

A GREEN'S FUNCTION PROOF OF THE POSITIVE MASS THEOREM

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ABSTRACT. In this paper, a new proof of the Positive Mass Theorem is established through a newly discovered monotonicity formula, holding along the level sets of the Green's function of an asymptotically flat 3-manifolds. In the same context and for $1 < p < 3$, a Geroch-type calculation is performed along the level sets of p -harmonic functions, leading to a new proof of the Riemannian Penrose Inequality under favourable assumptions. A new characterisation of scalar curvature lower bounds in terms of the monotonicity formulas is also given.

MSC (2020): 53C21, 31C12, 31C15, 53C24, 53Z05.

Keywords: monotonicity formulas, Green's function, ADM mass, geometric inequalities.

1. A MONOTONIC QUANTITY FOR THE GREEN'S FUNCTIONS

In the context of contemporary geometric analysis, monotonicity formulas are known to play a central role not only because of their tremendous implications, but also because they clarify the theory, eventually leading to a deeper understanding of the overall picture. Some of the most relevant examples come from the study of geometric flows such as, e.g., Huisken's monotonicity formulas for the Mean Curvature Flow [25], Perelman's entropy formulas for the Ricci Flow [37], or Geroch's monotonicity of the Hawking Mass along the Inverse Mean Curvature Flow [21, 26]. However, this is not the only scene where these formulas play a dominant role. For example, it must be noticed that a monotonicity formula is also at the core of comparison geometry, through the nowadays classical Bishop-Gromov Volume Comparison Theorem. Starting from this observation, Colding [16] and Colding-Minicozzi [17, 18] discovered in recent years analogous monotonic quantities on manifolds with nonnegative Ricci curvature, where the level set flow of the distance function is replaced by the level set flow of a harmonic function. This new class of monotonicity formulas, together with their extension to the case of p -harmonic functions, $1 < p < n$, has revealed to be extremely flexible and powerful, leading to the proof of new geometric inequalities [1, 9, 42] as well as to a new proof of classical results [4, 2]. More in general, level set methods for harmonic functions have been recently employed to investigate the geometry of asymptotically flat initial data in general relativity [3, 6, 14, 24]. Further comments on the latter point will be given in the next section. Here, we establish the following monotonicity result, holding along the level set flow of the Green's functions on complete nonparabolic 3-manifolds with nonnegative scalar curvature, whose topology is sufficiently simple. Recall that a complete Riemannian manifold is said to be *nonparabolic* if it admits a positive Green function, otherwise it is said to be *parabolic*.

Theorem 1.1. *Let (M, g) be a complete, noncompact, nonparabolic 3-dimensional Riemannian manifold with nonnegative scalar curvature and $H_2(M; \mathbb{Z}) = \{0\}$. Let u be the maximal distributional solution to*

$$\begin{cases} \Delta u = 4\pi\delta_o & \text{in } M, \\ u \rightarrow 1 & \text{at } \infty, \end{cases} \quad (1.1)$$

for some $o \in M$, and let $F : (0, +\infty) \rightarrow \mathbb{R}$ be the function defined as

$$F(t) := 4\pi t - t^2 \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} |\nabla u| H \, d\sigma + t^3 \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} |\nabla u|^2 \, d\sigma, \quad (1.2)$$

where $\text{Crit}(u)$ is the set of the critical points of u , and H is the mean curvature of the level set $\{u = 1 - 1/t\} \setminus \text{Crit}(u)$ computed with respect to the ∞ -pointing unit normal vector field $\nu = \nabla u / |\nabla u|$. Then, we have that

$$0 < s \leq t < +\infty \quad \Rightarrow \quad F(s) \leq F(t),$$

provided $1 - 1/s$ and $1 - 1/t$ are regular values of u .

More specifically, we are going to prove that, under the above assumptions, the function F admits a nondecreasing locally absolutely continuous representative, defined in $(0, +\infty)$ and still denoted by F , whose L^1_{loc} -weak derivative satisfies

$$F'(t) = 4\pi + \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} \left[-\frac{R^{\Sigma_t}}{2} + \frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma \quad (1.3)$$

a.e. in $(0, +\infty)$. Here, R^{Σ_t} and ∇^{Σ_t} denote the scalar curvature and the Levi-Civita connection of the metric induced by g on the regular part Σ_t of the level set $\{u = 1 - 1/t\}$. By h and H , we indicate the second fundamental form and the mean curvature of Σ_t , both computed with respect to the ∞ -pointing unit normal vector field $\nu = \nabla u / |\nabla u|$, whereas \mathring{h} represents the traceless second fundamental form of Σ_t . In particular, as soon as Σ_t is a connected regular level set of u , one can easily deduce that $F'(t)$ is nonnegative. In fact

$$4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} \, d\sigma = 4\pi - 2\pi\chi(\Sigma_t) \geq 0$$

by the Gauss-Bonnet Theorem, and all the remaining terms in (1.3) are manifestly nonnegative.

Remark 1.2. *It should be noted that, in a recent work [36], a new monotonicity formula holding along the level sets of the Green's function was discovered by Munteanu and Wang, working essentially in the same framework as in Theorem 1.1. However, both the monotonicity formulas and the geometric conclusions are considerably different from ours.*

Observe that the existence of the solution u to (1.1) is equivalent to the existence of the minimal positive Green's function \mathcal{G}_o vanishing at infinity and having a pole at o , and it is guaranteed in many reasonable frameworks, such as the one – particularly relevant to us – of asymptotically flat manifolds. Indeed, u and \mathcal{G}_o are related by the identity $u = 1 - 4\pi\mathcal{G}_o$.

As it is well known, u is smooth on $M \setminus \{o\}$ and proper, so that its level sets are compact. It follows then from [23, Theorem 1.7] that they also have finite 2-dimensional Hausdorff measure. For the reader's convenience, we also recall that the set $\text{Crit}(u)$ has locally finite 1-dimensional Hausdorff measure (see for instance [22, Theorem 1.1]), and that the set of the critical values of u has zero Lebesgue measure by Sard's Theorem, whereas the set of regular values of u is open, by the same argument as in [6, Theorem 2.3]. Building on the previous observations, it is not hard to realise that the function F given in (1.2) is well defined, as its summands come from the integration of bounded functions on sets with finite measure. To justify this latter sentence, one only needs to check that $|\langle \nabla u, \mathbf{H} \rangle|$ is bounded on $\{u = \tau\} \setminus \text{Crit}(u)$, for every $\tau \in (-\infty, 1)$. Since u is harmonic, \mathbf{H} can be expressed as

$$\mathbf{H} = -\frac{\nabla \nabla u(\nabla u, \nabla u)}{|\nabla u|^3} = -\frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|^2}, \quad (1.4)$$

away from $\text{Crit}(u)$, where angle brackets $\langle \cdot, \cdot \rangle$ denote the scalar product, taken with respect to the metric g . Consequently, one has that

$$|\langle \nabla u, \mathbf{H} \rangle| \leq |\nabla \nabla u(\nu, \nu)| \leq |\nabla \nabla u|, \quad (1.5)$$

whenever $|\nabla u| \neq 0$. We are now ready to prove Theorem 1.1. For the sake of exposition, we first give a proof under the favourable assumption that $\text{Crit}(u) = \emptyset$, and then we proceed with the proof of the full statement.

Proof in the absence of critical points. In this case, all of the level sets of u are regular and diffeomorphic to the 2-sphere (see the discussion after formula (1.7)), and in turn the function F is everywhere continuously differentiable in its domain of definition. We claim that $F'(t) \geq 0$ for every $t \in (0, +\infty)$. We start observing that

$$\begin{aligned} \frac{d}{dt} \int_{\{u=1-\frac{1}{t}\}} |\nabla u|^2 d\sigma &= -\frac{1}{t^2} \int_{\{u=1-\frac{1}{t}\}} |\nabla u| \mathbf{H} d\sigma, \\ \frac{d}{dt} \int_{\{u=1-\frac{1}{t}\}} |\nabla u| \mathbf{H} d\sigma &= -\frac{1}{t^2} \int_{\{u=1-\frac{1}{t}\}} |\nabla u| \left[\Delta_{\Sigma_t} \left(\frac{1}{|\nabla u|} \right) + \frac{|\mathbf{h}|^2 + \text{Ric}(\nu, \nu)}{|\nabla u|} \right] d\sigma, \end{aligned} \quad (1.6)$$

where Δ_{Σ_t} is the Laplace-Beltrami operator of the metric induced on $\Sigma_t = \{u = 1 - \frac{1}{t}\}$. Observe that formula (1.6) can be deduced via standard evolution equations for hypersurfaces moving in normal directions (see for example [27, Theorem 3.2]). Here, in particular, the vector velocity is given by $\nabla u / |\nabla u|^2$. With the help of the Gauss equation, the integrand on the right hand side of (1.6) can be expressed as

$$\begin{aligned} |\nabla u| \left[\Delta_{\Sigma_t} \left(\frac{1}{|\nabla u|} \right) + \frac{|\mathbf{h}|^2 + \text{Ric}(\nu, \nu)}{|\nabla u|} \right] &= \\ &= -\Delta_{\Sigma_t}(\log |\nabla u|) + \frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} + \frac{\mathbf{R}}{2} - \frac{\mathbf{R}^{\Sigma_t}}{2} + \frac{|\mathring{\mathbf{h}}|^2}{2} + \frac{3}{4} \mathbf{H}^2. \end{aligned}$$

