UNIVERSITÀ DEGLI STUDI DI NAPOLI
FEDERICO II

Dipartimento di Matematica e Applicazioni
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TESI DI DOTTORATO

ADM Mass and Linear Potential Theory

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Dottorato di Ricerca in
Matematica e Applicazioni
XXXIV Ciclo

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Notation and conventions

The term *smooth* means C^∞. All the objects we will consider are assumed to be smooth unless explicitly stated otherwise.

With $\mathbb{S}^{n-1}_r(x_0)$ and $B^n_r(x_0)$ we denote, respectively, the Euclidean sphere and open ball with radius r and center x_0 in $\mathbb{R}^n$. We simply write $\mathbb{S}^{n-1}$ for the unit sphere centered at the origin of $\mathbb{R}^n$ and we write $|\mathbb{S}^{n-1}|$ for its volume.

The term *manifold* means a connected, differentiable manifold of class C^∞, without boundary and of dimension greater or equal than 3. The term *manifold with boundary* means a connected manifold of class C^∞ of dimension greater or equal than 3, with (possibly nonconnected) boundary of class C^∞. The term *submanifold* without further qualifications means an embedded one without boundary.

A (k,l)–tensor field T on a manifold M is a smooth section of the vector bundle $T^k_l M$ and we write $T \in \Gamma(T^k_l M)$.

If (M,g) is a Riemannian manifold, we usually denote with μ its canonical measure.

We denote with g_{eucl} the standard Euclidean metric of $\mathbb{R}^n$, while, if (M,g) is a Riemannian manifold and $\big(U,(x^i,\dots,x^n)\big)$ is a coordinate chart of M, then g_e is the Riemannian metric on U defined as $g_e = \delta_{ij} dx^i \otimes dx^j$. The respective geometric quantities of g_{eucl} or g_e are labeled with the subscript e.

The Einstein convention of summing over the repeated indices is adopted in the whole thesis.

In all the computations, C denotes a general constant which may vary from line to line.

Introduction

Gravity is one of the four fundamental physical forces in our universe. Two major gravitational theories have evolved that are still relevant today. One of them is the Newtonian theory of gravity, the other is Albert Einstein's theory of general relativity, dating back to the early years of the twentieth century. In modern language, the first theory is often formulated in terms of a (Newtonian) potential U that satisfies the Poisson equation $\Delta U = 4\pi G\rho$, relating such potential to the density of matter ρ, via the gravitational constant G, in the Euclidean space $\mathbb{R}^3$. The general relativity theory uses instead a very different approach, avoiding the concept of "force" and unifying space and time into a curved Lorentzian 4–manifold $(\mathcal{M}, \mathbf{g})$ called *spacetime*. The *Einstein equation*

$$\mathbf{Ric} - \frac{1}{2}\mathbf{R}\mathbf{g} = k\mathbf{T} \qquad \text{with} \qquad k = \frac{8\pi G}{c^4}$$

then relates the curvatures $\mathbf{Ric}$ and $\mathbf{R}$ of the Lorentzian manifold to an "energy–momentum" or "matter" tensor field $\mathbf{T}$, which is the analogue of the classical mass density. Here, the constants G and c are the gravitational constant and the speed of light, respectively. Then, the spacetime $(\mathcal{M}, \mathbf{g})$ models a gravitational system, while the metric "represents" the gravitational field, which is influenced by the matter distribution and determines the dynamics: the trajectories of the freely falling point particles are (timelike) geodesics of $(\mathcal{M}, \mathbf{g})$.

Any "spacelike" hypersurface (M, g) (i.e. g, the metric induced by $\mathbf{g}$ on M, is Riemannian) in a spacetime $(\mathcal{M}, \mathbf{g})$ satisfies the *Einstein constraint equations*

$$\begin{aligned} \mathrm{R}_g - |K|_g^2 + \big(\mathrm{tr}_g K\big)^2 &= 2k\mu \\ \mathrm{div}_g K - d(\mathrm{tr}_g K) &= kJ \end{aligned}$$

where K is the second fundamental form induced by $\mathbf{g}$ on M, $\mu = \mathbf{T}(n,n)$ is the *local energy density* and $J = \mathbf{T}(n,\cdot)|_{\Gamma(TM)}$ is the *local momentum density*, being n the (local) *future–directed* unit normal to M. Indeed, these are simply the Gauss and Codazzi equations for a hypersurface, keeping into account the information about the "ambient" curvature, contained in the Einstein equation.

A triple (M, g, K), where (M, g) is a 3–dimensional Riemannian manifold and K is a symmetric $(0,2)$–tensor field on M satisfying the Einstein constraint equations is then usually called an *initial data set*. Moreover, (M, g, K) is said to be *time–symmetric*, if the tensor K vanishes everywhere (hence the hypersurface is totally geodesic) and that (M, g, K) satisfies the *dominant energy condition*, if $\mu \geq |J|_g$ everywhere (see [33] for the physical interpretation of this relation).

