Strong density results for manifold valued fractional Sobolev maps

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Abstract. We consider fractional Sobolev classes $W^{s,p}$ of maps defined in high dimensional domains and with values into compact smooth manifolds. The problem of strong density of smooth maps for s lower than one is discussed. An equivalent energy convergence defined through extensions in suitable weighted Sobolev spaces is exploited to obtain a new proof of the density of maps with "small" singular set. Moreover, a homotopy-type property is analyzed, yielding to a characterization of approximable maps through topological arguments. We then focus on maps taking values into high dimensional spheres, where homological tools allow to describe the singular set. For suitable values of the product sp, in fact, strong density of smooth maps is characterized by the triviality of the current of the singularities.

Keywords: fractional Sobolev spaces; weighted Sobolev spaces; maps between manifolds; singularities.

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1 Introduction

In this paper we deal with strong density of smooth functions in fractional Sobolev classes of maps $u : B^n \to \mathcal{Y}$ defined in the unit ball B^n of $\mathbb{R}^n$ and taking values into a manifold $\mathcal{Y}$.

We shall always assume $0 < s < 1$ and $p > 1$ real. In this case, the space $W^{s,p}(B^n)$ is given by the L^p functions $u : B^n \to \mathbb{R}$ with finite fractional Gagliardo semi-norm $|u|_{s,p}$, where

$$|u|^p_{s,p} := \int_{B^n}\int_{B^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\,.$$

It is Banach space when equipped with the norm $\|u\|_{s,p} := \|u\|_{L^p} + |u|_{s,p}$. A vector valued function $u : B^n \to \mathbb{R}^N$ belongs to the space $W^{s,p}(B^n, \mathbb{R}^N)$ if each component of u is in $W^{s,p}(B^n)$.

Let $\mathcal{Y}$ be a smooth, connected, and compact Riemannian manifold without boundary, isometrically embedded into $\mathbb{R}^N$. We equip $\mathcal{Y}$ with the metric induced by the Euclidean norm on the ambient space $\mathbb{R}^N$, and deal with the class:

$$W^{s,p}(B^n, \mathcal{Y}) := \{u \in W^{s,p}(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for a.e.} \quad x \in B^n\}\,.$$

For $1 < p < \infty$ and $0 < s < 1$, we correspondingly define

$$W^{s,p}_S(B^n, \mathcal{Y}) := \{u \in W^{s,p}(B^n, \mathcal{Y}) \quad | \quad \begin{array}{l}\text{there exists } \{u_h\} \subset C^\infty(B^n, \mathcal{Y}) \\ \text{such that } u_h \to u \text{ strongly in } W^{s,p}\}\,.\end{array} \tag{1.1}$$

Moreover, throughout the paper we shall always denote

$$d := [sp] + 1$$

$[q]$ being the integer part of $q \in \mathbb{R}$.

Density results. By the continuous embedding in the class VMO, see [8], it follows that

$$W^{s,p}_S(B^n, \mathcal{Y}) = W^{s,p}(B^n, \mathcal{Y}) \qquad \text{if } n = 1 \quad \text{or} \quad sp \geq n\,. \tag{1.2}$$

For $n \geq d \geq 1$, instead, following an idea by Bethuel [2], it turns out that a strongly dense class is given by maps that are smooth outside a singular set of codimension d. More precisely, we respectively denote by $R^\infty_{s,p}(B^n,\mathcal{Y})$ and $R^0_{s,p}(B^n,\mathcal{Y})$ the set of all maps $u\in W^{s,p}(B^n,\mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u)=\bigcup_{i=1}^r \Sigma_i\,,\qquad r\in\mathbb{N} \tag{1.3}$$

where Σ_i is a smooth $(n-d)$-dimensional subset of B^n with smooth boundary, if $n\geq d+1$, and Σ_i is a point, if $n=d$. The following density result was proved in [7].

Theorem 1.1 *For every $n\geq 2$, $0<s<1$, and $1<p<\infty$ such that $sp<n$, the class $R^\infty_{s,p}(B^n,\mathcal{Y})$ is dense in $W^{s,p}(B^n,\mathcal{Y})$.*

In Secs. 2, 3, and 4, we shall give a different proof of Theorem 1.1, this way answering to the question posed in [7, Rmk. 1.5]. Our proof is the starting point for obtaining the new results of this paper, and it relies on arguments from [3, 10, 17] and on the following strategy, that goes back to [19].

Roughly speaking, when $0<s<1$, strong convergence in $W^{s,p}$ can be reformulated in terms of energy convergence of extensions in suitable weighted Sobolev spaces, as described below. It allows us to overcome the difficulty of working with the non local $W^{s,p}$ semi-norm, as we did in the case of trace spaces analyzed in [17]. For the sake of clarity, we have preferred to give a complete proof of our results, since they are rather technical and it seems that it is not easy to recover them from the particular case of maps in $W^{1-1/p,p}$ analyzed in [17] and [18].

THE ENERGY. For $\gamma\in\mathbb{R}$ and $p>1$, denote by $W^{1,p}_\gamma(B^n\times(0,+\infty))$ the weighted Sobolev space given by the functions $U\in L^p((B^n\times(0,+\infty))$ whose distributional derivative DU is a measurable function satisfying

$$\int_{\Omega\times(0,+\infty)} t^\gamma|DU(x,t)|^p\,dx\,dt<\infty\,,\quad \Omega=B^n\,. \tag{1.4}$$

By interpolation theory, see e.g. [15], it turns out that when $0<s<1$, the fractional Sobolev space $W^{s,p}(B^n)$ agrees with the Besov space $B^s_{p,p}(B^n)$, for any $p>1$, and hence with the class of traces $u(x)=U(x,0)$ on $t=0$ of functions U in $W^{1,p}_\gamma(B^n\times(0,+\infty))$, say $\mathbf{T}(U)=u$, where

$$\gamma=\gamma(s,p):=p(1-s)-1\,,\quad p>1\,,\quad 0<s<1\,. \tag{1.5}$$

Notice that in case $s=1-1/p$, then $d=[p]$, $\gamma(1-1/p,p)=0$, and $W^{1-1/p,p}(B^n)$ is the trace space of $W^{1,p}(B^n\times(0,+\infty))$.

A particular case of interest is when $sp\in\mathbb{N}^+$, so that $d=sp+1\geq 2$ and $\gamma=p-d$. In that case, condition $0<s<1$ yields $p>d-1$, the fractional Gagliardo semi-norm becomes

$$|u|^p_{(d-1)/p,p}=\int_{B^n}\int_{B^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+d-1}}\,dx\,dy$$

and in low dimension $n=d-1$, when $\Omega=\mathbb{R}^{d-1}$, the energy (1.4) is scale invariant for any $p>d-1$.

Denote now by $\mathbf{C}^{n+1}$ the $(n+1)$-dimensional cylinder

$$\mathbf{C}^{n+1}:=B^n\times(0,1)$$

by $W^{1,p}_\gamma(\mathbf{C}^{n+1},\mathbb{R}^N)$ the class of vector valued functions $U:\mathbf{C}^{n+1}\to\mathbb{R}^N$ with components in $W^{1,p}_\gamma(\mathbf{C}^{n+1})$, and consider for $0<s<1$ and $p>1$ the *energy*

$$\mathcal{E}^p_{\gamma(s,p)}(U):=\int_{\mathbf{C}^{n+1}} t^{\gamma(s,p)}|DU(x,t)|^p\,dx\,dt\,,\qquad \gamma(s,p):=p(1-s)-1\,. \tag{1.6}$$

Let $u\in W^{s,p}(B^n,\mathcal{Y})$. Since $\mathcal{Y}$ is compact, we have $u\in L^\infty(B^n,\mathbb{R}^N)$. We thus denote by

$$U:=\operatorname{Ext}(u) \tag{1.7}$$

a bounded function that *minimizes the energy $\mathcal{E}^p_{\gamma(s,p)}(U)$ among all $U\in W^{1,p}_{\gamma(s,p)}(\mathbf{C}^{n+1},\mathbb{R}^N)\cap L^\infty$ such that $U(x,0)=u(x)$ on $B^n\times\{0\}$ in the sense of traces*, i.e., $\mathbf{T}(U)=u$.

Such a minimizer exists and is smooth in the interior of $\mathbf{C}^{n+1}$, by convexity of the functional $U \mapsto \mathcal{E}^p_{\gamma(s,p)}(U)$. In addition, see [15], if $\{u_h\} \subset W^{s,p}(B^n,\mathcal{Y})$ is a sequence converging a.e. in B^n to a function $u \in W^{s,p}(B^n,\mathcal{Y})$, it turns out that *strong convergence $u_h \to u$ in $W^{s,p}(B^n,\mathbb{R}^N)$ is equivalent to convergence $u_h \to u$ in $L^p(B^n,\mathbb{R}^N)$ joined with the energy convergence*

$$\lim_{h\to\infty} \mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u_h)) = \mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u)) \,. \tag{1.8}$$

Therefore, coming back to definition (1.1), we conclude that if $u \in W^{s,p}(B^n,\mathcal{Y})$ for some $0 < s < 1$ and $p > 1$, then $u \in W^{s,p}_S(B^n,\mathcal{Y})$ if and only if we can find a sequence $\{u_h\} \subset C^\infty(B^n,\mathcal{Y})$ strongly converging to u in $L^p(B^n,\mathbb{R}^N)$ and such that the energy convergence (1.8) holds.

Topological obstruction. For $j \geq 1$ integer, denote by $\pi_j(\mathcal{Y})$ the j-th free homotopy group of the target manifold $\mathcal{Y}$. Let $d := [sp] + 1 \geq 2$ and assume that $\pi_{d-1}(\mathcal{Y}) \neq 0$. Then, it is well-known that if $n \geq d$, the strict inclusion

$$W^{s,p}_S(B^n,\mathcal{Y}) \subsetneq W^{s,p}(B^n,\mathcal{Y})$$

holds. More precisely, there exist maps $u \in W^{s,p}(B^n,\mathcal{Y})$, actually in $R^\infty_{s,p}(B^n,\mathcal{Y})$, which cannot be approximated strongly in $W^{s,p}$ by sequences of smooth maps in $W^{s,p}(B^n,\mathcal{Y})$.

Example 1.2 Let $n \geq d := [sp] + 1 \geq 2$, and denote $x = (\widetilde{x},\widehat{x}) \in \mathbb{R}^d \times \mathbb{R}^{n-d} \simeq \mathbb{R}^n$. Assume that $\pi_{d-1}(\mathcal{Y}) \neq 0$, and let $u : B^n \to \mathcal{Y}$ the 0-homogeneous map

$$u(x) := \varphi\Big(\frac{\widetilde{x}}{|\widetilde{x}|}\Big)\,, \qquad x = (\widetilde{x},\widehat{x}) \in B^n \setminus \Sigma(u)$$

where $\varphi : \mathbb{S}^{d-1} \to \mathcal{Y}$ is a homotopically non-trivial smooth map and

$$\Sigma(u) := \{(\widetilde{x},\widehat{x}) \in \mathbb{R}^n \,:\, \widetilde{x} = 0\} \,.$$

Then $u \in R^\infty_{s,p}(B^n,\mathcal{Y})$, but $u \notin W^{s,p}_S(B^n,\mathcal{Y})$, see e.g. Theorem 1.5 below.

The non-triviality of the $(d-1)$-th homotopy group of the target manifold is the only obstruction to strong density of smooth maps, at least in case of standard domains. In fact, the following theorem was proved in [7].

Theorem 1.3 *Let $n \geq d := [sp] + 1 \geq 2$. Then $W^{s,p}_S(B^n,\mathcal{Y}) = W^{s,p}(B^n,\mathcal{Y})$ if and only if $\pi_{d-1}(\mathcal{Y}) = 0$.*

The analogous of Theorem 1.3 in case $s \geq 1$ was proved in [5], whereas equality $W^{s,p}_S = W^{s,p}$ for $(d-1)$-connected target manifolds was obtained in [6]. A different proof of Theorem 1.3 when $n = d$ is given in Sec. 5, as it reduces to remove point singularities, on account of Theorem 1.1. In high dimension $n > d$, the non-trivial implication in Theorem 1.3 is more involved. It is readily obtained by our characterization of strongly approximable $R^0_{s,p}$-maps, Theorem 1.5, the proof of which makes use of a $(d-1)$-*homotopy type* property of maps in $R^0_{s,p}(B^n,\mathcal{Y})$, Theorem 1.4.

Homotopy type property. Let X^k denote the k-skeleton of some finite cubeulation X of B^n. If $u \in W^{s,p}(B^n,\mathcal{Y})$, possibly slightly moving the faces of X we may assume that the restriction of u to F belongs to $W^{s,p}(F,\mathcal{Y})$ for every k-face F of X^k, where $k = d-1,\dots,n$. In this case, we say that X is in *generic position* with respect to u. Moreover, if $u \in R^0_{s,p}(B^n,\mathcal{Y})$, and $\Sigma(u)$ is the $(n-d)$-dimensional singular set of u, compare (1.3), we say that X is in *dual position* with respect to u if it is in generic position and in addition $X^{d-1} \cap \Sigma(u) = \emptyset$. Possibly slightly moving the faces of X^{d-1}, it turns out that the cubeulation X is in dual position with respect to u.

Using arguments from [3], that go back to [21], in Sec. 6 we prove the following:

Theorem 1.4 *Let $n > d := [sp]+1 \geq 2$. Let $u_\infty \in R^0_{s,p}(B^n,\mathcal{Y})$ and X a finite cubeulation of B^n in dual position with respect to u_∞. Let $\{u_i\} \subset W^{s,p}(X^{d-1},\mathcal{Y}) \cap C^\infty$ be a smooth sequence strongly converging in $W^{s,p}$ to the restriction $u_{\infty|X^{d-1}}$ of u_∞ to X^{d-1}. Then, we find $k_0 \in \mathbb{N}^+$ such that for every $i \geq k_0$ the maps u_i and $u_{\infty|X^{d-1}}$ are homotopic as maps from X^{d-1} to $\mathcal{Y}$.*

A CHARACTERIZATION. Using Theorem 1.4, in Sec. 7 we provide a characterization of strongly approximable $R^0_{s,p}$-maps:

Theorem 1.5 *Let $n \geq d := [sp] + 1 \geq 1$. Let $u \in R^0_{s,p}(B^n, \mathcal{Y})$ and X a finite cubeulation of B^n in dual position with respect to u. Then, u belongs to $W^{s,p}_S(B^n, \mathcal{Y})$, i.e., u is the strong $W^{s,p}$ limit of a smooth sequence in $C^\infty(B^n, \mathcal{Y})$, if and only if the restriction $u_{|X^{d-1}}$ of u to X^{d-1} can be extended to a continuous map from B^n into $\mathcal{Y}$.*

On account of the density theorem 1.1, Theorem 1.5 gives the non-trivial implication of Theorem 1.3, in any dimension $n \geq d$. Moreover, when $d = 1$ one recovers the following well-known fact:

Corollary 1.6 *Let $0 < s < 1$ and $p > 1$ such that $0 < sp < 1$. Then $H^{s,p}_S(B^n, \mathcal{Y}) = W^{s,p}(B^n, \mathcal{Y})$ for any $n \geq 1$.*

MORE GENERAL DOMAINS. The case of non-trivial domains, e.g. of maps $u : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} = \partial\mathcal{M}$ for some smooth, connected, and compact Riemannian $(n+1)$-manifold $\mathcal{M}$, can be treated in a similar way. In particular, Theorems 1.1, 1.4, and 1.5 continue to hold, with the obvious modifications. The same feature holds concerning Theorem 1.3 in low dimension $n = d$, since its proof relies on a local argument, too. When $n > d$, one has to follow some ideas due to Hang-Lin [13].

