REGULARITY OF CAPILLARITY DROPLETS WITH OBSTACLE

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Abstract. In this paper we study the regularity properties of Λ-minimizers of the capillarity energy in a half space with the wet part constrained to be confined inside a given planar region. Applications to a model for nanowire growth are also provided.

Contents

1. Introduction

Capillarity phenomena occur whenever two or more fluids are situated adjacent each other and do not mix. The separating interface is usually refereed to as a capillary surface. Since the pioneering works by Young and Laplace (see Finn's book [7] for an historical introduction) these phenomena have been the subject of countless studies in the mathematical and interdisciplinary literature. A modern treatment of the problem is based on Gauss' idea of describing equilibrium configurations as critical points or (local) minimizers of a free energy accounting for the area of the surface separating the fluids and the surrounding media, for the area of the *wet region* due to the adhesion between the fluids and the walls of the container, and for the possible presence of external fields acting on the system (such as gravity). The existence of minimizing configurations is easily obtained in the framework of sets of finite perimeter. While the regularity inside the container of such configurations reduces to the more classical study of minimal surfaces, a more specific question is related to the regularity of the contact line between the container and the fluid, see [18] for the physically relevant three dimensional case and [3] for a wide extension to more general anisotropic energies and to higher dimensions.

In this paper we study the regularity of the contact line for a capillarity problem where the 'container' is a half space H and the wet region is constrained to be confined inside

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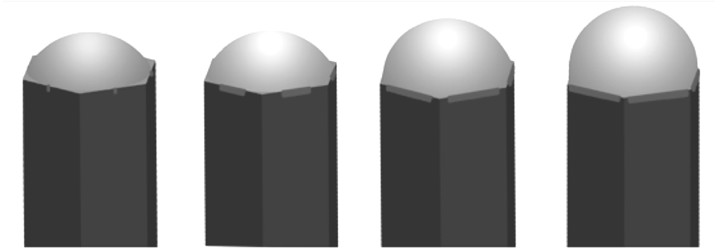

FIGURE 1. As the volume of the liquid drop increases the drop wets larger regions of the nanowire edge, but remains pinned at the top (reproduced from P. Krogstrup et al. [11])

a given planar domain $O \subset \partial H$. The motivation for this problem comes from the study of a mathematical model of vapor-liquid-solid (VLS) nanowire growth considered in the physical literature. We recall that during VLS growth a nanoscale liquid drop of catalyst deposited on the flat tip of the solid cylindrical nanowire feeds its vertical growth. In the experiments it is observed that the sharp edge of the nanowire produces a pinning effect and forces the wet part to remain confined inside the top face of the cylinder and the liquid drop to be contained in the upper half space, see Figure 1.

1.1. **Setting of the problem and main results.** Let us start by fixing some notations: We will work in $\mathbb{R}^N$ and we set

$$H = \{x \in \mathbb{R}^N : x_1 > 0\}.$$

Given $\sigma \in (-1, 1)$ we consider for a set of finite perimeter $E \subset H$ and an open set A (not necessarily contained in H) the *capillarity energy*

$$\begin{aligned}\mathcal{F}_\sigma(E; A) &= P(E; H \cap A) + \sigma P(E; \partial H \cap A)\\ &= \mathcal{H}^{N-1}(\partial^* E \cap H \cap A) + \sigma \mathcal{H}^{N-1}(\partial^* E \cap \partial H \cap A)\end{aligned}$$

where $\partial^* E$ is the reduced boundary of E, $P(E; G)$ is the perimeter of E in G (see the definitions at the beginning of Section 2) and $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure. In case $A = \mathbb{R}^N$ we will simply write $\mathcal{F}_\sigma(E)$.

We aim to impose a constraint on the *wet region* $\partial^* E \cap \partial H$ of E. To this end we consider a relatively open set $O \subset \partial H$ and we denote by

$$\mathcal{C}_O = \big\{E \subset H \text{ sets of locally finite perimeter such that } \partial^* E \cap \partial H \subset \overline{O}\big\}$$

the class of admissible competitors. We aim at studying the regularity properties of (local) minimizers of the variational problem

$$\min\{\mathcal{F}_\sigma(E) : E \in \mathcal{C}_O, |E| = m\}.$$

Note that classical variational arguments imply that if one assumes that $M := \overline{\partial E \cap H}$ is a smooth manifold with boundary, then any minimizer satisfies the following Euler-Lagrange conditions:

(i) (*Constant mean curvature*) There exists $\lambda > 0$ such that $H_M = \lambda$ in $M \cap H$, where H_M is the sum of the principle curvatures of M and more precisely coincides with the tangential divergence of the outer unit normal field ν_E to the boundary of E;
(ii) (*Young's inequality*) $\nu_E \cdot \nu_H \geq \sigma$ on $M \cap \partial H$;
(iii) (*Young's law inside O*) $\nu_E \cdot \nu_H = \sigma$ on $(M \cap \partial H) \setminus \partial_{\partial H} O$[1].

[1]Here and in the sequel for a set $U \subset \partial H$ we will denote by $\partial_{\partial H} U$ its relative boundary in ∂H

Note that (iii) above is the classical Young's law which holds true outside the *thin obstacle* $\partial_{\partial H} O$, while (ii) is a global inequality which should hold true on the whole free boundary $M \cap \partial H$.

As it is customary in Geometric Measure Theory, we will remove volume type constraints and we will deal with some perturbed minimality conditions. Note that this allows to treat several problems at once (volume constraints, potential terms, etc.). Concerning the regularity of the obstacle we introduce the following class of open subsets of ∂H satisfying a uniform inner and outer ball condition at every point of the (relative) boundary. More precisely, we give the following definition.

Definition 1.1. Let $R \in (0, +\infty)$. We denote by $\mathcal{B}_R$ the family of all relatively open subsets O of ∂H such that for every $x \in \partial_{\partial H} O$ there exist two $(N-1)$-dimensional relatively open balls B', $B'' \subset \partial H$ of radius R such that $B' \subset O$, $B'' \subset \partial H \setminus O$, and $\partial_{\partial H} B' \cap \partial_{\partial H} B'' = \{x\}$.

Morever, we set $\mathcal{B}_\infty = \cap_{R>0} \mathcal{B}_R$. Note that $\mathcal{B}_\infty$ is made up of all relatively open half-spaces of ∂H and of ∂H itself.

Remark 1.2. Note that if $O \in \mathcal{B}_R$, then O is of class $C^{1,1}$ with principal curvatures bounded by $\frac{1}{R}$, see [14, 2]. Therefore, if O_h is a sequence in $\mathcal{B}_{R_h}$ with $R_h \to R \in (0, +\infty]$, then there exists a (not relabelled) subsequence such that $\overline{O_h} \to \overline{O}$ in the Kuratowski sense, where $O \in \mathcal{B}_R$. Moreover, for all $\alpha \in (0,1)$ $O_h \to O$ in $C^{1,\alpha}_{loc}$ in the following sense: given $x \in \partial_{\partial H} O$, there exist a $(N-1)$-dimensional ball B centered at x, $\psi_h, \psi \in C^{1,1}(\mathbb{R}^{N-2})$ such that, up to a rotation in ∂H, $\partial_{\partial H} O_h \cap B$ coincides with the graph of ψ_h in B, $\partial_{\partial H} O \cap B$ coincides with the graph of ψ in B, and $\psi_h \to \psi$ locally in $C^{1,\alpha}(\mathbb{R}^{N-2})$.

Definition 1.3. Given $\Lambda \geq 0$, $r_0 > 0$, and $O \subset \partial H$ a relatively open set, we say that $E \in \mathcal{C}_O$ is a (Λ, r_0)-minimizer of $\mathcal{F}_\sigma$ with obstacle O if

$$\mathcal{F}_\sigma(E; B_{r_0}(x_0)) \leq \mathcal{F}_\sigma(F; B_{r_0}(x_0)) + \Lambda |E \Delta F| \qquad \text{for all } F \in \mathcal{C}_O \text{ such that } E \Delta F \Subset B_{r_0}(x_0)\,.$$

When E is a (Λ, r_0)-minimizer (with obstacle O) for every $r_0 > 0$ we will simply say that $E \subset H$ is a $(\Lambda, +\infty)$-minimizer (with obstacle O) or simply a Λ-minimizer (with obstacle O) of $\mathcal{F}_\sigma$.

We are in a position to state the main result of the paper, which establishes full regularity in three dimensions and partial regularity in any dimension, with an estimate on the Hausdorff dimension $\dim_{\mathcal{H}}$ of the singular set.

Theorem 1.4. *Let $O \subset \partial H$ be a relatively open set of class $C^{1,1}$ and let $E \in \mathcal{C}_O$ a (Λ, r_0)-minimizer with obstacle O of $\mathcal{F}_\sigma$. Then, the following conclusions hold true:*

(i) if $N = 3$, then $\overline{\partial E \cap H}$ is a surface with boundary of class $C^{1,\tau}$ for all $\tau \in (0, \frac{1}{2})$.

(ii) if $N \geq 4$, there exists a closed set $\Sigma \subset \overline{\partial E \cap H} \cap \partial H$, with $\dim_{\mathcal{H}}(\Sigma) \leq N-4$, such that $\overline{\partial E \cap H} \setminus \Sigma$ is (locally) a $C^{1,\tau}$ hypersurface with boundary for all $\tau \in (0, \frac{1}{2})$.

Moreover,

$$\begin{aligned} \nu_E \cdot \nu_H &\geq \sigma \quad \text{on} \quad M \cap \partial H; \\ \nu_E \cdot \nu_H &= \sigma \quad \text{on} \quad (M \cap \partial H) \setminus \partial_{\partial H} O, \end{aligned}$$

where we set $M := \overline{\partial E \cap H} \setminus \Sigma$.

The main new point of the above result is the regularity at the points of the free-boundary $\overline{\partial E \cap H} \cap \partial H$ lying on the thin obstacle $\partial_{\partial H} O$. Indeed the full regularity at the boundary points $\overline{\partial E \cap H} \cap O$ for $N = 3$ follows from the classical result of [18], while

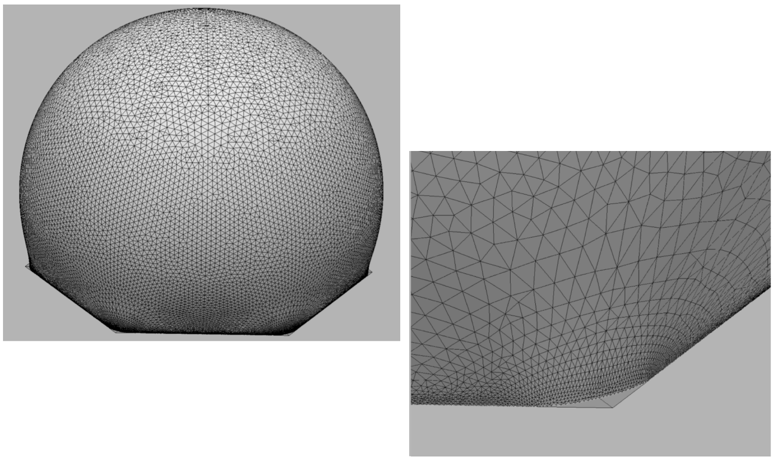

FIGURE 2. On the left a drop sitting on the tip of a nanowire with hexagonal section. The picture on the right shows the behaviour of the droplet near a corner (courtesy of B. Spencer)

the partial regularity for $N \geq 4$ from the more recent paper [3]. The idea the proof of Theorem 1.4 is based on the following dichotomy, inspired by the work of Fernández-Real and Serra, [6], where a thin obstacle problem for the area functional is studied, see also the work of Focardi and Spadaro [8] for the non parametric case. If in a r-neighborhood of $x \in \overline{\partial E \cap H} \cap \partial_{\partial H} O$ the boundary $\partial E \cap H$ is contained in a sufficiently thin strip with a slope sufficiently larger than the slope $\alpha(\sigma)$ given by the Young's law, then we show by a barrier argument that at scale $r/2$ the free boundary fully coincides with the thin obstacle $\partial_{\partial H} O$. If instead the slope of the strip is sufficiently close to $\alpha(\sigma)$, then a linearization procedure allows us to reduce to a Signorini type problem and to deduce from the decay estimates available for the latter a flatness improvement result at a smaller scale. Iterating this dichotomy argument leads to a *boundary ε-regularity* result, see Theorem 3.21, which, combined with a suitable monotonicity formula, finally yields the proof of Theorem 1.4. Since the barrier argument requires closeness in the "uniform norm", in order to make the dichotomy effective, we need to establish a decay of the excess in the same norm. For this step we follow the ideas introduced by Savin in [16] to deal with interior regularity of minimal surfaces and we rely on a partial Harnack inequality which is obtained by a nontrivial extension to the boundary of these ideas. The connection with the Signorini problem also explains why we cannot expect better regularity than $C^{1,1/2}$, see for instance [15].

1.2. **Applications to a model for nanowire growth.** As anticipated at the beginning of this introduction, we conclude by applying the above regularity theory to a model of vapor-liquid-solid (VLS) nanowire growth considered in the physical literature and studied in [9].

Following the work of several authors, see the references in [9], we consider a continuum framework for nanowire VLS growth. We model the nanowire as a semi-infinite cylinder $\mathbf{C} = \omega \times (-\infty, 0]$, where $\omega \subset \mathbb{R}^2$ is a bounded sufficiently regular domain, and the liquid drop as a set $E \subset \mathbb{R}^3 \setminus \mathbf{C}$ of finite perimeter. Typically the observed nanowires have either a regular or a polygonal section. Given $\sigma \in (-1, 1)$ we consider the free energy

$$J_{\sigma,\omega}(E) := \mathcal{H}^2(\partial^* E \setminus \mathbf{C}) + \sigma \mathcal{H}^2(\partial^* E \cap \mathbf{C}) .$$

The shape of the liquid drop is then described by (local) minimizers of $J_{\sigma,\omega}$ under a volume constraint. To this aim we say that $E \subset \mathbb{R}^3 \setminus \mathbf{C}$ is a *volume constrained local minimizer of* $J_{\sigma,\omega}$ if there exists $\varepsilon > 0$ such that $J_{\sigma,\omega}(E) \leq J_{\sigma,\omega}(F)$ for all $F \subset \mathbb{R}^3 \setminus \mathbf{C}$ with $|F| = |E|$ and $|E\Delta F| < \varepsilon$. Here we study local minimizing configurations corresponding to liquid drops sitting on the top $\mathbf{C}_{top} := \omega \times \{0\}$ of the cylinder and contained in the upper half space $\mathscr{H} := \{x_3 > 0\}$.

In the case where ω is a $C^{1,1}$ domain we prove the following regularity result.

Theorem 1.5. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$ and let $E \subset \mathscr{H}$ be a volume constrained local minimizer of $J_{\sigma,\omega}$. Then $\overline{\partial E \cap \mathscr{H}}$ is a surface with boundary of class $C^{1,\tau}$ for all $\tau \in (0, \frac{1}{2})$.*

In the experiments it is observed that for nanowires with a polygonal section, see for instance [11], the liquid drop never wets the corners of the polygon as illstrated in Figure 1.2. Here we give a rigorous proof of this fact when $\sigma < 0$ and, for a general σ, under the additional assumption that the contact between the liquid drop and $\gamma := \partial\omega \times \{0\}$ is nontangential (see Definition 4.6).

Theorem 1.6. *Let $\omega \subset \mathbb{R}^2$ be a convex polygon and let $E \subset \mathscr{H}$ be a volume constrained local minimizer of $J_{\sigma,\omega}$. If $\sigma < 0$, then $\overline{\partial E \cap \mathscr{H}}$ is a surface with boundary of class $C^{1,\tau}$ for all $\tau \in (0, \frac{1}{2})$ and the contact line $\overline{\partial E \cap \mathscr{H}} \cap \mathbf{C}_{top}$ does not contain any vertex of the polygon. If $\sigma \geq 0$ the same conclusion holds provided that E has a nontangential contact at all points of $\overline{\partial E \cap \mathscr{H}} \cap \gamma$.*

2. Density estimates and compactness

Given a Lebesgue measurable set $E \subset \mathbb{R}^N$, we say that E is of locally finite perimeter if there exists a $\mathbb{R}^N$-valued Radon measure μ_E such that

$$\int_E \nabla\varphi \, dx = \int_{\mathbb{R}^N} \varphi \, d\mu_E \qquad \text{for all } \varphi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N) \, .$$

If $G \subset \mathbb{R}^N$ is a Borel set we denote by $P(E; G) = |\mu_E|(G)$ the perimeter of E in G. For all relevant definitions and properties of sets of finite perimeter we shall refer to the book [12]. In the following we denote by $\partial^* E$ the reduced boundary of a set of finite perimeter and by $\partial^e E$ the *essential boundary*, which is defined as

$$\partial^e E := \mathbb{R}^N \setminus (E^{(0)} \cup E^{(1)}) \, ,$$

where $E^{(0)}$ and $E^{(1)}$ are the sets of points where the density of E is 0 and 1, respectively. Since the perimeter measure coincides with the $\mathcal{H}^{N-1}$ measure restricted to the reduced boundary $\partial^* E$, we will often write $\mathcal{H}^{N-1}(\partial^* E \cap \Omega)$ instead of $P(E; \Omega)$. Note that if $E \subset H$ the characteristic function of $\partial^* E \cap \partial H$ is the trace of 1_E intended as a BV function (for the definition of trace see [1, Th. 3.87]).

In the following, when dealing with a set of locally finite perimeter E, we will always assume that E coincides with a precise representative that satisfies the property $\partial E = \overline{\partial^* E}$, see [12, Remark 16.11]. A possible choice is given by $E^{(1)}$, for which one may check that

$$\partial E^{(1)} = \overline{\partial^* E} \, . \tag{2.1}$$

Given a set E we denote by $E_{x,r}$ the set $E_{x,r} := (E - x)/r$. A ball centered in x and of radius r is denoted by $B_r(x)$, if the center is 0 we simply set B_r. Note that if E is (Λ, r_0)-minimizer of $\mathcal{F}_\sigma$ with obstacle $O \in \mathcal{B}_R$ and $x \in \partial H$, $r > 0$ then $E_{x,r}$ is a $(\Lambda r, r_0/r)$

minimizer of $\mathcal{F}_\sigma$ with obstacle $O_{x,r} \in \mathcal{B}_{\frac{R}{r}}$. Hence we can assume without loss of generality that E is $(\Lambda, 1)$-minimizer with obstacle $O \in \mathcal{B}_R$, where

$$\Lambda + \frac{1}{R} \leq \boldsymbol{c_0} \leq 1\,, \tag{2.2}$$

with $\boldsymbol{c_0}$ is a small constant, depending on σ and N (to be chosen later). We will say, with a slight abuse of language, that a constant is *universal* if it only depends on Λ, σ and on the dimension.

We start by observing that the constraint on the wet part can be replaced by a suitable penalization. To this end we introduce the following functional

$$\mathcal{F}^O_\sigma(F; A) := P(F; A \setminus \overline{O}) + \sigma P(F; O \cap A)\,,$$

where $A \subset \mathbb{R}^N$ is an open set and $F \subset H$ is a set of finite perimeter. If $A = \mathbb{R}^N$ we simply write $\mathcal{F}^O_\sigma(F)$. We also denote by $\mathbf{C}_O$ the semi-infinite cylinder constructed over O, that is,

$$\mathbf{C}_O := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0 \text{ and } (x_2, \dots, x_N) \in O\}\,.$$

Lemma 2.1. *Assume that $E \subset H$ is a (Λ, r_0)-minimizer for $\mathcal{F}_\sigma$ in the sense of Definition 1.3. Then, E is a (Λ, r_0)-minimizer for $\mathcal{F}^O_\sigma$ without obstacle, that is,*

$$\mathcal{F}^O_\sigma(E; B_{r_0}(x_0)) \leq \mathcal{F}^O_\sigma(F; B_{r_0}(x_0)) + \Lambda|E\Delta F| \text{ for all } F \subset H \text{ such that } E\Delta F \Subset B_{r_0}(x_0)\,.$$

Proof. Since the argument is local we may assume without loss of generality that E is a set of finite perimeter.

Let F be as in the statement and let B be an open ball of radius r_0 such that $E\Delta F \Subset B$ and $\mathcal{H}^{N-1}(\partial^* F \cap \partial B) = 0$, and set $B^+ = B \cap H$. For every $\varepsilon > 0$ (sufficiently small) we set

$$F_\varepsilon := [(F \cap \{x_1 > \varepsilon\} \cap B^+) \setminus \mathbf{C}_O)] \cup (F \cap \mathbf{C}_O) \cup (F \setminus B^+)\,.$$

Note that F_ε is an admissible competitor in the sense of Definition 1.3 and thus

$$\mathcal{F}_\sigma(E) \leq \mathcal{F}_\sigma(F_\varepsilon) + \Lambda|F_\varepsilon \Delta E|\,. \tag{2.3}$$

Note that for a.e. ε

$$\begin{aligned}\mathcal{F}_\sigma(F_\varepsilon) \leq & P(F; (B^+ \cap \{x_1 > \varepsilon\}) \setminus \overline{\mathbf{C}}_O) + \mathcal{H}^{N-1}((F^{(1)} \cap B^+ \cap \{x_1 = \varepsilon\}) \setminus \overline{\mathbf{C}}_O) \\ & + P(F; B^+ \cap \overline{\mathbf{C}}_O) + P(F; H \setminus B^+) + \sigma P(F; O) \\ & + \mathcal{H}^{N-1}(F^{(1)} \cap B^+ \cap \partial\mathbf{C}_O \cap \{x_1 \leq \varepsilon\}) + \mathcal{H}^{N-1}(\partial B \cap F^{(1)} \cap \{x_1 \leq \varepsilon\})\,.\end{aligned}$$

By the continuity of the trace with respect to strict convergence in BV (see [1, Th. 3.88]) we have that

$$\mathcal{H}^{N-1}((F^{(1)} \cap B^+ \cap \{x_1 = \varepsilon_n\}) \setminus \overline{\mathbf{C}}_O) \to P(F; \partial H \setminus O)$$

for a sequence $\varepsilon_n \to 0^+$. Thus,

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{F}_\sigma(F_\varepsilon) \leq P(F; B^+ \setminus \overline{\mathbf{C}}_O) + P(F; \partial H \setminus O) + P(F; B^+ \cap \overline{\mathbf{C}}_O) \\ + P(F; H \setminus \overline{B}^+) + \sigma P(F; O) = \mathcal{F}^O_\sigma(F)\,.\end{aligned}$$

Then conclusion follows recalling (2.3). □

Remark 2.2. Observe that $\mathcal{F}_\sigma(\cdot, A)$ is lower semicontinuous with respect to the L^1_{loc} convergence in $\mathcal{C}_O$, see [12, Prop. 19.1]. The same proposition can be applied to show that also $\mathcal{F}^O_\sigma(\cdot, A)$ is lower semicontinuous with respect to L^1_{loc} in H. To this aim it is enough to observe that it is possible to construct an open set U of class $C^{1,1}$ contained in $\mathbb{R}^N \setminus \overline{H}$ such that $\partial U \cap \partial H = \overline{O}$ and then to apply [12, Prop. 19.1 and Prop. 1.3] to $\mathbb{R}^N \setminus \overline{U}$.

We now prove some useful volume and perimeter density estimates.

Proposition 2.3. *Let $\sigma \in (-1,1)$, let E be a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ with obstacle $O \in \mathcal{B}_R$ and assume that (2.2) is in force. Then there are universal positive constants c_1 and C_1 such that*

(i) *for all $x \in H$*

$$P(E; B_r(x)) \le C_1 r^{N-1};$$

(ii) *for all $x \in \partial E$*

$$|E \cap B_r(x)| \ge c_1 |B_r(x) \cap H|;$$

(iii) *for all $x \in \overline{\partial E \cap H}$*

$$P(E; B_r(x) \cap H) \ge c_1 r^{N-1};$$

(iv) *if $x \in \partial(H \setminus E)$ and $B_{2r}(x) \cap \partial H \subset O$, then*

$$|B_r(x) \setminus E| \ge c_1 |B_r(x) \cap H|$$

for every $r \le 1$. Finally, E is equivalent to an open set, still denoted by E, such that $\partial E = \partial^e E$, hence $\mathcal{H}^{N-1}(\partial E \setminus \partial^* E) = 0$.

