then define a map $\Psi_R : \mathcal{E}^{-L} \to M_k^{\mathrm{R}}$ simply by considering the composition

$$\mathcal{E}^{-L} \xrightarrow{\Psi} M_k \xrightarrow{R_*} M_k^{\mathrm{R}},$$

where R_* is the push-forward of measures induced by the retraction R. Therefore, we have the following result.

Lemma 4.4. *Suppose $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, there exists $L \gg 1$ and a continuous map $\Psi_R : \mathcal{E}^{-L} \to M_k^{R}$.*

The low sublevels are thus naturally described (at least partially) by M_k^{R}. As a matter of fact, we are going to construct a reverse map, mapping continuously M_k^{R} into $\mathcal{E}^{-L}$. This is done by suitable test functions on which the functional attains low values. The idea here is that, since M^{R} does not contain conical points q_j, we may consider a family of *regular* bubbles centered on M^{R}. We thus take a non-decreasing cut-off function χ_δ such that

$$\begin{cases} \chi_\delta(t) = t, & t \in [0, \delta], \\ \chi_\delta(t) = 2\delta, & t \geq 2\delta, \end{cases}$$

let $\lambda > 0$ and then define $\Phi : M_k^{\mathrm{R}} \to H^m(M)$ by

$$\Phi(\sigma) = \varphi_{\lambda,\sigma}, \quad \sigma = \sum_{i=1}^{k} t_i \delta_{p_i} \in M_k^{\mathrm{R}},$$

where

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{2m} \log \sum_{i=1}^{k} t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2(d(y, p_i))} \right)^{2m}.$$

Now, since we are considering bubbles centered on M^{R} which does not contain conical points, we can neglect the effect of the singularities and all the following estimates are carried out exactly as in the regular case, that is Lemma 4.5, Lemma 4.6 in [40] and references therein. To avoid technicalities, with a little abuse of notation we will write $o(1)$ to denote quantities which do not necessarily tend to zero but that can be made arbitrarily small.

Lemma 4.5. *Let $\varphi_{\lambda,\sigma}$ be as above. Then, for $\lambda \to +\infty$ it holds*

$$\int_M \varphi_{\lambda,\sigma} P_g^{2m} \varphi_{\lambda,\sigma}\, dvol_g \leq 2k\Lambda_m(1+o(1))\log\lambda,$$

$$\int_M \Big(Q_g^{2m} + \frac{\Lambda_m}{2|M|}\sum_{j=1}^N \alpha_j \Big) \varphi_{\lambda,\sigma}\, dvol_g = -\kappa_{g_v}(1+o(1))\log\lambda,$$

$$\log \int_M \widetilde{Q} e^{2m\varphi_{\lambda,\sigma}}\, dvol_g = O(1).$$

By the latter estimates we readily get the map we were looking for if we take $\kappa_{g_v} > k\Lambda_m$. Indeed, it is enough to observe that by Lemma 4.5 we have

$$\mathcal{E}(\Phi(\sigma)) \leq 4m(k\Lambda_m - \kappa_{g_v} + o(1))\log\lambda \to -\infty$$

as $\lambda \to +\infty$. Therefore, we can state the following result.

Proposition 4.6. *Suppose $\kappa_{g_v} > k\Lambda_m$ for some $k \in \mathbb{N}$. Then, for any $L > 0$ there exists $\lambda \gg 1$ such that $\Phi : M_k^{\mathrm{R}} \to \mathcal{E}^{-L}$.*

We are now in position to prove the existence result.

Proof of Theorem 1.1. Suppose $\kappa_{g_v} \in (k\Lambda_m, (k+1)\Lambda_m)$ for some $k \in \mathbb{N}$ and $\kappa_{g_v} \notin \Gamma$, where Γ is the critical set given in (1.8). The proof is based on a min-max argument relying on the set M_k^{R} which will keep track of the topological properties of the low sublveles of the functional $\mathcal{E}$, jointly with the compactness property in Theorem 1.3.

Step 1. Recalling Lemma 4.4, let $L \gg 1$ be such that there exists a continuous map $\Psi_R : \mathcal{E}^{-L} \to M_k^{\mathrm{R}}$. Then, by Proposition 4.6 we can take $\lambda \gg 1$ such that $\Phi : M_k^{\mathrm{R}} \to \mathcal{E}^{-L}$. Consider now the composition

$$\begin{array}{ccccc} M_k^{\mathrm{R}} & \xrightarrow{\Phi} & \mathcal{E}^{-L} & \xrightarrow{\Psi_R} & M_k^{\mathrm{R}} \\ \sigma & \mapsto & \varphi_{\lambda,\sigma} & \mapsto & \Psi_R\left(\frac{\widetilde{Q} e^{2m\varphi_{\lambda,\sigma}}\, dvol_g}{\int_M \widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, dvol_g} \right). \end{array}$$