We thus recall that $\mathcal{X}$ is said to satisfy the *k-extension property with respect to $\mathcal{Y}$*, where $k \in \mathbb{N}$, if for any given CW-complex X on $\mathcal{X}$, denoting by X^k its k-dimensional skeleton, any continuous map $f : X^{k+1} \to \mathcal{Y}$ is such that its restriction to X^k can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. Arguing exactly as e.g. in the proof of [17, Thm. 4], and using Theorem 1.5, one readily obtains the following result:

Theorem 1.7 *If $n > d := [sp] + 1 \geq 2$, smooth maps in $C^\infty(\mathcal{X}, \mathcal{Y})$ are sequentially dense in $W^{s,p}(\mathcal{X}, \mathcal{Y})$ if and only if $\pi_{d-1}(\mathcal{Y}) = 0$ and $\mathcal{X}$ satisfies the $(d-1)$-extension property with respect to $\mathcal{Y}$.*

If $d = 1$, then again $H^{s,p}_S(\mathcal{X}, \mathcal{Y}) = W^{s,p}(\mathcal{X}, \mathcal{Y})$ for any $n \geq 1$. Moreover, if n and d are as in Theorem 1.7 we deduce:

Corollary 1.8 *If $\pi_j(\mathcal{Y}) = 0$ for every integer $j = d-1, \dots, n-1$, then $H^{s,p}_S(\mathcal{X}, \mathcal{Y}) = W^{s,p}(\mathcal{X}, \mathcal{Y})$.*

Finally, using arguments from [21, Sec. 6], following [17, Cor. 2] we also obtain:

Corollary 1.9 *Let $k \in \{1, \dots, d-1\}$. If $\pi_i(\mathcal{X}) = 0$ for every $i = 0, \dots, k-1$ and $\pi_j(\mathcal{Y}) = 0$ for every $j = k, \dots, d-1$, then $H^{s,p}_S(\mathcal{X}, \mathcal{Y}) = W^{s,p}(\mathcal{X}, \mathcal{Y})$.*

For the sake of brevity, we omit the proof of the latter results, since it suffices to argue as in the case $s = 1 - 1/p$ from [17].

THE MODEL CASE. In Secs. 8 and 9, we shall restrict to the model case $\mathcal{Y} = \mathbb{S}^{N-1}$, where $N \geq 2$ and

$$\mathbb{S}^{N-1} := \{y \in \mathbb{R}^N \,:\, |y| = 1\}$$

is the unit $(N-1)$-sphere. Recalling the notation (1.1), since $\mathbb{S}^{N-1}$ is $(N-2)$-connected, using (1.2) and Theorem 1.3, it turns out that for any couple of exponents $0 < s < 1$ and $p > 1$ the strict inclusion

$$W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \subsetneq W^{s,p}(B^n, \mathbb{S}^{N-1})$$

holds provided that $N - 1 \leq sp < n$, a condition we shall assume from now on, whence $N \leq d \leq n$.

Notice in fact that a part from the case $N = 2$, when $N \geq 3$ the high order homotopy groups $\pi_j(\mathbb{S}^{N-1})$, for $j \geq N$, fail to be trivial, in general. Therefore, only in the case $N = 2$ one can restrict to the ranges $1 \leq sp < 2$, i.e., when $n \geq d = 2$. In particular, if $n > N \geq 3$, since $\pi_N(\mathbb{S}^{N-1}) \neq 0$, then B^n fails to satisfy the N-extension property with respect to $\mathbb{S}^{N-1}$.

Using some relevant estimates obtained by Bourgain-Brezis-Mironescu in [4], for each map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $n \geq N - 1 \geq 1$ and $p > N - 1$, we are able to find a smooth extension $V \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{R}^N)$ of u, so that $\mathbf{T}(V) = u$, satisfying:

$$\int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt \leq C \int_{\mathbf{C}^{n+1}} t^{p-N}\, |DU(x,t)|^p\, dx\, dt\,, \quad U = \mathrm{Ext}(u) \tag{1.9}$$

for some absolute constant $C > 0$, see Theorem 8.1.

As a consequence, when $n \geq N \geq 2$, $0 < s < 1$, $p > 1$, and $sp \geq N - 1$, for each map $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ we are able to construct an $(n - N)$-dimensional *current* $\mathbb{P}(u)$ in B^n that carries the information on the *homological singularities* of u. More precisely, following ideas from [12, 11, 19], for any compactly supported smooth $(n - N)$-form ϕ in B^n we define

$$\mathbb{P}(u)(\phi) := \frac{1}{\alpha_N} \int_{\mathbf{C}^{n+1}} d\,(\eta \wedge \phi) \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N)$$

where α_N is the measure of the unit ball $\mathbb{D}^N$ of dimension N, and $\eta : [0, 1] \to [0, 1]$ is a suitable cut-off function, with $\eta(0) = 1$. With this notation, in fact, in Proposition 8.5 we obtain

$$u \in W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,. \tag{1.10}$$

In a similar way, in dimension $n = N - 1$ a good notion of *degree* is analyzed, see (8.8).

For some ranges of sp, the converse implication holds in (1.10), whence the current $\mathbb{P}(u)$ retains all the information on the relevant singularities of u. More precisely, if $n > [sp] = N - 1$, when $N = 3$, or $n > [sp] \geq 1$, when $N = 2$, in Theorem 8.6 we shall prove the following characterization:

$$W^{s,p}_S(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}\,.$$

However, when $n > [sp] \geq N \geq 3$, *topological singularities* that cannot be seen by pure homological arguments come into play. In fact, differently to the case $N = 2$, where high order homotopy groups $\pi_j(\mathbb{S}^1)$, $j \geq 2$, are all trivial, when $N \geq 3$ we have $\pi_N(\mathbb{S}^{N-1}) \neq 0$, i.e., there exist homotopically non-trivial smooth maps $\varphi : \mathbb{S}^N \to \mathbb{S}^{N-1}$. When e.g. $N = 3$, we may choose $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$ equal to the Hopf fibration, a generator of the third homotopy group of the 2-sphere, $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$.

Example 1.10 Let $N \geq 3$ and $u : B^{N+1} \to \mathbb{S}^{N-1}$ the 0-homogeneous map

$$u(x) = \varphi\Big(\frac{x}{|x|}\Big)\,, \qquad x \in B^{N+1} \setminus \{0\} \tag{1.11}$$

where $\varphi : \mathbb{S}^N \to \mathbb{S}^{N-1}$ is a homotopically non-trivial smooth map.

Since $u \in W^{1,q}(B^{N+1}, \mathbb{S}^{N-1})$ for each $q < N+1$, and $u \in L^\infty$, by the Gagliardo-Nirenberg inequality $u \in W^{s,p}$ for each $0 < s < 1$ and $p > 1$ such that $N \leq sp < N+1$, whence we choose $d = [sp]+1 = N+1$. Moreover, $u \in R^\infty_{s,p}(B^{N+1}, \mathbb{S}^{N-1})$, as u is smooth outside the origin. Now, if X is a cubeulation of B^{N+1} such that $0 \notin X^N$, the restriction $u_{|X^N}$ cannot be extended to a continuous map from B^{N+1} into $\mathbb{S}^{N-1}$, by the non-triviality of the map φ. Therefore, Theorem 1.5, in case $d = N + 1$, yields that $u \notin W^{s,p}_S(B^{N+1}, \mathbb{S}^{N-1})$ if $N \leq sp < N+1$, i.e., one cannot find a smooth sequence $\{u_h\} \subset C^\infty(B^{N+1}, \mathbb{S}^{N-1})$ strongly converging to u in $W^{s,p}$. However, it can be checked that $\mathbb{P}(u) = 0$, see Example 9.6.

When $sp = N = 3$, and in the limiting case $s = 1$, topological connections of the singularities of maps in $W^{1,3}(B^4, \mathbb{S}^2)$ were firstly analyzed in [14] through new geometric tools called "bubbled scans".

In Sec. 9, we finally point out some other consequences of our estimate (1.9), outlining for $N \geq 3$ a striking difference from the case $N = 2$ already analyzed in [19]. Namely, if $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ for some $p > N-1$, in general $\mathrm{Ext}(u) \in W^{1,N-1}(\mathbf{C}^{n+1}, \mathbb{R}^N)$ provided that $N-1 < p < (N-1)^2/(N-2)$, see Proposition 9.2.

Therefore, for these reasons, and for other ones that we will not dwell on, the analysis of $W^{s,p}$ weak sequential density of smooth maps $C^\infty(B^n, \mathbb{S}^{N-1})$, when $N \geq 3$, appears to be much more difficult from the case $N = 2$ we tackled in [19], at least for large values of the exponent p or of the product sp.

2 The density theorem: part I

In this section, we begin the proof of Theorem 1.1. We first fix some notation. Since B^n is bilipschitz homeomorphic to the unit open n-cube

$$\mathcal{Q}^n :=]0, 1[^n$$

we reduce to prove Theorem 1.1 in case of maps defined in $\mathcal{Q}^n$. We thus denote

$$z = (x,t) = (x_1,\dots,x_n,t) \in \mathbb{R}^n \times \mathbb{R}$$

a point in the cylinder $\mathcal{Q}^n \times I$, where $I :=]-1,1[$, and we shall work with maps U in $W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$, where $\gamma = \gamma(s,p) = p(1-s)-1$. This means that $U \in L^p(\mathcal{Q}^n \times I, \mathbb{R}^N)$, the distributional derivative DU is a measurable function in $\mathcal{Q}^n \times I$, and U has finite energy:

$$\mathcal{E}^p_\gamma(U) := \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) = \int_{\mathcal{Q}^n \times I} |t|^\gamma |DU(x,t)|^p \, dx\, dt < \infty \,.$$

If A is a "smooth" $\mathcal{H}^k$-measurable k-dimensional subset of $\mathcal{Q}^n \times I$, where $\mathcal{H}^k$ is the Hausdorff measure, we also denote by

$$\mathcal{E}^p_\gamma(U, A) := \int_A |t|^\gamma |DU_{|A}|^p \, d\mathcal{H}^k$$

the energy integral of the restriction $U_{|A}$ of U to A. As before, we write $\mathbf{T}(U) = u$ if $u \in W^{s,p}(\mathcal{Q}^n, \mathbb{R}^N)$ is the trace of U on $\mathcal{Q}^n \times \{0\}$. If $v = (v_1,\dots,v_k) \in \mathbb{R}^k$, we also set

$$\|v\|_k := \max_{1\le i\le k} |v_i| \,.$$

Also, for $i = 1,\dots,n+1$ and $\lambda \in \mathbb{R}$, we denote by $P(\lambda, i)$ the restriction to $\mathcal{Q}^n \times I$ of the hyperplane of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ containing the point $\lambda\, e_i$ and orthogonal to e_i, where $\{e_1,\dots,e_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+1}$, i.e.,

$$P(\lambda, i) := \{z \in \mathcal{Q}^n \times I \mid (z - \lambda\, e_i \mid e_i)_{\mathbb{R}^{n+1}} = 0\} \,.$$

For $m \in \mathbb{N}^+$ and $a = (a_1,\dots,a_n) \in [1/(4m), 3/(4m)]^n$ we denote by $\mathcal{L}_m$ the grid of $\mathbb{R}^n \times \mathbb{R}$

$$\mathcal{L}_m := \bigcup_{i=1}^{n} \bigcup_{j=0}^{m-1} P(a_i + j/m, i) \tag{2.1}$$

and by $C^{(k)}_m$ the k-skeleton of the grid of $\mathcal{Q}^n$ given by the intersection of $\mathcal{L}_m$ with the n-space $\mathbb{R}^n \times \{0\}$. Moreover, we denote

$$\begin{array}{rcl} \mathcal{Q}^n_m & := & a + [0, (m-1)/m]^n \\ \Sigma^{(k)}_m & := & C^{(k)}_m \cap \mathcal{Q}^n_m \,, \qquad k = 0,\dots,n \end{array} \tag{2.2}$$

the closed n-cube of side $(m-1)/m$ inside $\mathcal{Q}^n$ and the part of the k-skeleton $C^{(k)}_m$ contained in $\mathcal{Q}^n_m$.

Remark 2.1 For future use, we denote by

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \operatorname{dist}(y, \mathcal{Y}) \le \varepsilon\}$$

the closure of the ε-neighborhood of $\mathcal{Y}$ in $\mathbb{R}^N$. Since $\mathcal{Y}$ is smooth and compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $\operatorname{Lip}(\Pi_\varepsilon) \le (1 + c\,\varepsilon) \to 1^+$ as $\varepsilon \to 0^+$. Moreover, the set $\mathcal{Y}_\varepsilon$ is equivalent to $\mathcal{Y}$ in the sense of algebraic topology, when $0 < \varepsilon \le \varepsilon_0$.

Let $u \in W^{s,p}(\mathcal{Q}^n, \mathcal{Y})$ and $U : \mathcal{Q}^n \times I \to \mathbb{R}^N$ be the extension $\operatorname{Ext}(u)$ of u, so that $\mathbf{T}(U) = u$ and $U \in W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$, with $\gamma = p(1-s)-1$. Notice that U is continuous outside $\mathcal{Q}^n \times \{0\}$. Moreover, we denote

$$U^{(m)} := U_{|C^{(d-1)}_m \times I} \tag{2.3}$$

the restriction of U to the d-skeleton $C^{(d-1)}_m \times I$, where $d = [sp] + 1$.

In order to prove Theorem 1.1, we first make use of the argument of [3, Sec. 2.1], that goes back to [20], and show that if the restriction $U^{(m)}$ belongs to $W^{1,p}_\gamma(C^{(d-1)}_m \times I, \mathbb{R}^N)$, then it can be approximated by continuous maps $U^{(m)}_h$ such that their traces take values in the neighborhood $\mathcal{Y}_{\varepsilon_0}$ of $\mathcal{Y}$.

Proposition 2.2 *Let n, s, and p as in Theorem 1.1. Assume that $U^{(m)} \in W^{1,p}_\gamma(C^{(d-1)}_m \times I, \mathbb{R}^N)$, where $\gamma = p(1-s)-1$ and $d = [sp]+1$. Then, there exists a sequence of continuous maps $\{U^{(m)}_h\}_h$ in $W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I, \mathbb{R}^N)$ such that $U^{(m)}_h \to U^{(m)}$ strongly in $W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I, \mathbb{R}^N)$ and the traces $\mathbf{T}(U^{(m)}_h) \in W^{s,p}(\Sigma^{(d-1)}_m, \mathcal{Y}_{\varepsilon_0})$ for every h.*

If $0 < sp < 1$, since $d = 1$, Proposition 2.2 holds true by taking $U^{(m)}_h = U^{(m)}$, see (2.3). Therefore, we assume $d \in \{2, \dots, n\}$ and follow the proof of [17, Prop. 2], where we had $s = 1 - 1/p$, whence $\gamma = 0$ and $d = [sp]+1 = [p]$. The main difference relies on the embedding of $W^{1,p}_\gamma$ into a suitable Hölder class, in low dimension $d-1$, see (2.8).

More precisely, if $u \in W^{s,p}(B^n)$, then $U := \mathrm{Ext}(u) \in W^{s+1/p,p}(\mathbf{C}^{n+1})$, and for "almost all" $(d-1)$-dimensional disks $D = D^{d-1}$ contained in $\mathbf{C}^{n+1}$, the restriction $U_{|D}$ of U to D belongs to $W^{s+1/p,p}(D)$. Recalling that $d = [sp]+1$, condition $0 < s < 1$ yields $p > d-1$ and hence $U_{|D} \in C^{0,\alpha}(D)$ with Hölder exponent

$$\alpha = s + \frac{1}{p} - \frac{d-1}{p} = \frac{2 + sp - d}{p} = \frac{1 + sp - [sp]}{p}\,. \tag{2.4}$$

Proof of Proposition 2.2: If $z = (x,t) \in \Sigma^{(d-1)}_m \times I$ and $0 < h < 1/(4m)$, we denote by

$$C(z,h) := \overline{B}^n(x, h/2) \times [t - h/2, t + h/2]$$

the cylinder centered at z over the ball of diameter h and height h, and by

$$\Sigma(z,h) := C(z,h) \cap (C^{(d-1)}_m \times I)$$

its intersection with the d-skeleton $C^{(d-1)}_m \times I$. Setting for $z \in \Sigma^{(d-1)}_m \times I$

$$U^{(m)}_h(z) := \fint_{\Sigma(z,h)} U^{(m)}(w)\, d\mathcal{H}^d(w) = \frac{1}{\mathcal{H}^d\big(\Sigma(z,h)\big)} \int_{\Sigma(z,h)} U^{(m)}(w)\, d\mathcal{H}^d(w)$$

then $U^{(m)}_h \in W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I, \mathbb{R}^N)$ is continuous, and $U^{(m)}_h \to U^{(m)}$ strongly in $W^{1,p}_\gamma$ as $h \to 0^+$.