Proof. The proof of this proposition follows the lines of [3, Lemma 2.8]; however, some modifications are needed. Given $x \in H$ and $r < 1$, we set $m(r) := |E \cap B_r(x)|$. Recall that for a.e. such r we have $m'(r) = \mathcal{H}^{N-1}(E^{(1)} \cap \partial B_r(x))$ and $\mathcal{H}^{N-1}(\partial^* E \cap \partial B_r(x)) = 0$. For any such r we set $F := E \setminus B_r(x)$. Then, using Definition 1.3, we have

$$\begin{aligned} P(E; B_r(x) \cap H) &\le \mathcal{H}^{N-1}(\partial B_r(x) \cap E^{(1)}) + \Lambda |E \cap B_r(x)| + |\sigma| P(E; \partial H \cap B_r(x)) \\ &\le C_1 r^{N-1} \end{aligned} \tag{2.4}$$

for a suitable universal constant C_1. Observe now that by an easy application of the divergence theorem we have

$$P(E; \partial H \cap B_r(x)) = P(E \cap B_r(x); \partial H) \le P(E \cap B_r(x); H).$$

Thus, using also (2.4), we have

$$\begin{aligned} P(E \cap B_r(x)) &= P(E \cap B_r(x); H) + P(E \cap B_r(x); \partial H) \\ &\le 2P(E \cap B_r(x); H) = 2P(E; B_r(x) \cap H) + 2m'(r) \\ &\le 4m'(r) + 2\Lambda m(r) + 2|\sigma| P(E; \partial H \cap B_r(x)) \\ &\le 4m'(r) + 2\Lambda m(r) + 2|\sigma| P(E \cap B_r(x); H). \end{aligned} \tag{2.5}$$

Comparing the first term in the second line with the fourth line of the previous chain of inequalities we have in particular that

$$P(E \cap B_r(x); H) \le \frac{1}{1-|\sigma|}(2m'(r) + \Lambda m(r)).$$

In turn, inserting the above estimate in (2.5) and using the isoperimetric inequality we get

$$\begin{aligned} N\omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} &\le P(E \cap B_r(x)) \le \frac{2}{1-|\sigma|}(2m'(r) + \Lambda m(r)) \\ &\le \frac{2}{1-|\sigma|}(2m'(r) + \Lambda r \omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}}) \\ &\le \frac{2}{1-|\sigma|}(2m'(r) + \mathbf{c_0} \omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}}), \end{aligned}$$

where in the last inequality we used (2.2). Now if $\frac{2\boldsymbol{c_0}}{1-|\sigma|} \leq 1$, then from the previous inequality we get

$$(N-1)\omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} \leq \frac{4}{1-|\sigma|} m'(r)\,.$$

Observe now that if in addition $x \in \partial^* E$, then $m(r) > 0$ for all r as above. Thus, we may divide the previous inequality by $m(r)^{\frac{N-1}{N}}$, and integrate the resulting differential inequality thus getting

$$|E \cap B_r(x)| \geq c_1 |B_r(x) \cap H|\,,$$

for a suitable positive constant c_1 depending only on N and $|\sigma|$.

To get the lower density estimate on the perimeter, let $x \in \partial^* E \cap H$ and $r < 1$. If $\mathrm{dist}(x, \partial_{\partial H} O) > \frac{r}{2}$, then either $B_\rho(x) \cap \partial H \subset \partial H \setminus \overline{O}$ for all $\rho \in (0, \frac{r}{2})$ or $B_\rho(x) \cap \partial H \subset O$ for all $\rho \in (0, \frac{r}{2})$.

In the first case, E is a $(\Lambda, r/2)$-minimizer of the standard perimeter in $B_{\frac{r}{2}}(x)$ under the constraint $E \subset H$. Then an easy truncation argument implies that E is also an unconstrained $(\Lambda, r/2)$-minimizer of the perimeter in the same ball. Then, the lower density estimate on the perimeter follows from classical results (see [12, Th. 21.11]).

In the second case, it follows that $H \setminus E$ is a $(\Lambda, r/2)$-minimizer of $\mathcal{F}_{-\sigma}$ in $B_{\frac{r}{2}}(x)$ and thus, arguing as above, we get

$$|E \cap B_{\frac{r}{2}}(x)| \leq (1-c_1)|B_{\frac{r}{2}}(x) \cap H|$$

and, in turn, by the relative isoperimetric inequality, setting for every $\tau \in (\frac{1}{2}, 1)$

$$\kappa(\tau) := \inf_{t \geq 0} \inf \Big\{ \frac{P(F; B_1(te_1) \cap H)}{|F|^{\frac{N-1}{N}}} : F \subset B_1(te_1) \cap H,\, |F| \leq \tau |B_1(te_1) \cap H| \Big\}$$

we obtain

$$P(E, B_{\frac{r}{2}}(x) \cap H) \geq \kappa(1-c_1)|E \cap B_{\frac{r}{2}}(x)|^{\frac{N-1}{N}} \geq c_1 \kappa(1-c_1)|B_{\frac{r}{2}}(x) \cap H|^{\frac{N-1}{N}}\,.$$

Assume now that $\mathrm{dist}(x, \partial_{\partial H} O) \leq \frac{r}{2}$. Then, there exists $y = (0, y') \in \partial_{\partial H} O$ such that the ball $B_{\frac{r}{2}}(y) \cap H$ is contained in $B_r(x) \cap H$. Assume that $|E \cap B_{\frac{r}{2}}(y)| \leq (1-\gamma)|B_{\frac{r}{2}}(y) \cap H|$ for some small $\gamma \in (0, \frac{1}{2})$ to be chosen later. By the relative isoperimetric inequality we get

$$P(E; B_r(x) \cap H) \geq \kappa_N \min\{|E \cap B_r(x)|, |(H \setminus E) \cap B_r(x)|\}^{\frac{N-1}{N}}\,.$$

If $\min\{|E \cap B_r(x)|, |(H \setminus E) \cap B_r(x)|\} = |E \cap B_r(x)|$, then (iii) follows from (ii). Otherwise

$$\begin{aligned} P(E; B_r(x) \cap H) &\geq \kappa_N |(H \setminus E) \cap B_r(x)|^{\frac{N-1}{N}} \\ &\geq \kappa_N |(H \setminus E) \cap B_{\frac{r}{2}}(y)|^{\frac{N-1}{N}} \geq \kappa_N \Big(\frac{\gamma \omega_N}{2^{N+1}}\Big)^{\frac{N-1}{N}} r^{N-1}\,. \end{aligned}$$

If instead

$$|E \cap B_{\frac{r}{2}}(y)| > (1-\gamma)|B_{\frac{r}{2}}(y) \cap H|\,, \tag{2.6}$$

then we denote by Π the orthogonal projection on ∂H of $\partial^* E \cap B_{\frac{r}{2}}(y) \cap H$. Set $D := \{(0, z') : |z' - y'| < \frac{r}{4}\} \setminus O$ and observe that

$$\mathcal{H}^{N-1}(D) \geq \tilde{c}\omega_{N-1}\Big(\frac{r}{4}\Big)^{N-1}\,,$$

for a universal constant $\tilde{c} > 0$. Note that in the above inequality we are using the assumption that $O \in \mathcal{B}_R$, with $\frac{1}{R} \leq \boldsymbol{c_0}$ (see (2.2)). Assume first that $\mathcal{H}^{N-1}(\Pi \cap D) \geq$

$\gamma\omega_{N-1}\big(\frac{r}{4}\big)^{N-1}$. In this case, we have

$$P(E;B_{\frac{r}{2}}(y)\cap H)\geq\gamma\omega_{N-1}\Big(\frac{r}{4}\Big)^{N-1}. \tag{2.7}$$

If instead $\mathcal{H}^{N-1}(\Pi\cap D)<\gamma\omega_{N-1}\big(\frac{r}{4}\big)^{N-1}$, then

$$|(H\setminus E)\cap B_{\frac{r}{2}}(y)|\geq\frac{r}{4}\mathcal{H}^{N-1}(D\setminus\Pi)\geq(\tilde c-\gamma)\omega_{N-1}\Big(\frac{r}{4}\Big)^{N}.$$

The last inequality contradicts (2.6) if γ is sufficiently small. This contradiction shows that (2.7) holds under the assumption (2.6). This completes the proof of (iii).

Property (iv) follows from the volume estimate (ii) recalling that $H\setminus E$ is a (Λ,r_0)-minimizer for $\mathcal{F}_{-\sigma}$ in $B_{2r}(x)$.

In order to show that $\partial E=\partial^e E$ it is enough to prove that $\partial E\subset\partial^e E$. To this aim fix $x\in\partial E$. If $x\in H$, from the density estimates (ii) and (iv) we have at once that $x\notin E^{(0)}\cup E^{(1)}$, that is $x\in\partial^e E$. On the other hand, if $x\in\partial H$, then $|B(x,r)\setminus E|\geq\frac{1}{2}|B(x,r)|$ and again, recalling (ii), we have that $x\in\partial^e E$. Hence $\mathcal{H}^{N-1}(\partial E\setminus\partial^* E)=\mathcal{H}^{N-1}(\partial^e E\setminus\partial^* E)=0$, where the last equality follows from Theorem 16.2 in [12].

Finally, since, see (2.1), $\partial E^{(1)}=\partial E=\partial^e E$, we have that $E^{(1)}\cap\partial E^{(1)}=\emptyset$, hence $E^{(1)}$ is an open set. □

We are now ready to prove the following compactness theorem for almost minimizers.

Theorem 2.4. *Let $\Lambda_h\geq 0$, $r_h,R_h\in[1,+\infty]$ satisfying*

$$\Lambda_h+\frac{1}{R_h}\leq\boldsymbol{c_0}\,,$$

and

$$\Lambda_h\to\Lambda_0\in[0,+\infty),\quad r_h\to r_0\in[1,+\infty],\quad R_h\to R_0\in[1,+\infty]\,,$$

with $\boldsymbol{c_0}$ as in (2.2). Let E_h be a (Λ_h,r_h)-minimizer of $\mathcal{F}_\sigma$ with obstacle $O_h\in\mathcal{B}_{R_h}$. Then there exist a (not relabelled) subsequence, a set $O\in\mathcal{B}_{R_0}$, and a set E of locally finite perimeter, such that $E_h\to E$ in $L^1_{loc}(\mathbb{R}^N)$, $O_h\to O$ in $C^{1,\alpha}_{loc}$ for all $\alpha\in(0,1)$, with the property that E is a (Λ_0,r_0)-minimizer of $\mathcal{F}^O_\sigma$. Moreover,

$$\mu_{E_h}\overset{*}{\rightharpoonup}\mu_E\,,\quad|\mu_{E_h}|\overset{*}{\rightharpoonup}|\mu_E|\,,$$

as Radon measures. In addition, the following Kuratowski convergence type properties hold:

(i) for every $x\in\partial E$ there exists $x_h\in\partial E_h$ such that $x_h\to x$;

(ii) if $x_h\in\overline{H\cap\partial E_h}$ and $x_h\to x$, then $x\in\overline{H\cap\partial E}$.

Finally, if $\Lambda_0=0$ and $\partial H\setminus\overline{O}$ is connected, then either $\partial E\cap(\partial H\setminus\overline{O})=\partial H\setminus\overline{O}$ or $\partial E\cap\partial H\subset\overline{O}$. In the latter case, we have that E is a $(0,r_0)$-minimizer with obstacle O,

$$|\mu_{E_h}|\llcorner H\overset{*}{\rightharpoonup}|\mu_E|\llcorner H$$

as Radon measures in $\mathbb{R}^N$, and

$$\partial E_h\cap\partial H\to\partial E\cap\partial H\quad\text{in } L^1_{loc}(\partial H)\,.$$

Proof. The proof goes along the lines of the proof of [3, Th. 2.9], with some nontrivial modifications that we explain here. First of all the existence of a non relabelled sequence E_h converging in $L^1_{loc}(\mathbb{R}^N)$ to some set E of locally finite perimeter and such that $\mu_{E_h}\overset{*}{\rightharpoonup}\mu_E$ as Radon measures follows at once since the sets E_h have locally equibounded perimeters, thanks to Proposition 2.3 (i). The convergence, up to a subsequence, of O_h to $O\in\mathcal{B}_{R_0}$ in $C^{1,\alpha}_{loc}$ follows from Remark 1.2.

We may assume without loss of generality that $r_0 < +\infty$. We now want to show that E is a (Λ_0, r_0)-minimizer of $\mathcal{F}_\sigma^O$. To this end, let us fix a ball $B_{r_0}(x)$, and consider a competitor F for E such that $E\Delta F \Subset B_{r_0}(x)$. Note that for a.e. $r < r_0$ such that $E\Delta F \Subset B_r(x)$, we have $\mathcal{H}^{N-1}(\partial^* E \cap \partial B_r(x)) = \mathcal{H}^{N-1}(\partial^* E_h \cap \partial B_r(x)) = 0$ for all h and $\mathcal{H}^{N-1}((E\Delta E_h)\cap\partial B_r(x)) \to 0$. Choose any such r and set $F_h := (F\cap B_r(x))\cup(E_h\setminus B_r(x))$. Denoting by $O^\delta := \{x \in \partial H : d_O(x) < -\delta \operatorname{sign}\sigma\}$ with $\delta > 0$ and d_O being the signed distance from ∂O restricted to ∂H and recalling that E_h is a (Λ_h, r_h)-minimizer of $\mathcal{F}_\sigma^{O_h}$, we have by Remark 2.2, if $\sigma \le 0$

$$
\begin{aligned}
\mathcal{F}_\sigma^{O^\delta}(E; B_r(x)) &\le \liminf_h \mathcal{F}_\sigma^{O^\delta}(E_h; B_r(x)) \le \liminf_h \mathcal{F}_\sigma^{O_h}(E_h; B_r(x)) \\
&\le \limsup_h \mathcal{F}_\sigma^{O_h}(E_h; B_{r_0}(x)) \le \lim_h \big[\mathcal{F}_\sigma^{O_h}(F_h; B_{r_0}(x)) + \Lambda_h |F_h \Delta E_h|\big] \\
&= \mathcal{F}_\sigma^{O}(F; B_{r_0}(x)) + \Lambda_0 |F\Delta E|\,.
\end{aligned}
$$

where we used the fact that $O_h \subset O^\delta$ for h sufficiently large. Instead, if $\sigma > 0$,

$$
\begin{aligned}
\mathcal{F}_\sigma^{O^\delta}(E; B_r(x)) &\le \liminf_h \mathcal{F}_\sigma^{O^\delta}(E_h; B_r(x)) \\
&\le \liminf_h \big(\mathcal{F}_\sigma^{O_h}(E_h; B_r(x)) + \mathcal{H}^{N-1}((O_h \setminus O^\delta) \cap B_r(x))\big) \\
&\le \limsup_h \mathcal{F}_\sigma^{O_h}(E_h; B_{r_0}(x)) + \mathcal{H}^{N-1}((O \setminus O^\delta) \cap B_r(x)) \\
&\le \lim_h \big[\mathcal{F}_\sigma^{O_h}(F_h; B_{r_0}(x)) + \mathcal{H}^{N-1}((O \setminus O^\delta) \cap B_r(x)) + \Lambda_h |F_h \Delta E_h|\big] \\
&= \mathcal{F}_\sigma^{O}(F; B_{r_0}(x)) + \mathcal{H}^{N-1}((O \setminus O^\delta) \cap B_r(x)) + \Lambda_0 |F\Delta E|\,.
\end{aligned}
$$

Letting $\delta \to 0^+$ we have in both cases that

$$
\begin{aligned}
\mathcal{F}_\sigma^{O}(E; B_{r_0}(x)) &\le \liminf_h \mathcal{F}_\sigma^{O_h}(E_h; B_{r_0}(x)) \le \limsup_h \mathcal{F}_\sigma^{O_h}(E_h; B_{r_0}(x)) \\
&\le \mathcal{F}_\sigma^{O}(F; B_{r_0}(x)) + \Lambda_0 |F\Delta E|\,.
\end{aligned}
$$

Thus, we have proved that E is a (Λ_0, r_0)-minimizer of $\mathcal{F}_\sigma^O$. Choosing $F = E$ in the previous inequality, we obtain

$$
\mathcal{F}_\sigma^{O_h}(E_h; B_{r_0}(x)) \to \mathcal{F}_\sigma^{O}(F; B_{r_0}(x))\,. \tag{2.8}
$$

Assume now that $\sigma \le 0$ and observe that by the lower semicontinuity of perimeter we get

$$
\begin{aligned}
P(E; B_{r_0}(x) \setminus \overline{O^\delta}) &\le \liminf_h P(E_h; B_{r_0}(x) \setminus \overline{O^\delta}) \\
&\le \liminf_h P(E_h; B_{r_0}(x) \setminus \overline{O_h}) = \liminf_h P(E_h; H \cap B_{r_0}(x))\,.
\end{aligned}
$$

Hence, letting $\delta \to 0^+$ we have

$$
P(E; B_{r_0}(x) \setminus \overline{O}) \le \liminf_h P\big(E_h; H \cap B_{r_0}(x)\big)\,. \tag{2.9}
$$

On the other hand, by similar arguments

$$
\begin{aligned}
\limsup_h P(E_h; O_h \cap B_{r_0}(x)) &= \limsup_h P(E_h; \overline{O_h} \cap B_{r_0}(x)) \\
&\le P(E; \overline{O} \cap B_{r_0}(x)) = P(E; O \cap B_{r_0}(x))\,.
\end{aligned} \tag{2.10}
$$

From the above inequalities and (2.8) and the fact that $\sigma \le 0$ we deduce that

$$
P(E_h; B_{r_0}(x)) \to P(E; B_{r_0}(x))\,. \tag{2.11}
$$

From this inequality we deduce that $|\mu_{E_h}| \overset{*}{\rightharpoonup} |\mu_E|$ as Radon measures in $\mathbb{R}^N$. If $\sigma > 0$, we observe that

$$\sigma P(E_h; O_h \cap B_{r_0}(x)) = \sigma \mathcal{H}^{N-1}(O_h \cap B_{r_0}(x)) - \sigma P(H \setminus E_h; O_h \cap B_{r_0}(x)) .$$

Therefore, arguing as in the proof of (2.10) and recalling (2.9) we conclude that (2.11) holds also in this case. The properties (i) and (ii) then follow by standard arguments using also the perimeter density estimate stated in Proposition 2.3 (iii), see also Step 4 of the proof of Theorem 2.9 in [12].

Let us now prove the last part the statement, and thus assume $\Lambda_0 = 0$ and that $\partial H \setminus \overline{O}$ is connected. Assume that there exists $x \in \partial E \cap (\partial H \setminus \overline{O})$. Since E is $(0, r_0)$-minimizer of the standard perimeter in $\mathbb{R}^N \setminus \overline{O}$ and E lies on one side of ∂H, we have that x is a regular point of ∂E. Indeed every blow-up is a minimizing cone contained in a half space and thus is a half space (see for instance [4, Lemma 3] for a proof of this well known fact). In turn this implies the local regularity. Then the strong maximum principle for the mean curvature equation implies that ∂E coincides with ∂H in the connected component of $\partial H \setminus \overline{O}$ containing x, that is, in the whole $\partial H \setminus \overline{O}$.

To conclude the proof observe now that if $\partial E \cap \partial H \subset \overline{O}$, then from (2.8) and (2.10), arguing as above, we conclude that

$$P(E_h; B_r(x) \cap H) \to P(E; B_r(x) \cap H)$$

and, in turn, $|\mu_{E_h}| \llcorner H \overset{*}{\rightharpoonup} |\mu_E| \llcorner H$ as Radon measures in $\mathbb{R}^N$. Now the very last part of the statement follows from [1, Th. 3.88]. □

3. ε-regularity

This section is devoted to the proof of the ε-regularity Theorem 3.21 for free boundary points lying on the thin obstacle $\partial_{\partial H} O$. Such a proof will split into two subsections.

3.1. **Partial Harnack inequality.**

In the following for any $a < b$ and $\alpha \in \mathbb{R}$ we set

$$S^\alpha_{a,b} := \Big\{ x \in \mathbb{R}^N : a < x_N - \alpha x_1 < b \Big\}.$$

When $\alpha = \frac{\sigma}{\sqrt{1-\sigma^2}}$ we shall simply write

$$S_{a,b} := \Big\{ x \in \mathbb{R}^N : a < x_N - \frac{\sigma x_1}{\sqrt{1-\sigma^2}} < b \Big\}.$$

In the following we shall write a generic point $x \in \mathbb{R}^N$ as (x', x_N) with $x' = (x_1, \dots, x_{N-1})$. With a slight abuse of notation we will also denote by x' the generic point of $\{x_N = 0\} \simeq \mathbb{R}^{N-1}$.

For $R > 0$, $x' \in \mathbb{R}^{N-1}$ we set $D_R(x') = \{y' \in \mathbb{R}^{N-1} : |x' - y'| < R\}$ and $\mathcal{C}_R(x') := D_R(x') \times \mathbb{R}$, $D^+_R(x') = D_R(x') \cap \{x_1 > 0\}$, $\mathcal{C}^+_R(x') = \mathcal{C}_R(x') \cap H$. When $x' = 0$ we will omit the center.

Lemma 3.1. *There exist two universal constants ε_0, $\eta_0 \in (0, 1/2)$, with the following properties: If $E \subset H$ is a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$, such that for $0 < r \le 1$*

$$\partial E \cap \mathcal{C}^+_{2r} \subset S_{a,b} \qquad \textit{with } (b-a) \le \varepsilon_0 r \,, \tag{3.1}$$

$$\Big\{ x_N < \frac{\sigma x_1}{\sqrt{1-\sigma^2}} + a \Big\} \cap \mathcal{C}^+_{2r} \subset E \,, \tag{3.2}$$

$\Lambda r^2 < \eta_0 (b-a)$ and

$$\frac{r^2}{R} \le \eta_0 (b-a) \,, \tag{3.3}$$

$$\partial_{\partial H} O \cap \mathcal{C}_r \cap S_{a,b} \neq \emptyset \,, \tag{3.4}$$

then there exist $a' \geq a$, $b' \leq b$ *with*

$$b' - a' \leq (1 - \eta_0)(b - a)$$

such that

$$\partial E \cap \mathcal{C}^+_{\frac{r}{2}} \subset S_{a',b'}.$$

The same conclusion holds if assumption (3.3) *is replaced by*

$$\partial_{\partial H} O \cap \mathcal{C}_{2r} \cap S_{a,b} = \emptyset \,. \tag{3.5}$$

Remark 3.2. We start noticing that if $E \subset H$ is a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$ such that, for $0 < r \leq 1$ (3.1) and (3.2) hold, then

$$\Big\{x_N < \frac{\sigma x_1}{\sqrt{1-\sigma^2}} + a\Big\} \cap \mathcal{C}^+_{2r} \subset E \cap \mathcal{C}^+_{2r} \subset \Big\{x_N < \frac{\sigma x_1}{\sqrt{1-\sigma^2}} + b\Big\} \cap \mathcal{C}^+_{2r} \tag{3.6}$$

provided ε_0 is small enough. Indeed if the above inequalities were not true then $E \cap \mathcal{C}^+_{2r}$ would be contained in $S_{a,b}$ thus violating the volume density estimates in Proposition 2.3.

We will investigate the consequences of the flatness condition:

$$\partial E \cap C^+_{2r} \subset S_{-\varepsilon r, \varepsilon r} \,.$$

Thanks to Remark 3.2 we may define two functions $u^\pm : D^+_{2r} \to \mathbb{R}$ as

$$u^+(x') = \max\{x_N : (x', x_N) \in \partial E\}$$
$$u^-(x') = \min\{x_N : (x', x_N) \in \partial E\}.$$

Note that u^+ is upper semicontinuous and u^- is lower semicontinuous. In particular we may define for every $x' \in D_{2r} \cap \{x_1 = 0\}$

$$u^-(x') = \inf\{\liminf_{h\to\infty} u^-(x'_h) : \, x'_h \in D^+_{2r}, \, x'_h \to x'\}$$

and, similarly, $u^+(x') = \sup\{\limsup_h u^-(x'_h) \, : \, x'_h \in D^+_{2r}, x'_h \to x'\}$. Observe that $\{(x', x_N) : x' \in D^+_{2r}, x_N < u^-(x')\} \subset E$ and thus from the above definition it follows that for

$$\{(x', x_N) : \, x' \in D_{2r} \cap \{x_1 = 0\}, \, x_N < u^-(x')\} \subset \partial E \cap \partial H \,.$$

In the following we recall the notions of viscosity super- and subsolutions.

