Strong density results for manifold valued fractional Sobolev maps

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Abstract. We consider fractional Sobolev classes $W^{s,p}$ of maps defined in high dimensional domains and with values into compact smooth manifolds. The problem of strong density of smooth maps for s lower than one is discussed. An equivalent energy convergence defined through extensions in suitable weighted Sobolev spaces is exploited to obtain a new proof of the density of maps with "small" singular set. Moreover, a homotopy-type property is analyzed, yielding to a characterization of approximable maps through topological arguments. We then focus on maps taking values into high dimensional spheres, where homological tools allow to describe the singular set. For suitable values of the product sp, in fact, strong density of smooth maps is characterized by the triviality of the current of the singularities.

Keywords: fractional Sobolev spaces; weighted Sobolev spaces; maps between manifolds; singularities.

AMS classification codes: 49Q20; 46E35; 28A75; 58D15.

1 Introduction

In this paper we deal with strong density of smooth functions in fractional Sobolev classes of maps $u : B^n \to \mathcal{Y}$ defined in the unit ball B^n of $\mathbb{R}^n$ and taking values into a manifold $\mathcal{Y}$.

We shall always assume $0 < s < 1$ and $p > 1$ real. In this case, the space $W^{s,p}(B^n)$ is given by the L^p functions $u : B^n \to \mathbb{R}$ with finite fractional Gagliardo semi-norm $|u|_{s,p}$, where

$$|u|_{s,p}^p := \int_{B^n}\int_{B^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy\,.$$

It is a Banach space when equipped with the norm $\|u\|_{s,p} := \|u\|_{L^p} + |u|_{s,p}$. A vector valued function $u : B^n \to \mathbb{R}^N$ belongs to the space $W^{s,p}(B^n, \mathbb{R}^N)$ if each component of u is in $W^{s,p}(B^n)$.

Let $\mathcal{Y}$ be a smooth, connected, compact Riemannian manifold without boundary, isometrically embedded into $\mathbb{R}^N$. We equip $\mathcal{Y}$ with the metric induced by the Euclidean norm on $\mathbb{R}^N$, and deal with the class:

$$W^{s,p}(B^n, \mathcal{Y}) := \{u \in W^{s,p}(B^n, \mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for a.e.} \quad x \in B^n\}\,.$$

For $1 < p < \infty$ and $0 < s < 1$, we correspondingly define

$$W_S^{s,p}(B^n, \mathcal{Y}) := \{u \in W^{s,p}(B^n, \mathcal{Y}) \mid \begin{array}{l}\text{there exists } \{u_h\} \subset C^\infty(B^n, \mathcal{Y}) \\ \text{such that } u_h \to u \text{ strongly in } W^{s,p}\}\,.\end{array} \tag{1.1}$$

Moreover, throughout the paper we shall always denote

$$d := [sp] + 1$$

$[q]$ being the integer part of $q \in \mathbb{R}$.

Density results. By the continuous embedding of $W^{s,p}$ in the class VMO when $sp \geq n$, see [9], and by Corollary 3.1 from [8] in the case of low dimension $n = 1$, it follows that

$$W_S^{s,p}(B^n, \mathcal{Y}) = W^{s,p}(B^n, \mathcal{Y}) \qquad \text{if } n = 1 \quad \text{or} \quad sp \geq n\,. \tag{1.2}$$

For $n \geq d \geq 1$, instead, following an idea by Bethuel [2], it turns out that a strongly dense class is given by maps that are smooth outside a singular set of codimension d. More precisely, we respectively denote by $R^\infty_{s,p}(B^n,\mathcal{Y})$ and $R^0_{s,p}(B^n,\mathcal{Y})$ the set of all maps $u\in W^{s,p}(B^n,\mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u)=\bigcup_{i=1}^r \Sigma_i\,,\qquad r\in\mathbb{N} \tag{1.3}$$

where Σ_i is a smooth $(n-d)$-dimensional subset of B^n with smooth boundary, if $n\geq d+1$, and Σ_i is a point, if $n=d$. The following density result was proved in [8].

Theorem 1.1 *For every $n\geq 2$, $0<s<1$, and $1<p<\infty$ such that $sp<n$, the class $R^\infty_{s,p}(B^n,\mathcal{Y})$ is dense in $W^{s,p}(B^n,\mathcal{Y})$.*

In Sec. 2, we shall give a different proof of Theorem 1.1, this way answering to the question posed in [8, Rmk. 1.5]. Our proof is the starting point for obtaining the new results of this paper, and it relies on arguments from [3, 12, 20] and on the following strategy, that goes back to [22].

Roughly speaking, when $0<s<1$, strong convergence in $W^{s,p}$ can be reformulated in terms of energy convergence of extensions in suitable weighted Sobolev spaces, as described below. It allows us to overcome the difficulty of working with the non local $W^{s,p}$ semi-norm, as we did in the case of trace spaces analyzed in [20]. For the sake of brevity, in Sec. 2 we only give some hints of our proof of Theorem 1.1, outlining the main differences from the one given in [20] for the particular case of maps in $W^{1-1/p,p}$.

THE ENERGY. For $\gamma\in\mathbb{R}$ and $p>1$, denote by $W^{1,p}_\gamma(B^n\times(0,+\infty))$ the weighted Sobolev space given by the functions $U\in L^p((B^n\times(0,+\infty))$ whose distributional derivative DU is a measurable function satisfying

$$\int_{\Omega\times(0,+\infty)} t^\gamma|DU(x,t)|^p\,dx\,dt<\infty\,,\quad \Omega=B^n\,. \tag{1.4}$$

By interpolation theory, see e.g. [18], it turns out that when $0<s<1$, the fractional Sobolev space $W^{s,p}(B^n)$ agrees with the Besov space $B^s_{p,p}(B^n)$, for any $p>1$, and hence with the class of traces $u(x)=U(x,0)$ on $t=0$ of functions U in $W^{1,p}_\gamma(B^n\times(0,+\infty))$, say $\mathbf{T}(U)=u$, where

$$\gamma=\gamma(s,p):=p(1-s)-1\,,\quad p>1\,,\quad 0<s<1\,. \tag{1.5}$$

Notice that in case $s=1-1/p$, then $d=[p]$, $\gamma(1-1/p,p)=0$, and $W^{1-1/p,p}(B^n)$ is the trace space of $W^{1,p}(B^n\times(0,+\infty))$.

A particular case of interest is when $sp\in\mathbb{N}^+$, so that $d=sp+1\geq 2$ and $\gamma=p-d$. In that case, condition $0<s<1$ yields $p>d-1$, the fractional Gagliardo semi-norm becomes

$$|u|^p_{(d-1)/p,p}=\int_{B^n}\int_{B^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+d-1}}\,dx\,dy$$

and in low dimension $n=d-1$, when $\Omega=\mathbb{R}^{d-1}$, the energy (1.4) is scale invariant for any $p>d-1$.

Denote now by $\mathbf{C}^{n+1}$ the $(n+1)$-dimensional cylinder

$$\mathbf{C}^{n+1}:=B^n\times(0,1)$$

by $W^{1,p}_\gamma(\mathbf{C}^{n+1},\mathbb{R}^N)$ the vector valued functions $U:\mathbf{C}^{n+1}\to\mathbb{R}^N$ with components in $W^{1,p}_\gamma(\mathbf{C}^{n+1})$, and consider for $0<s<1$ and $p>1$ the *energy*

$$\mathcal{E}^p_{\gamma(s,p)}(U):=\int_{\mathbf{C}^{n+1}} t^{\gamma(s,p)}|DU(x,t)|^p\,dx\,dt\,,\qquad \gamma(s,p):=p(1-s)-1\,. \tag{1.6}$$

Let $u\in W^{s,p}(B^n,\mathcal{Y})$. Since $\mathcal{Y}$ is compact, we have $u\in L^\infty(B^n,\mathbb{R}^N)$. We thus denote by

$$U:=\operatorname{Ext}(u) \tag{1.7}$$

a bounded function that *minimizes the energy $\mathcal{E}^p_{\gamma(s,p)}(U)$ among all $U\in W^{1,p}_{\gamma(s,p)}(\mathbf{C}^{n+1},\mathbb{R}^N)\cap L^\infty$ such that $U(x,0)=u(x)$ on $B^n\times\{0\}$ in the sense of traces*, i.e., $\mathbf{T}(U)=u$.

Such a minimizer exists and is smooth in the interior of $\mathbf{C}^{n+1}$, by convexity of the functional $U \mapsto \mathcal{E}^p_{\gamma(s,p)}(U)$. In addition, see [18], if $\{u_h\} \subset W^{s,p}(B^n,\mathcal{Y})$ is a sequence converging a.e. in B^n to $u \in W^{s,p}(B^n,\mathcal{Y})$, it turns out that *strong convergence* $u_h \to u$ *in* $W^{s,p}(B^n,\mathbb{R}^N)$ *is equivalent to convergence* $u_h \to u$ *in* $L^p(B^n,\mathbb{R}^N)$ *joined with the energy convergence*

$$\lim_{h\to\infty} \mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u_h)) = \mathcal{E}^p_{\gamma(s,p)}(\mathrm{Ext}(u))\,. \tag{1.8}$$

In conclusion, coming back to definition (1.1), if $u \in W^{s,p}(B^n,\mathcal{Y})$ for some $0 < s < 1$ and $p > 1$, then $u \in W^{s,p}_S(B^n,\mathcal{Y})$ if and only if we can find a sequence $\{u_h\} \subset C^\infty(B^n,\mathcal{Y})$ strongly converging to u in $L^p(B^n,\mathbb{R}^N)$ and such that the energy convergence (1.8) holds.

Topological obstruction. For $j \geq 1$ integer, denote by $\pi_j(\mathcal{Y})$ the j-th free homotopy group of the target manifold $\mathcal{Y}$. Let $d := [sp]+1 \geq 2$ and assume that $\pi_{d-1}(\mathcal{Y}) \neq 0$. Then, it is well-known that if $n \geq d$, the strict inclusion

$$W^{s,p}_S(B^n,\mathcal{Y}) \subsetneq W^{s,p}(B^n,\mathcal{Y})$$

holds. More precisely, there exist maps $u \in W^{s,p}(B^n,\mathcal{Y})$, actually in the class $R^\infty_{s,p}(B^n,\mathcal{Y})$, which cannot be approximated strongly in $W^{s,p}$ by sequences of smooth maps in $W^{s,p}(B^n,\mathcal{Y})$.

Example 1.2 Let $n \geq d := [sp]+1 \geq 2$, and denote $x = (\widetilde{x},\widehat{x}) \in \mathbb{R}^d \times \mathbb{R}^{n-d} \simeq \mathbb{R}^n$. Assume that $\pi_{d-1}(\mathcal{Y}) \neq 0$, and let $u : B^n \to \mathcal{Y}$ the 0-homogeneous map

$$u(x) := \varphi\Big(\frac{\widetilde{x}}{|\widetilde{x}|}\Big)\,, \qquad x = (\widetilde{x},\widehat{x}) \in B^n \setminus \Sigma(u)$$

where $\varphi : \mathbb{S}^{d-1} \to \mathcal{Y}$ is a homotopically non-trivial smooth map and

$$\Sigma(u) := \{(\widetilde{x},\widehat{x}) \in \mathbb{R}^n \,:\, \widetilde{x} = 0\}\,.$$

Then $u \in R^\infty_{s,p}(B^n,\mathcal{Y})$, but $u \notin W^{s,p}_S(B^n,\mathcal{Y})$, see e.g. Theorem 1.5 below.

The non-triviality of the $(d-1)$-th homotopy group of the target manifold is the only obstruction to strong density of smooth maps, at least in case of standard domains. In fact, the following theorem was proved in [8].

Theorem 1.3 *Let* $n \geq d := [sp]+1 \geq 2$. *Then* $W^{s,p}_S(B^n,\mathcal{Y}) = W^{s,p}(B^n,\mathcal{Y})$ *if and only if* $\pi_{d-1}(\mathcal{Y}) = 0$.

