# The second bifurcation branch for radial solutions of the Brezis-Nirenberg problem in dimension four*

Gianni ARIOLI – Filippo GAZZOLA

Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32, 20133 Milano (Italy)

Hans-Christoph GRUNAU – Edoardo SASSONE

Fakultät für Mathematik, Otto–von–Guericke–Universität Postfach 4120, 39016 Magdeburg (Germany)

**Abstract**

Existence results available for the semilinear Brezis-Nirenberg eigenvalue problem suggest that the compactness problems for the corresponding action functionals are more serious in small dimensions. In space dimension $n = 3$, one can even prove nonexistence of positive solutions in a certain range of the eigenvalue parameter. In the present paper we study a nonexistence phenomenon manifesting such compactness problems also in dimension $n = 4$.

We consider the equation $-\Delta u = \lambda u + u^3$ in the unit ball of $\mathbb{R}^4$ under Dirichlet boundary conditions. We study the bifurcation branch arising from the second radial eigenvalue of $-\Delta$. It is known that it tends asymptotically to the first eigenvalue as the $L^\infty$-norm of the solution tends to blow up. Contrary to what happens in space dimension $n = 5$, we show that it does not cross the first eigenvalue. In particular, the mentioned Dirichlet problem in $n = 4$ does not admit a nontrivial radial solution when $\lambda$ coincides with the first eigenvalue.

## 1 Introduction and main result

In their celebrated paper, Brezis-Nirenberg [9] studied the following semilinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain and $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent. Since they were interested in *positive* solutions of (1), they assumed that $0 < \lambda < \mu_1$, where $\mu_1$ denotes the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$. Subsequently, many other papers studying (1) appeared and it seems almost impossible to give a complete list of references. So, let us restrict our attention to *radial sign-changing* solutions in the case where $\Omega = \mathbf{B}$ (the unit ball). In this situation, (1) becomes an ordinary differential equation and the space dimension $n > 2$ may be considered as a real parameter. More precisely, putting $r := |x|$ (so that $0 < r < 1$) and assuming that $u = u(r)$, (1) reads

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda u(r) + |u(r)|^{4/(n-2)}u(r) = 0\ , \qquad u'(0) = u(1) = 0\ , \quad u(0) = \omega\ , \tag{2}$$

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where, for our convenience, we overdetermined the problem by adding the "shooting condition" $u(0) = \omega$. In general, (2) admits no solution since it involves 3 boundary conditions. However, for any $\omega > 0$ and for a suitable $\lambda = \lambda(\omega)$, problem (2) admits a solution $u_\omega$ with precisely one zero in $[0, 1)$, the second zero being at $r = 1$. We are here interested in studying the behaviour of the map $\omega \mapsto \lambda(\omega)$.

Let $\mu_1 = \mu_1(n)$ and $\mu_2 = \mu_2(n)$ denote the first two (positive) eigenvalues $\mu$ of the problem

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + \mu\psi(r) = 0 \quad (0 < r < 1) , \qquad \psi'(0) = \psi(1) = 0 ,$$

so that the eigenfunction corresponding to $\mu_1$ is positive whereas the eigenfunction corresponding to $\mu_2$ has exactly one zero in $[0, 1)$. If $n$ is an integer, $\mu_1$ and $\mu_2$ represent the first two radial eigenvalues of $-\Delta$ in $H^1_0(\mathbf{B})$. It is well-known (cf. e.g. Remark 3 in Section 3) that for any $n > 2$ we have

$$\lim_{\omega\to 0} \lambda(\omega) = \mu_2 .$$

Much richer appears the picture of the behaviour of $\lambda(\omega)$ as $\omega \to +\infty$. As we shall see, it strongly depends on the parameter $n$. Firstly, in "large dimensions" the bifurcation branch collapses to $\lambda = 0$. More precisely, we have

$$\text{if } n > 6 \qquad \text{then} \quad \lim_{\omega\to\infty} \lambda(\omega) = 0 . \tag{3}$$

Statement (3) was established by Atkinson-Peletier [8, Theorem 4 (b)], see also previous results by Cerami-Solimini-Struwe [11] for integer values of $n \geq 7$. Subsequently, Atkinson-Brezis-Peletier [5] proved that the behaviour changes for $n = 6$:

$$\text{if } n = 6 \qquad \text{then there exists } \mu \in (0, \mu_1) \text{ such that} \quad \lim_{\omega\to\infty} \lambda(\omega) = \mu . \tag{4}$$

Although the second bifurcation branch has not been explicitly studied for "small dimensions", the results by Atkinson-Peletier [6, 7] (about the *first* branch, relative to *positive* solutions of (2)) strongly suggest the *conjecture* that it does not reach $\mu_1$:

$$\text{if } 2 < n < 4 \qquad \text{then there exists } \mu \in (\mu_1, \mu_2) \text{ such that} \quad \lim_{\omega\to\infty} \lambda(\omega) = \mu . \tag{5}$$

But the most interesting cases seem to be when the bifurcation branch skips precisely one eigenvalue. As shown in [5], this occurs in the "intermediate dimensions". More precisely, we have

$$\text{if } 4 \leq n < 6 \qquad \text{then} \quad \lim_{\omega\to\infty} \lambda(\omega) = \mu_1 . \tag{6}$$

Unfortunately, (6) nothing says about the "asymptotic monotonicity" of the map $\omega \mapsto \lambda(\omega)$. This was studied in [13] where it was shown that if $4 \leq n \leq 2 + 2\sqrt{2}$ then $\lambda(\omega) > \mu_1$ for sufficiently large $\omega$, whereas if $2 + 2\sqrt{2} < n < 6$ then $\lambda(\omega) < \mu_1$ for sufficiently large $\omega$. Therefore, for any $n > 2+2\sqrt{2}$ the second bifurcation branch eventually goes below the first eigenvalue $\mu_1$. Since the number $n = 2 + 2\sqrt{2}$ plays a crucial role in the description of (1), it was conjectured in [13] that the second bifurcation branch does not cross $\mu_1$ if $n \leq 2 + 2\sqrt{2}$. The aim of this paper is to partly prove this conjecture. We show that the bifurcation branch in dimension $n = 4$ does not reach the first eigenvalue, namely that $\lambda(\omega) > \mu_1$ for all $\omega > 0$. We study dimension $n = 4$ for two crucial reasons. Firstly, because it is an integer dimension so that a corresponding result for the elliptic problem (1) is also obtained, see Corollary 1 below. Secondly, because in this case the nonlinearity $|u|^{2^*-2}u$ simply becomes $u^3$ which is analytic, and analytic nonlinearities are easier to tackle with computer assisted proofs.