Definition 3.3. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $v : \Omega \to \mathbb{R}$ be a lower (upper) semicontinuous function. Given $\xi_0 \in \mathbb{R}^d$, a constant $\gamma \in \mathbb{R}$ we say that v satisfies the inequality

$$\operatorname{div}\bigg(\frac{\nabla v + \xi_0}{\sqrt{1 + |\nabla v + \xi_0|^2}}\bigg) \leq \gamma \ (\geq \gamma) \tag{3.7}$$

in the viscosity sense if for any function $\varphi \in C^2(\Omega)$ such that $\varphi \leq v$ ($\varphi \geq v$) in a neighborhood of a point $x_0 \in \Omega$, $\varphi(x_0) = v(x_0)$ one has

$$\operatorname{div}\bigg(\frac{\nabla \varphi + \xi_0}{\sqrt{1 + |\nabla \varphi + \xi_0|^2}}\bigg)(x_0) \leq \gamma \ (\geq \gamma) \,. \tag{3.8}$$

Moreover, a lower (upper) semicontinuous function v in $\overline{\Omega}$ satisfies the Neumann boundary condition

$$\frac{\nabla v \cdot n}{\sqrt{1 + |\nabla v|^2}} \leq \gamma \ \ (\geq \gamma) \,, \qquad \text{on } \Gamma$$

in the viscosity sense, where Γ is a subset of $\partial\Omega$ and n stands for the inner normal to $\partial\Omega$, if for every $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi \le v$ ($\varphi \ge v$) in a neighborhood of a point $x_0 \in \Gamma$, $\varphi(x_0) = v(x_0)$, one has that

$$\frac{\nabla\varphi(x_0)\cdot n}{\sqrt{1+|\nabla\varphi(x_0)|^2}} \le \gamma \ \ (\ge \gamma)\,.$$

In the following we also need the following restricted notions of viscosity super- and subsolutions.

Definition 3.4. Let $v : \Omega \to \mathbb{R}$ be a lower (upper) semicontinuous function, $\xi_0 \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$. Given $\kappa > 0$, we say that v satisfies the inequality (3.7) in the κ-viscosity sense if (3.8) holds for any function $\varphi \in C^2(\Omega)$ such that $\varphi \le v$ ($\varphi \ge v$) in a neighborhood of a point $x_0 \in \Omega$, $\varphi(x_0) = v(x_0)$ and $|\nabla\varphi(x_0)| \le \kappa$.

We now recall a crucial result, which is essentially contained in [16].

Proposition 3.5. *Let $\xi \in \mathbb{R}^d$ with $|\xi| \le M$. There exist two constants $C_0 > 1$ and $\mu_0 \in (0,1)$, depending only on M and d, with the following properties: Let k be a positive integer and $\nu > 0$ such that $C_0^k\nu \le 1$ and let $v : \overline{B}_2 \to (0,\infty)$ be a lower semicontinuous function, bounded from above, satisfying*

$$\operatorname{div}\left(\frac{\nabla u+\xi}{\sqrt{1+|\nabla u+\xi|^2}}\right) \le \nu \tag{3.9}$$

in the $(C_0^k\nu)$-viscosity sense. If there exists a point $x_0 \in B_{1/2}$ such that $v(x_0) \le \nu$, then

$$|\{v \le C_0^k\nu\} \cap B_1| \ge (1-\mu_0^k)|B_1|\,.$$

Note that in [16] condition (3.9) is assumed to hold in the usual viscosity sense and not in the $(C_0^k\nu)$-viscosity sense considered here. However the proof of [16] extends to our framework without significant changes, see the Appendix where we provide the details for the reader's convenience.

We now prove a comparison lemma which is a variant of Lemma 2.12 in [3]. To this aim, in the following, given $0 < a < r$ we set

$$D_{r,a} := D_r \cap \{|x_1| < a\} \quad \text{and} \quad D^+_{r,a} := D^+_r \cap \{x_1 < a\}\,.$$

Lemma 3.6. *Let $E \subset H$ be a 0-minimizer of $\mathcal{F}_{-\sigma}$ with obstacle $O \in \mathcal{B}_R$ for some $R > 0$, let $0 < \eta < r$ and let $u_0 \in C^2(\overline{D^+_{r,\eta}})$ be such that*

$$\begin{aligned} &\operatorname{div}\left(\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\right) \ge 0 \qquad \text{in } D^+_{r,\eta},\\ &\frac{\partial_1 u_0}{\sqrt{1+|\nabla u_0|^2}} \ge \sigma \qquad \text{on } \partial D^+_{r,\eta} \cap \{x_1 = 0\}. \end{aligned} \tag{3.10}$$

Assume also that E is bounded from below,

$$E \cap [(\partial D^+_{r,\eta} \cap \{x_1 > 0\}) \times \mathbb{R}] \subset \{(x', x_N) \in (\partial D^+_{r,\eta} \cap \{x_1 > 0\}) \times \mathbb{R} : x_N \ge u_0(x')\} \tag{3.11}$$

and that

$$\mathcal{H}^{N-1}\big(\partial E \cap [(\partial D^+_{r,\eta} \cap \{x_1 > 0\}) \times \mathbb{R}]\big) = 0\,. \tag{3.12}$$

Then,

$$E \cap (D^+_{r,\eta} \times \mathbb{R}) \subset \{(x', x_N) \in D^+_{r,\eta} \times \mathbb{R} : x_N \ge u_0(x')\}\,.$$

Proof. We adapt the argument of [3, Lemma 2.12]. Denote

$$C := D^+_{r,\eta} \times \mathbb{R}$$

and set

$$F^{\pm} := \{(x', x_N) \in C : x_N \gtrless u_0(x')\}.$$

Consider the competitor given by

$$G := (E \setminus C) \cup (E \cap F^+),$$

which is an admissible compact perturbation of E since E is bounded from below. From the minimality of E we then obtain

$$\begin{aligned} &\mathcal{H}^{N-1}(\partial E \cap F^-) + \mathcal{H}^{N-1}(\partial E \cap \partial F^- \cap C \cap \{\nu_E = \nu_{F^-}\}) \\ &\qquad - \sigma \mathcal{H}^{N-1}(\partial E \cap \partial F^- \cap \partial H) \\ &\leq \mathcal{H}^{N-1}(E \cap \partial F^- \cap C). \end{aligned} \tag{3.13}$$

Denote

$$X(x) := \frac{1}{\sqrt{1 + |\nabla u_0(x')|^2}} (\nabla u_0(x'), -1),$$

and observe that $\operatorname{div} X \geq 0$ in C, thanks to the first assumption in (3.10). Then by the Divergence Theorem we obtain

$$\begin{aligned} 0 \leq \int_{E \cap F^-} \operatorname{div} X \, dx &= \int_{\partial E \cap F^-} X \cdot \nu_E \, d\mathcal{H}^{N-1} + \int_{\partial F^- \cap E \cap C} X \cdot \nu_{F^-} \, d\mathcal{H}^{N-1} \\ &\quad + \int_{\partial E \cap \partial F^- \cap C \cap \{\nu_E = \nu_{F^-}\}} X \cdot \nu_E \, d\mathcal{H}^{N-1} - \int_{\partial E \cap \partial F^- \cap \partial H} X \cdot e_1 \, d\mathcal{H}^{N-1}. \end{aligned}$$

Observing that $X \cdot \nu_{F^-} = -1$ on $\partial F^- \cap C$, from the previous inequality, we get, thanks to the second assumption in (3.10),

$$\begin{aligned} \sigma \mathcal{H}^{N-1}(\partial E \cap \partial F^- \cap \partial H) &\leq \int_{\partial E \cap \partial F^- \cap \partial H} \frac{\partial_1 u_0}{\sqrt{1 + |\nabla u_0|^2}} \, d\mathcal{H}^{N-1} \\ &\leq \int_{\partial E \cap F^-} X \cdot \nu_E \, d\mathcal{H}^{N-1} + \int_{\partial E \cap \partial F^- \cap C \cap \{\nu_E = \nu_{F^-}\}} X \cdot \nu_E \, d\mathcal{H}^{N-1} \\ &\quad - \mathcal{H}^{N-1}(\partial F^- \cap E \cap C) \\ &\leq \mathcal{H}^{N-1}(\partial E \cap F^-) + \mathcal{H}^{N-1}(\partial E \cap \partial F^- \cap C \cap \{\nu_E = \nu_{F^-}\}) \\ &\quad - \mathcal{H}^{N-1}(\partial F^- \cap E \cap C) \\ &\leq \sigma \mathcal{H}^{N-1}(\partial E \cap \partial F^- \cap \partial H), \end{aligned}$$

where the last inequality follows from (3.13). Thus all the inequalities above are equalities and, in turn, $X \cdot \nu_E = 1$ on $\partial E \cap F^-$. We now conclude by applying the Divergence Theorem in $E \cap F^-$ to the vector field $Y(x') := (x', x' \cdot \nabla u_0(x'))$. Since $Y \cdot \nu_{E \cap F^-} = 0$ $\mathcal{H}^{N-1}$-a.e. on $\partial E \cap F^-$ and on $\partial F^- \cap C$, recalling (3.11) and (3.12), we obtain

$$(N-1)|E \cap F^-| = \int_{E \cap F^-} \operatorname{div} Y \, dx = 0,$$

and the conclusion follows. □

Lemma 3.7. *Let $E \subset H$ is a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$. Assume also that (3.6) holds for some a, b and with $\frac{\sigma}{\sqrt{1-\sigma^2}}$ replaced by α for some $\alpha \in \mathbb{R}$, with $r = 1$, so that the functions $u^\pm$ are well defined on D_2^+. Then the functions u^+, u^- satisfy*

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u^+}{\sqrt{1+|\nabla u^+|^2}}\right) \geq -\Lambda \quad \text{in } D_2^+, \\ \operatorname{div}\left(\dfrac{\nabla u^-}{\sqrt{1+|\nabla u^-|^2}}\right) \leq \Lambda \quad \text{in } D_2^+, \\ \dfrac{\partial_1 u^+}{\sqrt{1+|\nabla u^+|^2}} \geq \sigma \quad \text{on } \Gamma, \\ \dfrac{\partial_1 u^-}{\sqrt{1+|\nabla u^-|^2}} \leq \sigma \quad \text{on } \Gamma \cap \{x' : (x', u^-(x')) \in O\}, \end{cases} \tag{3.14}$$

in the viscosity sense, where $\Gamma := D_2 \cap \{x_1 = 0\}$.

Proof. **Step 1.** We prove the statement first for u^-.

We fix $x_0' \in D_2^+$ and take a function $\varphi \in C^2(D_2^+)$ such that x_0' is a strict minimum point for $u^- - \varphi$ in D_2^+ and $u^-(x_0') = \varphi(x_0')$. Assume by contradiction that

$$\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) > \Lambda \tag{3.15}$$

in a neighborhood of x_0'. For $\eta > 0$ set

$$E_\eta := E \cup \{x \in E^c : x_N < \varphi(x') + \eta\}$$

and note that if η is sufficiently small is an admissible competitor for E. Note that by Λ-minimality, for all but countably many such η,

$$\begin{aligned} &\mathcal{F}_\sigma(E; \mathcal{C}_2) - \mathcal{F}_\sigma(E_\eta; \mathcal{C}_2) \\ &= \mathcal{H}^{N-1}(\partial E \cap \{x_N < \varphi(x') + \eta\}) - \mathcal{H}^{N-1}(E^c \cap \{x_N = \varphi(x') + \eta\}) \leq \Lambda |E_\eta \setminus E| \,. \end{aligned} \tag{3.16}$$

On the other hand, setting

$$X := \left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}, -\frac{1}{\sqrt{1+|\nabla\varphi|^2}}\right),$$

by (3.15) and (3.16) we have, if η is small enough,

$$\begin{aligned} \Lambda|E_\eta \setminus E| &< \int_{E_\eta \setminus E} \operatorname{div} X \, dx \\ &= -\int_{\partial E \cap \{x_N < \varphi(x') + \eta\}} X \cdot \nu_E \, d\mathcal{H}^{N-1} + \int_{E^c \cap \{x_N = \varphi(x') + \eta\}} X \cdot \nu_{E_\eta} \, d\mathcal{H}^{N-1} \\ &\leq \mathcal{H}^{N-1}(\partial E \cap \{x_N < \varphi(x') + \eta\}) - \mathcal{H}^{N-1}(E^c \cap \{x_N = \varphi(x') + \eta\}) \leq \Lambda |E_\eta \setminus E| \,, \end{aligned}$$

which yields a contradiction.

Step 2. Let $\varphi \in C^2(D_2)$ such that $\varphi \leq u^-$ in a neighborhood of $x_0' \in D_2 \cap \{x_1 = 0\}$ in D_2^+, $\varphi(x_0') = u^-(x_0')$ and $(x_0', u^-(x_0')) \in O$ and thus it lies strictly below the boundary of the obstacle. We claim that

$$\frac{\partial_1 \varphi(x_0')}{\sqrt{1+|\nabla\varphi(x_0')|^2}} \leq \sigma \,. \tag{3.17}$$

We argue by contradiction assuming that

$$\frac{\partial_1\varphi(x_0')}{\sqrt{1+|\nabla\varphi(x_0')|^2}} > \sigma\,. \tag{3.18}$$

In this case we set

$$E_n := n\big(E-(x_0',u^-(x_0'))\big),\quad \varphi_n(x') := n\Big[\varphi\Big(x_0'+\frac{x'}{n}\Big)-\varphi(x_0')\Big],\quad O_n := n\big(O-(x_0',u^-(x_0'))\big)\,.$$

Observe that E_n is a $(\Lambda/n,n)$-minimizer of $\mathcal{F}_\sigma$ with obstacle O_n. By Theorem 2.4 we may assume that up to a not relabelled subsequence $E_n\to E_\infty$ in $L^1_{loc}(\mathbb{R}^N)$, where E_∞ is a 0-minimizer of $\mathcal{F}_\sigma$ (with obstacle ∂H). Moreover, from property (ii) of Theorem 2.4 we have that

$$0\in\overline{\partial E_\infty\cap H}\,. \tag{3.19}$$

Note also that $\varphi_n\to\varphi_\infty$ locally uniformly, where $\varphi_\infty(x') := \nabla\varphi(x_0')\cdot x'$, and that

$$E_\infty\supset\{(x',x_N)\in H:\,x_N<\varphi_\infty(x')\}\,. \tag{3.20}$$

Set now for $\varepsilon>0$ small (to be chosen)

$$\psi_\varepsilon(x') := \varphi_\infty(x')-\varepsilon x_1-\frac{\varepsilon^2}{2}|x'|^2+\frac{\varepsilon}{2}x_1^2\,.$$

A direct calculation shows that

$$\frac{1}{\varepsilon}\operatorname{div}\Big(\frac{\nabla\psi_\varepsilon}{\sqrt{1+|\nabla\psi_\varepsilon|^2}}\Big)\to\frac{1+|\nabla\varphi(x_0')|^2-|\partial_1\varphi(x_0')|^2}{\big(1+|\nabla\varphi(x_0')|^2\big)^{3/2}}>0\,,$$

locally uniformly. Therefore, we may fix $\varepsilon>0$ so small that

$$\operatorname{div}\Big(\frac{\nabla\psi_\varepsilon}{\sqrt{1+|\nabla\psi_\varepsilon|^2}}\Big)>0\quad\text{in }\overline{D}_1^+ \tag{3.21}$$

and that, recalling (3.18),

$$\frac{\partial_1\psi_\varepsilon}{\sqrt{1+|\nabla\psi_\varepsilon|^2}}>\sigma\quad\text{on }\partial D_1^+\cap\{x_1=0\}. \tag{3.22}$$

Observe that we may now choose $\eta,r\in(0,1)$ such that

$$\psi_\varepsilon<\varphi_\infty\quad\text{on }\partial D_{r,\eta}\quad\text{and}\quad\mathcal{H}^{N-1}\big(\partial E_\infty\cap[(\partial D_{r,\eta}^+\cap\{x_1>0\})\times\mathbb{R}]\big)=0\,. \tag{3.23}$$

Finally, let $w\in C_c^\infty(D_{r,\eta})$, with $w(0)>0$, and note that by (3.21) and (3.22) we may choose $\delta\in(0,1)$ so small that, setting $\psi_{\varepsilon,\delta} := \psi_\varepsilon+\delta w$, we have

$$\begin{gathered}\operatorname{div}\Big(\frac{\nabla\psi_{\varepsilon,\delta}}{\sqrt{1+|\nabla\psi_{\varepsilon,\delta}|^2}}\Big)>0\quad\text{in }D_{r,\eta}^+,\\ \frac{\partial_1\psi_{\varepsilon,\delta}}{\sqrt{1+|\nabla\psi_{\varepsilon,\delta}|^2}}>\sigma\quad\text{on }\partial D_{r,\eta}^+\cap\{x_1=0\}.\end{gathered} \tag{3.24}$$

Recall that $H\setminus E_\infty$ is a 0-minimizer of $\mathcal{F}_{-\sigma}$. Note that this minimality property is a consequence of the fact that E_∞ is a 0-minimizer of $\mathcal{F}_\sigma$, which in turn follows from the assumption that $(x_0',u^-(x_0'))$ does not touch the boundary of O . Moreover, by (3.20) and (3.23), we have

$$(H\setminus E_\infty)\cap[(\partial D_{r,\eta}^+\cap\{x_1>0\})\times\mathbb{R}]\subset\{(x',x_N)\in(\partial D_{r,\eta}^+\cap\{x_1>0\})\times\mathbb{R}:\,x_N\ge\psi_{\varepsilon,\delta}(x')\}\,.$$

Therefore, taking into account also (3.24), we can apply Lemma 3.6, with E, u_0 replaced by $H\setminus E_\infty$, $\psi_{\varepsilon,\delta}$, respectively, to conclude that

$$(H\setminus E_\infty)\cap(D_{r,\eta}^+\times\mathbb{R})\subset\{(x',x_N)\in D_{r,\eta}^+\times\mathbb{R}:\,x_N\ge\psi_{\varepsilon,\delta}(x')\}\,.$$

In particular, since $\psi_{\varepsilon,\delta}(0) > 0$ the latter inclusion contradicts (3.19). This concludes the proof of (3.17).

Step 3. Concerning the proof of the statement for u^+, the case where $x_0' \in D_2^+$ is proved with the same argument used in Step 1. Instead, when $x_0' \in D_2 \cap \{x_1 = 0\}$ we argue as In Step 2, replacing E by $\widetilde{E} := \{(x', -x_N) : (x', x_N) \in E\}$ with the only difference that at the end of the proof we apply Lemma 3.6 to $\widetilde{E}_\infty$ instead of $H \setminus \widetilde{E}_\infty$. □

In the following for $x' \in D_{2r}^+$ we set

$$u_\sigma^\pm(x') := u^\pm(x') - \frac{\sigma x_1}{\sqrt{1-\sigma^2}}\,.$$

Lemma 3.8. *There exist two universal constants $C_1 > 1$, $\mu_1 \in (0,1)$ with the following property: Let $E \subset H$ be a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$. Assume that (3.6) holds for $a = -\varepsilon, b = \varepsilon$, $\varepsilon \in (0,1)$, $r = 1$, so that the functions $u^\pm$ are well defined on D_2^+. Assume also that $\Lambda < \eta\varepsilon$ with $\eta \in (0,1)$ and*

$$O \cap S_{-\varepsilon,\varepsilon} \cap \mathcal{C}_2 = \{x \in S_{-\varepsilon,\varepsilon} : x_1 = 0,\, x_N < \psi(x_2, \dots, x_{N-1})\} \cap \mathcal{C}_2\,,$$

where $\psi \in C^{1,1}(\partial H \cap D_2)$ with

$$\operatorname{div}\Big(\frac{\nabla\psi + \xi_0}{\sqrt{1+|\nabla\psi+\xi_0|^2}}\Big) \le \eta\varepsilon\,, \qquad \xi_0 := \frac{\sigma}{\sqrt{1-\sigma^2}} e_1\,.$$

Then the following holds:

(i) *If there exists $\bar{x}' \in D_{1/2}^+$ such that $u_\sigma^+(\bar{x}') \ge (1-\eta)\varepsilon$, then*

$$\mathcal{H}^{N-1}(\{x' \in D_1^+ : u_\sigma^+(x') \ge (1 - C_1^k\eta)\varepsilon\}) \ge (1-\mu_1^k)\mathcal{H}^{N-1}(D_1^+) \quad \text{for all } k \ge 1 \text{ such that } C_1^k\sqrt{\eta\varepsilon} \le 1\,.$$

(ii) *If there exists $\bar{x}' \in D_{1/2}^+$ such that $u_\sigma^-(\bar{x}') \le -(1-\eta)\varepsilon$, then*

$$\mathcal{H}^{N-1}(\{x' \in D_1^+ : \min\{u_\sigma^-(x'), \psi(x')\} \le -(1 - C_1^k\eta)\varepsilon\}) \ge (1-\mu_1^k)\mathcal{H}^{N-1}(D_1^+) \quad \text{for all } k \ge 1 \text{ such that } C_1^k\sqrt{\eta\varepsilon} \le 1\,,$$

where we have set $\psi(x') = \psi(x_2, \dots, x_{N-1})$ when $x_1 > 0$.

Proof. We start by proving (ii). To this aim we split the proof in three steps.

Step 1. Set $w^-(x') = \min\{u^-(x'), \psi(x') + \xi_0 \cdot x'\} = \min\big\{u^-(x'), \psi(x') + \frac{\sigma x_1}{\sqrt{1-\sigma^2}}\big\}$. We claim that w^- satisfies

$$\operatorname{div}\Big(\frac{\nabla w^-}{\sqrt{1+|\nabla w^-|^2}}\Big) \le \eta\varepsilon \qquad \text{in } D_2^+$$

in the viscosity sense. To this aim assume that $x_0' \in D_2^+$ is a strict minimum point for $w^- - \varphi$ in D_2^+, with $\varphi \in C^2(D_2^+)$ and $w^-(x_0') = \varphi(x_0')$. If $w^-(x_0') = \psi(x_0') + \xi_0 \cdot x_0'$, then clearly we also have that x_0' is a minimum point of $x' \to \psi(x') + \xi_0 \cdot x' - \varphi(x')$ and thus

$$\operatorname{div}\Big(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\Big)(x_0') \le \operatorname{div}\Big(\frac{\nabla\psi + \xi_0}{\sqrt{1+|\nabla\psi+\xi_0|^2}}\Big)(x_0') \le \eta\varepsilon\,.$$

If otherwise $w^-(x_0') = u^-(x_0')$ the claim follows from Lemma 3.7 since $\Lambda < \eta\varepsilon$.

Step 2. We now denote by w_σ^- the function defined on D_2 obtained by even reflection of

$$\min\{u_\sigma^-, \psi\} - L_k x_1, \qquad L_k := \Big(\frac{\sigma^+}{2\sqrt{1-\sigma^2}} + 1\Big) C_1^{2k}\eta^2\varepsilon^2\,,$$

with respect to ∂H, where $C_1 > 1$ will be chosen later and k is an integer such that $C_1^k\sqrt{\eta\varepsilon} \leq 1$. We claim that w_σ^- satisfies the inequality

$$\operatorname{div}\left(\frac{\nabla w_\sigma^- + \xi_0 + L_k e_1}{\sqrt{1+|\nabla w_\sigma^- + \xi_0 + L_k e_1|^2}}\right) \leq \eta\varepsilon \quad \text{in } D_2 \tag{3.25}$$

in the $(C_1^k\eta\varepsilon)$-viscosity sense, see Definition 3.4. To this aim let $\varphi \in C^2(D_2)$ such that $w_\sigma^- - \varphi$ has a minimum at $x_0' \in D_2$ and $|\nabla\varphi(x_0')| \leq C_1^k\eta\varepsilon$. We first show that $x_0' \notin \partial H$. Indeed, assume by contradiction the opposite and assume in addition that $u_\sigma^-(x_0') < \psi(x_0')$, that is $u^-(x_0') < \psi(x_0')$. Then, by Lemma 3.7, using as a test function $\varphi + \Big(\frac{\sigma}{\sqrt{1-\sigma^2}} + L_k\Big)x_1$, we infer that

$$\frac{\partial_1\varphi(x_0') + A_k}{\sqrt{1+(\partial_1\varphi(x_0') + A_k)^2 + |\nabla'\varphi(x_0')|^2}} \leq \sigma\,,$$

where we set $A_k := \frac{\sigma}{\sqrt{1-\sigma^2}} + L_k$ and $\nabla'\varphi = \nabla\varphi - (\partial_1\varphi)e_1$. From this inequality, using that $|\nabla'\varphi| \leq C_1^k\eta\varepsilon$, we easily get

$$\partial_1\varphi(x_0') + A_k \leq \frac{\sigma}{\sqrt{1-\sigma^2}} + \frac{\sigma^+}{\sqrt{1-\sigma^2}}\frac{C_1^{2k}\eta^2\varepsilon^2}{2}\,.$$

In turn, the last inequality implies that $\partial_1\varphi(x_0') < 0$.

