# DYNAMICAL SYSTEMS AND HAMILTON-JACOBI-BELLMAN EQUATIONS ON THE WASSERSTEIN SPACE AND THEIR $L^2$ REPRESENTATIONS

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ABSTRACT. Several optimal control problems in $\mathbb{R}^d$, like systems with uncertainty, control of flock dynamics, or control of multiagent systems, can be naturally formulated in the space of probability measures in $\mathbb{R}^d$. This leads to the study of dynamics and viscosity solutions to the Hamilton-Jacobi-Bellman equation satisfied by the value functions of those control problems, both stated in the Wasserstein space of probability measures. Since this space can be also viewed as the set of the laws of random variables in a suitable $L^2$ space, the main aim of the paper is to study such control systems in the Wasserstein space and to investigate the relations between dynamical systems in Wasserstein space and their representations by dynamical systems in $L^2$, both from the points of view of trajectories and of (first order) Hamilton-Jacobi-Bellman equations.

**Keywords:** Optimal control, optimal transport, Hamilton-Jacobi-Bellman equation, multiagent systems.
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## INTRODUCTION

During the last years, there has been an increasing interest in the control of the so-called multiagent systems. Such systems modelize dynamics where the number of interacting agents is so huge that only a statistical description is available. Under an assumption of *indistinguishability* of the agents, instead of studying the evolution of each invidual agent, it is preferable to consider the macroscopic evolution of a probability measure describing the fraction of the total amount of agents belonging to every set of the state space at each time. Such dynamics of measures naturally appear for instance in control systems or differential games with uncertainty [39], [19], [20], [22], [35], [36], in mean fields games [7], [38], [21], [24], [29], [32], in flock dynamics (see e.g. [41]), in pedestrian and vehicles dynamics (see e.g. [1] and references therein for an overview of the models).

We consider a multiagent controlled dynamical system at two levels

• The *microscopic scale*. Every agent, whose instantaneous position at time $t$ is $x(t) \in \mathbb{R}^d$, can choose his velocity in a set which depends on its own position and on a probability measure $\mu_t$ on $\mathbb{R}^d$ which describes the current distribution of all the other agents. For every (Borel) subset $A \subset \mathbb{R}^d$, $\mu_t(A)$ represents the fraction of the total number of agents that are present in $A$ at time $t$. In particular, the trajectory $x(\cdot)$ satisfies an equation of the form

$$\dot{x}(t) = f(t, x(t), u(t), \mu_t) \text{ for almost every } t \in [0, T],$$

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where $f : [0,T] \times \mathbb{R}^d \times U \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $u(\cdot) : [0,T] \mapsto U$ is the control function, $U$ is a subset of some finite dimensional space, and $\mathscr{P}_2(\mathbb{R}^d)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ with finite second order moment.

Notice that the case where $f$ is independent on the $\mu$-variable reduces to classical control dynamics. It is worth pointing out that in this model the indistinguishability assumption is encoded in the fact that, as expressed by the dynamics, each agent at position $x(t)$ does not interact individually with every other agents, which are indistinguable for him, but he interacts only with the total crowd of all the agents as an aggregate represented by the measure $\mu_t$. Throughout the paper we do not need an explicit form of the control, so we introduce the set-valued map $F(t,\mu,x) := \{(f(t,x,u,\mu),\ u \in U\}$, and we consider the microscopic dynamic satisfied by the trajectory $x(\cdot)$

$$\dot{x}(t) \in F(t,\mu_t,x(t)) \qquad \text{for almost every } t. \tag{1}$$

• The *macroscopic scale.* The probability measure $t \mapsto \mu_t$ evolves according the so called continuity equation

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0, \qquad \text{in the sense of distributions,} \tag{2}$$

which expresses that the total mass of the measure $\mu_t$ is preserved during the evolution (so the curve $t \mapsto \mu_t$ remains in the space of probability measures) and $v_t(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a time dependent vector field. The above continuity equation must be understood in the sense of distributions.

• The *link* between the macroscopic and the microscopic evolution is given by the vector field $v_t(\cdot)$ which has to satisfy

$$v_t(x) \in F(t,\mu_t,x) \text{ for } \mu_t\text{-almost every } x \in \mathbb{R}^d \text{ and for almost every } t. \tag{3}$$

which is constructed by taking the weighted average of the velocies of all the agent concurring in time $t$ at position $x$. Roughly speaking, this relation means that every point of the support of macroscopic variable $\mu_t$ has to evolve according to the microscopic scale equation. A different approach to dynamics in Wasserstein space is discussed in Remark 2.12.

Together with the above dynamical system, we consider an optimization problem of Bolza type, i.e., the minimization of a functional

$$\int_s^T \mathcal{L}(\mu_t)\,dt + \mathscr{G}(\mu_T) \in \mathbb{R} \cup \{+\infty\}, \tag{4}$$

on trajectories satisfying the above dynamical system with an initial datum $\mu_s = \mu$. It is natural to associate to this optimal control problem a value function obeying a dynamic programming principle, and one can expect to characterize it as the unique solution of a first-order Hamilton-Jacobi-Bellman equation (HJB in short) in the space of probability measures. Of course, since the value function is not smooth in general, a convenient notion of viscosity solution is needed to study this problem.

The study of a first order partial differential equation like HJB on the space of probability measures, which is not a normed space, is not an easy task. We focus now on two main ways. A direct approach requires to define suitable derivatives and sub/super-differentials for real value functions defined on the Wasserstein space; we refer the reader to various concepts in [3], [19], [4], [33], [37], [39]. Another possibility, commonly used for instance in the mean field theory, relies on the fact that any probability measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ could be represented as the law of a random variable $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ defined on some atomless probability space $(\Omega, \mathscr{B}(\Omega), \mathbb{P})$ equipped with its Borel $\sigma$-field (or equivalently $\mu$ is the image measure of the probability $\mathbb{P}$ by the function $X : \Omega \mapsto \mathbb{R}^d$: we will denote this by $X\sharp\mathbb{P} = \mu$). This allows to study derivatives and sub/super-differential for real valued functions defined on the Hilbert space $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, because a function $u : \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is immediately "lifted" to a function $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \mapsto \mathbb{R}$ defined by $U(X) := u(X\sharp\mathbb{P})$. We refer the reader to [21], [24], [33], [38]. By construction, $U(X)$ depends only on the law of $X$. A general function from $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ to $\mathbb{R}$ having this property is called *rearrangement invariant.*

In the framework of multiagent control problems, the above "representation" of a measure of $\mathscr{P}_2(\mathbb{R}^d)$ by random variables in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ leads to several questions. An immediate observation lies in the fact that the representation of the measure by a $L^2$ function is not unique: even if we fix from the beginning the probability space $(\Omega, B(\Omega), \mathbb{P})$, the same measure has multiple representatives in $L^2_{\mathbb{P}}(\Omega)$. One important contribution of the mean field approach lies on the fact that when the lift $U$ is smooth enough, the derivative at $X$ of $U$ depends only of the law $X\sharp\mathbb{P}$ of random variable $X$ (cf e.g [21], [24], [33]). However, the general validity of an analogous result for sub/super-subdifferential of nonsmooth function is not yet fully clear. Consequently, the comparison between viscosity solutions defined on $\mathscr{P}_2(\mathbb{R}^d)$ and viscosity solutions defined on $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ appears to be not straightforward. Another important question concerns the properties of the absolutely continuous curves in the two spaces: can any absolutely continuous trajectory in the Wasserstein space be represented by an absolutely continuous curves in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$? Conversely, do the laws of any absolutely continuous curve of random variables in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ provide an absolutely continuous trajectory in the Wasserstein space? Is it possible to establish quantitative estimates on the distance between a given absolutely continuous curve in $\mathscr{P}_2(\mathbb{R}^d)$ and the set of admissible trajectories of the dynamics in $\mathscr{P}_2(\mathbb{R}^d)$?

The goal of the present paper is to investigate the previous questions.

Before going further, we give an academic example of a multiagent evolution in the Wasserstein space which is not easily represented by an evolution in the $L^2_{\mathbb{P}}$ space. We define its microscopic dynamic as

$$\dot{x}(t) \in F(\mu) := B(0, \phi(\mu_t)) \text{ for almost every } t, \tag{5}$$

where $\phi : \mathscr{P}_2(\mathbb{R}^d) \mapsto [0, +\infty[$ is given by

$$\phi(\mu) = 1 \text{ if } \delta \leq \frac{\mu}{\mathscr{L}^d} \leq 1 \text{ and } \phi(\mu) = 0 \text{ otherwise,}$$

and $\frac{\mu}{\mathscr{L}^d}$ denotes the density - when it exists - of the measure $\mu$ with respect to the Lebesgue measure $\mathscr{L}^d$ on $\mathbb{R}^d$, and $\delta > 0$ is a fixed real number. The multi agent system is described by the above microscopic dynamics together with the macroscopic one (2) and the coupling (3). This could model, for instance, dynamics which are "frozen" as soon as the "density" becomes too big or not big enough, preventing the point to move in these cases. Clearly this kind of dynamics cannot easily be represented by a dynamics in $L^2_{\mathbb{P}}$ as we will discuss later on, see Remark 2.14.

From the point of view of trajectories in the Wasserstein and in the $L^2_{\mathbb{P}}$ spaces, our first main result says that an absolutely continuous curve in the Wasserstein space provides an absolutely continuous curve in $L^2_{\mathbb{P}}$ and conversely. We prove also that the $L^2_{\mathbb{P}}$ representation obeys an ordinary differential equation in $L^2_{\mathbb{P}}$ related to the vector field $v$ appearing in (2). In the framework of the multiagent control problems, the last one is a result of independent interest. However, given the above curve satisfying the above ordinary differential equation in $L^2_{\mathbb{P}}$, if we take another curve in $L^2_{\mathbb{P}}$ which points has the same law for any time, the second curve does not satisfy in general the differential equation. This somehow explains the limitations of representing trajectories of a dynamical system in the Wasserstein space by a dynamical system in $L^2_{\mathbb{P}}$.

From the point of view of optimal control of the multiagent system, an important issue is to prove a minimal regularity result (Lipschitz continuity) for the value function. This is usually done by a Grönwall-Filippov result. We provide a result of Filippov type, showing that any absolutely continuous curve $t \mapsto \mu_t$ in the Wasserstein space, can be approached by a trajectory of the multiagent system with a suitable quantitative estimate, by adapting a similar result holding for curves in the $L^2_{\mathbb{P}}$ space.

Concerning HJB equations, the value function associated to the multiagent system is expected to satisfy an HJB in a viscosity sense. As usual in control theory, a proper definition of viscosity solution must allow to prove a comparison theorem, and, consequently, to characterize the value function as the unique solution of an HJB equation. Indeed, the relevance of the notion of solution to an HJB lies precisely in the possibility of obtaining a comparison theorem. There are several available notions of viscosity solution defined directly in the Wasserstein space [3], [4], [19], [37], [39]. Others approaches consider a concept of viscosity solution through the representation in a $L^2_{\mathbb{P}}$ space [21], [33]: the nice structure of $L^2_{\mathbb{P}}$ allows to use the viscosity theory in Banach spaces [27], [28], where a definition of viscosity solution with smooth test functions is available. Both in Wasserstein and $L^2_{\mathbb{P}}$ spaces, some comparison theorems for HJB equations have been obtained in the quoted literature (an analysis of these comparison theorems is out of the aims of the present paper).

In analogy with the classical theory, given $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\varepsilon > 0$ it is possible to introduce a concept of $\varepsilon$-super/subtangent test function to $u(\cdot)$ at $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$: namely, $v : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is an $\varepsilon$-supertangent to $u(\cdot)$ at $\mu_0$ if $v$ is continuous, differentiable at $\mu_0$, $v(\mu_0) = u(\mu_0)$ and $u(\nu) \leq v(\nu) + \varepsilon W_2(\nu, \mu_0)$ in a neighborhood of $\mu_0$ (an analogous definition holds for $\varepsilon$-subtangent).

Applying the same idea to the lifted version $U(\cdot)$ of $u(\cdot)$, we can consider $\varepsilon$-super/subtangent test functions $V(\cdot)$ to $U(\cdot)$ at $X_0 \in L^2_{\mathbb{P}}(\Omega)$ with $X_0 \sharp \mathbb{P} = \mu_0$. Of course, a natural requirement for the consistency of the construction is to ask that $V(\cdot)$ is *rearrangement invariant.*

As usual, the notion of $\varepsilon$-sub/supertangency can be used as an alternative way to give a notion of viscosity solution for HJB equations, in $\mathscr{P}_2(\mathbb{R}^d)$ and in $L^2_{\mathbb{P}}(\Omega)$, respectively. Thus it is a natural question to compare this notion with the other ones defined by using sub/superdifferentials.

Our second main results says that, under minimal assumptions of the Hamiltonian, the first notion of viscosity sub/super solutions provided by using $\varepsilon$-sub/supertangent in $\mathscr{P}_2(\mathbb{R}^d)$, the second one provided by lifting HJB and using smooth rearrangement invariant $\varepsilon$-sub/supertest functions in $L^2_{\mathbb{P}}(\Omega)$, and the third one provided in [39] and [37] using a notion of $\varepsilon$-intrinsic sub/superdifferential, are all equivalent.

Throughout the paper we make the following simplification : although the value function associated with (4) and the dynamics (1), (2), (3) clearly depends both on the initial time $s$ and the initial measure $\mu$, we consider only the dependence in the $\mu$ variable. We will proceed as if the value would depend on $\mu$ only. This makes many expositions simpler and also the HJB equation be considered as if it was stationary.

The paper is organized as follows: in Section 1 we give some notation and background. Section 2 is devoted to trajectories of the multiagent control problem in the Wasserstein space and their $L^2_{\mathbb{P}}$ counterpart. Section 3 concerns viscosity solution to the HJB equations. In the last section we discuss the relevance to study a HJB equation either in $W^2$ or in $L^2$. We postponed to Appendix A some basic results and technical proofs to maintain the flow of the paper.

## 1. Preliminaries

1.1. **Definitions and Notations.** We will use the following notation.

| | |
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| $B(x,r)$ | (or $B_r(x)$) the open ball of radius $r$ of a metric space $(X, d_X)$; |
| $\overline{K}$ | the closure of a subset $K$ of a topological space $X$; |
| $I_K(\cdot)$ | the indicator function of $K$, i.e. $I_K(x) = 0$ if $x \in K$, $I_K(x) = +\infty$ if $x \notin K$; |
| $\chi_K(\cdot)$ | the characteristic function of $K$, i.e. $\chi_K(x) = 1$ if $x \in K$, $\chi_K(x) = 0$ if $x \notin K$; |
| $d_K(\cdot)$ | the distance function from a subset $K$ of a metric space $(X, d)$, i.e. $d_K(x) := \inf\{d(x,y) : y \in K\}$; |
| $C^0_b(X;Y)$ | the set of continuous bounded function from a Banach space $X$ to $Y$, endowed with $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$ (if $Y = \mathbb{R}$, $Y$ will be omitted); |
| $C^0_c(X;Y)$ | the set of compactly supported functions of $C^0_b(X;Y)$, with the topology induced by $C^0_b(X;Y)$; |
| $C^\infty_c(X;Y)$ | the space of smooth real functions with compact support in $\mathbb{R}^d$; |
| $\Gamma_I$ | the set of continuous curves from a real interval $I$ to $\mathbb{R}^d$; |
| $\Gamma_T$ | the set of continuous curves from $[0,T]$ to $\mathbb{R}^d$; |
| $e_t$ | the evaluation operator $e_t : \mathbb{R}^d \times \Gamma_I$ defined by $e_t(x,\gamma) = \gamma(t)$ for all $t \in I$; |
| $\mathscr{P}(X)$ | the set of Borel probability measures on a Banach space $X$, endowed with the weak$^*$ topology induced from $C^0_b(X)$; |
| $\mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ | the set of vector-valued Borel measures on $\mathbb{R}^d$ with values in $\mathbb{R}^d$, endowed with the weak$^*$ topology induced from $C^0_c(\mathbb{R}^d;\mathbb{R}^d)$; |
| $\|\nu\|$ | the total variation of a measure $\nu \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$; |
| $\ll$ | the absolutely continuity relation between measures |
| $\mathrm{m}_2(\mu)$ | the second moment of a probability measure $\mu \in \mathscr{P}(X)$; |
| $r\sharp\mu$ | the push-forward of the measure $\mu$ by the Borel map $r$; |
| $\mu \otimes \pi^x$ | the product measure of $\mu \in \mathscr{P}(X)$ with the Borel family of measures $\{\pi^x\}_{x\in X} \subseteq \mathscr{P}(Y)$ (see (50)); |
| $\mathrm{pr}_i$ | the $i$-th projection map $\mathrm{pr}_i(x_1,\dots,x_N) = x_i$; |
| $\Pi(\mu,\nu)$ | the set of admissible transport plans from $\mu$ to $\nu$; |
| $\Pi_o(\mu,\nu)$ | the set of optimal transport plans from $\mu$ to $\nu$; |
| $W_2(\mu,\nu)$ | the 2-Wasserstein distance between $\mu$ and $\nu$; |
| $\mathscr{P}_2(X)$ | the subset of the elements $\mathscr{P}(X)$ with finite second moment, |

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|---|---|
| | endowed with the 2-Wasserstein distance; |
| $\mathscr{L}^d$ | the Lebesgue measure on $\mathbb{R}^d$; |
| $\dfrac{\nu}{\mu}$ | the Radon-Nikodym derivative of the measure $\nu$ w.r.t. the measure $\mu$; |
| $\mathrm{Lip}(f)$ | the Lipschitz constant of a function $f$. |

Given a metric space $(X, d_X)$, an interval $I$ of $\mathbb{R}$, $p \geq 1$, we define

$$AC^p(I;X) := \Big\{\gamma : I \to X : \text{ there exists } m(\cdot) \in L^p(I) \text{ such that for all } s,t \in I \text{ with } s \leq t, \text{ it holds } d_X(\gamma(t),\gamma(s)) \leq \int_s^t m(\tau)\,d\tau\Big\}.$$

Given $\gamma \in AC^p(I;X)$, the *metric derivative* of $\gamma$ at $\tau$ is defined as

$$|\dot\gamma|(\tau) := \lim_{h\to 0} \frac{d_X(\gamma(\tau+h),\gamma(\tau))}{|\tau|}.$$

By Lebesgue Theorem, this limit exists at a.e. $\tau \in I$. Moreover, $|\dot\gamma|(\cdot)$ is the smallest function $m(\cdot)$ such that the inequality

$$d_X(\gamma(t),\gamma(s)) \leq \int_s^t m(\tau)\,d\tau$$

holds for every $s, t \in I$, $s \leq t$ (see [3] for further properties of metric derivative).

Given Banach spaces $X, Y$, we denote by $\mathscr{P}(X)$ the set of Borel probability measures on $X$ endowed with the weak$^*$ topology induced by the duality with the space $C^0_b(X)$ of the real-valued continuous bounded functions on $X$ with the uniform convergence norm. The second moment of $\mu \in \mathscr{P}(X)$ is denoted by $\mathrm{m}_2(\mu) = \int_X \|x\|^2_X\,d\mu(x)$, and we set $\mathscr{P}_2(X) = \{\mu \in \mathscr{P}(X) : \mathrm{m}_2(\mu) < +\infty\}$. For any Borel map $r : X \to Y$ and $\mu \in \mathscr{P}(X)$, we define the *push forward measure* $r\sharp\mu \in \mathscr{P}(Y)$ by setting $r\sharp\mu(B) = \mu(r^{-1}(B))$ for any Borel set $B$ of $Y$. The Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$ is equipped with the quadratic Wasserstein distance defined by for $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$

$$W_2(\mu,\nu) := \min_{\pi\in\Pi(\mu,\nu)} \left\{ \left( \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^2\,d\pi(x,y) \right)^{1/2} \right\}$$

where $\Pi(\mu,\nu) = \{\pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d),\ \mathrm{pr}_1\sharp\pi = \mu,\ \mathrm{pr}_2\sharp\pi = \nu\}$ is the set of *transport plans* between $\mu$ and $\nu$. We also denote by $\Pi_o(\mu,\nu)$ the set of *optimal transport plans* between $\mu$ and $\nu$, namely, the set of $\pi \in \Pi(\mu,\nu)$ achieving the mininimum in the above definition of $W_2(\mu,\nu)$. Recall that $\mathscr{P}_2(\mathbb{R}^d)$ endowed with the $W_2$-distance is a complete separable metric space.

1.2. **Basic facts on the Wasserstein space and an $L^2$ representation.** We fix some probability space $(\Omega, \mathscr{B}(\Omega), \mathbb{P})$ with $\Omega$ a Polish (metrizable, complete, separable) space, $\mathscr{B}(\Omega)$ its Borel $\sigma$-field, and $\mathbb{P}$ a probability measure with no atom. We denote by $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ (or $L^2_{\mathbb{P}}$ in short) the space of square integrable functions $X : \Omega \mapsto \mathbb{R}^d$ on the probability space $(\Omega, \mathscr{B}(\Omega), \mathbb{P})$.

We recall that for any $\mu$ in $\mathscr{P}_2(\mathbb{R}^d)$, there exists $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that $X\sharp\mathbb{P} = \mu$ (cf e.g. [42]) and

$$W_2(\mu,\nu) = \min\left\{ \|X - Y\|_{L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)} : X\sharp\mathbb{P} = \mu,\ Y\sharp\mathbb{P} = \nu \right\}. \tag{6}$$

The space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ can be identified with the quotient $(L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)/\sim)$ equipped with the quotient topology for the following equivalence relationship (cf appendix):

$$X \sim Y \Leftrightarrow X\sharp\mathbb{P} = Y\sharp\mathbb{P}.$$

We will make a constant use of the following

**Lemma 1.1.** *(Lemma 5.23 in* [24]*) Let $X, Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $X\sharp\mathbb{P} = Y\sharp\mathbb{P}$. Then, for any $\varepsilon > 0$, there exists $\tau : \Omega \to \Omega$ bijective satisfying:*
(i) *$\tau$ and $\tau^{-1}$ are measure-preserving that is $\tau\sharp\mathbb{P} = \tau^{-1}\sharp\mathbb{P} = \mathbb{P}$,*
(ii) $\|Y - X \circ \tau\|_{L^\infty_{\mathbb{P}}(\Omega, \mathbb{R}^d)} \leq \varepsilon$.

To a function $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ we associate its lift on $L^2_{\mathbb{P}}$ given by [38], [21], [24]

$$U : X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to u(X\sharp\mathbb{P}) \in \mathbb{R}.$$

By Corollary A.5, $u$ is continuous if and only if $U$ is continuous. Moreover $U$ is *rearrangement invariant* or *law dependent*. More precisely a map $V : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ is said to be *rearrangement invariant* if

$$\text{for all } (X, Y) \in (L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d))^2, \text{ it holds: } X\sharp\mathbb{P} = Y\sharp\mathbb{P} \Rightarrow V(X) = V(Y).$$

1.3. **About curves in the Wasserstein space.** We give some basic statements related to the dynamics of the macroscopic evolution.

Given a Borel vector field $(t, x) \mapsto v_t(x) \in \mathbb{R}^d$ such that

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x) \, dt < +\infty, \tag{7}$$

a continuous curve $\mu_t : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ is a solution to the continuity equation

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \text{ in } \mathbb{R}^d \times ]0, T[ \tag{8}$$

if an only if it holds in the sense of distributions on $[0, T] \times \mathbb{R}^d$ namely

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \varphi(t, x) + v_t(x) \cdot \nabla_x \varphi(x, t)) \; d\mu_t(x) \; dt = 0, \quad \forall \varphi \in C^\infty_c(\mathbb{R}^d \times ]0, T[)),$$

or equivalently in the sense of distributions in $[0, T]$ (see (8.1.3) in [3])

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t(x) = \int_{\mathbb{R}^d} \langle \nabla \varphi(x), v_t(x) \rangle \, d\mu_t(x), \text{ for all } \varphi \in C^1_c(\mathbb{R}^d).$$

According to Theorem 8.3.1 in [3], a continuous $\boldsymbol{\mu} = \{\mu_t\}_{t \in [0,T]} \in AC^2([0, T]; \mathscr{P}_2(\mathbb{R}^d))$ if and only if there exists a Borel vector field $v = v_t(x)$ satisfying (7) such that (8) holds.

We first recall the following useful result concerning solutions to the continuity equation (2) and their equivalent representation by a probability measure on $\mathbb{R}^d \times \Gamma_T$ where $\Gamma_T$ denotes the set of continuous functions from $[0, T]$ to $\mathbb{R}^d$.

**Proposition 1.2.** *[Superposition Principle cf. Theorem 8.2.1 of* [3]*] Consider $\mu_t : [0, T] \to \mathscr{P}_2(\mathbb{R}^d)$ a continuous solution of* (8) *for a Borel vector field $(t, x) \mapsto v_t(x)$ satisfying* (7)*. Then, there exists a probability measure $\eta$ on $\mathbb{R}^d \times \Gamma_T$ such that:*
(i) *$\eta$ is concentrated on the set of pairs $(x, \gamma) \in \mathbb{R}^d \times W^{1,2}([0, T], \mathbb{R}^d)$ such that:*

$$\dot{\gamma}(t) = v_t(\gamma(t)) \text{ for a.e. } t \in ]0, T[ \text{ with } \gamma(0) = x. \tag{9}$$

(ii) *we have $\mu_t = e_t\sharp\eta$ for all $t \in [0, T]$ with $e_t$ defined by*

$$e_t : (x, \gamma) \in \mathbb{R}^d \times \Gamma_T \mapsto \gamma(t) \in \mathbb{R}^d.$$

*Conversely if some* $\eta \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ *satisfies i) with*

$$\int_0^T \int_{\mathbb{R}^d \times \Gamma_T} |v_t(\gamma)|^2 \, d\eta(x,\gamma)dt < +\infty,$$

*then* $t \mapsto \mu_t := e_t \sharp \eta$ *solves the continuity equation* (8) *for some* $v$ *satisfying* (7).

1.4. **Assumptions on the multiagent control system.** Throughout the paper we suppose that the set valued map $F : \mathbb{R}^+ \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is Lipschitz continuous, with compact and convex images.