Substituting the latter expression into (1.6) and using standard manipulations, one easily arrives at

$$F'(t) = 4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} d\sigma + \int_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t} |\nabla u|^2|}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma. \quad (1.7)$$

Now, we notice that the last summand of the right hand side is always nonnegative, as the scalar curvature of (M, g) is nonnegative by assumption. The first two summands also give a nonnegative contribution, by virtue of Gauss-Bonnet Theorem combined with the observation that Σ_t is closed and connected for all $t \in (0, +\infty)$. It is worth pointing out that when critical points are not present, the connectedness of the level sets of u follows by rather elementary considerations, without any further assumption on the topology of (M, g) . In fact, on one hand, the asymptotic behaviour of u near the pole implies that in this region at least one level set of u is necessarily diffeomorphic to a 2-sphere. On the other hand, the same must be true for every level set of u , since ∇u never vanishes. \square

Proof. Let us consider the vector field Y , defined as

$$Y := \frac{\nabla |\nabla u|}{(1-u)^2} + \frac{|\nabla u|}{(1-u)^3} \nabla u, \quad (1.8)$$

where u is the solution to problem (1.1). The vector field Y is well defined and smooth on the open set $M_o \setminus \text{Crit}(u)$, where M_o is defined as

$$M_o := M \setminus \{o\}.$$

With the help of the Bochner formula, the divergence of Y on $M_o \setminus \text{Crit}(u)$ can be expressed as

$$\text{div}(Y) = \frac{|\nabla u|}{(1-u)^2} \left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{3\langle \nabla |\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|} + \frac{|\nabla \nabla u|^2 - |\nabla |\nabla u||^2 + \text{Ric}(\nabla u, \nabla u)}{|\nabla u|^2} \right],$$

where in the computation we took advantage of the fact that u is harmonic. Using the Gauss equation in combination with the standard identity $|\nabla \nabla u|^2 = |\nabla u|^2 |\mathring{h}|^2 + |\nabla |\nabla u||^2 + |\nabla^{\Sigma_t} |\nabla u|^2|^2$, one can work out an equivalent expression for $\text{div}(Y)$, adapted to (the regular portion of) the level sets of u , namely

$$\text{div}(Y) = \frac{|\nabla u|}{(1-u)^2} \left[-\frac{R^\Sigma}{2} + \frac{|\nabla^\Sigma |\nabla u|^2|}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right]. \quad (1.9)$$

Here, h, H, R^Σ , and ∇^Σ are all referred to the regular portion of the level set of u that passes through the point where $\text{div}(Y)$ is computed. Setting

$$\Phi(t) := F(t) - 4\pi t = -t^2 \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} |\nabla u| H d\sigma + t^3 \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} |\nabla u|^2 d\sigma = \int_{\{u=1-\frac{1}{t}\} \setminus \text{Crit}(u)} \left\langle Y, \frac{\nabla u}{|\nabla u|} \right\rangle d\sigma,$$

for every $t \in (0, +\infty)$, and using the Divergence Theorem, it is not hard to check that

$$\begin{aligned} \Phi(t) - \Phi(s) &= \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} \operatorname{div}(Y) d\mu = \\ &= \int_s^t d\tau \int_{\{u=1-\frac{1}{\tau}\}} \left[-\frac{R^{\Sigma_\tau}}{2} + \frac{|\nabla^{\Sigma_\tau}|\nabla u|^2|}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma, \end{aligned}$$

provided no critical values of u are contained in the interval $[1 - 1/s, 1 - 1/t]$. We aim at proving that a similar statement holds true also in the presence of critical values, showing $\Phi \in W_{loc}^{1,1}(0, +\infty)$. As a first step in this direction, we claim that

$$\operatorname{div}(Y) \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)} \in L_{loc}^1(M_o), \quad (1.10)$$

where $\mathbb{I}_{M_o \setminus \operatorname{Crit}(u)}$ denotes the characteristic function of $M_o \setminus \operatorname{Crit}(u)$. Let us observe that if K is a compact subset of M_o , then, by Sard's Theorem, K is contained in the set

$$E_s^t := \left\{ 1 - \frac{1}{s} < u < 1 - \frac{1}{t} \right\},$$

for some $0 < s < t$, such that both $1 - 1/s$ and $1 - 1/t$ are regular values of u . As we are dealing with the case where critical values of u are present in the interval $(1 - 1/s, 1 - 1/t)$, we have that necessarily the open subset $\{1 - 1/s < u < 1 - 1/t\}$ must contain critical points of u , so that the vector field Y is no longer smooth and well defined everywhere. To overcome this issue, we consider a sequence of cut-off functions $\{\eta_k\}_{k \in \mathbb{N}^+}$, where, for every $k \in \mathbb{N}^+$, the function $\eta_k : [0, +\infty) \rightarrow [0, 1]$ is smooth, nondecreasing, and obeying the following structural conditions:

$$\eta_k(\tau) \equiv 0 \quad \text{in} \quad \left[0, \frac{1}{2k} \right], \quad 0 \leq \eta_k'(\tau) \leq 2k \quad \text{in} \quad \left[\frac{1}{2k}, \frac{3}{2k} \right], \quad \eta_k(\tau) \equiv 1 \quad \text{in} \quad \left[\frac{3}{2k}, +\infty \right).$$

Using these cut-off functions, we define for every $k \in \mathbb{N}^+$ the vector field

$$Y_k := \eta_k \left(\frac{|\nabla u|}{1-u} \right) Y.$$

It is immediate to observe that for every $k \in \mathbb{N}^+$, Y_k is a smooth vector field defined in M_o . Moreover, on any given compact subset of $M_o \setminus \operatorname{Crit}(u)$, the vector field Y_k coincides with the vector field Y , provided k is large enough. For any such Y_k , the divergence is readily computed as follows

$$\begin{aligned} \operatorname{div}(Y_k) &= \frac{|\nabla u|}{(1-u)^2} \left\{ \eta_k \left(\frac{|\nabla u|}{1-u} \right) \left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{|\nabla \nabla u|^2 - |\nabla |\nabla u||^2}{|\nabla u|^2} \right] \right\} \\ &+ \frac{|\nabla u|}{(1-u)^2} \left\{ \eta_k \left(\frac{|\nabla u|}{1-u} \right) \left[\frac{3\langle \nabla |\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|} + \frac{\operatorname{Ric}(\nabla u, \nabla u)}{|\nabla u|^2} \right] \right\} \\ &+ \frac{|\nabla u|^2}{(1-u)^3} \eta_k' \left(\frac{|\nabla u|}{1-u} \right) \left| \frac{\nabla u}{1-u} + \frac{\nabla |\nabla u|}{|\nabla u|} \right|^2. \end{aligned} \quad (1.11)$$

An important feature of the above expression is that the last summand is always nonnegative. Taking into account these simple considerations and applying the Divergence Theorem, one gets

$$\Phi(t) - \Phi(s) = \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} \operatorname{div}(Y_k) d\mu \geq \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} P_k d\mu + \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} D_k d\mu, \quad (1.12)$$

where we set

$$P_k := \eta_k \left(\frac{|\nabla u|}{1-u} \right) P, \quad \text{with} \quad P := \frac{|\nabla u|}{(1-u)^2} \left[\frac{3|\nabla u|^2}{(1-u)^2} + \frac{|\nabla \nabla u|^2 - |\nabla |\nabla u||^2}{|\nabla u|^2} \right],$$

$$D_k := \eta_k \left(\frac{|\nabla u|}{1-u} \right) D, \quad \text{with} \quad D := \frac{|\nabla u|}{(1-u)^2} \left[\frac{3\langle \nabla |\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|} + \frac{\operatorname{Ric}(\nabla u, \nabla u)}{|\nabla u|^2} \right].$$

Concerning the functions D_k , we have that

$$|D_k| \leq \frac{|\nabla u|}{(1-u)^2} \left[\frac{3|\nabla \nabla u|}{1-u} + |\operatorname{Ric}| \right] \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)}.$$

Since the function on the right hand side belongs to $L^1_{loc}(M_o)$, Lebesgue's Dominated Convergence Theorem implies that $D \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)} \in L^1(E_s^t)$ and that

$$\lim_{k \rightarrow +\infty} \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} D_k d\mu = \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\} \setminus \operatorname{Crit}(u)} D d\mu < +\infty.$$

This fact, combined with inequality (1.12), implies that the sequence of the integrals of the functions P_k is uniformly bounded in k . On the other hand, the P_k 's are clearly nonnegative and they converge monotonically and pointwise to the function $P \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)}$. The Monotone Convergence Theorem thus yields

$$\lim_{k \rightarrow +\infty} \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\}} P_k d\mu = \int_{\{1-\frac{1}{s} < u < 1-\frac{1}{t}\} \setminus \operatorname{Crit}(u)} P d\mu < +\infty.$$

In particular, we have that $P \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)} \in L^1(E_s^t)$. Since $\operatorname{div}(Y) \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)} = (P+D) \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)}$, it follows then that $\operatorname{div}(Y) \mathbb{I}_{M_o \setminus \operatorname{Crit}(u)} \in L^1_{loc}(M_o)$, as desired.