It is not difficult to see that the latter composition is homotopic to the identity map on M_k^{R}. We have just to notice that, as $\lambda \to +\infty$, $\frac{\widetilde{Q} e^{2m\varphi_{\lambda,\sigma}}\, dvol_g}{\int_M \widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, dvol_g} \rightharpoonup \sigma$ in the sense of measures, that Ψ is a projection and that R is a retraction onto M^{R}. The homotopy is thus realized by letting $\lambda \to +\infty$. As a consequence, if we consider the induced maps between homology groups H_* we get that

$$H_*(M_k^{\mathrm{R}}) \hookrightarrow H_*\big(\mathcal{E}^{-L}\big) \text{ injectively.} \tag{4.4}$$

Now, since M^{R} is a closed manifold, it is well-known that M_k^{R} has non-trivial homology groups and hence, in particular, it is non-contractible. We refer the interested readers for example to Lemma 3.7 in [20] where the 4-dimensional case is considered. By the above discussion, this implies

$$\Phi(M_k^{\mathrm{R}}) \subset \mathcal{E}^{-L} \text{ is non-contractible.} \tag{4.5}$$

Step 2. We next consider the topological cone over M_k^{R} which is defined through the equivalence relation

$$\mathcal{C} = \frac{M_k^{\mathrm{R}} \times [0,1]}{M_k^{\mathrm{R}} \times \{0\}},$$

that is $M_k^{\mathrm{R}} \times \{0\}$ is collapsed to a single point which is the tip of the cone. We then define the min-max value

$$m = \inf_{f \in F} \sup_{\sigma \in \mathcal{C}} \mathcal{E}(f(\sigma)),$$

where

$$F = \Big\{ f : \mathcal{C} \to H^m(M) \text{ continuous } : f(\sigma) = \varphi_{\lambda,\sigma}, \ \forall \sigma \in \partial\mathcal{C} = M_k^{\mathrm{R}} \Big\},$$

which is non-empty since $t\Phi \in F$. Still by Proposition 4.6 we can take $\lambda \gg 1$ sufficiently large such that

$$\sup_{\sigma \in \partial\mathcal{C}} \mathcal{E}(f(\sigma)) = \sup_{\sigma \in M_k^{\mathrm{R}}} \mathcal{E}(\varphi_{\lambda,\sigma}) \leq -2L.$$

On the other hand, we claim that

$$m \geq -L.$$

To prove it, we just need to observe that $\partial\mathcal{C} = M_k^{\mathrm{R}}$ is contractible in $\mathcal{C}$ (by construction of the cone) and thus $\Phi(M_k^{\mathrm{R}})$ is contractible in $f(\mathcal{C})$ for any $f \in F$. Hence, we deduce by (4.5) that $f(\mathcal{C})$ can not be contained in $\mathcal{E}^{-L}$, which proves the claim.

We conclude that the functional $\mathcal{E}$ has a min-max geometry at the level m which in turn implies there exists a Palais-Smale sequence at this level.

Step 3. Since the Palais-Smale condition is not available in this framework, we can not directly pass to the limit to obtain a critical point. To overcome this problem we use the so-called monotonicity trick jointly with the compactness property in Theorem 1.3. This argument has been first introduced by Struwe in [44] and has been then applied by many authors, see for example [20, 40]. Therefore, we omit the details and just sketch the main ideas.

One considers a small perturbation $\mathcal{E}_\varepsilon$ of the functional so that the above min-max scheme applies uniformly. By using a monotonicity property of the perturbed min-max values m_ε it is possible to obtain a bounded Palais-Smale sequence which then converges to a solution of the perturbed problem. We then pass to the limit as $\varepsilon \to 0$ by using the compactness property in Theorem 1.3 to eventually recover a solution of the original problem. This concludes the proof. ☐

Finally, we present the proof of the multiplicity result in Theorem 1.2.

Proof of Theorem 1.2. Once the above analysis (needed to prove the existence result) is carried out, the multiplicity result is essentially a straightforward application of Morse inequalities. Thus, we will be sketchy and refer for example to [3, 19] for further details. Recall also that $\mathcal{E}$ is assumed to be a Morse functional. The (weak) Morse inequalities would assert that

$$\begin{aligned}\#\{\text{solutions of (1.6)}\} &\geq \sum_{q\geq 0} \#\{\text{critical points of } \mathcal{E} \text{ in } \{-L \leq \mathcal{E} \leq L\} \text{ with index } q\} \\ &\geq \sum_{q\geq 0} \dim H_q(\mathcal{E}^L, \mathcal{E}^{-L}),\end{aligned}$$

where $H_q(\mathcal{E}^L, \mathcal{E}^{-L})$ stands for the relative homology group of $(\mathcal{E}^L, \mathcal{E}^{-L})$, see for example Theorem 2.4 in [19]. Now, it is known that the high sublevels $\mathcal{E}^L$ are contractible. Roughly speaking, one can take $L \gg 1$ sufficiently large so that there are no critical points above the level L which then allows to construct a deformation retract of $\mathcal{E}^L$ onto $H^m(M)$, which is of course contractible, see for example the argument in [32]. Then, by the long exact sequence of the relative homology, it easily follows that

$$H_q(\mathcal{E}^L, \mathcal{E}^{-L}) \cong \widetilde{H}_q(\mathcal{E}^{-L}).$$