Our main result reads:

**Theorem 1** *Assume that $n = 4$ and let $\lambda(\omega)$ be defined as above. Then, for all $\omega > 0$ we have $\lambda(\omega) > \mu_1$.*

We prove Theorem 1 in three steps. In Section 2, by refining previous estimates in [5, 6, 7], we prove Theorem 1 for $\omega$ sufficiently large (exactly for $\omega \geq 349$). In Section 3, we use a comparison method and the variational characterization of eigenvalues in order to show that $\lambda(\omega) > \mu_1$ whenever $\omega \leq \sqrt{\mu_2 - \mu_1}$. Finally, in Section 4, we prove Theorem 1 for "intermediate" values of $\omega$ (i.e. for $\sqrt{\mu_2 - \mu_1} \leq \omega \leq 349$) with the assistance of a computer. We recall here a possible definition of computer assisted proof:

**Definition 1** *A proof is called* **computer assisted***, if it consists in finitely many elementary operations, but their number is so large that, although each step may be written down explicitly, it is only practical to perform such operations with a computer.*

As a straightforward consequence of Theorem 1 (see also Remark 1 below), we obtain

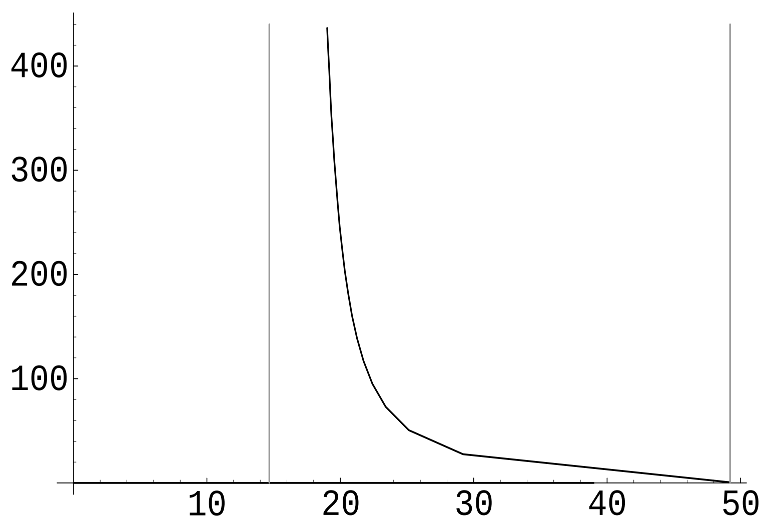
**Corollary 1** *Let $\mathbf{B}$ be the unit ball of $\mathbb{R}^4$ and let $\mu_1$ be the first (radial) eigenvalue of $-\Delta$ in $H^1_0(\mathbf{B})$. Then the problem*

$$\begin{cases} -\Delta u = \mu_1 u + u^3 & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}\ . \end{cases} \tag{7}$$

*admits no nontrivial radial solutions.*

Let us recall that (7) does admit a nontrivial (nonradial nonpositive!) solution, see [12]. This result, together with Corollary 1, complements [10, Theorem 0.1] where the proof was not complete in the particular case of dimension $n = 4$, when $\lambda$ belongs to the spectrum of $-\Delta$. Moreover, Corollary 1 shows that the very same proof cannot work in the class of radial functions and gives an explanation why the eigenvalues had to be skipped in [2, 14, 16].

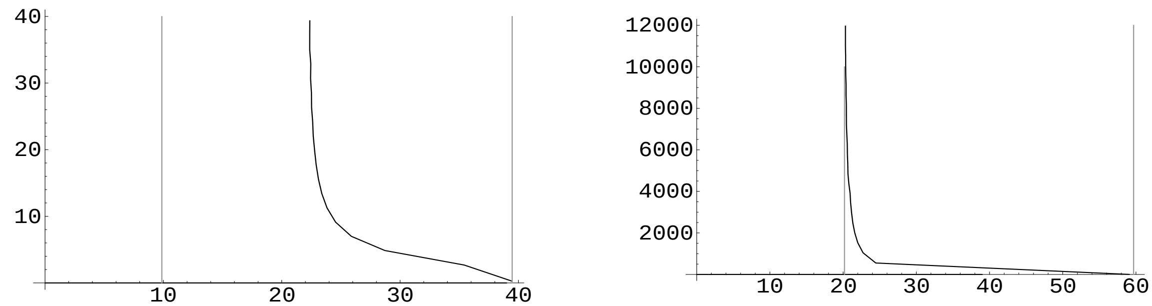
The above mentioned results (including Theorem 1) are illustrated in the Figure below, which is obtained numerically by means of the algorithm explained in Section 4.



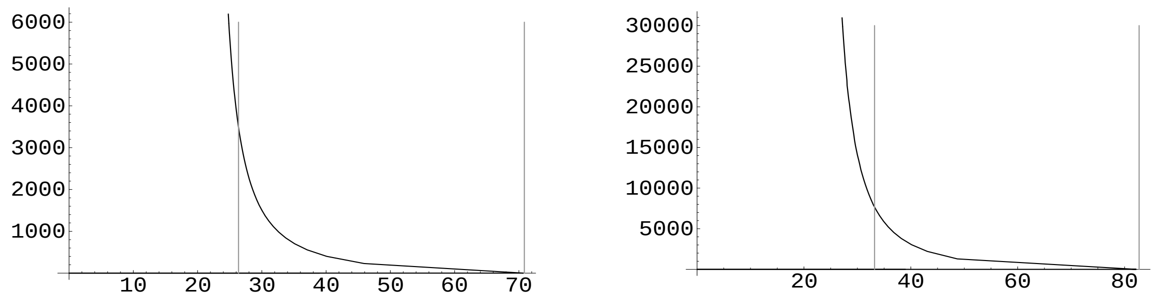

**Second radial bifurcation branch $\lambda \mapsto \omega(\lambda) = u(0)$ for $n = 4$**

With the same numerical procedure we obtained the following pictures concerning other values of $n$. For the reader's convenience, we also recall the values of $\mu_1$ and $\mu_2$, according to [1].

| $n$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| $\mu_1$ | $\pi^2$ | 14.68... | 20.19... | 26.37... | 33.22... |
| $\mu_2$ | $4\pi^2$ | 49.22... | 59.68... | 70.85... | 87.72... |




**Second radial bifurcation branch for $n = 3$ and $n = 5$**




**Second radial bifurcation branch for $n = 6$ and $n = 7$**

**Open Problem 1** It would be very interesting to give a proof of (5). Moreover, it would be nice to specify whether the branch approaches the number $\mu$ from the left or from the right. The latter correspondingly modified question is also interesting in dimension $n = 6$.

## 2 Proof of Theorem 1, part 1

In this section we prove:

**Proposition 1** *For all $\omega \geq 349$, we have $\lambda(\omega) > \mu_1$.*

Our proof of Proposition 1 consists in making more explicit several constants obtained in the estimates in [5, 6, 7]. As we are basing our analysis on these papers, we adopt their notation and we will often refer to formulas therein.

By means of scaling and of Emden-Fowler inversion $y(t) := \lambda^{-1/2} u\left(2\,\lambda^{-1/2}\,t^{-1/2}\right)$, equation (2) (for $n = 4$) becomes

$$y'' + t^{-3}(1 + y^2)y = 0 \quad (t > 0) \;, \qquad y(t) \to \gamma \text{ as } t \to \infty \tag{8}$$

where $\gamma = \omega\lambda^{-1/2} > 0$. In [5] it is shown that $y$ has infinitely many zeros $T_1(\gamma) > T_2(\gamma) > ...$, and that

$$\lim_{\gamma\to\infty} T_1(\gamma) = \infty \qquad \text{and} \qquad \lim_{\gamma\to\infty} T_j(\gamma) = \tau_{j-1} \quad \forall j \geq 2 \;. \tag{9}$$

Here $\tau_1 > \tau_2 > ...$ are the zeros of the function

$$\alpha(t) = \sqrt{t} J_1\left(\frac{2}{\sqrt{t}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} t^{-k} , \tag{10}$$

where $J_1$ is the first kind (regular) Bessel function of order 1. The first (smallest) zero of $J_1$ is $3.83170\ldots$ (see e.g. [1]) and therefore,

$$\tau_1 = 0.27244\ldots \ . \tag{11}$$

**Remark 1** The Emden-Fowler inversion generates a one-to-one correspondence between solutions of problems (2) and (8). In particular, by continuous dependence this shows that branches of solutions of (2) arising from an eigenvalue are connected. Moreover, by the unique continuation principle (uniqueness of solutions for the Cauchy problem), two different branches cannot intersect.