For time–symmetric initial data sets, the local momentum density J is zero, hence $\mathrm{R} = 16\pi\mu G/c^4$ and the dominant energy condition becomes equivalent to the requirement the nonnegativity of the scalar curvature R of (M, g).

Isolated systems and static systems are extensively studied gravitational systems. Isolated systems are stars or black holes that do not interact with other systems and cannot be reached by "external" gravitational waves, while static systems are individual stellar bodies or groups of stars and black holes that are not changing in time. Mathematically speaking, a static system is characterized by the existence of a special timelike Killing vector field in the spacetime, while an isolated system is modeled by the asymptotic “flatness” of its Lorenzian metric. More precisely, when describing isolated gravitational systems, one is then interested in *asymptotically flat* spacetimes (with one end), that is, spacetimes that far from the zone where the matter is concentrated approach the flat spacetime, i.e. the Minkowski spacetime [33, Section 5.1]. Indeed, from a physical point of view, one expects that when such a system is observed from great distance, its gravitational field should resemble the one of a point mass. Thus, the spacetime $(\mathcal{M}, \mathbf{g})$ modeling such system, should be asymptotically close to the *Schwarzschild solution of Einstein equation*, or simply *Schwarzschild spacetime* (see [33, Section 5.5]), modeling the gravitational field around a spherically symmetric, non–rotating, massive object. A time–symmetric initial data set $(M, g, 0)$ in an asymptotically flat spacetime satisfying the dominant energy condition is then *asymptotically flat* (see Definition 1.4.1) with nonnegative scalar curvature.

This discussion explains the great relevance of the theory of asymptotically flat Riemannian manifolds with nonnegative scalar curvature, which are the main objects of study in this thesis. Among them, the most famous example is the (exterior spatial) 3–dimensional *Schwarzschild manifold of mass* m $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$, described in Example 1.4.2. It is obtained by considering the $\{t = 0\}$–*spacelike slice* of the Schwarzschild spacetime mentioned above.

Any one–ended asymptotically flat manifold has the remarkable property of being equipped with a well–defined notion of “total” mass, called *ADM mass*, introduced in [3] by Arnowitt, Deser and Misner and denoted by m_{ADM} (see Definition 1.5.5). This coincides with the parameter m for the Schwarzschild manifolds in Example 1.4.2. Schoen and Yau proved in 1979 the celebrated *positive mass theorem* [76], stating that the ADM mass of a 3–dimensional, complete, one–ended asymptotically flat manifold (M, g) without boundary and with nonnegative scalar curvature is nonnegative and it is zero if and only if (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$. Later, Huisken and Ilmanen in [37] and Bray in [13] proved the *Riemannian Penrose inequality* for a 3–dimensional, complete, one–ended asymptotically flat manifold (M, g) with nonnegative scalar curvature and compact, connected, minimal boundary ∂M, assuming that M contains no other closed minimal surfaces:

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{\mathrm{Area}(\partial M)}{16\pi}} \tag{RPI}$$

with equality if and only if (M, g) is isometric to $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$.

The positive mass theorem asserts that a nonnegative local mass density ($\mathrm{R} \geq 0$) implies a nonnegative total mass ($m_{\mathrm{ADM}} \geq 0$). The proof of this natural physical property is actually highly nontrivial: Schoen and Yau’s proof [76,78] is based on a contradiction argument related to the existence of stable minimal hypersurfaces. Later, Witten [87] (see also [69]) gave an alternative proof in which the mass is directly expressed as the integral of a nonnegative quantity depending on an asymptotically constant harmonic spinor. Later, Lohkamp [53] explained how the positive mass theorem is the consequence of the nonexistence of positive scalar curvature metrics on the connected sum $N \# T$ of any closed 3–dimensional manifold N with a 3–dimensional torus T (which is a known result from [30,77]). In 2001, Huisken and Ilmanen [37] proved the theorem as a consequence of their proof of the Riemannian Penrose inequality, based on a weak version of the inverse mean curvature flow. In

2018 Li [52] gave a proof using Ricci flow, while in 2019, Bray, Kazaras, Khuri and Stern in [12] obtained another proof which makes use of asymptotically linear harmonic functions. Concerning to the positive mass theorem in higher dimensions, Witten's proof works for all spin manifolds, while Schoen and Yau were able to extend it up to dimension 7 [77], by a dimension–reduction argument. For the dimensions higher than 7, we refer the reader to the unpublished papers by Schoen and Yau [79] and Lohkamp [54]. Finally, we mention that there have been also proofs of various versions of the theorem for nonsmooth manifolds [44,45,55,61,63,80,81]. A very nice survey of many of these results may be found in [43].