In case of $(d-1)$-connected target manifolds $\mathcal{Y}$, when $s \geq 1$ and $sp < n$, strong density of smooth maps in $W^{s,p}(B^n,\mathcal{Y})$ was proved in [6], where the authors also obtained weak sequential density of smooth maps in the case sp integer. Moreover, when $0 < s < 1$, equality $W^{s,p}_S = W^{s,p}$ for $(d-1)$-connected target manifolds was obtained in [7]. A different proof of Theorem 1.3 when $n = d$ is given in Sec. 3, as it reduces to remove point singularities, on account of Theorem 1.1. In high dimension $n > d$, the non-trivial implication in Theorem 1.3 is more involved. It is readily obtained by our characterization of strongly approximable $R^0_{s,p}$-maps, Theorem 1.5, the proof of which makes use of a $(d-1)$-*homotopy type* property of maps in $R^0_{s,p}(B^n,\mathcal{Y})$, Theorem 1.4.

Homotopy type property. Let X^k denote the k-skeleton of some finite cubeulation X of B^n. If $u \in W^{s,p}(B^n,\mathcal{Y})$, possibly slightly moving the faces of X we may assume that the restriction of u to F belongs to $W^{s,p}(F,\mathcal{Y})$ for every k-face F of X^k, where $k = d-1,\dots,n$. In this case, we say that X is in *generic position* with respect to u. Moreover, if $u \in R^0_{s,p}(B^n,\mathcal{Y})$, and $\Sigma(u)$ is the $(n-d)$-dimensional singular set of u, compare (1.3), we say that X is in *dual position* with respect to u if it is in generic position and in addition $X^{d-1} \cap \Sigma(u) = \emptyset$. Possibly slightly moving the faces of X^{d-1}, it turns out that the cubeulation X is in dual position with respect to u.

Using arguments from [3], that go back to [24], in Sec. 3 we prove:

Theorem 1.4 *Let* $n > d := [sp]+1 \geq 2$. *Let* $u_\infty \in R^0_{s,p}(B^n,\mathcal{Y})$ *and* X *a finite cubeulation of* B^n *in dual position with respect to* u_∞. *Let* $\{u_i\} \subset W^{s,p}(X^{d-1},\mathcal{Y}) \cap C^\infty$ *be a smooth sequence strongly converging in* $W^{s,p}$ *to the restriction* $u_{\infty|X^{d-1}}$ *of* u_∞ *to* X^{d-1}. *Then, we find* $k_0 \in \mathbb{N}^+$ *such that for every* $i \geq k_0$ *the maps* u_i *and* $u_{\infty|X^{d-1}}$ *are homotopic as maps from* X^{d-1} *to* $\mathcal{Y}$.

Actually, Theorem 1.4 follows from results contained in [9], since strong convergence in $W^{s,p}(X^{d-1},\mathcal{Y})$ implies convergence in VMO, whereas homotopy in VMO between continuous maps is equivalent to standard homotopy.

A CHARACTERIZATION. Using Theorem 1.4, in Sec. 3 we provide a characterization of strongly approximable $R^0_{s,p}$-maps:

Theorem 1.5 *Let $n\geq d:=[sp]+1\geq 1$. Let $u\in R^0_{s,p}(B^n,\mathcal{Y})$ and X a finite cubeulation of B^n in dual position with respect to u. Then, u belongs to $W^{s,p}_S(B^n,\mathcal{Y})$, i.e., u is the strong $W^{s,p}$ limit of a smooth sequence in $C^\infty(B^n,\mathcal{Y})$, if and only if the restriction $u_{|X^{d-1}}$ of u to X^{d-1} can be extended to a continuous map from B^n into $\mathcal{Y}$.*

On account of the density theorem 1.1, Theorem 1.5 gives the non-trivial implication of Theorem 1.3, in any dimension $n\geq d$. Moreover, when $d=1$ one recovers the following well-known fact:

Corollary 1.6 *Let $0<s<1$ and $p>1$ such that $0<sp<1$. Then $W^{s,p}_S(B^n,\mathcal{Y})=W^{s,p}(B^n,\mathcal{Y})$ for any $n\geq 1$.*

As before, in Sec. 3 we only give some hints of the proof of our previous results, outlining the main differences from the ones given in [21] for the particular case of maps in $W^{1-1/p,p}$.

MORE GENERAL DOMAINS. The case of non-trivial domains, e.g. of maps $u:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}=\partial\mathcal{M}$ for some smooth, connected, and compact Riemannian $(n+1)$-manifold $\mathcal{M}$, can be treated in a similar way. In particular, Theorems 1.1, 1.4, and 1.5 continue to hold, with the obvious modifications. The same feature holds concerning Theorem 1.3 in low dimension $n=d$, since its proof relies on a local argument, too. When $n>d$, one has to follow some ideas due to Hang-Lin [15].

We thus recall that $\mathcal{X}$ is said to satisfy the *k-extension property with respect to $\mathcal{Y}$,* where $k\in\mathbb{N}$, if for any given CW-complex X on $\mathcal{X}$, denoting by X^k its k-dimensional skeleton, any continuous map $f:X^{k+1}\to\mathcal{Y}$ is such that its restriction to X^k can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. Arguing exactly as e.g. in the proof of [20, Thm. 4], and using Theorem 1.5, one readily obtains the following result:

Theorem 1.7 *If $n>d:=[sp]+1\geq 2$, smooth maps in $C^\infty(\mathcal{X},\mathcal{Y})$ are sequentially dense in $W^{s,p}(\mathcal{X},\mathcal{Y})$ if and only if $\pi_{d-1}(\mathcal{Y})=0$ and $\mathcal{X}$ satisfies the $(d-1)$-extension property with respect to $\mathcal{Y}$.*

If $d=1$, then again $W^{s,p}_S(\mathcal{X},\mathcal{Y})=W^{s,p}(\mathcal{X},\mathcal{Y})$ for any $n\geq 1$. Moreover, if n and d are as in Theorem 1.7 we deduce:

Corollary 1.8 *If $\pi_j(\mathcal{Y})=0$ for every integer $j=d-1,\dots,n-1$, then $W^{s,p}_S(\mathcal{X},\mathcal{Y})=W^{s,p}(\mathcal{X},\mathcal{Y})$.*

Finally, using arguments from [24, Sec. 6], following [20, Cor. 2] we also obtain:

Corollary 1.9 *Let $k\in\{1,\dots,d-1\}$. If $\pi_i(\mathcal{X})=0$ for every $i=0,\dots,k-1$ and $\pi_j(\mathcal{Y})=0$ for every $j=k,\dots,d-1$, then $W^{s,p}_S(\mathcal{X},\mathcal{Y})=W^{s,p}(\mathcal{X},\mathcal{Y})$.*

For the sake of brevity, we omit the proof of the latter results, since it suffices to argue as in the case $s=1-1/p$ from [20].

THE MODEL CASE. In Secs. 4–6, we shall restrict to the model case $\mathcal{Y}=\mathbb{S}^{N-1}$, where $N\geq 2$ and

$$\mathbb{S}^{N-1}:=\{y\in\mathbb{R}^N\ :|y|=1\}$$

is the unit $(N-1)$-sphere. Recalling the notation (1.1), since $\mathbb{S}^{N-1}$ is $(N-2)$-connected, using (1.2) and Theorem 1.3, it turns out that for any couple of exponents $0<s<1$ and $p>1$ the strict inclusion

$$W^{s,p}_S(B^n,\mathbb{S}^{N-1})\subsetneq W^{s,p}(B^n,\mathbb{S}^{N-1})$$

holds provided that $N-1\leq sp<n$, a condition we shall assume from now on, whence $N\leq d\leq n$.

Notice in fact that a part from the case $N=2$, when $N\geq 3$ the high order homotopy groups $\pi_j(\mathbb{S}^{N-1})$, for $j\geq N$, fail to be trivial, in general. Therefore, only in the case $N=2$ one can restrict to the ranges

$1 \le sp < 2$, i.e., when $n \ge d = 2$. In particular, if $n > N \ge 3$, since $\pi_N(\mathbb{S}^{N-1}) \neq 0$, then B^n fails to satisfy the N-extension property with respect to $\mathbb{S}^{N-1}$.

Using some relevant estimates obtained by Bourgain-Brezis-Mironescu in [4], for each map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $n \ge N-1 \ge 1$ and $p > N-1$, we are able to find a smooth extension $V \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{R}^N)$ of u, so that $\mathbf{T}(V) = u$, satisfying:

$$\int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt \le C \int_{\mathbf{C}^{n+1}} t^{p-N}\, |DU(x,t)|^p\, dx\, dt \tag{1.9}$$

where $U = \mathrm{Ext}(u)$, for some absolute constant $C > 0$, see Theorem 4.1.

As a consequence of our estimate (1.9), in Sec. 5 a good notion of *degree* is analyzed in low dimension $n = N-1$, see (5.1).

Moreover, when $n \ge N \ge 2$, $0 < s < 1$, $p > 1$, and $sp \ge N-1$, for each map $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ we are able to construct an $(n-N)$-dimensional *current* $\mathbb{P}(u)$ in B^n that carries the information on the *homological singularities* of u. More precisely, following ideas from [14, 13, 22], for any compactly supported smooth $(n-N)$-form ϕ in B^n we define

$$\mathbb{P}(u)(\phi) := \frac{1}{\alpha_N} \int_{\mathbf{C}^{n+1}} d\,(\eta \wedge \phi) \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N)$$

where α_N is the measure of the unit ball $\mathbb{D}^N$ of dimension N, and $\eta : [0,1] \to [0,1]$ is a suitable cut-off function, with $\eta(0) = 1$. With this notation, in fact, in Proposition 5.5 we obtain

$$u \in W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,. \tag{1.10}$$

Actually, for $n = N$ and $sp = N-1$ the construction of $\mathbb{P}(u)$ is inspired from [4]. Moreover, similar arguments to the previous ones appear in [5].

For some ranges of sp, the converse implication holds in (1.10), whence the current $\mathbb{P}(u)$ retains all the information on the relevant singularities of u. More precisely, if $n > [sp] = N-1$, when $N \ge 3$, or $n > [sp] \ge 1$, when $N = 2$, in Theorem 5.6 we shall prove the following characterization:

$$W^{s,p}_S(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}\,.$$

However, when $n > [sp] \ge N \ge 3$, *topological singularities* that cannot be seen by pure homological arguments come into play. In fact, differently to the case $N = 2$, where high order homotopy groups $\pi_j(\mathbb{S}^1)$, $j \ge 2$, are all trivial, when $N \ge 3$ we have $\pi_N(\mathbb{S}^{N-1}) \neq 0$, i.e., there exist homotopically non-trivial smooth maps $\varphi : \mathbb{S}^N \to \mathbb{S}^{N-1}$. When e.g. $N = 3$, we may choose $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$ equal to the Hopf fibration, a generator of the third homotopy group of the 2-sphere, $\pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$.

Example 1.10 Let $N \ge 3$ and $u : B^{N+1} \to \mathbb{S}^{N-1}$ given by

$$u(x) = \varphi\Big(\frac{x}{|x|}\Big)\,, \qquad x \in B^{N+1} \setminus \{0\} \tag{1.11}$$

where $\varphi : \mathbb{S}^N \to \mathbb{S}^{N-1}$ is a homotopically non-trivial smooth map.

Since $u \in W^{1,q}(B^{N+1}, \mathbb{S}^{N-1})$ for each $q < N+1$, and $u \in L^\infty$, by the Gagliardo-Nirenberg inequality $u \in W^{s,p}$ for each $0 < s < 1$ and $p > 1$ such that $N \le sp < N+1$, whence we choose $d = [sp]+1 = N+1$. Moreover, $u \in R^\infty_{s,p}(B^{N+1}, \mathbb{S}^{N-1})$, as u is smooth outside the origin. Now, if X is a cubeulation of B^{N+1} such that $0 \notin X^N$, the restriction $u_{|X^N}$ cannot be extended to a continuous map from B^{N+1} into $\mathbb{S}^{N-1}$, by the non-triviality of the map φ. Therefore, Theorem 1.5, in case $d = N+1$, yields that $u \notin W^{s,p}_S(B^{N+1}, \mathbb{S}^{N-1})$ if $N \le sp < N+1$, i.e., one cannot find a smooth sequence $\{u_h\} \subset C^\infty(B^{N+1}, \mathbb{S}^{N-1})$ strongly converging to u in $W^{s,p}$. However, it can be checked that $\mathbb{P}(u) = 0$, see Example 5.1.