Note that the function $\alpha$ defined in (10) satisfies the differential equation

$$\alpha'' + t^{-3}\alpha = 0. \tag{12}$$

As for the relative location of the respective zeros $\tau_k$ and $T_k$ of $\alpha$ and $y$, we observe:

**Lemma 1** *For any $\gamma > 0$ and every $k \in \mathbb{N}$ one has that $T_k > \tau_k$.*

*Proof.* For $k = 1$, the statement follows from the fact that (2) has positive solutions for some suitable $\omega > 0$ precisely when $\lambda \in (0, \mu_1)$. For $k \geq 2$, the statement follows from Sturm's comparison result applied to equations (8) and (12). $\square$

We now give a refinement of [5, (3.2)]:

**Lemma 2** *For all $t \in (0, T_1)$ we have $|y(t)| < \dfrac{2\gamma}{1+\gamma^2}(T_1 + 1 - t)$.*

*Proof.* Take $f(y) = y + y^3$ so that $f$ is as in [7, (2.6)] with $k = 3$, $q = 1$ and $p = 3$. Let $y$ be the solution of (8), which is none other than [7, (2.8)-(2.9)]. Hence, [7, Lemma 2.1] entails

$$\forall t \geq T_1 : \qquad y(t) < z(t) := \frac{2\gamma t}{1 + \gamma^2 + 2t} \ . \tag{13}$$

By [7, (2.12)], we know that $z$ satisfies the differential equation

$$z''(t) = -\frac{1+\gamma^2}{\gamma^2} \cdot \frac{1}{t^3} \cdot z^3(t). \tag{14}$$

Therefore, by making use of (8), we obtain

$$\forall t \geq T_1 : \qquad y'(t) = \int_t^{\infty} \frac{y(s) + y^3(s)}{s^3}\, ds < \frac{\gamma^2}{1+\gamma^2} z'(t) \ + \ \int_t^{\infty} \frac{2\gamma}{s^2(1+\gamma^2+2s)}\, ds.$$

By replacing the exact value of $z'(t)$ and taking into account that

$$\int_{T_1}^{\infty} \frac{ds}{s^2(1+\gamma^2+2s)} \ < \ \frac{1}{1+\gamma^2} \int_{T_1}^{\infty} \frac{ds}{s^2} \ = \ \frac{1}{1+\gamma^2} \frac{1}{T_1} \ ,$$

the previous inequality (when $t = T_1$) yields

$$y'(T_1) < \frac{2\gamma^3}{(1+\gamma^2)^2} + \frac{2\gamma}{1+\gamma^2}\frac{1}{T_1} < \frac{2\gamma}{1+\gamma^2} + \frac{2\gamma}{1+\gamma^2}\frac{1}{T_1} \ .$$

This estimate makes more precise the statement of [5, Lemma 4] (recall the limit in (9)). From the last inequality and from [5, Lemma 2] we get

$$\forall t \in (0, T_1): \quad |y(t)| < |y'(T_1)|(T_1 - t) < \left(\frac{2\gamma}{1+\gamma^2} + \frac{2\gamma}{1+\gamma^2}\frac{1}{T_1}\right)(T_1 - t) < \frac{2\gamma}{1+\gamma^2}(T_1 - t) + \frac{2\gamma}{1+\gamma^2} \ ,$$

which proves the statement. □

Our next goal is to provide a suitable upper bound for $T_1$. For this purpose we need an estimate of $y$ from below beyond $T_1$. By means of the differential equation (14) and integration by parts, we have:

$$\forall t: \qquad \int_t^\infty \frac{s-t}{s^3} z^3(s)\, ds = -\frac{\gamma^2}{1+\gamma^2}\int_t^\infty (s-t)z''(s)\, ds = \frac{\gamma^2}{1+\gamma^2}\left(\gamma - z(t)\right) = \frac{\gamma^3}{1+\gamma^2+2t}. \quad (15)$$

Furthermore, with a tedious calculation one can find

$$\forall t: \qquad \int_t^\infty \frac{s-t}{s^3}\, z(s)\, ds = -\frac{2\gamma}{1+\gamma^2} + \frac{2\gamma}{(1+\gamma^2)^2}(1+\gamma^2+2t)\log\left(\frac{1+\gamma^2+2t}{2t}\right) \ . \qquad (16)$$

Next, note that by $y(t) \to \gamma$ $(t \to \infty)$ and $y''(t) = -t^{-3}(y+y^3)$, one deduces that $|y'(t)| \le C(\gamma)t^{-2}$. Hence, we obtain for $t \ge T_1$:

$$\begin{aligned} y(t) &= \gamma - \int_t^\infty y'(s)\, ds = \gamma - \left[(s-t)y'(s)\right]_t^\infty + \int_t^\infty (s-t)y''(s)\, ds \\ &= \gamma - \int_t^\infty \frac{s-t}{s^3}\left(y(s) + y^3(s)\right)\, ds > \gamma - \int_t^\infty \frac{s-t}{s^3} z(s)\, ds - \int_t^\infty \frac{s-t}{s^3} z^3(s)\, ds \qquad (17) \\ &= \gamma + \frac{2\gamma}{1+\gamma^2} - \frac{\gamma^3}{1+\gamma^2+2t} - \frac{2\gamma}{(1+\gamma^2)^2}(1+\gamma^2+2t)\log\left(\frac{1+\gamma^2+2t}{2t}\right) \end{aligned}$$

where we used (15) and (16).

We now refine [6, Theorem 3 II] with the following:

**Lemma 3** *For all $\gamma \ge e^4$ we have $T_1 < 2\log\gamma$.*

*Proof.* It suffices to show that

$$\forall \gamma \ge e^4 \qquad \forall t \in [2\log\gamma, \infty): \qquad y(t) > 0. \qquad (18)$$

Let us rewrite (17) as

$$\forall t \ge T_1: \qquad \frac{y(t)}{\gamma} \ge \psi(t) := \frac{3+\gamma^2}{1+\gamma^2} - \frac{\gamma^2}{1+\gamma^2+2t} - \frac{2(1+\gamma^2+2t)}{(1+\gamma^2)^2}\log\left(\frac{1+\gamma^2+2t}{2t}\right). \qquad (19)$$

Since $y$ is positive at $\infty$ and $\gamma\psi$ is a lower bound for $y$ as long as $y$ is positive we have that $y$ is positive on any interval $[t, \infty)$ where $\psi > 0$. With some calculations one finds that

$$\lim_{t\to\infty} \psi(t) = 1 \qquad (20)$$

and

$$\psi''(t) = -\frac{8\gamma^2}{(1+\gamma^2+2t)^3} - \frac{2}{t^2(1+\gamma^2+2t)} < 0. \tag{21}$$