It remains to show that if $u^{(m)}_h := \mathbf{T}(U^{(m)}_h)$, possibly passing to a subsequence $u^{(m)}_h(\Sigma^{(d-1)}_m) \subset \mathcal{Y}_{\varepsilon_0}$ for every h. To this aim, for $\varepsilon > 0$ to be determined later, choose $h_\varepsilon > 0$ small so that for any $0 < h \le h_\varepsilon$

$$\int_{\Sigma(z,h)} |t|^\gamma |DU^{(m)}(w)|^p\, d\mathcal{H}^d(w) \le \varepsilon \qquad \forall\, z \in \Sigma^{(d-1)}_m \times I \tag{2.5}$$

where $w = (x,t)$ as above.

For a fixed $P_0 \in \Sigma^{(d-1)}_m \times \{0\}$, we observe that the connected set $\Sigma(P_0,h)$ always contains a d-cube R_1 of side h. More precisely, assume for example $P_0 = (x^1_0, \dots, x^n_0, 0)$, where $x^l_0 \in a_l + [0, (m-1)/m]$, if $l \in \{1, \dots, d-1\}$, and $x^i_0 = a_i + j_i/m$, if $i \in \{d, \dots, n\}$. Then we have

$$\Sigma(P_0,h) = R_1 \cup \bigcup_{i=2}^K R_i\,, \qquad K = \begin{pmatrix} n \\ d-1 \end{pmatrix}$$

where R_1 is the d-cube

$$R_1 := \left(\prod_{l=1}^{d-1} [x^l_0 - h/2, x^l_0 + h/2]\right) \times \{(x^d_0, \dots, x^n_0)\} \times [-h/2, h/2]$$

and $R_i := \widetilde{R}_i \times [-h/2, h/2]$ for $i = 2, \dots, K$, where $\widetilde{R}_i$ is a possibly degenerate $(d-1)$-parallelepiped of diameter lower than $\sqrt{d-1}\,h$, and edges parallel to the coordinate axes. In particular, we have

$$h^d \le \mathcal{H}^d(\Sigma(P_0,h)) \le c\, h^d \tag{2.6}$$

for some dimensional constant $c > 0$, not depending on P_0.

Slicing the d-cube R_1 with hyperplanes orthogonal to the direction e_1, for every $h \le h_\varepsilon$ we find $h_1 \in [x_0^1 - h/2, x_0^1 + h/2]$ such that by (2.5)

$$\begin{aligned} \mathcal{E}^p_\gamma(U^{(m)}, R_1 \cap P(h_1, 1)) &\le \frac{2}{h} \mathcal{E}^p_\gamma(U^{(m)}, R_1) \\ &\le \frac{2}{h} \int_{\Sigma(P_0,h)} |t|^\gamma |DU^{(m)}(w)|^p \, d\mathcal{H}^d(w) \le 2\, \frac{\varepsilon}{h} \,. \end{aligned} \tag{2.7}$$

We now choose $z_0 \in R_1 \cap P(h_1, 1) \cap (\Sigma_m^{(d-1)} \times \{0\})$ and set $y_h^{(m)} := U^{(m)}(z_0)$, so that $y_h^{(m)} \in \mathcal{Y}$. Due to the embedding of $W^{1,p}_\gamma$ into $C^{0,\alpha}$, where $\gamma = p(1-s) - 1$ and α is given by (2.4), since $R_1 \cap P(h_1, 1)$ is a $(d-1)$-cube of side h, it follows that

$$\max_{z \in R_1 \cap P(h_1,1)} |U^{(m)}(z) - y_h^{(m)}|^p \le c \cdot h^{1+sp-[sp]} \cdot \mathcal{E}^p_\gamma(U^{(m)}, R_1 \cap P(h_1, 1))$$

for some positive real constant $c = c(n, s, p)$, and hence by (2.7)

$$\max_{z \in R_1 \cap P(h_1,1)} |U^{(m)}(z) - y_h^{(m)}| \le c\, \varepsilon^{1/p} \,. \tag{2.8}$$

Moreover, we notice that

$$|U_h^{(m)}(P_0) - y_h^{(m)}| \le \fint_{\Sigma(P_0,h)} |U^{(m)}(w) - y_h^{(m)}| \, d\mathcal{H}^d(w) \,. \tag{2.9}$$

Let η be a positive number to be determined later. We slice the d-dimensional set $\Sigma(P_0, h)$ with hyperplanes orthogonal to the "vertical" direction e_{n+1}, and denote

$$\Omega_{h'} := \Sigma(P_0, h) \cap P(h', n+1) \,, \qquad h' \in [-h/2, h/2] \,.$$

Setting

$$A_h := \{h' \in [-h/2, h/2] \,:\, \mathcal{E}^p_\gamma(U^{(m)}, \Omega_{h'}) \le \varepsilon\, \eta/h\}$$

and $B_h := [-h/2, h/2] \setminus A_h$, by (2.5) we have $\mathcal{L}^1(B_h) \le h/\eta$. Moreover, for every h' the set $\Omega_{h'}$ is given by the connected union of $K = \binom{n}{d-1}$ parallelepipeds of dimension not greater than $d-1$ and diameter lower than $\sqrt{d-1}\, h$. Since $h^{(sp-[sp])/p} \le 1$, using again the embedding theorem, this time we obtain that for every $h' \in A_h$

$$\max_{z,y \in \Omega_{h'}} |U^{(m)}(z) - U^{(m)}(y)| \le c\, \eta^{1/p} \, \varepsilon^{1/p} \,, \quad c = c(n, s, p) \,.$$

Notice that $\Omega_{h'}$ intersects $R_1 \cap P(h_1, 1)$ for every h'. Therefore, combining with (2.8) we obtain

$$\max_{w \in \Omega_{h'}} |U^{(m)}(w) - y_h^{(m)}| \le c\, (\eta^{1/p} + 1)\, \varepsilon^{1/p} \qquad \forall\, h' \in A_h \,. \tag{2.10}$$

By Fubini's theorem we write

$$\begin{aligned} \int_{\Sigma(P_0,h)} |U^{(m)}(w) - y_h^{(m)}| \, d\mathcal{H}^d(w) &= \int_{B_h} \int_{\Omega_{h'}} |U^{(m)}(w) - y_h^{(m)}| \, d\mathcal{H}^{d-1} \, dh' \\ &+ \int_{A_h} \int_{\Omega_{h'}} |U^{(m)}(w) - y_h^{(m)}| \, d\mathcal{H}^{d-1} \, dh' \,. \end{aligned}$$

Since $\|U^{(m)}\|_\infty \le K_\infty < \infty$ by the compactness of $\mathcal{Y}$, whereas $\mathcal{L}^1(B_h) \le h/\eta$ and (2.6) holds, using (2.9) and (2.10) we get

$$|U_h^{(m)}(P_0) - y_h^{(m)}| \le c_1\, \frac{K_\infty}{\eta} + c_2\, (\eta^{1/p} + 1)\, \varepsilon^{1/p} \,. \tag{2.11}$$

Finally, taking first η large so that $c_1\, K_\infty/\eta < \varepsilon_0/2$, and then ε small so that $c_2\, (\eta^{1/p} + 1)\, \varepsilon^{1/p} < \varepsilon_0/2$, by the arbitrariness of P_0 in $\Sigma_m^{(d-1)} \times \{0\}$ we conclude that

$$\operatorname{dist}\big(u_h^{(m)}(x), \mathcal{Y}\big) < \varepsilon_0 \qquad \forall\, x \in \Sigma_m^{(d-1)}$$

for every $h \le h_\varepsilon$, which clearly yields the assertion. □

3 The density theorem: part II

In this section, we suitably modify the extension U in such a way that it agrees with $U_h^{(m)}$ on the d-skeleton $C_m^{(d-1)}\times I$. Recall that $\gamma=p(1-s)-1$ and $d=[sp]+1$.

Proposition 3.1 *Let n, s, and p as in Theorem 1.1. Assume that $U^{(m)}\in W^{1,p}_\gamma(C_m^{(d-1)}\times I,\mathbb{R}^N)$, see (2.3). Then there exists a sequence of maps $\{V_h^{(m)}\}_h$ in $W^{1,p}_\gamma(\mathcal{Q}^n_m\times I,\mathbb{R}^N)$, continuous out of $\mathcal{Q}^n_m\times\{0\}$, such that $V_h^{(m)}\to U_{|\mathcal{Q}^n_m\times I}$ strongly in $W^{1,p}_\gamma(\mathcal{Q}^n_m\times I,\mathbb{R}^N)$, with $V_h^{(m)}{}_{|\Sigma_m^{(d-1)}\times I}=U_h^{(m)}$, see Proposition 2.2. In particular we have*

$$\mathbf{T}(V_h^{(m)})_{|\Sigma_m^{(d-1)}}\in W^{s,p}(\Sigma_m^{(d-1)},\mathcal{Y}_{\varepsilon_0})\qquad\forall\,h\,.$$

PROOF: We follow the proof of [17, Prop. 3], and we first consider the case $n=d$, whence $[sp]\geq 1$.

The case $n=d$. Let $\mathcal{C}_m$ denote the family of all n-cubes Q of side $1/m$ with boundary contained in the $(n-1)$-grid $\Sigma_m^{(n-1)}$, i.e. $\partial Q\subset\Sigma_m^{(n-1)}$, so that

$$\cup\,\mathcal{C}_m=\mathcal{Q}^n_m\,.$$

Let $0<\varepsilon<1/2$ to be fixed later. On account of Proposition 2.2, if $Q\in\mathcal{C}_m$, we define $V_h^{(Q)}:Q\times I\to\mathbb{R}^N$ by setting for every $(x,t)\in Q\times I$

$$V_h^{(Q)}:=\begin{cases} U\Bigl(q+\dfrac{x-q}{1-\varepsilon},t\Bigr) & \text{if }\ \rho\leq\dfrac{1-\varepsilon}{2m}\\ S(\rho)\,U_h^{(m)}(y,t)+(1-S(\rho))\,U(y,t) & \text{if }\ \dfrac{1-\varepsilon}{2m}\leq\rho\leq\dfrac{1}{2m}\end{cases}\tag{3.1}$$

where $U^{(m)}$ is given by (2.3) and $U_h^{(m)}$ by Proposition 2.2. In the latter formula, $\rho=\rho(x):=\|x-q\|_n$, where q is the center of Q, so that $\rho(x)=1/(2m)$ if $x\in\partial Q$; moreover

$$y=y(x):=q+\frac{1}{2m}\,\frac{x-q}{\rho(x)}$$

and finally

$$S(\rho):=\frac{2m}{\varepsilon}\,\rho+\frac{\varepsilon-1}{\varepsilon}\tag{3.2}$$

so that $S\bigl(1/(2m)\bigr)=1$ and $S\bigl((1-\varepsilon)/(2m)\bigr)=0$. Trivially $V_h^{(Q)}$ is a function in $W^{1,p}_\gamma(Q\times I,\mathbb{R}^N)$, continuous out of $Q\times\{0\}$. Moreover, we estimate

$$\int_{\{\rho(x)\leq(1-\varepsilon)/(2m)\}\times I}|t|^\gamma|DV_h^{(Q)}|^p\,dx\,dt\leq(1-\varepsilon)^{n-p}\,\mathcal{E}^p_\gamma(U,Q\times I)$$

and

$$\begin{aligned}\int_{\{(1-\varepsilon)/(2m)\leq\rho(x)\leq 1/(2m)\}\times I}|t|^\gamma|DV_h^{(Q)}|^p\,dx\,dt&\leq c\,(m,p,s)\,\frac{1}{\varepsilon^{p-1}}\int_{\partial Q\times I}|t|^\gamma|U-U_h^{(m)}|^p\,d\mathcal{H}^n\\&\quad+c\,(m,p,s)\,\varepsilon\int_{\partial Q\times I}|t|^\gamma\bigl(|D_\tau U|^p+|D_\tau U_h^{(m)}|^p\bigr)\,d\mathcal{H}^n\end{aligned}$$

where τ is an orthonormal frame to $\Sigma_m^{(n-1)}\times I$ and $c\,(m,p,s)>0$ only depends on m, p, and s.

Define now $V_h^{(m)}:\mathcal{Q}^n_m\times I\to\mathbb{R}^N$ by $V_h^{(m)}(x,t):=V_h^{(Q)}(x,t)$ if $x\in Q$ for some $Q\in\mathcal{C}_m$. Then $\{V_h^{(m)}\}_h$ is a sequence in $W^{1,p}_\gamma(\mathcal{Q}^n_m\times I,\mathbb{R}^N)$, continuous out of $\mathcal{Q}^n_m\times\{0\}$, such that

$$\begin{aligned}\mathcal{E}^p_\gamma(V_h^{(m)},\mathcal{Q}^n_m\times I)\ &\leq\ (1-\varepsilon)^{n-p}\,\mathcal{E}^p_\gamma(U,\mathcal{Q}^n_m\times I)\\&\quad+\ c_1\,(m,p,s)\,\frac{1}{\varepsilon^{p-1}}\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|U^{(m)}-U_h^{(m)}|^p\,d\mathcal{H}^n\\&\quad+\ c_2\,(m,p,s)\,\varepsilon\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma\bigl(|D_\tau U^{(m)}|^p+|D_\tau U_h^{(m)}|^p\bigr)\,d\mathcal{H}^n\,.\end{aligned}$$

Moreover, since $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}_\gamma(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, see Proposition 2.2, there exists $\overline{h} \in \mathbb{N}$ such that for every $h \geq \overline{h}$

$$\int_{\Sigma_m^{(n-1)}\times I} |t|^\gamma |D_\tau U_h^{(m)}|^p \, d\mathcal{H}^n \leq 2 \int_{\Sigma_m^{(n-1)}\times I} |t|^\gamma |D_\tau U^{(m)}|^p \, d\mathcal{H}^n \,.$$

Now, for every $j \in \mathbb{N}^+$ we first choose $\varepsilon = \varepsilon_j \in (0, 1/2)$ small so that $\varepsilon_j \searrow 0$,

$$(1-\varepsilon_j)^{n-p}\, \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I) \leq \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I) + \frac{1}{j}$$

and

$$3\, c_2\,(m,p,s)\, \varepsilon_j \int_{\Sigma_m^{(n-1)}\times I} |t|^\gamma |D_\tau U^{(m)}|^p \, d\mathcal{H}^n \leq \frac{1}{j} \,.$$

Secondly, by the strong convergence of $U_h^{(m)}$ to $U^{(m)}$ in $W^{1,p}_\gamma(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, Hardy's inequality (see e.g. [15]) yields that

$$\lim_{h\to\infty} \int_{\Sigma_m^{(n-1)}\times I} |t|^\gamma |U^{(m)} - U_h^{(m)}|^p \, d\mathcal{H}^n = 0$$

and hence we can take $h = h_j \geq \overline{h}$ large so that $h_{j+1} > h_j$ and

$$c_1\,(m,p,s)\, \frac{1}{\varepsilon_j^{p-1}} \int_{\Sigma_m^{(n-1)}\times I} |t|^\gamma |U^{(m)} - U_{h_j}^{(m)}|^p \, d\mathcal{H}^n \leq \frac{1}{j} \qquad \forall\, j \,.$$

Finally, since by the previous estimates

$$\mathcal{E}^p_\gamma(V_{h_j}^{(m)}, \mathcal{Q}^n_m \times I) \leq \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I) + \frac{3}{j}$$

we relabel $\{V_j^{(m)}\}$ the subsequence $\{V_{h_j}^{(m)}\}$, where $\varepsilon = \varepsilon_j$ in (3.1). Using again the strong convergence of $U_h^{(m)}$ to $U^{(m)}$ in $W^{1,p}_\gamma(\Sigma_m^{(n-1)} \times I, \mathbb{R}^N)$, we obtain the strong L^p-convergence of $V_j^{(m)}$ to U and hence the assertion, by uniform convexity.