Now we give the precise definition of a trajectory of the multiagent system driven by $F$ on the time interval $I = [a, b]$

**Definition 1.3.** [37] A continuous curve $\boldsymbol{\mu} = \{\mu_t\}_{t\in I} \subseteq \mathscr{P}_2(\mathbb{R}^d)$, is an *admissible trajectory* driven by $F$ on $I$ if there exists $\boldsymbol{\nu} = \{\nu_t\}_{t\in I} \subseteq \mathscr{M}(\mathbb{R}^d; \mathbb{R}^d)$ such that

- $|\nu_t| \ll \mu_t$ for a.e. $t \in I$;
- $v_t(x) := \dfrac{\nu_t}{\mu_t}(x) \in F(t, \mu_t, x)$ for a.e. $t \in I$ and $\mu_t$-a.e. $x \in \mathbb{R}^d$, moreover the map $(t, x) \mapsto v_t(x)$ is Borel measurable;
- $\partial_t \mu_t + \operatorname{div} \nu_t = 0$ in the sense of distributions on $I \times \mathbb{R}^d$.

From the definition, it follows that $\boldsymbol{\mu} \in AC^2(I, \mathscr{P}_2(\mathbb{R}^d))$, i.e., there exists $m \in L^2(I; [0, +\infty[)$ such that

$$W_2(\mu_t, \mu_s) \le \int_s^t m(\tau)\, d\tau, \qquad \text{for all } t, s \in I \text{ with } s \le t.$$

Given $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, we denote by $A_I^F(\mu)$ the set of admissible trajectories on $I$ such that $\mu_a = \mu$. In [37], we have proved that the set $A_I^F(\mu)$ is nonempty, compact w.r.t. the natural uniform convergence metric on $C^0(I; \mathscr{P}_2(\mathbb{R}^d))$ defined as

$$d_{C^0}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}) = \sup_{t\in I} W_2(\mu_t^{(1)}, \mu_t^{(2)}),$$

for every $\boldsymbol{\mu}^{(i)} = \{\mu_t^{(i)}\}_{t\in I} \in C^0(I; \mathscr{P}_2(\mathbb{R}^d))$, $i = 1, 2$, and that any admissible trajectory can be equivalently represented by a probability measure on $\mathbb{R}^d \times \Gamma_I$ (cf also Theorem A.7 in Appendix).

## 2. Curves and trajectories in $\mathscr{P}_2(\mathbb{R}^d)$ and $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$

A natural question that arises is whether a dynamic in the Wasserstein space can be expressed as a dynamic in $L^2_{\mathbb{P}}$. We answer this question both for absolutely continuous curves and for trajectories of the multiagent system.

We denote by $\Gamma_T$ the set of continuous curves from $[0, T]$ to $\mathbb{R}^d$. Given a compact interval $I \subseteq \mathbb{R}$, we endow $C^0(I; \mathscr{P}_2(\mathbb{R}^d))$ with the structure of a complete metric space by defining the uniform convergence metric

$$d_{C^0}(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}) = \sup_{t\in I} W_2(\theta_t^{(1)}, \theta_t^{(2)}),$$

for every $\boldsymbol{\theta}^{(i)} = \{\theta_t^{(i)}\}_{t\in I} \in C^0(I; \mathscr{P}_2(\mathbb{R}^d))$, $i = 1, 2$. For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, the map $W_2^2(\mu, \cdot)$ is convex : given $\nu_i \in \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\pi}_i \in \Pi_o(\mu, \nu_i)$, $i = 0, 1$, $\lambda \in [0, 1]$, we set $\mu_\lambda := \lambda\mu_0 + (1 - \lambda)\mu_1$ and $\boldsymbol{\pi}_\lambda := \lambda\boldsymbol{\pi}_0 + (1 - \lambda)\boldsymbol{\pi}_1 \in \Pi(\mu, \mu_\lambda)$. Hence

$$W_2^2(\mu, \mu_\lambda) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\boldsymbol{\pi}_\lambda(x, y) = \lambda W_2^2(\mu, \mu_0) + (1 - \lambda) W_2^2(\mu, \mu_1).$$

So $W_2$-balls are convex, and $d_{C^0}$-balls around a curve $\boldsymbol{\theta}$ are convex.

2.1. **Absolutely continuous curves and trajectories.** Now we state the main result of this section comparing trajectories and curves in $\mathscr{P}_2(\mathbb{R}^d)$ and $L^2_{\mathbb{P}}$.

**Theorem 2.1** (Representation Theorem)**.**
*(i) Let* $\boldsymbol{\mu} : [0,T] \to \mathscr{P}_2(\mathbb{R}^d)$ *be a continuous solution of* (8) *for a Borel vector field* $(t,x) \mapsto v_t(x)$ *such that* (7) *holds true and let* $Y_0 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ *such that* $\mu_0 = Y_0\sharp\mathbb{P}$. *Then, for all* $\varepsilon > 0$, *there exists an absolutely continuous* $Z_\cdot \in W^{1,2}([0,T],L^2_{\mathbb{P}})$ *satisfying:*

$Z_t\sharp\mathbb{P} = \mu_t$ *for all* $t \in [0,T]$, $\dot{Z}_t = v_t(Z_t)$ *for a.e.* $t$, $\mathbb{P}$*-a.s. and* $\|Z_0 - Y_0\|_{L^\infty_{\mathbb{P}}(\Omega,\mathbb{R}^d)} \le \varepsilon$.

*(ii) Conversely, fix* $X_\cdot \in W^{1,2}([0,T];L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$. *Set* $\mu_t := X_t\sharp\mathbb{P}$ *and* $\nu_t \in \mathcal{M}(\mathbb{R}^d,\mathbb{R}^d)$ *defined as:*

$$\int_{\mathbb{R}^d} \Phi(x)\cdot d\nu_t(x) = \int_\Omega \Phi(X_t(\omega))\cdot \dot{X}_t(\omega)\,d\mathbb{P}(\omega), \quad \forall \Phi \in C_0(\mathbb{R}^d,\mathbb{R}^d). \tag{10}$$

*Then* $|\nu_t|$ *is absolutely continuous with respect to* $\mu_t$ *and setting* $v_t(\cdot) := \frac{\nu_t}{\mu_t}(\cdot)$ *for a.e.* $t \in ]0,T[$, $\mu_t$*-a.e, the curve* $\mu_t : [0,T] \to \mathscr{P}_2(\mathbb{R}^d)$ *is a continuous solution of* (8) *with* $v_t(x)$ *satisfying* (7).

*(iii) Let* $\mu_t : [0,T] \to \mathscr{P}_2(\mathbb{R}^d)$ *be a solution of the multiagent system driven by* $F$ *associated with the Borel vector field* $v_t(x)$ *(namely* $\{\mu_t\}_t$ *satisfying Definition 1.3 on* $[0,T]$*) and* $Y_0 \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ *such that* $\mu_0 = Y_0\sharp\mathbb{P}$. *Then there exists an absolutely continuous curve* $Z_\cdot \in W^{1,2}([0,T],L^2_{\mathbb{P}})$ *satisfying* $Z_t\sharp\mathbb{P} = \mu_t$ *(constructed in i) such that*

$$\dot{Z}_t(\omega) \in F(t, Z_t\sharp\mathbb{P}, Z_t(\omega)) \text{ for } \mathbb{P}\text{- a.e } \omega \text{ and for a.e. } t \text{ with } \|Z_0 - Y_0\|_{L^\infty_{\mathbb{P}}} \le \varepsilon. \tag{11}$$

*(iv) Conversely if* $X_\cdot \in W^{1,2}([0,T];L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ *satisfies* (11), *then there exists a Borel vector field* $v_t(x)$ *such that* $t \mapsto X_t\sharp\mathbb{P}$ *is an absolutely continuous curve satisfying* (8). *So* $\{X_t\sharp\mathbb{P}\}_{t\in[0,T]} \in A^F_{[0,T]}(X_0\sharp\mathbb{P})$.

Before proving this theorem, we start by defining, for every given $X \in L^2_{\mathbb{P}}$, the following subspace of $L^2_{\mathbb{P}}$

$$H_X := \{\Phi \circ X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d) : \Phi \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d,\mathbb{R}^d)\}. \tag{12}$$

The space $H_X$ is isometric to some $L^2_{\mathbb{P}}$ space :

**Lemma 2.2.** *(Lemma 5.10 in* [37]*)* $H_X$ *is a closed linear subspace of* $L^2_{\mathbb{P}}$. *Moreover, the map* $X_* : L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d) \to H_X$ *defined as* $X_*(\phi) = \phi \circ X$ *is a linear isometry.*

We denote by $\mathrm{pr}_{H_X} : L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d) \to H_X$ the projection on $H_X$.
The following result gives a characterization of the projection on $H_X$:

**Lemma 2.3.** *Let* $Z, X \in L^2_{\mathbb{P}}$, *we have*

$$\mathrm{pr}_{H_X}(Z) = p \circ X, \text{ with } \gamma = (X,Z)\sharp\mathbb{P} \text{ and } p(x) := \int z\,d\gamma^x(z).$$

*Proof.* (of Lemma 2.3) Indeed, for all $\phi \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d,\mathbb{R}^d)$

$$\begin{aligned}\int_\Omega (\phi\circ X)\cdot(p\circ X)\,d\mathbb{P} &= \int_{\mathbb{R}^d} \phi(x)\cdot p(x)\; d(X\sharp\mathbb{P})(x)\\ &= \int_{\mathbb{R}^d} \phi(x)\cdot\left[\int z\,d\gamma^x(z)\right]\; d(X\sharp\mathbb{P})(x)\end{aligned}$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\cdot z\ d\gamma^x(z)d(X\sharp\mathbb{P})(x)$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\cdot z\ d\gamma(x,z) = \int_\Omega (\phi\circ X)\cdot Z\,d\mathbb{P}.$$

□

*Remark* 2.4. Another expression of $p$ can be given by disintegrating $\mathbb{P}$ with respect to $X$ (see Theorem A.1). Indeed, $\mathbb{P}$ can be written $\mathbb{P}(\omega) = (X\sharp\mathbb{P})\otimes\mathbb{P}_x$ and then it can be proved that

$$p(x) = \int_{X^{-1}(x)} Z(\omega)\ d\mathbb{P}_x(\omega).$$

*Proof.* (of Theorem 2.1)

Proof of (i). Consider $\eta\in\mathscr{P}(\mathbb{R}^d\times\Gamma_T)$ associated with $\{\mu_t\}_{t\in[0,T]}$ by the superposition principle (Proposition 1.2). Since $\mathbb{P}$ has no atoms, there exists $T_\eta:\Omega\to\mathbb{R}^d\times\Gamma_T$ a Borel map such that $T_\eta\sharp\mathbb{P}=\eta$ (cf [42]). Set $X_t = e_t\circ T_\eta$ for all $t$. Then:

$$X_t\sharp\mathbb{P} =(e_t\circ T_\eta)\sharp\mathbb{P} = e_t\sharp(T_\eta\sharp\mathbb{P}) = e_t\sharp\eta = \mu_t, \qquad \|X_t\|^2_{L^2_\mathbb{P}} = \int |x|^2\,d\mu_t < +\infty,$$

so $X_t\in L^2_\mathbb{P}$. Moreover for all $Y\in L^2_\mathbb{P}$ and all $0\le s\le t\le T$, setting $\pi=(T_\eta\times Y)\sharp\mathbb{P}\in\Pi(\eta, Y\sharp\mathbb{P})$, using the superposition principle, we get:

$$\begin{aligned}\langle X_t, Y\rangle &= \int_\Omega e_t(T_\eta)\cdot Y\,d\mathbb{P} = \int_{(\mathbb{R}^d\times\Gamma_T)\times\mathbb{R}^d} e_t(x,\sigma)\cdot y\,d\pi((x,\sigma),y)\\
&= \int_{(\mathbb{R}^d\times\Gamma_T)\times\mathbb{R}^d} \sigma(t)\cdot y\,d\pi((x,\sigma),y)\\
&= \int_{(\mathbb{R}^d\times\Gamma_T)\times\mathbb{R}^d} \left(\sigma(s) + \int_s^t \dot\sigma(\tau)\ d\tau\right)\cdot y\,d\pi((x,\sigma),y)\\
&= \int \sigma(s)\cdot y\,d\pi((x,\sigma),y) + \int\left(\int_s^t v_\tau(\sigma(\tau))\cdot y\ d\tau\right)\,d\pi((x,\sigma),y)\\
&= \int e_s(x,\sigma)\cdot y\ d((T_\eta\times Y)\sharp\mathbb{P})(x,\sigma,y) + \int_s^t\left(\int v_\tau(\sigma(\tau))\cdot y\,d\pi((x,\sigma),y)\right)d\tau\end{aligned}$$

(using Fubini)

$$\begin{aligned}&= \int_\Omega e_s(T_\eta)\cdot Y\,d\mathbb{P} + \int_s^t\left(\int_\Omega v_\tau(e_\tau\circ T_\eta)\cdot Y\,d\mathbb{P}\right)\,d\tau\\
&= \int_\Omega X_s\cdot Y\ \ d\mathbb{P} + \int_s^t \langle v_\tau(X_\tau)\cdot Y\rangle d\tau = \langle X_s + \int_s^t v_\tau(X_\tau)d\tau\ ,\ Y\rangle\end{aligned}$$

using again Fubini. As this is true for any $Y\in L^2_\mathbb{P}$, we get for all $0\le s\le t\le T$:

$$X_t = X_s + \int_s^t v_\tau(X_\tau)\ d\tau.$$

So we can conclude $\{X_t\}_t$ is in $W^{1,1}([0,T];L^2_\mathbb{P}(\Omega,\mathbb{R}^d))$ and that $\dot X_t = v_t(X_t)$ a.e $t$, $\mathbb{P}$-a.s. We also have that $\{X_t\}_t$ is in $W^{1,2}([0,T];L^2_\mathbb{P}(\Omega,\mathbb{R}^d)$ because:

$$\int_0^T\int_\Omega |\dot X_t|^2\,d\mathbb{P}\ dt = \int_0^T\int_{\mathbb{R}^d} |v_t(x)|^2\ d\mu_t(x)\ dt < +\infty.$$

Now take $Y_0 \in L^2_{\mathbb{P}}$ and $\varepsilon > 0$ fixed such that $Y_0\sharp\mathbb{P} = X_0\sharp\mathbb{P}$. By lemma 1.1, there exists $\alpha$ measure preserving $\|X_0\circ\alpha - Y_0\|_{L^\infty(\Omega,\mathbb{R}^d)} \le \varepsilon$. Setting $Z_t = X_t\circ\alpha$, we have

$$\|Z_0 - Y_0\|_{L^\infty(\Omega,\mathbb{R}^d)} \le \varepsilon,\ \ Z_t\sharp\mathbb{P} = (X_t\circ\alpha)\sharp\mathbb{P} = X_t\sharp\mathbb{P} = \mu_t.$$

Moreover, repeating the same argument done for $X_t$, for all $Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ and all $0 \le s \le t \le T$, replacing $\pi$ by $\pi_\alpha = ((T_\eta\circ\alpha)\times Y)\sharp\mathbb{P} \in \Pi(\eta, Y\sharp\mathbb{P})$ leads:

$$\begin{aligned}\langle Z_t, Y\rangle =& \int (e_s\circ T_\eta\circ\alpha)\cdot Y\,d\mathbb{P} + \int_s^t \left(\int v_\tau(e_\tau\circ T_\eta\circ\alpha)\cdot Y\,d\mathbb{P}\right)\,d\tau\\ =&\langle Z_s + \int_s^t v_\tau(Z_\tau)d\tau\ ,\ Y\rangle.\end{aligned}$$

Again, this implies $\{Z_t\}_t \in W^{1,2}([0,T];L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d))$ and $\dot Z_t = v_t(Z_t)$ a.e $t$, $\mathbb{P}$-a.s.

Proof of (ii). From the definition of $\nu_t$, we get that $|\nu_t|$ is absolutely continuous with respect to $\mu_t$. By (10), we deduce:

$$v_t(X_t) = \mathrm{pr}_{H_X}(\dot X_t),\quad \mathbb{P}\text{-a.e., a.e. } t. \tag{13}$$

Then, since $\{X_t\}_t$ is in $W^{1,2}([0,T];L^2_{\mathbb{P}})$ one easily deduces (7). Since $W_2(\mu_s,\mu_t) \le \|X_s - X_t\|$, the curve $\{\mu_t\}_t$ is also continuous. To prove (8), taking $\varphi \in C_c^\infty(\mathbb{R}^d\times]0,T[)$ we have because of (13):

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}(\partial_t\varphi(t,x)+v_t(x)\cdot\nabla_x\varphi(x,t))\ d\mu_t(x)\ dt =\\ &\qquad= \int_0^T\int_\Omega (\partial_t\varphi(t,X_t)+v_t(X_t)\cdot\nabla_x\varphi(X_t,t))\ d\mu_t(x)\ dt =\\ &\qquad= \int_0^T\int_\Omega \Big(\partial_t\varphi(t,X_t)+\dot X_t\cdot\nabla_x\varphi(X_t,t)\Big)\ d\mu_t(x)\ dt =\\ &\qquad= \int_\Omega\int_0^T \frac{d}{dt}\varphi(X_t,t)\,dt\,d\mathbb{P} = \int_\Omega \varphi(X_t,T)-\varphi(X_t,0)\,dt\,d\mathbb{P} = 0.\end{aligned}$$

Proof of (iii). Consider the curve $\{Z_t\}_t$ given by $(i)$ associated to $\{\mu_t\}_{t\in[0,T]}$. Then

$$\int_0^T\int_\Omega I_{F(t,Z_t\sharp\mathbb{P},X_t(\omega))}(v_t(Z_t(\omega)))\,d\mathbb{P}(\omega)dt = \int_0^T\int_{\mathbb{R}^d} I_{F(t,\mu_t,x)}(v_t(x))\ d\mu_t(x)\ dt = 0$$

from Definition 1.3. Hence $Z.$ satisfies (11).

Proof of (iv). Consider the continuous curve $t\mapsto \mu_t := X_t\sharp\mathbb{P}$ and $v$ associated to $X.$ as in (ii). Since we already know that $v_t(X_t) = \mathrm{pr}_{H_X}(\dot X_t)$, setting $\gamma_t := (X_t,\dot X_t)\sharp\mathbb{P}$ a.e. $t$, by Lemma 2.3, it holds $v_t(x) = \int_{\mathbb{R}^d} y\ d\gamma_t^x(y)$ for $\mu_t$-a.e. $x$. Then, because $X.$ satisfies (11), using the convexity of the images of $F$ we get by Jensen's inequality:

$$\begin{aligned}0 \le& \int_0^T\int_{\mathbb{R}^d} I_{F(t,\mu_t,x)}(v_t(x))\ d\mu_t(x)\ dt = \int_0^T\int_{\mathbb{R}^d} I_{F(t,\mu_t,x)}\left(\int_{\mathbb{R}^d} y\ d\gamma_t^x(y)\right)\ d\mu_t(x)\ dt\\ \le& \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} I_{F(t,\mu_t,x)}(y)\ d\gamma_t(x,y)\ dt = \int_0^T\int_\Omega I_{F(t,X_t\sharp\mathbb{P},X_t(\omega))}(\dot X(t))\,d\mathbb{P} = 0.\end{aligned}$$

This proves that $\{\mu_t\}_{t\in[0,T]} \in A^F_{[0,T]}(X_0\sharp\mathbb{P})$. □

*Remark* 2.5. Part (i) of the previous theorem is a generalization of a result contained in [34] in dimension $d = 1$. In [34], the authors build each $Z_t$ as a unique map optimal such that $Z_t\sharp\mathbb{P} = \mu_t$ with $\mathbb{P}$ absolutely continuous with respect to the Lebesgue measure on $[-1/2, 1/2]$. The result holds for such choice of $Z_t$. Nevertheless, it is very specific to the dimension 1. Indeed for any $\nu$, $\mu$ in $\mathscr{P}_2(\mathbb{R})$, denoting by $X_\mu$ and $X_\nu$ the optimal transport maps from $\mathbb{P}$ to $\mu$ and $\nu$, it holds $W_2(\mu, \nu) = \|X_\mu - X_\nu\|_{L^2_{\mathbb{P}}(\Omega)}$. This equality is no longer true in higher dimension.
A. Tudorascu studied the results of [34] in dimension $d > 1$ (see [45]). Part (ii) of the above Theorem can be related to Proposition 3.3. of [45]. The author assumes again that $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and takes $X_t$ the optimal transport map from $\mathbb{P}$ to $\mu_t$. Then it is proved that, under an additionnal assumption, it exists a Borel vector field $(t, x) \mapsto v_t(x)$ such that:

$$\dot{X}_t = v_t(X_t) \text{ for a.e. } t,\ \mathbb{P}\text{-a.s.}$$

*Remark* 2.6. After the submission of the present paper, we discovered the work of Cavagnari, Lisini, Orrieri and Savaré ([25]), which is very much related to the previous Theorem in a slightly different setting. Thanks to [25], we understood that our first version of the proofs of (ii) and (iv) of Theorem 2.1 was not complete, and [25] helped us to fix the proof. We emphasize that [25] contains powerful results about the relation between Lagrangian and Eulerian formulations of the problem.

*Remark* 2.7. Observe that the fact the curve $\{X_t\}_t$ solves (11) does not imply that another curve $\{\bar{X}_t\}_t$ with the same law $\bar{X}_t\sharp\mathbb{P} = X_t\sharp\mathbb{P}$ solves (11). Consequently, the multiagent dynamical system cannot in general be studied in the space $L^2_{\mathbb{P}}$.

We already noticed that $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ is identified with $(L^2_{\mathbb{P}}/\sim)$. Easily, the equivalence classes are closed for the strong topology of $L^2_{\mathbb{P}}$. Nevertheless, they are neither convex nor closed for the weak topology of $L^2_{\mathbb{P}}$. So one needs to be very careful with the choice of the topology used when comparing continuity properties of curves in $\mathscr{P}_2(\mathbb{R}^d)$ and $L^2_{\mathbb{P}}$.

2.2. **Approximation of curves in $\mathscr{P}_2(\mathbb{R}^d)$ by trajectories of the multiagent system.** The goal of this section is to construct a trajectory of the multiagent system which approximates a given trajectory in $\mathscr{P}_2(\mathbb{R}^d)$. This is a crucial property to obtain regularity of the value of the control problem. So we obtain the following Grönwall-Filippov type result.

**Proposition 2.8.** *Take $I = [0, T]$. Let $\bar{\mu} \in \mathscr{P}_2(\mathbb{R}^d)$ be given, and $F : I \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^d$ be a Lipschitz continuous set-valued map with nonempty compact convex values. Let $\boldsymbol{\mu} = \{\mu_t\}_{t\in I} \subseteq AC^2(I; \mathscr{P}_2(\mathbb{R}^d))$ satisfying (8) for a Borel vector field $(t, x) \mapsto v_t(x)$ such that $\int_I \|v_t(\cdot)\|^2_{L^2_{\mu_t}}\, dt < +\infty$. Then there exists a trajectory $\tilde{\boldsymbol{\mu}} = \{\tilde{\mu}_t\}_{t\in I} \in A^F_I(\bar{\mu})$, such that for all $t \in I$*

$$W_2(\mu_t, \tilde{\mu}_t) \le e^{L't}\left(W_2(\mu_0, \bar{\mu}) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\mu_s,x)}(v_s(x))\, d\mu_s(x)\, ds\right), \tag{14}$$

*for some constant $L'$ depending only on $F$, $I$, and $\bar{\mu}$.*

This section is devoted to the proof of this proposition, according to the following outline.

(1) For a given curve $\boldsymbol{\theta} = \{\theta_t\}_{t\in I} \in \mathrm{Lip}(I; \mathscr{P}_2(\mathbb{R}^d))$, define the set $\Upsilon^{F,\boldsymbol{\theta}}_I(\bar{\mu})$ of solutions to the multiagent system associated to $(t, x) \mapsto F(t, \theta_t, x)$. Namely,

the set of $\boldsymbol{\mu} = \{\mu_t\}_{t\in I}$ satisfying (8) for a Borel vector field $(t,x) \mapsto v_t(x)$ such that $v_t(x) \in F(t,\theta_t,x)$ for $\mu_t$-a.e. $x \in \mathbb{R}^d$ and a.e. $t \in I$.

(2) We obtain a Filippov estimate for solutions in $\Upsilon_I^{F,\boldsymbol{\theta}}(\bar\mu)$ (see Proposition 2.10 below) by using our Representation Theorem 2.1 for the map $G^{\boldsymbol{\theta}} : I \times L^2_{\mathbb{P}}(\Omega) \rightrightarrows L^2_{\mathbb{P}}(\Omega)$ defined by

$$G^{\boldsymbol{\theta}}(t,X(\cdot)) := \{Y(\cdot) \in L^2_{\mathbb{P}}(\Omega) : Y(\omega) \in F(t,\theta_t,X(\omega)) \text{ for a.e. } \omega \in \Omega\} \tag{15}$$

with a Filippov Theorem in $L^2_{\mathbb{P}}$.

(3) We build the desired trajectory $\tilde{\boldsymbol{\mu}}$ as a fixed point of some submap of $\theta \mapsto \Upsilon_I^{F,\boldsymbol{\theta}}(\bar\mu)$ whose values satisfy (14).

We will need the following technical Lemma proved in the Appendix

**Lemma 2.9.** *Fix $\boldsymbol{\theta} = \{\theta_t\}_{t\in I} \in \mathrm{Lip}(I;\mathscr{P}_2(\mathbb{R}^d))$. Then $G^{\boldsymbol{\theta}} : I \times L^2_{\mathbb{P}}(\Omega) \rightrightarrows L^2_{\mathbb{P}}(\Omega)$, defined in* (15)*, is $\mathrm{Lip}\, F \cdot (1+\mathrm{Lip}\,\boldsymbol{\theta})$-Lipschitz continuous with closed images.*

**Proposition 2.10.** *In the assumptions of Proposition 2.8, fix $\varepsilon > 0$ and $\boldsymbol{\theta} = \{\theta_t\}_{t\in I} \in \mathrm{Lip}(I;\mathscr{P}_2(\mathbb{R}^d))$. Then there exists $\tilde\mu^{\boldsymbol{\theta}} = \{\tilde\mu_t^{\boldsymbol{\theta}}\}_{t\in I} \in \Upsilon_I^{F,\boldsymbol{\theta}}(\bar\mu)$ such that*

$$W_2(\mu_t,\tilde\mu_t^{\boldsymbol{\theta}}) \le e^{t\cdot \mathrm{Lip}\,F\cdot(1+\mathrm{Lip}\,\boldsymbol{\theta})}\left(W_2(\mu_0,\bar\mu) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\theta_s,x)}(v_s(x))\,d\mu_s(x)\,ds\right). \tag{16}$$

*Proof.* (of Proposition 2.10) Set $L := \mathrm{Lip}\,F\cdot(1+\mathrm{Lip}\,\boldsymbol{\theta})$.