On the other hand the symmetric argument in D_2^- shows that $\partial_1\varphi(x_0') > 0$, thus leading to a contradiction. If instead $u^-(x_0') = \psi(x_0')$, i.e., $u_\sigma^-(x_0') = \psi(x_0')$ then x_0' is a minimum for $\psi - L_k x_1 - \varphi$ in $\overline{D}_2^+$ and thus, in particular,

$$\partial_1\varphi(x_0') \leq \partial_1\psi(x_0') - L_k = -L_k < 0.$$

Arguing symmetrically in D_2^- we also get $\partial_1\varphi(x_0') > 0$, which is again a contradiction.

Thus, $x_0' \in D_2^+ \cup D_2^-$ and the fact that w_σ^- satisfies (3.25) in the viscosity sense now follows easily from Step 1, since on D_2^+ we have $w_\sigma^-(x') = w^-(x') - \big(\frac{\sigma}{\sqrt{1-\sigma^2}} + L_k\big)x_1$.

Step 3. Observe that from our assumptions we have that $-\varepsilon \leq \min\{u_\sigma^-, \psi\} \leq \varepsilon$. Thus, $0 < w_\sigma^- + \varepsilon + 2L_k$ in D_2. Moreover, by assumption we have

$$w_\sigma^-(\bar{x}') + \varepsilon + 2L_k \leq \Big(1 + \Big(\frac{\sigma^+}{\sqrt{1-\sigma^2}} + 2\Big)C_1^{2k}\eta\varepsilon\Big)\eta\varepsilon \leq \Big(3 + \frac{\sigma^+}{\sqrt{1-\sigma^2}}\Big)\eta\varepsilon := \nu\,.$$

Therefore from Step 2 and by Proposition 3.5, applied with $d = N-1$, we have that

$$\mathcal{H}^{N-1}(\{w_\sigma^- + \varepsilon + 2L_k \leq C_0^k\nu\} \cap D_1) \geq (1-\mu_0^k)\mathcal{H}^{N-1}(D_1)\,,$$

provided that $C_0^k\nu \leq C_1^k\eta\varepsilon \leq 1$, where μ_0 and C_0 are the constants provided by Proposition 3.5 corresponding to $M = \frac{|\sigma|}{\sqrt{1-\sigma^2}} + \frac{\sigma^+}{2\sqrt{1-\sigma^2}} + 1$. Note that the inequality $C_0^k\nu \leq C_1^k\eta\varepsilon$ is satisfied if we take $C_1 \geq C_0(3 + \sigma^+(1-\sigma^2)^{-1/2})$. Thus we finally have

$$\mathcal{H}^{N-1}(\{\min\{u_\sigma^-, \psi\} \leq -\varepsilon + L_k x_1 - 2L_k + C_0^k\nu\} \cap D_1^+) \geq (1-\mu_0^k)\mathcal{H}^{N-1}(D_1^+)\,,$$

from which the conclusion (ii) follows since $L_k x_1 - 2L_k + C_0^k\nu \leq C_1^k\eta\varepsilon$ in D_1^+.

Concerning the proof of (i) we argue as in the previous steps with w^- replaced by $-u^+$ and w_σ^- replaced by the even reflection of $-u_\sigma^+ - L_k x_1$. $\square$

Remark 3.9. Note that if $\partial_{\partial H}O \cap \mathcal{C}_2 \cap S_{-\varepsilon,\varepsilon} = \emptyset$ the conclusion of Lemma 3.8 holds with $\min\{u_\sigma^-, \psi\}$ replaced by u_σ^-.

Remark 3.10. Observe that the following interior version of the previous lemma holds: Let κ be a positive number. There exist a constant $C_1 > 1$ and $\mu_1 \in (0,1)$ depending only on κ with the following property: if $E \subset H$ is a $(\Lambda, 1)$-minimizer of the perimeter in $\mathbb{R}^N$ such that

$$\partial E \cap \mathcal{C}_2(y') \subset S^{\alpha}_{-\varepsilon,\varepsilon}\,, \subset E\,,$$

for some $y' = (y_1, \ldots, y_{N-1})$ with $y_1 \geq 2$, $\alpha \in [-\kappa, \kappa]$, $\varepsilon \in (0,1)$, $\Lambda < \eta\varepsilon$ with $\eta \in (0,1)$, then the following holds:

(i) If there exists $\bar{x}' \in D_{1/2}(y')$ such that $u^+(\bar{x}') - \alpha\bar{x}_1 \geq (1-\eta)\varepsilon$, then

$$\mathcal{H}^{N-1}(\{x' \in D_1(y') : u^+(x') - \alpha x_1 \geq (1 - C_1^k\eta)\varepsilon\}) \geq (1-\mu_1^k)\mathcal{H}^{N-1}(D_1^+)$$
$$\text{for all } k \geq 1 \text{ such that } C_1^k\sqrt{\eta\varepsilon} \leq 1.$$

(ii) If there exists $\bar{x}' \in D_{1/2}(y')$ such that $u^-(\bar{x}') - \alpha\bar{x}_1 \leq -(1-\eta)\varepsilon$, then

$$\mathcal{H}^{N-1}(\{x' \in D_1(y') : u^-(x') - \alpha x_1 \leq -(1 - C_1^k\eta)\varepsilon\}) \geq (1-\mu_1^k)\mathcal{H}^{N-1}(D_1^+)$$
$$\text{for all } k \geq 1 \text{ such that } C_1^k\sqrt{\eta\varepsilon} \leq 1\,.$$

This clearly follows from the interior version of the previous arguments. We only remark that the uniformity of the estimates with respect to α varying in a bounded interval relies on the estimates provided by Proposition 3.5 which are uniform with respect to ξ varying in a bounded set.

We need also the following lemma which relies on the classical interior regularity theory for (Λ, r_0)-minimizers of the perimeter.

Lemma 3.11. *For every $\delta \in (0,1)$, $\kappa > 0$, there exists $\varepsilon > 0$ such that if E is a $(\Lambda, 1)$-minimizer of the perimeter in C_2^+ with $\Lambda \leq 1$ such that*

$$\partial E \cap \mathcal{C}_2^+ \subset S^{\alpha}_{-\varepsilon,\varepsilon},$$

with $\alpha \in [-\kappa, \kappa]$, then the corresponding functions u^+ and u^- coincide in D_1^+ outside a set of measure less than δ.

Proof. It is enough to observe that given $\delta' \in (0,1)$, a sequence $\varepsilon_n > 0$ converging to zero and a sequence E_n of (Λ_n, r_n)-minimizers, with $\Lambda_n + \frac{1}{r_n} \leq 2$ and $\partial E_n \cap \mathcal{C}_2^+ \subset S^{\alpha_n}_{-\varepsilon_n,\varepsilon_n}$, with $\alpha_n \in [-\kappa, \kappa]$, $\alpha_n \to \alpha$, then from classical regularity results we have that $\partial E_n \cap C_1^+ \cap \{x_1 > \delta'\}$ converge in C^1 to the plane $\{x_N - \alpha x_1 = 0\} \cap C_1^+ \cap \{x_1 > \delta'\}$. □

Lemma 3.12. *For every $\delta > 0$ there exists $\eta = \eta(\delta) \in (0, 1/2)$ with the following property: Let $O \in \mathcal{B}_R$ be such that*

$$\partial H \cap \{x_N = -\varepsilon r\} \cap \mathcal{C}_{2r} \subset \overline{O}\,, \tag{3.26}$$

for some $\varepsilon \in (0,1]$, and for some $r > 0$ with $\frac{r}{R} \leq \eta(\delta)\varepsilon$. Assume also that there exists a point $x' \in \partial_{\partial H} O \cap \mathcal{C}_r \cap \{-\varepsilon r \leq x_N \leq \varepsilon r\}$. Then the connected component of $\partial_{\partial H} O \cap \mathcal{C}_{2r}$ containing x' is the graph of a function $\psi \in C^{1,1}(\partial H \cap D_{2r})$ such that

$$\|D\psi\|_{L^\infty(\partial H \cap D_{2r})} \leq \varepsilon, \quad r\|D^2\psi\|_{L^\infty(\partial H \cap D_{2r})} \leq \delta\varepsilon\,.$$

Proof. By rescaling it is enough to prove the statement with $r = 1$.

Observe that, under our assumptions, if $\eta(\delta)$ is sufficiently small, hence R is large, then the connected component of $\partial_{\partial H} O \cap \mathcal{C}_2$ containing x' is the graph of a $C^{1,1}(\partial H \cap D_2)$ function. We argue by contradiction assuming that there exist a sequence $\varepsilon_h \in (0,1]$, a sequence $O_h \in \mathcal{B}_{R_h}$, with $\frac{1}{R_h} \leq \frac{\varepsilon_h}{h}$ such that $\partial H \cap \{x_N = -\varepsilon_h\} \cap \mathcal{C}_2 \subset \overline{O}_h$ for all h and that there exist $x'_h \in \partial_{\partial H} O_h \cap \mathcal{C}_1 \cap \{-\varepsilon_h \leq x_N \leq \varepsilon_h\}$ such that

$$\|D\psi_h\|_{L^\infty(\partial H \cap D_2)} > \varepsilon_h \quad \text{or} \quad \|D^2\psi_h\|_{L^\infty(\partial H \cap D_2)} > \varepsilon_h \delta\,, \tag{3.27}$$

where the graph of $\psi_h \in C^{1,1}(\partial H \cap D_2)$ describes the connected component of $\partial_{\partial H} O_h \cap \mathcal{C}_2$.

We now set $\widetilde{O}_h = \Phi_h(O_h)$ where $\Phi_h(x) = (x_1, \dots, x_{N-1}, \varepsilon_h^{-1} x_N)$. Since Φ_h maps balls of radius 1 into ellipsoids with maximal semiaxis of length $\frac{1}{\varepsilon_h}$ it is easy to check that $\widetilde{O}_h \in \mathcal{B}_{\varepsilon_h R_h}$. Without loss of generality we may assume that $\Phi_h(x'_h)$ converges to a point $\bar{x}' \in \partial H \cap \overline{\mathcal{C}}_1$ with $-1 \leq \bar{x}_N \leq 1$. Then, recalling that $\varepsilon_h R_h \to +\infty$ and thus the uniform inner and outer ball condition for $\widetilde{O}_h$ hold for larger and larger radii, by a compactness argument we have that the connected components of $\partial_{\partial H} \widetilde{O}_h$ containing x'_h converge locally (up to a subsequence) in the Hausdorff sense to a $(N-2)$-dimensional plane π passing through $\bar{x}'$ and such that $\pi \cap \mathcal{C}_2 \subset \{x_N \geq -1\}$. Note that the latter inclusion follows from (3.26) applied to $\widetilde{O}_h$ and $\varepsilon = 1$. Note also that this inclusion together with the fact that $\bar{x}'_N \leq 1$ yields that the slope of π is strictly less than 1. Hence the functions $\frac{\psi_h}{\varepsilon_h}$ converge locally uniformly (actually in $C^{1,1}$) to an affine function whose gradient has norm strictly less than 1. This contradicts (3.27) for h sufficiently large. □

Proof of Lemma 3.1. By a simple rescaling argument it is enough to prove the statement for $r = 1$. Up to renaming the coordinates we can assume that $a + b = 0$, so that if we set $\varepsilon = (b-a)/2$ our assumption becomes

$$\partial E \cap \mathcal{C}_2^+ \subset S_{-\varepsilon,\varepsilon}.$$

Let $0 < \eta_0 < 1$ to be fixed in an universal way. If $\sup_{D_{1/2}^+} u_\sigma^+ \leq \varepsilon - \eta_0 \varepsilon$ (resp. if $\inf_{D_{1/2}^+} u_\sigma^- \geq -\varepsilon + \eta_0\varepsilon$) we are done by choosing $a' = a = -\varepsilon$ and $b' = \varepsilon - \eta_0 \varepsilon = b - \eta_0(b-a)/2$ (resp. $b' = b = \varepsilon$ and $a' = -\varepsilon + \eta_0 \varepsilon = a + \eta_0(b-a)/2$).

Hence we can assume by contradiction that there are $\bar{x}', \hat{x}' \in D_{1/2}^+$ such that

$$u_\sigma^+(\bar{x}') > \varepsilon - \eta_0 \varepsilon \qquad \text{and} \qquad u_\sigma^-(\hat{x}') < -\varepsilon + \eta_0 \varepsilon. \tag{3.28}$$

Assume that (3.4) holds. Then there exists $\bar{y}' \in \partial_{\partial H} O \cap \mathcal{C}_1 \cap \{-\varepsilon < y_N < \varepsilon\}$. Then, we may apply Lemma 3.12 to conclude that the connected component of $\partial_{\partial H} O \cap \mathcal{C}_2$ containing $\bar{y}'$ is the graph of a function $\psi \in C^{1,1}(\partial H \cap D_2)$, where $\|D^2\psi\|_{L^\infty(\partial H \cap D_2)} \leq \delta\varepsilon$ with $\delta = \delta(\sigma)$ so small that

$$\operatorname{div}\left(\frac{\nabla\psi + \xi_0}{\sqrt{1 + |\nabla\psi + \xi_0|^2}}\right) \leq \eta_0 \varepsilon\,.$$

This is certainly true provided that $O \in \mathcal{B}_R$ with $\frac{1}{R} \leq \eta(\delta(\sigma))\varepsilon$. Thus we can apply Lemma 3.8 with k_0 large to infer that

$$\mathcal{H}^{N-1}(\{x' \in D_1^+ : u_\sigma^+(x') \geq (1 - C_1^{k_0}\eta_0)\varepsilon\}) \geq (1 - \mu_1^{k_0})\mathcal{H}^{N-1}(D_1^+) \geq \frac{3}{4}\mathcal{H}^{N-1}(D_1^+)$$

and

$$\begin{aligned}\mathcal{H}^{N-1}(\{x' \in D_1^+ : \min\{u_\sigma^-, \psi\} \leq -(1 - C_1^{k_0}\eta_0)\varepsilon\})& \\ \geq (1 - \mu_1^{k_0})\mathcal{H}^{N-1}(D_1^+) \geq \frac{3}{4}\mathcal{H}^{N-1}(D_1^+),&\end{aligned}$$

provided that $C_1^{k_0}\sqrt{\eta_0} \leq 1$. If $\psi \geq -(1 - C_1^{k_0}\eta_0)\varepsilon$ in $D_1 \cap \partial H$ we are done since from Lemma 3.11 we may assume that $\mathcal{H}^{N-1}(\{u^+ \neq u^-\}) < \mathcal{H}^{N-1}(D_1^+)/4$ and therefore from the two previous inequalities we get

$$\mathcal{H}^{N-1}(\{x' \in D_1^+ : u_\sigma^+(x') \geq (1 - C_1^{k_0}\eta_0)\varepsilon\} \cap \{u^+ = u^-\}) > \frac{1}{2}\mathcal{H}^{N-1}(D_1^+)$$

and

$$\mathcal{H}^{N-1}(\{x' \in D_1^+ : \min\{u_\sigma^-, \psi\} \leq -(1 - C_1^{k_0}\eta_0)\varepsilon\} \cap \{u^+ = u^-\}) > \frac{1}{2}\mathcal{H}^{N-1}(D_1^+)$$

which is impossible.

Hence we only have to deal with the case where there exists a point in $D_1 \cap \partial H$ at which $\psi \leq -(1 - C_1^{k_0}\eta_0)\varepsilon \leq -3\varepsilon/4$ (provided η_0 is small). Thus, up to replacing O with $O + \frac{3\varepsilon}{4}e_N$, we may apply Lemma 3.12 in $\mathcal{C}_2$ with ε replaced by $\varepsilon/4$ (and taking $\eta(\delta)$ smaller if needed) to conclude that $\|\nabla\psi\|_{L^\infty(D_2)} \leq \frac{\varepsilon}{4}$. In turn this estimate implies that $\psi \leq 0$ in $D_2 \cap \partial H$.

Let $\Omega \subset \mathbb{R}^{N-1}$ be a smooth set such that $D_{4/3}^+ \subset \Omega \subset D_{3/2}^+$ and consider the solution to the following problem:

$$\begin{cases} \Delta w_\mu(1 + |\xi_0|^2) - D_{11}w_\mu|\xi_0|^2 = -\mu & \text{in } \Omega, \\ w_\mu = \varphi & \text{on } \partial\Omega, \end{cases}$$

where φ is a smooth function such that $\varphi \equiv 1$ on $\partial\Omega \cap H$ and $\varphi \equiv 1/4$ on $D_1 \cap \partial H$, and $\varphi \geq 1/4$ elsewhere on $\partial\Omega$ and $\mu > 0$ is to be chosen. Note that w_μ is smooth on $\overline{\Omega}$ and that it converges in $C^2(\overline{\Omega})$, as $\mu \to 0$, to the function w_0 such that $w_0 = \varphi$ on $\partial\Omega$ and

$$\Delta w_0(1 + |\xi_0|^2) - D_{11}w_0|\xi_0|^2 = 0 \qquad \text{in } \Omega,$$

By the maximum principle there exists $\tau \in (0, 1/2)$ such that $\max_{\overline{D_1^+}} w_0 \leq 1 - 2\tau$. Therefore there exists $\mu_0 > 0$ such that

$$\max_{\overline{D_1^+}} w_{\mu_0} \leq 1 - \tau\,. \tag{3.29}$$

We now claim that for $\varepsilon \leq \varepsilon_0$, with ε_0 depending only on w_0, the function $v_\varepsilon := \varepsilon w_{\mu_0}$ satisfies

$$\begin{aligned} &\operatorname{div}\left(\frac{\nabla v_\varepsilon + \xi_0}{\sqrt{1 + |\nabla v_\varepsilon + \xi_0|^2}}\right) \\ &= \frac{\varepsilon}{(1 + |\varepsilon\nabla w_{\mu_0} + \xi_0|^2)^{\frac{3}{2}}}[\Delta w_{\mu_0}(1 + |\varepsilon\nabla w_{\mu_0} + \xi_0|^2) - D^2 w_{\mu_0}(\varepsilon\nabla w_{\mu_0} + \xi_0)(\varepsilon\nabla w_{\mu_0} + \xi_0)] \\ &\qquad \leq \varepsilon\Big(- \mu_0 + C\varepsilon(1 + \|w_0\|^3_{C^2(\overline{\Omega})})\Big) < -\frac{\varepsilon\mu_0}{2} < -2\eta_0\varepsilon\,, \end{aligned}$$

with $C > 0$ universal, provided ε_0 and η_0 are chosen small enough.

By our assumptions, $u_\sigma^+ \leq v_\varepsilon$ on $\partial\Omega$. Thus, recalling the first inequality in (3.14) and the assumption on Λ, we get that

$$\operatorname{div}\left(\frac{\nabla u_\sigma^+ + \xi_0}{\sqrt{1 + |\nabla u_\sigma^+ + \xi_0|^2}}\right) \geq -2\eta_0\varepsilon$$

in the viscosity sense. By the comparison principle we conclude that $u_\sigma^+ \leq v_\varepsilon$ in Ω. In turn, recalling (3.29), we infer

$$\sup_{D_1^+} u_\sigma^+ \leq \sup_{D_1^+} v_\varepsilon \leq (1 - \tau)\varepsilon\,.$$

This is in contradiction with (3.28) if $\eta_0 < \tau$.

Finally, if (3.5) holds, we may argue as before taking $\psi \equiv \varepsilon$. □

Lemma 3.13. *There exist ε_1, $\eta_1 \in (0, 1/2)$ universal with the following property: if $E \subset H$ is a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$, such that*

$$\partial E \cap \mathcal{C}_{2r}^+(x') \subset S_{a,b} \qquad \textit{with } b - a \leq \varepsilon_1 r\,, \tag{3.30}$$

$$\Big\{x_N < \frac{\sigma x_1}{\sqrt{1-\sigma^2}} + a\Big\} \cap \mathcal{C}^+_{2r}(x') \subset E\,, \tag{3.31}$$

for some $x' \in \mathbb{R}^{N-1}$ *with* $x_1 \geq 0$, $0 < r \leq 1$, $\Lambda r^2 < \eta_1(b-a)$ *and*

$$\frac{r^2}{R} \leq \eta_1(b-a)\,,$$

then there exist $a' \geq a$, $b' \leq b$ *with*

$$b' - a' \leq (1-\eta_1)(b-a)$$

such that

$$\partial E \cap C^+_{\frac{r}{32}}(x') \subset S_{a',b'}.$$

Proof. We start by observing that a simplified version of the same arguments in the proof of Lemma 3.1, see Remark 3.10, (or, alternatively, the standard interior regularity results for (Λ, r_0)-minimizers of the perimeter) lead to the following interior version of the Lemma: there exist $\varepsilon_1 \leq \varepsilon_0$, $\eta_1 \leq \eta_0$, such that if (3.30) and (3.31) hold with $x_1 \geq 2r$ and $\Lambda r^2 < \eta_1(b-a)$, then there exist $a' \geq a$, $b' \leq b$ with $b' - a' \leq (1-\eta_1)(b-a)$ such that

$$\partial E \cap \mathcal{C}_{\frac{r}{2}}(x') \subset S_{a',b'}. \tag{3.32}$$

Therefore it is enough to prove the statement in the case $0 < x_1 < 2r$. By rescaling, we may assume $r = 1$.

If $0 < x_1 \leq \frac{1}{8}$, we may apply Lemma 3.1 with the origin replaced by the point $\overline{x}' = (0, x_2, \dots, x_N)$ and $r = \frac{1}{2}$, provided η_1 is sufficiently small, thus getting $\partial E \cap \mathcal{C}^+_{\frac{1}{4}}(\bar{x}') \subset S_{a',b'}$, hence in particular $\partial E \cap \mathcal{C}^+_{\frac{1}{8}}(x') \subset S_{a',b'}$.

Finally, if $x_1 \geq 1/8$ we just get the interior estimate (3.32) with $r = \frac{1}{16}$. $\square$

Remark 3.14. Let $\kappa > 0$ be given. Observe that there exist possibly smaller ε_1, η_1, depending on κ, such that if $D_{2r}(x') \subset \{x_1 > 0\}$ and

$$\partial E \cap \mathcal{C}_{2r}(x') \subset S^{\alpha}_{a,b} \qquad \text{with } b - a \leq \varepsilon_1 r\,,$$

for some $\alpha \in (-\kappa, \kappa)$ and $\Lambda r^2 < \eta_1(b-a)$, then

$$\partial E \cap \mathcal{C}_{\frac{r}{32}}(x') \subset S^{\alpha}_{a',b'},$$

with

$$b' - a' \leq (1-\eta_1)(b-a)\,.$$

Indeed this follows with the same arguments of the proof of Lemma 3.1, taking into account Remark 3.10 and Lemma 3.11 which holds uniformly with respect to $\alpha \in [-\kappa, \kappa]$.

3.2. **Flatness improvement and ε-regularity.** We start with the following crucial barrier argument, which forces the solution to coincide with the boundary of the obstacle if the solution is too close to a plane which does not satisfy the exact optimality condition, compare with [6] for a similar argument.

Lemma 3.15. *There exist universal constants* $\varepsilon_2, M_0 > 0$ *with the following property. Assume that* $E \subset H$ *is a* $(\Lambda, 1)$*-minimizer of* $\mathcal{F}_\sigma$ *in* $\mathbb{R}^N$ *with obstacle* $O \in \mathcal{B}_R$. *Assume also that*

$$\overline{\partial E \cap H} \cap \mathcal{C}_{2r} \subset \{|x_N - \alpha x_1| < \varepsilon r\}, \quad \{x_N < \alpha x_1 - \varepsilon r\} \cap \mathcal{C}^+_{2r} \subset E\,,$$

for some $0 < \varepsilon \leq \varepsilon_2$, $0 < r \leq 1$ *with* $\Lambda r < \varepsilon$ *and* $\frac{r}{R} \leq \eta(1)\varepsilon$ *(where* $\eta(1)$ *is as in Lemma 3.12), and for some* $\alpha \geq M_0\varepsilon + \frac{\sigma}{\sqrt{1-\sigma^2}}$. *Then*

$$\overline{\partial E \cap H} \cap \partial H \cap \mathcal{C}_{r/2} = \partial_{\partial H} O \cap \mathcal{C}_{r/2} \cap \{|x_N| < \varepsilon r\}.$$

Proof. Without loss of generality, by a rescaling argument, we may assume $r = 1$.