The case $n \geq d+1$. We first set $V_h^{(m)} = U_h^{(m)}$ on $\Sigma_m^{(d-1)} \times I$, according to Proposition 2.2. Arguing by induction on the dimension $k = d, \dots, n$, by the inductive hypothesis we have already defined $V_h^{(m)} : \Sigma_m^{(k-1)} \times I \to \mathbb{R}^N$ in such a way that $V_h^{(m)}$ converges to $U_{|\Sigma_m^{(k-1)}\times I}$ strongly in $W^{1,p}_\gamma(\Sigma_m^{(k-1)} \times I, \mathbb{R}^N)$.

We now extend $\{V_h^{(m)}\}$ to $\Sigma_m^{(k)} \times I$ as follows. Let F be a k-face of side $1/m$ of $\Sigma_m^{(k)}$, and hence with boundary contained in $\Sigma_m^{(k-1)}$. Without loss of generality, assume F oriented by $e_1 \wedge \cdots \wedge e_k$, and let

$$x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \,.$$

Similarly to (3.1), we define $V_h^{(F)} : F \times I \to \mathbb{R}^N$ by setting for $(x,t) \in F \times I$

$$V_h^{(F)} := \begin{cases} U\Big(\widetilde{q} + \dfrac{\widetilde{x} - \widetilde{q}}{1-\varepsilon}, \widehat{q}, t\Big) & \text{if } \ \rho \leq \dfrac{1-\varepsilon}{2m} \\ S(\rho)\, V_h^{(m)}\big(y, \widehat{q}, t\big) + \big(1 - S(\rho)\big)\, U\big(y, \widehat{q}, t\big) & \text{if } \ \dfrac{1-\varepsilon}{2m} \leq \rho \leq \dfrac{1}{2m} \,. \end{cases}$$

Here $\rho = \rho(\widetilde{x}) := \|\widetilde{x} - \widetilde{q}\|_k$, where $(\widetilde{q}, \widehat{q}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is the center of F; moreover

$$y = y(\widetilde{x}) := \widetilde{q} + \frac{1}{2m}\, \frac{\widetilde{x} - \widetilde{q}}{\rho(\widetilde{x})}$$

and $S(\rho)$ is given by (3.2).

We then extend $V_h^{(m)} : \Sigma_m^{(k)} \times I \to \mathbb{R}^N$ by setting $V_h^{(m)}(x,t) := V_h^{(F)}(x,t)$ if $x \in F$ for some k-face F as above. Similarly to the case $n = d$, using that $V_h^{(m)} \to U_{|\Sigma_m^{(k-1)}\times I}$ strongly in $W^{1,p}_\gamma(\Sigma_m^{(k-1)} \times I, \mathbb{R}^N)$, and by suitably choosing $\varepsilon = \varepsilon_j \searrow 0$, we infer that $\{V_h^{(m)}\}_h$ is a sequence in $W^{1,p}_\gamma(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$, continuous out of $\Sigma_m^{(k)} \times \{0\}$, such that, possibly passing to a subsequence, $V_{h_j}^{(m)} \to U_{|\Sigma_m^{(k)}\times I}$ strongly in $W^{1,p}_\gamma(\Sigma_m^{(k)} \times I, \mathbb{R}^N)$. The proof of Proposition 3.1 is complete. $\square$

4 The density theorem: part III

In this section, we conclude the proof of Theorem 1.1, that we state again:

Theorem 4.1 *For every $n \geq 2$, $0 < s < 1$, and $1 < p < \infty$ such that $sp < n$, the class $R^\infty_{s,p}(B^n, \mathcal{Y})$ is dense in $W^{s,p}(B^n, \mathcal{Y})$.*

Proof: Let $u \in W^{s,p}(\mathcal{Q}^n, \mathcal{Y})$ and $U : \mathcal{Q}^n \times I \to \mathbb{R}^N$ be the extension $\mathrm{Ext}(u)$ of u, so that $U \in W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$ and $\mathbf{T}(U) = u$. We proceed along the lines of [3, Lemma 5], and we first consider the case $n = d \geq 2$.

The case $n = d$. Let $m \in \mathbb{N}^+$. Since for $i = 1, \ldots, n$ we have

$$\begin{aligned} \int_{1/(4m)}^{3/(4m)} \sum_{j=0}^{m-1} \mathcal{E}^p_\gamma(U, P(\lambda + j/m, i))\, d\lambda &\leq \sum_{j=0}^{m-1} \mathcal{E}^p_\gamma(U, \{j/m \leq x_i \leq (j+1)/m\}) \\ &= \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) \end{aligned}$$

we find a vector $a = a(m) \in [1/(4m), 3/(4m)]^n$ such that

$$U_{|P(a_i + j/m, i)} \in W^{1,p}_\gamma(P(a_i + j/m, i), \mathbb{R}^N)$$

for every $i = 1, \ldots, n$ and $j = 0, \ldots, m-1$, and

$$\mathcal{E}^p_\gamma(U, C^{(n-1)}_m \times I) \leq c\, m\, \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I)\,. \tag{4.1}$$

We now apply Propositions 2.2 and 3.1 with $a = a(m)$. Slicing the cylinder $\mathcal{Q}^n_m \times I$ with hyperplanes $P(\lambda, n+1)$ orthogonal to the "vertical" direction e_{n+1}, since $\{V^{(m)}_h\}$ converges to $U_{|\mathcal{Q}^n_m \times I}$ strongly in $W^{1,p}_\gamma(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$, see Proposition 3.1, we may and do choose $a_{n+1} \in [1/(4m), 3/(4m)]$ so that

$$V^{(m)}_{h|P(a_{n+1} - j/m, n+1)} \in W^{1,p}_\gamma(P(a_{n+1} - j/m, n+1), \mathbb{R}^N)$$

for every h and for $j = 0, 1$, with

$$\sum_{j=0,1} \mathcal{E}^p_\gamma(V^{(m)}_h, P(a_{n+1} - j/m, n+1)) \leq c\, m\, \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I) \tag{4.2}$$

for every h. Let $\mathcal{F}_m$ denote the family of $(n+1)$-cubes of $\mathcal{Q}^n_m \times I$, of side $1/m$, whose boundary lies in the n-skeleton

$$\mathcal{L}_m \cup \bigcup_{j=0,1} P(a_{n+1} - j/m, n+1)\,,$$

compare (2.1), and let $\{C_l\}_{l=1}^{(m-1)^n}$ be a list of the $(n+1)$-cubes in $\mathcal{F}_m$. Notice that each C_l intersects the n-cube $\mathcal{Q}^n \times \{0\}$.

Recall that $V^{(m)}_{h\;|\Sigma^{(n-1)}_m \times I} = U^{(m)}_h$, where $U^{(m)}_h \to U^{(m)}$ strongly in $W^{1,p}_\gamma(\Sigma^{(n-1)}_m \times I, \mathbb{R}^N)$, see Propositions 2.2 and 3.1. Then, as in [3, Lemma 5], by refining the slicing arguments in (4.1) and (4.2) we in fact may and do choose $(a_1, \ldots, a_{n+1}) \in [1/(4m), 3/(4m)]^{n+1}$ in such a way that

$$\sum_{l=1}^{(m-1)^n} \mathcal{E}^p_\gamma(V^{(m)}_h, \partial C_l) \leq c\, m\, \mathcal{E}^p_\gamma(U, G_m) \qquad \forall\, h \geq \overline{h} \tag{4.3}$$

where

$$G_m := \mathcal{Q}^n \times] - 10m^{-1}, 10m^{-1}[\,.$$

For every l, choose a bilipschitz homeomorphism f_l between C_l and $[-1/(2m), 1/(2m)]^{n+1}$ such that

$$\begin{aligned} f_l(C_l \cap (\mathcal{Q}^n \times \{0\})) &= [-1/(2m), 1/(2m)]^n \times \{0\} \\ f_l(\partial C_l \cap (\mathcal{Q}^n \times \{0\})) &= \partial[-1/(2m), 1/(2m)]^n \times \{0\} \end{aligned}$$

and $\|Df_l\|_\infty \le K$, $\|Df_l^{-1}\|_\infty \le K$. Then, define $W_h^{(m)}$ on C_l by

$$W_h^{(m)}(z) := V_h^{(m)}\Big[f_l^{-1}\Big(\frac{f_l(z)}{2m\|f_l(z)\|_{n+1}}\Big)\Big] \tag{4.4}$$

so that

$$\mathcal{E}^p_\gamma(W_h^{(m)}, C_l) \le \frac{c}{m}\,\mathcal{E}^p_\gamma(V_h^{(m)}, \partial C_l)$$

for every l and hence, by (4.3),

$$\mathcal{E}^p_\gamma(W_h^{(m)}, \cup\mathcal{F}_m) \le C\,\mathcal{E}^p_\gamma(U, G_m)\,. \tag{4.5}$$

Setting

$$W_h^{(m)}(z) = V_h^{(m)}(z) \qquad \forall\, z \in (\mathcal{Q}^n_m \times I) \setminus \cup\mathcal{F}_m$$

the function $W_h^{(m)}$ is continuous on $\mathcal{Q}^n_m \times I$ except at one singular point on each C_l, which lies on $\mathcal{Q}^n_m \times \{0\}$. Moreover, $\{W_h^{(m)}\}$ is a sequence in $W^{1,p}_\gamma(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$ such that for h large enough

$$\mathcal{E}^p_\gamma(W_h^{(m)} - V_h^{(m)}, \mathcal{Q}^n_m \times I) \le C\,\mathcal{E}^p_\gamma(U, G_m)$$

and therefore, by Proposition 3.1,

$$\limsup_{h\to\infty} \mathcal{E}^p_\gamma(W_h^{(m)}, \mathcal{Q}^n_m \times I) \le \mathcal{E}^p_\gamma(U, \mathcal{Q}^n_m \times I) + C\,\mathcal{E}^p_\gamma(U, G_m)\,.$$

Remark 4.2 For every $(n+1)$-cube C_l in $\mathcal{F}_m$, we have $W_h^{(m)}{}_{|\partial C_l} = V_h^{(m)}{}_{|\partial C_l}$, with traces $\mathbf{T}(V_h^{(m)})_{|\Sigma_m^{(n-1)}}$ belonging to $W^{s,p}(\Sigma_m^{(n-1)}, \mathcal{Y}_{\varepsilon_0})$, see Proposition 3.1. As a consequence, by definition (4.4) we infer that the traces $\mathbf{T}(W_h^{(m)})$ are functions in $W^{s,p}(\mathcal{Q}^n_m, \mathcal{Y}_{\varepsilon_0})$ for every h.

Now, let $\psi_m : \mathcal{Q}^n \to \mathcal{Q}^n_m$ be an affine bijective function such that Lip $\psi_m = (m-1)/m$ and $\psi_m \to Id_{\mathcal{Q}^n}$ uniformly as $m \to \infty$. Setting $U_m(x,t) := W_{h_m}^{(m)}(\psi_m(x), t)$ for some increasing sequence $h_m \nearrow \infty$, since meas$(G_m) \to 0$ as $m \to \infty$ we easily infer that $\{U_m\}_m$ is a sequence of maps in $W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$, continuous out of a finite number of points, such that $U_m \to U$ strongly in $W^{1,p}_\gamma$. Moreover, by Remark 4.2 it follows that the traces $\mathbf{T}(U_m) \in W^{s,p}(\mathcal{Q}^n, \mathcal{Y}_{\varepsilon_0})$ for every m. Therefore, taking $u_m(x) := \Pi_{\varepsilon_0} \circ \mathbf{T}(U_m)(x)$, compare Remark 2.1, clearly $\{u_m\} \subset W^{s,p}(\mathcal{Q}^n, \mathcal{Y})$ is continuous out of a discrete set of points and $u_m \to u$ in $W^{s,p}$. Finally, e.g. as in [2, Appendix], every function u_m can be approximated by maps in $R^\infty_{s,p}(\mathcal{Q}^n, \mathcal{Y})$.

The case $n \ge d+1$. By applying iteratively Fubini's theorem, we fist observe that for a.e. $a = a(m)$ as above, the restriction of U to each k-face F of $C_m^{(k)}$ belongs to $W^{1,p}_\gamma(F, \mathbb{R}^N)$, for every $k = d-1, \dots, n$. We then may and do apply Propositions 2.2 and 3.1 with $a = a(m)$.

Let $\mathcal{F}_m^{(k)}$ be the k-dimensional skeleton of $\mathcal{F}_m$, i.e. the union of the k-faces of the $(n+1)$-cubes C_l of $\mathcal{F}_m$. Since $V_h^{(m)} \to U$ in $W^{1,p}_\gamma(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$, by using a more refined slicing argument as e.g. in [16, Sec. 4], we may and do choose $(a_1, \dots, a_{n+1}) \in [1/(4m), 3/(4m)]^{n+1}$ so that for every h sufficiently large the following holds:

(i) for every $k = d, \dots, n$ the restriction of $V_h^{(m)}$ to any k-face Q of $\mathcal{F}_m^{(k)}$ is a function in $W^{1,p}_\gamma(Q, \mathbb{R}^N)$;

(ii) there exists some absolute constant $c > 0$, not depending on h, such that for every $k = d, \dots, n$

$$\mathcal{E}^p_\gamma(V_h^{(m)}, \mathcal{F}_m^{(k)}) \le c\, m^{n+1-k}\,\mathcal{E}^p_\gamma(U, G_m)\,. \tag{4.6}$$

First we let $W_h^{(m)} \equiv V_h^{(m)}$ on $\mathcal{F}_m^{(d)}$. Arguing by induction on $k = d, \dots, n$, we now extend $W_h^{(m)}$ to $\mathcal{F}_m^{(k+1)}$. To this aim, for every $(k+1)$-face Q in $\mathcal{F}_m^{(k+1)}$, according to case $n \ge d+1$ in Proposition 3.1, we distinguish two cases.

If Q is "horizontal", i.e. the direction e_{n+1} is not spanned by the vector space underlying Q, we let

$$W_h^{(m)} \equiv V_h^{(m)} \qquad \text{on } Q\,. \tag{4.7}$$

If Q is not "horizontal", as in the case $n = d$ we choose a bilipschitz homeomorphism f_Q between Q and $[-1/(2m), 1/(2m)]^{k+1}$ such that

$$\begin{array}{rcl} f_Q(Q \cap (\mathcal{Q}^n \times \{0\})) & = & [-1/(2m), 1/(2m)]^k \times \{0\} \\ f_Q(\partial Q \cap (\mathcal{Q}^n \times \{0\})) & = & \partial[-1/(2m), 1/(2m)]^k \times \{0\} \end{array}$$

and $\|Df_Q\|_\infty \le K$, $\|Df_Q^{-1}\|_\infty \le K$. Since we have already defined $W_h^{(m)}$ on ∂Q, we extend $W_h^{(m)}$ to Q by setting

$$W_h^{(m)}(z) = W_h^{(m)}\Big[f_Q^{-1}\Big(\frac{f_Q(z)}{2m\|f_Q(z)\|_{k+1}}\Big)\Big] \tag{4.8}$$

so that

$$\mathcal{E}^p_\gamma(W_h^{(m)}, Q) \le \frac{c}{m}\,\mathcal{E}^p_\gamma(W_h^{(m)}, \partial Q)\,. \tag{4.9}$$

Repeating the argument for $k = d, \dots, n$, we then easily estimate

$$\mathcal{E}^p_\gamma(W_h^{(m)}, \cup\mathcal{F}_m) \le C(n,p,s)\sum_{k=d}^{n} \frac{1}{m^{n+1-k}}\mathcal{E}^p_\gamma(V_h^{(m)}, \mathcal{F}_m^{(k)}) \tag{4.10}$$

and hence, by (4.6), we obtain again (4.5). Setting then $W_h^{(m)}(z) = V_h^{(m)}(z)$ for every $z \in (\mathcal{Q}^n_m \times I)\setminus\cup\mathcal{F}_m$, this way $W_h^{(m)}$ is continuous on $\mathcal{Q}^n_m \times I$ outside an $(n-d)$-dimensional singular set, which lies on $\mathcal{Q}^n_m \times \{0\}$, given by the union of a finite number (depending on n, p, s, and m) of smooth subsets of affine $(n-d)$-planes parallel to the coordinate directions in $\mathbb{R}^n \times \{0\}$. Moreover, by the construction we infer that the traces $\mathbf{T}(W_h^{(m)}) \in W^{s,p}(\mathcal{Q}^n_m, \mathcal{Y}_{\varepsilon_0})$ for every m. The rest of the proof follows as in the case $n = d$. $\square$

5 Removing point singularities

In this section, we give the proof of Theorem 1.3 in dimension $n = d$, namely:

Theorem 5.1 *If $n = d := [sp] + 1 \ge 2$, then $W^{s,p}_S(B^n, \mathcal{Y}) = W^{s,p}(B^n, \mathcal{Y})$ if and only if $\pi_{d-1}(\mathcal{Y}) = 0$.*

Proof: We have to prove the non-trivial implication. Therefore, on account of Theorem 1.1, it suffices to prove that $R^\infty_{s,p}(B^n, \mathcal{Y}) \subset W^{s,p}_S(B^n, \mathcal{Y})$ provided that $\pi_{d-1}(\mathcal{Y}) = 0$. Moreover, since the argument is local, without loss of generality we reduce to the case where $u \in R^\infty_{s,p}(\mathcal{Q}^n, \mathcal{Y})$ is smooth outside the origin.