Step 1 Fix $\varepsilon > 0$. We first prove that there exists $\tilde\mu^{\boldsymbol{\theta}}$ such that

$$W_2(\mu_t,\tilde\mu_t^{\boldsymbol{\theta}}) \le e^{Lt}\left(W_2(\mu_0,\bar\mu) + \int_0^t\int_{\mathbb{R}^d} d_{F(s,\theta_s,x)}(v_s(x))\,d\mu_s(x)\,ds + \varepsilon + \varepsilon t\right), \tag{17}$$

Take $\tilde X_0, \tilde Y_0 \in L^2_{\mathbb{P}}(\Omega)$ such that $(\tilde X_0,\tilde Y_0)\sharp\mathbb{P} \in \Pi_o(\mu_0,\bar\mu)$. By Theorem 2.1 (i), there exists $t \mapsto X_t$ such that $\dot X_t = v_t \circ X_t$, $\|X_0 - \tilde X_0\|_{L^2_{\mathbb{P}}} \le \varepsilon$, and $X_t\sharp\mathbb{P} = \mu_t$ for all $t \in I$. By Theorem 1.2 in [30] applied[1] to $G^{\boldsymbol{\theta}}$, there exists $t \mapsto Y_t(\cdot)$ absolutely continuous such that $Y_0 = \tilde Y_0$, $Y_0\sharp\mathbb{P} = \bar\mu$, $\dot Y_t(\omega) \in F(t,\theta_t,Y(\omega))$ for a.e. $\omega$ and for all $t$

$$\begin{aligned}
\|X_t(\cdot) - Y_t(\cdot)\|_{L^2_{\mathbb{P}}} \le& e^{tL}\left(\|X_0(\cdot) - Y_0(\cdot)\|_{L^2_{\mathbb{P}}} + \int_0^t d_{G^{\boldsymbol{\theta}}(s,X_s(\cdot))}(v_s\circ X_s(\cdot))\,ds + \varepsilon t\right)\\
\le& e^{tL}\left(\|\tilde X_0(\cdot) - \tilde Y_0(\cdot)\|_{L^2_{\mathbb{P}}} + \int_0^t d_{G^{\boldsymbol{\theta}}(s,X_s(\cdot))}(v_s\circ X_s(\cdot))\,ds + \varepsilon + \varepsilon t\right)\\
=& e^{tL}\left(W_2(\mu_0,\bar\mu) + \int_0^t d_{G^{\boldsymbol{\theta}}(s,X_s(\cdot))}(v_s\circ X_s(\cdot))\,ds + \varepsilon + \varepsilon t\right).
\end{aligned}$$

Set $\tilde\mu_t^{\boldsymbol{\theta}} := Y_t\sharp\mathbb{P}$ for all $t$. Notice that $W_2(\mu_s,\tilde\mu_s^\theta) \le \|X_t(\cdot) - Y_t(\cdot)\|_{L^2_{\mathbb{P}}}$, and

$$d_{G^{\boldsymbol{\theta}}(s,X_s(\cdot))}(v_s\circ X_s(\cdot)) = d_{F(s,\theta_s,\cdot)}(v_s(\cdot))\circ X_s(\cdot),$$

therefore, after integrating in $\mathbb{P}$ and having switched the integrals in $ds$ and in $d\omega$, we obtain (17). In particular, by Theorem 2.1(iii), we have that $\boldsymbol{\mu}^\theta = \{\tilde\mu_t^{\boldsymbol{\theta}}\}_{t\in I}$ obeys the continuity equation

$$\partial_t\tilde\mu_t^{\boldsymbol{\theta}} + \mathrm{div}(w_t\tilde\mu_t^{\boldsymbol{\theta}}) = 0,$$

where $(t,x) \mapsto w_t(x)$ is a measurable selection of $(t,x) \mapsto F(t,\theta_t,x)$.

---

[1]Note that [30] concerns mild solutions which appear to be absolutely continuous solutions because the infinitesimal generator is $A = 0$ in our context.

Step 2 We claim that the map $\boldsymbol{\theta} \mapsto \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$ from $\{\boldsymbol{\theta} \in \mathrm{Lip}(I; \mathscr{P}_2(\mathbb{R}^d)) : \mathrm{Lip}\,\boldsymbol{\theta} \leq L\}$ to $C^0(I; \mathscr{P}_2(\mathbb{R}^d))$ is Lipschitz continuous with compact convex images and its Lipschitz constant is less than $e^{(b-a)\cdot(1+L)\mathrm{Lip}\,F} \cdot (b-a)\mathrm{Lip}\,F$.

Without loss of generality we assume $I = [0, T]$. Set $L' = (1 + L)\mathrm{Lip}\,F$. Let $\varepsilon > 0$, $\boldsymbol{\theta^{(i)}} \in \mathrm{Lip}(I; \mathscr{P}_2(\mathbb{R}^d))$ with $\mathrm{Lip}\,\boldsymbol{\theta} \leq L$ for $i = 1, 2$, and $\boldsymbol{\mu}^{(1)} = \{\mu_t^{(1)}\}_{t\in I} \in \Upsilon_I^{F,\boldsymbol{\theta^{(1)}}}(\bar{\mu})$. In particular, $\boldsymbol{\mu}^{(1)}$ solves the continuity equation with a Borel vector field $(t, x) \mapsto v_t(x)$ satisfying $v_t(x) \in F(t, \theta_t^{(1)}, x)$ for $\mu_t$-a.e. $x \in \mathbb{R}^d$ and a.e. $t \in I$. By the arguments of step 1, there exists $\boldsymbol{\mu}^{(2)} = \{\mu_t^{(2)}\}_{t\in I} \in \Upsilon_I^{F,\boldsymbol{\theta^{(2)}}}(\bar{\mu})$.

$$
\begin{aligned}
W_2(\mu_t^{(1)}, \mu_t^{(2)}) \leq & e^{tL'} \left( \int_0^t \int_{\mathbb{R}^d} d_{F(s,\theta_s^{(2)},x)}(v_s(x))\, d\mu_s^{(1)}(x)\, ds + \varepsilon + \varepsilon t \right), \\
\leq & e^{tL'} \left( \int_0^t \int_{\mathbb{R}^d} \mathrm{Lip}\,F \cdot W_2(\theta_s^{(1)}, \theta_s^{(2)})\, d\mu_s^{(1)}(x)\, ds + \varepsilon + \varepsilon t \right), \\
\leq & e^{TL'} \cdot \left( T\mathrm{Lip}\,F \cdot d_{C^0}(\boldsymbol{\theta^{(1)}}, \boldsymbol{\theta^{(2)}}) + \varepsilon + \varepsilon T \right).
\end{aligned}
$$

Thus

$$
d_{\Upsilon_I^{F,\boldsymbol{\theta^{(2)}}}(\bar{\mu})}(\boldsymbol{\mu}^{(1)}) \leq e^{TL'} \cdot \left( T\mathrm{Lip}\,F \cdot d_{C^0}(\boldsymbol{\theta^{(1)}}, \boldsymbol{\theta^{(2)}}) + \varepsilon + \varepsilon T \right).
$$

By letting $\varepsilon \to 0^+$ and interchanging $\boldsymbol{\theta^{(1)}}$ and $\boldsymbol{\theta^{(2)}}$, we obtain the Lipschitz continuity of $\boldsymbol{\theta} \mapsto \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$.

We show the convexity of $\Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$. Given $\lambda \in [0, 1]$, $\boldsymbol{\mu}^{(i)} = \{\mu_t^{(i)}\}_{t\in I} \in \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$, $i = 0, 1$ define $\boldsymbol{\mu}^{(\lambda)} = \{\mu_t^{(\lambda)}\}_{t\in I} := \lambda\boldsymbol{\mu}^{(0)} + (1 - \lambda)\boldsymbol{\mu}^{(1)}$. By linearity, we have

$$
\partial_t \mu_t^{(\lambda)} + \mathrm{div} \left( \lambda v_t^{(0)} \mu_t^{(0)} + (1 - \lambda) v_t^{(1)} \mu_t^{(1)} \right) = 0,
$$

where $v_t^{(i)}(x) \in F(t, \theta_t, x)$ for $\mu_t^{(i)}$-a.e. $x$ and a.e. $t$. Noticing that $\mu_t^{(i)} \ll \mu_t^{(\lambda)}$,

$$
\partial_t \mu_t^{(\lambda)} + \mathrm{div} \left( v_t^{(\lambda)} \mu_t^{(\lambda)} \right) = 0,
$$

where for $\mu_t^{(\lambda)}$-a.e. $x \in \mathbb{R}^d$ and a.e. $t \in I$ it holds

$$
\begin{aligned}
v_t^{\lambda}(x) := & \frac{\lambda v_t^{(0)} \mu_t^{(0)} + (1 - \lambda) v_t^{(1)} \mu_t^{(1)}}{\mu_t^{(\lambda)}}(x) \\
= & v_t^{(0)}(x) \frac{\lambda \mu_t^{(0)}}{\lambda \mu^{(0)} + (1 - \lambda)\mu^{(1)}}(x) + v_t^{(1)}(x) \frac{(1 - \lambda)\mu_t^{(1)}}{\lambda \mu^{(0)} + (1 - \lambda)\mu^{(1)}}(x).
\end{aligned}
$$

Therefore $v_t^{\lambda}(x) \in F(t, \theta_t, x)$ by convexity of the images of $F$. Thus $\boldsymbol{\mu}^{(\lambda)} \in \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$.

We notice that there exists $C_1^{\boldsymbol{\theta}}, C_2^{\boldsymbol{\theta}} > 0$ such that $F(t, \theta_t, x) \subseteq \overline{B(0, C_1^{\boldsymbol{\theta}} + C_2^{\boldsymbol{\theta}}|x|)}$. Indeed, take $M_{\boldsymbol{\theta}} = \max\{\mathrm{m}_2^{1/2}(\theta_s) : s \in I\}$, since

$$
F(t, \theta_t, x) \subseteq F(0, \delta_0, 0) + \overline{B(0, 1)} \cdot \mathrm{Lip}\,F \cdot (T + M_{\boldsymbol{\theta}} + |x|),
$$

we can take

$$
C_2^{\boldsymbol{\theta}} := \mathrm{Lip}\,F \cdot (T + M_{\boldsymbol{\theta}} + 1), \qquad C_1^{\boldsymbol{\theta}} := \max\{|v| : v \in F(0, \delta_0, 0)\} + C_2^{\boldsymbol{\theta}}.
$$

Thus

$$
|\gamma(t)| \leq |\gamma(0)| + \int_0^t (C_1^{\boldsymbol{\theta}} + C_2^{\boldsymbol{\theta}} |\gamma(s)|)\, ds,
$$

hence for the trajectories of $\dot\gamma(t)\in F(t,\theta_t,\gamma(t))$, we have by Grönwall inequality

$$|\gamma(t)|\le (|\gamma(0)|+TC_1^{\boldsymbol\theta})e^{C_2^{\boldsymbol\theta}T}.$$

So $F(t,\theta_t,\gamma(t))\subseteq \overline{B\left(0,C_1^{\boldsymbol\theta}+C_2^{\boldsymbol\theta}(|\gamma(0)|+TC_1^{\boldsymbol\theta})e^{C_2^{\boldsymbol\theta}T}\right)}$, in particular $\operatorname{Lip}\gamma\le C_1^{\boldsymbol\theta}+C_2^{\boldsymbol\theta}(|\gamma(0)|+TC_1^{\boldsymbol\theta})e^{C_2^{\boldsymbol\theta}T}$. Every $\boldsymbol\mu=\{\mu_t\}_{t\in I}\in\Upsilon_I^{F,\boldsymbol\theta}(\bar\mu)$ can be represented as $\mu_t=e_t\sharp\boldsymbol\eta$ with a measure $\boldsymbol\eta\in\mathscr{P}(\mathbb{R}^d\times\Gamma_I)$, concentrated on pairs $(x,\gamma)$ where $\gamma(0)=x$ and $\gamma$ is a trajectory of the differential inclusion. By integrating the above estimates, we obtain that the elements $\boldsymbol\mu$ of $\Upsilon_I^{F,\boldsymbol\theta}(\bar\mu)$ satisfy

- uniform boundedness of the images in $W^2$

$$W_2(\delta_0,\mu_t)=\left(\int_{\mathbb{R}^d}|\gamma(t)|^2\,d\boldsymbol\eta(x,\gamma)\right)^{1/2}\le(\mathrm{m}_2^{1/2}(\bar\mu)+TC_1^{\boldsymbol\theta})e^{C_2^{\boldsymbol\theta}T}.$$

- uniform Lipschitz continuity

$$\begin{aligned}W_2(\mu_t,\mu_s)\le&\|e_t-e_s\|_{L^2_{\mathbb{P}}}=\left(\int_{\mathbb{R}^d\times\Gamma_I}|\gamma(t)-\gamma(s)|^2\,d\boldsymbol\eta(x,\gamma)\right)^{1/2}\\ \le&|t-s|\cdot(C_1^{\boldsymbol\theta}+C_2^{\boldsymbol\theta}(\mathrm{m}_2^{1/2}(\bar\mu)+TC_1^{\boldsymbol\theta})e^{C_2^{\boldsymbol\theta}T}).\end{aligned}$$

- pointwise compactness in $W_2$: the uniform boundedness of the images in $W_2$ yields the pointwise narrow compactness, thus we have only to show the uniform integrability of the second-order moments. We notice that for $\boldsymbol\eta$-a.e. $(x,\gamma)$, if $|\gamma(t)|>r$, we have

$$s(r):=\max\{re^{-C_2^{\boldsymbol\theta}T}-TC_1^{\boldsymbol\theta},0\}<|\gamma(0)|.$$

Thus

$$\begin{aligned}\left(\int_{\mathbb{R}^d\setminus B(0,r)}|x|^2\,d\mu_t(x)\right)^{1/2}\le&\left(\iint_{(\mathbb{R}^d\setminus B(0,s(r)))\times\Gamma_I}|\gamma(t)|^2\,d\boldsymbol\eta(x,\gamma)\right)^{1/2}\\ \le&\left[\left(\int_{\mathbb{R}^d\setminus B(0,s(r))}|x|^2\,d\bar\mu(x)\right)^{1/2}+TC_1^{\boldsymbol\theta}\bar\mu(\mathbb{R}^d\setminus B(0,s(r)))\right]e^{C_2^{\boldsymbol\theta}T},\end{aligned}\tag{18}$$

and the right hand side tends to 0 as $r\to+\infty$, uniformly w.r.t. $\boldsymbol\mu\in\Upsilon_I^{F,\boldsymbol\theta}(\bar\mu)$.

By Ascoli-Arzelà theorem, $\Upsilon_I^{F,\boldsymbol\theta}(\bar\mu)$ is relatively compact in $C^0(I;\mathscr{P}_2(\mathbb{R}^d))$. We prove that it is closed. Given a sequence $\{\boldsymbol\mu^{(n)}\}_{n\in\mathbb{N}}\subseteq\Upsilon_I^{F,\boldsymbol\theta}(\bar\mu)$, converging to $\boldsymbol\mu$ in $C^0(I;\mathscr{P}_2(\mathbb{R}^d))$, we can find a sequence $\boldsymbol\eta^{\boldsymbol{(n)}}\subseteq\mathscr{P}(\mathbb{R}^d\times\Gamma_I)$ such that $\mu_t^{(n)}=e_t\sharp\boldsymbol\eta^{\boldsymbol{(n)}}$ for all $t\in I$ and $n\in\mathbb{N}$, where $\boldsymbol\eta^{(n)}$ is concentrated on pairs $(x,\gamma)$ where $\gamma$ is a trajectory of the differential inclusion and $\gamma(0)=x$. Since the functional

$$(x,\gamma)\mapsto\begin{cases}|x|^2+|\gamma(0)|^2+\|\dot\gamma\|_\infty^2, & \text{if }\gamma\in\operatorname{Lip}(I;\mathbb{R}^d),\\ +\infty, & \text{otherwise},\end{cases}$$

has compact sublevels, by using the estimates on the trajectories we obtain

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d\times\Gamma_I}(|x|^2+|\gamma(0)|^2+\|\dot\gamma\|_\infty^2)\,d\boldsymbol\eta^{\boldsymbol{(n)}}(x,\gamma)<+\infty.$$

By Remark 5.1.5 in [3] we extract a subsequence $\{\boldsymbol\eta^{(n_k)}\}_{k\in\mathbb{N}}$ narrowly convergent to $\boldsymbol\eta$. By the continuity of $e_t$, we have that $\mu_t=e_t\sharp\boldsymbol\eta$ for all $t$. Finally, for a.e. $(x,\gamma)$ in the support of $\boldsymbol\eta$ there exists $\{(x_n,\gamma_n)\}_{n\in\mathbb{N}}$ such that $x_n\to x$, $\|\gamma_n-\gamma\|_\infty\to 0$ and $\gamma_n$ is a trajectory of $\dot\gamma_n(t)\in F(t,\theta_t,\gamma_n(t))$ with $\gamma_n(a)=x$. Since the solution map of such differential inclusion has a compact graph ([6] Th. 3.5.2) $\gamma$ is a trajectory starting from $x$, and therefore $\boldsymbol{\mu} \in \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$. This proves Step 2.

Step 3 We now construct $\tilde{\boldsymbol{\mu}}^{\boldsymbol{\theta}}$ satisfying (16).

Consider a sequence $\varepsilon_n \to 0^+$. By Step 1, there exists a sequence $\{\tilde{\boldsymbol{\mu}}^{\boldsymbol{\theta},(\boldsymbol{n})}\}_{n\in\mathbb{N}} \subset \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$ satisfying

$$W_2(\mu_t, \tilde{\mu}_t^{\boldsymbol{\theta},(\boldsymbol{n})}) \le e^{Lt}\left(W_2(\mu_0,\bar{\mu}) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\theta_s,x)}(v_s(x))\, d\mu_s(x)\, ds + \varepsilon_n + \varepsilon_n t\right),$$

where $\tilde{\boldsymbol{\mu}}^{\boldsymbol{\theta},(\boldsymbol{n})} = \{\tilde{\mu}_t^{\boldsymbol{\theta},(\boldsymbol{n})}\}_{t\in I}$. By the compactness result obtained in Step 2, we can find a subsequence $\{\tilde{\boldsymbol{\mu}}^{\boldsymbol{\theta},(\boldsymbol{n_k})}\}_{k\in\mathbb{N}}$ converging in $d_{C^0}$ to $\tilde{\boldsymbol{\mu}}^{\boldsymbol{\theta}} = \{\tilde{\mu}_t^{\boldsymbol{\theta}}\}_{t\in I} \in \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu})$, satisfying (16). ☐

*Proof.* (of Proposition 2.8) Given $L, T > 0$, $\bar{\mu} \in \mathscr{P}_2(\mathbb{R}^d)$, and $M > \mathrm{m}_2^{1/2}(\bar{\mu})$, we set $I = [0,T]$,

$$C_2 := \mathrm{Lip}\, F \cdot (T + M + 1), \qquad C_1 := \max\{|v| : v \in F(0,\delta_0,0)\} + C_2,$$

and define $\mathscr{S}_{I,L,M}(\bar{\mu})$ to be the set of $\boldsymbol{\theta} = \{\theta_t\}_{t\in I} \in \mathrm{Lip}\,(I;\mathscr{P}_2(\mathbb{R}^d))$ satisfying $\theta_0 = \bar{\mu}$, $\mathrm{Lip}\,\boldsymbol{\theta} \le L$, $W_2(\delta_0,\theta_t) \le M$ and for all $r > 0$, $t \in I$

$$\left(\int_{\mathbb{R}^d\setminus\overline{B(0,r)}} |x|^2\, d\theta_t(x)\right)^{1/2} \le \tag{19}$$
$$\le \left[\left(\int_{\mathbb{R}^d\setminus B(0,\tilde{s}(r))} |x|^2\, d\bar{\mu}(x)\right)^{1/2} + TC_1\bar{\mu}(\mathbb{R}^d \setminus B(0,\tilde{s}(r)))\right] e^{C_2 T},$$

where $\tilde{s}(r) = \max\{re^{-C_2T} - TC_1, 0\}$.

We have that $\mathscr{S}_{I,L,M}(\bar{\mu})$ is uniformly bounded in $d_{C^0}$, thus we get the pointwise relative compactness of $\mathscr{S}_{I,L,M}(\bar{\mu})$ w.r.t. the narrow topology. We prove that $\mathscr{S}_{I,L,M}(\bar{\mu})$ is also pointwise relative compact in $W_2$. Indeed it is enough to show the uniform integrability of the second-order moments which comes from the fact from (19) we have

$$\left(\int_{\mathbb{R}^d\setminus B(0,2r)} |x|^2\, d\mu_t(x)\right)^{1/2} \le \left(\int_{\mathbb{R}^d\setminus\overline{B(0,r)}} |x|^2\, d\mu_t(x)\right)^{1/2} \le$$
$$\le \left[\left(\int_{\mathbb{R}^d\setminus B(0,\tilde{s}(r))} |x|^2\, d\bar{\mu}(x)\right)^{1/2} + TC_1\bar{\mu}(\mathbb{R}^d \setminus B(0,\tilde{s}(r)))\right] e^{C_2 T},$$

and that $\tilde{s}(r) \to +\infty$ as $r \to +\infty$.

The set $\mathscr{S}_{I,L,M}(\bar{\mu})$ is nonempty, since it contains the constant curves $\theta_t \equiv \bar{\mu}$. It is convex from the convexity of the $W_2$-ball. It is also closed in the $d_{C^0}$ topology, and hence compact by Ascoli-Arzelà theorem. Indeed, it is sufficient to recall that if $W_2(\xi_n,\xi) \to 0$ then we have

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^d\setminus\overline{B(0,r)}} |x|^2\, d\xi_n(x) \ge \int_{\mathbb{R}^d\setminus\overline{B(0,r)}} |x|^2\, d\xi(x).$$

Suppose now that

$M > \mathrm{m}_2^{1/2}(\bar{\mu}) + 1$,

$$L > \max\{|v| : v \in F(0, \delta_0, 0)\} + \mathrm{Lip}\, F \cdot (M + 2) + \mathrm{Lip}\, F \cdot (M + 2) \cdot (\mathrm{m}_2^{1/2}(\bar{\mu}) + 1)$$

Then we claim that for all $0 < T < 1$ small enough it holds $\Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu}) \subseteq \mathscr{S}_{I,L,M}(\bar{\mu})$ for all $\boldsymbol{\theta} \in \mathscr{S}_{I,L,M}(\bar{\mu})$.

We prove this claim. As in the proof of Proposition 2.10, given $\boldsymbol{\theta} \in \mathscr{S}_{I,L,M}(\bar{\mu})$ we get for every $\boldsymbol{\mu} \in \Upsilon_{[0,T]}^{F,\boldsymbol{\theta}}(\bar{\mu})$,

$$\mathrm{Lip}(\boldsymbol{\mu}) \leq C_1^{\boldsymbol{\theta}} + C_2^{\boldsymbol{\theta}}(\mathrm{m}_2^{1/2}(\bar{\mu}) + TC_1^{\boldsymbol{\theta}})e^{C_2^{\boldsymbol{\theta}}T},$$

where (recalling that $0 < T < 1$)

$$\begin{aligned} C_2^{\boldsymbol{\theta}} =& \mathrm{Lip}\, F \cdot (T + M_{\boldsymbol{\theta}} + 1) \leq C_3 := \mathrm{Lip}\, F \cdot (M + 2), \\ C_1^{\boldsymbol{\theta}} =& \max\{|v| : v \in F(0, \delta_0, 0)\} + C_2^{\boldsymbol{\theta}} \\ &\leq C_4 := \max\{|v| : v \in F(0, \delta_0, 0)\} + \mathrm{Lip}\, F \cdot (M + 2). \end{aligned}$$

Therefore $\mathrm{Lip}(\boldsymbol{\mu}) \leq f(T)$ where

$$f(T) = C_4 + C_3(\mathrm{m}_2^{1/2}(\bar{\mu}) + TC_4)e^{C_3 T}.$$

We easily get $f(0) < L$, and therefore, since $f(\cdot)$ is continuous, there exists $T_0 \in ]0, 1[$ such that $f(T) < L$ for $0 < T < T_0$, where $T_0$ depends only on $F$, $M$ and $\bar{\mu}$. In particular, we have $\mathrm{Lip}\, \boldsymbol{\mu} \leq L$. Moreover, we have

$$W_2(\delta_0, \mu_t) \leq \mathrm{m}_2^{1/2}(\bar{\mu}) + T\mathrm{Lip}(\boldsymbol{\mu}) \leq \mathrm{m}_2^{1/2}(\bar{\mu}) + LT,$$

therefore, by possibly further shrinking $T_0$, we have $W_2(\delta_0, \mu_t) \leq M$. Equation (19) follows from the estimate on $C_2^{\boldsymbol{\theta}}$ and from (18), our claim is proved.

Given an interval $J \subseteq [0, T]$, we define

$$\begin{aligned} Q_J(\boldsymbol{\mu}, \boldsymbol{\theta}) := \Big\{ \boldsymbol{\xi} = \{\xi_t\}_{t \in J} \in C^0(J; \mathscr{P}_2(\mathbb{R}^d)) : \text{ for all } t \in J \text{ it holds} \\ W_2(\mu_t, \xi_t) \leq e^{t \cdot \mathrm{Lip}\, F \cdot (1 + \mathrm{Lip}\, \boldsymbol{\theta})} \left( W_2(\mu_0, \bar{\mu}) + \int_0^t \int_{\mathbb{R}^d} d_{F(s, \theta_s, x)}(v_s(x))\, d\mu_s(x)\, ds \right) \Big\}, \end{aligned}$$

and we notice that $Q_I(\boldsymbol{\mu}, \boldsymbol{\theta})$ is a convex and $d_{C^0}$-closed set.