When $sp = N = 3$, and in the limiting case $s = 1$, topological connections of the singularities of maps in $W^{1,3}(B^4, \mathbb{S}^2)$ were firstly analyzed in [16] through new geometric tools called "bubbled scans".

In Sec. 6, we finally point out some other consequences of our estimate (1.9), outlining for $N \ge 3$ a striking difference from the case $N = 2$ already analyzed in [22]. Namely, if $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$

for some $p > N-1$, in general $\mathrm{Ext}(u) \in W^{1,N-1}(\mathbf{C}^{n+1}, \mathbb{R}^N)$ provided that $p < (N-1)^2/(N-2)$, see Proposition 6.2.

Therefore, for these reasons, and for other ones that we will not dwell on, the analysis of $W^{s,p}$ weak sequential density of smooth maps $C^\infty(B^n, \mathbb{S}^{N-1})$, when $N \geq 3$, appears much more difficult from the case $N = 2$ we tackled in [22], at least for large values of the exponent p or of the product sp.

2 The density theorem

In this section, we sketch the proof Theorem 1.1, showing how it can be obtained from the corresponding one from [20] for the case of trace spaces, i.e., when $\gamma(s,p) = 0$. We thus divide the proof into three steps, and we refer to the notation from [20, Sec. 2].

We thus let $u \in W^{s,p}(\mathcal{Q}^n, \mathcal{Y})$, where $\mathcal{Q}^n :=]0,1[^n$, and $U \in W^{1,p}_\gamma(\mathcal{Q}^n \times I, \mathbb{R}^N)$, with $\gamma = \gamma(s,p) = p(1-s) - 1$ and $I :=]-1,1[$, denote the extension $\mathrm{Ext}(u)$ of u, so that $\mathbf{T}(U) = u$. Then, $U \in L^p(\mathcal{Q}^n \times I, \mathbb{R}^N)$, the distributional derivative DU is measurable function in $\mathcal{Q}^n \times I$, and U has finite energy:

$$\mathcal{E}^p_\gamma(U) := \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) = \int_{\mathcal{Q}^n \times I} |t|^\gamma |DU(x,t)|^p \, dx \, dt < \infty \, .$$

If A is a "smooth" $\mathcal{H}^k$-measurable k-dimensional subset of $\mathcal{Q}^n \times I$, where $\mathcal{H}^k$ is the Hausdorff measure, we also let

$$\mathcal{E}^p_\gamma(U, A) := \int_A |t|^\gamma |DU_{|A}|^p \, d\mathcal{H}^k \, .$$

Moreover, we denote

$$U^{(m)} := U_{|C^{(d-1)}_m \times I} \tag{2.1}$$

the restriction of U to the d-skeleton $C^{(d-1)}_m \times I$, where $d = [sp] + 1$.

Step 1: We first make use of the argument of [3, Sec. 2.1], that goes back to [23], and prove the following

Proposition 2.1 *Let n, s, and p as in Theorem 1.1. Assume that $U^{(m)} \in W^{1,p}_\gamma(C^{(d-1)}_m \times I, \mathbb{R}^N)$, where $\gamma = p(1-s) - 1$ and $d = [sp] + 1$. Then, there exists a sequence of continuous maps $\{U^{(m)}_h\}_h$ in $W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I, \mathbb{R}^N)$ such that $U^{(m)}_h \to U^{(m)}$ strongly in $W^{1,p}_\gamma(\Sigma^{(d-1)}_m \times I, \mathbb{R}^N)$ and the traces $\mathbf{T}(U^{(m)}_h) \in W^{s,p}(\Sigma^{(d-1)}_m, \mathcal{Y}_{\varepsilon_0})$ for every h.*

Proof: If $0 < sp < 1$, since $d = 1$, we take $U^{(m)}_h = U^{(m)}$, see (2.1). If $d \in \{2, \dots, n\}$, we follow the proof of [20, Prop. 2], where we denoted $W^{1/p} := W^{1-1/p,p}$ and we had $s = 1 - 1/p$, whence $\gamma = 0$ and $d = [sp] + 1 = [p]$. The main difference relies on the embedding of $W^{1,p}_\gamma$ into a suitable Hölder class, in low dimension $d - 1$, see (2.4).

More precisely, if $u \in W^{s,p}(B^n)$, then $U := \mathrm{Ext}(u) \in W^{s+1/p,p}(\mathbf{C}^{n+1})$, and for "almost all" $(d-1)$-dimensional disks $D = D^{d-1}$ contained in $\mathbf{C}^{n+1}$, the restriction $U_{|D}$ of U to D belongs to $W^{s+1/p,p}(D)$. Recalling that $d = [sp] + 1$, condition $0 < s < 1$ yields $p > d - 1$ and hence $U_{|D} \in C^{0,\alpha}(D)$ with Hölder exponent

$$\alpha = s + \frac{1}{p} - \frac{d-1}{p} = \frac{2 + sp - d}{p} = \frac{1 + sp - [sp]}{p} \, . \tag{2.2}$$

Setting for $z = (x,t) \in \Sigma^{(d-1)}_m \times I$ and $0 < h < 1/(4m)$

$$\begin{aligned} C(z,h) &:= \overline{B}^n(x, h/2) \times [t - h/2, t + h/2] \\ \Sigma(z,h) &:= C(z,h) \cap (C^{(d-1)}_m \times I) \end{aligned}$$

we let

$$U^{(m)}_h(z) := \frac{1}{\mathcal{H}^d\big(\Sigma(z,h)\big)} \int_{\Sigma(z,h)} U^{(m)}(w) \, d\mathcal{H}^d(w)$$

so that $U_h^{(m)} \in W^{1,p}_\gamma(\Sigma_m^{(d-1)} \times I, \mathbb{R}^N)$ is continuous, and $U_h^{(m)} \to U^{(m)}$ strongly in $W^{1,p}_\gamma$ as $h \to 0^+$. It thus remains to show that if $u_h^{(m)} := \mathbf{T}(U_h^{(m)})$, possibly passing to a subsequence $u_h^{(m)}(\Sigma_m^{(d-1)}) \subset \mathcal{Y}_{\varepsilon_0}$ for every h.

By using the energy $\mathcal{E}^p_\gamma$ instead of $\mathcal{E}_p$, we get to the estimate:

$$\begin{aligned} \mathcal{E}^p_\gamma(U^{(m)}, R_1 \cap P(h_1,1)) &\leq \frac{2}{h} \mathcal{E}^p_\gamma(U^{(m)}, R_1) \\ &\leq \frac{2}{h} \int_{\Sigma(P_0,h)} |t|^\gamma |DU^{(m)}(w)|^p \, d\mathcal{H}^d(w) \leq 2\, \frac{\varepsilon}{h}\,. \end{aligned} \tag{2.3}$$

We then choose $z_0 \in R_1 \cap P(h_1,1) \cap (\Sigma_m^{(d-1)} \times \{0\})$ and set $y_h^{(m)} := U^{(m)}(z_0)$, so that $y_h^{(m)} \in \mathcal{Y}$. Due to the embedding of $W^{1,p}_\gamma$ into $C^{0,\alpha}$, where $\gamma = p(1-s)-1$ and α is given by (2.2), since $R_1 \cap P(h_1,1)$ is a $(d-1)$-cube of side h, it follows that

$$\max_{z \in R_1 \cap P(h_1,1)} |U^{(m)}(z) - y_h^{(m)}|^p \leq c \cdot h^{1+sp-[sp]} \cdot \mathcal{E}^p_\gamma(U^{(m)}, R_1 \cap P(h_1,1))$$

for some positive real constant $c = c(n,s,p)$, and hence by (2.3)

$$\max_{z \in R_1 \cap P(h_1,1)} |U^{(m)}(z) - y_h^{(m)}| \leq c\, \varepsilon^{1/p}\,. \tag{2.4}$$

Given $\eta > 0$ small, we slice the d-dimensional set $\Sigma(P_0,h)$ with hyperplanes orthogonal to the "vertical" direction e_{n+1}, and denote

$$\Omega_{h'} := \Sigma(P_0,h) \cap P(h', n+1)\,, \qquad h' \in [-h/2, h/2]\,.$$

Setting

$$A_h := \{h' \in [-h/2,h/2] \,:\, \mathcal{E}^p_\gamma(U^{(m)}, \Omega_{h'}) \leq \varepsilon\,\eta/h\}$$

and $B_h := [-h/2,h/2] \setminus A_h$, we again have $\mathcal{L}^1(B_h) \leq h/\eta$. Using that $h^{(sp-[sp])/p} \leq 1$, by the embedding theorem this time we obtain that for every $h' \in A_h$

$$\max_{z,y \in \Omega_{h'}} |U^{(m)}(z) - U^{(m)}(y)| \leq c\,\eta^{1/p}\,\varepsilon^{1/p}\,, \quad c = c(n,s,p)\,.$$

Since $\Omega_{h'}$ intersects $R_1 \cap P(h_1,1)$ for every h', combining with (2.4) we obtain

$$\max_{w \in \Omega_{h'}} |U^{(m)}(w) - y_h^{(m)}| \leq c\,(\eta^{1/p}+1)\,\varepsilon^{1/p} \qquad \forall\, h' \in A_h\,.$$

We then conclude as in [20, Prop. 2] □

Step 2: We now suitably modify the extension U in such a way that it agrees with $U_h^{(m)}$ on the d-skeleton $C_m^{(d-1)} \times I$.

Proposition 2.2 *Let n, s, and p as in Theorem 1.1. Assume that $U^{(m)} \in W^{1,p}_\gamma(C_m^{(d-1)} \times I, \mathbb{R}^N)$, see (2.1). Then there exists a sequence of maps $\{V_h^{(m)}\}_h$ in $W^{1,p}_\gamma(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$, continuous out of $\mathcal{Q}^n_m \times \{0\}$, such that $V_h^{(m)} \to U_{|\mathcal{Q}^n_m \times I}$ strongly in $W^{1,p}_\gamma(\mathcal{Q}^n_m \times I, \mathbb{R}^N)$, with ${V_h^{(m)}}_{|\Sigma_m^{(d-1)} \times I} = U_h^{(m)}$, see Proposition 2.1. In particular we have*

$$\mathbf{T}(V_h^{(m)})_{|\Sigma_m^{(d-1)}} \in W^{s,p}(\Sigma_m^{(d-1)}, \mathcal{Y}_{\varepsilon_0}) \qquad \forall\, h\,.$$

Proof: We follow the proof of [20, Prop. 3], and we only sketch the case $n = d$, whence $[sp] \geq 1$. In fact, the case $n \geq d+1$ is obtained exactly as in the cited proposition, but working with the space $W^{1,p}_\gamma$.

When $n = d$, this time we get the estimates

$$\int_{\{\rho(x) \leq (1-\varepsilon)/(2m)\} \times I} |t|^\gamma |DV_h^{(Q)}|^p \, dx\, dt \leq (1-\varepsilon)^{n-p}\, \mathcal{E}^p_\gamma(U, Q \times I)$$

and

$$\int_{\{(1-\varepsilon)/(2m)\le\rho(x)\le 1/(2m)\}\times I} |t|^\gamma |DV_h^{(Q)}|^p\,dx\,dt \le$$
$$c\,(m,p,s)\,\frac{1}{\varepsilon^{p-1}}\int_{\partial Q\times I} |t|^\gamma |U-U_h^{(m)}|^p\,d\mathcal{H}^n$$
$$+\,c\,(m,p,s)\,\varepsilon\int_{\partial Q\times I} |t|^\gamma\big(|D_\tau U|^p+|D_\tau U_h^{(m)}|^p\big)\,d\mathcal{H}^n$$

where τ is an orthonormal frame to $\Sigma_m^{(n-1)}\times I$ and $c\,(m,p,s)>0$ only depends on m, p, and s.