Having the claim (1.10) at hand, we are now going to prove that $\Phi \in W^{1,1}_{loc}(0, +\infty)$ with weak derivative given by

$$\Phi'(t) = \int_{\{u=1-\frac{1}{t}\} \setminus \operatorname{Crit}(u)} \left[-\frac{R^{\Sigma_t}}{2} + \frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma \quad (1.13)$$

a.e. in $(0, +\infty)$. The latter is in $L^1_{loc}(0, +\infty)$, thanks precisely to claim (1.10) coupled with the elementary properties of the integrals and the Coarea Formula. The following argument

is inspired by [9]. Let χ be a test function belonging to $C_c^\infty(0, +\infty)$. We have that

$$\begin{aligned} \int_0^{+\infty} \chi'(\tau) \Phi(\tau) d\tau &= \int_0^{+\infty} d\tau \int_{\left\{\frac{1}{1-u}=\tau\right\} \setminus \text{Crit}(u)} \chi'\left(\frac{1}{1-u}\right) \frac{\langle Y, \nabla u \rangle}{|\nabla u|} d\sigma = \int_{M_o \setminus \text{Crit}(u)} \left\langle Y, \nabla \left[\chi \left(\frac{1}{1-u} \right) \right] \right\rangle d\mu \\ &= \lim_{k \rightarrow +\infty} \int_{M_o} \left\langle Y_k, \nabla \left[\chi \left(\frac{1}{1-u} \right) \right] \right\rangle d\mu = - \lim_{k \rightarrow +\infty} \int_{M_o} \chi \left(\frac{1}{1-u} \right) \text{div}(Y_k) d\mu, \end{aligned}$$

where the second equality follows by the Coarea Formula, the third one by Lebesgue's Dominated Convergence Theorem, whereas the last one is a simple integration by parts. We now let $0 < s < t < +\infty$ be such that $\text{supp} \chi \subset (s, t)$. Setting

$$Q := \frac{|\nabla u|}{(1-u)^2} \left| \frac{\nabla u}{1-u} + \frac{\nabla |\nabla u|}{|\nabla u|} \right|^2$$

on $M_o \setminus \text{Crit}(u)$ and using formula (1.11), one gets

$$\int_{M_o} \chi \left(\frac{1}{1-u} \right) \text{div}(Y_k) d\mu = \int_{E_s^t} \chi \left(\frac{1}{1-u} \right) \left\{ P_k + D_k + \frac{|\nabla u|}{1-u} \eta'_k \left(\frac{|\nabla u|}{1-u} \right) Q \right\} d\mu.$$

Using the elementary inequality (1.5), it is not hard to show that

$$Q \mathbb{I}_{M_o \setminus \text{Crit}(u)} \leq 3P \mathbb{I}_{M_o \setminus \text{Crit}(u)} + 2 \frac{|\nabla u| |\nabla \nabla u|}{(1-u)^3},$$

so that $Q \mathbb{I}_{M_o \setminus \text{Crit}(u)} \in L_{loc}^1(M_o)$, whereas $|\nabla u|/(1-u) \eta'_k(|\nabla u|/(1-u))$ is always bounded. As $\lim_{k \rightarrow +\infty} \eta'_k(\tau) = 0$ for every $\tau \in (0, +\infty)$, the Dominated Convergence Theorem implies that

$$\lim_{k \rightarrow +\infty} \int_{E_s^t} \chi \left(\frac{1}{1-u} \right) \frac{|\nabla u|}{(1-u)} \eta'_k \left(\frac{|\nabla u|}{1-u} \right) Q d\mu = 0.$$

All in all we get

$$\begin{aligned} \int_0^{+\infty} \chi'(s) \Phi(s) ds &= - \lim_{k \rightarrow +\infty} \int_{M_o} \chi \left(\frac{1}{1-u} \right) \text{div}(Y_k) d\mu = - \int_{M_o \setminus \text{Crit}(u)} \chi \left(\frac{1}{1-u} \right) \text{div}(Y) d\mu \\ &= - \int_0^{+\infty} \chi(s) ds \int_{\left\{u=1-\frac{1}{s}\right\} \setminus \text{Crit}(u)} \left[-\frac{R^{\Sigma_s}}{2} + \frac{|\nabla^{\Sigma_s} |\nabla u||^2}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma, \end{aligned}$$

where in the last identity we used (1.9) together with the Coarea Formula. It is now clear that $\Phi \in W_{loc}^{1,1}(0, +\infty)$. As $F(t) = 4\pi t + \Phi(t)$, it is also clear that F belongs to $W_{loc}^{1,1}(0, +\infty)$ and that its weak derivative coincides a.e. with the expression in (1.3). In particular, the function F admits a (locally) absolutely continuous representative – still denoted by F –

satisfying the following identity, for any pair of positive real numbers $s < t$

$$\begin{aligned}
F(t) - F(s) &= \\
&= \int_s^t d\tau \left\{ 4\pi + \int_{\{u=1-\frac{1}{\tau}\} \setminus \text{Crit}(u)} \left[-\frac{R^{\Sigma_\tau}}{2} + \frac{|\nabla^{\Sigma_\tau} \nabla u|^2}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma \right\} \\
&= \int_{[s,t] \setminus \mathcal{N}} d\tau \left\{ 4\pi - \int_{\{u=1-\frac{1}{\tau}\}} \frac{R^{\Sigma_\tau}}{2} d\sigma + \int_{\{u=1-\frac{1}{\tau}\}} \left[\frac{|\nabla^{\Sigma_\tau} \nabla u|^2}{|\nabla u|^2} + \frac{R}{2} + \frac{1}{2} |\mathring{h}|^2 + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma \right\},
\end{aligned} \tag{1.14}$$

where \mathcal{N} is the set of the critical values of u and it is negligible, by Sard's Theorem. Since all the level sets corresponding to regular values of u are closed regular surfaces, the monotonicity of F follows by the very same considerations made after formula (1.7), with just one important exception. Indeed, in the present context, we need a different argument to ensure that the regular level sets of u are connected, so that, by the Gauss-Bonnet Theorem, one obtains the inequality $4\pi - 2\pi\chi(\{u = 1 - \frac{1}{\tau}\}) \geq 0$, for every $\tau \in (0, +\infty) \setminus \mathcal{N}$. Here is where the assumption $H_2(M; \mathbb{Z}) = \{0\}$ comes into play. To see this, suppose by contradiction that for some $\tau \in (0, +\infty) \setminus \mathcal{N}$ the (regular) level set $\Sigma = \{u = 1 - \frac{1}{\tau}\}$ is given by the disjoint union of at least two connected components Σ' and Σ'' , so that Σ' and Σ'' are two closed hypersurfaces. By the triviality of $H_2(M; \mathbb{Z})$, we have that each closed 2-dimensional surface in (M, g) is the boundary of a 3-dimensional bounded open domain. In particular, there exist two bounded connected and open subsets $\Omega', \Omega'' \subset M$ such that $\partial\Omega' = \Sigma'$ and $\partial\Omega'' = \Sigma''$. If o doesn't belong to Ω' , then $\overline{\Omega'}$ is contained in M_o , and by the Strong Maximum Principle u must then be constant in $\overline{\Omega'}$. But this is impossible, as all the level sets of u have finite 2-dimensional Hausdorff measure, since u is harmonic. The same argument obviously applies to Ω'' ; therefore, the pole o must belong to both Ω' and Ω'' , so that these latter turn out to have a nonempty intersection. Consequently, either $\Omega' \subset \Omega''$ or $\Omega'' \subset \Omega'$, since $\partial\Omega' = \Sigma'$ and $\partial\Omega'' = \Sigma''$ are disjoint. On the other hand, by the Strong Maximum Principle u must then be constant either in $\overline{\Omega'} \setminus \Omega'$ or in $\overline{\Omega''} \setminus \Omega''$ respectively, which is again a contradiction. \square

Combining the above theorem with some standard facts about the asymptotic behaviour of the Green's functions near the pole, one gets the following corollary, that should be regarded as an abstract Positive Mass Theorem, holding in a more general framework, where the total mass is replaced by the quantity $\lim_{t \rightarrow +\infty} F(t)$, or more likely, by $\sup_{o \in M} \lim_{t \rightarrow +\infty} F_o(t)$, where the index o here is reminiscent of the pole of the Green's function that we are considering.

Corollary 1.3. *Under the assumptions of Theorem 1.1, we have that*

$$0 \leq \lim_{t \rightarrow +\infty} F(t). \tag{1.15}$$

Moreover, if $\lim_{t \rightarrow +\infty} F(t) = 0$, then (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$.

Proof. We first claim that $\lim_{t \rightarrow 0^+} F(t) = 0$. To see this fact, we recall that u is related to the minimal positive Green's function \mathcal{G}_o of (M, g) with pole at o through the formula $u = 1 - 4\pi\mathcal{G}_o$. Consequently, there holds

$$\int_{\{u=1-\frac{1}{t}\}} |\nabla u| d\sigma \equiv 4\pi \quad (1.16)$$

for every $t \in (0, +\infty) \setminus \mathcal{N}$. On the other hand, it is well known (see for example [35]) that \mathcal{G}_o displays the following asymptotic behaviour near the pole:

$$\begin{aligned} \left| \mathcal{G}_o - \frac{1}{4\pi r} \right| &= o(r^{-1}), \\ \left| \nabla \mathcal{G}_o + \frac{1}{4\pi r^2} \nabla r \right| &= o(r^{-2}), \\ \left| \nabla \nabla \mathcal{G}_o - \frac{1}{4\pi r^2} \left(\frac{2}{r} dr \otimes dr - \nabla \nabla r \right) \right| &= o(r^{-3}), \end{aligned}$$

where r denotes the distance to the pole o . As a consequence, in a sufficiently small neighborhood of o , the function u is subject to the bounds

$$\frac{C_1}{r} \leq 1 - u \leq \frac{C_2}{r}, \quad \frac{C_3}{r^2} \leq |\nabla u| \leq \frac{C_4}{r^2}, \quad |\nabla \nabla u| \leq \frac{C_5}{r^3},$$

for some positive constants $C_i > 0$, $i = 1, \dots, 5$. Combining these bounds with (1.5) and (1.16), we conclude that

$$\begin{aligned} t^2 \int_{\{u=1-\frac{1}{t}\}} |\nabla u|^2 d\sigma &\leq \int_{\{u=1-\frac{1}{t}\}} \frac{C_4}{r^2(1-u)^2} |\nabla u| d\sigma \leq \frac{C_4}{C_1^2} \int_{\{u=1-\frac{1}{t}\}} |\nabla u| d\sigma \leq \frac{4\pi C_4}{C_1^2}, \\ t \int_{\{u=1-\frac{1}{t}\}} |H| |\nabla u| d\sigma &\leq \int_{\{u=1-\frac{1}{t}\}} \frac{|\nabla \nabla u|}{1-u} d\sigma \leq \frac{C_5}{C_1 C_3} \int_{\{u=1-\frac{1}{t}\}} |\nabla u| d\sigma \leq \frac{4\pi C_5}{C_1 C_3}. \end{aligned}$$

Plugging these estimates into the definition of F , it easily follows that $F(t) \rightarrow 0$, as $t \rightarrow 0^+$. Combining this fact with the nonmonotonicity of F yield (1.15).