We start by assuming that there exists $\bar{y}' \in \partial_{\partial H} O \cap \mathcal{C}_1 \cap S_{-\varepsilon,\varepsilon}$. From the assumption on O we get, thanks to Lemma 3.12 that the connected component of $\partial_{\partial H} O \cap \mathcal{C}_2$ containing $\bar{y}'$ is a graph of a $C^{1,1}$ function ψ such that $\|D\psi\|_{L^\infty(\partial H \cap D_2)} \leq \varepsilon$, $\|D^2\psi\|_{L^\infty(\partial H \cap D_2)} \leq \varepsilon$. In particular

$$\|\psi\|_{L^\infty(\partial H \cap D_1)} \leq 3\varepsilon\,. \tag{3.33}$$

As said, the proof is obtained via a barrier construction. To this end we fix $z' = (0, z_2, \ldots, z_{N-1})$ with $|z'| \leq 1/2$ and assume by contradiction that $u^-(z') < \psi(z')$. We define for $x' \in \mathbb{R}^{N-1}$

$$w(x') = \beta x_1 + \gamma\varepsilon(Lx_1^2 - |x' - z'|^2) + \psi(z') + \nabla\psi(z') \cdot (x' - z') + c$$

where

$$\beta = \frac{M_0\varepsilon}{2} + \frac{\sigma}{\sqrt{1-\sigma^2}}\,,$$

L and γ are positive constants to be chosen and c is the first constant such that the function w touches from below $u^-(x')$ in $\overline{D}^+_{1/2}(z')$. Observe that $c \leq 0$, since $w(z') = \psi(z') + c \leq u^-(z') \leq \psi(z')$, and that

$$\frac{1}{\varepsilon}\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) \to 2\gamma\frac{(L-N+1)(1+|\xi_0|^2) - (L-1)|\xi_0|^2}{(1+|\xi_0|^2)^{3/2}} \tag{3.34}$$

uniformly in D_2 as $\varepsilon \to 0$, where $\xi_0 = \sigma e_1/\sqrt{1-\sigma^2}$. Therefore, if L is chosen so large that the right hand side of (3.34) is bigger than 2, there exists ε_2 depending on M_0 such that if $0 < \varepsilon < \varepsilon_2$ then

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) > \varepsilon\,. \tag{3.35}$$

Observe that if $x' \in \partial D_{1/2}(z') \cap \{x_1 \geq 0\}$, then

$$w(x') \leq \beta x_1 + \frac{\gamma L\varepsilon}{2}x_1 - \frac{\gamma}{4}\varepsilon + \psi(z') + \nabla\psi(z') \cdot (x' - z') < \alpha x_1 - \varepsilon \leq u^-(x)\,,$$

provided $M_0 > \gamma L$ and $\gamma \geq 18$, where in the last inequality we used (3.33). We claim that w does not touch u^- at a point $x_0 \in D^+_{\frac{1}{2}}(z')$. Indeed if this is the case, by Lemma 3.7, recalling that Λ by assumption is smaller than ε we would get that

$$\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right)(x_0) \leq \varepsilon$$

a contradiction to (3.35). Thus we may conclude that w touches u^- at a point $\overline{x}' \in \{x_1 = 0\} \cap \partial D^+_{\frac{1}{2}}(z')$. Assume by contradiction that $u^-(\overline{x}') < \psi(\overline{x}')$, then from the last inequality in (3.14), observing that

$$|\nabla w(\overline{x}') - \partial_1 w(\overline{x}')\, e_1| \leq C\varepsilon\,,$$

for a constant depending only on N, we have

$$\partial_1 w(\overline{x}') \leq \frac{\sigma}{\sqrt{1-\sigma^2}} + \frac{\sigma^+}{\sqrt{1-\sigma^2}}\frac{C^2\varepsilon^2}{2}\,.$$

which is impossible, provided that M_0 is sufficiently large, since

$$\partial_1 w(\overline{x}') = \frac{M_0\varepsilon}{2} + \frac{\sigma}{\sqrt{1-\sigma^2}}\,.$$

Therefore, at the touching point we have

$$0 = u^-(\overline{x}') - w(\overline{x}') = \psi(\overline{x}') - w(\overline{x}')\,,$$

hence

$$\psi(\overline{x}') = \psi(z') + \nabla\psi(z')\cdot(\overline{x}'-z') + c - \gamma\varepsilon|\overline{x}'-z'|^2 .$$

On the other hand, recalling that $|D^2\psi|\le\varepsilon$, we have

$$\psi(\overline{x}') \ge \psi(z') + \nabla\psi(z')\cdot(\overline{x}'-z') - \varepsilon|\overline{x}'-z'|^2 .$$

Combining the two inequalities we get that $0\ge c\ge(\gamma-1)\varepsilon|\overline{x}'-z'|^2$. Therefore $\overline{x}'=z'$ hence $\psi(z')=u^-(z')$.

If instead $\partial_{\partial H}O\cap\mathcal{C}_1\cap S_{-\varepsilon,\varepsilon}=\emptyset$, we may repeat the same argument as before with ψ replaced by $\tilde\psi\equiv\varepsilon$, obtaining that $u^\pm=\tilde\psi=\varepsilon$ in $\partial H\cap\mathcal{C}_{1/2}$ which is impossible by assumption. □

Remark 3.16. Note that the assumptions of the previous lemma force the obstacle ψ to satisfy the inequality $|\psi|<\varepsilon r$ in $\partial H\cap\mathcal{C}_{r/2}$.

Lemma 3.17. *For all $\tau\in(0,1/2)$, $M>0$, there exist constants $\lambda_0=\lambda_0(M,\tau)>0$, $C_2=C_2(M,\tau)>0$ such that for all $\lambda\in(0,\lambda_0)$ one can find $\varepsilon_3=\varepsilon_3(M,\tau,\lambda)>0$, $\eta_2=\eta_2(M,\tau,\lambda)>0$ with the following property: Assume $E\subset H$ is a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O\in\mathcal{B}_R$. Assume also that $0\in\overline{\partial E\cap H}$ and*

$$\overline{\partial E\cap H}\cap\mathcal{C}_r\subset\{|x_N-\alpha x_1|<\varepsilon r\},\quad \{x_N<\alpha x_1-\varepsilon r\}\cap\mathcal{C}_r^+\subset E,$$

for some $0<\varepsilon\le\varepsilon_3$, $0<r\le1$ with $\Lambda r<\eta_2\varepsilon$ and $\frac{r}{R}\le\eta_2\varepsilon$, and for some $\frac{\sigma}{\sqrt{1-\sigma^2}}\le\alpha\le M\varepsilon+\frac{\sigma}{\sqrt{1-\sigma^2}}$. Then there exist $\bar\alpha\ge\frac{\sigma}{\sqrt{1-\sigma^2}}$, a rotation R about the x_1 axis, with $\|R-\mathrm{Id}\,\|\le C_2\varepsilon$, $|\bar\alpha-\alpha|\le C_2\varepsilon$, such that

$$R(\overline{\partial E\cap H})\cap\mathcal{C}_{\lambda r}\subset\{|x_N-\bar\alpha x_1|<\lambda^{1+\tau}\varepsilon r\}.$$

Proof. By rescaling it is enough to show the statement for $r=1$. We argue by contradiction and we assume that there exist τ and M and sequences $\epsilon_n\to0$, $\frac{\sigma}{\sqrt{1-\sigma^2}}\le\alpha_n\le M\epsilon_n+\frac{\sigma}{\sqrt{1-\sigma^2}}$, $(\Lambda_n,1)$-minimizers E_n with $0\in\overline{\partial E_n\cap H}$, $\overline{\partial E_n\cap H}\cap\mathcal{C}_1\subset\{|x_N-\alpha_n x_1|<\epsilon_n\}$ and $\{x_N<\alpha x_1-\varepsilon r\}\cap\mathcal{C}_r^+\subset E$, Λ_n, $O_n\in\mathcal{B}_{R_n}$, such that $\Lambda_n\le\frac{\epsilon_n}{n}$, $\frac{1}{R_n}\le\frac{\epsilon_n}{n}$ for which the conclusion fails for all $\bar\alpha$, R with $|\bar\alpha-\alpha_n|\le C_2\epsilon_n$, $\|R-\mathrm{Id}\,\|\le C_2\epsilon_n$, and for a certain $\lambda\le\lambda_0$. We will show that for a suitable choice of C_2 and λ_0 depending only on M, τ and the dimension this will lead to a contradiction.

We claim that there exist universal constants $C>0$, $\beta\in(0,1]$ and $\gamma_n>0$, $\gamma_n\to0$, such that for all $x',y'\in\overline{D}^+_{\frac12}$, with $|x'-y'|\ge\gamma_n$,

$$|u^\pm_{n,\sigma}(x')-u^\pm_{n,\sigma}(y')|\le C\epsilon_n|x-y|^\beta,\qquad |u^\pm_{n,\sigma}(x')-u^\mp_n(y')|\le C\epsilon_n|x-y|^\beta\,, \tag{3.36}$$

where $u^\pm_{n,\sigma}$ are the functions defined as $u^\pm_\sigma$ with E replaced by E_n.

To prove the claim we fix $x'\in\overline{D}^+_{\frac12}$. Note that our assumptions imply that if $0<\varrho\le\frac12$

$$\overline{\partial E_n\cap H}\cap\mathcal{C}_\varrho(x')\subset S_{a,b}\,,$$

with $b-a=2(1+M)\epsilon_n$. We now apply Lemma 3.13 to conclude that for all $y'\in\overline{D}^+_{\frac{\varrho}{64}}(x')$

$$|u^\pm_{n,\sigma}(x')-u^\pm_{n,\sigma}(y')|<2(1+M)(1-\eta_1)\epsilon_n,\quad |u^\pm_{n,\sigma}(x')-u^\mp_{n,\sigma}(y')|<2(1+M)(1-\eta_1)\epsilon_n, \tag{3.37}$$

provided that

$$4(1+M)\epsilon_n\le\varepsilon_1\varrho\quad\text{and}\quad \frac{\varrho^2}{n}\le8\eta_1(1+M)\,. \tag{3.38}$$

Thus, inequalities (3.37) hold for n sufficiently large. Note that (3.37) implies that

$$\overline{\partial E_n\cap H}\cap\mathcal{C}_{\frac{\varrho}{64}}(x')\subset S_{a',b'}\quad\text{with }\ b'-a'\le2(1+M)(1-\eta_1)\epsilon_n.$$

Therefore, by applying Lemma 3.13 $m_n - 1$ more times we conclude that for all $y' \in \overline{D}^+_{\frac{\varrho}{64^{j+1}}}(x')$, $j = 0, 1, \dots, m_n - 1$,

$$
\begin{aligned}
&|u^\pm_{n,\sigma}(x') - u^\pm_{n,\sigma}(y')| < 2(1+M)(1-\eta_1)^{j+1}\epsilon_n, \\
&|u^\pm_{n,\sigma}(x') - u^\mp_{n,\sigma}(y')| < 2(1+M)(1-\eta_1)^{j+1}\epsilon_n,
\end{aligned}
\tag{3.39}
$$

provided that

$$
4 \cdot 64^j (1+M)(1-\eta_1)^j \epsilon_n \le \varepsilon_1 \varrho, \quad \frac{\varrho^2}{64^{2j} n} \le 8\eta_1 (1-\eta_1)^j (1+M)\,.
$$

Note that the last inequality follows from the second one in (3.38), since $\eta_1 < \frac{1}{2}$, while the first holds provided m_n is the largest integer for which

$$
4 \cdot 64^{m_n - 1}(1+M)(1-\eta_1)^{m_n-1}\epsilon_n \le \varepsilon_1 \varrho\,.
$$

In particular, taking $\varrho = \frac{1}{2}$ and applying (3.39) with $y \in \overline{D}^+_{\frac{\varrho}{64^j}}(x') \setminus D^+_{\frac{\varrho}{64^{j+1}}}(x')$ and $j = 1, \dots m_n - 1$ we easily get that if $y' \in \overline{D}^+_{1/128}(x') \setminus D^+_{\gamma_n}(x')$, with $\gamma_n = \frac{1}{2 \cdot 64^{m_n}}$, then

$$
|u^\pm_{n,\sigma}(x') - u^\pm_{n,\sigma}(y')| \le C\epsilon_n |x'-y'|^\beta, \qquad |u^\pm_{n,\sigma}(x') - u^\mp_{n,\sigma}(y')| \le C\epsilon_n |x'-y'|^\beta\,,
$$

for a constant C depending only on M and $\beta = -\frac{\log(1-\eta_1)}{\log 64}$. This proves (3.36) when $x' \in \overline{D}^+_{\frac{1}{2}}$ and $y' \in D^+_{\frac{1}{128}}(x')$, with $|x'-y'| \ge \gamma_n$. Clearly, $\gamma_n \to 0$, since $m_n \to \infty$.

In particular if we consider the functions

$$
v^\pm_n(x') = \frac{u^\pm_{n,\sigma}(x') - \Big(\alpha_n - \frac{\sigma}{\sqrt{1-\sigma^2}}\Big)x_1}{\epsilon_n} = \frac{u^\pm_n(x') - \alpha_n x_1}{\epsilon_n}
$$

they converge (up to a subsequence) to the same Hölder continuous function v defined on $D^+_{1/2}$ (note that $\big|\alpha_n - \frac{\sigma}{\sqrt{1-\sigma^2}}\big| \le M\epsilon_n$). Furthermore $\|v\|_\infty \le 1$ and $v(0) = 0$, since the assumption $0 \in \overline{\partial E_n \cap H}$ implies that $u^-_{n,\sigma}(0) \le 0 \le u^+_{n,\sigma}(0)$. We also assume that (up to further subsequence),

$$
\frac{1}{\epsilon_n}\Big(\alpha_n - \frac{\sigma}{\sqrt{1-\sigma^2}}\Big) \to \gamma \in [0, M]\,.
\tag{3.40}
$$

We now consider two possible cases. Let us assume first that for all n there exists $y'_n \in \partial_{\partial H} O_n \cap \mathcal{C}_{\frac{1}{2}} \cap \{-\epsilon_n < x_N < \epsilon_n\}$. Then by Lemma 3.12, we have that for n large the connected component of $\partial_{\partial H} O_n \cap \mathcal{C}_1$ containing y'_n is a graph of a $C^{1,1}$ function ψ_n such that

$$
\|D\psi_n\|_{L^\infty(\partial H \cap D_1)} \le 2\varepsilon_n, \quad \|D^2\psi_n\|_{L^\infty(\partial H \cap D_1)} \le \delta_n \varepsilon_n
$$

for a suitable sequence δ_n converging to zero. Recall that

$$
\frac{\psi_n(y'_n)}{\epsilon_n} \in (-1, 1)\,.
$$

Therefore we may assume that, up to a subsequence,

$$
\frac{\psi_n(x')}{\epsilon_n} \to \psi_\infty + \omega \cdot x' \quad \text{uniformly with respect to } x' \in D_1 \cap \partial H
$$

for some $\psi_\infty \in [-2, 2]$ and $\omega \in \mathbb{R}^{N-1}$ of the form $(0, \omega_2, \dots, \omega_{N-1})$ with $|\omega| \le 2$.

We now claim that $v - \omega \cdot x'$ satisfies

$$
\begin{cases}
Lw := \Delta w(1+|\xi_0|^2) - D_{11}w|\xi_0|^2 = 0 & \text{in } D^+_{1/2} \\
w \le \psi_\infty & \text{on } D_{1/2} \cap \{x_1 = 0\} \\
\partial_1 w \ge -\gamma & \text{on } D_{1/2} \cap \{x_1 = 0\} \\
(\psi_\infty - w)(\partial_1 w + \gamma) = 0 & \text{on } D_{1/2} \cap \{x_1 = 0\}
\end{cases}
\tag{3.41}
$$

in the viscosity sense. To this aim observe that the functions $u_n^\pm$, are subsolutions, respectively supersolutions, of

$$
\operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = \mp\frac{\epsilon_n}{n}
$$

thanks to Lemma 3.7. Therefore v_n^- and v_n^+ are supersolutions, respectively subsolutions, of

$$
L_n w := \Delta w - \frac{D^2 w(\alpha_n e_1 + \epsilon_n \nabla w)(\alpha_n e_1 + \epsilon_n \nabla w)}{1 + |\alpha_n e_1 + \epsilon_n \nabla w|^2} = \pm\frac{1}{n}\sqrt{1+|\alpha_n e_1 + \epsilon_n \nabla w|^2}\,.
$$

Passing to the limit in the previous equation, recalling that $v_n^\pm$ converge to v uniformly in $D^+_{\frac{1}{2}}$ and using the stability of viscosity super- and subsolutions, we get that v, hence $v - \omega \cdot x'$, is a viscosity solution of the first equation in (3.41). Recalling that $v_n^\pm \le \frac{\psi_n}{\epsilon_n}$ on $D_{\frac{1}{2}} \cap \partial H$, we get that $v(x') \le \psi_\infty + \omega \cdot x'$ for all $x' \in D_{\frac{1}{2}} \cap \partial H$, hence the second inequality in (3.41) follows for $v - \langle \omega, \cdot \rangle$.

To prove the third inequality we take a C^2 test function φ touching v from above in $D^+_{\frac{1}{2}}$ at a point $\overline{x}' \in D_{\frac{1}{2}} \cap \{x_1 = 0\}$. Without loss of generality we may assume that $\overline{x}'$ is the unique touching point. We argue by contradiction assuming that $\partial_1\varphi(\overline{x}') < -\gamma$. If this is the case the function

$$
\tilde{\varphi}(x') = -\tilde{\gamma}x_1 - bx_1^2 + \varphi(0, x_2, \dots, x_{N-1}), \qquad \text{with } \tilde{\gamma} \in (\gamma, -\partial_1\varphi(\overline{x}'))
\tag{3.42}
$$

and $b > 0$ to be chosen, stays above φ and hence above v in a neighborhood of $\overline{x}'$. In particular $\overline{x}'$ is the unique touching point between $\tilde{\varphi}$ and v in such a neighborhood. Therefore there exists a sequence $x_n' \in \overline{D^+_{\frac{1}{2}}}$ converging to $\overline{x}'$ such that x_n' is a local maximizer of $v_n^+ - \tilde{\varphi}$. If $x_n' \in D^+_{\frac{1}{2}}$ for infinitely many n, thus recalling the subsolution property of v_n^+ we have

$$
L_n\tilde{\varphi}(x_n') \ge -\frac{1}{n}\sqrt{1+|\alpha_n e_1 + \epsilon_n \nabla\tilde{\varphi}(x_n')|^2}.
$$

Thus, passing to the limit, $L\tilde{\varphi}(\overline{x}') \ge 0$ which is impossible if we choose b so large that $L\tilde{\varphi}(\overline{x}') < 0$. Otherwise, $x_n' \in D_{\frac{1}{2}} \cap \{x_1 = 0\}$ for infinitely many n. In particular for all such n the function $u_n^+ - (\epsilon_n\tilde{\varphi} + \alpha_n x_1)$ has a local maximum in x_n'. Hence by the third inequality in (3.14) we have

$$
\epsilon_n\partial_1\tilde{\varphi}(x_n') + \alpha_n \ge \frac{\sigma}{\sqrt{1-\sigma^2}} - \frac{\sigma^-}{\sqrt{1-\sigma^2}}\frac{\epsilon_n^2\|\nabla\tilde{\varphi}\|^2_{L^\infty(D_1^+)}}{2}\,.
$$

Dividing the previous inequality by ϵ_n and recalling the definition of γ we have

$$
-\tilde{\gamma} + \gamma \ge 0\,,
$$

which is a contradiction to (3.42). This shows the third inequality in (3.41).

To show the last equality we take a test function φ such that $v - \langle \omega, \cdot \rangle - \varphi$ has a strict local minimum at a point $\overline{x}' \in D_{\frac{1}{2}} \cap \{x_1 = 0\}$ such that $v(\overline{x}') - \omega \cdot \overline{x}' < \psi_\infty$. Then, arguing by contradiction as before and using u_n^- and the fourth inequality in (3.14) in place of u_n^+

and the third inequality in (3.14) we infer that $\partial_1\varphi(\overline{x}') \le -\gamma$, thus getting also the last equality in (3.41) in the viscosity sense.

Thus, the function $\overline{w}(x') := \psi_\infty - v(x') + \omega\cdot x' - \gamma x_1$ solves the following Signorini type problem

$$\begin{cases} \Delta\overline{w}(1+|\xi_0|^2) - D_{11}\overline{w}|\xi_0|^2 = 0 & \text{in } D^+_{1/2} \\ \overline{w} \ge 0 & \text{on } D_{1/2}\cap\{x_1=0\} \\ \partial_1\overline{w} \le 0 & \text{on } D_{1/2}\cap\{x_1=0\} \\ \overline{w}\,\partial_1\overline{w} = 0 & \text{on } D_{1/2}\cap\{x_1=0\} \end{cases}$$

in the viscosity sense. In particular by the regularity estimates proved in [13] (see also [10]) we infer that there exist a universal constant C, such that for all $\lambda < 1/4$

$$\sup_{D^+_\lambda}\big|v(x') - \nabla v(0)\cdot x'\big| = \sup_{D^+_\lambda}\big|\overline{w}(x') - \overline{w}(0) - \nabla\overline{w}(0)\cdot x'\big| \le C\lambda^{\frac32}\|\overline{w}\|_{L^\infty(D^+_{\frac12})} \le C(1+M)\lambda^{\frac32}$$

and

$$|\nabla v(0)| \le C\|\overline{w}\|_{L^\infty(D^+_{\frac12})} \le C(1+M)\,.$$

We first choose λ_0 so that $C(1+M)\lambda^{\frac32} < \frac14\lambda^{1+\tau}$ for all $\lambda\le\lambda_0$. Therefore, by the above estimate and uniform convergence of $v_n^\pm$ to v we get for n large, recalling that $v(0)=0$,

$$\begin{aligned} &(\epsilon_n\partial_1 v(0) + \alpha_n)x_1 + \epsilon_n(\nabla v(0) - \partial_1 v(0)e_1)\cdot x' - \frac14\lambda^{1+\tau}\epsilon_n \\ &\quad < u_n^-(x') \le u_n^+(x') < (\epsilon_n\partial_1 v(0) + \alpha_n)x_1 + \epsilon_n(\nabla v(0) - \partial_1 v(0)e_1)\cdot x' + \frac14\lambda^{1+\tau}\epsilon_n\,. \end{aligned}$$

In particular, setting

$$\bar\alpha_n = \epsilon_n\Big(\partial_1 v(0) + \frac{\lambda^{1+\tau}}{4}\Big) + \alpha_n$$

and

$$\boldsymbol{v}_n = \epsilon_n(\nabla v(0) - \partial_1 v_1(0)e_1)$$

$\partial E_n\cap\mathcal{C}_\lambda$ is contained in the strip

$$S_n := \Big\{|x_N - \bar\alpha_n x_1 - \boldsymbol{v}_n\cdot x'| < \frac12\lambda^{1+\tau}\epsilon_n\Big\}\,.$$

Note that, since by (3.41) $\partial_1 v(0) \ge -\gamma$, recalling (3.40), we have that for n sufficiently large $\bar\alpha_n > \frac{\sigma}{\sqrt{1-\sigma^2}}$. Hence, if R_n is the rotation around x_1 which maps the vector

$$\frac{e_N - \boldsymbol{v}_n}{|e_N - \boldsymbol{v}_n|}$$

in e_N, we conclude that

$$R_n(\overline{\partial E_n\cap H})\cap\mathcal{C}_\lambda \subset \{|x_N - \bar\alpha_n x_1| < \lambda^{1+\tau}\epsilon_n\}\,,$$

thus giving a contradiction in this case.