Notice that the set-valued map $\boldsymbol{\theta} \mapsto Q_I(\boldsymbol{\mu}, \boldsymbol{\theta})$, defined on $C^0([0, T]; \mathscr{P}_2(\mathbb{R}^d))$, has closed graph since for all $\boldsymbol{\theta}^{(i)} = \{\theta_t^{(i)}\}_{t \in I}$, $i = 1, 2$, and every $v \in \mathbb{R}^d$ we have

$$d_{F(t, \theta_t^{(1)}, x)}(v) \leq d_{F(t, \theta_t^{(2)}, x)}(v) + \mathrm{Lip}\, F d_{C^0}(\boldsymbol{\theta}^1, \boldsymbol{\theta}^2),$$

and $W_2(\mu_t, \cdot)$ is continuous.

We consider the map $\mathscr{S}_{I,L,M}(\bar{\mu}) \rightrightarrows \mathscr{S}_{I,L,M}(\bar{\mu})$ defined as $\boldsymbol{\theta} \mapsto \Upsilon_I^{F,\boldsymbol{\theta}}(\bar{\mu}) \cap Q_I(\boldsymbol{\mu}, \boldsymbol{\theta})$. As this map is not suitable to apply Kakutani fixed point theorem, we will use an embedding given by the following Lemma, whose proof is postponed to the Appendix

**Lemma 2.11.** *Let $\mathscr{S} \subseteq C^0(I; \mathscr{P}_2(\mathbb{R}^d))$ be compact and convex, endowed with the topology induced by $d_{C^0}$. Then there exists a locally convex topological vector space $\mathcal{L}$ and an homeomorphism $h : (\mathscr{S}, d_{C^0}) \to \mathcal{L}$ such that*

$$h(\lambda \boldsymbol{\mu}^{(1)} + (1 - \lambda)\boldsymbol{\mu}^{(2)}) = \lambda h(\boldsymbol{\mu}^{(1)}) + (1 - \lambda)h(\boldsymbol{\mu}^{(2)})$$

*for all $\lambda \in [0, 1]$ and $\boldsymbol{\mu}^{(i)} \in \mathscr{S}$, $i = 1, 2$, mapping $(\mathscr{S}, d_{C^0})$ to a compact convex subset of $\mathcal{L}$.*

The map $\boldsymbol{\theta} \mapsto \Upsilon_I^{F,\boldsymbol{\theta}}(\bar\mu) \cap Q_I(\boldsymbol{\mu}, \boldsymbol{\theta})$ has closed graph, and nonempty convex images. Its graph is contained in a compact set, so it is upper semicontinuous. According to Lemma 2.11, there is an affine homeomorphism $h : \mathscr{S}_{I,L,f}(\bar\mu) \to \mathcal{L}$, where $\mathcal{L}$ is a topological vector space. In particular, we can consider

$$h(\mathscr{S}_{I,L,f}(\bar\mu)) \ni h(\boldsymbol{\theta}) \mapsto h(\Upsilon_I^{F,\boldsymbol{\theta}}(\bar\mu) \cap Q_I(\boldsymbol{\mu}, \boldsymbol{\theta})) \subseteq h(\mathscr{S}_{I,L,f}(\bar\mu)).$$

Recalling that $h$ is an affine homeomorphism, we have that $h(\mathscr{S}_{I,L,f}(\bar\mu))$ is again compact and convex, and the above set-valued map is upper semicontinuous with compact convex images. By Kakutani-Fan-Glicksberg fixed point theorem (see e.g. Theorem 13.1 in [40]), this set-valued map admits a fixed point, i.e., there exists $\ell \in h(\mathscr{S}_{I,L,f}(\bar\mu))$ such that $\ell \in h(\Upsilon_I^{F,h^{-1}(\ell)}(\bar\mu) \cap Q_I(\boldsymbol{\mu}, h^{-1}(\ell)))$. In particular, there exists $\tilde{\boldsymbol{\mu}} := h^{-1}(\ell) \in \Upsilon_I^{F,\tilde{\boldsymbol{\mu}}}(\bar\mu) \cap Q_I(\boldsymbol{\mu}, \boldsymbol{\eta})$ thus $\tilde{\boldsymbol{\mu}} \in \Upsilon_I^{F,\tilde{\boldsymbol{\mu}}}(\bar\mu)$ and $\tilde{\boldsymbol{\mu}} \in Q_I(\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}})$, thus

$$W_2(\mu_t, \tilde\mu_t) \le e^{t\cdot \mathrm{Lip}\, F\cdot(1+L)} \left( W_2(\mu_0, \bar\mu) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\tilde\mu_s,x)}(v_s(x))\, d\mu_s(x)\, ds \right), \tag{20}$$

for some constant $L$ depending only on $F$, $I$, and $\bar\mu$. Recalling the Lipschitz continuity of $F$, this implies

$$W_2(\mu_t, \tilde\mu_t) \le$$
$$e^{t\cdot \mathrm{Lip}\, F\cdot(1+L)} \left( W_2(\mu_0, \bar\mu) + \int_0^t \int_{\mathbb{R}^d} \left[ d_{F(s,\mu_s,x)}(v_s(x)) + \mathrm{Lip}\, F \cdot W_2(\mu_s, \tilde\mu_s) \right] \, d\mu_s(x)\, ds \right).$$

Set

$$g(t) := e^{t\cdot \mathrm{Lip}\, F\cdot(1+L)} \left( W_2(\mu_0, \bar\mu) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\mu_s,x)}(v_s(x))\, d\mu_s(x)\, ds \right),$$

we have (recalling $0 \le t \le T < 1$)

(21)

$$\begin{aligned} &W_2(\mu_t, \tilde\mu_t) \\ &\le e^{t\cdot \mathrm{Lip}\, F\cdot(1+L)} \left( W_2(\mu_0, \bar\mu) + \int_0^t \int_{\mathbb{R}^d} d_{F(s,\mu_s,x)}(v_s(x))\, d\mu_s(x)\, ds + \mathrm{Lip}\, F \cdot \int_0^t W_2(\mu_s, \tilde\mu_s)\, ds \right) \\ &\le g(t) + \mathrm{Lip}\, F \cdot e^{\mathrm{Lip}\, F\cdot(1+L)} \int_0^t W_2(\mu_s, \tilde\mu_s)\, ds \end{aligned}$$

(22)

Grönwall's inequality yields the desired estimate (21) with

$$L' = \mathrm{Lip}\, F \cdot e^{\mathrm{Lip}\, F\cdot(1+L)} + \mathrm{Lip}\, F \cdot (1+L),$$

yielding Proposition 2.8 for $T$ small enough.

We prove now the case of possibly large $T > 0$. To this aim, we apply Zorn's lemma to the set

$$\mathscr{Z} := \{(\tau, \tilde{\boldsymbol{\mu}} = \{\tilde\mu_t\}_{t\in[0,\tau]}) : \tau \in [0,T],\, \tilde\mu_0 = \bar\mu, \tilde{\boldsymbol{\mu}} \in \Upsilon_{[0,\tau]}^{F,\tilde{\boldsymbol{\mu}}} \cap Q_J(\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}})\}$$

with the following partial order: $(\tau^{(1)}, \tilde{\boldsymbol{\mu}}^{(1)}) \preceq (\tau^{(2)}, \tilde{\boldsymbol{\mu}}^{(2)})$ if and only if $\tau^{(1)} \le \tau^{(2)}$ and $\tilde\mu_t^{(1)} = \tilde\mu_t^{(2)}$ for all $t \in [0, \tau^{(1)}]$. Given a totally ordered chain $A$, set $T_A = \sup\{\tau \in [0,T] : (\tau, \tilde{\boldsymbol{\mu}}) \in A\}$ and define $\hat{\boldsymbol{\mu}}^{(A)} = \{\hat\mu_t^{(A)}\}_{t\in[0,T_A[}$ by setting $\hat\mu_t = \tilde\mu_t$ for all $t \in \tau$, $(\tau, \tilde{\boldsymbol{\mu}}) \in A$.

Notice that given $(\tau, \tilde{\boldsymbol{\mu}})$ there exists $\tilde{\boldsymbol{\eta}} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_{[0,\tau]})$, supported on the pairs $(x, \gamma)$ satisfying $\dot{\gamma}(t) \in F(t, \tilde{\mu}_t, \gamma(t))$ for a.e. $t \in [0, \tau]$ and $\gamma(0) = x$, such that $\tilde{\mu}_t = e_t \sharp \tilde{\boldsymbol{\eta}}$ for all $t \in [0, \tau]$. In particular, we have for $\tilde{\boldsymbol{\eta}}$-a.e. $(x, \gamma) \in \mathbb{R}^d \times \Gamma_{[0,\tau]}$

$$F(t, \tilde{\mu}_t, \gamma(t)) \leq F(0, \delta_0, 0) + \overline{B(0,1)} \cdot \operatorname{Lip} F \cdot (t + \mathrm{m}_2^{1/2}(\tilde{\mu}_s) + |\gamma(t)|).$$

Therefore, since $C_1 = \max\{|v| : v \in F(0, \delta_0, 0)\}$

$$|\gamma(t)| \leq |\gamma(0)| + \int_0^t \left[ C_1 + (s + \mathrm{m}_2^{1/2}(\tilde{\mu}_s) + |\gamma(s)|) \right] ds.$$

By taking the $L^2_{\tilde{\boldsymbol{\eta}}}$-norm and using Jensen's inequality

$$\mathrm{m}_2^{1/2}(\tilde{\mu}_t) \leq \mathrm{m}_2^{1/2}(\bar{\mu}) + \int_0^t \left[ C_1 + \operatorname{Lip} F \cdot (s + 2\mathrm{m}_2^{1/2}(\tilde{\mu}_s)) \right] ds.$$

By Grönwall's inequality, and recalling that $\tau \leq T$

$$\mathrm{m}_2^{1/2}(\tilde{\mu}_t) \leq \mathrm{m}_2^{1/2}(\bar{\mu}) + \tau C_1 + \operatorname{Lip} F \frac{T^2}{2} \cdot e^{2T \operatorname{Lip} F}$$

Arguing as above, $\tilde{\boldsymbol{\mu}}$ is Lipschitz continuous, with Lipschitz constant depending only on $\bar{\mu}$, $T$, and $F$. Since $(\tau, \tilde{\boldsymbol{\mu}}) \in A$ are arbitrary, $\tilde{\boldsymbol{\mu}}^{(A)}$ is Lipschitz on $[0, T_A[$ and therefore it can be uniquely extended to a Lipschitz function on $[0, T_A]$ still denoted $\tilde{\boldsymbol{\mu}}^{(A)}$. One gets easily that $\tilde{\boldsymbol{\mu}}^{(\boldsymbol{A})} \in \Upsilon^{F, \tilde{\boldsymbol{\mu}}^{(\boldsymbol{A})}}_{[0,T_A]}(\bar{\mu})$, and $\tilde{\boldsymbol{\mu}}^{(A)} \in Q_{[0,T_A]}(\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}})$. Therefore $(T_A, \tilde{\boldsymbol{\mu}}^{(\boldsymbol{A})}) \in \mathscr{Z}$ majorizes every element of $A$. By Zorn's lemma there exist $(T', \hat{\boldsymbol{\mu}}) \in A)$ a maximal element. If $T' < T$, by applying the first part of the proof to extend $\hat{\boldsymbol{\mu}}$ on $[T', T' + \varepsilon]$ for some $\varepsilon > 0$, we contradict the maximality of $\hat{\boldsymbol{\mu}}$.

In particular, we obtain $\tilde{\boldsymbol{\mu}} \in \Upsilon^{F, \tilde{\boldsymbol{\mu}}}_I(\bar{\mu}) \cap Q_I(\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}})$, and we can conclude by Grönwall's inequality as in the case of small $T$. □

*Remark* 2.12. In a series of recent papers ([8], [9], [10], [11], [12], [13], [14], [15]), optimization problems in the Wasserstein space driven by a controlled continuity equation were studied in the *Cauchy-Lipschitz framework*, i.e., assuming a local Lipschitz regularity in space of the (possibly nonlocal) driving vector field. It is well known that in this case the continuity equation is well-posed, and moreover its unique solution is given by the push-forward of the initial measure along the flow: in particular mass splitting along the trajectories is not possible. The concept of trajectory used in the Cauchy-Lipschitz framework yields a powerful tool to extend the classical finite-dimensional theory to the Wasserstein space, at the price of restricting the set of available trajectories for the agents (by adding a hidden interaction between the velocities of close agents, which must be selected to be closed). A short comparison of the concept of trajectory used in this paper and the Cauchy-Lipschitz framework was outlined also in Remark 6 of [12].

*Remark* 2.13. Another Filippov-like theorem was obtained in [12] with a different notion of solution to (1)-(2), under more smoothness assumption on the vector field. Also when $\boldsymbol{\mu}$ is itself a solution to (1)-(2), a Grönwall-Filippov result was obtained in [37].

Thanks to Theorem 2.1, we can write the value function $\mathcal{V}(\cdot)$ associated to (4) on admissible trajectories in two different ways: we can write

$$\mathcal{V}(\mu, T) := \inf \left\{ \int_s^T \mathcal{L}(\mu_t)\, dt + \mathscr{G}(\mu_T) : \boldsymbol{\mu} \in A^F_{[0,T]}(\mu) \right\},$$

and, setting $L(Y) := \mathcal{L}(Y\sharp\mathbb{P})$ and $G(Y) = \mathscr{G}(Y\sharp\mathbb{P})$ for all $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, we can write also

$$\mathcal{V}(\mu, T) := \inf \Big\{ \int_s^T L(X_t)\, dt + G(X_T) : \, X_\cdot \in W^{1,2}([0,T], \mathbb{R}^d),\ X_0\sharp\mathbb{P} = \mu,$$
$$\dot{X}_t(\omega) \in F(X_t\sharp\mathbb{P}, X_t(\omega)) \text{ for a.e. } t \in [0,T] \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega \Big\}.$$

It has been shown in [37] that $\mathcal{V}$ is a solution of a HJB equation in $\mathscr{P}_2(\mathbb{R}^d)$ of the type:

$$\frac{\partial u}{\partial t} + \mathcal{H}(\mu, D_\mu u(t,\mu)) = 0. \tag{23}$$

In the next section, we explore some properties of this equation and its meaning in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$. (We reduce our study to the sationary equation in order to simplify). Moreover, we expect that, setting $V(X) := \mathcal{V}(X\sharp\mathbb{P})$ for all $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$, $V$ should be a solution of a HJB equation in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$. In subsection 3.4, we give some insights on the difficult question of studying (23) as a classic HJB equation in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$.

*Remark* 2.14. One of the most relevant drawback in the $L^2$-representation for evolutions in the Wasserstein space is due to to the difficult to model *density constraints*. Indeed, in a quite general form, a density constraint on $\mathscr{P}(\mathbb{R}^d)$ can be expressed as follows: given a reference measure $\gamma$ and a density penalization function $f$, we define the integral functional $\mathscr{F} : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$

$$\mathscr{F}(\mu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} f\left(x, \frac{\mu}{\gamma}(x)\right) d\gamma(x), & \text{if } \mu \ll \gamma, \\ +\infty, & \text{otherwise.} \end{cases}$$

the constraint requires that, during an evolution $\boldsymbol{\mu} = \{\mu_t\}_{t\in I}$, it holds $\mathscr{F}(\mu_t) \le 0$ for a.e. $t \in I$.

Under quite general assumptions on $f$ (basically convexity and superlinearity of $f(x,\cdot)$, see Theorem 3.4.1 in [17] and [16]), the functional $\mathscr{F}(\cdot)$ turns out to be w*-l.s.c., and therefore the set $\{\mu \in \mathscr{P}(\mathbb{R}^d) : \mathscr{F}(\mu) \le 0\}$ where the constraint is satisfied is closed, and so the set of continuous curves satisfying the constraint is closed w.r.t. uniform convergence.

A basic ingredient to represent this constraint in the $L^2$-setting is the possibility to easily compute the density of the pushforward w.r.t. a given measure: indeed, we have $\mu_t = X_t\sharp\mathbb{P}$, and therefore we need to compute $\omega \mapsto \dfrac{X_t\sharp\mathbb{P}}{\gamma}(X_t(\omega))$.

Even in the case of $\gamma = \mathbb{P} = \mathscr{L}^d$, an explicit description of the density of the pushforward measure requires quite strong assumption on $X_t(\cdot)$ (namely, existence of an approximate differential and essential injectivity, see e.g. Lemma 5.5.3 in [3]), which prevents the possibility to use them for generic trajectories in the Wasserstein space (in particular in presence of the constraint on the velocities given by the differential inclusion). Indeed, such representation formulas are mostly used for the geodesics of the space, which meet additional properties.

# 3. Hamilton Jacobi Bellman Equations

The Lipschitz value functions of multiagent control problem should satisfy a HJB equation in the Wasserstein space in suitable senses [4], [19], [33], [37]. The relevance of the notions of viscosity solutions proposed in the previous references lies in the fact that the value function can be *characterized* by a HJB equation. This needs comparison principles discussed in [19], [26], [33], [39], [37]. Here we investigates several super/subdifferential needed to obtain viscosity solutions on $\mathscr{P}_2(\mathbb{R}^d)$.

We consider the Hamilton-Jacobi equation satisfied by a function $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$

$$\text{(HJ)} \qquad \mathcal{H}(\mu, D_\mu u(\mu)) = 0, \qquad \mu \in \mathscr{P}_2(\mathbb{R}^d),$$

with the following hamiltonian

$$\mathcal{H} : (\mu_0, p) \in \mathscr{P}_2(\mathbb{R}^d) \times L^2_{\mu_0}(\mathbb{R}^d) \to \mathcal{H}(\mu_0, p) \in \mathbb{R}.$$

Since $D_\mu u$ has not yet been defined and because $u$ may not be regular, the meaning of this equation has to be considered in the viscosity sense, by replacing the derivatives by suitable super/subdifferentials.

## 3.1. **Super/sub differential in $\mathscr{P}_2(\mathbb{R}^d)$.**

Now we introduce the following notion of superdifferential

**Definition 3.1** (Superdifferentials in $\mathscr{P}_2$ cf [37])**.** Consider $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and $\varepsilon \geq 0$. The $\varepsilon$-superdifferential of $u$ at $\mu_0$ is the set $D^+_\varepsilon u(\mu_0)$ of elements $p \in L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ such that $p \in dis^+(\mu_0)$ and

$$u(\nu) - u(\mu_0) \leq \int p(x) \cdot (y - x) \, d\gamma(x, y) + \varepsilon W_2(\mu_0, \nu) + o\left(W_2(\mu_0, \nu)\right) \tag{24}$$

for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, $\gamma \in \Pi_o(\mu_0, \nu)$. The set $dis^+(\mu_0) \subset L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ is the convex cone generated by optimal "anti"-displacements namely:

$dis^+(\mu_0) := \{\lambda(Id - T) : \ \lambda > 0,\ T$ an optimal transport map between $\mu_0$ and $T\sharp\mu_0\}$.

When $\varepsilon = 0$ we write $D^+ u(\mu_0)$ for $D^+_0 u(\mu_0)$.

*Remark* 3.2. The definition above is not exactly equivalent to those of [39], [37]. The difference is that in [39], [37], the elements of $D^+_\varepsilon u(\mu_0)$ are optimal anti-displacements. The set of optimal anti-displacement is not stable under multiplication by a non-negative real number. Indeed, let $\mu_0$ be the Lebesgue measure restricted to a ball centered at $0_{\mathbb{R}^d}$ of measure 1. Then $\mathrm{Id}_{\mathbb{R}^d} = \mathrm{Id}_{\mathbb{R}^d} - 0_{\mathbb{R}^d}$ is an optimal anti-displacement as $x \mapsto 0_{\mathbb{R}^d}$ is an optimal transport map from $\mu_0$ to $\delta_{0_{\mathbb{R}^d}}$. But $2\mathrm{Id}_{\mathbb{R}^d}$ is not an optimal displacement as $-\mathrm{Id}_{\mathbb{R}^d}$ is not an optimal transport map (it is not cyclically monotone- see [44]).

Various concepts of super/subdifferentials have been proposed [3], [10], [19], [33], [37], [39]. In our control framework, a "good" super/subdifferentials should enable us to prove that the value function is the unique viscosity solution of some HJB equation. Namely it should allow to obtain a comparison principle. The above definition provides such a comparison principle (proved in less restrictive assumptions in [39] and [37]).

Symmetrically we can define the $\varepsilon$-subdifferential:

**Definition 3.3.** [Subdifferentials in $\mathscr{P}_2$] The $\varepsilon$-subdifferential or $u$ at $\mu_0$ is the set $D^-_\varepsilon u(\mu_0)$ of elements $p \in L^2_{\mu_0}(\mathbb{R}^d,\mathbb{R}^d)$ such that $p \in dis^-(\mu_0)$ and

$$u(\nu) - u(\mu_0) \geq \int p(x)\cdot(y-x)\ d\gamma(x,y) - \varepsilon W_2(\mu_0,\nu) + o\left(W_2(\mu_0,\nu)\right)$$

for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, $\gamma \in \Pi_o(\mu_0,\nu)$, where
$dis^-(\mu_0) := \{\lambda(T - \mathrm{Id}_{\mathbb{R}^d}) :\ \lambda > 0,\ T$ an optimal transport map between $\mu_0$ and $T\sharp\mu_0\}$.
Again $D^-u(\mu_0) := D^-_0(\mu_0)$.

We will discuss several alternative definitions of the superdifferential. Before doing this, we recall the definition of tangent space to $\mathscr{P}_2(\mathbb{R}^d)$ at $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ (cf [3])

$$\mathcal{T}_\mu(\mathbb{R}^d) := \overline{\{\nabla\varphi :\ \varphi \in C^\infty_c(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}, \tag{25}$$

which is related to optimal displacement thanks to the following relation proved in [3]

$$\mathcal{T}_\mu(\mathbb{R}^d) = \overline{dis^+(\mu)}^{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)} = \overline{dis^-(\mu)}^{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}. \tag{26}$$

We recall an equivalent definition of superdifferential (later on we will use both definitions without citing this equivalence result).

**Proposition 3.4** (Equivalent definition of superdifferential [37])**.** *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a map, let $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, $\varepsilon \geq 0$ and $p \in dis^+(\mu_0)$. Then $p$ is in $D^+_\varepsilon u(\mu_0)$ iff*

$$\begin{aligned} u(\nu) - u(\mu_0) \leq \int p(x)\cdot(y-x)\ d\gamma(x,y) + \varepsilon\left[\int |y-x|^2 d\gamma(x,y)\right]^{1/2} + \\ + o\left(\left[\int |y-x|^2 d\gamma(x,y)\right]^{1/2}\right), \qquad \forall \nu \in \mathscr{P}_2(\mathbb{R}^d) \text{ and } \gamma \in \Pi(\mu,\nu). \end{aligned} \tag{27}$$

Indeed, the proof in [37] shows a more general result: $p \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$ satisfies

$$u(\nu) - u(\mu_0) \geq \int p(x)\cdot(y-x)\ d\gamma(x,y) + \varepsilon W_2(\mu_0,\nu) + o\left(W_2(\mu_0,\nu)\right)$$

for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and all $\gamma \in \Pi_o(\mu,\nu)$, if and only if it satisfies (27) for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and all $\gamma \in \Pi(\mu,\nu)$.

Now we provide a simpler definition of $D^+_\varepsilon u(\mu)$ for atomless $\mu$. Indeed take $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ with $\mu$ having no atom, then we know [42]

$$W_2(\mu,\nu) = \inf_{T \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)} \left\{\left(\int |Tx - x|^2 d\mu(x)\right)^{1/2} :\quad T\sharp\mu = \nu\right\}.$$

This implies that there exists $\{T_n\}_n \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ such that

$$T_n\sharp\mu = \nu, \quad \lim_{n\to+\infty}\left(\int |T_n x - x|^2 d\mu(x)\right)^{1/2} = W_2(\mu,\nu). \tag{28}$$

Possibly extracting a subsequence, by compactness of $\Pi(\mu,\nu)$ , we have also that

$$\gamma_n := (\mathrm{Id}_{\mathbb{R}^d} \times T_n)\sharp\mu \overset{*}{\rightharpoonup} \gamma \in \Pi_o(\mu,\nu). \tag{29}$$

Note that $\gamma$ is optimal because $\pi \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \mapsto \displaystyle\int |y-x|^2\, d\pi(x,y)$ is l.s.c.

This suggests that, for atomless $\mu$, we could restrict the definition of $D^+_\varepsilon u(\mu)$ to transport plans supported on the graph of transport maps.

**Proposition 3.5.** *Let* $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ *and* $\varepsilon \geq 0$.

*Assume* $\mu$ *has no atom and* $p$ *belongs to* $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$. *Then:*

*(a) if for all* $\Phi \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ *the function* $p$ *satisfies*

$$(30) \quad u(\Phi\sharp\mu) - u(\mu) \leq \\ \leq \int_{\mathbb{R}^d} p(x) \cdot (\Phi(x) - x)\ d\mu(x) + \varepsilon \|\Phi - \mathrm{Id}\|_{L^2_\mu} + o\left(\|\Phi - \mathrm{Id}\|_{L^2_\mu}\right),$$

*then it satisfies* (24).

*(b) if* $p$ *satisfies* (30) *then the projection on* $\mathcal{T}_\mu(\mathbb{R}^d)$ *of* $p$ *satisfies* (27).

*(c) if* $p \in dis^+(\mu)$, *then* $p \in D^+_\varepsilon u(\mu)$ *if and only if it satisfies* (30).