Following the proof of [17, Thm. 2], for $0 < r < 1$ we denote

$$Q_r := [-r, r]^{n+1}\,, \qquad F_r := Q_r \cap (\mathbb{R}^n \times \{0\})\,.$$

Let $U = \mathrm{Ext}(u) \in W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$ be the extension of u, where $\gamma = p(1-s) - 1$. For every fixed $\varepsilon > 0$ let $0 < R = R(\varepsilon) \ll 1$ be such that $\mathcal{E}^p_\gamma(U, Q_R) \le \varepsilon$. Since

$$\mathcal{E}^p_\gamma(U, Q_R \setminus Q_{R/2}) = \int_{R/2}^{R} dr \int_{\partial Q_r} |t|^\gamma |DU|^p \, d\mathcal{H}^n\,,$$

there exists $r = r(\varepsilon) \in [R/2, R]$ such that

$$\mathcal{E}^p_\gamma(U, \partial Q_r) := \int_{\partial Q_r} |t|^\gamma |DU|^p \, d\mathcal{H}^n \le \frac{2}{R}\,\mathcal{E}^p_\gamma(U, Q_R \setminus Q_{R/2}) \le \frac{2\,\varepsilon}{R}\,. \tag{5.1}$$

Since $u_{|\partial F_r} : \partial F_r \to \mathcal{Y}$ is a smooth map in $W^{s,p}(\partial F_r, \mathcal{Y})$ and $\pi_{n-1}(\mathcal{Y}) = 0$, there exists a smooth extension $u_r : F_r \to \mathcal{Y}$ of u with finite $W^{1,p}_\gamma$-norm.

Let now $Q_r^\pm := \{z=(x,t)\in Q_r \mid \pm t\geq 0\}$ be the upper and lower half $(n+1)$-cubes of Q_r. Moreover, let $V_r^\pm : Q_r^\pm \to \mathbb{R}^N$ be a function that minimizes the $\mathcal{E}^p_\gamma$-energy on $Q_r^\pm$ among all maps in $W^{1,p}_\gamma(Q_r^\pm,\mathbb{R}^N)$ satisfying the boundary condition

$$\begin{cases} V_r^\pm = U & \text{on} \quad \partial Q_r^\pm \cap \{(x,t)\mid \pm t>0\} \\ V_r^\pm = u_r & \text{on} \quad F_r \end{cases}$$

and let $V_r : Q_r \to \mathbb{R}^N$ be given by $V_r(z) = V_r^\pm(z)$ if $z\in Q_r^\pm$. Define then $W_r : \mathcal{Q}^n\times I \to \mathbb{R}^N$ by

$$W_r(z) := \begin{cases} V_r\Big(\dfrac{r}{\delta} z\Big) & \text{if} \quad \|z\|_{n+1}\leq \delta \\ U\Big(\dfrac{r\,z}{\|z\|_{n+1}}\Big) & \text{if} \quad \delta\leq \|z\|_{n+1}\leq r \\ U(z) & \text{if} \quad \|z\|_{n+1}\geq r \end{cases}$$

for a suitable $0<\delta<r$. Since $V_r^\pm$ is continuous, $W_r\in W^{1,p}_\gamma(\mathcal{Q}^n\times I,\mathbb{R}^N)$ is continuous and with trace $\mathbf{T}(W_r)\in W^{s,p}(\mathcal{Q}^n,\mathcal{Y})$. We also estimate

$$\mathcal{E}^p_\gamma(W_r,\mathcal{Q}^n\times I) \leq \mathcal{E}^p_\gamma(U,\mathcal{Q}^n\times I) + c\,r\,\mathcal{E}^p_\gamma(U,\partial Q_r) + \Big(\frac{\delta}{r}\Big)^{n+1+\gamma-p}\mathcal{E}^p_\gamma(V_r,Q_r)$$

for some absolute constant $c>0$, depending on n, s, and p. Recalling that $\gamma = p(1-s)-1$ and $n=d=[sp]+1$, we have:

$$n+1+\gamma-p = 1+[sp]-sp>0\,.$$

Therefore, by (5.1), and since $r<R$,

$$\begin{aligned} \mathcal{E}^p_\gamma(W_r,\mathcal{Q}^n\times I) &\leq \mathcal{E}^p_\gamma(U,\mathcal{Q}^n\times I) + 2\,c\,\varepsilon + \Big(\frac{\delta}{r}\Big)^{1+[sp]-sp}\mathcal{E}^p_\gamma(V_r,Q_r) \\ &\leq \mathcal{E}^p_\gamma(U,\mathcal{Q}^n\times I) + (2c+1)\,\varepsilon \end{aligned}$$

taking $\delta=\delta(r,\varepsilon)$ sufficiently small. Letting $\varepsilon\to 0$ we infer that $W_{r(\varepsilon)}\to U$ in $W^{1,p}_\gamma(\mathcal{Q}^n\times I,\mathbb{R}^N)$ and hence that $\mathbf{T}(W_{r(\varepsilon)})\to u$ in $W^{s,p}(\mathcal{Q}^n,\mathcal{Y})$. Since the trace $\mathbf{T}(W_r)\in W^{s,p}(\mathcal{Q}^n,\mathcal{Y})$ is continuous, then in a standard way it can be approximated by smooth maps, as required. □

6 Homotopy type of $W^{s,p}$ maps

Let $n\geq d := [sp]+1\geq 2$. In this section, arguing as in [17, Prop. 4], we obtain the proof of Theorem 1.4 as a consequence of the following

Theorem 6.1 *Let $u\in W^{s,p}(\mathcal{Q}^n,\mathcal{Y})$ and X a finite cubeulation of $\mathcal{Q}^n$ in generic position with respect to u. For any smooth sequence $\{u_i\}\subset W^{s,p}(X^{d-1},\mathcal{Y})\cap C^\infty$ strongly converging to $u_{|X^{d-1}}$ in $W^{s,p}$, we find $k_0\in\mathbb{N}^+$ such that for every $i,j\geq k_0$ the maps u_i and u_j are homotopic as maps from X^{d-1} to $\mathcal{Y}$.*

Proof: Following the notation from Sec. 2, we shall give the proof in the case when $X^k := \Sigma_m^{(k)}$, see (2.2), making use of the argument from [3, Lemma 1], that goes back to [21].

Let $U_i\in W^{1,p}_\gamma(\Sigma_m^{(d-1)}\times I,\mathbb{R}^N)$ be such that $\mathbf{T}(U_i)=u_i$ and U_i minimizes the $\mathcal{E}^p_\gamma$-energy among all maps $V\in W^{1,p}_\gamma(\Sigma_m^{(d-1)}\times I,\mathbb{R}^N)$ such that $\mathbf{T}(V)=u_i$. Let $\sigma_0>0$ to be chosen. By the strong convergence of u_i to u, we can find $k_0\in\mathbb{N}$ such that

$$\|U_i-U_j\|_{W^{1,p}_\gamma(\Sigma_m^{(d-1)}\times I,\mathbb{R}^N)} < \sigma_0 \qquad \forall\, i,j\geq k_0\,. \tag{6.1}$$

If $z=(x,t)\in\Sigma_m^{(d-1)}\times I$ and $0<h<1/(4m)$, we let

$$U_i(z,h) := \fint_{\Sigma(z,h)} U_i(w)\,d\mathcal{H}^d(w) := \frac{1}{\mathcal{H}^d\big(\Sigma(z,h)\big)}\int_{\Sigma(z,h)} U_i(w)\,d\mathcal{H}^d(w)$$

where the d-dimensional set $\Sigma(z,h)$ is defined as in the proof of Proposition 2.2. Moreover, let $u_i(\cdot,h) := \mathbf{T}(U_i(\cdot,h)) \in W^{s,p}(\Sigma_m^{(d-1)},\mathbb{R}^N)$. For every i, we infer that $U_i(z,h)$ is continuous, whereas $U_i(\cdot,h)$ converges to U_i and $u_i(\cdot,h)$ to u_i uniformly, as $h\to 0$. Let $\varepsilon_1>0$ to be chosen. By the strong convergence of u_i to u, we also may and do fix a positive number $h_0<1/(4m)$ such that for every $z\in\Sigma_m^{(d-1)}\times I$, and for any $0<h\le h_0$,

$$\mathcal{E}^p_\gamma(U_i,\Sigma(z,h))\le\varepsilon_1 \qquad \forall\, i\,. \tag{6.2}$$

If $\xi:=(x,0)\in\Sigma_m^{(d-1)}\times\{0\}$, for $i\ne j$ we estimate

$$\begin{aligned}
|u_i(x,h_0)-u_j(x,h_0)| &= |U_i(\xi,h_0)-U_j(\xi,h_0)| \\
&= \Big(\fint_{\Sigma(\xi,h_0)}|U_i(\xi,h_0)-U_j(\xi,h_0)|^p\,d\mathcal{H}^d(w)\Big)^{1/p} \\
&\le \Big(\fint_{\Sigma(\xi,h_0)}|U_i(\xi,h_0)-U_i(w)|^p\,d\mathcal{H}^d(w)\Big)^{1/p} \\
&+ \Big(\fint_{\Sigma(\xi,h_0)}|U_j(\xi,h_0)-U_j(w)|^p\,d\mathcal{H}^d(y)\Big)^{1/p} \\
&+ \Big(\fint_{\Sigma(\xi,h_0)}|U_i(w)-U_j(w)|^p\,d\mathcal{H}^d(w)\Big)^{1/p} \\
&=: I_1+I_2+I_3\,.
\end{aligned}$$

Using (6.2) and the Poincaré inequality, we have

$$I_1+I_2\le c\,h_0^{(p-\gamma-d)/p}\,{\varepsilon_1}^{1/p}\,,\quad \gamma=p(1-s)-1$$

whereas by (6.1), using that $\mathcal{H}^d(\Sigma(\xi,h_0))\ge h_0^d$, we infer that if $i,j\ge k_0$

$$I_3\le h_0^{-d/p}\,\|U_i-U_j\|_{W^{1,p}_\gamma(\Sigma_m^{(d-1)}\times I)}\le C\,h_0^{-d/p}\,\sigma_0\,.$$

Since $p-\gamma-d=sp-[sp]\ge 0$ and $h_0<1$, we then obtain for every $x\in\Sigma_m^{(d-1)}$, and for $i,j\ge k_0$,

$$|u_i(x,h_0)-u_j(x,h_0)|\le c_3\,{\varepsilon_1}^{1/p}+c_4\,h_0^{-d/p}\,\sigma_0\,. \tag{6.3}$$

Let now $\varepsilon_0>0$ given by Remark 2.1. As in the proof of Proposition 2.2, see (2.11), taking first η large so that $c_1\,K_\infty/\eta<\varepsilon_0/2$, we infer that if ε_1 satisfies

$$c_2\,(\eta^{1/p}+1)\,{\varepsilon_1}^{1/p}<\varepsilon_0/2 \tag{6.4}$$

by (6.2) we obtain that for $0<h\le h_0$ and for every i

$$\mathrm{dist}(u_i(x,h),\mathcal{Y})<\varepsilon_0 \qquad \forall\, x\in\Sigma_m^{(d-1)}\,. \tag{6.5}$$

We now fix ε_1 so that both (6.4) and $c_3\,{\varepsilon_1}^{1/p}\le\varepsilon_0/2$ hold true, and determine h_0 by condition (6.2). We then choose $\sigma_0>0$ small in such a way that $c_4\,h_0^{-d/p}\,\sigma_0\le\varepsilon_0/2$, and select k_0. By (6.3) we obtain

$$|u_i(x,h_0)-u_j(x,h_0)|<\varepsilon_0 \qquad \forall\, x\in\Sigma_m^{(d-1)}\,,\quad \forall\, i,j\ge k_0\,. \tag{6.6}$$

Setting $u_i(\cdot,0)=u_i$, on account of (6.5), for every $i\ge k_0$ the homotopy maps

$$H_i:\Sigma_m^{(d-1)}\times[0,h_0]\to\mathcal{Y}_{\varepsilon_0}\,,\qquad H_i(x,h):=u_i(x,h)$$

are well defined. Therefore, the functions u_i and $u_i(h_0,\cdot)$ are homotopic, as maps from $\Sigma_m^{(d-1)}$ into $\mathcal{Y}_{\varepsilon_0}$. Moreover, (6.6) says that $u_i(\cdot,h_0)$ and $u_j(\cdot,h_0)$ are homotopic in the same sense, for $i,j\ge k_0$. This yields that u_i and u_j are homotopic, too, for $i,j\ge k_0$, and hence the assertion, by projecting $\mathcal{Y}_{\varepsilon_0}$ onto $\mathcal{Y}$. □

Proof of Theorem 1.4: Let U_∞, $U_\infty(z,h)$, and $u_\infty(\cdot,h)$ defined as in the proof of Theorem 6.1, but for $u=u_\infty$. With our hypotheses, it turns out that $U_\infty(z,h)$ is continuous, whereas $U_\infty(\cdot,h)$ converges to U_∞ and $u_\infty(\cdot,h)$ to u_∞ uniformly, as $h\to 0$. Moreover, we can assume that both (6.1) and (6.2) hold true also for $i=\infty$. The assertion readily follows. □

7 A characterization

In this section, arguing as in [17, Thm. 3] we prove Theorem 1.5, namely:

Theorem 7.1 *Let $n \geq d := [sp]+1 \geq 1$. Let $u \in R^0_{s,p}(B^n,\mathcal{Y})$ and X a finite cubeulation of B^n in dual position with respect to u. Then, u belongs to $W^{s,p}_S(B^n,\mathcal{Y})$ if and only if the restriction $u_{|X^{d-1}}$ of u to X^{d-1} can be extended to a continuous map from B^n into $\mathcal{Y}$.*

PROOF: Assume first that $u \in R^0_{s,p}(B^n,\mathcal{Y})$ is the strong $W^{s,p}$ limit of a sequence of smooth maps $\{u_i\}$ in $C^\infty(B^n,\mathcal{Y})$. Let X be a finite cubeulation of B^n in dual position with respect to u. Let $d \geq 2$. Then, denoting by $\widetilde{u}_i \in W^{s,p}(X^{d-1},\mathcal{Y})$ the restriction of u_i to X^{d-1}, possibly slightly moving the faces of X, by Fubini's theorem we have that $\widetilde{u}_i$ strongly converges to $\widetilde{u} := u_{|X^{(d-1)}}$ in $W^{s,p}$. Moreover, by Theorem 1.4 we infer that for i sufficiently large $\widetilde{u}_i$ is homotopically equivalent to $\widetilde{u}$, as maps from X^{d-1} to $\mathcal{Y}$. Since each $\widetilde{u}_i$ is the restriction of a smooth map from B^n to $\mathcal{Y}$, and (B^n,X^{d-1}) satisfies the so called *homotopy extension property*, see e.g. [13, Prop. 2.1], this yields that $\widetilde{u}$ can be extended to a continuous map from B^n into $\mathcal{Y}$. The same conclusion holds for any cubeulation X in dual position with respect to u. Finally, if $d=1$ the conclusion trivially follows.