Let us now focus our attention on the rigidity statement, so that, in what follows, we will assume that $\lim_{t \rightarrow +\infty} F(t) = 0$. By the above discussion, we have that u behaves like $1 - 1/r$ and ∇u behaves like $\nabla r/r^2$ in a sufficiently small neighborhood of the pole o . In particular, there exists a maximal time $T > 0$ such that $\nabla u \neq 0$ in $u^{-1}(-\infty, 1 - 1/T)$. It follows that F is continuously differentiable on the interval $(0, T)$. Moreover, using (1.15), one easily gets that $F' \equiv 0$ in $(0, T)$, so that all the positive summands in formula (1.7) are forced to vanish for every $t \in (0, T)$. This fact has very strong implications. First of all $\nabla^{\Sigma_t} |\nabla u| \equiv 0$ implies that $|\nabla u| = f(u)$, for some positive function $f : (0, T) \rightarrow (0, +\infty)$. It turns out that such a function can be made explicit. Indeed, from (1.7) one also has that $H = 2f(u)/(1-u)$. On the other hand, it follows from (1.4) that $H = -\partial|\nabla u|/\partial u = -f'(u)$. All in all, we have that f obeys the ODE

$$f'(u) = -\frac{2f(u)}{1-u}.$$

Now, the only solution to this ODE which is compatible with the asymptotic behaviour of u and $|\nabla u|$, as $u \rightarrow -\infty$, is given by $f(u) = (1 - u)^2$. Since $u < 1$ on the whole manifold, f never vanishes, so that $T = +\infty$ and $|\nabla u| \neq 0$ everywhere. In particular, all the level sets of u are regular and diffeomorphic to each other. More precisely, by the vanishing of the Gauss-Bonnet term in (1.7), they are all diffeomorphic to a 2-sphere and M is diffeomorphic to \mathbb{R}^3 . So far we have that the metric g can be written on $M \setminus \{o\}$ as

$$g = \frac{du \otimes du}{(1 - u)^4} + g_{ij}(u, \vartheta) d\vartheta^i \otimes d\vartheta^j,$$

where $g_{ij}(u, \vartheta) d\vartheta^i \otimes d\vartheta^j$ represents the metric induced by g on the level sets of u . Exploiting the vanishing of the traceless second fundamental form of the level sets in (1.7), it turns out that the coefficients $g_{ij}(u, \vartheta)$ obey the following first order system of PDE's

$$\frac{\partial g_{ij}}{\partial u} = \frac{2 g_{ij}}{1 - u}.$$

Arguing as in [15, Theorem 1.1, Case 2], one can deduce that $g_{ij}(u, \vartheta) d\vartheta^i \otimes d\vartheta^j = (1 - u)^{-2} g_{\mathbb{S}^2}$, so that g takes the form

$$g = \frac{du \otimes du}{(1 - u)^4} + \frac{g_{\mathbb{S}^2}}{(1 - u)^2}.$$

The prescription $R \equiv 0$, which also follows from (1.7), implies that $u = 1 - 1/r$ and $g = g_{\mathbb{R}^3}$, if used in combination with the asymptotic behaviour of u and $|\nabla u|$, as $u \rightarrow -\infty$. \square

Remark 1.4. *The validity of both Theorem 1.1 and Corollary 1.3 might suggest the interpretation of the monotonic quantity F as a potential-theoretic notion of quasi-local mass. This impression is confirmed in Section 3, where the small sphere limit of the monotonic quantities is analysed in details.*

2. PROOF OF THE POSITIVE MASS THEOREM

Building on Theorem 1.1, we present in this section a new proof of the (Riemannian) Positive Mass Theorem, originally due to Schoen and Yau [38, 39] and Witten [41]. To be more precise, we will present a complete and self-contained treatment of this famous inequality in the case where the underlying manifold is diffeomorphic to \mathbb{R}^3 and has Schwarzschildian asymptotics. On the other hand, thanks to a number of preparatory lemmata, which are now collected in [14, Proposition 2.1], this is sufficient to deduce the general statement. It goes as follows:

Theorem 2.1 (Positive Mass Theorem). *Let (M, g) be a complete, asymptotically flat, 3-dimensional Riemannian manifold with nonnegative scalar curvature. Then, the total ADM-mass of (M, g) is nonnegative. In symbols,*

$$m_{ADM}(M, g) \geq 0.$$

Moreover, $m_{ADM}(M, g) = 0$ if and only if (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$.

In what follows we just discuss the case where (M, g) has only one end, from which the general statement can be deduced by nowadays standard arguments. For the reader's convenience, we recall that a complete, 3-dimensional, one-ended Riemannian manifold (M, g) is said to be *asymptotically flat (of order τ)* if there exists a compact set K such

that $M \setminus K$ is diffeomorphic to the exterior of a closed ball $\mathbb{R}^3 \setminus \overline{B}$, through a so called *asymptotically flat coordinate chart* $x = (x^1, x^2, x^3)$. In such a chart, the metric coefficients obey the expansion

$$g_{ij}(x) = \delta_{ij} + O_2(|x|^{-\tau}), \quad (2.1)$$

for some constant $\tau > 1/2$. Moreover, we assume that the scalar curvature is integrable on M . These latter conditions guarantee that the total ADM-mass of (M, g) , which can be computed through the chart x as

$$m_{ADM}(M, g) = \lim_{r \rightarrow +\infty} \frac{1}{16\pi} \int_{\{|x|=r\}} \sum_{i,j=1}^3 \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right) \frac{x^i}{|x|} d\sigma_{\text{eucl}}, \quad (2.2)$$

is a well defined geometric invariant, whose value does not depend on the particular *asymptotically flat coordinate chart* employed in the above computation (see e.g. [8]).

Before proceeding with our proof of Theorem 2.1, let us mention that since the original work of Schoen and Yau, several other approaches have been proposed to prove this crucial result. Far from being complete and referring the reader to [30] for a comprehensive survey on this topic, we just mention that the first alternative proof was found by Witten [41], using harmonic spinors. Another route to the Positive Mass Theorem was subsequently provided by the Huisken-Ilmanen's theory of the Weak Inverse Mean Curvature Flow [26], in combination with the Monotonicity of the Hawking Mass observed by Geroch [21]. Yet another proof of the Positive Mass Theorem has been recently proposed by Li [32], using the Ricci Flow. Finally, in a recent paper, Bray, Kazaras, Kuhri and Stern [14] were able to provide a new argument, based on the study of the level sets of linearly growing harmonic functions combined with integral identities, deduced via the Bochner technique. This latter approach, inspired by [40], turned out to be flexible enough to also allow for the treatment of the space-time case [24, 13], and, together with some of the computations carried out in [29, 28], is probably the method that displays the largest number of analogy with ours. On this regard, it should be mentioned that level set methods, combined with Bochner technique and integral identities, have recently found some applications to the study of static metrics in general relativity [3, 6], even when the function under consideration is not necessarily harmonic [11, 12, 10].

Proof. For the sake of notation, let us set $m = m_{ADM}(M, g)$, and let us focus on the first part of the positive mass statement, i.e., $m \geq 0$. By the very accurate discussion given in [14, Section 2], and in particular from [14, Proposition 2.1], the analysis can be reduced to the case where the underlying manifold M is diffeomorphic to \mathbb{R}^3 and there exists a distinguished asymptotically flat coordinate chart $x = (x^1, x^2, x^3)$ – called *Schwarzschildian coordinate chart* – in which the metric g can be expressed as

$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} dx^i \otimes dx^j. \quad (2.3)$$

A manifold of this kind clearly fulfills all the assumptions made in Theorem 1.1. In particular, there exists the solution u to (1.1), and consequently the function F defined as in (1.2). Also, in virtue of the monotonicity of F established in Theorem 1.1, we have that

$$\lim_{t \rightarrow 0^+} F(t) \leq \lim_{t \rightarrow +\infty} F(t). \quad (2.4)$$

As we have seen in the proof of Corollary 1.3, it is a general fact that $\lim_{t \rightarrow 0^+} F(t) = 0$. We now claim that $\lim_{t \rightarrow +\infty} F(t) = 8\pi m$. It is clear that combining this claim with (2.4), one easily gets $m \geq 0$. In order to compute the limit of $F(t)$ as $t \rightarrow +\infty$, it is useful to understand the asymptotic behaviour of u at infinity. This is done in the following lemma.