To prove this result, we need the following lemma:

**Lemma 3.6.** *Let* $\mu$, $\{\nu_k\}_{k\in\mathbb{N}}$ *in* $\mathscr{P}_2(\mathbb{R}^d)$ *such that* $\lim_{k\to+\infty} W_2(\mu, \nu_k) = 0$. *Assume for some* $\varepsilon \geq 0$, $p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ *and fixed* $\gamma_k \in \Pi_o(\mu, \nu_k)$ *we have:*

$$(31) \qquad \limsup_{k\to+\infty} \frac{u(\nu_k) - u(\mu) - \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)\ d\gamma_k(x,y)}{W_2(\mu,\nu_k)} \leq \varepsilon.$$

*Then, taking another sequence* $\{\bar\gamma_k\}_k$ *in* $\Pi_o(\mu, \nu_k)$, *we also have:*

$$(32) \qquad \limsup_{k\to+\infty} \frac{u(\nu_k) - u(\mu) - \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)\ d\bar\gamma_k(x,y)}{W_2(\mu,\nu_k)} \leq \varepsilon.$$

*Proof.* (of the Lemma) Denote by $q$ the projection of $p$ on $\mathcal{T}_\mu(\mathbb{R}^d)$. We first remark that, since $\bar\gamma_k$ is an optimal transport plan for any $k$, by Lemma A.2, $x \mapsto \int y d\gamma^x(y) - x$ is in $dis^-(\mu) \subset \mathcal{T}_\mu(\mathbb{R}^d)$, and this yields

$$(33) \qquad \begin{aligned} \iint_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)\, d\bar\gamma_k(x,y) &= \int_{\mathbb{R}^d} p(x)\cdot\left[\int_{\mathbb{R}^d} y d\bar\gamma^x_k(y) - x\right] d\mu(x) \\ &= \int_{\mathbb{R}^d} q(x)\cdot\left[\int_{\mathbb{R}^d} y\, d\bar\gamma^x_k(y) - x\right] d\mu(x) \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} q(x)\cdot(y-x)\ d\bar\gamma_k(x,y). \end{aligned}$$

By definition of $\mathcal{T}_\mu(\mathbb{R}^d)$, for all $\delta > 0$, there exists $\varphi_\delta \in C^\infty_c(\mathbb{R}^d)$ such that $\|\nabla\varphi_\delta - q\|_{L^2_\mu} \leq \delta$. Then, by using Lemma 3.3. p.10 of [33] and Cauchy-Schwartz inequality, we have

$$\begin{aligned} \int q(x)\cdot(y-x)\ d\bar\gamma_k(x,y) &\leq \int \nabla\varphi_\delta(x)\cdot(y-x)\ d\bar\gamma_k(x,y) + \delta W_2(\mu,\nu_k) \\ &\leq \int \nabla\varphi_\delta(x)\cdot(y-x)\ d\gamma_k(x,y) + \|D^2\varphi_\delta\|_\infty W_2^2(\mu,\nu_k) + \delta W_2(\mu,\nu_k) \\ &\leq \int q(x)\cdot(y-x)\ d\gamma_k(x,y) + \|D^2\varphi_\delta\|_\infty W_2^2(\mu,\nu_k) + 2\delta W_2(\mu,\nu_k). \end{aligned}$$

Applying the same argument as in (33), we get:

$$\int q(x)\cdot(y-x)\,d\bar\gamma_k(x,y) \le \int p(x)\cdot(y-x)\,d\gamma_k(x,y)+\|D^2\varphi_\delta\|_\infty W_2^2(\mu,\nu_k)+2\delta W_2(\mu,\nu_k).$$

Finally, for all $\delta>0$, by (31):

$$\begin{aligned}&\limsup_{k\to+\infty}\frac{u(\nu_k)-u(\mu)-\int p(x)\cdot(y-x)\,d\bar\gamma_k(x,y)}{W_2(\mu,\nu_k)}\\ &\le \limsup_{k\to+\infty}\frac{u(\nu_k)-u(\mu)-\int p(x)\cdot(y-x)\,d\gamma_k(x,y)}{W_2(\mu,\nu_k)}+\|D^2\varphi_\delta\|_\infty W^2(\mu,\nu_k)+2\delta\\ &\le \varepsilon+2\delta,\end{aligned}$$

and letting $\delta\to 0$ yields (32). □

*Proof.* (of Proposition 3.5) Assume for simplicity that $\varepsilon=0$.

*Proof of a)*: Let $(\nu_k)_k$ a sequence of $\mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_k\in\Pi_o(\mu,\nu_k)$ such that:

$$\lim_{k\to+\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,d\gamma_k(x,y)=\lim_{k\to+\infty}W_2(\nu_k,\mu)^2=0.$$

We aim to prove that:

$$\limsup_{k\to+\infty}\frac{u(\nu_k)-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d}p(x)\cdot(y-x)\,d\gamma_k(x,y)}{W_2(\mu,\nu_k)}\le 0. \tag{34}$$

Set $r_k=W_2(\mu,\nu_k)$. Take $\Phi_k\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ such that:

$$\|p-\Phi_k\|_{L^2_\mu}\le\|y-x\|_{L^2_{\gamma_k}}=r_k. \tag{35}$$

By (28) and (29), for all $k\in\mathbb{N}$, there exists a sequence $(T_{k,n})_n$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ such that:

$$T_{k,n}\sharp\mu=\nu_k,\quad \gamma_{k,n}=(\mathrm{Id}_{\mathbb{R}^d},T_{k,n})\sharp\mu\overset{*}{\rightharpoonup}\bar\gamma_k\in\Pi_o(\mu,\nu_k),\quad \lim_{n\to+\infty}\int|\mathrm{Id}_{\mathbb{R}^d}-T_{k,n}|^2\,d\mu=r_k^2.$$

It is worth pointing out that $\bar\gamma_k$ may be different from $\gamma_k$. Fix $k$ in $\mathbb{N}$. Note that, uniformly in $n$, $(\bar\gamma_{k,n})_n$ has uniformly integrable moments of order 2. Then, since for all $x$ and $y$ we have $|\Phi_k(x)\cdot(y-x)|\le\|\Phi_k\|_\infty(|y|+|x|)$, it holds (cf e.g. Lemma 5.1.7. of [3]):

$$\lim_{n\to\infty}\int\Phi_k(x)\cdot(y-x)\,d\bar\gamma_{k,n}(x,y)=\int\Phi_k(x)\cdot(y-x)\,d\bar\gamma_k(x,y).$$

As a consequence, for all $k\in\mathbb{N}$, we can choose $T_k\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ such that:

$$T_k\sharp\mu=\nu_k,\qquad \left|\|\mathrm{Id}-T_k\|_{L^2_\mu}-\|y-x\|_{L^2_{\bar\gamma_k}}\right|\le r_k^2, \tag{36}$$

$$\left|\int_{\mathbb{R}^d}\Phi_k(x)\cdot(T_k(x)-x)\,d\mu(x)-\int_{\mathbb{R}^d}\Phi_k(x)\cdot(y-x)\,d\bar\gamma_k(x,y)\right|\le r_k^2. \tag{37}$$

Then, using (35), (36), (37) and Cauchy-Schwarz inequality, for $k$ large enough we have

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}p(x)\cdot(y-x)\,d\bar\gamma_k(x,y)\le\int\Phi_k(x)\cdot(y-x)\,d\bar\gamma_k(x,y)+r_k^2\\ &\int_{\mathbb{R}^d\times\mathbb{R}^d}\Phi_k(x)\cdot(T_k(x)-x)\,d\mu(x)+2r_k^2\le\int_{\mathbb{R}^d\times\mathbb{R}^d}p(x)\cdot(T_k(x)-x)\,d\mu(x)+4r_k^2.\end{aligned}$$

This yields

$$\frac{u(\nu_k) - u(\mu) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)\; d\bar{\gamma}_k(x,y)}{r_k} \leq$$

$$\leq \frac{u(\nu_k) - u(\mu) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(T_k(x)-x)\; d\mu(x)}{r_k} + 4r_k.$$

Since when $k$ tends to $+\infty$, we have $r_k \to 0$ and $\displaystyle\int_{\mathbb{R}^d} |\mathrm{Id} - T_k|^2\, d\mu(x) \to 0$, we get the desired relation (34) with $\bar{\gamma}_k$ instead of $\gamma_k$. The conclusion follows by use of Lemma 3.6.

The proof of b) follows from a), using the same argument as in (33) and a similar proof to the Proposition 3.4. The proof of c) follows from (26). □

The following example shows that the result is no longer true when $\mu$ has atoms.

*Example* 3.7. Again we set $\varepsilon = 0$. Set $d = 1$ and $u(\mu) := \left[1 - \displaystyle\int_{\mathbb{R}} \varphi(x)\, d\mu(x)\right]^{1/2}$ with $\varphi \in C_b^0(\mathbb{R})$ such that $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. So we have $u(\delta_0) = 0$ and $D^+u(\delta_0) = \emptyset$, but $p = 0$ satisfies (30). Indeed, $\delta_0$ can only be transported to some $\delta_x$ by transport maps, moreover $W_2(\delta_x, \delta_0) = |x|$ and $\displaystyle\limsup_{x\to 0} \frac{u(\delta_x)}{|x|} = 0$. We show that $D^+u(\delta_0) = \emptyset$. Let us remark that

$$D^+u(\delta_0) = \left\{ a \in \mathbb{R} : \limsup_{\int_{\mathbb{R}} |x|^2\, d\nu(x) \to 0} \frac{u(\nu) - \displaystyle\int_{\mathbb{R}} ax\, d\nu(x)}{\left(\displaystyle\int_{\mathbb{R}} |x|^2\, d\nu(x)\right)^{1/2}} \leq 0 \right\}.$$

Fix $a \in \mathbb{R}$. Then, taking $\nu_n = (1 - \frac{1}{n^2})\delta_0 + \frac{1}{n^2}\delta_2$ leads to

$$\limsup_{n\to+\infty} \frac{u(\nu_n) - \displaystyle\int_{\mathbb{R}} ax\; d\nu_n(x)}{\left(\displaystyle\int_{\mathbb{R}} |x|^2 d\nu_n(x)\right)^{1/2}} = \frac{1}{2}.$$

From that we deduce $D^+u(\delta_0) = \emptyset$.

*Remark* 3.8. Proposition 3.5 makes the link with viscosity solutions in the Wasserstein space introduced in [19] for HJB related to differential games. More precisely, the super-differential $D^+_{CQ,\varepsilon}u(\mu)$ defined in [19] is the set of $p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ such that for all $T \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$

$$u(T\sharp\mu) - u(\mu) \leq \int p(x)\cdot(Tx - x)d\mu(x) + \varepsilon\|\mathrm{Id} - T\|_{L^2_\mu} + o(\|\mathrm{Id} - T\|_{L^2_\mu}).$$

By Proposition 3.5, we have for any $\mu$ without atom:

$$D^+_{CQ,\varepsilon}u(\mu) \cap dis^+(\mu) = D^+_\varepsilon u(\mu).$$

The previous example shows that this equality is no longer true when $\mu$ has atoms.

Now we provide a result showing that somehow the atomic part and nonatomic part of $\mu$ can be considered separately.

**Proposition 3.9.** *Let $u : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $\varepsilon \geq 0$ and $p \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$. Denote by $\mu_0$ the non atomic part of $\mu$ and $\mu_\sharp$ the atomic part of $\mu$. We consider the following property :*

(38)

$$u(\Phi\sharp\mu_0 + \nu) - u(\mu) \leq \int p(x) \cdot (\Phi(x) - x)\, d\mu_0(x) + \int p(x) \cdot (y - x)\, d\gamma(x,y) + \\ + \varepsilon \left( \|\Phi - \mathrm{Id}\|^2_{L^2_\mu} + \|y - x\|^2_{L^2_\gamma} \right)^{1/2} + o\left( \left( \|\Phi - \mathrm{Id}\|^2_{L^2_{\mu_0}} + \|y - x\|^2_{L^2_\gamma} \right)^{1/2} \right),$$

*for all $\Phi \in L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$, $\nu$ positive measure with $\mu_\sharp(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\gamma \in \Pi(\mu_\sharp, \nu)$.*
*Then*

a) *if $p$ satisfies* (38) *then it satisfies* (24),
b) *if $p$ satisfies* (38) *then the projection on $\mathcal{T}_\mu(\mathbb{R}^d)$ of $p$ satisfies* (27),
c) *if $p \in dis^+(\mu)$, then $p \in D^+_\varepsilon u(\mu)$ if and only if it satisfies* (38).

*Proof.* We show (38) $\Rightarrow$ (24). Let $m \in \mathscr{P}_2(\mathbb{R}^d)$ and $\pi \in \Pi(\mu, m)$. By disintegration:

$$\pi(x,y) = \pi^x(y) \otimes \mu(x) = \pi^x(y) \otimes \mu_0(x) + \pi^x(y) \otimes \mu_\sharp(x).$$

Denote by $\nu$ the second marginal of $\gamma := \pi^x \otimes \mu_\sharp$ and by $\nu_0$ the second marginal of $\gamma_0 = \pi^x \otimes \mu_0 \in \Pi(\mu_0, \nu_0)$. The first marginal of $\gamma_0$ has no atom. Arguing as in the proof of Proposition 3.5, we get the conclusion. ☐

In the definition of the super-differential, we can restrict the variations $\nu$:

**Lemma 3.10.** *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $\varepsilon \geq 0$ and $u : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ continuous. Let $A \in \mathcal{P}_2(\mathbb{R}^d)$ be dense. Assume $p \in \mathscr{P}_2(\mathbb{R}^d)$ satisfies for all $\nu \in A$*

$$u(\nu) - u(\mu) \leq \int p(x) \cdot (y - x)\, d\gamma(x,y) + \varepsilon \|y - x\|_{L^2_\gamma} + o\left( \|y - x\|_{L^2_\gamma} \right) \tag{39}$$

*for all $\gamma \in \Pi(\mu, \nu)$. Then* (39) *is satisfied for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \Pi(\mu, \nu)$. If moreover $p \in dis^+(\mu)$, then $p \in D^+_\varepsilon u(t, \mu)$.*

*Proof.* Take again $\varepsilon = 0$. Let $(\nu_k)_{k\in\mathbb{N}}$, $\mu$ in $\mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_k \in \Pi(\mu, \nu_k)$ such that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^{2d}} |y - x|^2 d\gamma_k(x,y) = 0. \tag{40}$$

As $A$ is dense and $u$ is continuous, we can choose $\bar{\nu}_k \in A$ such that:

$$W_2(\bar{\nu}_k, \nu_k) \leq \|y - x\|^2_{L^2_{\gamma_k}}, \quad |u(\bar{\nu}_k) - u(\nu_k)| \leq \|y - x\|_{L^2_{\gamma_k}}. \tag{41}$$

Let $\bar{\gamma}_k \in \Pi_o(\nu_k, \bar{\nu}_k)$. We disintegrate $\gamma_k$ and $\bar{\gamma}_k$ and glue them to get a transport plan $\pi_k \in \Pi(\mu, \bar{\nu}_k)$:

$$\gamma_k(x,y) = \gamma_k^y(x) \otimes \nu_k(y), \quad \bar{\gamma}_k(y,z) = \bar{\gamma}_k^y(z) \otimes \nu_k(y),$$
$$\pi_k(x,z) = \int_{\mathbb{R}^d} \gamma_k^y(x) \otimes \bar{\gamma}_k^y(z)\, d\nu_k(y).$$

Then we have

$$\left( \int_{\mathbb{R}^{2d}} |z - x|^2\, d\pi_k(x,z) \right)^{1/2} = \left( \int_{\mathbb{R}^{3d}} |z - y + y - x|^2 d\gamma_k^y(x)\, d\bar{\gamma}_k^y(z) d\nu_k(y) \right)^{1/2}$$

$$\leq \left( \int_{\mathbb{R}^{2d}} |z-y|^2 d\bar{\gamma}_k(y,z) \right)^{1/2} + \left( \int_{\mathbb{R}^{2d}} |y-x|^2 \; d\gamma_k(x,y) \right)^{1/2}.$$

From (41) and the definition of $\bar{\gamma}_k$

(42)
$$\left( \int_{\mathbb{R}^{2d}} |z-x|^2 d\pi_k(x,z) \right)^{1/2} \leq \|y-x\|_{L^2_{\gamma_k}} \left( 1 + \|y-x\|_{L^2_{\gamma_k}} \right).$$

Now, we have by (41):

$$\begin{aligned}
&\frac{1}{\|y-x\|_{L^2_{\gamma_k}}} (u(\nu_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(y-x) \; d\gamma_k(x,y)) \\
&\leq \frac{1}{\|y-x\|_{L^2_{\gamma_k}}} (u(\bar{\nu}_k) - u(\mu) - \int_{\mathbb{R}^{3d}} p(x)\cdot(y-z+z-x) \; d\gamma_k^y(x) d\bar{\gamma}_k^y(z) d\nu_k(y)) \\
&\qquad + \|y-x\|_{L^2_{\gamma_k}} \\
&\leq \frac{u(\bar{\nu}_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(z-x)\, d\pi_k(x,z)}{\|y-x\|_{L^2_{\gamma_k}}} + \frac{\|p\|_{L^2_\mu} \|z-y\|_{L^2_{\bar{\gamma}_k}}}{\|y-x\|_{L^2_{\gamma_k}}} + \|y-x\|_{L^2_{\gamma_k}} \\
&\leq \frac{u(\bar{\nu}_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(z-x)\, d\pi_k(x,z)}{\|y-x\|_{L^2_{\gamma_k}}} + (1+\|p\|_{L^2_\mu}) \|y-x\|_{L^2_{\gamma_k}}
\end{aligned}$$

(by (41) and the definition of $\bar{\gamma}_k$)

$$\leq (1+\|y-x\|_{L^2_{\gamma_k}}) \cdot \frac{u(\bar{\nu}_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(z-x)\, d\pi_k(x,z)}{\|z-x\|_{L^2_{\pi_k}}} + C\|y-x\|_{L^2_{\gamma_k}}.$$

(by (42), setting $C := 1 + \|p\|_{L^2_\mu}$). Then using (40), (42) and the assumption of the lemma:

$$\begin{aligned}
&\limsup_{k\to+\infty} \frac{1}{\|y-x\|_{L^2_{\gamma_k}}} (u(\nu_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(y-x) \; d\gamma_k(x,y)) \\
&\qquad \leq \limsup_{k\to+\infty} (1+\|y-x\|_{L^2_{\gamma_k}}) \cdot \frac{u(\bar{\nu}_k) - u(\mu) - \int_{\mathbb{R}^{2d}} p(x)\cdot(z-x)\, d\pi_k(x,z)}{\|z-x\|_{L^2_{\pi_k}}} \leq 0.
\end{aligned}$$

□

*Remark* 3.11. The previous result may be used with $A \subseteq \mathscr{P}_2(\mathbb{R}^d)$ the set of probability measures whose support is a finite set, another example is the set of absolutely continuous probability measures. Recall that, when $\nu$ is absolutely continuous then $\Pi_o(\mu,\nu) = \{(T\times \mathrm{Id})\sharp\nu)\}$ for some $T \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ such that $T\sharp\nu = \mu$. With this remark, it is easily seen that $p \in dis^+(\mu)$ belongs to $D^+_\varepsilon u(\mu)$ if and only if, for all $\nu$ absolutely continuous and all $T \in L^2_\nu(\mathbb{R}^d,\mathbb{R}^d)$ such that $\mu = T\sharp\nu$:

$$u(\nu) - u(\mu) \leq \int_{\mathbb{R}^d} p(Ty)\cdot(y-Ty) \; d\nu(y) + \varepsilon\|\mathrm{Id}-T\|_{L^2_\nu} + o\left(\|\mathrm{Id}-T\|_{L^2_\nu}\right).$$

3.2. **Differentiability in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ and $\mathscr{P}_2(\mathbb{R}^d)$.** In what follows the scalar product $\langle\cdot,\cdot\rangle_{L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)}$ is shortly denoted by $\langle\cdot,\cdot\rangle_{L^2_{\mathbb{P}}}$ and $\|\cdot\|_{L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)}$ is abbreviated in $\|\cdot\|$.

Consider $u:\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}$ and its lift $U: X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\to u(X\sharp\mathbb{P})$. Following [21] and [38], we say that $u$ is differentiable at $\mu_0\in\mathscr{P}_2(\mathbb{R}^d)$ if its lift $U$ is differentiable in $L^2_{\mathbb{P}}$ at one $X_0\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law $\mu_0$. As already known in [21], [24] for continuous differentiable function and in [33], the Fréchet gradient of $U$, denoted by $DU(X)$, has a specific structure:

**Proposition 3.12.** *Assume that $U: L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\to\mathbb{R}$ is rearrangement invariant and differentiable at $X_0$ of law $X_0\sharp\mathbb{P}=\mu_0$. Then there exists $p\in\mathcal{T}_{\mu_0}(\mathbb{R}^d)$ such that $DU(X_0)=p\circ X_0$. Moreover if $X_1$ is also of law $\mu_0$ then $DU(X_1)=p\circ X_1$.*

This allows to introduce the following definition:

**Definition 3.13.** We say that $u:\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}$ is differentiable at $\mu_0\in\mathscr{P}_2(\mathbb{R}^d)$ if its lift $U$ is differentiable at one $X_0$ of law $\mu_0$. That is, there exists some $p\in\mathcal{T}_{\mu_0}(\mathbb{R}^d)$ such that for any $\nu\in\mathscr{P}_2(\mathbb{R}^d)$ and any $\gamma\in\Pi_o(\mu_0,\nu)$:

$$u(\nu)-u(\mu_0)=\int_{\mathbb{R}^d\times\mathbb{R}^d}p(x)\cdot(y-x)\,d\gamma(x,y)+o(W_2(\mu_0,\nu)).$$

We denote by $D_\mu u(\mu_0):=p$.

We refer to [21] and [2] for examples. In this section, we aim to provide a new proof of Proposition 3.12. As in [33], the proof is based on the following proposition that will be proved, together with Proposition 3.12, at the end of this section:

**Proposition 3.14.** *Let $X\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$. Assume that $U: L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\to\mathbb{R}$ is rearrangement invariant. Let $Z\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ belong to the Fréchet superdifferential of $U$ at $X$, namely, it satisfies for all $Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:*

$$U(Y)-U(X)\le\langle Z,Y-X\rangle_{L^2_{\mathbb{P}}}+o\left(\|Y-X\|\right). \tag{43}$$

*Then $\mathrm{pr}_{H_X}(Z)$ also belongs to the Fréchet superdifferential of $U$ at $X$: for all $Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$,*

$$U(Y)-U(X)\le\langle\mathrm{pr}_{H_X}(Z),Y-X\rangle_{L^2_{\mathbb{P}}}+o\left(\|Y-X\|\right),$$

*moreover if $\mathrm{pr}_{H_X}(Z)=p\circ X$ and $q$ is the projection on $\mathcal{T}_{X\sharp P}(\mathbb{R}^d)$ of $p$, we have again:*

$$U(Y)-U(X)\le\langle q\circ X,Y-X\rangle_{L^2_{\mathbb{P}}}+o\left(\|Y-X\|\right).$$

We refer to (12) for the definition of $H_X$ and to Lemma 2.3 for the characterization of the projection on $H_X$ denotes by $\mathrm{pr}_{H_X}$.

*Remark* 3.15. Proposition 3.12 has been proved in [33] with some extra assumptions on the probability space $(\Omega,B(\Omega),\mathbb{P})$. The proof relies on the Proposition 3.14 and uses a very technical result in [23]. We provide a different and simpler proof only requiring that $\Omega$ is Polish, without using [23].

3.2.1. *Preliminary results.*

**Lemma 3.16.** *Let $U: L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)\to\mathbb{R}$ be rearrangement invariant and let $X,Z\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ such that for all $Y\in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$:*

$$U(Y)-U(X)\le\langle Z,Y-X\rangle_{L^2_{\mathbb{P}}}+o\left(\|Y-X\|\right). \tag{44}$$

*Then for any couple* $(X', Z') \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)^2$ *such that* $(X', Z')\sharp\mathbb{P} = (X, Z)\sharp\mathbb{P}$*:*

$$U(Y) - U(X') \leq \langle Z', Y - X\rangle_{L^2_{\mathbb{P}}} + o\left(\|Y - X'\|\right)$$

*for all* $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$*.*

*Proof.* Let $(Y_n)_n$ be a sequence in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $\lim_{n\to+\infty} \|Y_n - X'\| = 0$. By Lemma 1.1, it exists $\tau_n : \Omega \to \Omega$ one to one such that $\tau_n\sharp\mathbb{P} = \tau_n^{-1}\sharp\mathbb{P} = \mathbb{P}$ and:

$$\|(Z, X) - (Z', X') \circ \tau_n\| \leq \|Y_n - X'\|^2. \tag{45}$$

Then, as $U$ is rearrangement invariant and using Cauchy-Schwarz:

$$\begin{aligned}
U(Y_n) - U(X') - \langle Z', Y_n - X'\rangle_{L^2_{\mathbb{P}}} &= U(Y_n \circ \tau_n) - U(X) - \langle Z' \circ \tau_n, Y_n \circ \tau_n - X' \circ \tau_n\rangle_{L^2_{\mathbb{P}}} \\
&\leq U(Y_n \circ \tau_n) - U(X) - \langle Z, Y_n \circ \tau_n - X' \circ \tau_n\rangle_{L^2_{\mathbb{P}}} + \|Y_n \circ \tau_n - X' \circ \tau_n\| \cdot \|Z - Z'\|\tau_n\| \\
&\leq U(Y_n \circ \tau_n) - U(X) - \langle Z, Y_n \circ \tau_n - X\rangle_{L^2_{\mathbb{P}}} + \|Z\|\|X - X' \circ \tau_n\| + \|Y_n - X'\|^3 \\
&\leq U(Y_n \circ \tau_n) - U(X) - \langle Z, Y_n \circ \tau_n - X\rangle_{L^2_{\mathbb{P}}} + \|Y_n - X'\|^2(\|Z\| + \|Y_n - X'\|).
\end{aligned}$$

Moreover, using again (45), leads to

$$\|Y_n \circ \tau_n - X\| \leq \|Y_n \circ \tau_n - X' \circ \tau_n\| + \|X - X' \circ \tau_n\| \leq \|Y_n - X'\|(1 + \|Y_n - X'\|). \tag{46}$$

This gives

$$\begin{aligned}
&\frac{U(Y_n) - U(X') - \langle Z', Y_n - X'\rangle_{L^2_{\mathbb{P}}}}{\|Y_n - X'\|} \\
&\quad\leq \frac{U(Y_n \circ \tau_n) - U(X) - \langle Z, Y_n \circ \tau_n - X\rangle_{L^2_{\mathbb{P}}}}{\|Y_n \circ \tau_n - X\|}(1 + \|Y_n - X'\|) + \varepsilon(\|Y_n - X'\|),
\end{aligned}$$

where $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\lim_{t\to 0} \varepsilon(t) = 0$.