The Riemannian Penrose inequality (RPI) can be seen as a physically natural refinement of the positive mass theorem. Indeed, we can imagine the minimal boundary as an event horizon hiding a black hole, which actually should give a contribution to the total mass. Moreover, it is expected that such contribution depends on the area of the event horizon, as in the case of $(M_{\mathsf{Sch}(m)}, g_{\mathsf{Sch}(m)})$, which represents the simplest model of a vacuum "exterior region" of a black hole. Under this point of view, inequality (RPI) tell us that the mass of our initial data set is at least equal to the mass of the Schwarzschild manifold whose boundary has the same area of ∂M. The first proof of this fact was given by Huisken and Ilmanen in [37], making rigorous an argument based on the inverse mean curvature flow, suggested by Geroch [26] and Jang–Wald [38]. An alternative approach was followed by Bray [13], by means of a conformal flow of metrics and the application of the positive mass theorem. We mention that Bray was able to extend the inequality also to the case of a disconnected boundary. Moreover, while the proof of Huisken–Ilmanen is essentially 3–dimensional, the approach of Bray can be generalized to higher dimensions. Indeed, using the same technique, Bray and Lee [14] proved the conclusion in any dimension $n \leq 7$, as in the work of Schoen and Yau on the positive mass theorem (the obstruction to the generalization of the proof of Schoen and Yau to dimension larger than 7 is given by the lack of regularity of the minimal hypersurfaces, if $n > 7$). We remark that a proof of the Riemannian Penrose inequality in the case where the initial data set is not time–symmetric presents considerable difficulties. In [70], using a heuristic argument, Penrose suggested what should be the natural form of inequality (RPI) for general initial data sets, however, only very partial results are known on the validity of such statement. We mention that the proof of such Penrose conjecture would give an indirect evidence of the validity of the *cosmic censorship conjecture* (we refer the interested reader to the survey [59]).

The aim of our work is to obtain geometric inequalities involving the ADM mass and the capacity of the boundary (if present), via monotonicity formulas holding along the level sets of appropriate harmonic functions. An advantage of this approach is that the proofs are simpler and more direct.

In Chapter 2 we will consider a triple (M, g, u), satisfying the following conditions,

(a) (M, g) is a complete, one–ended asymptotically flat manifold of dimension n, $n \geq 3$, with compact boundary ∂M;

(b) $u \in C^\infty(M)$ satisfies the system

$$\begin{cases} u\mathrm{Ric} - \nabla du \geq 0 & \text{in } M \\ \Delta u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases} \qquad (\star)$$

Moreover, sometimes we will also consider condition

(c) the boundary ∂M is connected.

In dimension 3, if assumptions (a) and (b) hold, the asymptotically flat spacetime $(\mathcal{M},\mathbf{g})$, given by $\mathcal{M} := \mathbb{R}\times\big(M\setminus\{u=0\}\big)$ with the Lorentzian metric $\mathbf{g} := -u^2 dt\otimes dt + g$, satisfies the so called *null convergence condition* [85], i.e. $\mathbf{Ric}(\mathbf{V},\mathbf{V})\geq 0$ for every $\mathbf{V}\in\Gamma(T\mathcal{M})$ such that $\mathbf{g}(\mathbf{V},\mathbf{V})=0$, which is the curvature assumption in Penrose's celebrated *singularity theorem* [33, p. 263, Theorem 1]. Moreover, in the special case that the first inequality in the above system is an equality, the asymptotically flat spacetime $(\mathcal{M},\mathbf{g})$ solves the *vacuum Einstein equation*, i.e. the Einstein equation with $\mathbf{T}=0$ or equivalently $\mathbf{Ric}=0$ (we refer to [33] for further details about the vacuum Einstein equation and the study of some famous solutions). Then, $(\mathcal{M},\mathbf{g})$ is a *standard static spacetime* having u as *lapse function*, which can be interpreted as a function describing the "flowing of time t" with respect to a canonical observer in the corresponding 4–dimensional Lorentzian manifold.

In dimension 3, a classic result of Bunting and Masood–ul–Alam [16] states that, if (M,g,u) satisfies assumptions (a) and (b), this latter with equality at the first line of the system, then (M,g) is isometric to a spacelike slice $\big(\mathbb{R}^3\setminus B_{\frac{m}{2}}(0),(1+\frac{m}{2|x|})^4 g_{\mathbb{R}^3}\big)$ of a Schwarzschild spacetime with positive mass m. Assuming a strong enough decay rate at infinity of the scalar curvature, we are able to extend this rigidity conclusion in Theorem 2.3.1 to all triples (M,g,u) satisfying assumptions (a) and (b) (without requiring the equality $u\mathrm{Ric}-\nabla du=0$), in all the dimensions such that the positive mass theorem holds.