Lemma 2.2. *The function u satisfies the asymptotic expansion at infinity*

$$u = 1 - \frac{1}{|x|} + \frac{1}{2|x|^2} (m + \phi(x/|x|)) + O_2(|x|^{-3+\alpha}), \quad (2.5)$$

where $x = (x^1, x^2, x^3)$ is a Schwarzschildian coordinate chart at infinity, ϕ satisfies $\Delta_{\mathbb{S}^2} \phi = -2\phi$, and $0 < \alpha < 1/2$ is a fixed real number that can be chosen as small as desired.

Proof of Lemma 2.2. First of all we notice that by virtue of the Schwarzschildian asymptotics (2.3), our metric is asymptotically flat of order $\tau = 1$. In particular, by [34, Appendix A], we know that

$$u(x) = 1 - \frac{\mathcal{C}}{|x|} + h(x), \quad \text{with } h(x) = O_2(|x|^{-2+\alpha}), \quad (2.6)$$

for some fixed $0 < \alpha < 1/2$, that can be chosen as small as desired. Moreover, we have that $\mathcal{C} = 1$, in view of (1.16). Our aim is to investigate the structure of the remainder h , eventually proving that

$$h(x) = \frac{1}{2|x|^2} (m + \phi(x/|x|)) + O_2(|x|^{-3+\alpha}).$$

Writing the equation $\Delta u = 0$ in terms of the background Euclidean metric $g_{\mathbb{R}^3}$, it is readily seen that h satisfies the equation

$$\Delta_{\mathbb{R}^3} h(x) - \mathcal{U}^{-1} \frac{m}{|x|^2} \left\langle Dh(x), \frac{x}{|x|} \right\rangle_{\mathbb{R}^3} = \mathcal{U}^{-1} \frac{m}{|x|^4}, \quad (2.7)$$

where we have set $\mathcal{U} = 1 + \frac{m}{2|x|}$, for the ease of the reader. To proceed, we plug into (2.7) the *ansatz*

$$h(x) = \mathcal{U}^{-1} \frac{m}{2|x|^2} + f(x), \quad (2.8)$$

with $f(x) = O_2(|x|^{-2+\alpha})$ and find out that the new remainder f satisfies the equation

$$\Delta_{\mathbb{R}^3} f(x) = \mathcal{U}^{-1} \frac{m}{|x|^2} \left\langle Df(x), \frac{x}{|x|} \right\rangle_{\mathbb{R}^3}. \quad (2.9)$$

Observing that $\mathcal{U} \rightarrow 1$ as $|x| \rightarrow +\infty$, our claim on f is that

$$f(x) = \frac{\phi(x/|x|)}{2|x|^2} + O_2(|x|^{-3+\alpha}).$$

To prove this claim, we observe that, since $f(x) = O_2(|x|^{-2+\alpha})$, the right hand side of (2.9) can be estimated by an $O_1(|x|^{-5+\alpha})$. Moreover, among the elements lying in the kernel of $\Delta_{\mathbb{R}^3}$ which are compatible with the condition $f(x) = O_2(|x|^{-2+\alpha})$, the ones with the slowest decay rate are of the form $\phi(x/|x|)/(2|x|^2)$, for some ϕ as in the statement of the lemma, whereas the other ones can be estimated by an $O_2(|x|^{-3})$. The claim, as well as the thesis of the lemma, follows now from the standard theory for elliptic PDEs. \square

With Lemma 2.2 at hand, we can now compute the limit on the right hand side of (2.4). To this aim, it is convenient to rewrite F as

$$F(t) = \int_{\{u=1-\frac{1}{t}\}} \frac{1}{1-u} \left[1 + \frac{\langle \nabla|\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2} \right] |\nabla u| d\sigma.$$

By virtue of Lemma 2.2, we have that

$$\begin{aligned} |\nabla u| &= \frac{1}{|x|^2} \left[1 - \frac{1}{|x|} (2m + \phi(x/|x|)) + O(|x|^{-2+\alpha}) \right], \\ \frac{\langle \nabla|\nabla u|, \nabla u \rangle}{|\nabla u|^2} &= -\frac{2}{|x|} \left[1 - \frac{1}{2|x|} (4m + \phi(x/|x|)) + O(|x|^{-2+\alpha}) \right], \end{aligned}$$

so that

$$\lim_{|x| \rightarrow +\infty} \frac{1}{1-u} \left[1 - \frac{\langle \nabla|\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2} \right] = 2m.$$

In particular, for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that whenever $u \geq 1 - 1/t_\varepsilon$ one has that

$$2m - \varepsilon \leq \frac{1}{1-u} \left[1 - \frac{\langle \nabla|\nabla u|, \nabla u \rangle}{(1-u)|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2} \right](x) \leq 2m + \varepsilon.$$

Using this fact in combination with (1.16), we deduce that, for every $t \geq t_\varepsilon$, it holds

$$4\pi(2m - \varepsilon) \leq F(t) \leq 4\pi(2m + \varepsilon).$$

Therefore, we have that $\lim_{t \rightarrow +\infty} F(t) = 8\pi m$ and in turn that $m \geq 0$.

The rigidity statement, i.e. the fact that (M, g) and $(\mathbb{R}^3, g_{\mathbb{R}^3})$ are isometric if $m = 0$, can be deduced from the validity of the inequality $m \geq 0$, through the nowadays standard argument proposed in the original Schoen-Yau's paper [38] (see also [30, pp.95-97 and p.102]). \square

3. SMALL SPHERE LIMITS AND NONNEGATIVE SCALAR CURVATURE

As mentioned in Remark 1.4, Theorem 1.1 and Corollary 1.3 support the notion of the quantity F , introduced in (1.1), as a quasi-local mass. This section provides additional evidence in favour of this idea. Specifically, we will calculate the small sphere limits of our quantities and observe that they can be utilized to accurately determine, with only a constant factor discrepancy, the scalar curvature's precise value. It is important to note that in General Relativity this geometric invariant coincides for time-symmetric initial data sets with mass/energy density. Therefore, it is reassuring to contemplate that the very same quantities that allow us to establish the non-negativity of the total mass are inherently connected to mass density. Before delving into the explicit computations, let us briefly review the well-established case of the Hawking mass, which serves as a prominent example of quasi-local mass. The Hawking mass is defined as follows:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma \right),$$

where Σ is a smooth closed surface embedded in a 3-dimensional Riemannian manifold (M, g) . In [19] the following small-sphere limit of the Hawking mass was computed, leading to

$$m_H(\partial B_q(t)) = \frac{R(q)}{12} t^3 + \left(\frac{\Delta R(q)}{120} - \frac{R^2(q)}{144} \right) t^5 + O(t^6), \quad \text{as } t \rightarrow 0^+,$$

where q is a point in M , $B_q(t)$ is the ball of radius t centred at q , and R denotes the scalar curvature of (M, g) . We are going to prove that a similar expansion holds for (a localised version of) the monotonic quantities introduced in Section 1. To this aim, let us consider a local Green's function \mathcal{G}_q with pole at $q \in M$, i.e., the unique distributional solution (see [7]) to the problem

$$\begin{cases} \Delta \mathcal{G}_q = -\delta_q & \text{in } B_q(r_q), \\ \mathcal{G}_q = 0 & \text{on } \partial B_q(r_q), \end{cases} \quad (3.1)$$

where we agree that $r_q = \text{inj}(q)/2$ if $\text{inj}(q) < +\infty$, and $r_q = 1$ otherwise. Here, of course, $\text{inj}(q)$ denotes the injectivity radius at q . In particular, the distance function $d_g(q, \cdot)$ is smooth on the punctured ball $B_q^*(r_q) = B_q(r_q) \setminus \{q\}$. Following [33, Proposition B.1], it is immediate to check that if (x^1, x^2, x^3) is a normal coordinate system centred at q , then \mathcal{G}_q satisfies the asymptotic expansion

$$\mathcal{G}_q = \frac{1}{4\pi|x|} + A_q + O_2(|x|^{1-\tau}), \quad \text{as } |x| \rightarrow 0, \quad (3.2)$$

where the exponent τ can be chosen arbitrarily in $(0, 1)$. Notice that, since $|x(\cdot)| = d_g(q, \cdot)$, the constant A_q remains the same regardless of the set of normal coordinates centered at q that is being used. To introduce the localised analog of the monotonic quantity F defined in (1.1), it is convenient to consider, on the punctured (closed) ball $\overline{B}_r^*(r_q)$, the function

$$u_q := 1 + 4\pi A_q - 4\pi \mathcal{G}_q. \quad (3.3)$$

By the maximum principle, the range of u_q coincides with the interval $(-\infty, 1 + 4\pi A_q]$. In analogy with (1.1), we set

$$F_q(t) = 4\pi t - t^2 \int_{\{u_q=1-\frac{1}{t}\}} |\nabla u_q| \mathbb{H} \, d\sigma + t^3 \int_{\{u_q=1-\frac{1}{t}\}} |\nabla u_q|^2 \, d\sigma, \quad (3.4)$$

for $t \in I_q$, where the set I_q depends on the sign of the constant A_q . More precisely, if $A_q < 0$, we let $I_q = (0, -1/4\pi A_q)$, whereas if $A_q \geq 0$, we let $I_q = (0, +\infty)$. Notice that this choice might appear slightly restrictive in the case where $A_q > 0$, as it corresponds to consider only the level sets of u_q with values in $(-\infty, 1)$. Indeed, it would be possible to extend the definition of F_q to the set $(-\infty, -1/4\pi A_q)$ in order to cover the whole natural range of u_q . However, since we are only interested in the small sphere limit of our quantities, this will result in an unnecessary complication. With these notations at hand we are now ready to state the main result of this section.