Therefore, the sequence $\{V_h^{(m)}\}_h$ belongs to $W^{1,p}_\gamma(\mathcal{Q}^n_m\times I,\mathbb{R}^N)$, is continuous out of $\mathcal{Q}^n_m\times\{0\}$, and

$$\begin{aligned}\mathcal{E}^p_\gamma(V_h^{(m)},\mathcal{Q}^n_m\times I) &\le (1-\varepsilon)^{n-p}\,\mathcal{E}^p_\gamma(U,\mathcal{Q}^n_m\times I)\\ &\quad + c_1\,(m,p,s)\,\frac{1}{\varepsilon^{p-1}}\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|U^{(m)}-U_h^{(m)}|^p\,d\mathcal{H}^n\\ &\quad + c_2\,(m,p,s)\,\varepsilon\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma\big(|D_\tau U^{(m)}|^p+|D_\tau U_h^{(m)}|^p\big)\,d\mathcal{H}^n\,.\end{aligned}$$

Since moreover $U_h^{(m)}\to U^{(m)}$ strongly in $W^{1,p}_\gamma(\Sigma_m^{(n-1)}\times I,\mathbb{R}^N)$, see Proposition 2.1, there exists $\overline{h}\in\mathbb{N}$ such that for every $h\ge\overline{h}$

$$\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|D_\tau U_h^{(m)}|^p\,d\mathcal{H}^n\le 2\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|D_\tau U^{(m)}|^p\,d\mathcal{H}^n\,.$$

Now, for every $j\in\mathbb{N}^+$ we first choose $\varepsilon=\varepsilon_j\in(0,1/2)$ small so that $\varepsilon_j\searrow 0$,

$$(1-\varepsilon_j)^{n-p}\,\mathcal{E}^p_\gamma(U,\mathcal{Q}^n_m\times I)\le\mathcal{E}^p_\gamma(U,\mathcal{Q}^n_m\times I)+\frac{1}{j}$$

and

$$3\,c_2\,(m,p,s)\,\varepsilon_j\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|D_\tau U^{(m)}|^p\,d\mathcal{H}^n\le\frac{1}{j}\,.$$

Secondly, by the strong convergence of $U_h^{(m)}$ to $U^{(m)}$ in $W^{1,p}_\gamma(\Sigma_m^{(n-1)}\times I,\mathbb{R}^N)$, Hardy's inequality (see e.g. [18]) yields that

$$\lim_{h\to\infty}\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|U^{(m)}-U_h^{(m)}|^p\,d\mathcal{H}^n=0$$

and hence we can take $h=h_j\ge\overline{h}$ large so that $h_{j+1}>h_j$ and

$$c_1\,(m,p,s)\,\frac{1}{\varepsilon_j^{p-1}}\int_{\Sigma_m^{(n-1)}\times I}|t|^\gamma|U^{(m)}-U_{h_j}^{(m)}|^p\,d\mathcal{H}^n\le\frac{1}{j}\qquad\forall\,j\,.$$

Finally, since by the previous estimates

$$\mathcal{E}^p_\gamma(V_{h_j}^{(m)},\mathcal{Q}^n_m\times I)\le\mathcal{E}^p_\gamma(U,\mathcal{Q}^n_m\times I)+\frac{3}{j}$$

we relabel $\{V_j^{(m)}\}$ the subsequence $\{V_{h_j}^{(m)}\}$, where $\varepsilon=\varepsilon_j$ in the definition of $V_h^{(Q)}$. Using again the strong convergence of $U_h^{(m)}$ to $U^{(m)}$ in $W^{1,p}_\gamma(\Sigma_m^{(n-1)}\times I,\mathbb{R}^N)$, we obtain the strong L^p-convergence of $V_j^{(m)}$ to U and hence the assertion, by uniform convexity. □

Step 3: Finally, the proof of Theorem 1.1 is obtained by arguing exactly as in the proof of [20, Thm. 1], with $W^{1,p}_\gamma$ and $\mathcal{E}^p_\gamma$ instead of $W^{1/p}$ and $\mathcal{E}_p$, where we proceeded along the lines of [3, Lemma 5], but using this time Propositions 2.1 and 2.2. We omit any further detail.

3 Topological arguments

In this section, we prove Theorem 1.3 in dimension $n = d$, Theorem 1.4, and Theorem 1.5.

Removing point singularities. When $n = d$, Theorem 1.3 becomes:

Theorem 3.1 *If $n = d := [sp] + 1 \geq 2$, then $W^{s,p}_S(B^n, \mathcal{Y}) = W^{s,p}(B^n, \mathcal{Y})$ if and only if $\pi_{d-1}(\mathcal{Y}) = 0$.*

Proof: We have to prove the non-trivial implication. Therefore, on account of Theorem 1.1, it suffices to show that $R^\infty_{s,p}(B^n, \mathcal{Y}) \subset W^{s,p}_S(B^n, \mathcal{Y})$ provided that $\pi_{d-1}(\mathcal{Y}) = 0$. Moreover, since we are going to modify the extension of a map in $R^\infty_{s,p}$ in a small neighborhood of each singular point, without loss of generality we reduce to the case where $u \in R^\infty_{s,p}(\mathcal{Q}^n, \mathcal{Y})$ is smooth outside the origin.

We thus follow the lines of the proof of [20, Thm. 2], with $\mathcal{E}^p_\gamma$, $W^{1,p}_\gamma$, and $W^{s,p}$ instead of $\mathcal{E}_p$, $W^{1,p}$, and $W^{1-1/p,p}$, respectively. In fact, this time we obtain the estimate

$$\mathcal{E}^p_\gamma(W_r, \mathcal{Q}^n \times I) \leq \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) + c\, r\, \mathcal{E}^p_\gamma(U, \partial Q_r) + \left(\frac{\delta}{r}\right)^{n+1+\gamma-p} \mathcal{E}^p_\gamma(V_r, Q_r)$$

for some absolute constant $c > 0$, depending on n, s, and p. Recalling that $\gamma = p(1-s) - 1$ and $n = d = [sp] + 1$, we have:

$$n + 1 + \gamma - p = 1 + [sp] - sp > 0\,.$$

Therefore, since $r < R$ this time we get

$$\begin{aligned} \mathcal{E}^p_\gamma(W_r, \mathcal{Q}^n \times I) &\leq \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) + 2\, c\, \varepsilon + \left(\frac{\delta}{r}\right)^{1+[sp]-sp} \mathcal{E}^p_\gamma(V_r, Q_r) \\ &\leq \mathcal{E}^p_\gamma(U, \mathcal{Q}^n \times I) + (2c + 1)\, \varepsilon \end{aligned}$$

taking $\delta = \delta(r, \varepsilon)$ sufficiently small, as required. □

Homotopy type of $W^{s,p}$ maps. Arguing as in the proof of [20, Prop. 1], Theorem 1.4 is an immediate consequence of the following

Proposition 3.2 *Let $n \geq d := [sp] + 1 \geq 2$. Let $u \in W^{s,p}(\mathcal{Q}^n, \mathcal{Y})$ and X a finite cubeulation of $\mathcal{Q}^n$ in generic position with respect to u. For any smooth sequence $\{u_i\} \subset W^{s,p}(X^{d-1}, \mathcal{Y}) \cap C^\infty$ strongly converging to $u_{|X^{d-1}}$ in $W^{s,p}$, we find $k_0 \in \mathbb{N}^+$ such that for every $i, j \geq k_0$ the maps u_i and u_j are homotopic as maps from X^{d-1} to $\mathcal{Y}$.*

As already observed in the introduction, Proposition 3.2 can be obtained by using results contained in [9] concerning homotopy theory in VMO. Our proof relies on the argument from [3, Lemma 1], that goes back to [24].

Proof of Proposition 3.2: We follow the lines of the proof of [20, Prop. 4], to which we refer for the notation. We thus assume $X^k := \Sigma^{(k)}_m$ and denote $u_i(\cdot, h) := \mathbf{T}(U_i(\cdot, h)) \in W^{s,p}(\Sigma^{(d-1)}_m, \mathbb{R}^N)$. For every i, we infer that $U_i(z, h)$ is continuous, whereas $U_i(\cdot, h)$ converges to U_i and $u_i(\cdot, h)$ to u_i uniformly, as $h \to 0$. Let $\varepsilon_1 > 0$ to be chosen. By the strong convergence of u_i to u, working with $W^{s,p}$ and $W^{1,p}_\gamma$ instead of $W^{1/p} := W^{1-1/p,p}$ and $W^{1,p}$, respectively, we in particular obtain for every $z \in \Sigma^{(d-1)}_m \times I$ and $0 < h \leq h_0$

$$\mathcal{E}^p_\gamma(U_i, \Sigma(z, h)) \leq \varepsilon_1 \qquad \forall\, i\,. \tag{3.1}$$

If $\xi := (x, 0) \in \Sigma^{(d-1)}_m \times \{0\}$, for $i \neq j$ we again estimate

$$\begin{aligned} |u_i(x, h_0) - u_j(x, h_0)| &\leq \left(\fint_{\Sigma(\xi, h_0)} |U_i(\xi, h_0) - U_i(w)|^p \, d\mathcal{H}^d(w) \right)^{1/p} \\ &+ \left(\fint_{\Sigma(\xi, h_0)} |U_j(\xi, h_0) - U_j(w)|^p \, d\mathcal{H}^d(y) \right)^{1/p} \\ &+ \left(\fint_{\Sigma(\xi, h_0)} |U_i(w) - U_j(w)|^p \, d\mathcal{H}^d(w) \right)^{1/p} \\ &=: I_1 + I_2 + I_3\,. \end{aligned}$$

We now apply the Poincaré inequality in the weighted Sobolev space $W^{1,p}_\gamma$, whose validity follows from e.g. Remark 2.3 in [10]. By (3.1) we have

$$I_1 + I_2 \le c\, h_0^{(p-\gamma-d)/p}\, {\varepsilon_1}^{1/p}\,, \quad \gamma = p(1-s) - 1\,.$$

Moreover, using that $\mathcal{H}^d(\Sigma(\xi,h_0)) \ge h_0^d$, for any $i,j \ge k_0$ we get the bound

$$I_3 \le h_0^{-d/p}\, \|U_i - U_j\|_{W^{1,p}_\gamma(\Sigma_m^{(d-1)}\times I)} \le C\, h_0^{-d/p}\, \sigma_0\,.$$

Since $p - \gamma - d = sp - [sp] \ge 0$ and $h_0 < 1$, we then obtain for every $x \in \Sigma_m^{(d-1)}$, and for $i,j \ge k_0$,

$$|u_i(x,h_0) - u_j(x,h_0)| \le c_3\, {\varepsilon_1}^{1/p} + c_4\, h_0^{-d/p}\, \sigma_0\,.$$

The rest of the proof is equal to the one of [20, Prop. 4]. □

A CHARACTERIZATION. Arguing as in [20, Thm. 3], we finally give the:

PROOF OF THEOREM 1.5: Assume first that $u \in R^0_{s,p}(B^n,\mathcal{Y})$ is the strong $W^{s,p}$ limit of a sequence of smooth maps $\{u_i\}$ in $C^\infty(B^n,\mathcal{Y})$. Let X be a finite cubeulation of B^n in dual position with respect to u, and let $d \ge 2$. It clearly suffices to show the extension property of u for a cubeulation $\widetilde{X}$ in dual position and obtained by slightly moving the faces of X. Denoting by $\widetilde{u}_i \in W^{s,p}(X^{d-1},\mathcal{Y})$ the restriction of u_i to X^{d-1}, by Fubini's theorem we thus may and do assume without loss of generality that $\widetilde{u}_i$ strongly converges to $\widetilde{u} := u_{|X^{(d-1)}}$ in $W^{s,p}$. Therefore, we can argue as in the proof of [20, Thm. 3], on account of Theorem 1.4 instead of [20, Prop. 1]. Finally, if $d = 1$ the conclusion trivially follows.

We now prove the converse implication, and assume that the restriction $\widetilde{u} := u_{|X^{(d-1)}}$ can be extended to a continuous map from B^n into $\mathcal{Y}$.