This, together with (20), proves (18) provided that

$$\forall \gamma \geq e^4 : \qquad \psi\left(2\log\gamma\right) > 0. \tag{22}$$

We have

$$\psi\left(2\log\gamma\right) = \frac{3+3\gamma^2+12\log\gamma+4\gamma^2\log\gamma}{(1+\gamma^2)(1+\gamma^2+4\log\gamma)} - \frac{2(1+\gamma^2+4\log\gamma)}{(1+\gamma^2)^2}\log\left(\frac{1+\gamma^2+4\log\gamma}{4\log\gamma}\right)$$

so that (22) holds if and only if

$$\Psi_1(\gamma) := \left(\frac{3}{\gamma^2}+3+12\frac{\log\gamma}{\gamma^2}+4\log\gamma\right)\left(1+\frac{1}{\gamma^2}\right) - 2\log\left(1+\frac{1+\gamma^2}{4\log\gamma}\right)\left(\frac{1}{\gamma^2}+1+4\frac{\log\gamma}{\gamma^2}\right)^2 > 0$$

for all $\gamma \geq e^4$. Since we assume $\gamma \geq e^4$ one has

$$\log\left(1+\frac{1+\gamma^2}{4\log\gamma}\right) \leq \log\left(\frac{17+\gamma^2}{16}\right) \leq 2\log\left(\frac{51}{200}\gamma\right) \leq 2\log\gamma - \frac{5}{2}$$

and may conclude:

$$\begin{aligned}
\Psi_1(\gamma) \quad &\geq \quad 8+\frac{16}{\gamma^2}+\frac{8}{\gamma^4}+48\frac{\log\gamma}{\gamma^2}+48\frac{\log\gamma}{\gamma^4}+48\frac{\log^2\gamma}{\gamma^4}-32\frac{\log^2\gamma}{\gamma^2}-64\frac{\log^3\gamma}{\gamma^4}\\
&\geq \quad 8-32\left(\frac{4}{e^4}\right)^2+\frac{16}{\gamma^2}\left(1-4\frac{\log^3\gamma}{\gamma^2}\right)\\
&\geq \quad 7+\frac{16}{\gamma^2}\left(1-\frac{4^4}{e^8}\right) > 0.
\end{aligned}$$

We see that (22) indeed holds, so that (18) also follows and the lemma is proved. $\square$

Next, we prove a lower bound for $y'(T_1)$:

**Lemma 4** *For all $\gamma \geq 110$ we have $y'(T_1) > \dfrac{1.69}{\gamma}$.*

*Proof.* Since $\gamma \geq 110$, in view of Lemma 3 we also have $\gamma > 2\log\gamma > T_1$. Beyond $T_1$, the solution $y$ is concave and we obtain $y'(T_1) > \frac{1}{\gamma}[y(\gamma) - z(T_1)]$. We make use of [7, Lemma 2.2], according to which

$$y(t) > \frac{\gamma^2}{1+\gamma^2}\left(z(t) - \frac{2}{\gamma}\log\left(1+\frac{1+\gamma^2}{2t}\right)\right)$$

and arguing as on p.156 in [7] (case $q = k-2$) we get

$$\gamma y'(T_1) > \frac{\gamma^2}{1+\gamma^2}\left[\frac{2\gamma^2}{(1+\gamma)^2} - \frac{2}{\gamma}\log\frac{(1+\gamma)^2}{2\gamma}\right] - \frac{2\gamma T_1}{1+\gamma^2+2T_1}\ . \tag{23}$$

In turn, by Lemma 3, this implies

$$\gamma y'(T_1) > \frac{2\gamma^4}{(1+\gamma^2)(1+\gamma)^2} - \frac{2\gamma}{1+\gamma^2}\log\frac{(1+\gamma)^2}{2\gamma} - \frac{4\gamma\log\gamma}{1+\gamma^2}$$

so that we have to prove that

$$\forall \gamma \geq 110: \qquad \frac{2\gamma^4}{(1+\gamma^2)(1+\gamma)^2} - \frac{2\gamma}{1+\gamma^2}\log\frac{(1+\gamma)^2}{2\gamma} - \frac{4\gamma\log\gamma}{1+\gamma^2} > 1.69.$$

This is equivalent to show that for all $\gamma \geq 110$:

$$H_1(\gamma) := 2\gamma^4 - 4\gamma(1+\gamma)^2\log(1+\gamma) - 2\gamma(1+\gamma)^2\log\gamma + (2\log 2)\,\gamma(1+\gamma)^2 - 1.69(1+\gamma)^2(1+\gamma^2) > 0.$$

Since we assume $\gamma \geq 110$, we have

$$\frac{\log(1+\gamma)}{\gamma} \leq \frac{\log 111}{110} \leq \frac{1}{23.3}, \quad \frac{\log\gamma}{\gamma} \leq \frac{\log 110}{110} \leq \frac{1}{23.4}, \quad \frac{1+\gamma}{\gamma} \leq \frac{111}{110}, \quad \frac{1+\gamma^2}{\gamma^2} \leq \frac{12101}{12100}.$$

We may conclude

$$\begin{aligned} H_1(\gamma) &\geq 2\left(\frac{110}{111}\right)^2\gamma^2(1+\gamma)^2 - \frac{4}{23.3}\gamma^2(1+\gamma)^2 - \frac{2}{23.4}\gamma^2(1+\gamma)^2 - 1.69\frac{12101}{12100}(1+\gamma)^2\gamma^2 \\ &\geq \frac{1}{100}(1+\gamma)^2\gamma^2 > 0, \end{aligned}$$

and the statement follows. □

For $\alpha$ as in (10), define the function $b$ as in [5, (4.6)].

$$b(t) := y'(t)\alpha(t) - y(t)\alpha'(t). \tag{24}$$

Then

$$b(T_1) \;=\; y'(T_1)\alpha(T_1) \;=\; y'(T_1)\sqrt{T_1}\,J_1\left(\frac{2}{\sqrt{T_1}}\right). \tag{25}$$

Lemma 4 combined with (25) enables us to refine [5, (4.10)] with the following

$$\forall \gamma \geq 110: \qquad b(T_1) > \frac{1.69}{\gamma}\,\alpha(T_1). \tag{26}$$

Observe that $\alpha(T_1) > 0$ by Lemma 1.

As in [5, (4.12)] we now conclude from the differential equations (8) for $y$ and (12) for $\alpha$ that

$$b(\tau_1) = b(T_1) + \int_{\tau_1}^{T_1} \frac{y^3(s)}{s^3}\,\alpha(s)\,ds\ . \tag{27}$$

Since $\tau_1 < T_1$ by Lemma 1 and hence $0 < \alpha(t) < \alpha(T_1)$ for all $t \in (\tau_1, T_1)$, an estimate of the integral in the right hand side of (27) by using Lemmas 2-3 and (11), yields

$$\begin{aligned} \forall \gamma \geq 110: \qquad &\left|\int_{\tau_1}^{T_1} \frac{y^3(s)}{s^3}\,\frac{\alpha(s)}{\alpha(T_1)}\,ds\right| \leq \int_{\tau_1}^{T_1} \frac{|y(s)|^3}{s^3}\,ds \leq \left(\frac{2\gamma}{(1+\gamma^2)}(T_1+1-\tau_1)\right)^3 \int_{\tau_1}^{\infty} \frac{ds}{s^3} < \\ &< \frac{54\gamma^3}{(1+\gamma^2)^3}(T_1+1-\tau_1)^3 < \frac{54\gamma^3}{(1+\gamma^2)^3}\,(0.72756 + 2\log\gamma)^3\,. \end{aligned} \tag{28}$$

Inserting (28) and (26) into (27) yields

$$b(\tau_1) > \alpha(T_1)\left[\frac{1.69}{\gamma} - \frac{54\gamma^3}{(1+\gamma^2)^3}\,(0.72756 + 2\log\gamma)^3\right] > \alpha(T_1)\left[\frac{1.69}{\gamma} - \frac{54}{\gamma^3}\,(0.72756 + 2\log\gamma)^3\right] > 0$$

the last inequality being true for all $\gamma \geq 222$. By (24) we get $b(\tau_1) = -y(\tau_1)\alpha'(\tau_1)$. Since $\alpha'(\tau_1) > 0$, we have so proved the following implications:

$$\gamma \geq 222 \implies b(\tau_1) > 0 \implies y(\tau_1) < 0 \, . \tag{29}$$

Since we wish to prove (29) for smaller values of $\gamma$, we need to improve some of the previous estimates. Firstly, we complement Lemma 3 with