We now prove the converse implication, and assume that the restriction $\widetilde{u} := u_{|X^{(d-1)}}$ can be extended to a continuous map from B^n into $\mathcal{Y}$. We distinguish two cases.

The case $n = d := [sp]+1$. The map $u \in R^0_{s,p}(B^n,\mathcal{Y})$ is continuous outside a discrete set $\Sigma(u)$. Since the argument is local, without loss of generality we assume that $u \in R^0_{s,p}(\mathcal{Q}^n,\mathcal{Y})$ and u is smooth outside the origin. We then argue as in the proof of Theorem 1.3 from Sec. 5, where $n=d$. In fact, this time we infer that $u_{|\partial F_r} : \partial F_r \to \mathcal{Y}$ is a continuous map in $W^{s,p}(\partial F_r,\mathcal{Y})$ for which we can find a continuous extension $u_r : F_r \to \mathcal{Y}$ with finite $W^{s,p}$ norm, as required.

The case $n \geq d+1$. We use a local argument and return to the proof of Theorem 1.1 from Sec. 4. Recall that the singular set of the approximating maps $W^{(m)}_h$ is contained in $\mathcal{Q}^n_m \times \{0\}$ and intersects every not "horizontal" $(k+1)$-cube Q in $\mathcal{F}^{(k+1)}_m$, for $k = d,\dots,n$, on a $(k-d)$-dimensional set obtained by the "homogeneous" extension (4.8) of the restriction of $W^{(m)}_h$ to the boundary of Q. In order to remove the singular set, working by induction on $k = d,\dots,n$, it then suffices to modify the definition (4.8) to (7.1) below, where $V_Q : Q \to \mathbb{R}^N$ is a suitable smooth extension of the boundary datum.

To this aim, we now recall that ${V^{(m)}_h}_{|\Sigma^{(d-1)}_m \times I} = U^{(m)}_h$, where $\{U^{(m)}_h\} \subset W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I,\mathbb{R}^N) \cap C^\infty$ is such that $U^{(m)}_h \to U^{(m)}$ strongly in $W^{1,p}_\gamma$, see Propositions 2.2 and 3.1, and the traces $\mathbf{T}(U^{(m)}_h) \in W^{s,p}(\Sigma^{(d-1)}_m,\mathcal{Y}_{\varepsilon_0}) \cap C^0$. Since the cubeulation given by $C^{(k)}_m$ is in dual position with respect to u, by Theorem 1.4, applied this time with $\mathcal{Y}_{\varepsilon_0}$ instead of $\mathcal{Y}$, see Remark 2.1, we infer that for h large enough $\mathbf{T}(U^{(m)}_h)$ is homotopically equivalent to the restriction $u_{|\Sigma^{(d-1)}_m}$ of u to $\Sigma^{(d-1)}_m$, as maps from $\Sigma^{(d-1)}_m$ into $\mathcal{Y}_{\varepsilon_0}$. Moreover, by the hypothesis $u_{|\Sigma^{(d-1)}_m}$ can be extended to a continuous map from $\mathcal{Q}^n$ into $\mathcal{Y}$. As a consequence, the trace $\mathbf{T}(U^{(m)}_h)_{|\Sigma^{(d-1)}_m}$ of $U^{(m)}_h$ on $\Sigma^{(d-1)}_m$ can be extended to a continuous map $v_h : \mathcal{Q}^n \to \mathcal{Y}_{\varepsilon_0}$ such that the restriction of v_h to every k-face of $\Sigma^{(k)}_m$ has finite $W^{s,p}$-norm, for every $k = d,\dots,n$.

First we let $W^{(m)}_h \equiv V^{(m)}_h$ on $\mathcal{F}^{(d)}_m$. Arguing by induction on $k = d,\dots,n$, we now extend $W^{(m)}_h$ to $\mathcal{F}^{(k+1)}_m$ as follows.

If Q is a "horizontal" $(k+1)$-cube in $\mathcal{F}^{(k+1)}_m$, we define $W^{(m)}_h$ as in (4.7).

If Q is not "horizontal", we let

$$F := Q \cap (\mathbb{R}^n \times \{0\})$$

be the k-face in $\Sigma^{(k)}_m$ given by the intersection of Q with $\mathcal{Q}^n \times \{0\}$, see (2.2). Moreover, let $u_{h,F} : F \to \mathcal{Y}_{\varepsilon_0}$ be given by the restriction of v_h to F, so that $u_{h,F} \in W^{s,p}(F,\mathcal{Y}_{\varepsilon_0}) \cap C^0$.

Let $Q^\pm := \{z = (x,t) \in Q \mid \pm t \geq 0\}$ be the upper and lower half $(k+1)$-cubes of Q, denote by $V^\pm_Q : Q^\pm \to \mathbb{R}^N$ the function that minimizes the $\mathcal{E}^p_\gamma$-energy on $Q^\pm$ among all maps in $W^{1,p}_\gamma(Q^\pm,\mathbb{R}^N)$

satisfying the boundary condition

$$\begin{cases} V_Q^{\pm} = W_h^{(m)} & \text{on} \quad \partial Q^{\pm} \cap \{(x,t) \mid \pm t > 0\} \\ V_Q^{\pm} = u_{h,F} & \text{on} \quad F \end{cases}$$

and let $V_Q : Q \to \mathbb{R}^N$ be given by $V_Q(z) = V_Q^{\pm}(z)$ if $z \in Q^{\pm}$. If f_Q is the bilipschitz homeomorphism between Q and $[-1/(2m), 1/(2m)]^{k+1}$ defined in the proof of Theorem 1.1, we modify the definition (4.8) of $W_h^{(m)}$ on Q by setting for every $z \in Q$

$$\widetilde{W}_h^{(m)} := \begin{cases} V_Q\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\delta}\Big)\Big] & \text{if} \quad \|f_Q(z)\|_{k+1} \le \delta \\ W_h^{(m)}\Big[f_Q^{-1}\Big(\dfrac{f_Q(z)}{2m\|f_Q(z)\|_{k+1}}\Big)\Big] & \text{if} \quad \delta \le \|f_Q(z)\|_{k+1} \le \dfrac{1}{2m}\,. \end{cases} \tag{7.1}$$

Similarly to the proof of Theorem 1.3, we easily infer that (4.9) holds again if $0 < \delta < 1/(2m)$ is sufficiently small, whereas this time $\widetilde{W}_h^{(m)}$ is continuous on Q and with trace $\mathbf{T}(\widetilde{W}_h^{(m)})$ in $W^{s,p}(F, \mathcal{Y}_{\varepsilon_0})$.

We then obtain again (4.10) and hence, by (4.6), we conclude again with (4.5). The rest of the proof is similar to the one of Theorem 1.1 from Sec. 4. □

8 Maps into spheres

In this section and in the next one, we let $\mathcal{Y} = \mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$, where $N \ge 2$. As noticed in the introduction, we shall assume that $0 < s < 1$ and $p > 1$ satisfy $N - 1 \le sp < n$. Recalling that $d = [sp] + 1$, we thus reduce to the ranges $N \le d \le n$.

A special case is when $sp = N - 1$, so that $p > N - 1$ and $\gamma(s,p) = p - N$. If $sp > N - 1$, we shall apply the continuous embedding

$$W^{s,p}(B^n, \mathbb{S}^{N-1}) \subset W^{s,(N-1)/s}(B^n, \mathbb{S}^{N-1})\,. \tag{8.1}$$

When $sp = N-1$, we prove a relevant estimate inspired by arguments from [4], see Theorem 8.1. As a consequence, we introduce a notion of degree, see (8.8), showing that it is strongly continuous and integer valued, Proposition 8.3. For $N-1 \le sp < n$ and $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$, we then define an $(n-N)$-current $\mathbb{P}(u)$ in B^n that describes the relevant singularities of u, at least when $[sp] = N - 1$, i.e., when $d = N$.

In fact, recalling that $W_S^{s,p}(B^n, \mathbb{S}^{N-1})$ denotes the strong closure of smooth maps $u \in C^\infty(B^n, \mathbb{S}^{N-1})$ in the $W^{s,p}$-norm, in Proposition 8.5 we show that for any $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$

$$u \in W_S^{s,p}(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,.$$

When $[sp] = N - 1$, the converse implication holds true. In fact, in Theorem 8.6 we show that

$$[sp] = N - 1 \quad \Longrightarrow \quad W_S^{s,p}(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}$$

whereas in case $N = 2$ the same conclusion holds true whenever $sp \ge 1$.

However, the converse implication fails to hold when $3 \le N \le [sp] < n$. More precisely, in that case we show existence of maps u in $W^{s,p}(B^N, \mathbb{S}^{N-1})$, actually in $R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$, such that $\mathbb{P}(u) = 0$ but $u \notin W_S^{s,p}(B^N, \mathbb{S}^{N-1})$.

A RELEVANT ESTIMATE. Assume $sp = N-1$, so that $d = N$ and $\gamma(s,p) = p-N$. Let $n \ge N-1 \ge 1$, and recall that condition $0 < s < 1$ implies $p > N - 1$.

Let $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ and $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{R}^N)$ the *harmonic extension* of u to $\mathbf{C}^{n+1} := B^n \times (0,1)$, so that U takes values into the unit N-ball $\mathbb{D}^N$ of the target space,

$$\mathbb{D}^N := \{y \in \mathbb{R}^d \,:\, |y| \le 1\}\,.$$

Following [4, Lemma 1.2], we denote

$$G := \{(x,t) \in \mathbf{C}^{n+1} \, : \, |U(x,t)| \le 1/2\}$$

and let $\mathbf{d} : B^n \to]0,1/2]$ the function such that $\mathbf{d}(x) := 1/2$ if $|U(x,t)| \ge 1/2$ for each $t \in (0,1/2)$, and

$$\mathbf{d}(x) := \min\{t \in (0,1/2) \, : \, |U(x,t)| \le 1/2\}$$

otherwise. Using that $|DU(x,t)| \le c/t$ for some absolute constant c, for any exponent $\alpha > 1$ one has

$$\int_G |DU(x,t)|^\alpha \, dx \, dt \le c \int_{B^n} \Big(\int_{\mathbf{d}(x)}^1 t^{-\alpha} \, dt \Big) \, dx \le C \int_{B^n} \frac{1}{\mathbf{d}(x)^{\alpha-1}} \, dx \, .$$

In a similar way to the case $\alpha = N$, using that $t > \mathbf{d}(x)$ if $(x,t) \in G$, for each $p > N-1$ we estimate

$$\int_G t^{p-N} |DU(x,t)|^p \, dx \, dt \le \int_G \frac{C}{t^N} \, dx \, dt \le C \int_{B^n} \frac{1}{\mathbf{d}(x)^{N-1}} \, dx \tag{8.2}$$

where $C = C(n,p,N)$. Moreover, as in [4, Lemma 1.3], since $U \in W^{N/p,p}(B^n \times I, \mathbb{R}^N)$, where $I = (0,1/2)$, using the embedding of $W^{N/p,p}(I)$ in the Hölder class $C^{0,(N-1)/p}(I)$, it turns out that for a.e. $x \in B^n$ the function $\varphi_x(t) := U(x,t)$ belongs to $W^{N/p,p}(I,\mathbb{R}^N)$, whence to $C^{0,(N-1)/p}(I,\mathbb{R}^N)$, so that we have:

$$\frac{1}{2} \le |\varphi_x(\mathbf{d}(x)) - \varphi_x(0)| \le C \, \mathbf{d}(x)^{(N-1)/p} \|\varphi_x\|_{C^{0,(N-1)/p}(I)} \le C \, \mathbf{d}(x)^{(N-1)/p} \, \|\varphi_x\|_{W^{N/p,p}(I,\mathbb{R}^N)}$$

and hence

$$\frac{1}{\mathbf{d}(x)^{N-1}} \le C \, \|\varphi_x\|^p_{W^{N/p,p}(I,\mathbb{R}^N)} \, . \tag{8.3}$$

Therefore, using the inequality on Besov-type spaces

$$\int_{B^n} \|\varphi_x\|^p_{W^{N/p,p}(I,\mathbb{R}^N)} \, dx = \int_{B^n} \|U(x,\cdot)\|^p_{W^{N/p,p}(I,\mathbb{R}^N)} \, dx \le C \, \|U\|^p_{W^{N/p,p}(\mathbf{C}^{n+1},\mathbb{R}^N)} \le C \, \|u\|^p_{W^{(N-1)/p,p}(B^n,\mathbb{R}^N)}$$

by (8.2) and (8.3) one gets the estimate

$$\int_G t^{p-N} |DU(x,t)|^p \, dx \, dt \le C_1 \int_{B^n} \frac{1}{\mathbf{d}(x)^{N-1}} \, dx \le C_2 \int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx \, dt \tag{8.4}$$

for some positive constants C_1, C_2 only depending on n, p, and N.

In the sequel, we choose a smooth function $\Phi : \mathbb{R}^N \to \mathbb{D}^N$ such that $\Phi(y) = y/|y|$ if $|y| \ge 1/2$, where $y = (y_1, \ldots, y_N)$, and Φ is a bi-Lipschitz map from $\{y \in \mathbb{R}^N \, : \, |y| \le 1/2\}$ onto $\mathbb{D}^N$.

Setting $V := \Phi \circ U$, we clearly have:

$$|DV(x,t)| \le C_1 \, |DU(x,t)| \quad \forall \, (x,t) \in \mathbf{C}^{n+1} \, , \quad |DU(x,t)| \le C_2 \, |DV(x,t)| \qquad \forall \, (x,t) \in G \, . \tag{8.5}$$

Denote now by $V^\#(dy^1 \wedge \cdots \wedge dy^N)$ the N-form in $\mathbf{C}^{n+1}$ given by the pull-back by V of the N-form $dy^1 \wedge \cdots \wedge dy^N$. One has

$$|V^\#(dy^1 \wedge \cdots \wedge dy^N)| = J_V \tag{8.6}$$

where J_V is the Jacobian of the map V, so that $J_V(x,t)^2$ is the sum of the squares of all the $N \times N$ minors of the gradient matrix $DV(x,t)$. Therefore, by the area formula one has $J_V(x,t) = 0$ if $(x,t) \in G$ whereas by the parallelogram inequality one gets the general estimate $J_V(x,t) \le C \, |DV(x,t)|^N$, where $C = C(n,N)$. These are the main facts that led Bourgain-Brezis-Mironescu [4] to obtain the estimate

$$|\deg g| \le C_p \, \|g\|^p_{(N-1)/p,p} \qquad \forall \, p > 1$$

on the degree $\deg g$ of maps $g \in W^{(N-1)/p,p}(\mathbb{S}^{N-1}, \mathbb{S}^{N-1})$, see also (8.8).

We similarly obtain in any dimension the following relevant estimate:

Theorem 8.1 *Let $n \geq N-1 \geq 1$ integers and $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ for some $p > N-1$. Then*

$$\int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt \leq C \int_{\mathbf{C}^{n+1}} t^{p-N} \, |DU(x,t)|^p \, dx \, dt \tag{8.7}$$

for some real constant $C > 0$ only depending on n, p, and N.

Proof: By the previous facts, using (8.5) inequality (8.7) readily follows when $p = N$, and hence for $N-1 < p < N$, by the continuous embedding $W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1}) \subset W^{(N-1)/N,N}(B^n, \mathbb{S}^{N-1})$. When $p > N$, letting $\alpha = \alpha(p, N) = N(p-N)/p$, by Hölder inequality with exponents $q = p/N$ and $q' = p/(p-N)$ we get:

$$\begin{aligned} \int_G |DV(x,t)|^N \, dx \, dt \leq & \; C \int_G (t^\alpha \, |DU(x,t)|^N) \, t^{-\alpha} \, dx \, dt \\ \leq & \; C \Big(\int_G t^{p-N} |DU(x,t)|^p \, dx \, dt \Big)^{N/p} \cdot \Big(\int_G t^{-N} \, dx \, dt \Big)^{(p-N)/p} \end{aligned}$$

where by (8.2) and (8.4) we can estimate

$$\Big(\int_G t^{-N} \, dx \, dt \Big)^{(p-N)/p} \leq C_{n,N} \Big(\int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx \, dt \Big)^{(p-N)/p} .$$

Since by (8.5) and (8.6)

$$\int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt = \int_G J_V(x,t) \, dx \, dt \leq C \int_G |DV(x,t)|^N \, dx \, dt$$

the assertion readily follows. □

Remark 8.2 By the continuous embedding (8.1), it turns out that our estimate (8.7) extends to maps in $W^{s,p}(B^n, \mathbb{S}^{N-1})$ for any $0 < s < 1$ and $p > 1$ such that $sp > N-1$.