Theorem 3.1 (Small Sphere Limit). *Let (M, g) be a complete, 3-dimensional Riemannian manifold. Then, at every point $q \in M$, the following expansion holds*

$$\frac{F_q(t)}{8\pi} = \frac{R(q)}{12} t^3 + o(t^3), \quad \text{as } t \rightarrow 0^+, \quad (3.5)$$

where R is the scalar curvature of g and F_q is the quantity defined in (3.4).

An immediate consequence of the aforementioned statement is the following characterization of the non-negativity of the scalar curvature in terms of the monotonicity of the functions F_q , for $q \in M$.

Corollary 3.2 (Scalar Curvature Lower Bound). *Let (M, g) be a complete, 3-dimensional Riemannian manifold. Then, the scalar curvature R of the metric g is nonnegative on M if and only if for every $q \in M$ the function F_q is either nonnegative or monotonically non decreasing on an initial portion $(0, \varepsilon_q]$ of I_q , for some $\varepsilon_q > 0$.*

It is worth noticing that the above characterisation might suggest a plausible notion of synthetic scalar curvature lower bound, to be employed in the context of nonsmooth differential and Riemannian geometry.

Proof of Theorem 3.1. As the point q will remain unchanged throughout the proof, we drop it from the notation whenever it is possible. Our first claim is that

$$\lim_{t \rightarrow 0^+} \frac{F(t)}{|\Omega(t)|} = \frac{R(q)}{2}, \quad (3.6)$$

where $\Omega(t) = \{u < 1 - 1/t\}$ and $|\Omega(t)|$ denote its measure. In a given normal coordinates system (x^1, x^2, x^3) centred at q , so that (3.2) hold, we have that the following expansions are in force

$$1 - u = \frac{1}{|x|} (1 + O(|x|^{2-\tau})), \quad (3.7)$$

$$\partial_\alpha u = \frac{1}{|x|^2} \left[\frac{x_\alpha}{|x|} + O(|x|^{2-\tau}) \right], \quad (3.8)$$

$$|\nabla u| = \frac{1}{|x|^2} (1 + O(|x|^{2-\tau})), \quad (3.9)$$

$$\nabla_\alpha \nabla_\beta u = \frac{1}{|x|^3} \left[\delta_{\alpha\beta} - \frac{3x_\alpha x_\beta}{|x|^2} + O(|x|^{2-\tau}) \right], \quad (3.10)$$

$$H = \frac{2}{|x|} (1 + O(|x|^{2-\tau})), \quad (3.11)$$

where the function u is defined as in (3.3), and we have set $x_\alpha = \delta_{\alpha\beta} x^\beta$. Moreover, as far as distance, area and volume are concerned, it is easily seen that, for every $p \in \Sigma_t = \{u = 1 - 1/t\}$,

$$A_1 t \leq |x(p)| \leq A_2 t \quad \text{and} \quad B_1 t^2 \leq |\Sigma_t| \leq B_2 t^2 \quad \text{and} \quad C_1 t^3 \leq |\Omega(t)| \leq C_2 t^3, \quad (3.12)$$

as expected. Arguing as in the proof of Corollary 1.3, it is not hard to show that $\lim_{t \rightarrow 0^+} F(t) = 0$. By l'Hôpital's rule, we have that Claim (3.6) is equivalent to

$$\lim_{t \rightarrow 0^+} \frac{F'(t)}{|\Omega(t)|'} = \frac{R(q)}{2}. \quad (3.13)$$

Notice that, by virtue of (3.9), we can always select a small initial interval of times such that the level sets Σ_t of u are regular. In the same interval, both the function $t \mapsto F(t)$ and

$t \mapsto |\Omega(t)|$ turn out to be continuously differentiable. In particular, we can compute

$$F'(t) = \int_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t} |\nabla u|^2|}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma,$$

$$|\Omega(t)|' = \frac{1}{t^2} \int_{\Sigma_t} \frac{d\sigma}{|\nabla u|} = \int_{\Sigma_t} \frac{(1-u)^2}{|\nabla u|} d\sigma,$$

where the second formula is an easy consequence of the Coarea Formula, whereas the first one follows from (1.7), taking into account that, for small enough t , all the level sets Σ_t are spherical, and thus

$$4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} d\sigma = 0,$$

by the Gauss-Bonnet Theorem. To compute the limit in (3.13), we first observe that

$$|\Omega(t)|' = |\Sigma_t| (1 + O(t^{2-\tau})), \quad (3.14)$$

since, by formulas (3.7) and (3.9), one has

$$\frac{(1-u)^2}{|\nabla u|} = 1 + O(|x|^{2-\tau}). \quad (3.15)$$

On the other hand, the simple Taylor expansion of the scalar curvature at q gives

$$R = R(q) + O(|x|),$$

which implies

$$\int_{\Sigma_t} R d\sigma = |\Sigma_t| (R(q) + O(t)). \quad (3.16)$$

Combinig (3.14) with (3.16), one gets

$$\lim_{t \rightarrow 0^+} \frac{\frac{1}{2} \int_{\Sigma_t} R d\sigma}{|\Omega(t)|'} = \frac{R(q)}{2}. \quad (3.17)$$

Hence, Claim (3.13) is proven, if we can show that

$$\int_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t} |\nabla u|^2|}{|\nabla u|^2} + \frac{|\mathring{h}|^2}{2} + \frac{3}{4} \left(\frac{2|\nabla u|}{1-u} - H \right)^2 \right] d\sigma = O(t^{4-2\tau}). \quad (3.18)$$

Actually, we are going to prove that the function in square brackets is of order $O(|x|^{2-2\tau})$, as $|x| \rightarrow 0$. The desired estimate will then be a consequence of (3.12). To this end, we first observe that the expansions (3.7), (3.9) and (3.11) imply that

$$\frac{2|\nabla u|}{1-u} - H = O(|x|^{1-\tau}), \quad (3.19)$$

and thus the last summand in the square brackets is of order $O(|x|^{2-2\tau})$. To evaluate the first summand, we recall identity (1.4) and we observe that

$$\begin{aligned}\nabla_\alpha^{\Sigma_t} |\nabla u| &= \nabla_\alpha |\nabla u| - \langle \nabla |\nabla u|, \nu \rangle \nu_\alpha = \nabla_\alpha |\nabla u| + \mathbb{H} \nabla_\alpha u = \frac{\nabla \nabla u (\nabla u, \partial_\alpha)}{|\nabla u|} + \mathbb{H} \partial_\alpha u \\ &= -2 \frac{x_\alpha}{|x|^4} + O(|x|^{-1-\tau}) + 2 \frac{x_\alpha}{|x|^4} + O(|x|^{-1-\tau}) = O(|x|^{-1-\tau}),\end{aligned}$$

where in the last row we used the asymptotic expansions (3.8), (3.9), (3.10) and (3.11). It follows that

$$\frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} = O(|x|^{2-2\tau}).$$

Finally, we need to prove that also $|\mathring{h}|^2 = O(|x|^{2-2\tau})$. To see this, let $\varepsilon > 0$ be a positive real number such that $\varepsilon^2 < 1/3$ and let U_α be the open set defined by

$$U_\alpha = \left\{ p \in B_q^*(r_q) : \frac{|x^\alpha|}{|x|}(p) > \varepsilon \text{ and } |\nabla u|(p) \neq 0 \right\}, \quad (3.20)$$

for $\alpha = 1, \dots, 3$. Due to our choice of $\varepsilon > 0$, it is easy to realize that

$$U_1 \cup U_2 \cup U_3 = \{p \in B_q^*(r_q) : |\nabla u|(p) \neq 0\}. \quad (3.21)$$

Without loss of generality, let us focus on the open set U_1 and prove the desired estimate $|\mathring{h}|^2 = O(|x|^{2-2\tau})$ on it. To perform our computations, it is convenient to consider on U_1 , the frame field given by $\{\nabla u/|\nabla u|, X_2, X_3\}$, where the vector fields X_2 and X_3 are defined as

$$X_2 = -\partial_2 u \partial_1 + \partial_1 u \partial_2, \quad \text{and} \quad X_3 = -\partial_3 u \partial_1 + \partial_1 u \partial_3.$$