Then, since $\lim\limits_{n\to+\infty} \|Y_n \circ \tau_n - X\| = 0$ by (46), by letting $n \to +\infty$ in the previous inequality we get the result. □

*Remark* 3.17. *Note that, applying the previous Lemma with* $Z = p \circ X$*, (27) holds true for all* $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ *and all* $\gamma \in \Pi(\mu, \nu)$ *iff*

$$U(Y) - U(X) \leq \int_\Omega (p \circ X) \cdot (Y - X)\, d\mathbb{P} + \varepsilon\|Y - X\|_{L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)} + o(\|Y - X\|_{L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)})$$

*for all* $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ *of law* $\mu_0$ *and all* $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$*. Indeed,*

- *given* $\gamma \in \Pi(\mu_0, \nu)$*, there exist* $X', Y' \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ *s.t.* $(X', Y')\sharp\mathbb{P} = \gamma$*,*
- *given* $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ *of law* $\mu_0$*, and* $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$*, then* $\gamma = (X, Y)\sharp\mathbb{P} \in \Pi(\mu_0, Y\sharp\mathbb{P})$*.*

Next we will use the following specific notations:

$$\pi_x : (x, y, z) \in \mathbb{R}^3 \mapsto x, \qquad \pi_y : (x, y, z) \in \mathbb{R}^3 \mapsto z,$$
$$\pi_{x,y} : (x, y, z) \in \mathbb{R}^3 \mapsto (x, y), \qquad \pi_{x,z} : (x, y, z) \in \mathbb{R}^3 \mapsto (x, z).$$

**Corollary 3.18.** *Assume that* $X, Z$ *satisfy (44). Set* $\gamma = (X, Z)\sharp\mathbb{P}$ *and* $\mu = X\sharp\mathbb{P}$ *and* $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *associated to* $U$*. Then, for any* $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ *and any tri-plan* $\varpi \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ *such that* $\pi_{x,z}\sharp\varpi = \gamma$ *and* $\pi_y\sharp\varpi = \nu$*, it holds:*

$$u(\nu) - u(\mu) \leq \int_{(\mathbb{R}^d)^3} z \cdot (y - x)\ d\varpi(x, y, z) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\varpi(x, y))^{1/2} \right). \tag{47}$$

*Conversely, let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\mu \in P_2(\mathbb{R}^d)$. Assume $\gamma \in \mathscr{P}_2(\mathbb{R}^d)$ satisfying $\pi_x\sharp\gamma = \mu$ is such that (47) holds for any for any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and any tri-plan $\varpi \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $\pi_{x,z}\sharp\varpi = \gamma$ and $\pi_y\sharp\varpi = \nu$. Then, denoting by $U$ the lift of $u$, the assumptions of Lemma 3.16 holds for any $X, Z \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $(X, Z)\sharp\mathbb{P} = \gamma$.*

*Proof.* The first assertion is easily proved by noticing that there exist $X', Y, Z' \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $(X', Y, Z')\sharp\mathbb{P} = \varpi$ and, consequently:

$$\nu = \pi_y\sharp\varpi = Y, \qquad \mu = \pi_x\sharp\varpi = X', \qquad (X, Z)\sharp\mathbb{P} = \gamma = \pi_{x,z}\sharp\varpi = (X', Z')\sharp\mathbb{P},$$

and by Lemma 3.16:

$$\begin{aligned} u(\nu) - u(\mu) =& U(Y) - U(X') \leq \langle Z', Y - X'\rangle_{L^2_{\mathbb{P}}} + o\left(\|Y - X'\|\right) \\ &\leq \int_{(\mathbb{R}^d)^3} z \cdot (y - x)\ d\varpi(x, y, z) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\varpi(x, y))^{1/2} \right). \end{aligned}$$

The converse is similar. □

*Remark* 3.19. Taking $u$ and $U$ as in Corollary 3.18, the previous result makes a direct link between the notion of super-differential in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ (namely $Z \in D^+U(X)$ iff the assumption of the Lemma 3.16 holds) and the notion of strong Fréchet super-differential introduced in [3] (namely $\gamma \in \partial^+u(\mu)$ iff (47) holds for any $\nu$ and $\varpi$ as in the corollary). Precisely, if $(X, Z)\sharp\mathbb{P} = \gamma$ and $X\sharp\mathbb{P} = \mu = \pi_x\sharp\gamma$ we have

$$\gamma \in \partial^+u(\mu) \Leftrightarrow Z \in D^+U(X).$$

3.2.2. *Proof of Proposition 3.14 and Proposition 3.12.*

*Proof.* (of Proposition 3.14) Let $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ arbitrary. We set:

$$\gamma := (X, Z)\sharp\mathbb{P}, \qquad \mu := X\sharp\mathbb{P}, \qquad \nu = Y\sharp\mathbb{P}, \qquad \rho = (X, Y)\sharp\mathbb{P}.$$

By disintegration, $\gamma$ and $\rho$ write as

$$\gamma(x, z) = \gamma^x(z) \otimes \mu(x), \quad \rho(x, y) = \rho^x(y) \otimes \mu(x).$$

Then, setting $\varpi(x, y, z) = \gamma^x(z) \otimes \rho^x(y) \otimes \mu(x)$ we get a tri-plan satisfying:

$$\pi_{x,y}\sharp\varpi = \rho, \quad \pi_y\sharp\varpi = \pi_y\sharp\rho = \nu, \quad \pi_{x,z}\sharp\varpi = \gamma.$$

We apply the first assertion of Corollary 3.18:

$$\begin{aligned} U(Y) - U(X) =& u(\nu) - u(\mu) \\ &= \int_{(\mathbb{R}^d)^3} z \cdot (y - x)\ d\varpi(x, y, z) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\varpi(x, y, z))^{1/2} \right) \\ &\leq \int_{(\mathbb{R}^d)^3} z \cdot (y - x)\ d\gamma^x(z) d\rho^x(y)\ d\mu(x) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\rho(x, y))^{1/2} \right) \\ &= \int_{(\mathbb{R}^d)^2} [\int_{\mathbb{R}^d} z d\gamma^x(z)] \cdot (y - x)\ d\rho(x, y) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\rho(x, y))^{1/2} \right). \end{aligned}$$

Recalling Lemma 2.3, we have $p(x) = [\int_{\mathbb{R}^d} z d\gamma^x(z)]$ with $p \circ X = pr_{H_X}(Z)$, and as $(X, Y)\sharp\mathbb{P} = \rho$:

$$\begin{aligned} U(Y) - U(X) \leq& \int_{(\mathbb{R}^d)^2} p(x) \cdot (y - x)\ d\rho(x, y) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\rho(x, y))^{1/2} \right) \\ =& \langle p \circ X, Y - X\rangle_{L^2_{\mathbb{P}}} + o\left(\|Y - X\|\right). \end{aligned}$$

The first assertion of the Proposition is proved.

To prove the second assertion, notice that by the computation above, for any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, an any optimal $\rho \in \Pi_o(\mu, \nu)$:

$$u(\nu) - u(\mu) \leq \int_{(\mathbb{R}^d)^2} p(x) \cdot (y - x)\ d\rho(x, y) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\rho(x, y))^{1/2} \right).$$

Now, $x \mapsto \int y\ d\rho^x(y) - x$ being an optimal displacement, it is in $\mathcal{T}_\mu(\mathbb{R}^d)$ and:

$$\begin{aligned} u(\nu) - u(\mu) &\leq \int_{\mathbb{R}^d} p(x) \cdot \left[ \int y\ d\rho^x(y) - x \right]\ d\mu(x) + o(W_2(\mu, \nu)) \\ &\leq \int_{\mathbb{R}^d} \mathrm{pr}_{\mathcal{T}_\mu}(p)(x) \cdot \left[ \int y\ d\rho^x(y) - x \right]\ d\mu(x) + o(W_2(\mu, \nu)) \\ &= \int_{\mathbb{R}^d} \mathrm{pr}_{\mathcal{T}_\mu}(p)(x) \cdot (y - x)\ d\rho(x, y) + o\left( (\int_{(\mathbb{R}^d)^2} |y - x|^2 d\rho(x, y))^{1/2} \right). \end{aligned}$$

The conclusion follows by Proposition 3.4 and Remark 3.17 . □

*Proof.* (of Proposition 3.12). We have that, for all $Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:

$$U(Y) - U(X_0) = \langle DU(X_0), Y - X_0 \rangle_{L^2_{\mathbb{P}}} + o\left( \|Y - X_0\| \right),$$

so that by Proposition 3.14, we also have:

$$U(Y) - U(X_0) \leq \langle p \circ X_0, Y - X_0 \rangle_{L^2_{\mathbb{P}}} + o\left( \|Y - X_0\| \right),$$

where $p$ is the projection on $\mathcal{T}_{X_0 \sharp \mathbb{P}}(\mathbb{R}^d)$ of $\tilde{p} \circ X =: \mathrm{pr}_{H_{X_0}}(DU(X_0))$. In a symmetric way, we could get:

$$U(Y) - U(X_0) \geq \langle p \circ X_0, Y - X_0 \rangle_{L^2_{\mathbb{P}}} + o\left( \|Y - X_0\| \right),$$

so that $DU(X) = p \circ X$ and $p$ is in $\mathcal{T}_{X_0 \sharp \mathbb{P}}(\mathbb{R}^d)$. The last assertion follows from Lemma 3.16. □

3.3. **Viscosity solutions.** We recall the definition of viscosity sub and supersolution associated to the previous definitions of sub and superdifferential (see [37]).

**Definition 3.20** (Viscosity Solutions)**.** A function $w : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is

- a *subsolution* of (HJ) if $w$ is upper semicontinuous and there exists $C > 0$ such that for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p \in D^+_\varepsilon w(\mu)$, and $\varepsilon > 0$
$$\mathcal{H}(\mu, p) \geq -C\varepsilon.$$
- a *supersolution* of (HJ) if $w$ is lower semicontinuous and a constant $C > 0$ exists such that $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p \in D^-_\varepsilon w(t, \mu)$, and $\varepsilon > 0$
$$\mathcal{H}(\mu, p) \leq C\varepsilon.$$
- a *solution* of (HJ) if $w$ is both a supersolution and a subsolution.

We refer to [39], [37] for comparison principle using these notions with some quite weak assumptions.

We will assume some regularity for the Hamiltonian associated with (HJ):

**(A)** For all $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, the map $p \in \mathcal{T}_{\mu_0}(\mathbb{R}^d) \mapsto \mathcal{H}(\mu_0, p)$ is continuous in $L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$.

We also introduce a Hamiltonian on the set

$$\{(X, p \circ X) : \ X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d), \ p \in \mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d)\}$$

by $H(X, p \circ X) := \mathcal{H}(X\sharp\mathbb{P}, p)$, and the corresponding Hamilton Jacobi equation:

$$(\overline{HJ}) \qquad H(X, DU(X)) = 0.$$

3.3.1. *Properties of the superdifferential.* We provide some properties of $D^+_\varepsilon u(\mu_0)$ and relations with the following superdifferential introduced in [4]: $p \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$ belongs to $D^+_{AG}u(\mu_0)$ if it satisfies for all $\gamma \in \Pi_o(\mu_0, \nu)$:

$$u(\nu) - u(\mu_0) \leq \int p(x) \cdot (y - x) \ d\gamma(x, y) + o\left(W_2(\mu_0, \nu)\right).$$

*Remark* 3.21. By Proposition 3.5 and Remark 3.8, when $\mu$ has no atom, taking $\varepsilon = 0$

$$\mathrm{pr}_{\mathcal{T}_\mu(\mathbb{R}^d)}(D^+_{CQ,0}u(\mu)) = D^+_{AG}u(\mu).$$

We prove that superdifferentials are nonempty when $\mu_0$ belongs to some dense set. Moreover we give a link between superdifferentials of [39],[37],[33].

**Proposition 3.22.** *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous. Then there exists $\mathcal{A}$ a dense subset of $\mathscr{P}_2(\mathbb{R}^d)$ such that we have:*

(i) *$D^+_{AG}u(\mu_0)$ is non empty for any $\mu_0 \in \mathcal{A}$,*
(ii) *for all $\varepsilon > 0$, $D^+_\varepsilon u(\mu_0)$ is non empty for any $\mu_0 \in \mathcal{A}$,*
(iii) *it holds $D^+_{AG}u(\mu_0) = \{p = \lim_{\varepsilon\to 0} p_\varepsilon : \ p_\varepsilon \in D^+_\varepsilon u(\mu_0)\}$.*

*The last assertion is true even when u is not continuous.*

*Proof.* We prove (i) and (iii), assertion (ii) follows.

(i) Let $U$ the lift of $u$ and $X_0 \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ a random variable of law $\mu_0$. Note that $U$ is also continuous. Take also $R > 0$, $\varepsilon > 0$ and consider the following function:

$$V_\varepsilon(Y) := \begin{cases} U(Y) - \dfrac{\|Y - X_0\|^2}{\varepsilon}, & \text{if } \|Y - X_0\| \leq R, \\ -\infty, & \text{otherwise.} \end{cases}$$

This function being upper semi-continuous, by Stegall's variational principle (Theorem 8.8 p 55 of [21]), there exists $\xi \in L^2_{\mathbb{P}}$ with $\|\xi\| \leq \varepsilon$ and such that $V_\varepsilon - \langle \xi, \cdot \rangle$ attains its maximum at some $X^*$. By definition of $V_\varepsilon$, we have $\|X^* - X_0\| \leq R$. Since $V_\varepsilon(X_0) - \langle \xi, X_0 \rangle \leq V_\varepsilon(X^*) - \langle \xi, X^* \rangle$ we get

$$\|X^* - X_0\|^2 \leq \varepsilon(U(X^*) - U(X_0)) + \varepsilon^2\|X^* - X_0\|.$$

Then, for $\varepsilon$ small, $\|X^* - X_0\| < R$, and for $Y$ in a neighborhood of $X_0$, $V_\varepsilon(Y) - \langle \xi, Y \rangle \leq V_\varepsilon(X^*) - \langle \xi, X^* \rangle$ implies

$$U(Y) - U(X^*) \leq \langle \frac{2}{\varepsilon}(X^* - X_0) + \xi, Y - X^* \rangle + \frac{\|Y - X^*\|^2}{\varepsilon}.$$

This means $\frac{2}{\varepsilon}(X^* - X_0) + \xi$ satisfies the condition of Proposition 3.14. Then, by applying the proposition, we get some $p \in \mathcal{T}_{X^*\sharp\mathbb{P}}(\mathbb{R}^d)$ such that:

$$U(Y) - U(X^*) \leq \langle p \circ X^*, Y - X^* \rangle + o(\|Y - X^*\|).$$

The conclusion follows using Lemma 3.16 and Remark 3.17.

(iii) First we show $D^+_{AG}u(\mu_0) \subset \{p = \lim_{\varepsilon\to 0} p_\varepsilon \ : \ \ p_\varepsilon \in D^+_\varepsilon u(\mu_0)\}$. Let $p \in D^+_{AG}u(\mu_0)$, by definition of $\mathcal{T}_\mu(\mathbb{R}^d)$, there exists $\{p_\varepsilon\}_\varepsilon$ in $dis^+(\mu_0)$ such that $\|p_\varepsilon - p\|_{L^2_{\mu_0}} \leq \varepsilon$. Then for all $\nu$ in $\mathscr{P}_2(\mathbb{R}^d)$ and all $\gamma \in \Pi_o(\mu_0, \nu)$:

$$\begin{aligned} u(\nu) - u(\mu_0) &\leq \int p(x)\cdot(y-x)\ d\gamma(x,y) + o\left(W_2(\mu_0,\nu)\right) \\ &\leq \int p_\varepsilon(x)\cdot(y-x)\ d\gamma(x,y) + \|p_\varepsilon - p\|_{L^2_{\mu_0}} W_2(\mu_0,\nu) + o\left(W_2(\mu_0,\nu)\right) \\ &\leq \int p_\varepsilon(x)\cdot(y-x)\ d\gamma(x,y) + \varepsilon W_2(\mu_0,\nu) + o\left(W_2(\mu_0,\nu)\right). \end{aligned}$$

So we have the desired inclusion. We prove the converse. Let $p = \lim_{\varepsilon\to 0} p_\varepsilon$ with $p_\varepsilon \in D^+_\varepsilon u(\mu_0)$. As, for any $\varepsilon$, $p_\varepsilon \in dis^+(\mu_0)$ we have $p \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$. Now take $\{\nu_k\}_k$ a sequence of $\mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_k \in \Pi_o(\mu,\nu_k)$ such that:

$$\lim_{k\to+\infty} W_2^2(\mu,\nu_k) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^2\ d\gamma_k(x,y) = 0.$$

We have by Cauchy-Schwarz:

$$\begin{aligned} u(\nu_k) - u(\mu_0) &- \int p(x)\cdot(y-x)d\gamma_k(x,y) \\ &\leq u(\nu_k) - u(\mu_0) - \int p_\varepsilon(x)\cdot(y-x)d\gamma_k(x,y) + \|p_\varepsilon - p\|_{L^2_{\mu_0}}. \end{aligned}$$

So for every $\varepsilon > 0$:

$$\limsup_{k\to+\infty} \frac{u(\nu_k) - u(\mu_0) - \displaystyle\int p(x)\cdot(y-x)d\gamma_k(x,y)}{W_2(\nu_k,\mu_0)} \leq \varepsilon + \|p_\varepsilon - p\|_{L^2_{\mu_0}}.$$

Letting $\varepsilon \to 0^+$ yields the result. □

**Proposition 3.23.** *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be $k$-Lipschitz, $\mu$ in $\mathscr{P}_2(\mathbb{R}^d)$. Then:*

(i) *For all $\varepsilon > 0$, and all $q \in D^+_\varepsilon u(\mu)$ we have $\|q\|_{L^2_\mu} \leq k + \varepsilon$.*
(ii) *Take $p_\varepsilon \in D^+_\varepsilon u(\mu)$ for all $\varepsilon > 0$. Up to a subsequence $\{p_\varepsilon\}_\varepsilon$ admits an $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$-weak limit $p$ as $\varepsilon$ tend to $0$. Moreover $p \in D^+_{AG}(\mu)$.*

*Proof.* Denote by $U$ the lift of $u$ and let $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ of law $\mu$. By Corollary A.5 3), the map $U$ is $k$-Lipschitz. We have for all $Y \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$,

$$U(Y) - U(X) \leq \langle q\circ X, Y - X\rangle + \varepsilon\|Y - X\| + o(\|Y - X\|).$$

Then, applying this inequality with $Y = -t(q\circ X)$ with $t \in \mathbb{R}$ leads:

$$U(-t(q\circ X)) - U(X) \leq -t\|q\circ X\|^2 + \varepsilon t\|q\circ X\| + o(t)$$

and using the Lipschitz property of $U$:

$$t\|q\circ X\|^2 \leq (k+\varepsilon)t\|q\circ X\| + o(t).$$

The property (i) follows by dividing by $t$ and letting $t \to 0^+$.

Then, take the sequence $\{p_\varepsilon\}_\varepsilon$ defined in (ii). Up to a subsequence (similarly denoted) $\{p_\varepsilon\}_\varepsilon$ admits an $L^2_{\mathbb{P}}$-weak limit $p$ as $\varepsilon \to 0^+$. Now let $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_n \in \Pi_o(\mu,\nu_n)$. Possibly extracting a subsequence, we may assume that

$$\limsup_{n\to+\infty} \frac{u(\nu_n)-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)d\gamma_n(x,y)}{W_2(\mu,\nu_n)} = \lim_{n\to+\infty} \frac{u(\nu_n)-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)d\gamma_n(x,y)}{W_2(\mu,\nu_n)}.$$

Setting $r_n := W_2(\mu,\nu_n)$, by Jensen's inequality, it holds:

$$\left(\int_{\mathbb{R}^d}\left|\frac{1}{r_n}\left(\int_{\mathbb{R}^d} y\,d\gamma_n^x(y)-x\right)\right|^2 d\mu(x)\right)^{1/2} = \left(\int_{\mathbb{R}^d}\left|\frac{1}{r_n}\int_{\mathbb{R}^d} y-x\,d\gamma_n^x(y)\right|^2 d\mu(x)\right)^{1/2} \le \frac{\left(\int_{\mathbb{R}^d}|y-x|^2\,d\gamma(x,y)\right)^{1/2}}{W_2(\mu,\nu_n)} \le 1.$$

Then, set $q_n : x \mapsto \frac{1}{r_n}\left(\int_{\mathbb{R}^d} y\,d\gamma_n^x(y) - x\right)$, the sequence $\{q_n\}_{n\in\mathbb{N}}$ is bounded, so we can extract a subsequence $\{q_{n_k}\}_{k\in\mathbb{N}}$ weakly convergent to some $q\in L^2_\mu$. For all $\varepsilon>0$ we have

$$\begin{aligned}
&\limsup_{n\to+\infty}\frac{1}{r_n}\cdot\left[u(\nu_n)-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)d\gamma_n(x,y)\right]=\\
&=\lim_{k\to+\infty}\frac{1}{r_{n_k}}\cdot\left[u(\nu_{n_k})-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d} p(x)\cdot(y-x)d\gamma_{n_k}(x,y)\right]\\
&=\lim_{k\to+\infty}\frac{1}{W_2(\mu,\nu_{n_k})}\cdot\left[u(\nu_{n_k})-u(\mu)-\int_{\mathbb{R}^d\times\mathbb{R}^d} p_\varepsilon(x)\cdot(y-x)d\gamma_{n_k}(x,y)\right]+\\
&\quad+\frac{1}{r_{n_k}}\cdot\left[\int(p_\varepsilon-p)(x)\cdot(y-x)\,d\gamma_{n_k}(x,y)\right]\\
&\le\varepsilon+\lim_k\int(p_\varepsilon-p)(x)\cdot\frac{1}{r_{n_k}}\left(\int y d\gamma^x_{n_k}(y)-x\right)\,d\mu(x)\\
&=\varepsilon+\lim_k\int(p_\varepsilon-p)(x)\cdot q_{n_k}(x)\,d\mu(x)=\varepsilon+\int(p_\varepsilon-p)(x)\cdot q(x)\,d\mu(x).
\end{aligned}$$

By letting $\varepsilon\to0^+$, we obtain (ii). □

Proposition 3.22 provides some links between notions of subsolutions:

**Corollary 3.24.** *Assume assumption **(A)** holds. Let $u:\mathscr{P}_2(\mathbb{R}^d)\to\mathbb{R}$ be a subsolution of (HJ) and let $C>0$ the constant appearing in definition 3.20. Then*
*(i) $\mathcal{H}(\mu,p)\ge0$ for all $p\in D^+_{AG}u(\mu_0)$,*
*(ii) For all $\varepsilon>0$ and all $p\in\mathcal{T}_\mu(\mathbb{R}^d)$ such that for all $\nu\in\mathscr{P}_2(\mathbb{R}^d)$, all $\gamma\in\Pi_o(\mu_0,\nu)$*

$$u(\nu)-u(\mu_0)\le\int p(x)\cdot(y-x)\,d\gamma(x,y)+\varepsilon W_2(\mu_0,\nu)+o(W_2(\mu_0,\nu)),$$

*we have $\mathcal{H}(\mu,p)\ge -C\varepsilon$.*

*Proof.* We only prove $(ii)$. Arguing as in the previous proof, for any $\delta>0$ there exists $p_\delta\in D^+_{\varepsilon+\delta}u(\mu_0)$ such that $\|p_\delta-p\|_{L^2_{\mu_0}}\le\delta$ and

$$\mathcal{H}(\mu,p)\ge -C(\varepsilon+\delta).$$

The result follows by letting $\delta\to0^+$. □

3.3.2. *Test Functions.* We want to express the notion of Hamilton-Jacobi solution in $\mathscr{P}_2(\mathbb{R}^d)$ in terms of test functions, defined as follows.

**Definition 3.25.** Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$.
$v : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is an $\varepsilon$-*supertest* function for $u$ at $\mu_0$ if it is continuous, differentiable at $\mu_0$ and there exists $r > 0$ such that: $u(\mu_0) = v(\mu_0)$ and

$$u(\nu) \leq v(\nu) + \varepsilon W_2(\mu_0, \nu) \quad \forall \nu \in \mathscr{P}_2(\mathbb{R}^d) \text{ such that } W_2(\mu_0, \nu) < r.$$

$v$ is an $\varepsilon$-*subtest* function for $u$ at $\mu_0$ if $-v$ is a $\varepsilon$-*supertest* function for $-u$ at $\mu_0$

We also have similar $\varepsilon$-test functions in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:

**Definition 3.26.** Let $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$, $X_0 \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ and $\varepsilon > 0$.
$V : L^2_{\mathbb{P}}(\mathbb{R}^d) \to \mathbb{R}$ is an $\varepsilon$-supertest function for $U$ at $X_0$ if it is continuous, differentiable at $X_0$ and there exists $r > 0$ such that:

$$U(X_0) = V(X_0)$$

$$U(Y) \leq V(Y) + \varepsilon \|Y - X_0\| \quad \forall Y \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \text{ such that } \|Y - X_0\| < r.$$

$V$ is an $\varepsilon$-subtest function for $U$ at $X_0$ if $-V$ is $\varepsilon$-supertest function for $U$ at $X_0$.

We wish to give a result comparing both above notions

**Theorem 3.27.** *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous and $U : L^2_{\mathbb{P}} \to \mathbb{R}$ be its lift.*
*Assume **(A)** holds. Then the following assertions are equivalent:*
(i) *$u$ is a viscosity subsolution of (HJ),*
(ii) *there exists $C > 0$ such that for all $\varepsilon > 0$, all $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and all $\varepsilon$-supertest function $v$ of $u$ at $\mu_0$ it holds*

$$\mathcal{H}(\mu_0, D_\mu v(\mu_0)) \geq -C\varepsilon.$$

(iii) *there exists $C > 0$ such that for all $\varepsilon > 0$, all $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and all rearrangement invariant $\varepsilon$-supertest function $V$ of $U$ at some $X_0$ of law $\mu_0$ it holds*

$$H(X_0, DV(X_0)) \geq -C\varepsilon.$$

To prove this theorem we need some preliminary results.