The case $n = d := [sp] + 1$. The map $u \in R^0_{s,p}(B^n,\mathcal{Y})$ is continuous outside a discrete set $\Sigma(u)$. Since we are going to modify the extension of a map in $R^0_{s,p}$ in a small neighborhood of each singular point, without loss of generality we assume that $u \in R^0_{s,p}(\mathcal{Q}^n,\mathcal{Y})$ and u is smooth outside the origin. We then argue as in the proof of Theorem 1.3 from Sec. 2, where $n = d$. In fact, this time we infer that $u_{|\partial F_r} : \partial F_r \to \mathcal{Y}$ is a continuous map in $W^{s,p}(\partial F_r,\mathcal{Y})$ for which we can find a continuous extension $u_r : F_r \to \mathcal{Y}$ with finite $W^{s,p}$ norm, as required.

The case $n - 1 \ge d$. It suffices to follow the lines from the corresponding case in the proof of [20, Thm. 3]. We thus omit any further detail. □

4 Maps into spheres

In this section and in the next ones, we let $\mathcal{Y} = \mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$, where $N \ge 2$. As noticed in the introduction, we shall assume that $0 < s < 1$ and $p > 1$ satisfy $N - 1 \le sp < n$. Recalling that $d = [sp] + 1$, we thus reduce to the ranges $N \le d \le n$.

A special case is when $sp = N - 1$, so that $p > N - 1$ and $\gamma(s,p) = p - N$. If $sp > N - 1$, we shall apply the continuous embedding

$$W^{s,p}(B^n,\mathbb{S}^{N-1}) \subset W^{s,(N-1)/s}(B^n,\mathbb{S}^{N-1})\,. \tag{4.1}$$

Our estimate in Theorem 4.1 is inspired by some relevant arguments obtained by Bourgain-Brezis-Mironescu in [4].

A RELEVANT ESTIMATE. Assume $sp = N - 1$, so that $d = N$ and $\gamma(s,p) = p - N$. Let $n \ge N - 1 \ge 1$, and recall that condition $0 < s < 1$ implies $p > N - 1$. Let $u \in W^{(N-1)/p,p}(B^n,\mathbb{S}^{N-1})$ and $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1},\mathbb{R}^N)$ the *harmonic extension* of u to $\mathbf{C}^{n+1} := B^n \times (0,1)$, so that U takes values into the unit N-ball $\mathbb{D}^N$ of the target space,

$$\mathbb{D}^N := \{y \in \mathbb{R}^d \,:\, |y| \le 1\}\,.$$

Following [4, Lemma 1.2], we denote

$$G := \{(x,t) \in \mathbf{C}^{n+1} \,:\, |U(x,t)| \le 1/2\}$$

and let $\mathbf{d} : B^n \to]0,1/2]$ the function such that $\mathbf{d}(x) := 1/2$ if $|U(x,t)| \ge 1/2$ for each $t \in (0,1/2)$, and

$$\mathbf{d}(x) := \min\{t \in (0,1/2) \,:\, |U(x,t)| \le 1/2\}$$

otherwise. Using that $|DU(x,t)| \le c/t$ for some absolute constant c, for any exponent $\alpha > 1$ one has

$$\int_G |DU(x,t)|^\alpha \, dx\, dt \le c \int_{B^n} \Big(\int_{\mathbf{d}(x)}^1 t^{-\alpha}\, dt \Big)\, dx \le C \int_{B^n} \frac{1}{\mathbf{d}(x)^{\alpha-1}}\, dx\,.$$

In a similar way to the case $\alpha = N$, using that $t > \mathbf{d}(x)$ if $(x,t) \in G$, for each $p > N-1$ we estimate

$$\int_G t^{p-N} |DU(x,t)|^p \, dx\, dt \le \int_G \frac{C}{t^N}\, dx\, dt \le C \int_{B^n} \frac{1}{\mathbf{d}(x)^{N-1}}\, dx \tag{4.2}$$

where $C = C(n,p,N)$. Since moreover $U \in W^{N/p,p}(B^n \times I, \mathbb{R}^N)$, where $I = (0,1/2)$, as in [4, Lemma 1.3], using the embedding of $W^{N/p,p}(I)$ in the Hölder class $C^{0,(N-1)/p}(I)$, it turns out that for a.e. $x \in B^n$ the function $\varphi_x(t) := U(x,t)$ belongs to $W^{N/p,p}(I,\mathbb{R}^N)$, whence to $C^{0,(N-1)/p}(I,\mathbb{R}^N)$, so that:

$$\begin{aligned} \frac{1}{2} &\le |\varphi_x(\mathbf{d}(x)) - \varphi_x(0)| \le C\, \mathbf{d}(x)^{(N-1)/p} \|\varphi_x\|_{C^{0,(N-1)/p}(I)} \\ &\le C\, \mathbf{d}(x)^{(N-1)/p}\, \|\varphi_x\|_{W^{N/p,p}(I,\mathbb{R}^N)} \end{aligned}$$

and hence

$$\frac{1}{\mathbf{d}(x)^{N-1}} \le C\, \|\varphi_x\|^p_{W^{N/p,p}(I,\mathbb{R}^N)}\,. \tag{4.3}$$

Therefore, using the inequality on Besov-type spaces

$$\begin{aligned} \int_{B^n} \|\varphi_x\|^p_{W^{N/p,p}(I,\mathbb{R}^N)}\, dx &= \int_{B^n} \|U(x,\cdot)\|^p_{W^{N/p,p}(I,\mathbb{R}^N)}\, dx \\ &\le C\, \|U\|^p_{W^{N/p,p}(\mathbf{C}^{n+1},\mathbb{R}^N)} \le C\, \|u\|^p_{W^{(N-1)/p,p}(B^n,\mathbb{R}^N)} \end{aligned}$$

by (4.2) and (4.3) one gets the estimate

$$\int_G t^{p-N} |DU|^p \, dx\, dt \le C_1 \int_{B^n} \frac{1}{\mathbf{d}(x)^{N-1}}\, dx \le C_2 \int_{\mathbf{C}^{n+1}} t^{p-N} |DU|^p \, dx\, dt \tag{4.4}$$

for some positive constants C_1, C_2 only depending on n, p, and N.

In the sequel, we choose a smooth function $\Phi : \mathbb{R}^N \to \mathbb{D}^N$ such that $\Phi(y) = y/|y|$ if $|y| \ge 1/2$, where $y = (y_1, \dots, y_N)$, and Φ is a bi-Lipschitz map from $\{y \in \mathbb{R}^N \,:\, |y| \le 1/2\}$ onto $\mathbb{D}^N$.

Setting $V := \Phi \circ U$, we clearly have:

$$\begin{aligned} |DV(x,t)| &\le C_1\, |DU(x,t)| \quad \forall\, (x,t) \in \mathbf{C}^{n+1}\,, \\ |DU(x,t)| &\le C_2\, |DV(x,t)| \quad \forall\, (x,t) \in G\,. \end{aligned} \tag{4.5}$$

Denote now by $V^\#(dy^1 \wedge \cdots \wedge dy^N)$ the N-form in $\mathbf{C}^{n+1}$ given by the pull-back by V of the N-form $dy^1 \wedge \cdots \wedge dy^N$. One has

$$|V^\#(dy^1 \wedge \cdots \wedge dy^N)| = J_V \tag{4.6}$$

where J_V is the Jacobian of the map V, so that $J_V(x,t)^2$ is the sum of the squares of all the $N \times N$ minors of the gradient matrix $DV(x,t)$. Therefore, by the area formula one has $J_V(x,t) = 0$ if $(x,t) \in G$ whereas by the parallelogram inequality one gets the general estimate $J_V(x,t) \le C\, |DV(x,t)|^N$, where $C = C(n,N)$. These are the main facts that led Bourgain-Brezis-Mironescu [4] to obtain the estimate

$$|\deg g| \le C_p\, \|g\|^p_{(N-1)/p,p} \qquad \forall\, p > 1$$

on the degree $\deg g$ of maps $g \in W^{(N-1)/p,p}(\mathbb{S}^{N-1}, \mathbb{S}^{N-1})$, see also (5.1).

We similarly obtain in any dimension the following relevant estimate:

Theorem 4.1 *Let $n \geq N-1 \geq 1$ and $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ for some $p > N-1$. Then*

$$\int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt \leq C \int_{\mathbf{C}^{n+1}} t^{p-N}\, |DU(x,t)|^p\, dx\, dt \tag{4.7}$$

for some real constant $C > 0$ only depending on n, p, and N.

Proof: By the previous facts, using (4.5) inequality (4.7) readily follows when $p = N$, and hence for $N-1 < p < N$, by the continuous embedding $W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1}) \subset W^{(N-1)/N,N}(B^n, \mathbb{S}^{N-1})$. When $p > N$, letting $\alpha = \alpha(p, N) = N(p-N)/p$, by Hölder inequality with exponents $q = p/N$ and $q' = p/(p-N)$ we get:

$$\begin{aligned}\int_G |DV(x,t)|^N\, dx\, dt &\leq C \int_G (t^\alpha\, |DU(x,t)|^N)\, t^{-\alpha}\, dx\, dt \\ &\leq C\Big(\int_G t^{p-N} |DU(x,t)|^p\, dx\, dt\Big)^{N/p} \cdot \Big(\int_G t^{-N}\, dx\, dt\Big)^{(p-N)/p}\end{aligned}$$

where by (4.2) and (4.4) we can estimate

$$\Big(\int_G t^{-N}\, dx\, dt\Big)^{(p-N)/p} \leq C_{n,N} \Big(\int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p\, dx\, dt\Big)^{(p-N)/p}.$$

Since by (4.5) and (4.6)

$$\int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt = \int_G J_V(x,t)\, dx\, dt \leq C \int_G |DV(x,t)|^N\, dx\, dt$$

the assertion readily follows. □

Remark 4.2 By the continuous embedding (4.1), it turns out that our estimate (4.7) extends to maps in $W^{s,p}(B^n, \mathbb{S}^{N-1})$ for any $0 < s < 1$ and $p > 1$ such that $sp > N-1$.

5 Degree, currents, and homological singularities

In this section, as a consequence of Theorem 4.1, we first introduce a notion of degree, see (5.1), showing that it is strongly continuous and integer valued, Proposition 5.2.

For $N-1 \leq sp < n$ and $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$, we then define an $(n-N)$-current $\mathbb{P}(u)$ in B^n that describes the relevant singularities of u, at least when $[sp] = N-1$, i.e., when $d = N$, see (5.2).

In fact, recalling that $W^{s,p}_S(B^n, \mathbb{S}^{N-1})$ denotes the strong closure of smooth maps $u \in C^\infty(B^n, \mathbb{S}^{N-1})$ in the $W^{s,p}$-norm, in Proposition 5.5 we show that for any $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$

$$u \in W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,.$$

When $[sp] = N-1$, the converse implication holds true. In fact, in Theorem 5.6 we show that if $N \geq 3$

$$[sp] = N-1 \quad \Longrightarrow \quad W^{s,p}_S(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}$$

whereas in case $N = 2$ the same conclusion holds true whenever $sp \geq 1$.

However, the converse implication fails to hold when $3 \leq N \leq [sp] < n$. More precisely, in that case we show existence of maps u in $W^{s,p}(B^N, \mathbb{S}^{N-1})$, actually in $R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$, such that $\mathbb{P}(u) = 0$ but $u \notin W^{s,p}_S(B^N, \mathbb{S}^{N-1})$.

Example 5.1 Let $N \geq 3$, $0 < s < 1$, and $p > 1$ such that $N \leq sp < N+1$, whence $d = [sp] + 1 = N+1$. We have already seen that the zero-degree homogeneous map u given by Example 1.10 belongs to $W^{s,p}(B^{N+1}, \mathbb{S}^{N-1})$, but $u \notin W^{s,p}_S(B^{N+1}, \mathbb{S}^{N-1})$, due to the topological singularity at the origin. However, since $u \in W^{1,q}(B^{N+1}, \mathbb{S}^{N-1})$ for each $q < N+1$, by Example 6.5 below it turns out that $\mathbb{P}(u) = 0$.