It is immediate to check that $\nabla u/|\nabla u|$ is orthogonal to both X_2 and X_3 and that the latter are linearly independent on U_1 . To complete the set-up, we let $\{du/|\nabla u|, \xi^2, \xi^3\}$ be the dual co-frame of $\{\nabla u/|\nabla u|, X_2, X_3\}$. In this framework, using the expansions (3.8), (3.9) and (3.10), we can write

$$\begin{aligned}g_\Sigma &= g_{ij}^\Sigma \xi^i \otimes \xi^j = g(X_i, X_j) \xi^i \otimes \xi^j = \frac{1}{|x|^4} \left[\frac{x_i x_j}{|x|^2} + \frac{(x^1)^2}{|x|^2} \delta_{ij} + O(|x|^{2-\tau}) \right] \xi^i \otimes \xi^j, \\ \mathring{h} &= h_{ij} \xi^i \otimes \xi^j = \frac{\nabla \nabla u(X_i, X_j)}{|\nabla u|} \xi^i \otimes \xi^j = \frac{1}{|x|^5} \left[\frac{x_i x_j}{|x|^2} + \frac{(x^1)^2}{|x|^2} \delta_{ij} + O(|x|^{2-\tau}) \right] \xi^i \otimes \xi^j,\end{aligned}$$

where $i, j \in \{2, 3\}$. From the first of above expressions and by the fact that $|x^1| > \varepsilon|x|$, it follows that $g_\Sigma^{ij} = O(|x|^4)$ on U_1 , and in turn that

$$|\mathring{h}|^2 = g_\Sigma^{ik} g_\Sigma^{jl} \left(h_{il} - \frac{\mathbb{H}}{2} g_{il}^\Sigma \right) \left(h_{jk} - \frac{\mathbb{H}}{2} g_{jk}^\Sigma \right) = O(|x|^{2-2\tau}), \quad (3.22)$$

where we also took advantage of expansion (3.11) in order to get the desired cancellation. Since the same estimate holds on U_2 and U_3 , we finally have that (3.18) is proven, so that in turn also Claim (3.6) is proven. Using the third estimate in (3.12), we have that Claim (3.6) can be reformulated as follows

$$\frac{F_q(t)}{8\pi} = \frac{\mathbb{R}(q)}{16\pi} |\Omega(t)| + o(t^3).$$

Therefore, in order to obtain the expansion (3.5), it is sufficient to show that

$$\lim_{t \rightarrow 0^+} \frac{|\Omega(t)|}{\frac{4\pi}{3} t^3} = 1.$$

By l'Hôpital's rule, this reduces to proving that

$$\lim_{t \rightarrow 0^+} \frac{|\Omega(t)|'}{4\pi t^2} = 1.$$

The latter statement follows from (3.14), combined with

$$|\Sigma_t| = \int_{\Sigma_t} d\sigma = \int_{\Sigma_t} \frac{|\nabla u|}{(1-u)^2} d\sigma + O(t^{4-\tau}) = 4\pi t^2 + O(t^{4-\tau}),$$

where in the last two identities we used (3.15) and the fact that $\int_{\Sigma_t} |\nabla u| d\sigma = 4\pi$. \square

4. FURTHER DIRECTIONS: GEROCH-TYPE CALCULATION FOR p -HARMONIC FUNCTIONS.

In this last section, we extend the monotonicity formula (1.7) to the nonlinear potential theoretic setting, where harmonic functions are replaced by p -harmonic functions, with $1 < p < 3$. Subsequently, we show how these new monotonicity formulas can be employed to deduce the Riemannian Penrose Inequality under favourable assumptions. The treatment of the general case is beyond the purposes of the present note and it is deferred to [5]. We consider, for every $1 < p < 3$, the unique solution u to the problem

$$\begin{cases} \Delta_p u = 0 & \text{in } M, \\ u = 0 & \text{on } \partial M, \\ u \rightarrow 1 & \text{at } \infty, \end{cases} \quad (4.1)$$

where (M, g) is a 3-dimensional, complete, asymptotically flat Riemannian manifold with nonnegative scalar curvature and smooth, connected, minimal boundary. Assume that $|\nabla u| \neq 0$ everywhere, so that u is smooth.

Remark 4.1. *Notice that the condition $|\nabla u| \neq 0$, though far from being optimal, is automatically satisfied by relevant classes of boundary geometries. For example, following [31], one can show that, in the Euclidean setting, mean convex and star-shaped boundaries give rise to p -capacitary potentials whose gradient is never vanishing. It is also worth noticing that the monotonicity of the functions F_p , defined below, still holds if the set of the critical values of the p -harmonic function u is negligible and the regular level sets of u are connected.*

To introduce our monotonic quantities, we recall that the p -capacity of ∂M is defined as

$$\text{Cap}_p(\partial M) := \inf \left\{ \int_M |\nabla v|^p d\mu : v \in C_c^\infty(M), v = 1 \text{ on } \partial M \right\}, \quad (4.2)$$

which is related to u through the well known identities

$$\text{Cap}_p(\partial M) = \int_M |\nabla u|^p d\mu = \int_{\{u=t\}} |\nabla u|^{p-1} d\sigma. \quad (4.3)$$

In analogy with (1.2), we define, for every $1 < p < 3$, the function

$$F_p(t) = 4\pi t - \frac{t^{\frac{2}{p-1}}}{c_p} \int_{\{u=\alpha_p(t)\}} |\nabla u| H d\sigma + \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int_{\{u=\alpha_p(t)\}} |\nabla u|^2 d\sigma, \quad (4.4)$$

where

$$c_p = \left(\frac{\text{Cap}_p(\partial M)}{4\pi} \right)^{\frac{1}{p-1}}, \quad \alpha_p(t) = 1 - c_p \frac{p-1}{3-p} t^{-\frac{3-p}{p-1}}, \quad (4.5)$$

and the variable t ranges in $\left[\left(c_p \frac{p-1}{3-p} \right)^{\frac{p-1}{3-p}}, +\infty \right)$. Proceeding in the same spirit as in the smooth proof of Theorem 1.1, we compute

$$\begin{aligned} \frac{d}{dt} \int_{\{u=\alpha_p(t)\}} |\nabla u|^2 d\sigma &= -\frac{3-p}{p-1} \frac{c_p}{t^{\frac{2}{p-1}}} \int_{\{u=\alpha_p(t)\}} |\nabla u| H d\sigma, \\ \frac{d}{dt} \int_{\{u=\alpha_p(t)\}} |\nabla u| H d\sigma &= \frac{c_p}{t^{\frac{2}{p-1}}} \int_{\{u=\alpha_p(t)\}} \left\{ \frac{p-2}{p-1} H^2 - |\nabla u| \left[\Delta_{\Sigma_t} \left(\frac{1}{|\nabla u|} \right) + \frac{|\mathring{h}|^2 + \text{Ric}(\nu, \nu)}{|\nabla u|} \right] \right\} d\sigma, \end{aligned}$$

where $\Sigma_t = \{u = \alpha_p(t)\}$. Using the traced Gauss equation, as in Section 1, we obtain

$$\begin{aligned} \frac{p-2}{p-1} H^2 - |\nabla u| \left[\Delta_{\Sigma_t} \left(\frac{1}{|\nabla u|} \right) + \frac{|\mathring{h}|^2 + \text{Ric}(\nu, \nu)}{|\nabla u|} \right] \\ = \Delta_{\Sigma_t}(\log |\nabla u|) - \frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} - \frac{R}{2} + \frac{R^{\Sigma_t}}{2} - \frac{|\mathring{h}|^2}{2} - \frac{5-p}{4(p-1)} H^2. \end{aligned}$$

By standard manipulations, we finally get

$$F_p'(t) = 4\pi - \int_{\Sigma_t} \frac{R^{\Sigma_t}}{2} d\sigma + \int_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t} |\nabla u||^2}{|\nabla u|^2} + \frac{R}{2} + \frac{|\mathring{h}|^2}{2} + \frac{5-p}{4(p-1)} \left[\frac{2(p-1)}{3-p} \frac{|\nabla u|}{1-u} - H \right]^2 \right] d\sigma. \quad (4.6)$$

Again, the right hand side is nonnegative by virtue of Gauss-Bonnet Theorem and by the assumption $R \geq 0$.

Let us illustrate how the above monotonicity result might be employed to deduce the Riemannian Penrose Inequality. We start observing that the monotonicity of F_p implies

$$F_p(\beta_p) \leq \lim_{t \rightarrow +\infty} F_p(t), \quad (4.7)$$

where we have used the short-hand notation $\beta_p = \left(c_p \frac{p-1}{3-p} \right)^{\frac{p-1}{3-p}}$. By the minimality of ∂M , we immediately get

$$F_p(\beta_p) = 4\pi\beta_p + \frac{\beta_p^{\frac{5-p}{p-1}}}{c_p^2} \int_{\partial M} |\nabla u|^2 d\sigma \geq 4\pi\beta_p. \quad (4.8)$$

To compute the limit on the right hand side of (4.7), we first observe that F_p rewrites as

$$F_p(t) = \int_{\{u=\alpha_p(t)\}} I_p |\nabla u|^{p-1} d\sigma,$$

where

$$I_p := c_p^{\frac{3p-7}{3-p}} \left(\frac{p-1}{3-p} \frac{1}{1-u} \right)^{\frac{p-1}{3-p}} \left[c_p^{3-p} + \left(\frac{p-1}{3-p} \frac{c_p}{1-u} \right)^2 |\nabla u|^{3-p} - \frac{p-1}{3-p} \frac{c_p^2}{1-u} |\nabla u|^{2-p} H \right].$$

To evaluate the asymptotic behaviour of I_p , we assume that, in a distinguished Schwarzschildian coordinate chart (x^1, x^2, x^3) at infinity, the function u satisfies

$$u = 1 - \frac{p-1}{3-p} \frac{c_p}{|x|^{\frac{3-p}{p-1}}} + \frac{3-p}{2} \frac{c_p}{|x|^{\frac{2}{p-1}}} (m + \phi(x/|x|)) + o_2(|x|^{-\frac{2}{p-1}}), \quad (4.9)$$

for some ϕ satisfying $\Delta_{\mathbb{S}^2} \phi = -2\phi$. The above expansion implies in particular that

$$|\nabla u| = \frac{c_p}{|x|^{\frac{2}{p-1}}} \left[1 - \frac{2}{p-1} \frac{m}{|x|} - \frac{3-p}{p-1} \frac{\phi(x/|x|)}{|x|} + o(|x|^{-1}) \right],$$

$$H = -(p-1) \frac{\nabla \nabla u(\nabla u, \nabla u)}{|\nabla u|^3} = \frac{2}{|x|} \left[1 - \frac{2m}{|x|} - \frac{3-p}{2} \frac{\phi(x/|x|)}{|x|} + o(|x|^{-1}) \right].$$