If instead for infinitely many n we have that $\partial_{\partial H}O_n\cap\mathcal{C}_{\frac12}\cap\{-\epsilon_n < x_N < \epsilon_n\} = \emptyset$, then reasoning as above we may infer that the function $\tilde w(x') := v(x') + \gamma x_1$ solves the Neumann problem

$$\begin{cases} \Delta\tilde w(1+|\xi_0|^2) - D_{11}\tilde w|\xi_0|^2 = 0 & \text{in } D^+_{1/2} \\ \partial_1\tilde w = 0 & \text{on } D_{1/2}\cap\{x_1=0\}\,. \end{cases}$$

The same argument, now using the more standard elliptic estimates for the Neumann problem, leads to a contradiction also in this case. $\square$

Next Lemma is the interior case of the above estimate and the proof follows from the interior version of the previous arguments taking into account Remark 3.14.

Lemma 3.18. *For all $\tau \in (0,1)$, $\kappa > 0$ there exist constants $\lambda_1 = \lambda_1(\tau,\kappa) > 0$, $C_3 = C_3(\tau,\kappa) > 0$ such that for all $\lambda \in (0,\lambda_1)$ there exist $\varepsilon_4 = \varepsilon_4(\tau,\kappa,\lambda) > 0$, $\eta_3 = \eta_3(\tau,\kappa,\lambda) > 0$ with the following property. If $E \subset H$ is a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle O such that if $\bar{x} \in \partial E$, $D_r(\bar{x}') \subset \{x_1 > 0\}$, $0 < r \le 1$,*

$$\partial(E-\bar{x}) \cap \mathcal{C}_r \subset \{|x_N - \alpha x_1| < \varepsilon r\}\,,$$

with $\varepsilon \le \varepsilon_4$, for some $\alpha \in [-\kappa,\kappa]$, $\Lambda r < \eta_3 \varepsilon$, then there exist $\bar{\alpha} \in \mathbb{R}$, a rotation R about the x_1 axis, with $\|R - \mathrm{Id}\| \le C_3\varepsilon$, $|\bar{\alpha} - \alpha| \le C_3 \varepsilon$, such that

$$\partial R(E-\bar{x}) \cap \mathcal{C}_{\lambda r} \subset \{|x_N - \bar{\alpha} x_1| < \lambda^{1+\tau}\varepsilon r\}.$$

Next Lemma deals with the case when $\partial_{\partial H} E$ fully coincides with the obstacle and it is a simple application of the boundary regularity result. This would be needed for the situations where Lemma 3.15 applies.

Lemma 3.19. *For all $\tau \in (0,1)$, $\kappa > 0$, there exist constants $\lambda_2 = \lambda_2(\tau,\kappa) > 0$, $C_4 = C_4(\kappa) > 0$ such that for all $\lambda \in (0,\lambda_2)$ one can find $\varepsilon_5 = \varepsilon_5(\tau,\kappa,\lambda) > 0$, with the following property: Assume that $E \subset H$ is a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$. Assume also that $\bar{x} \in \partial H \cap \overline{\partial E \cap H}$ and*

$$(\partial E - \bar{x}) \cap \mathcal{C}_r^+ \subset \{|x_N - \alpha x_1| < \varepsilon r\}, \quad \{x_N < \alpha x_1 - \varepsilon r\} \cap \mathcal{C}_r^+ \subset E - \bar{x}\,,$$

for some $\varepsilon \le \varepsilon_5$, $0 < r \le 1$ with $\Lambda r < \varepsilon$ and $\frac{r}{R} < \eta(1)\varepsilon$ (where $\eta(1)$ is the constant provided in Lemma 3.12), and for some $\alpha \in [-\kappa,\kappa]$. Finally assume that

$$\partial H \cap \overline{(\partial E - \bar{x}) \cap H} \cap \mathcal{C}_r = \partial_{\partial H}(O - \bar{x}) \cap \mathcal{C}_r \cap \{-\varepsilon r < x_N < \varepsilon r\}\,.$$

Then there exist $\bar{\alpha} \ge \frac{\sigma}{\sqrt{1-\sigma^2}}$, a rotation R about the x_1 axis, with $\|R - \mathrm{Id}\| \le C_4\varepsilon$, $|\bar{\alpha} - \alpha| \le C_4\varepsilon$, such that

$$R(\partial E - \bar{x}) \cap \mathcal{C}_{\lambda r}^+ \subset \{|x_N - \bar{\alpha} x_1| < \lambda^{1+\tau}\varepsilon r\}.$$

Proof. By rescaling and translating we may assume $r = 1$ and $\bar{x} = 0$. Denote by $\mathcal{E}$ the cylindrical excess

$$\mathcal{E}(E;r) = \frac{1}{r^{N-1}} \int_{\partial E \cap \mathcal{C}_r^+} |\nu_E(x) - \nu_\alpha|^2 \, d\mathcal{H}^{N-1}(x)\,,$$

where ν_α is the normal to the hyperplane $\{x_N - \alpha x_1 = 0\}$ pointing upward. Fix $\delta \in (0,1)$. We claim that there exists ε_5 such that if the assumptions are satisfied for $r = 1$ then

$$\mathcal{E}\Big(E;\frac{1}{2}\Big) \le \delta\,. \tag{3.43}$$

To prove this claim we observe that if E_n is a sequence of $(\Lambda_n,1)$-minimizers satisfying the assumptions with $\varepsilon = \varepsilon_n \to 0$, then by Theorem 2.4 we have that, up to subsequence, $|(E_n \Delta F) \cap \mathcal{C}^+_{\frac{1}{2}}| \to 0$, $|\mu_{E_n}| \llcorner \mathcal{C}^+_{\frac{1}{2}} \xrightarrow{*} |\mu_F| \llcorner \mathcal{C}^+_{\frac{1}{2}}$, where $F = \{x_N - \alpha x_1 \le 0\}$. Thus,

$$\mathcal{E}\Big(E_n;\frac{1}{2}\Big) \to \mathcal{E}\Big(F;\frac{1}{2}\Big) = 0\,.$$

Hence (3.43) follows by a compactness argument.

We recall that, thanks to Lemma 3.12, there exists $\eta(1) > 0$ such that if $\frac{1}{R} < \eta(1)\varepsilon$, $\partial_{\partial H} O \cap \mathcal{C}_1$ is the graph of a function $\psi \in C^{1,1}(D_1)$ such that

$$\|\nabla \psi\|_{L^\infty(\partial H \cap D_1)} \le 2\varepsilon, \quad \|D^2 \psi\|_{L^\infty(\partial H \cap D_1)} \le 2\varepsilon\,.$$

Observe that, there exists $\mu(\kappa) \in (0,1)$ such that if $\Sigma \subset \mathcal{C}^+_{\frac{1}{4}}$ can be described as a graph over $\{x_N - \alpha x_1 = 0\}$ of a C^1 function g such that $\|\nabla g\|_\infty \leq \mu(\kappa)$, then Σ can be also written as a graph over $D^+_{\frac{1}{4}}$ of a C^1 function f with $\|\nabla f\|_\infty \leq 2\kappa$.

By [5, Theorem 6.1][2] we have that there exists $\bar{\delta} = \bar{\delta}(\kappa)$ such that if

$$\mathcal{E}\Big(E; \frac{1}{2}\Big) + \Lambda + \|D^2\psi\|_\infty \leq \bar{\delta}\,,$$

then $\partial E \cap \mathcal{C}^+_{\frac{1}{4}}$ is a graph with respect to the hyperplane $\{x_N - \alpha x_1 = 0\}$ of a $C^{1,\gamma}$ function g with $\|\nabla g\|_{C^{0,\gamma}} \leq \mu(\kappa)$, for some universal $\gamma > 0$. In particular $\partial E \cap \mathcal{C}^+_{\frac{1}{4}}$ is the graph over $D^+_{\frac{1}{4}}$ of a $C^{1,\gamma}$ function f with $\|\nabla f\|_{C^{0,\gamma}} \leq C(\kappa)$. Then, observing that the function $f(x') - \alpha x_1$ is a solution of

$$\operatorname{div}\left(\frac{\nabla w + \alpha e_1}{\sqrt{1 + |\nabla w + \alpha e_1|^2}}\right) = h$$

with $|h| \leq \Lambda$, standard regularity estimates for solutions of the mean curvature equation imply that for all $s \in (0,1)$ there exists a constant $C_{s,\kappa}$, depending only on s and κ such that

$$\|f - \alpha x_1\|_{C^{1,s}(D^+_{\frac{1}{8}})} \leq C_{s,\kappa}\big(\|f - \alpha x_1\|_{L^\infty(D^+_{\frac{1}{4}})} + \|\psi\|_{C^{1,s}(D^+_{\frac{1}{4}})}\big) \leq C'_{s,\kappa}\varepsilon\,, \tag{3.44}$$

where the last inequality follows from the fact that $\|\psi\|_{C^{1,1}(D^+_{\frac{1}{4}})} \leq C(\|D^2\psi\|_{L^\infty(D^+_{\frac{1}{4}})} + \|\psi\|_{L^\infty(D^+_{\frac{1}{4}})})$ for a universal constant C. Let us fix $\tau \in (0,1)$ and take $s = (1+\tau)/2$. From the previous estimate we have that for all $\lambda < 1/8$, since $f(0) = 0$,

$$\sup_{D^+_\lambda}\big|f(x') - \nabla f(0)\cdot x'\big| \leq C'_{s,\kappa}\lambda^{1+s}\varepsilon < \frac{1}{4}\lambda^{1+\tau}\varepsilon\,,$$

provided that $\lambda < \lambda_2(\tau,\kappa)$. We take

$$\tilde{\alpha} = \partial_1 f(0)$$

and

$$\boldsymbol{v} = \nabla f(0) - \partial_1 f(0) e_1 = \nabla\psi(0)\,,$$

where the last equality follows from the fact that by assumption $f = \psi$ on $\partial H \cap \mathcal{C}_1$. Thus $R(\partial E) \cap \mathcal{C}^+_\lambda$ is contained in the strip

$$S := \{|x_N - \tilde{\alpha} x_1| < \frac{1}{2}\lambda^{1+\tau}\varepsilon\}\,,$$

where R is the rotation around x_1 which maps the vector

$$\frac{e_N - \boldsymbol{v}}{|e_N - \boldsymbol{v}|}$$

in e_N, provided ε, hence ε_5, is sufficiently small, depending on λ. Note that, recalling that $|\boldsymbol{v}| = |\nabla\psi(0)| \leq C\varepsilon$, the choice of $\tilde{\alpha}$ and (3.44), we have

$$\|R - \mathrm{Id}\,\| \leq C_4\varepsilon, \quad |\tilde{\alpha} - \alpha| \leq C_4\varepsilon\,,$$

for a sufficiently large C_4 depending only on κ.

[2] Theorem 6.1 in [5] is stated and proved for almost minimizing currents. However it is well known to the experts that the methods of the proof extend without significant changes also to the framework of almost minimizing sets of finite perimeter.

From the fact that E is a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathcal{C}^+_{\frac{1}{4}}$ of class $C^{1,\tau}$ up to the boundary via a standard first variation argument, see also Lemma 3.7, we get that

$$\frac{\partial_1 f(0)}{\sqrt{1+|\nabla f(0)|^2}} \geq \sigma\,.$$

From this inequality we get

$$\tilde{\alpha} = \partial_1 f(0) \geq \frac{\sigma}{\sqrt{1-\sigma^2}} - \frac{\sigma^-}{\sqrt{1-\sigma^2}}\frac{|\nabla\psi(0)|^2}{2}\,,$$

since $|\nabla f(0) - \tilde{\alpha}e_1| = |\nabla\psi(0)| \leq 2\varepsilon$. From the last inequality, setting $\bar{\alpha} = \max\big\{\tilde{\alpha}, \frac{\sigma}{\sqrt{1-\sigma^2}}\big\}$, we finally get

$$R(\partial E)\cap\mathcal{C}^+_\lambda \subset S\cap\mathcal{C}^+_\lambda \subset \Big\{|x_N - \bar{\alpha}x_1| < \frac{1}{2}\lambda^{1+\tau}\varepsilon + \frac{2\sigma^-\varepsilon^2}{\sqrt{1-\sigma^2}}\Big\} \subset \{|x_N - \bar{\alpha}x_1| < \lambda^{1+\tau}\varepsilon\}\,,$$

provided ε_5 is chosen sufficiently small. □

We can now prove the following

Lemma 3.20. *Let $\tau \in (0,1/2)$. There exist $\bar{\lambda} = \bar{\lambda}(\tau) \in (0,1/2)$ and $\bar{C} = \bar{C}(\tau)$ such that for all $\lambda \in (0,\bar{\lambda})$ it is possible to find $\bar{\varepsilon} = \bar{\varepsilon}(\lambda,\tau) \in (0,\frac{1}{2})$, $\bar{\eta} = \bar{\eta}(\lambda,\tau) \in (0,\frac{1}{2})$ with the following property: Assume $E \subset H$ is a $(\Lambda,1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$ and let $y \in \overline{\partial E\cap H}$ be such that*

$$(\overline{\partial E\cap H} - y)\cap\mathcal{C}_\varrho \subset S^\alpha_{-\varepsilon\varrho,\varepsilon\varrho}, \qquad \{x_N < \alpha x_1 - \varepsilon r\}\cap\mathcal{C}_\varrho \subset E - y\,,$$

for some $0 < \varrho \leq 1$, $0 < \varepsilon \leq \bar{\varepsilon}$, $\alpha \in [\frac{\sigma}{\sqrt{1-\sigma^2}} - 1, \frac{\sigma}{\sqrt{1-\sigma^2}} + 1]$ (and $\alpha \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ if $0 \leq y_1 < \frac{\lambda\varrho}{16}$), with $\Lambda\varrho < \bar{\eta}\varepsilon$, $\frac{\varrho}{R} \leq \bar{\eta}\varepsilon$. Then there exist a sequence of rotations $R_k = R_k(y)$, $R_0 = I$, a sequence $\alpha_k = \alpha_k(y) \in \mathbb{R}$, $\alpha_0 = \alpha$, such that, setting $\varrho_k = \frac{\lambda^{k+1}\varrho}{16}$,

$$\|R_{k+1} - R_k\| \leq \frac{\bar{C}}{\lambda}\lambda^{k\tau}\varepsilon, \qquad |\alpha_{k+1} - \alpha_k| \leq \frac{\bar{C}}{\lambda}\lambda^{k\tau}\varepsilon \tag{3.45}$$

and

$$R_k(\overline{\partial E\cap H} - y)\cap\mathcal{C}_{\varrho_k} \subset \big\{|x_N - \alpha_k x_1| < \lambda^{k(1+\tau)}\varrho\varepsilon\big\}. \tag{3.46}$$

Moreover $\alpha_k(y) \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ whenever $0 \leq y_1 < \frac{\lambda^k\varrho}{16}$.

Proof. By rescaling we may assume $\varrho = 1$. We fix $\tau \in (0,1/2)$.

Step 1. We first assume that $y \in \partial H$ and that $\alpha \leq \frac{\sigma}{\sqrt{1-\sigma^2}} + M_0\varepsilon$, where M_0 is the constant in Lemma 3.15, $\varepsilon \leq \bar{\varepsilon}$. Thus we may apply Lemma 3.17 taking $\lambda \leq \lambda_0(M_0,\tau)$ assuming that $\bar{\varepsilon} \leq \varepsilon_3$ and $\bar{\eta} \leq \eta_2$. We recall that both these constants depend on M_0, τ and λ. Then we get a rotation R_1 and $\alpha_1 \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ satisfying

$$\|R_1 - I\| \leq C_2\varepsilon, \qquad |\alpha_1 - \alpha| \leq C_2\varepsilon\,,$$

$$R_1(\overline{\partial E\cap H} - y)\cap\mathcal{C}_\lambda \subset \big\{|x_N - \alpha_1 x_1| < \lambda^{1+\tau}\varepsilon\big\}.$$

If $\alpha_1 \leq \frac{\sigma}{\sqrt{1-\sigma^2}} + M_0\lambda^\tau\varepsilon$, we can apply Lemma 3.17 again, with $r = \lambda$ and ε replaced by $\lambda^\tau\varepsilon$, to get a rotation $\tilde{R}$ such that $\|\tilde{R} - I\| \leq C_2\lambda^\tau\varepsilon$ and $\alpha_2 \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ such that, setting $R_2 = \tilde{R}\circ R_1$, $\|R_2 - R_1\| \leq C_2\lambda^\tau\varepsilon$, $|\alpha_2 - \alpha_1| \leq C_2\lambda^\tau\varepsilon$ and

$$R_2(\overline{\partial E\cap H} - y)\cap\mathcal{C}_{\lambda^2} \subset \big\{|x_N - \alpha_2 x_1| < \lambda^{2(1+\tau)}\varepsilon\big\}.$$

We may now iterate this procedure and get a sequence of rotations R_k and a sequence $\alpha_k \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ such that

$$\|R_{k+1} - R_k\| \leq C_2\lambda^{k\tau}\varepsilon, \qquad |\alpha_{k+1} - \alpha_k| \leq C_2\lambda^{k\tau}\varepsilon$$

and

$$R_{k+1}(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\lambda^{k+1}} \subset \big\{|x_N - \alpha_{k+1}x_1| < \lambda^{(k+1)(1+\tau)}\varepsilon\big\}.$$

hold as long as $\alpha_k \leq \frac{\sigma}{\sqrt{1-\sigma^2}} + M_0\lambda^{k\tau}\varepsilon$. If the latter inequality is satisfied for every k, we get the conclusion with $\frac{\bar{C}}{\lambda}$ replaced by C_2 and with ϱ_k replaced by λ^{k+1} and thus also by $\lambda^{k+1}/16$.

Otherwise let $\bar{k} \in \mathbb{N} \cup \{0\}$ be the first integer such that $\alpha_{\bar{k}} > \frac{\sigma}{\sqrt{1-\sigma^2}} + M_0\lambda^{\bar{k}\tau}\varepsilon$. In this case assuming that $\bar{\varepsilon} \leq \varepsilon_2$, Lemma 3.15 yields that

$$R_{\bar{k}}(\overline{\partial E \cap H} - y) \cap \partial H \cap \mathcal{C}_{\frac{\lambda^{\bar{k}}}{4}} = \partial_{\partial H} O \cap \mathcal{C}_{\frac{\lambda^{\bar{k}}}{4}} \cap \{|x_N| < \lambda^{\bar{k}(1+\tau)}\varepsilon\},$$

provided that we also enforce $\bar{\eta} \leq \eta(1)$. Observe that from the previous iteration argument we know in particular that

$$R_{\bar{k}}(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\frac{\lambda^{\bar{k}}}{4}} \subset \big\{|x_N - \alpha_{\bar{k}}x_1| < \lambda^{\bar{k}(1+\tau)}\varepsilon\big\}.$$

We may now use Lemma 3.19 with $\tilde{\varepsilon} = 4\lambda^{\bar{k}\tau}\varepsilon$ and $r = \lambda^{\bar{k}}/4$, provided that $4\bar{\varepsilon} \leq \varepsilon_5$ and that we have chosen from the beginning $\lambda \leq \lambda_2(\tau, \kappa)$, where $\kappa := \frac{|\sigma|}{\sqrt{1-\sigma^2}} + 2$. Indeed, since $\alpha \leq \frac{|\sigma|}{\sqrt{1-\sigma^2}} + 1$, we have that $\alpha_{\bar{k}} \in [-\kappa, \kappa]$, since

$$|\alpha_{\bar{k}} - \alpha| \leq C_2\bar{\varepsilon}\sum_{n=0}^{\infty}\lambda^{\tau n} < 1 \tag{3.47}$$

by taking $\bar{\varepsilon}$ smaller if needed. So we get there exist a rotation $R_{\bar{k}+1}$ with $\|R_{\bar{k}+1} - R_{\bar{k}}\| \leq 4C_4\lambda^{\bar{k}\tau}\varepsilon$, $|\alpha_{\bar{k}+1} - \alpha_{\bar{k}}| \leq 4C_4\lambda^{\bar{k}\tau}\varepsilon$, $\alpha_{\bar{k}+1} \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$, such that

$$R_{\bar{k}+1}(\overline{\partial E \cap H} - y) \cap \mathcal{C}^+_{\frac{\lambda^{\bar{k}+1}}{4}} \subset \big\{|x_N - \bar{\alpha}x_1| < \lambda^{(\bar{k}+1)(1+\tau)}\varepsilon\big\}.$$

At this point we keep iterating the previous argument by applying Lemma 3.19 to get for all $k > \bar{k}$ a sequence R_k and a sequence $\alpha_k \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ satisfying (3.46) (even with ϱ_k replaced by $\lambda^k/4$) and (3.45) with $\frac{\bar{C}}{\lambda}$ replaced by $4C_4$. Note that, arguing as for (3.47), we may ensure that during this iteration process $\alpha_k \in [-\kappa, \kappa]$ provided that we choose $\bar{\varepsilon}$ smaller if needed, depending on λ and κ.

Step 2. Let us now assume that $y \in \partial E \cap H$. If $y_1 \geq \frac{\lambda}{16}$, since by assumption we have

$$\partial(E - y) \cap \mathcal{C}_{\frac{\lambda}{16}} \subset \big\{|x_N - \alpha x_1| < \varepsilon\big\},$$

we may apply iteratively Lemma 3.18 choosing $\lambda < \lambda_1(\tau, \kappa)$, with κ as above, taking $\bar{\varepsilon} \leq \frac{\lambda}{16}\varepsilon_4$ and $\bar{\eta} \leq \eta_3$, where we recall that the constants η_3, ε_4 depend on τ, κ and λ. In this way we get the conclusion $\varrho_k = \frac{\lambda^{k+1}}{16}$ and with a sequence of rotations R_k, and a sequence α_k such that $\|R_{k+1} - R_k\| \leq \frac{16}{\lambda}C_3\lambda^{k\tau}\varepsilon$, $|\alpha_{k+1} - \alpha_k| \leq \frac{16}{\lambda}C_3\lambda^{k\tau}\varepsilon$. Note that, arguing as above, choosing $\bar{\varepsilon}$ smaller if needed, we may ensure that all the α_k remain in the interval $[-\kappa, \kappa]$.

Step 3. If $y_1 < \frac{\lambda}{16}$, we denote by $\hat{k}$ the last integer such that $y_1 < \frac{\lambda^{\hat{k}}}{16}$. We denote by $\bar{y}$ the point $\bar{y} = (0, y_2 \ldots, y_{N-1}, y_N - \alpha y_1)$. Observe that by assumption this point satisfies

$$(\overline{\partial E \cap H} - \bar{y}) \cap \mathcal{C}_{\frac{3}{4}} \subset S^{\alpha}_{-\varepsilon,\varepsilon}.$$

Hence from Step 1 we have that for all $k = 0, 1, \dots, \hat{k}$ there exist a radius $r_k \in \{\frac{3}{16}\lambda^k, \frac{3}{4}\lambda^k\}$, a rotation R_k and a number α_k, such that

$$\|R_{k+1} - R_k\| \le \frac{4}{3}\bar{C}\lambda^{k\tau}\varepsilon, \qquad |\alpha_{k+1} - \alpha_k| \le \frac{4}{3}\bar{C}\lambda^{k\tau}\varepsilon,$$

with $\bar{C} = \max\{C_2, 4C_4\}$, and

$$R_k(\overline{\partial E \cap H} - \bar{y}) \cap \mathcal{C}_{r_k} \subset \{|x_N - \alpha_k x_1| < \lambda^{k(1+\tau)}\varepsilon\}.$$

In particular we have that that for $k = 1, \dots, \hat{k}$

$$R_k(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\frac{\lambda^k}{8}} \subset \{|x_N - \alpha_k x_1| < \lambda^{k(1+\tau)}\varepsilon\}\,.$$

In particular we have that

$$R_{\hat{k}}(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\frac{\lambda^{\hat{k}+1}}{16}} \subset \{|x_N - \alpha_{\hat{k}} x_1| < \lambda^{\hat{k}(1+\tau)}\varepsilon\}\,.$$

Note that, as already observed in Step1, $\alpha_k \ge \frac{\sigma}{\sqrt{1-\sigma^2}}$ for all $k = 1, \dots, \hat{k}$. Observing that the cylinder $\mathcal{C}_{\frac{\lambda^{\hat{k}+1}}{16}}(y) \subset H$, we may start from this cylinder arguing as in the proof of Step 2 to conclude. $\square$

Theorem 3.21 (ε-regularity theorem)**.** *There exists $\widehat{\varepsilon} > 0$ with the following property. If $E \subset H$ is a $(\Lambda, 1)$-minimizer of $\mathcal{F}_\sigma$ in $\mathbb{R}^N$ with obstacle $O \in \mathcal{B}_R$, $\bar{x} \in \overline{\partial E \cap H} \cap \partial H$ such that*

$$(\overline{\partial E \cap H} - \bar{x}) \cap \mathcal{C}_r \subset S_{-\widehat{\varepsilon}r, \widehat{\varepsilon}r} \quad \left\{x_N < \frac{\sigma x_1}{\sqrt{1-\sigma^2}} - \widehat{\varepsilon}r\right\} \cap \mathcal{C}_r^+ \subset E - \bar{x}\,,$$

where $0 < r \le 1$, $\Lambda r < \widehat{\varepsilon}$, $\frac{r}{R} \le \widehat{\varepsilon}$, then $M := \overline{\partial E \cap H} \cap \mathcal{C}_{\frac{r}{2}}(\bar{x}')$ is a hypersurface (with boundary) of class $C^{1,\tau}$ for all $\tau \in (0, \frac{1}{2})$. Moreover,

$$\begin{aligned} &\nu_E \cdot \nu_H \ge \sigma \quad \text{on} \quad M \cap \partial H; \\ &\nu_E \cdot \nu_H = \sigma \quad \text{on} \quad (M \cap \partial H) \setminus \partial_{\partial H} O, \end{aligned}$$

Proof. We may assume $\bar{x} = 0$.