Degree. For $0 < s < 1$ and $p > 1$ such that $sp \geq N-1$, denote by $W^{s,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ the class of locally summable maps $u : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ such that $u(x) - P_u \in L^p(\mathbb{R}^{N-1}, \mathbb{R}^N)$ for some point $P_u \in \mathbb{S}^{N-1}$, and $|u|_{s,p} < \infty$, where

$$|u|_{s,p}^p := \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|u(x)-u(y)|^p}{|x-y|^{N-1+sp}}\, dx\, dy\,.$$

The class $W^{s,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ is equipped with the norm $\|u - P_u\|_{L^p} + |u|_{s,p}$. We define the *degree* of a map u in $W^{s,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ through the formula

$$\deg u := \frac{1}{\alpha_N} \int_{\mathbb{R}^N_+} V^{\#}(dy^1 \wedge \cdots \wedge dy^N)\,, \quad \alpha_N := |\mathbb{D}^N| \tag{5.1}$$

where $\mathbb{R}^N_+ := \{(x,t) \in \mathbb{R}^N \mid t > 0\}$ denotes the upper N-space, $U \in W^{1,p}_{\gamma(s,p)}(\mathbb{R}^N_+, \mathbb{D}^N)$ the harmonic extension of u, and $V := \Phi \circ U$, as before.

Proposition 5.2 *If $0 < s < 1$ and $p > 1$ satisfy $sp \geq N-1$, the degree of maps in $W^{s,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ is strongly continuous, and $\deg u \in \mathbb{Z}$ for each $u \in W^{s,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$.*

Proof: By the continuous embedding (4.1), we reduce to the case when $sp = N-1$, so that $\gamma(s,p) = p - N$. Let $u \in W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$. Arguing as in the proof of Theorem 4.1, we have:

$$\int_{\mathbb{R}^N_+} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt \leq C(N,p) \int_{\mathbb{R}^N_+} t^{p-N}\, |DU(x,t)|^p\, dx\, dt\,.$$

Let $\{u_h\} \subset W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ such that $u_h \to u$ in $W^{(N-1)/p,p}$. For each h, denote $V_h := \Phi \circ U_h$, where $U_h \in W^{1,p}_{p-N}(\mathbb{R}^N_+, \mathbb{D}^N)$ is the harmonic extension of u_h. The strong convergence $u_h \to u$ in $W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{R}^N)$ implies the strong convergence $V_h \to V$ in $W^{1,p}_{p-N}(\mathbb{R}^N_+, \mathbb{R}^N)$. Therefore, by the above estimate, the dominated convergence theorem yields

$$\lim_{h\to\infty} \int_{\mathbb{R}^N_+} V_h^{\#}(dy^1 \wedge \cdots \wedge dy^N) = \int_{\mathbb{R}^N_+} V^{\#}(dy^1 \wedge \cdots \wedge dy^N)$$

whence $\deg u_h \to \deg u$. Since moreover $n = N-1$, there exists a sequence $\{u_h\} \subset C^1(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$ such that $u_h \to u$ strongly in $W^{(N-1)/p,p}$, see [9]. Let $\varepsilon > 0$ small. Then, by means of a cut-off argument, for each h we can find a smooth map $W_h : \mathbb{R}^N \to \mathbb{D}^N$ and a point $P_h \in \mathbb{S}^{N-1}$ such that $W_h(x,t) - P_h$ has compact support contained in $\mathbb{R}^N_+$ and

$$\int_{\mathbb{R}^N_+} |W_h^{\#}(dy^1 \wedge \cdots \wedge dy^N) - V_h^{\#}(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt < \varepsilon\,.$$

It is then readily checked that

$$\int_{\mathbb{R}^N_+} W_h^{\#}(dy^1 \wedge \cdots \wedge dy^N) = \mathbf{d}_h \cdot \alpha_N$$

for some $\mathbf{d}_h \in \mathbb{Z}$. Therefore, we get $\deg u_h = \mathbf{d}_h$ for each h, whence $\deg u \in \mathbb{Z}$, as $\deg u_h \to \deg u$. $\square$

Proposition 5.2 is in accordance with a similar result from [9], see also [17]. By the previous argument, it turns out that if u is smooth, our notion of degree (5.1) is equivalent to the classical one. Therefore the degree of a smooth map $v : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ agrees with the one of the map $u = v \circ \Pi : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$, where $\Pi : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ is the inverse of the stereographic projection. For future use, we also point out the following fact.

Remark 5.3 Let $u \in R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$, where $n \geq N \geq 2$ and $sp \geq N-1$, and let X a finite cubeulation of B^n in dual position with respect to u. Assume in addition that $\mathbb{P}(u) = 0$. Then, by a slicing argument, it turns out that (by possibly slightly moving the faces of X) the restriction $u_{|F}$ of u to each $(N-1)$-simplex F of X^{N-1} is a smooth map into $\mathbb{S}^{N-1}$ with zero degree. More precisely, there exists a bilipschitz homeomorphism $\Phi : \mathbb{S}^{N-1} \to F$ such that $u \circ \Phi : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ has zero degree in the previous sense, hence $u \circ \Phi$ can be extended to a smooth map from B^N to $\mathbb{S}^{N-1}$.

Of course, due to the well-known bubbling phenomenon, the degree fails to be continuous w.r.t. the weak sequential convergence in $W^{(N-1)/p,p}$.

CURRENTS. Let $0 \leq k \leq m$ integers and $\Omega \subset \mathbb{R}^m$ an open set. The space $\mathcal{D}_k(\Omega)$ of *k-currents* in Ω is the strong dual of the space $\mathcal{D}^k(\Omega)$ of compactly supported smooth k-forms. Weak convergence $T_h \rightharpoonup T$ in $\mathcal{D}_k(\Omega)$ is defined by duality through the formula

$$T_h(\omega) \to T(\omega) \qquad \forall\, \omega \in \mathcal{D}^k(\Omega)\,.$$

The *mass* of a current $T \in \mathcal{D}_k(\Omega)$ is defined by

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(\Omega),\ \|\omega\| \leq 1\}$$

where $\|\omega\|$ is the *comass* norm of ω. Therefore, the mass functional is lower semicontinuous w.r.t. the weak convergence. The *boundary* of a current T in $\mathcal{D}_k(\Omega)$, when $k \geq 1$, is defined by duality as

$$\partial T(\eta) := T(d\eta)\,, \qquad \eta \in \mathcal{D}^{k-1}(\Omega)$$

yielding to a current ∂T in $\mathcal{D}_{k-1}(\Omega)$.

In particular, when $\Omega = A \times \mathbb{R}^N$, where $A \subset \mathbb{R}^k$ is a bounded domain, and $v : A \to \mathbb{R}^N$ is a sufficiently smooth function, the k-current G_v carried by the graph of v acts on k-forms $\omega \in \mathcal{D}^k(A \times \mathbb{R}^N)$ as

$$G_v(\omega) = ((Id \bowtie v)_{\#}[\![A]\!], \omega) := \int_A (Id \bowtie v)^{\#}\omega$$

where $(Id \bowtie v)(x) := (x, v(x))$ and $(Id \bowtie v)^{\#}\omega$ is the pull-back of ω. By the area formula one then computes

$$\mathbf{M}(G_v) = \int_A J_{Id \bowtie v}(x)\, dx$$

where $J_{Id \bowtie v}$ is the Jacobian of the graph map. If e.g. $k \geq N \geq 2$, one has

$${J_{Id \bowtie v}}^2 = 1 + |Dv|^2 + |M_2(Dv)|^2 + \cdots + |M_N(Dv)|^2$$

where $|M_j(Dv)|^2$ is the sum of the square of the $j \times j$ minors of the gradient matrix Dv. In particular, if $v \in W^{1,N-1}(A, \mathbb{R}^N)$ and $|M_N(Dv)| \in L^1(A)$, then G_v is an *i.m. rectifiable* current in $\mathcal{R}_k(A \times \mathbb{R}^N)$, with finite mass, compare Proposition 6.2. We refer e.g. to the treatise [11] for further details.

HOMOLOGICAL SINGULARITIES. Let $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $n \geq N \geq 2$ and $p > N-1$. We can define the current $\mathbb{P}(u) \in \mathcal{D}_{n-N}(B^n)$ that carries the relevant information on the singular set of u. Choose a smooth decreasing cut-off function $\eta : [0,1] \to [0,1]$ such that $\eta(t) = 1$ for $t \in [0,1/4]$ and $\eta(t) = 0$ for $t \in [3/4, 1]$, and for any k-form $\phi \in \mathcal{D}^k(B^n)$ denote by $\widetilde{\phi}$ the k-form in $\mathbf{C}^{n+1}$ given by $\widetilde{\phi} := \phi \wedge \eta$. We let

$$\mathbb{P}(u)(\phi) := \frac{1}{\alpha_N} \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N)\,, \quad \phi \in \mathcal{D}^{n-N}(B^n) \tag{5.2}$$

where $\alpha_N := |\mathbb{D}^N|$ and $V = \Phi \circ U$, with U the harmonic extension of u, as before.

Remark 5.4 Again by the continuous embedding (4.1), it turns out that definition (5.2) extends to maps in $W^{s,p}(B^n, \mathbb{S}^{N-1})$ for any $0 < s < 1$ and $p > 1$ such that $sp > N-1$.

With this notation, we have:

Proposition 5.5 *Let $0 < s < 1$ and $p > 1$ satisfy $1 \leq N-1 \leq sp < n$, and let $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$. Then*

$$u \in W^{s,p}_S(B^n, \mathbb{S}^{N-1}) \quad \Longrightarrow \quad \mathbb{P}(u) = 0\,.$$

PROOF: Let $\{u_h\} \subset C^1(B^n, \mathbb{S}^{N-1})$ a smooth sequence strongly converging to u in $W^{s,p}(B^n, \mathbb{R}^N)$. In case $sp > N-1$, by the continuous embedding (4.1) we infer that $u_h \to u$ strongly in $W^{s,(N-1)/s}(B^n, \mathbb{R}^N)$. As a consequence, if $U_h := \mathrm{Ext}(u_h)$ is the harmonic extension of u_h, then $U_h \to U := \mathrm{Ext}(u)$ strongly in $W^{1,q}_{q-N}(B^n, \mathbb{R}^N)$ for some $q > N-1$. Therefore, if $V_h = \Phi \circ U_h$, by (4.7) and the dominated convergence theorem

$$\lim_{h\to\infty} \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V_h^{\#}(dy^1 \wedge \cdots \wedge dy^N) = \int_{\mathbf{C}^{n+1}} d\widetilde{\phi} \wedge V^{\#}(dy^1 \wedge \cdots \wedge dy^N) \tag{5.3}$$

for every $\phi \in \mathcal{D}^{n-N}(B^n)$, i.e., $\mathbb{P}(u_h) \rightharpoonup \mathbb{P}(u)$ weakly in $\mathcal{D}_{n-N}(B^n)$, whereas $\mathbb{P}(u_h) = 0$ for each h, by the smoothness property. □

For some ranges of sp, the converse implication holds, too:

Theorem 5.6 *Let $0 < s < 1$ and $p > 1$. Assume that $n > [sp] = N-1$, when $N \geq 3$, or $n > [sp] \geq 1$, when $N = 2$. Then*

$$W^{s,p}_S(B^n, \mathbb{S}^{N-1}) = \{u \in W^{s,p}(B^n, \mathbb{S}^{N-1}) \mid \mathbb{P}(u) = 0\}\,.$$

PROOF: On account of Proposition 5.5, it suffices to show that if $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ and $\mathbb{P}(u) = 0$, then u is the strong $W^{s,p}$ limit of a smooth sequence in $C^1(B^n, \mathbb{S}^{N-1})$. We make use of arguments taken from [21] and reduce to the case of maps in $R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$ by means of the following result, the proof of which is postponed to the end of the last section.