**Lemma 3.28.** *Let $U : L^2_{\mathbb{P}} \to \mathbb{R}$ be rearrangement invariant, $X_0 \in L^2_{\mathbb{P}}$ and $\varepsilon > 0$. Let $V$ be a rearrangement invariant $\varepsilon$-supertest function of $U$ at $X_0$. Let $r > 0$ be the constant appearing in the definition of supertest function. Then $V$ is an $\varepsilon$-supertest function of $U$ at any $X \in L^2_{\mathbb{P}}$ with the same law of $X_0$ and the constant $r$.*

*Proof.* By Lemma 1.1, for any $n \in \mathbb{N}^*$, there exists $\tau_n : \Omega \to \Omega$, (measurable, invertible with $\tau_n \sharp \mathbb{P} = \tau_n^{-1} \sharp \mathbb{P} = \mathbb{P}$) such that: $\|X - X_0 \circ \tau_n\| \leq \frac{1}{n}$. Let $Y \in L^2_{\mathbb{P}}$ satisfies $\|X - Y\| < r$, then for $n$ big enough we have:

$$\|X_0 - Y \circ \tau_n^{-1}\| = \|X_0 \circ \tau_n - Y\| \leq \|X - Y\| + \|X_0 \circ \tau_n - X\| < r.$$

Then as $V$ is an $\varepsilon$-supertest function at $X_0$, we have:

$$U(Y \circ \tau_n^{-1}) \leq V(Y \circ \tau_n^{-1}) + \varepsilon \|X_0 - Y \circ \tau_n^{-1}\|.$$

As $U$ and $V$ are rearrangement invariant, this leads to

$$U(Y) \leq V(Y) + \varepsilon \|X_0 - Y \circ \tau_n^{-1}\| \leq V(Y) + \varepsilon \|X_0 \circ \tau_n - Y\| \leq V(Y) + \varepsilon \|X - Y\| + \frac{\varepsilon}{n}.$$

Letting $n$ tend to the infinity gives the result. ☐

The representation of Wasserstein distance (6) gives immediately

**Corollary 3.29.** *Let $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ its lift.*

a) *Let $\varepsilon > 0$ and $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and $v$ an $\varepsilon$-supertest function of $u$ at $\mu_0$. Then the lift $V$ of $v$ is an $\varepsilon$-supertest function of $U$ at any $X_0$ of law $\mu_0$.*

b) *Let $\varepsilon > 0$, $X_0 \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ and $V$ an $\varepsilon$-supertest function of $U$ at $X_0$. Assume $V$ is rearrangement invariant. Given any $\nu \in \mathscr{P}_2(\mathbb{R}^d, \mathbb{R}^d)$, set*

$$v(\nu) := V(Y) \text{ for any } Y \text{ of law } \nu.$$

*Then the map $v$ is an $\varepsilon$-supertest function of the lift $u$ at $\mu_0$ the law of $X_0$.*

**Proposition 3.30.** *Take $u$ continuous on $P_2(\mathbb{R}^d)$, $\varepsilon > 0$ and $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$. Then:*

a) *if $v$ an $\varepsilon$-supertest function of $u$ at $\mu_0$, its gradient $D_\mu v(\mu_0)$ can be approximate in $L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ by a sequence $\{p_n\}_n$ such that $p_n \in D^+_{\varepsilon+1/n} u(\mu_0)$.*

b) *if $p \in L^2_{\mu_0}(\mathbb{R}^d, \mathbb{R}^d)$ belongs to $D^+_\varepsilon u(\mu_0)$, there exists a sequence $\{v_n\}_n$ of $(\varepsilon + 1/n)$-supertest functions of $u$ at $\mu_0$ such that:*

$$\lim_{n\to+\infty} \|D_\mu v_n(\mu_0) - p\| = 0.$$

We need a technical Lemma whose proof is very similar to Lemma 3.1.8 in [18]

**Lemma 3.31.** *Let $R > 0$ and $\omega : ]0, R] \to \mathbb{R}$ be a lower semicontinuous such that $\lim_{t\to 0^+} \omega(t) = 0$. Then there exists $\omega_0 : [0, \frac{R}{2}] \to \mathbb{R}$ such that:*

a) *$\omega(\tau) \le \omega_0(\tau)$ for all $\tau \in ]0, \frac{R}{2}]$,*

b) *$\omega_0$ is continuous on $[0, \frac{R}{2}[$,*

c) *$\omega_0(\tau) = 0$.*

*Proof.* (of Proposition 3.30)

a) As $D_\mu v(\mu_0) \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$, there exists $\{p_n\}_{n\in\mathbb{N}^*}$ in $dis^+_0(\mu_0)$ such that:

$$\|p_n - D_\mu v(\mu_0)\|_{L^2_{\mu_0}(\mathbb{R}^d,\mathbb{R}^d)} \le \frac{1}{n}.$$

Then, for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and all $\gamma \in \Pi_0(\mu_0, \nu)$:

$$\begin{aligned} u(\nu) - u(\mu_0) &\le v(\nu) - v(\mu_0) + \varepsilon W_2(\mu_0, \nu) \le \\ &\le \left[\int D_\mu v(\mu_0)(x) \cdot (y - x) d\gamma(x, y) + o\left(W_2(\mu_0, \nu)\right)\right] + \varepsilon W_2(\mu_0, \nu) \\ &\le \int p_n(x) \cdot (y - x) d\gamma(x, y) + (\varepsilon + 1/n) W_2(\mu_0, \nu) + o\left(W_2(\mu_0, \nu)\right). \end{aligned}$$

b) As $p \in \mathcal{T}_{\mu_0}(\mathbb{R}^d)$, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}^*}$ in $C^\infty_c(\mathbb{R}^d)$ such that:

$$\|\nabla\varphi_n - p\|_{L^2_{\mu_0}} \le \frac{1}{n}. \tag{48}$$

Then setting $w_n(\nu) = \int \varphi_n(x) d\nu$ for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, it is continuous and

$$D_\mu w_n(\mu_0) = \nabla\varphi_n, \quad \|D_\mu w_n(\mu_0) - p\|_{L^2_{\mu_0}} \le \frac{1}{n}.$$

Moreover as $p \in D^+_\varepsilon u(\mu_0)$ we have in view of (48)

$$\limsup_{\substack{W_2(\mu_0,\nu)\to 0\\ \gamma\in\Pi_o(\mu_0,\nu)}} \frac{u(\nu)-u(\mu_0)-\displaystyle\int \nabla\varphi_n(x)\cdot(y-x)\;d\gamma(x,y)}{W_2(\mu_0,\nu)} \leq$$

$$\leq \limsup_{\substack{W_2(\mu_0,\nu)\to 0\\ \gamma\in\Pi_o(\mu_0,\nu)}} \frac{u(\nu)-u(\mu_0)-\displaystyle\int p(x)\cdot(y-x)\;d\gamma(x,y)}{W_2(\mu_0,\nu)} + \frac{1}{n} \leq \left(\varepsilon+\frac{1}{n}\right).$$

Since $D_\mu w_n(\mu_0) = \nabla\varphi_n$

$$\alpha := \limsup_{W_2(\mu_0,\nu)\to 0} \frac{u(\nu)-u(\mu_0)-w_n(\nu)+w_n(\mu_0)}{W_2(\mu_0,\nu)} - (\varepsilon+1/n) \leq 0$$

If $\alpha < 0$ then setting $v(\nu) := u(\mu_0) + w_n(\nu) - w_n(\mu_0)$ the proof is concluded.

Assume $\alpha = 0$. Then set for all $r > 0$

$$\omega(r) = \sup_{W_2(\mu_0,\nu)\leq r} \frac{u(\nu)-u(\mu_0)-w_n(\nu)+w_n(\mu_0)}{W_2(\mu_0,\nu)} - (\varepsilon+1/n)$$

This function is non-decreasing, bounded on some $]0,R[$, and it satisfies $\lim_{r\to 0^+}\omega(r) = 0$. Assume that this function is measurable (we will prove it later). Then we use the previous lemma and set for all $\nu$ with $W_2(\mu_0,\nu) < \frac{R}{2}$

$$v_n(\nu) := u(\mu_0) + w_n(\nu) - w_n(\mu_0) + W_2(\mu_0,\nu)\omega_0(W_2(\mu_0,\nu)).$$

Moreover we have

$$\lim_{W_2(\mu_0,\nu)\to 0} \frac{W_2(\mu_0,\nu)\omega_0(W_2(\mu_0,\nu)) - W_2(\mu_0,\mu_0)\omega_0(W_2(\mu_0,\mu_0))}{W_2(\mu_0,\nu)} = 0.$$

Thus $v_n$ is continuous and differentiable at $\mu_0$ and

$$D_\mu v_n(\mu_0) = \nabla\varphi_n \text{ and } v_n(\mu_0) = u(\mu_0)$$

$$\forall\nu \text{ such that } W_2(\mu_0,\nu) < \frac{R}{2}: \quad u(\nu) \leq v_n(\nu) + (\varepsilon+1/n)W_2(\mu_0,\nu).$$

The result is proved.

It remains to prove that $\omega$ is l.s.c., hence measurable. Indeed let $\rho_0 \in ]0,R[$ and take $\rho_k \to \rho_0$ such that $\liminf_{\rho\to\rho_0}\omega(\rho) = \lim_{k\to+\infty}\omega(\rho_k)$. We want to show that $\lim_{k\to+\infty}\omega(\rho_k) \geq \omega(\rho_0)$. If $(\rho_k)_k$ admits a non-increasing sub-sequence we are done because $\omega$ is non-decreasing.

Let us assume $(\rho_k)_k$ is non-decreasing. Let $\delta > 0$ and $\nu$ be $\delta$-optimal for $\omega(\rho_0)$. Take $t\in[0,1]\to\nu_t$ a geodesic curve joining $\nu$ and $\mu_0$. For $k$ big enough we can find $\nu_{t_k}$ such that $W_2(\mu_0,\nu_{t_k}) = \rho_k$. Then:

$$(\varepsilon+1/n) + \omega(\rho_k) \geq \frac{u(\nu_{t_k})-u(\mu_0)-w_n(\nu_{t_k})+w_n(\mu_0)}{\rho_k}.$$

As $W_2(\nu_{t_k},\nu) = W_2(\mu_0,\nu) - W_2(\nu_{t_k},\mu_0) = (\rho_0-\rho_k)$, by continuity of $u$ and $w_n$,

$$\lim_{k\to+\infty}\omega(\rho_k) \geq \frac{u(\nu)-u(\mu_0)-w_n(\nu)+w_n(\mu_0)}{\rho_0} - (\varepsilon+1/n) \geq \omega(\rho_0) - \delta.$$

By the arbitrariness of $\delta > 0$, we get the desired regularity. $\square$

*Proof.* (of Theorem 3.27) First we show $(i) \Rightarrow (ii)$. Let $C$ be the constant appearing in definition 3.20. We take $\varepsilon > 0$, $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and $v$ any $\varepsilon$-supertest function of $u$ at $\mu_0$. By Proposition 3.30, a), there exists $(p_n)_n$ in $D^+_{\varepsilon+1/n}u(\mu_0)$ such that

$$\lim_{n\to+\infty} \|p_n - D_\mu v(\mu_0)\|_{L^2_{\mu_0}} = 0.$$

Then, by (i),

$$\mathcal{H}(\mu_0, p_n) \geq -C(\varepsilon + 1/n)$$

and using **(A)** and letting $n \to +\infty$

$$\mathcal{H}(\mu_0, D_\mu v(\mu_0)) \geq -C\varepsilon.$$

Using Corollary 3.29 and Theorem 3.12, we have $(ii) \Leftrightarrow (iii)$.

The proof of $(ii) \Rightarrow (i)$ follows from Proposition 3.30, b). □

3.4. **To study Hamilton Jacobi equations in $L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$ or in $\mathscr{P}_2(\mathbb{R}^d)$?** It is a natural question to ask weather HJB can be studied as an equation in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ with the usual notion of viscosity solution in $L^2_{\mathbb{P}}$. This leads to several questions:

1) In order to give a definition of $H(X, DV(X))$ for any test function $V : L^2_{\mathbb{P}} \to \mathbb{R}$, we need to extend $H$ to the whole $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)^2$.

2) The extension $\tilde{H}$ should be chosen in order to get some equivalences between $L^2_{\mathbb{P}}$-solutions of the extended equation and $\mathscr{P}_2$-solutions of $(HJ)$. More precisely, provided $\tilde{H}$ is rearrangement invariant (see definition 3.32), it is easily seen that any rearrangement invariant $L^2_{\mathbb{P}}$-solution of the extended equation is the lift of a $\mathscr{P}_2$-solution of $(HJ)$. The opposite property is more involved.

3) As we would like to apply usual results in $L^2_{\mathbb{P}}$ to the extended equation, we want $\tilde{H}$ to preserve the regularity of $\mathcal{H}$.

We would like to share our reflections on the subject.

In this section, we will use consider the following sets

$$\begin{aligned}\mathcal{F}_2(\mathbb{R}^d) :=& \{(\mu,p) :\ p \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d),\ \mu \in \mathscr{P}_2(\mathbb{R}^d)\},\\ F_2 :=& \{(X,p) : X\sharp\mathbb{P} \in \mathscr{P}_2(\mathbb{R}^d),\ p \in L^2_{X\sharp\mathbb{P}}(\mathbb{R}^d;\mathbb{R}^d)\}.\end{aligned}$$

We also give the definition:

**Definition 3.32** (Rearrangement invariance of Hamiltonians)**.** Given $D \subseteq L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)\times L^2_{\mathbb{P}}(\Omega;\mathbb{R}^d)$, a function $\hat{H} : D \to \mathbb{R}$ is called *rearrangement invariant* on $D$ if $\hat{H}(X,\xi) = \hat{H}(Y,\zeta)$ for all $(X,\xi)$, $(Y,\zeta) \in D$ satisfying $(X,\xi)\sharp\mathbb{P} = (Y,\zeta)\sharp\mathbb{P}$.

Note that the lift $H$ is rearrangement invariant on $F_2$.

3.4.1. *Comparing convergences in $L^2_{\mathbb{P}}$ and $\mathscr{P}_2(\mathbb{R}^d)$.* For any sequence $\{(X_n, p\circ X_n)\}_n$ in $F_2$, we consider here two natural types of convergences in $L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)^2$: the strong/weak convergence and the strong/strong convergence. In this section we study the corresponding notions in $\mathcal{F}_2(\mathbb{R}^d)$. First, we introduce the following distance in $\mathcal{F}_2(\mathbb{R}^d)$:

$$d_{\mathcal{F}_2}((\mu_1,p_1),(\mu_2,p_2)) := W_2((\mathrm{Id}_{\mathbb{R}^d},p_1)\sharp\mu_1, (\mathrm{Id}_{\mathbb{R}^d},p_2)\sharp\mu_2).$$

In addition to the topology induced by $d_{\mathcal{F}_2}$, following [3], we introduce the following notion of convergence.

**Definition 3.33** (Strong/weak onvergence in $\mathcal{F}_2(\mathbb{R}^d)$)**.** Let $\{(\mu_n,p_n)\}_{n\in\mathbb{N}}$ and $(\mu,p)$ be in $\mathcal{F}_2(\mathbb{R}^d)$. We say that $\{(\mu_n,p_n)\}_{n\in\mathbb{N}}$ converges strong/weak (s/w-converges in short) to $(\mu,p)$ if

- $\lim_{n\to+\infty} W_2(\mu_n, \mu) = 0$;
- $p_n\mu_n \overset{*}{\rightharpoonup} p\mu$ as a sequence of vector-valued measures, i.e., for all $\Phi \in C^0_b(\mathbb{R}^d, \mathbb{R}^d)$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \Phi(x) \cdot p_n(x)\, d\mu_n(x) = \int_{\mathbb{R}^d} \Phi(x) \cdot p(x)\, d\mu(x);$$

- $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |p_n|^2 d\mu_n < +\infty$.

By [3] Theorem 5.4.4. p127, $\{(\mu_n, p_n)\}_{n\in\mathbb{N}}$ converges to $(\mu, p)$ for $d_{\mathcal{F}_2}$ iff $\{(\mu_n, p_n)\}_{n\in\mathbb{N}}$ converges s/w to $(\mu, p)$ and satisfies $\limsup_{n\to+\infty} \int |p_n|^2\, d\mu_n \le \int |p|^2\, d\mu$.
The following Lemma gives the correspondence with converges in $L^2_{\mathbb{P}}$:

**Lemma 3.34** (Alternative characterization for convergence)**.**

i) *Given $\{(\mu_n, p_n)\}_{n\in\mathbb{N}}$ and $(\mu, p)$ in $\mathcal{F}_2(\mathbb{R}^d)$, the following are equivalent*
   (1) $W_2\left((\mathrm{Id}_{\mathbb{R}^d}, p_n)\sharp\mu_n, (\mathrm{Id}_{\mathbb{R}^d}, p)\sharp\mu\right) \to 0$*;*
   (2) *there exists $\{X_n\}_{n\in\mathbb{N}} \subseteq L^2_{\mathbb{P}}(\Omega)$, $X \in L^2_{\mathbb{P}}(\Omega)$ such that $X_n\sharp\mathbb{P} = \mu_n$ for all $n \in \mathbb{N}$, $X_\sharp\mathbb{P} = \mu$ and $X_n \to X$, $p_n \circ X_n \to p \circ X$ strongly in $L^2_{\mathbb{P}}$.*

ii) *Let $\{X_n\}_{n\in\mathbb{N}}, \{\xi_n\}_{n\in\mathbb{N}} \subseteq L^2_{\mathbb{P}}$ and $X, \xi \in L^2_{\mathbb{P}}$. Suppose that $X_n \to X$ strongly and $\xi_n \rightharpoonup \xi$ weakly in $L^2_{\mathbb{P}}$. Then, set $\mu_n = X_n\sharp\mathbb{P}$, $\mu = X\sharp\mathbb{P}$, $\mathrm{pr}_{H_{X_n}}(\xi_n) = p_n \circ X_n$, $\mathrm{pr}_{H_X}(\xi) = p \circ X$, the sequence $\{(\mu_n, p_n)\}_{n\in\mathbb{N}}$ s/w converges to $(\mu, p)$.*

iii) *Let $\{(\mu_n, p_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}_2(\mathbb{R}^d)$ s/w converging to $(\mu, p)$.*
   *Then there exist $\{(\mu_{n_k}, p_{n_k})\}_{k\in\mathbb{N}}$, and $\{X_{n_k}\}_{k\in\mathbb{N}} \subseteq L^2_{\mathbb{P}}$, $X, \xi \in L^2_{\mathbb{P}}$ satisfying $X_{n_k}\sharp\mathbb{P} = \mu_{n_k}$, $X\sharp\mathbb{P} = \mu$, $\mathrm{pr}_{H_X}(\xi) = p \circ X$, with $X_{n_k} \to X$ strongly and $p_{n_k} \circ X_{n_k} \rightharpoonup \xi$ weakly in $L^2_{\mathbb{P}}$.*

*Proof.* i) follows from Lemma A.3.

ii) The convergence of the Wasserstein distance in Definition 3.33 (1) follows easily from (6). Moreover, by weak convergence of $\xi_n$:

$$\sup_{n\in\mathbb{N}} \|p_n\|_{L^2_{\mu_n}} = \sup_{n\in\mathbb{N}} \|p_n \circ X_n\|_{L^2_{\mathbb{P}}} \le \sup_{n\in\mathbb{N}} \|\xi_n\|_{L^2_{\mathbb{P}}} < +\infty.$$

To get the second assertion, note that, setting $\pi_n = (X, X_n)\sharp\mathbb{P}$:

$$W_2(\pi_n, (\mathrm{Id}_{\mathbb{R}^d}, \mathrm{Id}_{\mathbb{R}^d})\sharp\mu) \le \|(X, X_n) - (X, X)\|_{L^2_{\mathbb{P}}} \to 0,$$

thus for all $\Phi \in C^0_b(\mathbb{R}^d, \mathbb{R}^d)$, it holds

$$\int_\Omega |\Phi(X(\omega)) - \Phi(X_n(\omega))|^2\, d\mathbb{P}(\omega) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |\Phi(x) - \Phi(y)|^2\, d\pi_n(x, y) = 0.$$

Finally:

$$\begin{aligned}\lim_{n\to+\infty} \int_{\mathbb{R}^d} p(x) \cdot \Phi(x)\, d\mu_n(x) &= \lim_{n\to+\infty} \int_\Omega (p_n \circ X_n(\omega)) \cdot (\Phi \circ X_n(\omega))\, d\mathbb{P}(\omega) \\ &= \lim_{n\to+\infty} \langle \xi_n, \Phi \circ X_n\rangle_{L^2_{\mathbb{P}}} = \langle \xi, \Phi \circ X\rangle_{L^2_{\mathbb{P}}} = \int_{\mathbb{R}^d} p(x) \cdot \Phi(x)\, d\mu(x).\end{aligned}$$

iii) By Lemma A.3, there exists $X_n$, $X$ in $L^2_{\mathbb{P}}$ such that $X_n \to X$ in $L^2_{\mathbb{P}}$, $\mu_n = X_n\sharp\mathbb{P}$, $\mu = X\sharp\mathbb{P}$. Moreover, since $\sup_{n\in\mathbb{N}} \|p_n \circ X_n\|_{L^2_{\mathbb{P}}} = \sup_{n\in\mathbb{N}} \|p_n\|_{L^2_{\mu_n}} < +\infty$, there exists $\{p_{n_k} \circ X_{n_k}\}_{k\in\mathbb{N}}$ weakly converging in $L^2_{\mathbb{P}}$ to some $\xi$. Then for any regular $\Phi$, we have:

$$\langle \xi, \Phi \circ X\rangle_{L^2_{\mathbb{P}}} = \lim_{k\to+\infty} \langle p_{n_k} \circ X_{n_k}, \Phi \circ X\rangle_{L^2_{\mathbb{P}}} = \lim_{k\to+\infty} \int_{\mathbb{R}^d} p_{n_k}(x) \cdot \Phi(x)\; d\mu_{n_k}(x)$$

$$= \int_{\mathbb{R}^d} p(x) \cdot \Phi(x) d\mu(x) = \langle p \circ X, \Phi \circ X \rangle_{L^2_{\mathbb{P}}}.$$

hence $\mathrm{pr}_{H_X}(\xi) = p \circ X$. □

Lemma 3.34 provides some consequences on the regularity of the Hamiltonian.

**Corollary 3.35.** *a ) Hamiltonian $\mathcal{H}$ is Lipschitz (respectively continuous) w.r.t. to $d_{\mathcal{F}_2}$ on $\mathcal{F}_2(\mathbb{R}^d)$ iff its lift $H$ is Lipschitz (respectively continuous) w.r.t. the strong topology on $F_2$.*
*b) If $\mathcal{H}$ is s/w continuous in $\mathcal{F}_2(\mathbb{R}^d)$ then $H$ is s/w continuous on $F_2$.*

3.4.2. *Some insights on the regularity of the extension proposed in* [33]. The authors of [33] propose to consider the following Hamiltonian on $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:

$$\hat{H}(X, \xi) = \mathcal{H}(X\sharp\mathbb{P}, \mathrm{pr}_{\mathcal{T}_{X\sharp\mathbb{P}}}(p)) \text{ with } \mathrm{pr}_{H_X}(\xi) = p \circ X,$$

together the following extended HJB in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$

$$(\widehat{HJ}) \quad \hat{H}(X, DU(X)) = 0.$$

The Hamiltonian $\hat{H}$ is rearrangement invariant and satisfies:

$$\hat{H}(X, p \circ X) = \mathcal{H}(X\sharp\mathbb{P}, p)$$

for all $(X, p \circ X) \in TF_2 := \{(X, p \circ X) : \ p \in \mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d),\ X\sharp\mathbb{P} \in \mathscr{P}_2(\mathbb{R}^d)\}$.

The interest of this extension, as emphasized in [33], is that the lift of any solution of $(HJ)$ in the $\mathscr{P}_2$-sense is a solution of $(\widehat{HJ})$ in the $L^2_{\mathbb{P}}$-sense (using $\varepsilon$-subdifferential an using the definition of $\mathscr{P}_2$-viscosity solutions of the present paper.) This result is a consequence of Proposition 3.14.

Here we want to determine whether $\hat{H}$ is regular if $\mathcal{H}$ is so. As pointed out previously, the regularity of $\hat{H}$ is crucial in order to apply the $L^2_{\mathbb{P}}$-theory of viscosity solution.

**Lemma 3.36.**

(i) *The map $P_H : (L^2_{\mathbb{P}}, \|\cdot\|_{L^2_{\mathbb{P}}}) \times (L^2_{\mathbb{P}}, \sigma) \to (\mathcal{F}_2(\mathbb{R}^d), s/w)$, defined by $(X, \xi) \mapsto (X\sharp\mathbb{P}, p)$ where $\mathrm{pr}_{H_X}(\xi) = p \circ X$, is continuous ($\sigma$ denotes the weak topology).*

(ii) *The map $P_H : (L^2_{\mathbb{P}}, \|\cdot\|_{L^2_{\mathbb{P}}}) \times (L^2_{\mathbb{P}}, \|\cdot\|_{L^2_{\mathbb{P}}}) \to (\mathcal{F}_2(\mathbb{R}^d), d_{\mathcal{F}_2})$, defined by $(X, \xi) \mapsto (X\sharp\mathbb{P}, p)$ where $\mathrm{pr}_{H_X}(\xi) = p \circ X$, is not continuous.*

(iii) *The map $F : (L^2_{\mathbb{P}}, \|\cdot\|_{L^2_{\mathbb{P}}}) \times (L^2_{\mathbb{P}}, \|\cdot\|_{L^2_{\mathbb{P}}}) \to (\mathcal{F}_2(\mathbb{R}^d), d_{\mathcal{F}_2})$, defined by $(\xi, X) \mapsto (\mathrm{pr}_{\mathcal{T}_{X\sharp\mathbb{P}}(\mathbb{R}^d)}(p), X\sharp\mathbb{P})$, where $\mathrm{pr}_{H_X}(\xi) = p \circ X$ is not continuous.*

*Proof.* (i) is an immediate consequence of Lemma 3.34.
(ii)-(iii) Let $\bar{x} \in \mathbb{R}^d$ and $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{P}_2(\mathbb{R}^d)$ with $\mu_n \ll \mathscr{L}^d$ satisfying $\mu_n \to \delta_{\bar{x}}$. Let $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ be without atoms , and choose $\pi_n = (\mathrm{Id}_{\mathbb{R}^d}, T_n)\sharp\mu_n \in \Pi_o(\mu_n, \nu)$. Up to a subsequence, we can assume that $\pi_n \to \pi_0 = \delta_{\bar{x}} \otimes \nu$ which is the unique element of $\Pi(\delta_{\bar{x}}, \nu)$. Then, by A.3, there exist $\{X_n\}_{n\in\mathbb{N}} \subseteq L^2_{\mathbb{P}}$, $Y \in L^2_{\mathbb{P}}$, such that

$$X_n\sharp\mathbb{P} = \mu_n, \quad (X_n, T_n \circ X_n)\sharp\mathbb{P} = \pi_n, \quad (\bar{x}, Y)\sharp\mathbb{P} = \pi_0,$$
$$\lim_n \|(X_n, T_n \circ X_n) - (\bar{x}, Y)\|_{L^2_{\mathbb{P}} \times L^2_{\mathbb{P}}} = 0.$$

Then the sequence $\{(X_n, T_n \circ X_n - X_n)\}_{n\in\mathbb{N}}$ strongly converges to $(\bar{x}, Y - \bar{x})$ but

$$P_H(X_n, T_n \circ X_n - X_n) = F(X_n, T_n \circ X_n - X_n) = (\mu_n, T_n - \mathrm{Id}_{\mathbb{R}^d})$$

does not converge s/s to

$$P_H(\bar{x}, Y-\bar{x}) = F(\bar{x}, Y-\bar{x}) =: (\delta_{\bar{x}}, p).$$

Indeed, set $\gamma$ the transport plan defined by:

$$\int \varphi(x,y)d\gamma(x,y) = \int \varphi(\bar{x}, y-\bar{x})d\nu(y) = \int \varphi(\bar{x}, Y-\bar{x})\, d\mathbb{P} \text{ for any regular } \varphi : \mathbb{R}^{2d} \to \mathbb{R}.$$

Then, recalling that $\nu$ has no atom, clearly $\gamma \neq (Id \times p)\sharp\delta_{\bar{x}}$ and

$$\lim_{n\to+\infty} W_2((T_n - \mathrm{Id}_{\mathbb{R}^d})\sharp\mu_n, \gamma) = 0 \neq \lim_{n\to+\infty} W_2((T_n - \mathrm{Id}_{\mathbb{R}^d})\sharp\mu_n, (Id \times p)\sharp\delta_{\bar{x}}).$$

□

*Remark* 3.37.