Step 1. We claim that given τ there exists $\hat{\varepsilon} = \hat{\varepsilon}(\tau)$ such that if the assumptions are satisfied for such $\hat{\varepsilon}$ then ∂E is of class $C^{1,\tau}$ in $\mathcal{C}^+_{\frac{r}{2}}(\bar{x}')$ uniformly up to ∂H.

To this aim we may assume without loss of generality that $r = 1$. We fix $\tau \in (0, 1/2)$ and let $\bar{\lambda}$ and $\bar{C}$ as in Lemma 3.20. Fix $\lambda \in (0, \bar{\lambda})$. Let $\bar{\varepsilon}$ and $\bar{\eta}$ be the corresponding constants provided once again by Lemma 3.20 and set $\hat{\varepsilon} = \frac{1}{2}\bar{\eta}\tilde{\varepsilon}$, with $\tilde{\varepsilon} \le \bar{\varepsilon}$ to be chosen.

We fix $y \in \partial E \cap H \cap \mathcal{C}_{\frac{1}{2}}$, $y = (y', y_N)$ and observe that

$$(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\frac{1}{2}} \subset S_{-\frac{\tilde{\varepsilon}}{2}, \frac{\tilde{\varepsilon}}{2}}$$

and that $\frac{1}{2}\Lambda < \bar{\eta}\tilde{\varepsilon}$, $\frac{1}{2R} \le \bar{\eta}\tilde{\varepsilon}$. Therefore, from (3.45) and (3.46) we have that there exist a sequence of rotations $R_k(y)$ converging to $R(y)$ and a sequence $\alpha_k(y)$ converging to $\alpha(y)$, for suitable $R(y)$ and $\alpha(y)$ such that

$$\|R_k(y) - R(y)\| \le C(\lambda, \tau)\lambda^{k\tau}\tilde{\varepsilon}, \qquad |\alpha_k(y) - \alpha(y)| \le C(\lambda, \tau)\lambda^{k\tau}\tilde{\varepsilon}\,, \tag{3.48}$$

with $C(\lambda, \tau) = \frac{\overline{C}}{\lambda(1-\lambda^\tau)}$ and

$$R_k(y)(\overline{\partial E \cap H} - y) \cap \mathcal{C}_{\varrho_k} \subset \left\{|x_N - \alpha_k(y) x_1| < \lambda^{k(1+\tau)}\frac{\tilde{\varepsilon}}{2}\right\},$$

with $\varrho_k = \frac{\lambda^{k+1}}{32}$. Note now that by the classical interior regularity results $\partial E \cap H$ is a locally $C^{1,\gamma}$-hypersurface for all $\gamma \in (0,1)$, provided $\tilde{\varepsilon}$ is sufficiently small. Therefore the hyperplane

$$y + R(y)^{-1}(\{x_N - \alpha(y)x_1 = 0\})$$

coincides with the tangent plane to ∂E at y.

Let now $y, z \in \partial E \cap H \cap \mathcal{C}_{\frac{1}{2}}$ with $0 < |y - z| < \frac{\lambda}{32}$ and let h an integer $h \geq 0$ such that $\frac{\varrho_{h+1}}{2} \leq |y - z| < \frac{\varrho_h}{2}$. Assume that $0 < y_1 \leq z_1$. Since $\mathcal{C}_{\frac{\varrho_h}{2}}(z') \subset \mathcal{C}_{\varrho_h}(y')$, we have

$$R_h(y)(\overline{\partial E \cap H} - z) \cap \mathcal{C}_{\frac{\varrho_h}{2}} \subset \left\{|x_N - \alpha_h(y)x_1| < \lambda^{h(1+\tau)}\tilde{\varepsilon}\right\}.$$

Thus we may apply Lemma 3.20 with $\varrho = \frac{\varrho_h}{2}$, $\varepsilon = \frac{64\lambda^{h\tau}}{\lambda}\tilde{\varepsilon} \leq \bar{\varepsilon}$ provided we have chosen $\tilde{\varepsilon}$ sufficiently small. Thus we get for $k \geq h$ a sequence of radii $r_k = \frac{\lambda^{k-h+1}\varrho_h}{32} = \frac{\lambda^{k+2}}{32^2}$, a sequence of rotations $S_k(z)$ converging to $S(z)$ and a sequence $\beta_k(z)$ converging to $\beta(z)$ such that

$$S_k(z)(\overline{\partial E \cap H} - z) \cap \mathcal{C}_{r_k} \subset \{|x_N - \beta_k(z)x_1| < \lambda^{k(1+\tau)}\tilde{\varepsilon}\}.$$

Clearly $S(z) = R(z)$ and $\beta(z) = \alpha(z)$ by the uniqueness of the tangent plane. Note also that, arguing as for (3.48), we have

$$\|S_k(z) - R(z)\| \leq C(\lambda, \tau)\lambda^{k\tau}\tilde{\varepsilon}, \qquad |\beta_k(z) - \alpha(z)| \leq C(\lambda, \tau)\lambda^{k\tau}\tilde{\varepsilon},$$

for a possibly larger constant $C(\lambda, \tau)$. Therefore, since $R_h(y) = S_h(z)$, and since $\frac{\lambda^{h+2}}{64} \leq |y - z|$ by our choice of h, we have

$$\|R(y) - R(z)\| \leq \|R(y) - R_h(y)\| + \|S_h(z) - R(z)\| \leq 2C(\lambda, \tau)\lambda^{h\tau}\tilde{\varepsilon} \leq \widetilde{C}(\lambda, \tau)\tilde{\varepsilon}|y - z|^\tau.$$

A similar estimate holds also for $|\alpha(y) - \alpha(z)|$, showing that both the maps α and R are τ-Hölder continuous uniformly up to ∂H. This proves that ∂E is of class $C^{1,\tau}$ up to ∂H, where τ is the exponent fixed at the beginning of Step 1.

Finally observe that if $y \in \overline{\partial E \cap H} \cap \partial H$, we may choose a sequence $y_k \in \partial E \cap H$ converging to y and such that $y_k \cdot e_1 < \frac{\lambda^k}{32}$. Then, from Lemma 3.20 we have that $\alpha_k(y_k) \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ and in turn using (3.48) and passing to the limit we get that also

$$\alpha(y) \geq \frac{\sigma}{\sqrt{1 - \sigma^2}}. \tag{3.49}$$

Step 2. Let us now show that if $\overline{\partial E \cap H}$ is of class $C^{1,\tau}$ in $\mathcal{C}_{\frac{1}{2}}$ for some $\tau \in (0, \frac{1}{2})$, then it is also of class $C^{1,\gamma}$ for all $\gamma \in (0, 1/2)$. To this aim we take a point $y \in \overline{\partial E \cap H} \cap \partial H \cap \mathcal{C}_{\frac{1}{2}}$ and consider two cases.

Assume first that $\alpha(y) > \frac{\sigma}{\sqrt{1-\sigma^2}}$. Exploiting the C^1 regularity of $\overline{\partial E \cap H}$ up to ∂H we may find $\varepsilon < \varepsilon_2$ and ϱ so small that $\alpha(y) > \frac{\sigma}{\sqrt{1-\sigma^2}} + M_0\varepsilon$, where M_0 and ε_2 are the constants of Lemma 3.15, and that

$$(\partial E - y) \cap \mathcal{C}^+_{2\varrho} \subset \{|x_N - \alpha x_1| < \varepsilon\varrho\}.$$

Then we have that $\overline{(\partial E - y) \cap H} \cap \partial H \cap \mathcal{C}_{\varrho/2} = \partial_{\partial H} O \cap \mathcal{C}_{\varrho/2} \cap \{|x_N| < \varepsilon\varrho\}$. Therefore, see for instance [5], we may conclude that $\overline{\partial E \cap H}$ is of class $C^{1,\gamma}$ in $\mathcal{C}_{\varrho/2}(y')$ for all $\gamma \in (0, 1)$.

Otherwise, recalling (3.49), we have $\alpha(y) = \frac{\sigma}{\sqrt{1-\sigma^2}}$. In this case, given $\gamma \in (0, 1/2)$, we may choose ϱ so small that the assumptions of the claim in Step 1 are satisfied in $\mathcal{C}_\varrho(y')$ with $\hat{\varepsilon}(\tau)$ replaced by $\hat{\varepsilon}(\gamma)$ and the conclusion follows from Step 1. $\square$

4. A monotonicity formula and proof of the main results

In this section, in view of the applications to the model for nanowire growth discussed in Subsection 1.2, we consider also the case of convex polyhedral obstacles. Thus, in order to deal at the same time with convex and smooth obstacles, we introduce the following definition where we identify O as a subset of $\mathbb{R}^{N-1}$.

Definition 4.1. Let $O \subset \partial H \approx \mathbb{R}^{N-1}$, we say that O is locally *semi-convex* at scale $\bar{r} > 0$ and with constant $C \geq 0$ such that, with if for all $\bar{x} \in \partial_{\partial H} O$ there exists a radius C-semiconvex function $\psi : \mathbb{R}^{N-2} \to \mathbb{R}$ with $\psi(0) = 0$ such that, up to a change of of coordinates:

$$O \cap \partial H \cap B_{\bar{r}}(\bar{x}) = \Big(\bar{x} + \big\{(0, x'', x_N) : x_N \geq \psi(x'')\big\}\Big) \cap B_{\bar{r}}(\bar{x}). \tag{4.1}$$

Recall that a function $\psi : \mathbb{R}^{N-2} \to \mathbb{R}$ is said to be C semiconvex if the function $\psi(x'') + C|x''|^2/2$ is convex. In particular the sub-differential of $\partial\psi(x'')$ for all x'' is non-empty, where

$$\partial\psi(x'') = \Big\{p \in \mathbb{R}^{N-2} : \psi(y'') \geq \psi(x'') + p \cdot (y'' - x'') - \frac{C|x'' - y''|^2}{2} \text{ for all } y'' \in \mathbb{R}^{N-2}\Big\}.$$

Note in particular that if $\psi(0) = 0$ and $p \in \partial\psi(x'')$ then

$$0 \geq \psi(x'') - p \cdot x'' - \frac{C|x''|^2}{2}. \tag{4.2}$$

By taking O as above we define for $x = (0, x'', x_N) \in \partial_{\partial H} O$ the normal and the tangent cones at x in ∂H as

$$N_x O = \{\lambda(0, p, -1) : \, p \in \partial\psi(x''), \lambda \in [0, +\infty)\}$$

and

$$T_x O = \{v \in \mathbb{R}^N : \, v \cdot e_1 = 0, \, v \cdot \nu \leq 0 \quad \text{for all } \nu \in N_x O\}.$$

It is well known that for $x = (0, x'', x_N) \in \partial_{\partial H} O$, the sets

$$O_{x,r} = \frac{O - x}{r} \to T_x O$$

as $r \to 0$ where the convergence is the Kuratowski sense. We start with the following technical lemma.

Lemma 4.2. *Let $O \subset \partial H$ be such that* (4.1) *is satisfied. Then there exist a smooth vector field $X : \mathbb{R}^N \to \mathbb{R}^N$ such that*

(i) $X(x) \cdot e_1 = 0$ for all $x \in \partial H$
(ii) $X(x) \cdot \nu \geq 0$ for all $x \in \partial_H O \cap B_{\bar{r}}(\bar{x})$ and all $\nu \in N_x O$
(iii) For all $x \in B_{\bar{r}}(\bar{x})$ it holds:

$$X(x) = x + O(|x|^2)\,, \tag{4.3}$$

$$\nabla X(x) = \mathrm{Id} + O(|x|)\,. \tag{4.4}$$

Proof. We may assume that $\bar{x} = 0$ and we define

$$X(x) = \Big(x_1, x'', x_N - \frac{C|x''|^2}{2}\Big)$$

where C is the semiconvexity constant of ψ. Clearly (i) and (iii) are satisfied. To check (ii), note that if $\nu \in N_x O$ then

$$\nu = \lambda(0, p, -1)$$

for $p \in \partial\psi(x')$ and $\lambda \geq 0$ so that

$$X(x) \cdot \nu = \lambda\Big(p \cdot x' - \psi(x') + \frac{C|x'|^2}{2}\Big) \geq 0$$

by (4.2). $\square$

We can now state the desired monotonicity formula for (Λ, r_0) minimizers.

Theorem 4.3. *Let $E \subset H$ be a (Λ, r_0)-minimizer of $\mathcal{F}_\sigma^O$ with obstacle O. Assume that $\bar{x} \in \partial_{\partial H} O$ and that (4.1) is satisfied with $0 < \bar{r} \leq r_0$ and a C-semiconvex function ψ with $\psi(0) = 0$. Then there exists a constant $c_0 = c_0(C, \Lambda, N)$ such that for all $0 < s < r$, with $r < \bar{r}$ sufficiently small,*

(4.5)
$$e^{c_0 r}\frac{\mathcal{F}_\sigma^O(E, B_r(\bar{x}))}{r^{N-1}} - e^{c_0 s}\frac{\mathcal{F}_\sigma^0(E, B_s(\bar{x}))}{s^{N-1}} \geq \frac{1}{2}\int_s^r t^{1-N} e^{c_0 t}\frac{d}{dt}\Big[\int_{\partial E \cap H \cap B_t(\bar{x})} \frac{(x \cdot \nu_E)^2}{|x|^2}\Big]\, dt\,.$$

Proof. In this proof by $O(r)$ we mean any function bounded by cr for r small, where the constant c depends only on Λ, N and the semiconvexity constant C.

We assume that $\bar{x} = 0$ and we let $\phi \in C_c^\infty([0, 1); [0, 1))$ be a smooth decreasing function with $\phi(0) = 1$ and we consider for $r \leq \bar{r}/2$ the vector field $T : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$T(x) = -\phi\left(\frac{|x|}{r}\right) X(x)\,,$$

where X is as in Lemma 4.2. By (i) and (ii) of the lemma, if we let φ_t be the flow generated by T, then for all $t > 0$

$$\varphi_t(\partial H) = \partial H, \qquad \varphi_t(O) \subset O \qquad \text{and} \qquad \varphi_t(x) = x \quad \text{for } x \notin B_r\,.$$

In particular $\varphi_t(E)\Delta E \Subset B_{\bar{r}}$ and thus the set $\varphi_t(E)$ is a competitor for the (Λ, r_0)-minimality of E. Hence

$$\begin{aligned} \mathcal{F}_\sigma^O(E; B_{\bar{r}}) &\leq \mathcal{F}_\sigma^O(\varphi_t(E); B_{\bar{r}}) + \Lambda|\varphi_t(E)\Delta E| \\ &\leq P(\varphi_t(\partial E \setminus O)) + \sigma P(\varphi_t(\partial E \cap O)) + \Lambda|\varphi_t(E)\Delta E| \end{aligned} \tag{4.6}$$

for all $t > 0$. By using the coarea formula and recalling that $|X(x)| = O(r)$ on $\operatorname{spt} T$, it is easy to check that for $t > 0$ small enough

$$|\varphi_t(E)\Delta E| = t\int_{\partial E \cap H} |T(x) \cdot \nu_E(x)|\, d\mathcal{H}^{N-1} + o(t) \leq tO(r)\int_{\partial E \cap H} \phi\left(\frac{|x|}{r}\right) d\mathcal{H}^{N-1}\,.$$

Therefore, differentiating the inequality in (4.6) we get

$$O(r)\int_{\partial E \cap H} \phi\left(\frac{|x|}{r}\right) \mathcal{H}^{N-1} \leq \int_{\partial E \setminus O} \operatorname{div}_\tau T d\mathcal{H}^{N-1} + \sigma\int_{\partial E \cap O} \operatorname{div}_\tau T d\mathcal{H}^{N-1}\,, \tag{4.7}$$

where

$$\operatorname{div}_\tau T = \operatorname{div} T - \nabla T[\nu_E] \cdot \nu_E$$

is the tangential divergence of T. Since

$$\nabla T = -\phi\left(\frac{|x|}{r}\right)\nabla X(x) - \phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\frac{x}{|x|} \otimes \frac{X(x)}{|x|}$$

by exploiting (4.3), (4.4) we have

$$\begin{aligned} \operatorname{div}_\tau T = &- \phi\left(\frac{|x|}{r}\right)(N-1) - \phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r} \\ &+ \phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\left(\frac{(x \cdot \nu_E)^2}{|x|^2}\right) + O(r)\phi\left(\frac{|x|}{r}\right) + O(r)\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}. \end{aligned}$$

which can be written as

$$\begin{aligned}\operatorname{div}_\tau T = (1+O(r))r^N \frac{d}{dr}\left(r^{1-N}\phi\left(\frac{|x|}{r}\right)\right)\\
+\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\left(\frac{(x\cdot\nu_E)^2}{|x|^2}\right)+O(r)\phi\left(\frac{|x|}{r}\right)\end{aligned}$$

Combining the above inequality with (4.7) we infer, after easy computations,

$$\begin{aligned}(1+O(r))\frac{d}{dr}&\left(r^{1-N}\int_{\partial E\setminus O}\phi\left(\frac{|x|}{r}\right)\mathcal{H}^{N-1}+\sigma r^{1-N}\int_{\partial E\cap O}\phi\left(\frac{|x|}{r}\right)d\mathcal{H}^{N-1}\right)\\
&\geq -c\left(r^{1-N}\int_{\partial E\setminus O}\phi\left(\frac{|x|}{r}\right)\mathcal{H}^{N-1}+\sigma r^{1-N}\int_{\partial E\cap O}\phi\left(\frac{|x|}{r}\right)d\mathcal{H}^{N-1}\right)\\
&\quad -r^{-N}\int_{\partial E\setminus O}\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\left(\frac{(x\cdot\nu_E)^2}{|x|^2}\right)-\sigma r^{-N}\int_{\partial E\cap O}\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\left(\frac{(x\cdot\nu_E)^2}{|x|^2}\right).\end{aligned}$$

In turn, noticing that

$$\int_{\partial E\cap H}\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\frac{(x\cdot\nu_E)^2}{|x|^2}=0\,,$$

setting

$$\begin{aligned}h(r)&=r^{1-N}\int_{\partial E\setminus O}\phi\left(\frac{|x|}{r}\right)d\mathcal{H}^{N-1}+\sigma r^{1-N}\int_{\partial E\cap O}\phi\left(\frac{|x|}{r}\right)d\mathcal{H}^{N-1},\\
k(r)&=-r^{-N}\int_{\partial E\setminus O}\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\frac{(x\cdot\nu_E)^2}{|x|^2}-\sigma r^{-N}\int_{\partial E\cap O}\phi'\left(\frac{|x|}{r}\right)\frac{|x|}{r}\frac{(x\cdot\nu_E)^2}{|x|^2}\\
&=r^{1-N}\frac{d}{dr}\left[\int_{\partial E\cap H}\phi\left(\frac{|x|}{r}\right)\frac{(x\cdot\nu_E)^2}{|x|^2}\,d\mathcal{H}^{N-1}\right],\end{aligned}$$

the inequality above implies that

$$h'(r)\geq -c_0h(r)+\frac{1}{2}k(r)$$

provided that $2\geq 1+O(r)\geq 1/2$, where c_0 is a costant depending only on Λ, N and C. Note also that $k(r)\geq 0$. Multiplying both sides of this inequality by e^{c_0r} and integrating in (s,r), we have

$$h(r)e^{c_0r}-h(s)e^{c_0s}\geq\frac{1}{2}\int_s^r e^{c_0t}k(t)\,dt\,.$$

Then we conclude by letting $\phi\to 1$. □

By classical argument, the above monotonicity formula allows for the study of blow-ups of Λ-minimizers. To this aim we preliminary observe that the following compactness property for blow-ups in the case of convex polyhedral domains holds.

Theorem 4.4. *Let $E\subset H$ be a (Λ, r_0)-minimizer of $\mathcal{F}_\sigma$ with obstacle O, where O is a convex polyhedron. Let $\bar{x}\in\partial_H O\cap\partial E$ and set*

$$E_h=\frac{E-\bar{x}}{r_h}\,,$$

where $r_h \to 0^+$. Then there exist a (not relabelled) subsequence and a set E_∞ of locally finite perimeter, such that $E_h \to E_\infty$ in $L^1_{loc}(\mathbb{R}^N)$ with the property that E_∞ is a 0-minimizer of $\mathcal{F}_\sigma^{\tilde{O}}$, where $\tilde{O} = T_{\bar{x}}O$. Moreover,

$$\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E\,, \quad |\mu_{E_h}| \overset{*}{\rightharpoonup} |\mu_E|\,,$$

as Radon measures. In addition, the following Kuratowski convergence type properties hold:

(i) for every $x \in \partial E_\infty$ there exists $x_h \in \partial E_h$ such that $x_h \to x$;

(ii) if $x_h \in \overline{\partial E_h \cap H}$ and $x_h \to x$, then $x \in \overline{\partial E_\infty \cap H}$.

Finally, either $\partial E_\infty \cap (\partial H \setminus \tilde{O}) = \partial H \setminus \tilde{O}$ or $\partial E_\infty \cap \partial H \subset \overline{\tilde{O}}$.

The proof of this theorem follows as in the proof of Theorem 2.4 observing that the density estimates proved in Proposition 2.3 still hold when O is a convex polyhedron.

Proposition 4.5. *Let $E \subset H$ be a (Λ, r_0)-minimizer of $\mathcal{F}_\sigma$ with obstacle O, where O is either of class $C^{1,1}$ or a convex polyhedron. If $\bar{x} \in \partial_{\partial H} O \cap \partial E$ then the sets*

$$E_{\bar{x},r} = \frac{E - \bar{x}}{r}$$

are pre-compact in L^1 and every limit point E_∞ as $r \to 0$ is a conical minimizer of $\mathcal{F}_\sigma^{\tilde{O}}$ with obstacle $\tilde{O} = T_{\bar{x}}O$. Moreover if $N = 3$ either $\partial E_\infty = \partial H$ or, after a rotation of coordinates in ∂H,

$$\partial E_\infty \cap H = \big\{x : x_N = \alpha x_1\big\} \cap H$$

with

$$\alpha \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$$

and $\tilde{O}$ is a half space.

Proof. Let E_∞ be a limit point of of $E_{\bar{x},r}$ as $r \to 0$. By Theorem 2.4 or Theorem 4.4 E_∞ is a 0-minimizer of $\mathcal{F}_\sigma^{\tilde{O}}$. Observe that by Theorem 4.3 there exists

$$\lim_{r\to 0} \frac{\mathcal{F}_\sigma(E, B_r(\bar{x}))}{r^{N-1}}$$

and it is finite. From this a standard argument, see for instance the proof of Theorem 28.6 in [12], shows that

$$\frac{\mathcal{F}_\sigma^{\tilde{O}}(E_\infty, B_r(\bar{x}))}{r^{N-1}}$$

is constant with respect to r. Thus, the right-hand side of (4.5) with E_∞ in place of E is zero. This immediately implies that $x \cdot \nu_{E_\infty}(x) = 0$ for $\mathcal{H}^{N-1}$-a.e. x, hence, see [12, Prop. 28.8], E_∞ is a cone.