**Lemma 5** *For all $\gamma \in [e^{9/2}, 222]$ we have $T_1 < \frac{3}{2}\log\gamma$.*

*Proof.* Let $\psi$ be as in (19). By (20) and (21) it suffices to show that for all $\gamma \in [e^{9/2}, 222]$:

$$\psi\left(\frac{3}{2}\log\gamma\right) = \frac{3+3\gamma^2+9\log\gamma+3\gamma^2\log\gamma}{(1+\gamma^2)\,(1+\gamma^2+3\log\gamma)} - \frac{2\,(1+\gamma^2+3\log\gamma)}{(1+\gamma^2)^2}\log\left(1+\frac{1+\gamma^2}{3\log\gamma}\right) > 0. \tag{30}$$

And (30) holds true if and only if

$$\Psi_2(\gamma) := \left(\frac{3}{\gamma^2}+3+9\frac{\log\gamma}{\gamma^2}+3\log\gamma\right)\left(1+\frac{1}{\gamma^2}\right) - 2\log\left(1+\frac{1+\gamma^2}{3\log\gamma}\right)\left(\frac{1}{\gamma^2}+1+3\frac{\log\gamma}{\gamma^2}\right)^2 > 0$$

for all $\gamma \in [e^{9/2}, 222]$. Since we assume $\gamma \geq e^{9/2}$ one has

$$\log\left(1+\frac{1+\gamma^2}{3\log\gamma}\right) \leq \log\left(\frac{14.5+\gamma^2}{13.5}\right) \leq 2\log(0.28\gamma) \leq 2\log\gamma - 2.5.$$

and may conclude by using also $\gamma \leq 222$

$$\begin{aligned}
\Psi_2(\gamma) &\geq -\log\gamma + 8 - 24\frac{\log^2\gamma}{\gamma^2} + 34\frac{\log\gamma}{\gamma^2} + \frac{16}{\gamma^2} - 36\frac{\log^3\gamma}{\gamma^4} + 21\frac{\log^2\gamma}{\gamma^4} + 35\frac{\log\gamma}{\gamma^4} + \frac{8}{\gamma^4} \\
&\geq 2.5 - 24\frac{4.5^2}{e^9} + \frac{1}{\gamma^2}\left(169 - 36\frac{\log^3\gamma}{\gamma^2}\right) \geq 2.4 + \frac{1}{\gamma^2}\left(169 - 36\frac{4.5^3}{e^9}\right) \geq 2.4 + \frac{168}{\gamma^2} > 0.
\end{aligned}$$

Hence, we see that (30) indeed holds on $\gamma \in [e^{9/2}, 222]$, so that the lemma is proved. □

We now extend the statement of Lemma 4 to smaller values of $\gamma$:

**Lemma 6** *For all $\gamma \in [91, 222]$ we have $y'(T_1) > \frac{1.69}{\gamma}$.*

*Proof.* Since $\gamma \in [91, 222]$ and $91 > e^{9/2}$, in view of Lemma 5 we have $\gamma > \frac{3}{2}\log\gamma > T_1$. Therefore, the same arguments used in Lemma 4 lead to (23). Combining Lemma 5 with (23) yields

$$y'(T_1) > \frac{2\gamma^3}{(1+\gamma^2)(1+\gamma)^2} - \frac{2}{1+\gamma^2}\log\frac{(1+\gamma)^2}{2\gamma} - \frac{3\log\gamma}{1+\gamma^2} \, .$$

That means that we have to show that

$$H_2(\gamma) := 2\gamma^4 - 4\gamma(1+\gamma)^2\log(1+\gamma) - \gamma(1+\gamma)^2\log\gamma + (2\log 2)\,\gamma(1+\gamma)^2 - 1.69(1+\gamma)^2(1+\gamma^2) > 0.$$

Since we assume $\gamma \geq 91$, we have

$$\frac{\log(1+\gamma)}{\gamma} \leq \frac{\log 92}{91} \leq \frac{1}{20.1}, \quad \frac{\log \gamma}{\gamma} \leq \frac{\log 91}{91} \leq \frac{1}{20.1}, \quad \frac{1+\gamma}{\gamma} \leq \frac{92}{91}, \quad \frac{1+\gamma^2}{\gamma^2} \leq \frac{8282}{8281}.$$

We may conclude

$$\begin{aligned} H_2(\gamma) &\geq 2\left(\frac{91}{92}\right)^2 \gamma^2(1+\gamma)^2 - \frac{4}{20.1}\gamma^2(1+\gamma)^2 - \frac{1}{20.1}\gamma^2(1+\gamma)^2 - 1.69\frac{8282}{8281}(1+\gamma)^2\gamma^2 \\ &\geq \frac{1}{100}(1+\gamma)^2\gamma^2 > 0, \end{aligned}$$

The lemma is proved. □

Lemma 6 combined with (25) enables us to complement (26) with the following

$$\forall \gamma \in [91, 222]: \qquad b(T_1) > \frac{1.69}{\gamma}\,\alpha(T_1). \tag{31}$$

Recalling again the fact that $0 < \alpha(t) < \alpha(T_1)$ for all $t \in (\tau_1, T_1)$, if we estimate the integral in the right hand side of (27) by using (11) and Lemmas 2 and 5, we get:

$$\forall \gamma \in [91, 222]: \qquad \left| \int_{\tau_1}^{T_1} \frac{y^3(s)}{s^3}\,\frac{\alpha(s)}{\alpha(T_1)}\,ds \right| < \frac{54\gamma^3}{(1+\gamma^2)^3}\left(0.72756 + \frac{3}{2}\log\gamma\right)^3. \tag{32}$$

Inserting (32) and (31) into (27) yields

$$b(\tau_1) > \frac{\alpha(T_1)}{\gamma}\left[1.69 - \frac{54}{\gamma^2}\left(0.72756 + \frac{3}{2}\log\gamma\right)^3\right] > 0$$

the last inequality being true for all $\gamma \in [129, 222]$ (it suffices to show that the term inside square brackets is positive when $\gamma = 129$). By (24) we get $b(\tau_1) = -y(\tau_1)\alpha'(\tau_1)$. Since $\alpha'(\tau_1) > 0$, we have now proved the following implications:

$$\gamma \in [129, 222] \implies b(\tau_1) > 0 \implies y(\tau_1) < 0\,. \tag{33}$$

A third iteration of this procedure is in order:

**Lemma 7** *For all $\gamma \in [e^{9/2}, 129]$ we have $T_1 < \frac{5}{4}\log\gamma$.*

*Proof.* Let $\psi$ be as in (19). By (20) and (21) it suffices to show that for all $\gamma \in [e^{9/2}, 129]$:

$$\psi\left(\frac{5}{4}\log\gamma\right) = \frac{3+3\gamma^2+\frac{15}{2}\log\gamma+\frac{5}{2}\gamma^2\log\gamma}{(1+\gamma^2)\left(1+\gamma^2+\frac{5}{2}\log\gamma\right)} - \frac{2\left(1+\gamma^2+\frac{5}{2}\log\gamma\right)}{(1+\gamma^2)^2}\log\left(1+\frac{1+\gamma^2}{\frac{5}{2}\log\gamma}\right) > 0. \tag{34}$$