Degree. For $N \geq 2$ and $p > N-1$, denote now by $W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ the class of locally summable maps $u : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ such that $u(x) - P_u \in L^p(\mathbb{R}^{N-1}, \mathbb{R}^N)$ for some point $P_u \in \mathbb{S}^{N-1}$, and $|u|_{(N-1)/p,p} < \infty$, where

$$|u|^p_{(N-1)/p,p} := \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|u(x) - u(y)|^p}{|x-y|^{2(N-1)}} \, dx \, dy \, .$$

The class $W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ is equipped with the norm $\|u - P_u\|_{L^p} + |u|_{(N-1)/p,p}$.

We define the *degree* of a map u in $W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ through the formula

$$\deg u := \frac{1}{\alpha_N} \int_{\mathbb{R}^N_+} V^{\#}(dy^1 \wedge \cdots \wedge dy^N) \, , \quad \alpha_N := |\mathbb{D}^N| \tag{8.8}$$

where $\mathbb{R}^N_+ := \{(x,t) \in \mathbb{R}^N \mid t > 0\}$ denotes the upper N-space, $U \in W^{1,p}_{p-N}(\mathbb{R}^N_+, \mathbb{D}^N)$ is the harmonic extension of u, and $V := \Phi \circ U$, as before. We have:

Proposition 8.3 *The degree of maps in $W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ is strongly continuous, and $\deg u \in \mathbb{Z}$ for each $u \in W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$.*

Proof: Let $u \in W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$. Arguing as in the proof of Theorem 8.1, we have:

$$\int_{\mathbb{R}^N_+} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt \leq C \int_{\mathbb{R}^N_+} t^{p-N} \, |DU(x,t)|^p \, dx \, dt$$

for some real constant $C > 0$ depending on p and N. Let $\{u_h\} \subset W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ such that $u_h \to u$ strongly in $W^{(N-1)/p,p}$. For each h, denote $V_h := \Phi \circ U_h$, where $U_h \in W^{1,p}_{p-N}(\mathbb{R}^N_+, \mathbb{D}^N)$ is the harmonic extension of u_h. The strong convergence $u_h \to u$ in $W^{(N-1)/p,p}(\mathbb{R}^{N-1},\mathbb{R}^N)$ implies the strong convergence $V_h \to V$ in $W^{1,p}_{p-N}(\mathbb{R}^N_+,\mathbb{R}^N)$. Therefore, by the above estimate, the dominated convergence theorem yields

$$\lim_{h\to\infty}\int_{\mathbb{R}^N_+} V_h^{\#}(dy^1\wedge\cdots\wedge dy^N)=\int_{\mathbb{R}^N_+} V^{\#}(dy^1\wedge\cdots\wedge dy^N)$$

whence $\deg u_h \to \deg u$. Since moreover $n=N-1$, there exists a sequence $\{u_h\}\subset C^1(\mathbb{R}^{N-1},\mathbb{S}^{N-1})$ such that $u_h \to u$ strongly in $W^{(N-1)/p,p}$, see [8]. Choose $\varepsilon>0$ small. Then, by means of a cut-off argument, for each h we readily find a smooth map $W_h:\mathbb{R}^N\to\mathbb{D}^N$ and a point $P_h\in\mathbb{S}^{N-1}$ such that $W_h(x,t)-P_h$ has compact support contained in $\mathbb{R}^N_+$ and

$$\int_{\mathbb{R}^N_+} |W_h^{\#}(dy^1\wedge\cdots\wedge dy^N)-V_h^{\#}(dy^1\wedge\cdots\wedge dy^N)|\,dx\,dt<\varepsilon\,.$$

It is then readily checked that

$$\int_{\mathbb{R}^N_+} W_h^{\#}(dy^1\wedge\cdots\wedge dy^N)=\mathbf{d}_h\cdot\alpha_N$$

for some $\mathbf{d}_h\in\mathbb{Z}$. Therefore, we get $\deg u_h=\mathbf{d}_h$ for each h, whence $\deg u\in\mathbb{Z}$, as $\deg u_h\to\deg u$. □

Of course, due to the well-known bubbling phenomenon, the degree fails to be continuous w.r.t. the weak sequential convergence in $W^{(N-1)/p,p}$.

CURRENTS. Let $0\le k\le m$ integers and $\Omega\subset\mathbb{R}^m$ an open set. The space $\mathcal{D}_k(\Omega)$ of k-currents in Ω is the strong dual of the space $\mathcal{D}^k(\Omega)$ of compactly supported smooth k-forms. Weak convergence $T_h\rightharpoonup T$ in $\mathcal{D}_k(\Omega)$ is defined by duality through the formula

$$T_h(\omega)\to T(\omega)\qquad\forall\,\omega\in\mathcal{D}^k(\Omega)\,.$$

The *mass* of a current $T\in\mathcal{D}_k(\Omega)$ is defined by

$$\mathbf{M}(T):=\sup\{T(\omega)\mid \omega\in\mathcal{D}^k(\Omega),\ \|\omega\|\le 1\}$$

where $\|\omega\|$ is the *comass* norm of ω. Therefore, the mass functional is lower semicontinuous w.r.t. the weak convergence. The *boundary* of a current T in $\mathcal{D}_k(\Omega)$, when $k\ge 1$, is defined by duality as

$$\partial T(\eta):=T(d\eta)\,,\qquad \eta\in\mathcal{D}^{k-1}(\Omega)$$

yielding to a current ∂T in $\mathcal{D}_{k-1}(\Omega)$.

In particular, when $\Omega=A\times\mathbb{R}^N$, where $A\subset\mathbb{R}^k$ is a bounded domain, and $v:A\to\mathbb{R}^N$ is a sufficiently smooth function, the k-current G_v carried by the graph of v acts on k-forms $\omega\in\mathcal{D}^k(A\times\mathbb{R}^N)$ as

$$G_v(\omega)=((Id\bowtie v)_{\#}[\![A]\!],\omega):=\int_A (Id\bowtie v)^{\#}\omega$$

where $(Id\bowtie v)^{\#}\omega$ is the pull-back of ω through the graph map $(Id\bowtie v)(x):=(x,v(x))$. By the area formula one then computes

$$\mathbf{M}(G_v)=\int_A J_{Id\bowtie v}(x)\,dx$$

where $J_{Id\bowtie v}$ is the Jacobian of the graph map. If e.g. $k\ge N\ge 2$, one has

$${J_{Id\bowtie v}}^2=1+|Dv|^2+|M_2(Dv)|^2+\cdots+|M_N(Dv)|^2$$

where $|M_j(Dv)|^2$ is the sum of the square of the $j\times j$ minors of the gradient matrix Dv. In particular, if $v\in W^{1,N-1}(A,\mathbb{R}^N)$ and $|M_N(Dv)|\in L^1(A)$, then G_v is an *i.m. rectifiable* current in $\mathcal{R}_k(A\times\mathbb{R}^N)$, with finite mass, compare Proposition 9.2. We refer e.g. to the treatise [9] for further details.

Homological singularities. Let $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $n \geq N \geq 2$ and $p > N-1$. We can define the current $\mathbb{P}(u) \in \mathcal{D}_{n-N}(B^n)$ that carries the relevant information on the singular set of u. Choose a smooth decreasing cut-off function $\eta : [0,1] \to [0,1]$ such that $\eta(t) = 1$ for $t \in [0, 1/4]$ and $\eta(t) = 0$ for $t \in [3/4, 1]$, and for any k-form $\phi \in \mathcal{D}^k(B^n)$ denote by $\widetilde{\phi}$ the k-form in $\mathbf{C}^{n+1}$ given by $\widetilde{\phi} := \phi \wedge \eta$. We let

$$\mathbb{P}(u)(\phi) := \frac{1}{\alpha_N} \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N), \quad \phi \in \mathcal{D}^{n-N}(B^n) \tag{8.9}$$

where $\alpha_N := |\mathbb{D}^N|$ and $V = \Phi \circ U$, with U the harmonic extension of u, as before.

Remark 8.4 Again by the continuous embedding (8.1), it turns out that definition (8.9) extends to maps in $W^{s,p}(B^n, \mathbb{S}^{N-1})$ for any $0 < s < 1$ and $p > 1$ such that $sp > N-1$.

With this notation, we have:

Proposition 8.5 *Let $0 < s < 1$ and $p > 1$ satisfy $1 \leq N-1 \leq sp < n$, and let $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$. Then*

$$u \in W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,.$$

Proof: Let $\{u_h\} \subset C^1(B^n, \mathbb{S}^{N-1})$ a smooth sequence strongly converging to u in $W^{s,p}(B^n, \mathbb{R}^N)$. In case $sp > N-1$, by the continuous embedding (8.1) we infer that $u_h \to u$ strongly in $W^{s,(N-1)/s}(B^n, \mathbb{R}^N)$. As a consequence, if $U_h := \mathrm{Ext}(u_h)$ is the harmonic extension of u_h, then $U_h \to U := \mathrm{Ext}(u)$ strongly in $W^{1,q}_{q-N}(B^n, \mathbb{R}^N)$ for some $q > N-1$. Therefore, if $V_h = \Phi \circ U_h$, by (8.7) and the dominated convergence theorem we get

$$\lim_{h\to\infty} \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V_h^{\#}(dy^1 \wedge \cdots \wedge dy^N) = \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N) \quad \forall\, \phi \in \mathcal{D}^{n-N}(B^n) \tag{8.10}$$

i.e., $\mathbb{P}(u_h) \rightharpoonup \mathbb{P}(u)$ weakly in $\mathcal{D}_{n-N}(B^n)$, whereas $\mathbb{P}(u_h) = 0$ for each h, by the smoothness property. □

For some ranges of sp, the converse implication holds, too:

Theorem 8.6 *Let $0 < s < 1$ and $p > 1$. Assume that $n > [sp] = N-1$, when $N = 3$, or $n > [sp] \geq 1$, when $N = 2$. Then*

$$W^{s,p}_S(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}\,.$$

Proof: On account of Proposition 8.5, it suffices to show that if $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ and $\mathbb{P}(u) = 0$, then u is the strong $W^{s,p}$ limit of a smooth sequence in $C^1(B^n, \mathbb{S}^{N-1})$. We make use of arguments taken from [18] and reduce to the case of maps in $R^0_{s,p}(B^n, \mathbb{S}^{N-1})$ by means of the following result, the proof of which is postponed to the end of the last section.

Proposition 8.7 *Let $0 < s < 1$, $p > 1$, and $1 \leq N-1 \leq sp < n$, and let $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ satisfying $\mathbb{P}(u) = 0$. Then, u is the strong $W^{s,p}$ limit of a sequence $\{u_h\} \subset R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$ satisfying $\mathbb{P}(u_h) = 0$ for each h.*

Assume first $N \geq 3$, so that $d = [sp] + 1 = N \leq n$. Now, if $u \in R^0_{s,p}(B^n, \mathbb{S}^{N-1})$ is such that $\mathbb{P}(u) = 0$, and X is a cubeulation of B^n in dual position with respect to u, by Theorem 1.5 it suffices to show that the restriction $u_{|X^{N-1}}$ has a continuous extension $g : B^n \to \mathbb{S}^{N-1}$. Moreover, by condition $\mathbb{P}(u) = 0$, without loss of generality we can assume that the restriction of u to each $(N-1)$-simplex of X^{N-1} has zero degree. This yields that $u_{|X^{N-1}}$ has a continuous extension $f : X^N \to \mathbb{S}^{N-1}$, and proves the claim in low dimension $n = N$. If $n \geq N+1$, instead, since B^n is N-connected, arguing as in [21, Sec. 6], we find a continuous map $\phi : B^n \to \mathbb{S}^{N-1}$ homotopic to the identity map and such that the restriction $\phi_{|X^N}$ is constant. Then $f \circ \phi$ is homotopic to f and $f \circ \phi_{|X^N}$ is constant. Whence, $f_{|X^N}$ can be extended to a continuous map, as required.

In case $N = 2$, recall that we can restrict to the case $1 \leq sp < 2$, whence $d = N$ and therefore we argue as before. □

9 Further results

In this final section, we collect some other consequences of Theorem 8.1. We then conclude with the proof of Proposition 8.7.

COAREA FORMULA. Following Almgren-Browder-Lieb [1], we have:

Proposition 9.1 *Let $n \geq N \geq 2$ integers and $p > N-1$. For every map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ there exists a smooth extension $V \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ and a regular value $y \in \mathbb{D}^N$ for V such that*

$$\mathcal{H}^{n+1-N}(V^{-1}(\{y\})) \leq C \int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx \, dt \tag{9.1}$$

for some real constant C only depending on n, p, and N.

PROOF: Choose $V := \Phi \circ U$, where $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ is the harmonic extension of u. We have

$$\int_{\mathbb{D}^N} \mathcal{H}^{n+1-N}(V^{-1}(\{y\})) \, d\mathcal{H}^N(y) = \int_{\mathbf{C}^{n+1}} J_V(x,t) \, dx \, dt = \int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt$$

and hence we can find a regular value $y \in \mathbb{D}^N$ such that

$$\mathcal{H}^{n+1-N}(V^{-1}(\{y\})) \leq \frac{1}{\alpha_N} \cdot \int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt$$

where $\alpha_N := |\mathbb{D}^N|$. The assertion follows from Theorem 8.1. □

GRADIENT SUMMABILITY. For $N \geq 2$ integer, denote now

$$p(N) := \begin{cases} +\infty & \text{if } N = 2 \\ \dfrac{(N-1)^2}{N-2} & \text{if } N \geq 3 \,. \end{cases} \tag{9.2}$$

On account of Theorem 8.1, we also obtain:

Proposition 9.2 *Let $n \geq N-1 \geq 1$ integers and $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $N-1 < p < p(N)$. If $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ is the harmonic extension of u, then $U \in W^{1,N-1}(\mathbf{C}^{n+1}, \mathbb{D}^N)$. As a consequence, letting $V := \Phi \circ U$, then the graph current G_V is i.m. rectifiable in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1} \times \mathbb{R}^N)$, with finite mass bounded by*

$$\mathbf{M}(G_V) = \int_{\mathbf{C}^{n+1}} J_{Id \bowtie V} \, dz \leq c \Big(1 + \int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dz\Big) \tag{9.3}$$

for some constant $c > 0$, not depending on u, and G_V satisfies the null-boundary condition:

$$(\partial G_V) \llcorner \mathbf{C}^{n+1} \times \mathbb{R}^N = 0 \,. \tag{9.4}$$

PROOF: Letting $\alpha := (p-N)(N-1)/p$, by Hölder inequality with $q := p/(N-1)$ and $q' = p/(p-N+1)$ we get:

$$\begin{aligned} \int_{\mathbf{C}^{n+1}} |DU(z)|^{N-1} \, dz &= \int_{\mathbf{C}^{n+1}} (t^\alpha \, |DU(x,t)|^{N-1}) \, t^{-\alpha} \, dx \, dt \\ &\leq \Big(\int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx \, dt\Big)^{(N-1)/p} \cdot \Big(\int_{\mathbf{C}^{n+1}} t^{-\alpha p/(p-N+1)} \, dx \, dt\Big)^{(p-N+1)/p} . \end{aligned}$$

However, $-\alpha p/(p-N+1) > -1$ if and only if $(p-N)(N-2) < 1$, i.e., $p < p(N)$. Equivalently, for $N \geq 3$, we recall that $W^{(N-2)/(N-1),N-1}$ is the trace space of $W^{1,N-1}$, whereas by the Gagliardo-Nirenberg and Sobolev inequalities

$$W^{(N-1)/p,p} \cap L^\infty \subset W^{(N-2)/(N-1),N-1} \iff \frac{N-1}{p} > \frac{N-2}{N-1} \iff p < p(N) \,.$$

We now recall that $|DV| \le C\,|DU|$ and by the parallelogram inequality $|M_k(DV)| \le c\cdot|DV|^k$ for $k=1,\dots,N-1$, whereas $|V^\#(dy^1\wedge\cdots\wedge dy^N)| = |M_N(DV)|$. Therefore, V is locally Lipschitz, the mass estimate (9.3) follows from (8.7), and the null boundary condition (9.4) is readily checked by a standard density argument, on account of (9.3) and of the dominated convergence theorem. □

For e.g. $N=3$, there exist maps $u\in W^{1/2,4}(B^n,\mathbb{S}^2)\setminus W^{1/2,2}(B^n,\mathbb{S}^2)$. Therefore, if U is the harmonic extension of u in $W^{1,4}_1(\mathbf{C}^{n+1},\mathbb{R}^3)$, since $\mathbf{T}(U)=u$, then $U\notin W^{1,2}(B^n,\mathbb{R}^3)$. As a consequence, differently to the case $N=2$ analyzed in [19], when $N\ge 3$ it is possible to give a good notion of current G_u carried by the graph of a map $u\in W^{(N-1)/p,p}(B^n,\mathbb{S}^{N-1})$ only for some ranges of $p>N-1$.