Moreover, following similar ideas in [2], in the case that we cannot have such conclusion by the lack of a strong decay of the scalar curvature and assuming also condition (c) above, on the connectedness of the boundary ∂M (besides conditions (a) and (b)), we will show that for a triple (M,g,u) there holds

$$\mathcal{C}\geq\frac{1}{2}\bigg(\frac{\mathrm{Volume}(\partial M)}{\mathrm{Volume}(\mathbb{S}^{n-1})}\bigg)^{\frac{n-2}{n-1}}$$

where $\mathcal{C}$ is the *boundary capacity* of ∂M (see Definition 1.32), with equality holding if and only if (M,g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$. A key point of the proof is to show that the functions $F_\beta : [1,+\infty)\to[0,+\infty)$, defined by

$$F_\beta(\tau)=(1+\tau)^{\beta\,\frac{n-1}{n-2}}\int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}}|\nabla u|^{\beta+1}\,d\sigma\,,$$

for every $\beta>\frac{n-2}{n-1}$, are nonincreasing (Theorem 2.1.1). Then, the above inequality follows by computing and comparing the limit of F_β at $+\infty$ and the value $F_\beta(1)$.

We underline that in our line of proof of this inequality and the same for all the results in the thesis, a key technical point is dealing with the possible presence of non regular level sets (hence, also with the possibly nonempty set of the critical points of the harmonic functions that we consider), where the normal is possibly not everywhere well–defined and the derivatives of the functions could not exist, thus making difficult showing the monotonicity of the integrals, "crossing" such level sets.

In Chapter 3, the main goal is to obtain a new proof of the positive mass inequality, Theorem 3.1.1. We briefly describe here the key steps.

Given a complete, one–ended asymptotically flat, 3–dimensional Riemannian manifold (M, g) with nonnegative scalar curvature, Bray, Kazaras, Khuri and Stern in [12] proved that, given any $\varepsilon > 0$, there exists another complete, asymptotically flat, 3–dimensional Riemannian manifold $(\overline{M}, \overline{g})$, with nonnegative scalar curvature $\overline{\mathrm{R}} \geq 0$, satisfying the following properties,

(i) $\overline{M}$ is diffeomorphic to $\mathbb{R}^3$;

(ii) the ADM mass $\overline{m}_{\mathrm{ADM}}$ of $(\overline{M}, \overline{g})$ satisfies $|m_{\mathrm{ADM}} - \overline{m}_{\mathrm{ADM}}| < \varepsilon$;

(iii) there exists a coordinate chart (x^1, x^2, x^3), such that, for $|x|$ large enough, there holds
$$\overline{g} = \left(1 + \frac{\overline{m}_{\mathrm{ADM}}}{2|x|}\right)^4 \delta_{ij}\, dx^i \otimes dx^j .$$

Hence, by this result, we can clearly limit ourselves to prove the positive mass inequality for complete, asymptotically flat, 3–dimensional Riemannian manifolds (M, g) with nonnegative scalar curvature and satisfying conditions (i) and (iii).
We then consider an "appropriate" function $u \in C^\infty(M \setminus \{o\})$, solution of the system

$$\begin{cases} \Delta u = 4\pi\delta_o & \text{in } M \\ u \to 1 & \text{at } \infty \end{cases}$$

for some point $o \in M$ (see Chapter 2 for a discussion of the existence of such function u and its properties). Condition (i) implies that the function $F : (0, +\infty) \to \mathbb{R}$, defined as

$$F(t) \;=\; 4\pi t \;-\; t^2 \int\limits_{\{u = 1 - \frac{1}{t}\}} |\nabla u|\, \mathrm{H}\, d\sigma \;+\; t^3 \int\limits_{\{u = 1 - \frac{1}{t}\}} |\nabla u|^2\, d\sigma\,,$$

is nondecreasing, for t in the intervals such that $1 - 1/t$ is a regular value of u (Theorem 3.1.2). Here, H is the mean curvature of the Riemannian (connected or unconnected) smooth surface $\{u = 1 - 1/t\} \setminus \mathrm{Crit}(u)$, where $\mathrm{Crit}(u)$ is the set of the critical points of u, computed with respect to the ∞–pointing unit normal vector field $\nu = \nabla u / |\nabla u\,|$ and σ is the 2–dimensional Hausdorff measure of (M, g). The topological assumption that M is diffeomorphic to $\mathbb{R}^3$ (actually, $H_2(M; \mathbb{Z}) = 0$ is sufficient) and the harmonicity guarantee the connectedness of the regular level sets of u, then the monotonicity is a consequence of Gauss–Bonnet theorem. By means of condition (iii), a careful description of the behavior of u at infinity leads to the computation of the limit of $F(t)$ as $t \to +\infty$, then the comparison of such limit with the limit as $t \to 0$, which follows by the monotonicity (taking care of the critical values by means of Sard's theorem) implies the positive mass inequality, as

$$\lim_{t \to 0^+} F(t) = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} F(t) = 8\,\pi\, m_{\mathrm{ADM}}\,.$$