In turn, we get

$$\lim_{|x| \rightarrow +\infty} I_p(x) = 2m c_p^{1-p}. \quad (4.10)$$

In other words, for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that, whenever $u(x) \geq \alpha_p(t_\varepsilon)$, it holds

$$2m c_p^{1-p} - \varepsilon \leq I_p(x) \leq 2m c_p^{1-p} + \varepsilon.$$

Therefore, for every $t \geq t_\varepsilon$, we get that

$$8\pi m - 4\pi c_p^{p-1} \varepsilon \leq F_p(t) \leq 8\pi m + 4\pi c_p^{p-1} \varepsilon,$$

where we have used (4.3) and (4.5), and in turn that

$$\lim_{t \rightarrow +\infty} F_p(t) = 8\pi m.$$

Combining this limit with the inequalities in (4.7) and (4.8) yields $2m \geq \beta_p$. By definition of β_p and c_p , this means that

$$2m \geq \left(\frac{p-1}{3-p} \right)^{\frac{p-1}{3-p}} \left(\frac{\text{Cap}_p(\partial M)}{4\pi} \right)^{\frac{1}{3-p}}.$$

Finally, passing to the limit as $p \rightarrow 1^+$ with the help of [20, Theorem 1.2], we get that

$$m \geq \sqrt{\frac{|\partial M^*|}{16\pi}},$$

where ∂M^* denotes the strictly outward minimising hull of ∂M and is defined as in [20]. Assuming now that ∂M is an outermost minimal surface, then it is also strictly outward minimising and thus $|\partial M^*| = |\partial M|$, which implies the validity of the desired inequality.

Acknowledgements. *The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), which is part of the Istituto Nazionale di Alta Matematica (INdAM). Francesca Oronzio is supported by the ERC-STG Grant 759229: HiCoS "Higher Co-dimension Singularities: Minimal Surfaces and the Thin Obstacle Problem". The authors thank G. Ascione, L. Benatti and C. Mantegazza for various useful discussions during the preparation of the manuscript.*

Data availability statement. *The authors declare that the present manuscript has no associated data.*

Conflict of interest statement. *The authors declare that they have no conflict of interest.*

REFERENCES

1. V. Agostiniani, M. Fogagnolo, and L. Mazzieri, *Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature*, *Inventiones mathematicae* **222** (2020), no. 3, 1033–1101.
2. ———, *Minkowski inequalities via nonlinear potential theory*, *Archive for Rational Mechanics and Analysis* **244** (2022), no. 1, 51–85.
3. V. Agostiniani and L. Mazzieri, *On the geometry of the level sets of bounded static potentials*, *Communications in Mathematical Physics* **355** (2017), no. 1, 261–301.
4. ———, *Monotonicity formulas in potential theory*, *Calculus of Variations and Partial Differential Equations* **59** (2020), no. 1, 32 pp.
5. V. Agostiniani, L. Mazzieri, C. Mantegazza and F. Oronzio, *Riemannian Penrose Inequality via nonlinear potential theory*, *ArXiv Preprint Server – arXiv:2205.11642v1*, 2022.
6. V. Agostiniani, L. Mazzieri and F. Oronzio, *A geometric capacity inequality for sub-static manifolds with harmonic potentials*, *Mathematics in Engineering* **4** (2022), no. 2, 40 pp.
7. T. Aubin, *Nonlinear analysis on manifolds. Monge–Ampère equations*, Springer–Verlag, 1982.
8. R. Bartnik, *The mass of an asymptotically flat manifold*, *Communications on Pure and Applied Mathematics* **39** (1986), no. 5, 661–693.
9. L. Benatti, M. Fogagnolo, and L. Mazzieri, *Minkowski inequality on complete Riemannian manifolds with nonnegative Ricci curvature*, to appear in *Analysis and PDE*, *ArXiv Preprint Server – arXiv:2101.06063v5*, 2022.
10. S. Borghini, P.T. Chruściel, and L. Mazzieri, *On the uniqueness of Schwarzschild-deSitter Spacetime*, *Archive for Rational Mechanics and Analysis* **247** (2023), no. 2, 35 pp.
11. S. Borghini and L. Mazzieri, *On the mass of static metrics with positive cosmological constant: I*, *Classical and Quantum Gravity* **35** (2018), no. 12, 43 pp.
12. ———, *On the mass of static metrics with positive cosmological constant: II*, *Communications in Mathematical Physics* **377** (2020), no. 3, 2079–2158.
13. H. Bray, S. Hirsch, D. Kazaras, M. Khuri, and Y. Zhang, *Spacetime harmonic functions and applications to mass*, *Perspectives in scalar curvature. Vol. 2*, World Scientific Publishing Co. Pte. Ltd., 2023, 593–639.
14. H. Bray, D. Kazaras, M. Khuri, and D. Stern, *Harmonic functions and the mass of 3-dimensional asymptotically flat Riemannian manifolds*, *The Journal of Geometric Analysis* **32** (2022), no. 6, 29 pp.
15. G. Catino, C. Mantegazza, and L. Mazzieri, *On the global structure of conformal gradient solitons with nonnegative Ricci tensor*, *Communications in Contemporary Mathematics* **14** (2012), no. 6, 12 pp.
16. T. H. Colding, *New monotonicity formulas for Ricci curvature and applications. I*, *Acta Mathematica* **209** (2012), no. 2, 229–263.
17. T. H. Colding and W. P. Minicozzi, *Monotonicity and its analytic and geometric implications*, *Proceedings of the National Academy of Sciences* **110** (2013), no. 48, 19233–19236.
18. ———, *Ricci curvature and monotonicity for harmonic functions*, *Calculus of Variations and Partial Differential Equations* **49** (2014), no. 3, 1045–1059.
19. X.-Q. Fan, Y. Shi and L.-F. Tam, *Large-sphere and small-sphere limits of the Brown-York mass*, *Communications in Analysis and Geometry* **17** (2009), no. 1, 37–72.

20. M. Fogagnolo and L. Mazziere, *Minimising hulls, p -capacity and isoperimetric inequality on complete Riemannian manifolds*, Journal of Functional Analysis **283** (2022), no. 9, 49 pp.
21. R. Geroch, *Energy extraction*, Annals of the New York Academy of Sciences **224** (1973), no. 1, 108–117.
22. R. Hardt, H. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nasirashvili, *Critical sets of solutions to elliptic equations*, Journal of Differential Geometry **51** (1999), no. 2, 359–373.
23. R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*, Journal of Differential Geometry **30** (1989), no. 2, 505–522.
24. S. Hirsch, D. Kazaras, and M. Khuri, *Spacetime harmonic functions and the mass of 3-dimensional asymptotically flat initial data for the Einstein equations*, Journal of Differential Geometry **122** (2022), no. 2, 223–258.
25. G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, Journal of Differential Geometry **31** (1990), no. 1, 285–299.
26. G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, Journal of Differential Geometry **59** (2001), no. 3, 353–437.
27. G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of Variations and Geometric Evolution Problems: Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro, Italy, June 15–22, 1996, Springer Berlin Heidelberg (1999), 45–84.
28. J. Jezierski, *Positivity of mass for certain spacetimes with horizons*, Classical Quantum Gravity **6** (1989), no. 11, 1535–1539.
29. J. Jezierski and J. Kijowski, *Positivity of total energy in general relativity*, Physical Review. D. Third Series **36** (1987), no. 4, 1041–1044.
30. D. Lee, *Geometric relativity*, Graduate Studies in Mathematics, Vol. 201, American Mathematical Society, 2019.
31. J.L. Lewis, *Capacitary functions in convex rings*, Archive for Rational Mechanics and Analysis **66** (1977), no. 3, 201–224.
32. Y. Li, *Ricci flow on asymptotically Euclidean manifolds*, Geometry & Topology **22** (2018), no. 3, 1837–1891.
33. Y. Li and M. Zhu, *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries*, Communications on Pure and Applied Mathematics **50** (1997), no. 5, 449–487.
34. C. Mantoulidis, P. Miao, and L.-F. Tam, *Capacity, quasi-local mass, and singular fill-ins*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2020** (2020), no. 768, 55–92.
35. L. Mari, M. Rigoli, and A. G. Setti, *On the $1/H$ -flow by p -Laplace approximation: new estimates via fake distances under Ricci lower bounds*, American Journal of Mathematics **144** (2022), no. 3, 779–849.
36. O. Munteanu and J. Wang, *Comparison theorems for 3D manifolds with scalar curvature bounds*, International Mathematics Research Notices. IMRN **2023** (2023), no. 3, 2215–2242.
37. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv Preprint Server – arXiv:math/0211159, 2002.
38. R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Communications in Mathematical Physics **65** (1979), no. 1, 45–76.
39. ———, *The energy and the linear momentum in of space-times in general relativity*, Communications in Mathematical Physics **79** (1981), no. 1, 47–51.
40. D. Stern, *Scalar curvature and harmonic maps to \mathbb{S}^1* , Journal of Differential Geometry **122** (2022), no. 2, 259–269.
41. E. Witten, *A new proof of the positive energy theorem*, Communications in Mathematical Physics **80** (1981), no. 3, 381–402.
42. C. Xia and J. Yin, *Anisotropic p -capacity and anisotropic Minkowski inequality*, Science China Mathematics, **65** (2022), no. 3, 559–582.

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