And (34) holds if and only if

$$\Psi_3(\gamma) := \left(\frac{3}{\gamma^2}+3+\frac{15\log\gamma}{2\gamma^2}+\frac{5}{2}\log\gamma\right)\left(1+\frac{1}{\gamma^2}\right) - 2\log\left(1+\frac{1+\gamma^2}{\frac{5}{2}\log\gamma}\right)\left(\frac{1}{\gamma^2}+1+\frac{5\log\gamma}{2\gamma^2}\right)^2 > 0$$

for all $\gamma \in [e^{9/2}, 129]$. Since we assume $\gamma \geq e^{9/2}$ one has

$$\log\left(1 + \frac{1+\gamma^2}{\frac{5}{2}\log\gamma}\right) \leq \log\left(\frac{12.25+\gamma^2}{11.25}\right) \leq 2\log(0.3\gamma) \leq 2\log\gamma - 2.4.$$

and may conclude by using also $\gamma \leq 129$

$$\begin{aligned}
\Psi_3(\gamma) &\geq -\frac{3}{2}\log\gamma + 7.8 - 20\frac{\log^2\gamma}{\gamma^2} + 26\frac{\log\gamma}{\gamma^2} + \frac{15.6}{\gamma^2} - 25\frac{\log^3\gamma}{\gamma^4} + 10\frac{\log^2\gamma}{\gamma^4} + 27.5\frac{\log\gamma}{\gamma^4} + \frac{7.8}{\gamma^4} \\
&\geq 0.51 - 20\frac{4.5^2}{e^9} + \frac{1}{\gamma^2}\left(132.6 - 25\frac{\log^3\gamma}{\gamma^2}\right) \geq 0.45 + \frac{1}{\gamma^2}\left(132.6 - 25\frac{4.5^3}{e^9}\right) \geq 0.45 + \frac{132}{\gamma^2} \\
&> 0.
\end{aligned}$$

Hence, we see that (34) indeed holds on $\gamma \in [e^{9/2}, 129]$, so that the lemma is proved. □

Using now Lemma 7, complementing (32) we obtain

$$\forall\gamma \in [91, 129] : \qquad \left|\int_{\tau_1}^{T_1} \frac{y^3(s)}{s^3}\,\frac{\alpha(s)}{\alpha(T_1)}\,ds\right| < \frac{54\gamma^3}{(1+\gamma^2)^3}\left(0.72756 + \frac{5}{4}\log\gamma\right)^3. \tag{35}$$

Inserting (35) and (31) into (27) yields

$$b(\tau_1) > \frac{\alpha(T_1)}{\gamma}\left[1.69 - \frac{54}{\gamma^2}\left(0.72756 + \frac{5}{4}\log\gamma\right)^3\right] > 0$$

the last inequality being true for all $\gamma \in [91, 129]$. Similarly as above we have the following implications:

$$\gamma \in [91, 129] \implies b(\tau_1) > 0 \implies y(\tau_1) < 0\ . \tag{36}$$

Summarizing, if we combine (29)-(33)-(36) we have

$$\gamma \geq 91 \implies y(\tau_1) < 0\ .$$

On the other hand, by (9) and continuity of the maps $\gamma \mapsto T_j(\gamma)$ ($j \geq 1$), this shows that

$$\gamma \geq 91 \implies \tau_1 > T_2 \implies \lambda > \mu_1\ . \tag{37}$$

We may now prove Proposition 1, namely that $\lambda > \mu_1$ whenever $\omega \geq 349$. Assume for contradiction that $\lambda \leq \mu_1$. Then, using (37), we have

$$\sqrt{\lambda}\,\gamma = \omega \geq 349 > 91\sqrt{\mu_1} \geq 91\sqrt{\lambda} \implies \gamma \geq 91 \implies \lambda > \mu_1\ ,$$

a contradiction! □

**Remark 2** One could gain the impression that with (finitely or possibly infinitely many) further iterations, one could finally show that $\lambda(\omega) > \mu_1$ for arbitrary $\omega > 0$. However, some numerical experiments show that this does not seem to be the case, therefore it seemed convenient to let the computer complete the proof for $\omega < 349$, except for the case $\omega \in (0, 5.87\ldots)$, see the next section.

## 3 Proof of Theorem 1, part 2

In this section we prove:

**Proposition 2** *For all $\omega \leq \sqrt{\mu_2 - \mu_1}$, we have $\lambda(\omega) > \mu_1$.*

As above, $\mu_1$ and $\mu_2$ denote the first two radial eigenvalues of $-\Delta$ in $H_0^1(\mathbf{B})$.

We begin with a simple observation on solutions of the equation

$$u''(r) + \frac{3}{r}u'(r) + \lambda u(r) + u^3(r) = 0 \quad \text{for } r > 0. \tag{38}$$

**Lemma 8** *Let $\lambda \geq 0$ and $u$ be a nontrivial solution of* (38)*, with $u'(0) = 0$, then*

$$\forall r > 0 : \qquad |u(r)| < |u(0)|.$$

*Proof.* We may assume that $u(0) > 0$. Consider the energy function

$$E(r) := \frac{1}{2}u'(r)^2 + \frac{\lambda}{2}u^2(r) + \frac{1}{4}u^4(r),$$

so that, using (38),

$$E'(r) = u'(r)\left(u''(r) + \lambda u(r) + u^3(r)\right) = -\frac{3}{r}u'(r)^2.$$

This tells us that $r \mapsto E(r)$ is decreasing. Since we also have $E(r) \geq 0$ for all $r$, the solution $u$ is globally bounded. Moreover, in any further critical point $R > 0$ of the solution of (38), we have

$$\frac{\lambda}{2}u^2(R) + \frac{1}{4}u^4(R) = E(R) < E(0) = \frac{\lambda}{2}u^2(0) + \frac{1}{4}u^4(0).$$

This immediately gives $|u(R)| < u(0)$ and the statement follows. □

As a straightforward consequence of Lemma 8, for all solutions of (38) one has

$$\omega = \max_{[0,1]} |u| > |u(r)| \qquad \forall r \in (0, 1]. \tag{39}$$

*Proof of Proposition 2.* Let $\omega \leq \sqrt{\mu_2 - \mu_1}$ and let $u_\omega$ be a solution of (38) with precisely one zero in the interval $[0, 1)$. This means that $u_\omega = \varphi$ is the second radial eigenfunction of

$$\begin{cases} -\Delta\varphi = \lambda\varphi + u_\omega^2\varphi & \text{in } \mathbf{B} \\ \varphi = 0 & \text{on } \partial\mathbf{B} \end{cases}$$

with eigenvalue $\lambda = \lambda(\omega)$. In what follows $\mathcal{H}_r$ denotes the space of radially symmetric functions in $H_0^1(\mathbf{B})$. By means of the variational characterization of eigenvalues and (39) we have

$$\begin{aligned} \lambda(\omega) &= \min_{\substack{V \subset \mathcal{H}_r \\ \dim V = 2}} \max_{\substack{\varphi \in V \\ \|\varphi\|_{L^2(\mathbf{B})} = 1}} \left(\int_{\mathbf{B}} |\nabla\varphi|^2 \, dx - \int_{\mathbf{B}} u_\omega^2 \varphi^2 \, dx\right) \\ &> \min_{\substack{V \subset \mathcal{H}_r \\ \dim V = 2}} \max_{\substack{\varphi \in V \\ \|\varphi\|_{L^2(\mathbf{B})} = 1}} \left(\int_{\mathbf{B}} |\nabla\varphi|^2 \, dx - \omega^2\right) \\ &= \mu_2 - \omega^2 \\ &\geq \mu_1 \end{aligned}$$

since we assumed initially that $\omega \leq \sqrt{\mu_2 - \mu_1}$. This completes the proof of Proposition 2. □

**Remark 3** The above proof may be extended to any space dimension $n \geq 3$. In particular, it states that $\lambda(\omega) \geq \mu_2 - \omega^2$ for all $\omega$ sufficiently small. In turn, Lemma 1 states that $\lambda(\omega) < \mu_2$ for all $\omega$. Therefore, $\lim_{\omega \to 0} \lambda(\omega) = \mu_2$.