In his proof of the Riemannian Penrose inequality (RPI), Bray in [13] also obtained the sharp inequality $m_{\mathrm{ADM}} \geq \mathcal{C}$, comparing the ADM mass with the boundary capacity $\mathcal{C}$ of ∂M, mentioned above (see Definition 1.32). Such inequality was later applied by Miao in [64] to generalize the previously discussed Bunting and Masood–ul–Alam's rigidity theorem to the triples (M, g, u) with ∂M only minimal (that is, with zero mean curvature), not requiring that $u = 0$ on the boundary in the system ($\star$), indeed, such condition and the equality in the first line imply that ∂M

is totally geodesic (that is, its second fundamental form is zero). Then, Bray and Miao asked in [15] whether a similar ADM mass/capacity inequality holds in general, for every complete, one–ended asymptotically flat, three–dimensional Riemannian manifold (M, g) with $\mathrm{R} \geq 0$ and compact, connected boundary not necessarily minimal (that is, ∂M could have nonzero mean curvature), in order to get further rigidity results. In such paper, they actually were able to obtain it under the assumption that ∂M has nonnegative *Hawking mass* (another concept of mass, actually local, see [26, 37] for details) and also derived a sharp upper bound for the capacity of the boundary in terms of its area and its *Willmore energy*, given by the functional $\int_{\partial M} \mathrm{H}^2\, d\sigma$. Anyway, we underline that topological assumptions on M are necessary for the conclusion. Xiao [88] then showed the analogues of Bray and Miao's inequalities, replacing the capacity with the *p–harmonic capacity*, with $p \in (0, 3)$. Finally, we mention that another positive answer to the question of Bray and Miao was given (in all dimensions such that the positive mass theorem holds) in the paper by Hirsch and Miao [35], under different hypotheses on the boundary.

In Chapter 4 we reprove with our methods the upper bound on the boundary capacity obtained by Bray and Miao in [15] and Bray's ADM mass/capacity inequality in [13], underlining that the hypotheses in our case and the ones in such papers are actually independent.

Precisely, we will show that in a 3–dimensional, complete, one–ended asymptotically flat manifold (M, g) with minimal, compact and connected boundary, nonnegative scalar curvature and vanishing first Betti number, there hold

$$\mathcal{C} \leq \sqrt{\frac{\mathrm{Area}(\partial M)}{16\pi}} \qquad \text{and} \qquad m_{\mathrm{ADM}} \geq \mathcal{C},$$

where $\mathcal{C} > 0$ is the boundary capacity of ∂M (Theorem 4.2.2). Moreover, if the equality holds in one of these two inequalities, then (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$ (Example 1.4.2).

About the second inequality, we actually obtain the following quantitative estimate

$$m_{\mathrm{ADM}} - \mathcal{C} \geq \frac{\mathcal{C}}{4\pi}\left[\pi - \int_{\partial M} |\nabla u|^2\, d\sigma\right] \geq 0$$

where the function $u \in C^\infty(M)$ is the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases}$$

and if the equality holds in the first or second inequality, then (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$ (see Section 4.1).

To show this, we will consider in this case two functions $G, \widehat{G} : [\mathcal{C}/2, +\infty) \to \mathbb{R}$ defined as follows,

$$G(t) = -\frac{\pi\mathcal{C}^2}{t} + \frac{t}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^4 \int_{\Sigma_t} |\nabla u|^2\, d\sigma\,,$$

$$\widehat{G}(t) = 4\pi t + \frac{t^3}{\mathcal{C}^2}\left(1 + \frac{\mathcal{C}}{2t}\right)^3\left(1 - \frac{3\mathcal{C}}{2t}\right) \int_{\Sigma_t} |\nabla u|^2\, d\sigma - \frac{t^2}{\mathcal{C}}\left(1 + \frac{\mathcal{C}}{2t}\right)^2 \int_{\Sigma_t} |\nabla u|\, \mathrm{H}\, d\sigma\,,$$

where Σ_t denotes the level set $\{u = (1 - \frac{\mathcal{C}}{2t})/(1 + \frac{\mathcal{C}}{2t})\}$, H is the mean curvature of $\Sigma_t \setminus \mathrm{Crit}(u)$ with respect to the ∞–pointing unit normal vector field $\nu = \nabla u/|\nabla u|$ (being $\mathrm{Crit}(u)$ the set of the critical points of u) and σ is the 2–dimensional Hausdorff measure of (M, g).
Similarly to the case of the positive mass inequality, we get that the function $\widehat{G}$ is nondecreasing for t in the intervals such that $(1 - \frac{\mathcal{C}}{2t})/(1 + \frac{\mathcal{C}}{2t})$ is a regular value of u and this in turn implies that the same for the function G. Then, the computation and the consequent comparison of the limits at infinity with the values at $t = \mathcal{C}/2$ of the functions $\widehat{G}$ and G, prove the above inequalities.