But we already now from (4.4) that

$$H_*(M_k^{\mathrm{R}}) \hookrightarrow H_*\big(\mathcal{E}^{-L}\big) \text{ injectively,}$$

thus

$$\dim \widetilde{H}_q(\mathcal{E}^{-L}) \geq \dim \widetilde{H}_q(M_k^{\mathrm{R}})$$

and we are done. □

Remark 4.1. *As already pointed out, for simplicity, all the argument has been carried out in the case $P_g^{2m} \geq 0$. In general, one needs to modify the Adams-Trudinger-Moser inequality and its improvements by adding a bound to the component u_- of the function u lying in the direct sum of the negative eigenspaces of P_g^{2m}. As a consequence, in the low subleveles $\mathcal{E}^{-L}$ either the function u concentrates or u_- tends to infinity, or both alternative can hold. To express this alternative one can use the topological join between M_k^R and a set representing the negative eigenvalues of P_g^{2m}. We refer the interested reader to* [20].

5. Appendix: Pohozaev identity

Here we state a Pohozaev-type identity which is used in the blow-up argument.

Lemma 5.1. *Let $B_r(0)$ be a ball in $\mathbb{R}^{2m}$, we have the following identities:*

(a). *If $m = 2m_0,\ m_0 \geq 1$, then*

(5.1)
$$\begin{aligned}
&\int_{B_r(0)} x \cdot \nabla f (-\Delta)^{2m_0} f dx \\
&= \sum_{i=2}^{m_0} \int_{\partial B_r(0)} 2(i-1) \left((-\Delta)^{m-i} f \frac{\partial (-\Delta)^{i-1} f}{\partial \nu} - \frac{\partial (-\Delta)^{m-i} f}{\partial \nu} (-\Delta)^{i-1} f \right) ds \\
&\quad + \sum_{i=1}^{m_0} \int_{\partial B_r(0)} \left((-\Delta)^{m-i} f \partial_\nu \langle x, \nabla (-\Delta)^{i-1} f \rangle - \langle x, \nabla (-\Delta)^{i-1} f \rangle \frac{\partial (-\Delta)^{m-i} f}{\partial \nu} \right) ds \\
&\quad + \int_{\partial B_r(0)} \frac{1}{2} |x| ((-\Delta)^{m_0} f)^2 ds.
\end{aligned}$$

(b). *If $m = 2m_0 + 1,\ m_0 \geq 1$, then*

(5.2)
$$\begin{aligned}
&\int_{B_r(0)} x \cdot \nabla f (-\Delta)^{2m_0+1} f \\
&= \sum_{i=2}^{m_0} \int_{\partial B_r(0)} 2(i-1) \left((-\Delta)^{m-i} f \frac{\partial (-\Delta)^{i-1} f}{\partial \nu} - \frac{\partial (-\Delta)^{m-i} f}{\partial \nu} (-\Delta)^{i-1} f \right) ds \\
&\quad + \sum_{i=1}^{m_0} \int_{\partial B_r(0)} \left((-\Delta)^{m-i} f \partial_\nu \langle x, \nabla (-\Delta)^{i-1} f \rangle - \langle x, \nabla (-\Delta)^{i-1} f \rangle \frac{\partial (-\Delta)^{m-i} f}{\partial \nu} \right) ds \\
&\quad + \int_{\partial B_r(0)} \frac{1}{2} |x| (\nabla (-\Delta)^{m_0} f)^2 ds - 2m_0 \int_{\partial B_r(0)} \partial_\nu (-\Delta)^{m_0} f (-\Delta)^{m_0} f ds \\
&\quad - \int_{\partial B_r(0)} \partial_\nu (-\Delta)^{m_0} f \langle x, \nabla (-\Delta)^{m_0} f \rangle ds.
\end{aligned}$$

Here ν is outward normal vector along the boundary $\partial B_r(0)$.

Proof. The proof is based on the following identity

$$(-\Delta) \langle x, \nabla (-\Delta)^i f \rangle = 2(-\Delta) f + \langle x, \nabla (-\Delta)^{i+1} f \rangle.$$

Using the second Green's identity repeatedly, we can get above formula by straightforward computations. We omit the details. □

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