# 4 Proof of Theorem 1, part 3

In this section we prove:

**Proposition 3** *For all $\sqrt{\mu_2 - \mu_1} \leq \omega \leq 349$, we have $\lambda(\omega) > \mu_1$.*

Since $\sqrt{\mu_2 - \mu_1} = 5.8767\ldots$, we prove Proposition 3 for all $\omega \in [5, 349]$.

## 4.1 Transformation

In this subsection we transform the equation (2) (with $\omega = u(0)$) in order to make it suitable for the computer assisted proof when $n = 4$ and for the numerical study of the dimensions $n = 3, 5, 6, 7$.

Let $t = \omega^{\frac{2}{n-2}} r$ and $w(t) = \omega^{-1} u(r)$ so that $u'(r) = \omega^{\frac{n}{n-2}} w'(t)$, $u''(r) = \omega^{\frac{n+2}{n-2}} w''(t)$. Then, (2) becomes

$$\begin{cases} w''(t) + \frac{n-1}{t} w'(t) + \gamma^{\frac{4}{2-n}} w(t) + |w(t)|^{\frac{4}{n-2}} w(t) = 0 & t \in (0, \infty) \\ w(0) = 1 \\ w'(0) = 0 \end{cases} \tag{40}$$

where $\gamma = \omega \lambda^{\frac{2-n}{4}}$ and we want to determine the second zero $z$ of the solution of (40) as a function of $\gamma$. Note that $z = \omega^{\frac{2}{n-2}} = \gamma^{\frac{2}{n-2}} \sqrt{\lambda}$ so that $\lambda = z^2 \gamma^{\frac{4}{2-n}}$ and $\omega = z^{\frac{n-2}{2}} = \gamma \lambda^{\frac{n-2}{4}}$.

Summarizing, in the case $n = 4$ we need to show that

$$5 \leq \gamma\sqrt{\lambda} \leq 349 \implies z > \gamma\sqrt{\mu_1}\,. \tag{41}$$

Since we already know that $\gamma\sqrt{\lambda} \leq 5\ldots$ and $\gamma\sqrt{\lambda} \geq 349$ imply $\lambda > \mu_1$, by continuity (41) follows if we prove the following

**Proposition 4** *For all $\gamma$ satisfying $5 \leq \gamma\sqrt{\mu_1} \leq 349$, the second positive zero $z$ of the solution of (40) satisfies $z > \gamma\sqrt{\mu_1}$.*

In order to prove Proposition 4, we solve the initial value problem (40) with a rigorous computer assisted method, introduced in [3]. We describe here the peculiarities of this equation and we refer to the above mentioned paper for the details. We remark that equation (40) has also been used to make the numerical experiments leading to the pictures concerning the cases $n = 3, 5, 6, 7$ displayed in the introduction.

## 4.2 Technical lemmas

In this subsection we recall the functional analytic background introduced in [3], to which we refer for the proofs. Let $R > 0$, let $\mathcal{H}_R$ be the space of analytic functions in the open disk $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let $\mathcal{X}_R$ and $\mathcal{Y}_R$ be the subspaces of $\mathcal{H}_R$ with finite norm

$$\|u\|_{\mathcal{X}_R} = \sum_{k=0}^{\infty} |u_k|\, R^k \quad \text{and} \quad \|u\|_{\mathcal{Y}_R} = \sup_{t \in D_R} |u(t)|$$

respectively, where

$$u(t) = \sum_{k=0}^{\infty} u_k t^k \tag{42}$$

and $u_k \in \mathbb{R}$. In the sequel, we denote by $\mathcal{Z}_R$ either $\mathcal{X}_R$ or $\mathcal{Y}_R$, and by $||\cdot||_{\mathcal{Z}_R}$ the respective norm. The following lemma is straightforward:

**Lemma 9** *The spaces $\mathcal{Z}_R$ are Banach algebras, i.e. for all $u, v \in \mathcal{Z}_R$ we have $uv \in \mathcal{Z}_R$ and $||uv||_{\mathcal{Z}_R} \le ||u||_{\mathcal{Z}_R} ||v||_{\mathcal{Z}_R}$.*

**Remark 4** In particular, this implies that $||u^m||_{\mathcal{Z}_R} \le ||u||_{\mathcal{Z}_R}^m$ for all $m \in \mathbb{N}$ and $||e^u||_{\mathcal{Z}_R} \le e^{||u||_{\mathcal{Z}_R}}$.

The derivative operator $D_R : \mathcal{Z}_R \to \mathcal{H}_R$ is unbounded, but if we choose $R' < R$ we may define $D_{R,R'} : \mathcal{Z}_R \to \mathcal{Z}_{R'}$ and we have the following

**Lemma 10** *$||D_{R,R'}|| \le C_{R,R'}$, where $C_{R,R'} = \left(eR' \log \frac{R}{R'}\right)^{-1}$ when $\mathcal{Z}_R = \mathcal{X}_R$ and $C_{R,R'} = (R - R')^{-1}$ when $\mathcal{Z}_R = \mathcal{Y}_R$.*

Since we want the computer to handle functions in $\mathcal{Z}_R$, we need to represent such functions by using only a finite set of representable numbers [15]. Our choice is to write functions in $\mathcal{Z}_R$ as

$$u(t) = \sum_{k=0}^{N-1} u_k t^k + t^N E_u(t) \tag{43}$$

where $E_u \in \mathcal{Z}_R$. We store $2N + 1$ representable numbers: $N$ pairs represent lower and upper bounds for the value of the (real) coefficients $\{u_k\}$, while the last number is an upper estimate of the norm of $E_u$.

**Lemma 11** *Let $0 < R' < R$. If $u \in \mathcal{Z}_R$ is represented as in (43), then $u' \in \mathcal{Z}_R$ is represented as*

$$u'(t) = \sum_{k=0}^{N-1} v_k t^k + t^N E_v(t)\,,$$

*where $v_k = (k+1)u_{k+1}$ for $k = 0, \ldots, N-2$, $v_{N-1} = [-N||E_u||_{\mathcal{Z}_R}, N||E_u||_{\mathcal{Z}_R}]$, $||E_v||_{\mathcal{X}_R} \le ||E_u||_{\mathcal{X}_R}(N/R + C_{R,R'})$ and $||E_v||_{\mathcal{Y}_R} \le ||E_u||_{\mathcal{Y}_R}(2N/R + C_{R,R'})$.*

### 4.3 The first step

An easy computation shows that, when $\gamma \ge 1$, the solution of (40) can be extended analytically at least to the disk centered at 0 of radius $R = 1$. For this reason, for the first step we set $R = 11/10$ and

$$\tilde{\mathcal{X}}_R = \{w \in \mathcal{X}_R \,:\, w(0) = 1,\, w(t) = w(-t)\}\,.$$