At the moment we are not able to improve the results of Chapter 4 in order to drop the assumption of the minimality of the boundary, as in the question/conjecture of Bray and Miao in [15].

A natural future development of our work is extending our computations and results to appropriate p–harmonic functions, in order to use them for monotonicity arguments, having in mind as a main goal a simpler proof of the Riemannian Penrose inequality (RPI). Two immediate and clear obstacles to this line of research are dealing with the structure of the set of the critical values (which, for instance, a priori could have positive Lebesgue measure, due to the possible "bad behavior" of p–harmonic functions) and obtaining a careful description of the decay at infinity of these functions. The first point is related to showing the monotonicity of the appropriate quantities along the level sets, the second one is a key step in getting the limit of such quantities, as the level sets go to infinity.

Chapter 1

Preliminaries

In this chapter we introduce some basic notations and results about Riemannian manifolds and their submanifolds. Then, we discuss the behavior "at infinity" of some harmonic functions and of some relevant quantities in complete asymptotically flat manifolds with one end. Finally, we discuss the so called *ADM mass* of an asymptotically flat manifold, after the names of R. Arnowitt, S. Deser and C. W. Misner, who introduced such concept in [3].

1.1 Riemannian manifolds and curvature

A Riemannian manifold (M, g) of dimension n is an n–dimensional smooth manifold M with a positive definite and symmetric $(0,2)$–tensor field g.
By means of such metric, we can define the lengths of the curves and the distance on M as the infimum of the length of the curves joining two points, giving M a metric space structure (compatible with the original topology).
The Levi–Civita connection ∇, symmetric and compatible with the Riemannian metric g is uniquely defined, allowing a "differential calculus" for tensor fields and other geometric objects on M, whose "deviation" from the usual differential calculus in $\mathbb{R}^n$ is "measured" by the Riemann curvature tensor.
Finally, one has a "natural" canonical measure μ from a given Riemannian metric (see [29, Section 3.4] and [84, Chapter 12]), hence we have a well defined notion of integral for some real functions on M.
For sake of completeness, we are going to briefly recall these notions, however we refer the reader to [47, 72, 74] for a detailed treatment (see [68] for the Lorentzian setting and [46] as a general reference of differential geometry).

Taking a local chart $\big(U, (x^1, \dots, x^n)\big)$ of M, we have the coordinate vector fields and the coordinate 1–forms, respectively given by $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^j\}$, which in each $p \in U$ give a basis of T_pM and T_p^*M.
Then a (local or global) vector field X and and a (local or global) 1–form ω can be written in a coordinate chart as $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_j dx^j$ respectively and a general (k, l)–tensor field T as

$$T = T^{i_1 \dots i_k}_{j^1 \dots j^l} dx^{j^1} \otimes \dots \otimes dx^{j^l} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \,.$$

The metric g of M extended to tensors (of the same type) is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l} \,,$$

where (g_{ij}) is the matrix of the coefficients of g in local coordinates and (g^{ij}) is its inverse. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T,T)}\,.$$

We can associate to each $X \in \Gamma(TM)$ the 1–form $X^\flat$ on M satisfying

$$X^\flat(Y) = g(X,Y), \qquad\qquad X^\flat = g_{ij}X^j dx^i$$

in local coordinates. Similarly, we can also associate to each 1–form ω the vector field $\omega^\sharp$ on M satisfying

$$\omega(Y) = g(\omega^\sharp, Y), \qquad\qquad \omega^\sharp = g^{ij}\omega_j \frac{\partial}{\partial x^i}$$

in local coordinates. In particular, the *gradient* ∇f of a smooth function f is defined as $\nabla f = df^\sharp$.

A subset $E \subseteq M$ is said *measurable* if $\varphi(E \cap U) \subseteq \mathbb{R}^n$ is Lebesgue–measurable for every chart (U,φ) of M. The family $\mathcal{M}(M)$ of the measurable sets of M is a σ–algebra which clearly contains the Borel sets $\mathcal{B}(M)$. Then, one can define uniquely the *canonical (or volume) measure* μ on the measurable space $(M, \mathcal{M}(M))$ by imposing in any chart (U,φ) that $d\mu = \sqrt{\det g_{ij}}\, d\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure in U. The measure μ is then a complete and regular, Radon measure, that is, finite on the compact subsets of M (and positive on any open subset), moreover, we mention that the measures μ and $\mathcal{H}^n$, the Hausdorff n–dimensional measure, coincide.