Note that if $N = 3$ the only cones with zero mean curvature are planes and this forces ∂E_∞ to be a union of planes. These planes can not intersect in H by the interior regularity theory. In particular since $0 \in \partial E_\infty$ (which holds true since $\bar{x} \in \partial E$), either $\partial E_\infty = \partial H$ or, after a rotation in the hyperplane H,

$$\partial E_\infty \cap H = \bigcup_{i=1}^{k} \big\{x : x_1 = \alpha_i x_N\big\} \cap H.$$

Minimality forces $k = 1$ and $\partial E_\infty \cap \partial H = \tilde{O}$ and thus $\tilde{O}$ is a half space. $\square$

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. We start by proving part (i). In view of the result [3, Theorem 1.2] it is enough to prove the regularity in a neighborhood of a point $\bar{x} \in \partial E \cap \partial_{\partial H} O$. Given such a point, by Proposition 4.5 we know that there exists a sequence $E_h = E_{\bar{x}, r_h} = \frac{E - \bar{x}}{r_h}$, $r_h \to 0$, of blow-ups of E converging in L^1 to a 0-minimizer E_∞ of $\mathcal{F}_\sigma^{\tilde{O}}$, where $\tilde{O} = T_{\bar{x}} O$. Moreover, either $\partial E_\infty = \partial H$ or, after a rotation of coordinates in ∂H, $\partial E_\infty \cap H = \{x : x_1 = \alpha x_N\} \cap H$ with $\alpha \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$. Note that if $E_\infty = \partial H$ or if $\alpha > \frac{\sigma}{\sqrt{1-\sigma^2}}$ by conclusion (ii) of Theorem 2.4 we get that $\overline{\partial E_h \cap H}$ satisfies the assumptions of Lemma 3.15 in a neighborhood of $\bar{x}$ and thus $\overline{\partial E \cap H} \cap \partial H$ coincides with $\partial_{\partial H} O$ in such a neighborhood. In turn, the conclusion follows by [5, Theorem 6.1]. If instead $\alpha = \frac{\sigma}{\sqrt{1-\sigma^2}}$, using again (ii) of Theorem 2.4 we have that for E_h satisfies the assumptions of Theorem 3.21 for h sufficiently large. Hence the conclusion follows also in this case.

Now the proof of part (ii) follows combining (i) with the classical Federer's dimension reduction argument, see for instance [17, Appendix A] or [12, Sections 28.4-28.5]. We leave the details to the reader. □

We conclude the section by proving Theorems 1.5 and 1.6. In the following we make use of the notation introduced in Subsection 1.2.

Proof of Theorem 1.5. We start by recalling that by a standard argument, see for instance Example 21.3 in [12], there exists $\Lambda, r_0 > 0$ such that

$$J_{\sigma,\omega}(E) \leq J_{\sigma,\omega}(F) + \Lambda ||E| - |F||$$

for all $F \subset \mathbb{R}^3 \setminus \mathbf{C}$ with $\operatorname{diam}(E \Delta F) < r_0$. It easily follows that E is a (Λ, r_0)-minimizer of $J_{\sigma,\omega}$ with obstacle ω in the sense introduced in the previous sections. Thus the conclusion follows from Theorem 1.4. □

Before proving Theorem 1.6 we set the following definition.

Definition 4.6. We say that $E \subset \mathbb{R}^3 \setminus \mathbf{C}$ has *a nontangential contact with* $\mathbf{C}_{top}$ *at a point* $\bar{x} \in \overline{\partial E \cap \mathscr{H}} \cap \gamma$ if for any subsequence $E_{\bar{x}, r_h}$ of blow-ups of E with $r_h \to 0$, converging to E_∞, we have that ∂E_∞ does not contain the plane $\{x_3 = 0\}$.

Proof of Theorem 1.6. We start by observing that, arguing as in the proof of Theorem 1.5 we have that E is (Λ, r_0)-minimizer of $J_{\sigma,\omega}$ with ostacle ω.

Assume first that E has a nontangential contact at all points of $\overline{\partial E \cap \mathscr{H}} \cap \gamma$. Fix a point $\bar{x} \in \overline{\partial E \cap \mathscr{H}} \cap \gamma$ and consider a sequence $E_{\bar{x}, r_h}$ of blow-ups of E with $r_h \to 0$, converging to some set E_∞ of locally finite perimeter. By Proposition 4.5 it follows that E_∞ is a 0-minimizer of $J_{\sigma,\omega}$ with obstacle $T_{\bar{x}}\omega$, that after a rotation in the plane $\{x_3 = 0\}$

$$\partial E_\infty \cap \{x_3 > 0\} = \{x : x_1 = \alpha x_3\} \cap \{x_3 > 0\},$$

with $\alpha \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$ and thus $\partial E_\infty \cap \{x_3 = 0\}$ is a half plane. In turn $T_{\bar{x}}\omega$ is a half plane and thus it cannot be a vertex of the polygon. Hence, thanks to Theorem 1.4, $\overline{\partial E \cap \mathscr{H}}$ is of class $C^{1,\tau}$ for all $\tau \in (0, \frac{1}{2})$ in a neighborhood of $\bar{x}$.

Finally, we claim that if $\sigma < 0$ then E has a nontangential contact at all points of $\overline{\partial E \cap \mathscr{H}} \cap \gamma$. Indeed assume by contradiction that $E_\infty = \{x_3 > 0\}$ for some point $\bar{x} \in \overline{\partial E \cap \mathscr{H}} \cap \gamma$ and some sequence of blow-ups $E_{\bar{x}, r_h}$ with $r_h \to 0$. Then, arguing similarly as for Theorem 4.4, see also [3, Theorem 2.9], we get that E_∞ is a 0-minimizer of $J_{\sigma, T_{\bar{x}}\omega}$, that is

$$J_{\sigma, T_{\bar{x}}\omega}(E_\infty) \leq J_{\sigma, T_{\bar{x}}\omega}(F)$$

for all $F \subset \mathbb{R}^3 \setminus (T_{\bar{x}}\omega \times (-\infty, 0])$ with $E_\infty \Delta F$ bounded. From this it easily follows that $F_\infty := (\mathbb{R}^3 \setminus (T_{\bar{x}}\omega \times (-\infty, 0])) \setminus E_\infty$ is a 0-minimizer of $J_{-\sigma, T_{\bar{x}}\omega}$. By a rotation in the plane $\{x_3 = 0\}$ we may assume without loss of generality that the half line $\ell = \{(0, x_2, 0) : x_2 >$

$0\}$ is contained in $\partial T_{\bar{x}}\omega \times \{0\}$ and that F_∞ (locally around each point of ℓ) is contained in the half space $H = \{x_1 > 0\}$. Hence, locally around any point $y \in \ell$, F_∞ is a 0 minimizer of $\mathcal{F}_{-\sigma}$ with obstacle $O = \{x_3 < 0\} \cap \partial H$. Thus, by the third inequality in (3.14) (with σ replaced by $-\sigma$) it follows that ∂F_∞ must form with the vertical plane ∂H an angle which is strictly larger then $\pi/2$, thus leading to a contradiction. □

5. Appendix

Here we give the proof of Proposition 3.5. We shall closely follow the proofs of Lemmas 2.1 and 2.2 and of Theorem 1.1 in [16] with the necessary changes.

We denote by $\mathbb{M}^{d\times d}_{\text{sym}}$ the space of symmetric $d \times d$ matrices and by $\mathrm{Tr}\,(A)$ the trace of A. Let $F : \mathbb{M}^{d\times d}_{\text{sym}} \times \mathbb{R}^d \to \mathbb{R}$ be the function

$$F(A,p) := \frac{\mathrm{Tr}\,(A)}{(1+|p+\xi|^2)^{1/2}} - \sum_{i,j=1}^{d} \frac{A_{ij}(p_i+\xi_i)(p_j+\xi_j)}{(1+|p+\xi|^2)^{3/2}}\,,$$

where $\xi \in \mathbb{R}^d$ is a vector, with $|\xi| \le M$. Observe that there exist two positive constants $\tilde{\lambda}, \lambda > 0$, depending only on M and d such that if $|p| \le 1$, $A \in \mathbb{M}^{d\times d}_{\text{sym}}$ and $A \ge 0$, then

$$\tilde{\lambda}\|A\| \ge F(A,p) \ge \lambda\|A\|\,,$$

where $\|A\| := \sqrt{\sum_{i,j} A_{ij}^2}$. Since throughout this section M will be fixed, in the following with a slight abuse of language we will say that a constant is universal if it depends only on d and M. The next result is essentially as [16, Lemma 2.1].

Lemma 5.1. *Let $r \in (0,1)$ and $a \in (0,1/2)$. Let $v : B_1 \to (0,\infty)$ be a viscosity $(2a)$-supersolution of the equation*

$$\mathrm{div}\left(\frac{\nabla u + \xi}{\sqrt{1+|\nabla u+\xi|^2}}\right) = a\,, \tag{5.1}$$

bounded from above. There exists a universal constant c_0 with the following property: Let $\overline{B}_r(x_0) \subset B_1$ and let $B \subset \overline{B}_1$ be a closed set. For every $y \in B$ consider the paraboloid

$$-\frac{a}{2}|x-y|^2 + c_y \tag{5.2}$$

staying below v in $\overline{B}_r(x_0)$ and touching v in $\overline{B}_r(x_0)$. Assume that the set A of all such touching points for $y \in B$ is contained in $B_r(x_0)$. Then $|A| \ge c_0|B|$.

Proof. Observe that since v is a bounded lower semicontinuous function, then A is a closed set.

Assume that v is semi-concave in $B_r(x_0)$, that is there exists $b > 0$ such that $v - b|x|^2$ is concave in $B_r(x_0)$. Note that this implies in particular that v is differentiable at all touching points $z \in A$. Moreover it is not difficult to show that Dv is Lipschitz in A with a Lipschitz constant depending only on a and b. By assumption if $z \in A$ there exists a paraboloid of vertex $y \in B$ and opening a as in (5.2) touching v at z from below. Moreover,

$$y = z + \frac{1}{a}Dv(z)\,. \tag{5.3}$$

Let us denote by Z the set of points in $B_r(x_0)$ such that v is twice differentiable at z. By Alexandrov theorem $|B_r(x_0) \setminus Z| = 0$. Fix $z \in Z$. For all $x \in B_r(x_0)$ we have

$$v(x) = P(x;z) + o(|x-z|^2) = v(z) + Dv(z)\cdot(x-z) + \frac{1}{2}(x-z)^T D^2 v(z)(x-z) + o(|x-z|)^2\,.$$

We claim that there exists a universal constant $C > 0$ such that

$$-aI \le D^2 v(z) \le CaI \qquad \text{for all } z \in A \cap Z\,.$$

The left inequality follows from the fact that the paraboloid in (5.2) touches v from below. To prove the second inequality assume that there exists a unit vector e such that

$$D^2v(z) \geq Ca\, e\otimes e - aI\,.$$

We will prove that if C is sufficiently large this leads to a contradiction.

Consider now the test function $\varphi(x) = P(x;z) - \frac{\varepsilon}{2}|x-z|^2$, with $\varepsilon > 0$. Note that φ lies below v in a neighborhood of z and touches v at z and that by (5.3) $|D\varphi(z)| = |Dv(z)| = a|y-z| \leq 2a$. Therefore φ is an admissible test function and we have

$$\begin{aligned} a &\geq F(D^2\varphi(z), D\varphi(z)) = F(D^2v(z) - \varepsilon I, D\varphi(z)) \geq F(Ca\, e\otimes e - (a+\varepsilon)I, D\varphi(z)) \\ &= F(Ca\, e\otimes e, D\varphi(z)) - (a+\varepsilon)F(I, D\varphi(z)) \geq Ca\lambda - (a+\varepsilon)\tilde{\lambda}\sqrt{n} \end{aligned}$$

which is a contradiction if $C > (\tilde{\lambda}\sqrt{n}+1)/\lambda$ and ε is sufficiently small.

Now, from the area formula, using (5.3) we obtain

$$|B| \leq \int_A \Big|\det\Big(I + \frac{1}{a}D^2v(x)\Big)\Big|\, dx \leq C|A|, \tag{5.4}$$

where C is another universal constant.

If v is not semi-concave for $\varepsilon > 0$ we define the inf-convolution, setting for $x \in \overline{B}_r(x_0)$

$$v_\varepsilon(x) = \inf_{y\in \overline{B}_r(x_0)} \Big\{v(y) + \frac{1}{\varepsilon}|y-x|^2\Big\}.$$

It is easily checked that v_ε is semi-concave: Since v is lower semicontinuous and bounded we have also that v_ε converges pointwise to v in $\overline{B}_r(x_0)$. Moreover each v_ε is a viscosity $(2a)$-supersolution of the equation (5.1) in $B_r(x_0)$. In fact, if $\varphi \in C^2(B_r(x_0))$ lies below v_ε in a neighborhood of $\overline{x} \in B_r(x_0)$ and touches v_ε at $\overline{x}$, let $\overline{y}$ the point in $\overline{B}_r(x_0)$ such that

$$\inf_{y\in \overline{B}_r(x_0)} \Big\{v(y) + \frac{1}{\varepsilon}|y-\overline{x}|^2\Big\} = v(\overline{y}) + \frac{1}{\varepsilon}|\overline{y}-\overline{x}|^2.$$

Then the function

$$\varphi(x + \overline{x} - \overline{y}) + v(\overline{y}) - \varphi(\overline{x})$$

touches v from below at $\overline{y}$ and thus

$$F(D^2\varphi(\overline{x}), D\varphi(\overline{x})) \leq a.$$

Observe that for ε small enough the contact set A_ε of v_ε is contained in $B_r(x_0)$. Indeed if this is not true there exist a sequence ε_n converging to 0 and points $x_n \in A_{\varepsilon_n}$, the contact set of v_{ε_n}, such that $x_n \in \partial B_r(x_0)$. Therefore there exist $y_n \in B$ and $c_n \in \mathbb{R}$ such that

$$-\frac{a}{2}|x-y_n|^2 + c_n \leq v_{\varepsilon_n}(x)\ \ \forall x \in \overline{B}_r(x_0), \quad -\frac{a}{2}|x_n - y_n|^2 + c_n = v_{\varepsilon_n}(x_n).$$

Since B is closed and the v_{ε_n} are equibounded we may assume that $y_n \to \bar{y} \in B$, $c_n \to \bar{c}$ and $x_n \to \bar{x} \in \partial B_r(x_0)$. Thus, from the first inequality above we have that the paraboloid $-\frac{a}{2}|x-\bar{y}|^2 + \bar{c}$ stays below v in $B_r(x_0)$. Moreover, by lower semicontinuity we have

$$\begin{aligned} -\frac{a}{2}|\bar{x}-\bar{y}|^2 + \bar{c} &= \lim_{n\to\infty} v_{\varepsilon_n}(x_n) \\ &= \lim_{n\to\infty} \Big(v(z_n) + \frac{1}{\varepsilon_n}|z_n - x_n|^2\Big) \geq \liminf_{z\to\bar{x}} \geq v(z) \geq v(\bar{x}). \end{aligned}$$

Hence $\bar{x} \in A \cap \partial B_r(x_0)$ which is impossible by assumption. In particular we may assume that (5.4) holds with A replaced by A_ε for ε sufficiently small.

Using the fact that B is closed and the pointwise convergence of v_ε to v, a similar argument shows

$$\bigcap_{h=1}^{\infty}\bigcup_{k=h}^{\infty} A_{1/k} \subset A.$$

From the above inclusion and (5.4) we then conclude that $|B| \le C|A|$. □

Let $v : \overline{B}_1 \to (0,\infty)$ be a lower semicontinuous function. If $a > 0$ we denote by A_a the set of points where v can be touched from below by a paraboloid of opening a and vertex in $\overline{B}_1$ and where the value of v is also smaller than a.

$$A_a := \Big\{x \in \overline{B}_1 :\ v(x) \le a \text{ and there exists } y \in \overline{B}_1 \text{ such that} \\ \min_{z\in\overline{B}_1}\Big(v(z) + \frac{a}{2}|z-y|^2\Big) = v(x) + \frac{a}{2}|x-y|^2\Big\}.$$

Lemma 5.2. *There exist two constants $c_1 > 0$, $C_1 > 1$, with the following properties: Let $0 < a < 1/C_1$ and let $v : \overline{B}_1 \to (0,\infty)$ be a viscosity $(C_1 a)$-supersolution of the equation* (5.1)*, bounded from above. If*

$$\overline{B}_r(x_0) \subset B_1, \qquad A_a \cap \overline{B}_r(x_0) \neq \emptyset\,,$$

then

$$\big|A_{C_1 a} \cap B_{r/8}(x_0)\big| \ge c_1|B_r|\,.$$

Proof. Up to replace r with $r+\varepsilon$ and then letting $\varepsilon \to 0^+$ we may assume that there exists

$$x_1 \in A_a \cap B_r(x_0)\,.$$

Thus there exists $y_1 \in \overline{B}_1$ such that the paraboloid

$$P(x;y_1) = v(x_1) + \frac{a}{2}|x_1-y_1|^2 - \frac{a}{2}|x-y_1|^2$$

touches v in x_1 from below. We claim that there exists a universal constant C such that there exists a point $z \in \overline{B}_{r/16}(x_0)$ such that

$$v(z) \le P(z;y_1) + Car^2\,. \tag{5.5}$$

If $x_1 \in \overline{B}_{r/16}(x_0)$ then we may take trivially $z = x_1$. Otherwise, we consider the function

$$\varphi(x) = \begin{cases} \alpha^{-1}(|x|^{-\alpha} - 1) & \text{if } \dfrac{1}{16} \le |x| \le 1, \\ \alpha^{-1}(16^{\alpha} - 1) & \text{if } |x| \le \dfrac{1}{16}\,, \end{cases}$$

with $\alpha > 0$, universal, to be chosen. Then, for $x \in B_r(x_0)$, we set

$$\psi(x) = P(x;y_1) + ar^2\varphi\Big(\frac{x-x_0}{r}\Big)\,.$$

Note that $|D\psi| \le a(16^{\alpha+1} + 2) \le 1$ if a is small enough and that in the annulus $B_r(x_0) \setminus \overline{B}_{r/16}(x_0)$ we have

$$F(D^2\psi, D\psi) = aF\Big(D^2\varphi\Big(\frac{x-x_0}{r}\Big), D\psi\Big) - aF(I, D\psi) \ge a\lambda(\alpha+1) - a\tilde{\lambda}\sqrt{n} > a, \tag{5.6}$$

provided $\alpha > 0$ is chosen large enough.

Let us now denote by z a minimum point of $v - \psi$ in $\overline{B}_r(x_0)$. Since $v(x_1) - \psi(x_1) = -ar^2\varphi(\frac{x_1-x_0}{r}) < 0$, the minimum of $v - \psi$ is strictly negative. Therefore $z \notin \partial B_r(x_0)$ since $v - \psi \ge 0$ on $\partial B_r(x_0)$. On the other hand, if $r/16 < |z - x_0| < r$, observing that $|D\psi| \le a(16^{\alpha+1} + 2) \le C_1 a$, if we choose C_1 large enough, we would have that ψ is an admissible test function for v in a neighborhood of z satisfying (5.6), which is impossible. Therefore $z \in \overline{B}_{r/16}(x_0)$ and we have

$$v(z) \leq \psi(z) = P(z; y_1) + ar^2\alpha^{-1}(16^\alpha - 1)\,,$$

thus proving (5.5).

To conclude the proof, consider for every $y \in \overline{B}_{r/64}(z)$ the paraboloid

$$P(x; y_1) - C'\frac{a}{2}|x-y|^2 + c_y\,,$$

where C' is a universal constant to be chosen and c_y is such that the above paraboloid touches v from below. Note that the above paraboloid has opening $(C'+1)a$ and vertex at

$$\frac{y_1}{C'+1} + \frac{C'y}{C'+1}\,.$$

Observe that

$$P(z; y_1) - C'\frac{a}{2}|z-y|^2 + c_y \leq v(z) \leq P(z; y_1) + Car^2\,,$$

hence $c_y \leq Car^2 + \frac{C'a}{2}\left(\frac{r}{64}\right)^2$. On the other hand, if $|x - z| \geq \frac{r}{16}$, we have

$$\begin{aligned} P(x; y_1) - C'\frac{a}{2}|x-y|^2 + c_y &\leq P(x; y_1) - C'\frac{a}{2}\Big(\frac{3r}{64}\Big)^2 + Car^2 + \frac{C'a}{2}\Big(\frac{r}{64}\Big)^2 \\ &< P(x; y_1) \leq v(x)\,, \end{aligned}$$

provided that we choose C' large enough independently of a and r. Thus the contact point x_y belongs to the ball $B_{r/16}(z) \subset B_{r/8}(x_0)$. Note that for $y \in \overline{B}_{r/64}(z)$ the vertex $\frac{y_1}{C'+1} + \frac{C'y}{C'+1}$ spans the ball with center $\frac{y_1}{C'+1} + \frac{C'z}{C'+1}$ and radius $C'r/64(C'+1)$, which is contained in B_1, provided that C' is large enough. Moreover, the gradient of the function $x \mapsto P(x; y_1) - C'\frac{a}{2}|x-y|^2 + c_y$ is smaller than $2(C'+1)a$ and

$$v(x_y) = P(x_y; y_1) - C'\frac{a}{2}|x_y - y|^2 + c_y \leq a + 2a + c_y \leq C''a$$

for a sufficiently large, universal, constant $C'' > 2(C'+1)a$. Therefore, by applying Lemma 5.1 with $B_r(x_0)$ and B replaced respectively by $B_{r/8}(x_0)$ and the ball of radius $C'r/64(C'+1)$ and center $\frac{y_1}{C'+1} + \frac{C'z}{C'+1}$, with a replaced by $C''a$, we have that, if $3C''a \leq 1$

$$\big|A_{C''a} \cap B_{r/8}(x_0)\big| \geq c_0\Big(\frac{C'}{C'+1}\Big)^n |B_{r/64}|\,,$$

from which (5.2) follows with $C_1 = 3\max\{C'', 16^{\alpha+1} + 2\}$. □

We can now give the

Proof of Proposition 3.5. Let $x_0 \in B_{1/2}$ be a point where $v(x_0) \leq \nu$. Then the function $u(x) = v(x_0 + x)$ is a positive viscosity $(C_0^k\nu)$-supersolution of (3.9) in $\overline{B}_1$ with $u(0) \leq \nu$. Consider the paraboloid with vertex at the origin and opening 20ν touching u from below and observe that since u is positive this paraboloid touches u in $B_{1/3}$. Therefore $A_{20\nu} \cap \overline{B}_{1/3} \neq \emptyset$. Moreover, setting $D_0 = A_{20\nu} \cap \overline{B}_{1/3}$, from Lemma 5.2 we know that if $B_r(x) \subset B_1$, $B_{r/8}(x) \subset B_{1/3}$ and $D_0 \cap B_r(x) \neq \emptyset$, then

$$|(A_{20C_1\nu} \cap \overline{B}_{1/3}) \cap B_{r/8}(x)| = |A_{20C_1\nu} \cap B_{r/8}(x)| \geq c_1|B_r|\,,$$

provided $20C_1\nu \leq 1$ and v, hence u, is a viscosity $(20C_1\nu)$-supersolution. By applying the same lemma to $D_1 = A_{20C_1\nu} \cap \overline{B}_{1/3}$ and proceeding by induction, we have that the sets

$D_0 \subset D_1 \subset \cdots \subset D_k \subset B_{1/3}$, where $D_i = A_{20C_1^i\nu} \cap \overline{B}_{1/3}$ for $i = 1, \ldots, k$, have all the property that if $B_r(x) \subset B_1$, $B_{r/8}(x) \subset B_{1/3}$ and $D_i \cap B_r(x) \neq \emptyset$, for $i \leq k$, then

$$|D_{i+1} \cap B_{r/8}(x)| \geq c_1 |B_r|,$$

provided $20C_1^k\nu \leq 1$ and v, hence u, is a viscosity $(20C_1^k\nu)$-supersolution. Then, using Lemma 2.3 in [16] and setting $C_2 = 20C_1$, it follow that for a suitable μ depending only on c_1, hence universal,

$$\left|\{x \in B_{1/3}(x_0) : v(x) \leq C_2^k\nu\}\right| \leq (1 - \mu^k)|B_{1/3}|.$$

From this inequality the conclusion then follows by a standard rescaling and covering argument. □

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