- According to the previous result, even if $\mathcal{H}$ is Lipschitz for the distance $d_{\mathcal{F}_2}$, in general the extension $\tilde{H}$ may fail to be continuous for the $L^2_{\mathbb{P}} \times L^2_{\mathbb{P}}$ norm.
- If $\mathcal{H}$ is s/w regular, and supposing that the composition with the projection on $\mathcal{T}_\mu(\mathbb{R}^d)$ preserves this regularity, then $\tilde{H}$ is also s/w regular.
- It is still an open problem to establish if $\{(\mu, p) :\ p \in \mathcal{T}_\mu(\mathbb{R}^d) :\ \mu \in \mathscr{P}_2(\mathbb{R}^d)\}$ is s/w or $d_{\mathcal{F}_2}$ closed, and the regularity of $(\mu, p) \in \mathcal{F}_2(\mathbb{R}^d) \mapsto (\mu, \mathrm{pr}_{\mathcal{T}_\mu(\mathbb{R}^d)}(p))$. w.r.t. these types of convergence.

Even assuming $\mathcal{H}$ is quite regular, it seems a very difficult question to find, in the general case, a regular extension giving equivalence of solutions in $L^2_{\mathbb{P}}$ and $\mathscr{P}_2(\mathbb{R}^d)$. Nevertheless, in some cases, this can be done in a quite natural way as shown in the next example.

3.4.3. *Example.* As in [19], we consider the following Hamiltonian:

$$\mathcal{H}(\mu, p) := \inf_{u\in U} \sup_{v\in V} \int_{\mathbb{R}^d} f(x, u, v) \cdot p(x)\ d\mu(x),$$

with $f : \mathbb{R}^d \times U \times V \to \mathbb{R}^d$ where $f$ is bounded, continuous and Lipschitz in its first variable. The sets $U$ and $V$ are compact subsets of some finite dimensional spaces.

We also consider the following time dependent HJB (cf [19]) :

$$(HJ_t) \quad \begin{cases} \partial_t u + \mathcal{H}(\mu, D_\mu u) = 0 & \text{on } [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \\ u(T, \mu) = G(\mu) & \text{on } \mathscr{P}_2(\mathbb{R}^d)\}. \end{cases}$$

where $G$ is Lipschitz and bounded. We slightly modify the notion of solution: $(p_t, p_\mu) \in \mathbb{R} \times L^2_\mu(\mathbb{R}^d, \mathbb{R}^d) \in D^+_\varepsilon u(t, \mu)$ iff $p_\mu \in dis^+(\mu)$ and for all $\gamma \in \Pi(\mu, \nu)$:

$$\begin{aligned} u(s,\nu) - u(t,\mu) \leq & \int p_\mu(x) \cdot (y-x)\ d\gamma(x,y) + p_t(s-t) \\ & - \varepsilon\sqrt{(t-s)^2 + W_2^2(\mu_0, \nu)} + o\left(\sqrt{(t-s)^2 + W_2^2(\mu_0,\nu)}\right). \end{aligned}$$

A natural $L^2_{\mathbb{P}}$-extension of $\mathcal{H}$ is $\tilde{H}(X, Y) = \inf_{u\in U} \sup_{v\in V} \int_\Omega f(X, u, v) \cdot Y\, d\mathbb{P}$. We set:

$$(\widetilde{HJ_t}) \quad \begin{cases} \partial_t U + \tilde{H}(X, DU) = 0, & \text{on } [0,T] \times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)\} \\ u(T, X\sharp\mathbb{P}) = G(X\sharp\mathbb{P}) & \text{on } L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)\}. \end{cases}$$

Note that:

$$\tilde{H}(X, Y) = \tilde{H}(X, pr_{H_X}(Y)). \tag{49}$$

Then,

- $\tilde{H}$ is rearrangement invariant and continuous (so in $L^2_{\mathbb{P}}$, we can consider approximate superdifferentials or subdifferentials);
- $\tilde{H}$ and $\mathcal{H}$ both satisfy the assumptions needed to obtain a comparison principle (cf Theorem 2 of [27], Theorem 5.6. of [22] also Lemma 6 in [19]);
- using Proposition 4.5. of [33], if $(\widetilde{HJ_t})$ has a unique solution, it is rearrangement invariant.

From all these considerations, we can deduce that, assuming $(\widetilde{HJ_t})$ has a bounded uniformly continuous solution, it is unique, rearrangement invariant, and it is also the unique solution of $(HJ_t)$. Then, it can easily be seen (using for instance (49), Proposition 3.14, and Proposition 3 of [19]) that the lift of the value function $\mathcal{V}$ of [19] is the unique solution of $(\widetilde{HJ_t})$ and $\mathcal{V}$ is the unique solution of $(HJ_t)$. In this case solving $(\widetilde{HJ_t})$ or $(HJ_t)$ is equivalent.

## Appendix A

A.1. **Measure Theory.** Let $X, Y$ be a complete metric space. Given $\mu \in \mathscr{P}(X)$ and a Borel family $\{\nu^x\}_{x\in X} \subseteq \mathscr{P}(X\times Y)$ of probability measures (i.e., $x \mapsto \nu^x(B)$ is a Borel map for every Borel set $B \subseteq X$). The product measure $\mu\otimes\nu^x \in \mathscr{P}(X\times Y)$ is defined (see e.g. Section 5.3 in [3]) by setting for all $f \in C^0_b(X\times Y)$

$$\iint_{X\times Y} f(x,y)\,d(\mu\otimes\nu^x)(x,y) = \int_X\int_Y f(x,y)\,d\nu^x(y)\,d\mu(x) \tag{50}$$

**Theorem A.1** (Disintegration Theorem, Th 5.3.1 in [3])**.** *Given a measure $\mu \in \mathscr{P}(\mathbb{X})$ and a Borel map $r : \mathbb{X} \to X$, there exists a family of probability measures $\{\mu_x\}_{x\in X} \subseteq \mathscr{P}(\mathbb{X})$, uniquely defined for $r\sharp\mu$-a.e. $x \in X$, such that $\mu^x(\mathbb{X}\setminus r^{-1}(x)) = 0$ for $r\sharp\mu$-a.e. $x \in X$, and for any Borel map $\varphi : X\times Y \to [0,+\infty]$ we have*

$$\int_{\mathbb{X}} \varphi(z)\,d\mu(z) = \int_X \left[\int_{r^{-1}(x)} \varphi(z)\,d\mu^x(z)\right] d(r\sharp\mu)(x).$$

*We will write $\mu = (r\sharp\mu)\otimes\mu^x$. If $\mathbb{X} = X\times Y$ and $r^{-1}(x) \subseteq \{x\}\times Y$ for all $x \in X$, we can identify each measure $\mu^x \in \mathscr{P}(X\times Y)$ with a measure on $Y$.*

We recall a characterization of optimal displacement of $dis^-(\mu)$ ( definition 3.3).

**Lemma A.2** ([37] lemma 5.2)**.** *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d)$. The map $p$ is an optimal displacement in $dis^-(\mu)$ iff there exists $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\gamma \in \Pi_o(\mu,\nu)$ such that $p(x) = \int_{\mathbb{R}^d} y\,d\gamma^x(y) - x$, where $\gamma = \mu\otimes\gamma^x$ is a disintegration of $\gamma$.*

A.2. **Wasserstein space and $L^2_{\mathbb{P}}$.** In section 1.2, we have already defined the relation $\sim$ allowing to identify $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ with the quotient $(L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)/\sim)$ equipped with the quotient topology. Let us denote by $[X]$ the equivalence class of $X \in L^2_{\mathbb{P}}$, it is clear that the following map is one to one:

$$[X] \in \left(L^2_{\mathbb{P}}(\Omega, R^d)/\sim\right) \mapsto X\sharp\mathbb{P} \in \mathcal{P}_2(\mathbb{R}^d).$$

Consider also $\mathrm{pr} : X \mapsto [X]$ the canonical projection on the quotient space.

It is well-known that if sequence $\{X_n\}_{n\in\mathbb{N}}$ converge to $X$ in $L^2_{\mathbb{P}}$ then it converges also in law i.e $W_2(X_n\sharp\mathbb{P}, X\sharp\mathbb{P}) \to 0$ (while the converse is false). On the other hand, we have

**Lemma A.3.** *If a sequence $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ converge to $\mu$ for the distance $W_2$ then for any $\varepsilon_n \to 0$ there exist $\{X_n\}_{n\in\mathbb{N}}$, $X$ in $L^2_{\mathbb{P}}$ such that $X\sharp\mathbb{P} = \mu$ and*

$$W_2(\mu_n, \mu) \leq \|X_n - X\|_{L^2_{\mathbb{P}}} \leq W_2(\mu_n, \mu) + \varepsilon_n.$$

*Proof.* Take $X$ such that $X\sharp\mathbb{P} = \mu$. There exists $Y_n$, $X$ such that:

$$W_2(\mu, \mu_n) = \|Y_n - Z_n\|_{L^2_{\mathbb{P}}}, \quad Y_n\sharp\mathbb{P} = \mu, \quad Z_n\sharp\mathbb{P} = \mu_n.$$

Then, arguing as in Lemma 1.1, there exists $\tau_n$ one to one such that $\tau_n$ and $\tau_n^{-1}$ are measure preserving such that $\|X - Y_n \circ \tau_n\|_{L^\infty_{\mathbb{P}}} \leq \varepsilon_n$. Then we have:

$$\begin{aligned} W_2(\mu, \mu_n) \leq &\|Z_n \circ \tau_n - X\|_{L^2_{\mathbb{P}}} = \|Z_n - X \circ \tau_n^{-1}\|_{L^2_{\mathbb{P}}} \leq \|Z_n - Y_n\|_{L^2_{\mathbb{P}}} + \|Y_n - X \circ \tau_n^{-1}\|_{L^2_{\mathbb{P}}} \\ =&W_2(\mu, \mu_n) + \|Y_n \circ \tau_n \ - X\|_{L^2_{\mathbb{P}}} \leq W_2(\mu, \mu_n) + \varepsilon_n. \end{aligned}$$

So with $X_n := Z_n \circ \tau_n$ the proof is complete. □

We recall a useful known result:

**Proposition A.4.** *A subset $F$ of the quotient space is closed if and only if for all $([X_n])_n$ in $F$ and $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$,*

$$\lim_{n\to+\infty} W_2(X_n\sharp\mathbb{P}, X\sharp\mathbb{P}) = 0 \Rightarrow [X] \in F.$$

The previous results then easily imply the useful

**Corollary A.5.** a) *Take $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to R$ a rearrangement invariant and set $u(X\sharp\mathbb{P}) = U(X)$. Then*

*$U$ is continuous for the $L^2_{\mathbb{P}}$- norm $\Leftrightarrow$ $u$ is continuous for the distance $W_2$.*

b) *Let $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ its lift. Then*

*$U$ is continuous for the $L^2_P$- norm $\Leftrightarrow$ $u$ is continuous for the distance $W_2$.*

c) *Let $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $U : L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ its lift. Let $k > 0$, then:*

*$U$ is $k$-Lipschitz $\Leftrightarrow$ $u$ is $k$-Lipschitz.*

Now we show a very close relationship between geodesics in the Wasserstein space and goedesics in the space $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$. Recall that $t \in [0, 1] \to \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ is a constant speed geodesic if

$$W_2(\mu_t, \mu_s) = (t - s)W_2(\mu_t, \mu_0) \quad \forall\ 0 \leq s \leq t \leq 1.$$

**Proposition A.6.** *Let $\{\mu_t\}_{t\in[0,1]}$ be a constant speed geodesic and $(T_0, T_2)$ two random variables of $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ such that $\gamma := (T_0, T_1)\sharp\mathbb{P}$ is an optimal transport plan from $\mu_0$ to $\mu_1$. Then:*

$$\mu_t = [(1 - t)T_0 + tT_1]\sharp P, \ \forall\ t \in [0, 1]$$
$$W_2(\mu_s, \mu_t) = \|T_t \circ S - T_s \circ S\|_{L^2_{\mathbb{P}}}, \ \forall\ s, t \in [0, 1]$$

*where $S$ is any map in $L^2_{\mathbb{P}}$ such that $S\sharp\mathbb{P} = \mu_0$. In particular, $(T_t \circ S)_{t\in[0,1]}$ is a geodesic in $L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$.*

*Proof.* By Lemma 7.2.1. p158 of [3], denoting by $\pi_0, \pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ the projections on the first and second variable, it holds: $\mu_t = [(1 - t)\pi_0 + t\pi_1]\sharp\gamma$. Then as $\gamma = (T_0, T_1)\sharp\mathbb{P}$, we get the first equality. To prove the second equality just notice:

$$W_2(\mu_s, \mu_t) = |t - s|W_2(\mu_0, \mu_1) = |t - s|\|T_1 - T_0\|_{L^2_{\mathbb{P}}}.$$

□

We also recall a result concerning the existence and representation of solution of the multiagent control system

**Theorem A.7** (Theorem 3.6 in [37]). *Consider a Lipschitz continuous set valued map $F : \mathbb{R}^+ \times \mathscr{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with compact and convex images. Then for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \subseteq \mathscr{P}_2(\mathbb{R}^d) \in \mathscr{A}^F_{[0,T]}(\mu)$ an admissible trajectory driven by $F$. Moreover, there exists $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ such that*

*(1) $\mu_t = e_t\sharp\boldsymbol{\eta}$ for all $t \in [0,T]$;*

*(2) for $\boldsymbol{\eta}$-a.e. $(x,\gamma) \in \mathbb{R}^d \times \Gamma_T$, we have*

$$\gamma(0) = x \text{ and } \dot\gamma(t) \in F(e_t\sharp\boldsymbol{\eta}, \gamma(t)), \text{ for a.e. } t \in [0,T].$$

*Conversely, if $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ satisfies (2) above, then $\boldsymbol{\mu} := \{\mu_t := e_t\sharp\boldsymbol{\eta}\}_{t\in[0,T]} \in \mathscr{A}^F_{[0,T]}(\mu)$ is an admissible trajectory driven by $F$, with $\boldsymbol{\nu} = \{v_t\mu_t\}_{t\in[0,T]}$ and for a.e. $t \in [0,T]$ and $\mu_t$-a.e. $y \in \mathbb{R}^d$*

$$v_t(y) = \int_{e_t^{-1}(y)} \dot\gamma(t)\, d\eta_t^y(x,\gamma),$$

*and $\eta_t^y$ is given by the disintegration $\boldsymbol{\eta} = \mu_t \otimes \eta_t^y$.*

## A.3. **Technical proofs.**

*Proof.* (of Lemma 2.9) Given $\{Y_n(\cdot)\}_{n\in\mathbb{N}} \subseteq L^2_{\mathbb{P}}(\Omega)$ and $Y(\cdot) \in L^2_{\mathbb{P}}(\Omega)$ such that $Y_n \to Y$ in $L^2_{\mathbb{P}}$ and $Y_n(\cdot) \in G^{\boldsymbol{\theta}}(t, X(\cdot))$, we can extract a subsequence $\{Y_{n_k}(\cdot)\}_{k\in\mathbb{N}}$ satisfying $Y_{n_k}(\omega) \to Y(\omega)$ for a.e. $\omega \in \Omega$, and therefore we conclude that $Y(\omega) \in F(t, \theta_t, X(\omega))$ for a.e. $\omega \in \Omega$ by the closedness of $F(t,\theta_t,X(\omega))$. Thus $Y(\cdot) \in G^{\boldsymbol{\theta}}(t, X(\cdot))$.

Given $t_1 \in I$ and $X_1(\cdot) \in L^2_{\mathbb{P}}(\Omega)$, consider the set-valued map $G^{\boldsymbol{\theta}}_{t_1,X_1} : \Omega \rightrightarrows \mathbb{R}^d$ defined as $G^{\boldsymbol{\theta}}_{t_1,X_1}(\omega) = F(t_1, \theta_{t_1}, X_1(\omega))$.

Since the map $x \mapsto F(t_1, \theta_{t_1}, x)$ is continuous with compact convex nonempty images, there exists a countable family of continuous map $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n(x) \in F(t_1,\theta_{t_1},x)$ and $F(t_1,\theta_{t_1},x) = \overline{\bigcup_{n\in\mathbb{N}} f_n(x)}$ for all $x \in \mathbb{R}^d$. In particular, we have that

$$G^{\boldsymbol{\theta}}_{t_1,X_1}(\omega) = F(t_1,\theta_{t_1},X_1(\omega)) = \overline{\bigcup_{n\in\mathbb{N}} f_n(X_1(\omega))}.$$

Since $f_n \circ X_1(\cdot)$ are measurable, $G^{\boldsymbol{\theta}}_{t_1,X_1}$ is a measurable. Given $t_2 \in I$, $X_2(\cdot) \in L^2_{\mathbb{P}}$ and $Y_2(\cdot) \in G^{\boldsymbol{\theta}}(t_2, X_2(\cdot))$. By Corollary 8.2.13 in [5], since $Y_2(\cdot)$ is measurable there exists a measurable selection $Y_1(\cdot)$ of $G^{\boldsymbol{\theta}}_{t_1,X_1}$ such that

$$|Y_2(\omega) - Y_1(\omega)| = d_{G^{\boldsymbol{\theta}}_{t_1,X_1}(\omega)}(Y_2(\omega)) = d_{F(t_1,\theta_{t_1},X_1(\omega))}(Y_2(\omega)).$$

So by the Lipschitz continuity of $F$ we get easily

$$\|Y_1 - Y_2\|_{L^2_{\mathbb{P}}} \le \operatorname{Lip} F \cdot (1 + \operatorname{Lip}\boldsymbol{\theta}) \cdot (|t_1 - t_2| + \|X_1 - X_2\|_{L^2_{\mathbb{P}}}).$$

Interchanging $X_1$ and $X_2$, we have $\operatorname{Lip} G^{\boldsymbol{\theta}} \le \operatorname{Lip} F \cdot (1 + \operatorname{Lip}\boldsymbol{\theta})$. □

*Proof.* (of Lemma 2.11) According to [31] and [43] it is enough to show that every point of $\mathscr{S}$ has a fundamental system of open convex neighborhoods and that the convex structure on $\mathscr{S}$ is compatible in a suitable sense with the topology induced on $\mathscr{S}$ by $d_{C^0}$, more precisely that the function $\psi : C^0(I;\mathscr{P}_2(\mathbb{R}^d)) \times C^0(I;\mathscr{P}_2(\mathbb{R}^d)) \times [0,1] \to C^0(I;\mathscr{P}_2(\mathbb{R}^d))$ defined by $\psi(\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \lambda) = \lambda\boldsymbol{\theta}^{(1)} + (1-\lambda)\boldsymbol{\theta}^{(2)}$ is continuous. Intersecting each element of the $d_{C^0}$-open balls of positive rational radius around $\boldsymbol{\theta} \in \mathscr{S}$ provides a fundamental system of open convex neighborhoods of $\boldsymbol{\theta}$ w.r.t. the topology induced by $d_{C^0}$ on $\mathscr{S}$.

For $i = 0, 1$, let $\boldsymbol{\theta}^{i,n} = \{\theta_t^{i,n}\}_{t\in I}$ be a sequence $d_{C^0}$-converging to $\boldsymbol{\theta}^i = \{\theta_t^i\}_{i\in I}$, and $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ converging to $\lambda$. Let

$$\begin{aligned}\boldsymbol{\theta}^\lambda :=& \{\theta_t^\lambda\}_{t\in I}, \text{ where } \theta_t^\lambda = \lambda\theta_t^0 + (1-\lambda)\theta_t^1,\\ \boldsymbol{\theta}^{\lambda_n,n} :=& \{\theta_t^{\lambda_n,n}\}_{t\in I}, \text{ where } \theta_t^{\lambda_n,n} := \lambda_n\theta_t^{0,n} + (1-\lambda_n)\theta_t^{1,n},\end{aligned}$$

and, chosen $\pi^{i,n} \in \Pi_o(\theta_t^{i,n}, \theta_s^{i,n})$, $i = 0, 1$, for all $n \in \mathbb{N}$, $t, s \in I$, set

$$\pi_{t\to s}^n = \lambda_n\pi^{0,n} + (1-\lambda_n)\pi^{1,n} \in \Pi(\theta_t^{\lambda_n,n}, \theta_s^{\lambda_n,n}).$$

We show that $\{\boldsymbol{\theta}^{\lambda_n,n}\}_{n\in\mathbb{N}}$ are equibounded. Since $\{\boldsymbol{\theta}^{i,n}\}_{n\in\mathbb{N}}$ are $d_{C^0}$-converging for $i = 0, 1$, in particular they are bounded therefore, considering for instance the constant curve $\boldsymbol{\xi} = \{\xi_t\}_{t\in I}$ with $\xi_t \equiv \delta_0$, there exists $R > 0$ such that $d_{C^0}(\boldsymbol{\theta}^{i,n}, \boldsymbol{\xi}) \leq R$, $i = 1, 2$. From the convexity of the $d_{C^0}$-ball we have that $d_{C^0}(\boldsymbol{\theta}^{\lambda_n,n}, \boldsymbol{\xi}) \leq R$.

We show that $\{\boldsymbol{\theta}^{\lambda_n,n}\}_{n\in\mathbb{N}}$ are equicontinuous. It holds

$$\begin{aligned}W_2^2(\theta_t^{\lambda_n,n}, \theta_s^{\lambda_n,n}) \leq& \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\, d\pi_{t\to s}^n(x,y)\\ =&\lambda_n W_2^2(\theta_t^{0,n}, \theta_s^{0,n}) + (1-\lambda_n)W_2^2(\theta_t^{1,n}, \theta_s^{1,n}) \leq W_2^2(\theta_t^{0,n}, \theta_s^{0,n}) + W_2^2(\theta_t^{1,n}, \theta_s^{1,n}).\end{aligned}$$

Since for $i = 0, 1$ the set $\{\boldsymbol{\theta}^{i,n}\}_{n\in\mathbb{N}} \cup \{\boldsymbol{\theta}^i\}$ in $d_{C^0}$-compact, in particular it is equicontinuous, therefore there exists a continuous increasing $\omega_i : [0, +\infty[\to [0, +\infty[$ satisfying $\omega_i(0) = 0$ and $W_2^2(\theta_t^{i,n}, \theta_s^{i,n}) \leq \omega_i(|t-s|)$ for all $t, s \in I$, $n \in \mathbb{N}$, $i = 0, 1$. Therefore $\{\boldsymbol{\theta}^{\lambda_n,n}\}_{n\in\mathbb{N}}$ are equicontinuous with modulus $\omega(\cdot) := \sqrt{\omega_0^2(\cdot) + \omega_1^2(\cdot)}$.

We show that $\{\boldsymbol{\theta}^{\lambda_n,n}\}_{n\in\mathbb{N}}$ pointwise converges to $\boldsymbol{\theta}^\lambda$. Indeed, given any $\varphi \in C_b^0(\mathbb{R}^d)$, $t \in I$, we have

$$\int_{\mathbb{R}^d} \varphi(x)\, d\theta_t^{\lambda_n,n}(x) = \lambda_n \int_{\mathbb{R}^d} \varphi(x)\, d\theta_t^{0,n}(x) + (1-\lambda_n)\int_{\mathbb{R}^d} \varphi(x)\, d\theta_t^{1,n}(x),$$

and by passing to the limit on $n$ and recalling that $|\lambda_n| \leq 1$ and that the convergence in $d_0$ implies that $\theta_t^{i,n}$ converges in $W_2$ and narrowly to $\theta_t^i$ for $i = 0, 1$, we have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\theta_t^{\lambda_n,n}(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\theta_t^\lambda(x),$$

and so we have narrow pointwise convergence. We prove the uniform integrability of the second order moments, indeed

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B(0,R)} |x|^2\, d\theta_t^{\lambda_n,n}(x) =&\lambda_n \int_{\mathbb{R}^d\setminus B(0,R)} |x|^2\, d\theta_t^{0,n}(x) + (1-\lambda_n)\int_{\mathbb{R}^d\setminus B(0,R)} |x|^2\, d\theta_t^{1,n}(x),\\ \leq& \int_{\mathbb{R}^d\setminus B(0,R)} |x|^2\, d\theta_t^{0,n}(x) + \int_{\mathbb{R}^d\setminus B(0,R)} |x|^2\, d\theta_t^{1,n}(x),\end{aligned}$$

and uniform integrability of the second order moments follows from the uniform integrability of the second moments for $\{\theta_t^{i,n}\}_{n\in\mathbb{N}}$, which are $W_2$-converging sequences.

By Ascoli-Arzela theorem we conclude that $d_{C^0}(\boldsymbol{\theta}^{\lambda_n,n}, \boldsymbol{\theta}^\lambda) \to 0$, thus $\Psi$ is continuous and so the assumptions of [31] and [43] are satisfied. □

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