If Σ is a k–dimensional Riemannian submanifold of (M,g), the canonical measure σ on Σ with the induced Riemannian metric, the k–dimensional Hausdorff measure on (Σ, d_Σ), where d_Σ the distance function of Σ, the k–dimensional Hausdorff measure on (Σ, d_M), where d_M is the distance function of M (restricted to Σ) and finally $\mathcal{H}^k_M \llcorner \Sigma$, the k–dimensional Hausdorff measure of M restricted to Σ, all coincide.

If $f : M \to N$ is a function of class C^k between Riemannian manifolds with $k \geq \max\{1, \dim M - \dim N + 1\}$, then the set of the critical values of f has zero measure zero in N. This is known as *Sard's theorem*.

Another useful result is the so called *coarea formula*, it says that if $f \in C^n(M)$, where n is the dimension of M, then for any measurable function $u : M \to \mathbb{R}$, which is everywhere nonnegative or it is in $L^1(M)$, one has

$$\int_M u|\nabla f|\, d\mu = \int_{\mathbb{R}} dt \int_{f^{-1}(t)} u\, d\sigma\,, \tag{1.1}$$

where σ is the canonical measure of $f^{-1}(t)$ which is a hypersurface for almost every $t \in \mathbb{R}$ (by Sard's theorem). We refer the reader to [67,82] for a detailed discussion of the coarea formula.

The *Levi–Civita connection* is the unique linear connection $\nabla : (X,Y) \in \Gamma(TM) \times \Gamma(TM) \mapsto \nabla_X Y \in \Gamma(TM)$ which is symmetric, i.e.

$$\nabla_X Y - \nabla_Y X = [X,Y]\,,$$

and compatible with the metric g, i.e.

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,.$$

Here $[\cdot,\cdot]$ denotes the so called *Lie bracket*, i.e. $[X,Y]$ is the vector field given by $XY - YX$ for every $X,Y \in \Gamma(TM)$. Taking a local chart $\big(U,(x^1,\dots,x^n)\big)$ of M, then the *Christoffel symbols* Γ^k_{ij} are the smooth functions defined on U by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}\,.$$

They satisfy the following conditions

$$\begin{aligned}\Gamma^k_{ij} &= \Gamma^k_{ji}\,,\\ \frac{\partial g_{ij}}{\partial x^k} &= \Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{li}\end{aligned}$$

for every $i,j,k \in \{1,\dots,n\}$, by virtue of the symmetry and compatibility of ∇ with respect to g, respectively. By a straightforward computation, it follows that

$$\Gamma^l_{ij} = \frac{g^{lk}}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\,.$$

Then, the covariant derivative of a vector field X with respect to $Y \in \Gamma(TM)$ can be written as

$$\nabla_X Y = X^i\left(\frac{\partial Y^k}{\partial x^i} + Y^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}.$$

One can extend uniquely the Levi–Civita connection to every tensor bundle (by defining it in a natural way on $C^\infty(M)$ and by imposing the Leibniz rule and the commutativity with any contraction). Then, in local coordinates the covariant derivative of a tensor $T \in \Gamma(T^k_l M)$ with respect to $X \in \Gamma(TM)$ has coefficients

$$\big(\nabla_X T\big)^{i_1\dots i_r}_{j_1\dots j_s} = X^k\left[\frac{\partial}{\partial x^k}T^{i_1\dots i_r}_{j_1\dots j_s} - \sum_{p=1}^{s}\Gamma^{l_p}_{kj_p}T^{i_1\dots i_r}_{j_1\dots j_{p-1}l_pj_{p+1}\dots j_s} + \sum_{q=1}^{r}\Gamma^{i_q}_{kl_q}T^{i_1\dots i_{q-1}l_qi_{q+1}\dots i_r}_{j_1\dots j_s}\right].$$

It follows that ∇T is a $(k,l+1)$–tensor field for every $T \in \Gamma(T^k_l M)$ and we will write $\nabla^m T$ for the m–th iterated covariant derivative of T.
In particular, the *Hessian* of a smooth function f is $\nabla^2 f = \nabla df$ and it is a symmetric $(0,2)$–tensor field.

The *divergence* $\mathrm{div}X$ of a vector field X is defined by

$$\mathrm{div}X = \mathrm{tr}(Z \mapsto \nabla_Z X) = \nabla_i X^i = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{ij}X^j = \frac{1}{\sqrt{\det g_{kl}}}\partial_i\Big(\sqrt{\det g_{kl}}\,X^i\Big)\,.$$

In particular, the divergence of the gradient of a smooth function f is called *Laplacian* of f and is denoted by Δf.
If X is a vector field with compact support on a Riemannian manifold (M,g) with or without boundary, then the following *divergence theorem* holds

$$\int_M \mathrm{div}X\,d\mu = \begin{cases} 0 & \text{if } M \text{ is without boundary}\\ \displaystyle\int_{\partial M} g(X,\nu)\,d\sigma & \text{if } M \text{ has boundary}\end{cases}$$

where in the second case, ν is the outward–pointing unit normal vector field along ∂M, μ and σ are the canonical measures of the Riemannian manifolds (M,g) and

$(\partial M, \iota^*_{\partial M} g)$, respectively (being $\iota_{\partial M}$ the inclusion map).