Graph currents. If n, N, and p are as in Proposition 9.2, to any map $u\in W^{(N-1)/p,p}(B^n,\mathbb{S}^{N-1})$ we can associate an n-current G_u in $\mathcal{D}_n(B^n\times\mathbb{S}^{N-1})$ by setting

$$G_u := (-1)^{n-N+1}(\partial G_V)\llcorner((B^n\times\{0\})\times\mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^n(B^n\times\mathbb{S}^{N-1})\,, \tag{9.5}$$

where $V:=\Phi\circ U$ and $U\in W^{1,p}_{p-N}(\mathbf{C}^{n+1},\mathbb{D}^N)$ is the harmonic extension of u.

Remark 9.3 In formula (9.5), the boundary ∂G_V is seen by extending the action of the current G_V to forms in $\mathcal{D}^{n+1}(\mathbb{R}^{n+1}\times\mathbb{R}^N)$. Moreover, by Federer's support theorem, see e.g. Thm. 5 in [9, Sec. 5.1.3], it turns out that the current G_u belongs to the class $\mathcal{D}_n(B^n\times\mathbb{S}^{N-1})$. Notice however that in general G_u is not i.m. rectifiable, and fails to satisfy the null-boundary condition $(\partial G_u)\llcorner B^n\times\mathbb{S}^{N-1}=0$, when $n\ge N$. However, in low dimension $n=N-1$, the null-boundary condition $(\partial G_u)\llcorner B^{N-1}\times\mathbb{S}^{N-1}=0$ holds true as a consequence of the strong density of smooth maps.

In these cases, we can write the current of the singularities $\mathbb{P}(u)$ in terms of the graph current G_u. For this purpose, we let $\omega_{\mathbb{S}^{N-1}}$ denote the *normalized volume* $(N-1)$*-form* in $\mathbb{S}^{N-1}$

$$\omega_{\mathbb{S}^{N-1}} := \frac{1}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}\sum_{j=1}^N (-1)^{j-1}y^j\,dy^1\wedge\cdots\wedge dy^{j-1}\wedge dy^{j+1}\wedge\cdots\wedge dy^N\,. \tag{9.6}$$

Recalling that $\mathcal{H}^{N-1}(\mathbb{S}^{N-1}) = N\cdot\alpha_N$, where $\alpha_N:=|\mathbb{D}^N|$, by Stokes theorem we get

$$\int_{\mathbb{S}^{N-1}}\omega_{\mathbb{S}^{N-1}} = \int_{\partial\mathbb{D}^N}\omega_{\mathbb{S}^{N-1}} = \int_{\mathbb{D}^N} d\omega_{\mathbb{S}^{N-1}} = \frac{1}{\alpha_N}\int_{\mathbb{D}^N} dy^1\wedge\cdots\wedge dy^N = 1\,.$$

Moreover, let $\pi_1: A\times\mathbb{R}^N\to A$ and $\pi_2: A\times\mathbb{R}^N\to\mathbb{R}^N$ denote the orthogonal projections onto the two factors, where $A=B^n$ or $A=\mathbf{C}^{n+1}$.

Proposition 9.4 *If n, N, and p are as in Proposition 9.2, for any $u\in W^{(N-1)/p,p}(B^n,\mathbb{S}^{N-1})$ we have*

$$\mathbb{P}(u)(\phi) = \partial G_u(\pi_1^\#\phi\wedge\pi_2^\#\omega_{\mathbb{S}^{N-1}}) \qquad \forall\,\phi\in\mathcal{D}^{n-N}(B^n)\,.$$

Proof: Denote by $\widehat{\omega}_{\mathbb{S}^{N-1}}$ an $(N-1)$-form in $\mathcal{D}^{N-1}(\mathbb{R}^N)$ that agrees with the right-hand side of (9.6) on $\mathbb{D}^N$. Since $V(\mathbf{C}^{n+1})\subset\mathbb{D}^N$, then

$$V^\#\,d\widehat{\omega}_{\mathbb{S}^{N-1}} = \frac{1}{\alpha_N}\,V^\#(dy^1\wedge\cdots\wedge dy^N)\,.$$

On account of Proposition 9.2, by (8.9) we have:

$$\mathbb{P}(u)(\phi) = G_V(\pi_1^\# d\widetilde{\phi}\wedge\pi_2^\# d\widehat{\omega}_{\mathbb{S}^1}) = G_V(\pi_1^\# d\widetilde{\phi}\wedge d\pi_2^\#\widehat{\omega}_{\mathbb{S}^1}) = (-1)^{n-N+1}\partial G_V(\pi_1^\# d\widetilde{\phi}\wedge\pi_2^\#\widehat{\omega}_{\mathbb{S}^1}) \tag{9.7}$$

for every $\phi\in\mathcal{D}^{n-N}(B^n)$. Moreover, by definition (9.5), using that V satisfies the null-boundary condition (9.4) and that $\eta(t)=1$ for $t\in[0,1/4]$ and $\eta(t)=0$ for $t\in[3/4,1]$, we have

$$\begin{aligned}(-1)^{n-N+1}\partial G_V(\pi_1^\# d\widetilde{\phi}\wedge\pi_2^\#\widehat{\omega}_{\mathbb{S}^{N-1}}) &= G_u(\pi_1^\#(d_x\widetilde{\phi}+d_t\widetilde{\phi})_{|t=0}\wedge\pi_2^\#\omega_{\mathbb{S}^{N-1}})\\ &= G_u(\pi_1^\# d\phi\wedge\pi_2^\#\omega_{\mathbb{S}^{N-1}})\,.\end{aligned}$$

Finally, since $d\pi_2^\# \omega_{\mathbb{S}^{N-1}} = \pi_2^\# d\omega_{\mathbb{S}^{N-1}} = 0$, as $\omega_{\mathbb{S}^{N-1}}$ is a closed $(N-1)$-form in $\mathbb{S}^{N-1}$, we compute

$$G_u(\pi_1^\# d\phi \wedge \pi_2^\# \omega_{\mathbb{S}^{N-1}}) = G_u(d\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^{N-1}}) = \partial G_u(\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^{N-1}})$$

that clearly concludes the proof. □

Example 9.5 Of course, the conclusion in Proposition 9.4 extends to the limiting case of maps u in the Sobolev class $W^{1,N-1}(B^n, \mathbb{S}^{N-1})$. In fact, in that case the graph current G_u is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$. If e.g. $u : B^n \to \mathbb{S}^{N-1}$ is given by $u(x) = \varphi(x/|x|)$ for some smooth map $\varphi : \partial B^n \to \mathbb{S}^{N-1}$, then $u \in W^{1,n-1}(B^n, \mathbb{S}^{N-1})$, and one has

$$(\partial G_u) \llcorner B^n \times \mathbb{S}^{N-1} = -\delta_0 \times \varphi_\# [\![\partial B^n]\!]$$

where δ_0 is the unit Dirac mass at the origin, see Ex. 2 in [9, Sec. 3.2.2]. Therefore, if $n = N$ we get

$$\mathbb{P}(u) = -(\deg \varphi)\, \delta_0$$

where $\deg \varphi$ is the degree of $\varphi : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$. If $n \geq N+1$, instead, since $\varphi(\partial B^n) \subset \mathbb{S}^{N-1}$, we infer that $\varphi_\# [\![\partial B^n]\!] = 0$, whence the graph current G_u has no inner boundary, and finally $\mathbb{P}(u) = 0$.

Example 9.6 Let $N \geq 3$, $0 < s < 1$, and $p > 1$ such that $N \leq sp < N+1$, whence $d = [sp] + 1 = N+1$. We have already seen that the zero-degree homogeneous map u given by Example 1.10 belongs to $W^{s,p}(B^{N+1}, \mathbb{S}^{N-1})$, but $u \notin W^{s,p}_S(B^{N+1}, \mathbb{S}^{N-1})$, due to the topological singularity at the origin. However, since $u \in W^{1,q}(B^{N+1}, \mathbb{S}^{N-1})$ for each $q < N+1$, by the previous example it turns out that $\mathbb{P}(u) = 0$.

Remark 9.7 If $1 \leq N-1 < p < p(N)$, see (9.2), and $u \in W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$, arguing as before we can define the $(N-1)$-current G_u in $\mathcal{D}_{N-1}(\mathbb{R}^{N-1} \times \mathbb{S}^{N-1})$ by setting

$$G_u := (\partial G_V) \llcorner ((\mathbb{R}^{N-1} \times \{0\}) \times \mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^{N-1}(\mathbb{R}^{N-1} \times \mathbb{S}^{N-1}) .$$

Using a cut-off argument on the function V, it can be checked that in that case the definition (8.8) of degree yields to equation

$$\deg u = \frac{1}{\alpha_N} G_u(\pi_2^\# \omega_{\mathbb{S}^{N-1}}) .$$

It thus remains to give the

Proof of Proposition 8.7: We come back to the proof of Theorem 1.1, in case $\mathcal{Y} = \mathbb{S}^{N-1}$, where some improvements are in order.

We first point out that due to the "almost Dirichlet principle", see e.g. [15, Thms. 1.9 and 1.11], in definition (1.7) we can choose $U := \mathrm{Ext}(u)$ as the harmonic extension, and in the proof of Theorem 1.1 we can take $U(x,t) = U(x,-t)$ if $t < 0$. Also, recall that we have chosen a smooth function $\Phi : \mathbb{R}^N \to \mathbb{D}^N$ such that $\Phi(y) = y/|y|$ if $|y| \geq 1/2$, and Φ is a bi-Lipschitz map from $\{y \in \mathbb{R}^N : |y| \leq 1/2\}$ to $\mathbb{D}^N$. Therefore, on account of the gradient estimates (8.5), in the sequel we shall replace maps W with values into the ε_0-neighborhood $\mathbb{S}^{N-1}_{\varepsilon_0}$ of $\mathbb{S}^{N-1}$ with their projections $\Phi \circ W$ onto $\mathbb{S}^{N-1}$, where we choose $0 < \varepsilon_0 < 1/2$. These operations do not affect our previous estimates.

Moreover, we point out that if $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$, with $n \geq N+1$, and $\mathbb{P}(u) = 0$, for a.e. hyperplane Π orthogonal to a coordinate direction in $\mathbb{R}^n$, with $\Pi \cap B^n$ non-empty, it turns out that the restriction v of u to $\Pi \cap B^n$ is a map in $W^{s,p}$ satisfying $\mathbb{P}(v) = 0$. In addition, when $n = N$ and F is an N-cube in B^n, by possibly slightly moving the faces of F, it turns out that the restriction $u_{|\partial F}$ of u to the boundary of F is a map in $W^{s,p}(\partial F, \mathbb{S}^{N-1})$, hence in VMO, its degree $\deg(u_{|\partial F})$ is well-defined and equal to zero. Finally, arguing in a way similar to the proof of Proposition 8.3, it turns out that the degree is continuous w.r.t. the $W^{s,p}$-convergence of maps from ∂F into $\mathbb{S}^{N-1}$, see also [8].

Recalling that $2 \leq N \leq d \leq n$, we can thus further improve the slicing argument at the beginning of the proof of Theorem 1.1 in Sec. 4, when $\mathcal{Y} = \mathbb{S}^{N-1}$, by choosing for every $m \in \mathbb{N}^+$ the grid of size $1/m$ in such a way that the following additional properties hold true:

iii) the restriction u_F to each k-face F of the k-skeleton $C_m^{(k)}$ satisfies $\mathbb{P}(u_F) = 0$, for $k = d \vee N, \dots, n$;

iv) $\deg(u_{|\partial F}) = 0$ for each N-face F in $C_m^{(N)}$, if $d = N$.

The case $n = d$. We recall that $\{C_l\}_{l=1}^{(m-1)^d}$ is a list of the $(d+1)$-cubes in $\mathcal{F}_m$. We denote by F_l the d-cube given by the interesection of C_l with $\mathcal{Q}^n \cap \{0\}$. We also recall formula (4.4), where $V_h^{(m)}$ is given by Proposition 3.1 in correspondence to a grid satisfying i) and ii).

The assertion follows if we show that we can find a sequence $\{h_j\} \searrow 0$ such that the traces $\mathbf{T}(W_{h_j}^{(m)})$ of the approximating maps $W_{h_j}^{(m)}$ are homologically trivial in the previous sense.

Now, looking at the proof of Proposition 2.2, we observe that $\Sigma(P_0, h)$ intersects the $(d-1)$-skeleton $\Sigma_m^{(d-1)} \times \{0\}$ if $P_0 \in \Sigma_m^{(d-1)} \times]-h/2, h/2[$. Therefore, the same argument yields that the approximating sequence $\{U_h^{(m)}\}_h$ actually satisfies $U_h^{(m)}(x,t) \in \mathbb{S}_{\varepsilon_0}^{N-1}$ for every $(x,t) \in \partial F_l \times]-h/2, h/2[$ and for every l, provided that $h < h_\varepsilon$. By a left composition with Φ, we thus get $U_h^{(m)}(x,t) \in \mathbb{S}^{N-1}$. As a consequence, the approximating sequence $\{V_h^{(m)}\}_h$ satisfies $V_h^{(m)}(\partial F_l \times]-h/2, h/2[) \subset \mathbb{S}^{N-1}$ for every l, if $h < h_\varepsilon$.

Setting $v_h^{(m)} := \mathbf{T}(V_h^{(m)})$, on account of formula (4.4) it then suffices to show the existence of a sequence $\{h_j\} \searrow 0$ such that every l and j the $(d-1)$-cycle $v_{h_j\,|\partial F_l}^{(m)}$ is homologically trivial, i.e.

$$v_{h_j\,\#}^{(m)} [\![\partial F_l]\!](\omega_{\mathbb{S}^{N-1}}) = 0 \tag{9.8}$$

where $\omega_{\mathbb{S}^{N-1}}$ is given by (9.6).

If $d > N$, property (9.8) is automatically satisfied, see Example 9.5.

If $d = N$, instead, by the strong convergence of $V_h^{(m)}{}_{|\partial F_l \times [0,1]}$ to $U_l := U_{|\partial F_l \times [0,1]}$, using iv) and the continuity of the degree it turns out that $v_{h\,|\partial F_l}^{(m)}$ has zero degree, definitely on h. Therefore, a diagonal argument on $l = 1, \dots, (m-1)^n$ gives (9.8).

The case $n \geq d+1$. Firstly, when extending $W_h^{(m)}$ to the $(d+1)$-cubes of the grid, we argue as in the case $n = d$. Moreover, when extending $W_h^{(m)}$ to the $(k+1)$-cubes of the grid, for $k = d+1, \dots, n$, since the traces $\mathbf{T}(W_h^{(m)})$ take values into $\mathbb{S}^{N-1}$, it turns out that *no boundary is "produced"*, i.e., the traces $\mathbf{T}(W_h^{(m)})$ are homologically trivial in the previous sense. The validity of the latter statement can be verified as a consequence of the case $n > N$ analyzed in Example 9.5. Further details are omitted. □

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