Let $L : \tilde{\mathcal{X}}_R \to \mathcal{H}_R$ and $f_\gamma : \tilde{\mathcal{X}}_R \to \mathcal{X}_R$ be defined by

$$Lw = w'' + \frac{3}{t}w' \qquad \text{and} \qquad f_\gamma(w) = -\gamma^{-2}w - w^3\,,$$

and consider the operator

$$F_\gamma := (L^{-1} f_\gamma) : \tilde{\mathcal{X}}_R \to \tilde{\mathcal{X}}_R\,.$$

The following lemma is straightforward:

**Lemma 12** *The operator $L$ is invertible and solutions of equation (40) in the interval $(0,R)$ (more precisely, their analytic extension in $D_R$) correspond to fixed points of the operator $F_\gamma$.*

If

$$w(t)=\sum_{k=0}^{\infty} w_k t^k\,,$$

with $w_{2k+1}=0$ for all integers $k$, then

$$Lw=\sum_{k=0}^{\infty}(k+2)(k+4)w_{k+2}t^k\,;$$

inverting this relation we get

$$L^{-1}w=1+\sum_{k=0}^{\infty}\frac{w_k t^{k+2}}{(k+2)(k+4)}\,.$$

Let $B(0,K)=\{w\in\tilde{\mathcal{X}}_R\,:\,||w||_{\mathcal{X}_R}\le K\}$, then

**Lemma 13** *The Lipschitz constant of $F_\gamma$ restricted to $B(0,K)$ is at most $\frac{R^2}{8}\left(\gamma^{-2}+3K^2\right)$.*

*Proof.* We have

$$\left\|L^{-1}w\right\|_{\mathcal{X}_R}=\sum_{k=0}^{\infty}\frac{|w_k|\,R^{k+2}}{(k+2)(k+4)}\le\frac{R^2}{8}\sum_{k=0}^{\infty}|w_k|\,R^k=\frac{R^2}{8}\,\|w\|_{\mathcal{X}_R}\,.$$

The statement follows considering that $f'_\gamma(w)=-\gamma^{-2}-3w^2$ and that $\mathcal{X}_R$ is a Banach algebra. □

Assume that we have an approximate solution $\bar{w}(t)=\sum_{k=0}^{N-1}\bar{w}_k t^k$, where $\{\bar{w}_k\}$ are interval values satisfying $\bar{w}_0=[1,1]$ and $\bar{w}_{2k+1}=[0,0]$ for all $k=0,\ldots,N/2-1$ (since 0 and 1 are representable numbers, cf. [15], we may choose intervals of width 0 for $\bar{w}_0$ and $\bar{w}_1$). The following lemma yields a true solution close to $\bar{w}$:

**Lemma 14** *Let $\bar{w}(t)=\sum_{k=0}^{N-1}\bar{w}_k t^k$. If there exist $\varepsilon,\rho>0$ such that $||F_\gamma(\bar{w})-\bar{w}||_{\mathcal{X}_R}<\varepsilon$ and the restriction of $F_\gamma$ to the ball $B(\bar{w},\rho)$ has Lipschitz constant $L(F_\gamma)\le 1-\varepsilon/\rho$, then there exists a fixed point of $F_\gamma$ in $B(\bar{w},\rho)$.*

**Remark 5** Typical values of the constants mentioned above are as follows: $K\simeq 1.2$, $L\simeq 0.8$, $\varepsilon,\rho\simeq 10^{-7}$. The actual values of the constant occurring in Lemma 14 can be obtained from the function *Basics.Integrate* of the Ada program.

### 4.4 Second step

By applying Lemmas 10 and 11 we rigorously compute $W_0:=w(1)$ and $W_1:=w'(1)$. To proceed, it is convenient to make another change of variable. Let $V(s):=tw(t)$ where $s=\log t$. The differential equation (40) together with the initial conditions in $t=1$ transforms into

$$\begin{cases} V''=(1-\gamma^{-2}e^{2s})V-V^3\\ V(0)=W_0\\ V'(0)=W_0+W_1\end{cases}\tag{44}$$

Fix $R>0$ and consider the space $\mathcal{Y}_R$; let

$$\hat{\mathcal{Y}}_R=\{V\in\mathcal{Y}_R : V(0)=W_0,\ V'(0)=W_0+W_1\}$$

and let $C_\gamma:\hat{\mathcal{Y}}_R\to\hat{\mathcal{Y}}_R$ be defined by

$$C_\gamma(V)=D^{-2}[(1-\gamma^{-2}e^{2s})V-V^3]\,,$$

where $D^{-2}:\mathcal{Y}_R\to\hat{\mathcal{Y}}_R$ is the inverse of the second derivative. It is clear that the analytic extension in $D_R$ of the solution of the initial value problem (44) is a fixed point of the operator $C_\gamma$. The analogue of Lemma 14 reads:

**Lemma 15** *Let $\bar{v}(t)=\sum_{k=0}^{N-1}\bar{v}_k t^k$. If there exists $\varepsilon,\rho>0$ such that $||C_\gamma(\bar{v})-\bar{v}||_{\mathcal{Y}_R}<\varepsilon$ and the restriction of $C_\gamma$ to the ball $B(\bar{v},\rho)$ has Lipschitz constant $L(C_\gamma)\le 1-\varepsilon/\rho$, then there exists a fixed point of $C_\gamma$ in $B(\bar{v},\rho)$.*

To proceed, we need an upper bound for the Lipschitz constant of $C_\gamma$:

**Lemma 16** *Let $B_K=\{v\in\hat{\mathcal{Y}}_R\,,\,||v||_{\mathcal{Y}_R}\le K\}$. The Lipschitz constant $L(C_\gamma)$ of the operator $C_\gamma$ restricted to $B_K$ satisfies*

$$L(C_\gamma)\le\left[\max_{|s|\le R}\left(1-\gamma^{-2}e^{2s}\right)+3K^2\right]\frac{R^2}{2}\,.$$

*Proof.* The statement follows when considering $||D^{-2}||=\frac{R^2}{2}$,

$$\frac{\partial}{\partial V}\left((1-\gamma^{-2}e^{2s})V-V^3\right)=(1-\gamma^{-2}e^{2s})-3V^2$$

and the definition of the norm in $\mathcal{Y}_R$. □

In order to solve equation (44), we proceed as follows. We compute an approximate solution $\bar{v}$ as a truncated power series, we compute its norm and by Lemma 16 we estimate $R$ in such a way that $C_\gamma$ has Lipschitz constant not larger than 0.95 in a ball of radius equal to the norm of the approximate solution. Then we compute an upper bound for $||C_\gamma(\bar{v})-\bar{v}||_{\mathcal{Y}_R}$ and we choose $\rho>0$ such that the assumptions of Lemma 15 are satisfied. Finally, by using again Lemmas 10 and 11 we compute $V(T)$ and $V'(T)$ for some $T$ close to, but less than $R$.

### 4.5 Successive steps and proof of Proposition 4

We can now proceed by setting $V_0=V(T)$ and $V_1=V'(T)$ and by solving

$$\begin{cases} V''=(1-\gamma^{-2}e^{2(T+s)})V-V^3 \\ V(0)=V_0 \\ V'(0)=V_1 \end{cases}$$

with the method described above (up to small adjustments). It is straightforward to iterate the procedure as many times as necessary, in order to obtain a lower bound for the second zero of the solution.

Finally, we partition the interval $[5,349]$ into the union of small intervals. For each such interval we solve the equation (40) as described above, until we reach the second zero $z$ and we check the inequality $z>\gamma\sqrt{\mu_1}$, which proves Proposition 4. See the Ada files [4] for the details of the proof.

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