Proposition 5.7 *Let $0 < s < 1$, $p > 1$, and $1 \leq N-1 \leq sp < n$, and let $u \in W^{s,p}(B^n, \mathbb{S}^{N-1})$ satisfying $\mathbb{P}(u) = 0$. Then, u is the strong $W^{s,p}$ limit of a sequence $\{u_h\} \subset R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$ satisfying $\mathbb{P}(u_h) = 0$ for each h.*

Assume first $N \geq 3$, so that $d = [sp] + 1 = N \leq n$. Now, if $u \in R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$ is such that $\mathbb{P}(u) = 0$, and X is a cubeulation of B^n in dual position with respect to u, by Theorem 1.5 it suffices to show that the restriction $u_{|X^{N-1}}$ has a continuous extension $g : B^n \to \mathbb{S}^{N-1}$. Moreover, by condition $\mathbb{P}(u) = 0$, on account of Remark 5.3 we may and do assume that the restriction of u to each $(N-1)$-simplex of X^{N-1} has zero degree. This yields that $u_{|X^{N-1}}$ has a continuous extension $f : X^N \to \mathbb{S}^{N-1}$, and proves the claim in low dimension $n = N$. If $n \geq N+1$, instead, since B^n is N-connected, arguing as in [24, Sec. 6], we find a continuous map $\phi : B^n \to \mathbb{S}^{N-1}$ homotopic to the identity map and such that the restriction $\phi_{|X^N}$ is constant. Then $f \circ \phi$ is homotopic to f and $f \circ \phi_{|X^N}$ is constant. Whence, $f_{|X^N}$ can be extended to a continuous map, as required.

In case $N = 2$, recall that we can restrict to the case $1 \leq sp < 2$, whence $d = N$ and therefore we argue as before. □

6 Further results

In this final section, we collect some other consequences of Theorem 4.1. We then conclude with the proof of Proposition 5.7.

COAREA FORMULA. Following Almgren-Browder-Lieb [1], we have:

Proposition 6.1 *Let $n \geq N \geq 2$ integers and $p > N-1$. For every map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ there exists a smooth extension $V \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ and a regular value $y \in \mathbb{D}^N$ for V such that*

$$\mathcal{H}^{n+1-N}(V^{-1}(\{y\})) \leq C \int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx\, dt \tag{6.1}$$

for some real constant C only depending on n, p, and N.

PROOF: Choose $V := \Phi \circ U$, where $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ is the harmonic extension of u. We have

$$\begin{aligned} \int_{\mathbb{D}^N} \mathcal{H}^{n+1-N}(V^{-1}(\{y\}))\, d\mathcal{H}^N(y) &= \int_{\mathbf{C}^{n+1}} J_V(x,t)\, dx\, dt \\ &= \int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)|\, dx\, dt \end{aligned}$$

and hence we can find a regular value $y \in \mathbb{D}^N$ such that

$$\mathcal{H}^{n+1-N}(V^{-1}(\{y\})) \leq \frac{1}{\alpha_N} \cdot \int_{\mathbf{C}^{n+1}} |V^{\#}(dy^1 \wedge \cdots \wedge dy^N)| \, dx \, dt$$

where $\alpha_N := |\mathbb{D}^N|$. The assertion follows from Theorem 4.1. □

Gradient summability. For $N \geq 2$ integer, denote now

$$p(N) := \begin{cases} +\infty & \text{if } N = 2 \\ \dfrac{(N-1)^2}{N-2} & \text{if } N \geq 3 \,. \end{cases} \tag{6.2}$$

On account of Theorem 4.1, we also obtain:

Proposition 6.2 *Let $n \geq N-1 \geq 1$ and $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$, where $N-1 < p < p(N)$. If $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ is the harmonic extension of u, then $U \in W^{1,N-1}(\mathbf{C}^{n+1}, \mathbb{D}^N)$. As a consequence, letting $V := \Phi \circ U$, then the graph current G_V is i.m. rectifiable in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1} \times \mathbb{R}^N)$, with finite mass bounded by*

$$\mathbf{M}(G_V) = \int_{\mathbf{C}^{n+1}} J_{Id \bowtie V} \, dz \leq c \Big(1 + \int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dz \Big) \tag{6.3}$$

for some constant $c > 0$, not depending on u, and G_V satisfies the null-boundary condition:

$$(\partial G_V) \llcorner \mathbf{C}^{n+1} \times \mathbb{R}^N = 0 \,. \tag{6.4}$$

Proof: Letting $\alpha := (p-N)(N-1)/p$, by Hölder inequality with $q := p/(N-1)$ and $q' = p/(p-N+1)$ we get:

$$\begin{aligned} \int_{\mathbf{C}^{n+1}} |DU(z)|^{N-1} \, dz &= \int_{\mathbf{C}^{n+1}} (t^\alpha \, |DU(x,t)|^{N-1}) \, t^{-\alpha} \, dx \, dt \\ &\leq \Big(\int_{\mathbf{C}^{n+1}} t^{p-N} |DU(x,t)|^p \, dx \, dt \Big)^{(N-1)/p} \Big(\int_{\mathbf{C}^{n+1}} t^{-\alpha p/(p-N+1)} \, dx \, dt \Big)^{(p-N+1)/p} . \end{aligned}$$

However, $-\alpha p/(p-N+1) > -1$ if and only if $(p-N)(N-2) < 1$, i.e., $p < p(N)$. Equivalently, for $N \geq 3$, we recall that $W^{1-1/(N-1),N-1}$ is the trace space of $W^{1,N-1}$, whereas by the Gagliardo-Nirenberg and Sobolev inequalities

$$W^{(N-1)/p,p} \cap L^\infty \subset W^{1-1/(N-1),N-1} \iff \frac{N-1}{p} > \frac{N-2}{N-1} \iff p < p(N).$$

Recall that $|DV| \leq C \, |DU|$ and $|M_k(DV)| \leq c \cdot |DV|^k$ for $1 \leq k \leq N-1$, by the parallelogram inequality, whereas $|V^{\#}(dy^1 \wedge \cdots \wedge dy^N)| = |M_N(DV)|$. Therefore, V is locally Lipschitz, and the mass estimate (6.3) follows from (4.7). Finally, we observe that by Stokes' theorem the null boundary condition (6.4) holds if $u \in W^{(N-1)/p,p}(B^n, \mathbb{R}^N)$ is smooth and bounded, whence (6.4) is readily checked by a standard density argument, on account of (6.3) and of the dominated convergence theorem. □

For e.g. $N = 3$, there exist maps $u \in W^{1/2,4}(B^n, \mathbb{S}^2) \setminus W^{1/2,2}(B^n, \mathbb{S}^2)$. Therefore, if U is the harmonic extension of u in $W^{1,4}_1(\mathbf{C}^{n+1}, \mathbb{R}^3)$, since $\mathbf{T}(U) = u$, then $U \notin W^{1,2}(B^n, \mathbb{R}^3)$. As a consequence, differently to the case $N = 2$ analyzed in [22], when $N \geq 3$ it is possible to give a good notion of current G_u carried by the graph of a map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ only for some ranges of $p > N-1$.

Graph currents. If n, N, and p are as in Proposition 6.2, to any map $u \in W^{(N-1)/p,p}(B^n, \mathbb{S}^{N-1})$ we can associate an n-current G_u in $\mathcal{D}_n(B^n \times \mathbb{S}^{N-1})$ through the map $V := \Phi \circ U$, where $U \in W^{1,p}_{p-N}(\mathbf{C}^{n+1}, \mathbb{D}^N)$ is the harmonic extension of u, by setting

$$G_u := (-1)^{n-N+1} (\partial G_V) \llcorner ((B^n \times \{0\}) \times \mathbb{R}^N) \quad \text{on} \quad \mathcal{D}^n(B^n \times \mathbb{S}^{N-1}) \,. \tag{6.5}$$

Remark 6.3 In formula (6.5), the boundary ∂G_V is seen by extending the action of the current G_V to forms in $\mathcal{D}^{n+1}(\mathbb{R}^{n+1}\times\mathbb{R}^N)$. Moreover, by Federer's support theorem, see e.g. Thm. 5 in [11, Sec. 5.1.3], it turns out that the current G_u belongs to the class $\mathcal{D}_n(B^n\times\mathbb{S}^{N-1})$. Notice however that in general G_u is not i.m. rectifiable, and fails to satisfy the null-boundary condition $(\partial G_u)\llcorner B^n\times\mathbb{S}^{N-1}=0$, when $n\geq N$. However, in low dimension $n=N-1$, the null-boundary condition $(\partial G_u)\llcorner B^{N-1}\times\mathbb{S}^{N-1}=0$ holds true as a consequence of the strong density of smooth maps.

In these cases, we can write the current of the singularities $\mathbb{P}(u)$ in terms of the graph current G_u. For this purpose, recalling that $\mathcal{H}^{N-1}(\mathbb{S}^{N-1})=N\cdot\alpha_N$, where $\alpha_N:=|\mathbb{D}^N|$, we let $\omega_{\mathbb{S}^{N-1}}$ denote the *normalized volume* $(N-1)$*-form* in $\mathbb{S}^{N-1}$

$$\omega_{\mathbb{S}^{N-1}}:=\frac{1}{N\cdot\alpha_N}\sum_{j=1}^{N}(-1)^{j-1}y^j\,dy^1\wedge\cdots\wedge dy^{j-1}\wedge dy^{j+1}\wedge\cdots\wedge dy^N\,. \tag{6.6}$$

By Stokes' theorem we in fact have:

$$\int_{\mathbb{S}^{N-1}}\omega_{\mathbb{S}^{N-1}}=\int_{\partial\mathbb{D}^N}\omega_{\mathbb{S}^{N-1}}=\int_{\mathbb{D}^N}d\omega_{\mathbb{S}^{N-1}}=\frac{1}{\alpha_N}\int_{\mathbb{D}^N}dy^1\wedge\cdots\wedge dy^N=1\,.$$

Moreover, let $\pi_1:A\times\mathbb{R}^N\to A$ and $\pi_2:A\times\mathbb{R}^N\to\mathbb{R}^N$ denote the orthogonal projections onto the two factors, where $A=B^n$ or $A=\mathbf{C}^{n+1}$.

Proposition 6.4 *If n, N, and p are as in Proposition* 6.2*, for any $u\in W^{(N-1)/p,p}(B^n,\mathbb{S}^{N-1})$ we have*

$$\mathbb{P}(u)(\phi)=\partial G_u(\pi_1^{\#}\phi\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})\qquad\forall\,\phi\in\mathcal{D}^{n-N}(B^n)\,.$$

Proof: Denote by $\widehat{\omega}_{\mathbb{S}^{N-1}}$ an $(N-1)$-form in $\mathcal{D}^{N-1}(\mathbb{R}^N)$ that agrees with the right-hand side of (6.6) on $\mathbb{D}^N$. Since $V(\mathbf{C}^{n+1})\subset\mathbb{D}^N$, then

$$V^{\#}\,d\widehat{\omega}_{\mathbb{S}^{N-1}}=\frac{1}{\alpha_N}\,V^{\#}(dy^1\wedge\cdots\wedge dy^N)\,.$$

On account of Proposition 6.2, by (5.2) we have:

$$\begin{aligned}\mathbb{P}(u)(\phi)=&\ G_V(\pi_1^{\#}d\widetilde{\phi}\wedge\pi_2^{\#}d\widehat{\omega}_{\mathbb{S}^{N-1}})\\ =&\ G_V(\pi_1^{\#}d\widetilde{\phi}\wedge d\pi_2^{\#}\widehat{\omega}_{\mathbb{S}^{N-1}})=(-1)^{n-N+1}\partial G_V(\pi_1^{\#}d\widetilde{\phi}\wedge\pi_2^{\#}\widehat{\omega}_{\mathbb{S}^{N-1}})\end{aligned}$$

for every $\phi\in\mathcal{D}^{n-N}(B^n)$. Moreover, by definition (6.5), using that V satisfies the null-boundary condition (6.4) and that $\eta(t)=1$ for $t\in[0,1/4]$ and $\eta(t)=0$ for $t\in[3/4,1]$, we have

$$\begin{aligned}(-1)^{n-N+1}\partial G_V(\pi_1^{\#}d\widetilde{\phi}\wedge\pi_2^{\#}\widehat{\omega}_{\mathbb{S}^{N-1}})=&\ G_u(\pi_1^{\#}(d_x\widetilde{\phi}+d_t\widetilde{\phi})_{|t=0}\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})\\ =&\ G_u(\pi_1^{\#}d\phi\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})\,.\end{aligned}$$

Finally, since $d\pi_2^{\#}\omega_{\mathbb{S}^{N-1}}=\pi_2^{\#}d\omega_{\mathbb{S}^{N-1}}=0$, as $\omega_{\mathbb{S}^{N-1}}$ is a closed $(N-1)$-form in $\mathbb{S}^{N-1}$, we compute

$$G_u(\pi_1^{\#}d\phi\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})=G_u(d\pi_1^{\#}\phi\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})=\partial G_u(\pi_1^{\#}\phi\wedge\pi_2^{\#}\omega_{\mathbb{S}^{N-1}})$$

that clearly concludes the proof. □

Example 6.5 Of course, the conclusion in Proposition 6.4 extends to the limiting case of maps u in the Sobolev class $W^{1,N-1}(B^n,\mathbb{S}^{N-1})$. In fact, in that case the graph current G_u is i.m. rectifiable in $\mathcal{R}_n(B^n\times\mathbb{R}^N)$. If e.g. $u:B^n\to\mathbb{S}^{N-1}$ is given by $u(x)=\varphi(x/|x|)$ for some smooth map $\varphi:\partial B^n\to\mathbb{S}^{N-1}$, then $u\in W^{1,n-1}(B^n,\mathbb{S}^{N-1})$, and one has

$$(\partial G_u)\llcorner B^n\times\mathbb{S}^{N-1}=-\delta_0\times\varphi_{\#}[\![\,\partial B^n\,]\!]$$

where δ_0 is the unit Dirac mass at the origin, see Ex. 2 in [11, Sec. 3.2.2]. Therefore, if $n=N$ we get

$$\mathbb{P}(u)=-(\deg\varphi)\,\delta_0$$

where $\deg\varphi$ is the degree of $\varphi:\mathbb{S}^{N-1}\to\mathbb{S}^{N-1}$. If $n\geq N+1$, instead, since $\varphi(\partial B^n)\subset\mathbb{S}^{N-1}$, we infer that $\varphi_{\#}[\![\,\partial B^n\,]\!]=0$, whence the graph current G_u has no inner boundary, and finally $\mathbb{P}(u)=0$.

Remark 6.6 If $u \in W^{(N-1)/p,p}(\mathbb{R}^{N-1}, \mathbb{S}^{N-1})$, where $1 \le N-1 < p < p(N)$, see (6.2), arguing as before we can define the $(N-1)$-current G_u in $\mathcal{D}_{N-1}(\mathbb{R}^{N-1} \times \mathbb{S}^{N-1})$ by setting

$$G_u := (\partial G_V) \llcorner ((\mathbb{R}^{N-1} \times \{0\}) \times \mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^{N-1}(\mathbb{R}^{N-1} \times \mathbb{S}^{N-1}) .$$

Using a cut-off argument on the function V, it can be checked that in that case the definition (5.1) of degree yields to equation

$$\deg u = \frac{1}{\alpha_N} G_u(\pi_2^{\#} \omega_{\mathbb{S}^{N-1}}) .$$

It thus remains to give the

Proof of Proposition 5.7: We outline the main differences w.r.t. the proof of [21, Thm. 3]. Therefore, we come back to the proof of Theorem 1.1, in case $\mathcal{Y} = \mathbb{S}^{N-1}$, where some improvements are in order.

Due to the "almost Dirichlet principle", see e.g. [18, Thms. 1.9 and 1.11], in definition (1.7) we can choose $U := \text{Ext}(u)$ as the harmonic extension, and in the proof of Theorem 1.1 we can take $U(x,t) = U(x,-t)$ if $t < 0$. Also, recall that we have chosen a smooth function $\Phi : \mathbb{R}^N \to \mathbb{D}^N$ such that $\Phi(y) = y/|y|$ if $|y| \ge 1/2$, and Φ is a bi-Lipschitz map from $\{y \in \mathbb{R}^N \,:\, |y| \le 1/2\}$ to $\mathbb{D}^N$. Therefore, on account of the gradient estimates (4.5), we can replace maps W with values into the ε_0-neighborhood $\mathbb{S}^{N-1}_{\varepsilon_0}$ of $\mathbb{S}^{N-1}$ with their projections $\Phi \circ W$ onto $\mathbb{S}^{N-1}$, where we choose $0 < \varepsilon_0 < 1/2$, without affecting our previous estimates. Moreover, we point out the following:

Remark 6.7 With the notation from the Remark 5.3, assume now that $\{u_h\} \subset R^\infty_{s,p}(B^n, \mathbb{S}^{N-1})$ is such that the restriction $u_{h|F}$ strongly converges in $W^{s,p}$ to the restriction $u_{|F}$ of some map u. If $\mathbb{P}(u) = 0$, by possibly slightly moving the faces of X, by a slicing argument it turns out that

$$\lim_{h\to\infty} u_{h\#}[\![\partial F]\!](\omega_{\mathbb{S}^{N-1}}) = 0 .$$

Since moreover $u_{h\#}[\![\partial F]\!](\omega_{\mathbb{S}^{N-1}})$ is the degree of $u_{h|F} : \partial F \to \mathbb{S}^{N-1}$, we thus infer that for h sufficiently large the restriction $u_{h|F}$ has zero degree. In addition, by the continuity of the degree w.r.t. the $W^{s,p}$-convergence of maps from ∂F into $\mathbb{S}^{N-1}$, we also conclude that $\deg(u_{|\partial F}) = 0$.

Recalling that $2 \le N \le d \le n$, we can thus further improve the slicing argument at the beginning of the proof of Theorem 1.1 in Sec. 2, when $\mathcal{Y} = \mathbb{S}^{N-1}$, by choosing for every $m \in \mathbb{N}^+$ the grid of size $1/m$ in such a way that the following additional properties hold true:

iii) the restriction u_F to each k-face F of the k-skeleton $C^{(k)}_m$ satisfies $\mathbb{P}(u_F) = 0$, for $k = d \vee N, \dots, n$;

iv) $\deg(u_{|\partial F}) = 0$ for each N-face F in $C^{(N)}_m$, if $d = N$, see Remark 6.7.

The case $n = d$. Referring to the notation from the proof of [20, Thm. 1], as in the proof of [20, Prop. 5] it suffices to show the existence of a sequence $\{h_j\} \searrow 0$ such that for every l and j the $(d-1)$-cycle $v^{(m)}_{h_j\,|\partial F_l}$ is homologically trivial, i.e.

$$v^{(m)}_{h_j\,\#}[\![\partial F_l]\!](\omega_{\mathbb{S}^{N-1}}) = 0 \tag{6.7}$$

where $\omega_{\mathbb{S}^{N-1}}$ is given by (6.6).

If $d > N$, property (6.7) is automatically satisfied, see Example 6.5.

If $d = N$, recall that in Proposition 2.2, case $n = d$, we have proved the strong convergence of $V^{(m)}_h{}_{|\partial F_l \times [0,1]}$ to $U_l := U_{|\partial F_l \times [0,1]}$. By using iv) and on account of Remark 6.7, we thus infer that $v^{(m)}_{h\,|\partial F_l}$ has zero degree, definitely on h. Therefore, a diagonal argument on $l = 1, \dots, (m-1)^d$ gives (6.7).

The case $n \ge d+1$. Firstly, when extending $W^{(m)}_h$ to the $(d+1)$-cubes of the grid, we argue as in the case $n = d$. Moreover, when extending $W^{(m)}_h$ to the $(k+1)$-cubes of the grid, for $k = d+1, \dots, n$, since the traces $\mathbf{T}(W^{(m)}_h)$ take values into $\mathbb{S}^{N-1}$, it turns out that *no boundary is "produced"*, i.e., the traces $\mathbf{T}(W^{(m)}_h)$ are homologically trivial in the previous sense. The validity of the latter statement can be verified as a consequence of the case $n \ge N+1$ analyzed in Example 6.5. For that reason, further details are omitted. □

Acknowledgments. I wish to thank A. Appel for some useful discussions. I also warmly thank the referee for his/her careful reading of the paper which helped me to improve the first version. The author is a member of the GNAMPA of INDAM.

References

[1] F. Almgren, W. Browder & E.H. Lieb, "Co-area, liquid crystals, and minimal surfaces", In: *Partial differential equations* (Tianjin, 1986), 1–22, Lecture Notes in Math. 1306, Springer, Berlin, 1988.

[2] F. Bethuel, "The approximation problem for Sobolev maps between manifolds", *Acta Math.* **167** (1992) 153–206.

[3] F. Bethuel, "Approximations in trace spaces defined between manifolds", *Nonlinear Analysis* **24** (1995), 121–130.

[4] J. Bourgain, H. Brezis & P. Mironescu, "Lifting, degree and distributional jacobian revisited", *Comm. Pure Appl. Math.* **58** (2005), 529–551.

[5] P. Bousquet & P. Mironescu, "Prescribing the Jacobian in critical spaces", *J. Anal. Math.* **122** (2014), 317–373.

[6] P. Bousquet, A. C. Ponce & J. Van Schaftinghen, "Density of smooth maps for fractional Sobolev spaces $W^{s,p}$ into l simply connected manifolds when $s \geq 1$", *Confluentes Math.* **5** (2013), no. 2, 3–24.

[7] P. Bousquet, A. C. Ponce & J. Van Schaftinghen, "Strong approximation of fractional Sobolev spaces", *J. Fixed Point Theory Appl.* **15** (2014), 133–153.

[8] H. Brezis & P. Mironescu, "Density in $W^{s,p}(\Omega;N)$", *J. Funct. Anal.* **269** (2015), 2045–2109.

[9] H. Brezis & L. Nirenberg, "Degree theory and BMO; Part I: compact manifolds without boundaries", *Selecta Math. N. S.* **1** (1995), 197–263.

[10] A. Cianchi, D. E. Edmunds & P. Gurka, "On weighted Poincaré inequalities", *Math. Nachr.* **180** (1996), 15–41.

[11] M. Giaquinta, G. Modica & J. Souček: *Cartesian currents in the calculus of variations*, Ergebnisse Math. Grenzgebiete (III Ser), vol. 37, Springer, Berlin, 1998.

[12] M. Giaquinta & D. Mucci, "Density results for the $W^{1/2}$ energy of maps into a manifold", *Math. Z.* **251** (2005), 535–549.

[13] M. Giaquinta & D. Mucci, "On sequences of maps into a manifold with equibounded $W^{1/2}$-energies", *J. Funct. Anal.* **225** (2005), 94–146.

[14] F. Hang & F. Lin, "A remark on the Jacobians", *Comm. Contemp. Math.* **2** (2000), 35–46.

[15] F. Hang & F. Lin, "Topology of Sobolev mappings. II", *Acta Math.* **191** (2003), 55–107.

[16] F. Hardt & T. Rivière, "Connecting topological Hopf singularities", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **2** (2003), 287–344.

[17] P. Mironescu, "Sobolev maps on manifolds: degree, approximation, lifting. " Perspectives in nonlinear partial differential equations, 413–436, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, 2007.

[18] P. Mironescu & E. Russ, "Traces of weighted Sobolev spaces. Old and new", *Nonl. Anal.* **119** (2015), 354–381.

[19] D. Mucci, "A characterization of graphs which can be approximated in area by smooth graphs", *J. Eur. Math. Soc. (JEMS)* **3** (2001), 1–38.

[20] D. Mucci, "Strong density results in trace spaces of maps between manifolds", *Manuscripta Math.* **128** (2009), no. 4, 421–441.

[21] D. Mucci, "The homological singularities of maps in trace spaces between manifolds", *Math. Z.* **266** (2010), 817–849.

[22] D. Mucci, "The relaxed energy of fractional Sobolev maps with values into the circle", *Preprint 2021*, https://cvgmt.sns.it/paper/5406/

[23] R. Schoen & K. Uhlenbeck, "A regularity theory for harmonic maps", *J. Diff. Geom.* **17** (1982), 307–335.

[24] B. White, "Infima of energy functionals in homotopy classes", *J. Diff. Geom.* **23** (1986), 127–142.