Remark 1.1.1. A useful application of the divergence theorem is the following formula/argument, that we will use repeatedly in the whole thesis.
Given a Riemannian manifold (M, g) with or without boundary and a function $f \in C^\infty(M)$, having the boundary ∂M as a level set in the case $\partial M \neq \emptyset$, if $s, S \in \mathbb{R}$ are regular values of f such that $s < S$ and $\{s \leq f \leq S\}$ is compact, then $\{s \leq f \leq S\}$ is a Riemannian submanifold with boundary, given by $\{f = s\} \cup \{f = S\}$ and for every X vector field on $\{s \leq f \leq S\}$ we have

$$\begin{aligned} \int\limits_{\{s<f<S\}} \mathrm{div} X \, d\mu &= \int\limits_{\{s\leq f\leq S\}} \mathrm{div} X \, d\mu = \int\limits_{\{f=S\}} g(X, \nu) \, d\sigma + \int\limits_{\{f=s\}} g(X, \nu) \, d\sigma \\ &= \int\limits_{\{f=S\}} g\Big(X, \frac{\nabla f}{|\nabla f|}\Big) \, d\sigma - \int\limits_{\{f=s\}} g\Big(X, \frac{\nabla f}{|\nabla f|}\Big) \, d\sigma \,, \end{aligned} \tag{1.2}$$

where the first equality follows since the two level sets have zero μ–measure, the second one by applying the divergence theorem to the vector field X on each connected component of $\{s \leq f \leq S\}$ and the last one by observing that the outward–pointing unit normal vector field along $\{f = S\}$ is $\nabla f / |\nabla f|$ while along $\{f = s\}$ is $-\nabla f / |\nabla f|$, being s and S regular values of f. In several occasions, with a small abuse of language we will talk from now on of "applying the divergence theorem to the vector field X on $\{s < f < S\}$", in referring to formula (1.2).

The notion of curvature is given by the *Riemann operator*, which is the $(1, 3)$–tensor field, given by

$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for every $X, Y, Z \in \Gamma(TM)$. In a local chart $\big(U, (x^1, \dots, x^n)\big)$, the smooth functions R^l_{ijk} on U, defined by

$$\mathrm{R}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} = \mathrm{R}^l_{ijk} \frac{\partial}{\partial x^l},$$

can be expressed in terms of the Christoffel symbols as

$$\mathrm{R}^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^h_{jk}\Gamma^l_{ih} - \Gamma^h_{ik}\Gamma^l_{jh}.$$

From the operator of Riemann we can obtain the $(0, 4)$–tensor field, called *curvature (or Riemann) tensor*, defined by

$$\mathrm{Riem}(X, Y, Z, W) = g(\mathrm{R}\,(X, Y)\, Z, W)$$

for every $X, Y, Z, W \in \Gamma(TM)$. In a local chart $\big(U, (x^1, \dots, x^n)\big)$, one has

$$\mathrm{R}_{ijkh} = \mathrm{Riem}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)$$

and

$$\mathrm{R}_{ijkh} = g_{lh} \mathrm{R}^l_{ijk}.$$

The Riemann tensor satisfies the following well known symmetries/properties,

$$\begin{aligned}\mathrm{Riem}(X,Y,Z,W) &= -\mathrm{Riem}(Y,X,Z,W)\ ,\\ \mathrm{Riem}(X,Y,Z,W) &= -\mathrm{Riem}(X,Y,W,Z)\ ,\\ \mathrm{Riem}(X,Y,Z,W) &= \mathrm{Riem}(Z,W,X,Y)\ ,\end{aligned}$$

and

$$\mathrm{Riem}(X,Y,Z,W)+\mathrm{Riem}(Y,Z,X,W)+\mathrm{Riem}(Z,X,Y,W)=0\,, \qquad \text{(\textit{Bianchi's first identity})}$$

$$\nabla_X\mathrm{Riem}(Y,Z,W,T)+\nabla_Y\mathrm{Riem}(Z,X,W,T)+\nabla_Z\mathrm{Riem}(X,Y,W,T)=0\,. \qquad \text{(\textit{Bianchi's second identity})}$$

for every $X,Y,Z,W,T\in\Gamma(TM)$.
The knowledge of the curvature tensor is equivalent of the knowledge of the sectional curvatures of all 2–planes contained in every tangent space, where the *sectional curvature* of a 2–plane $\pi\subseteq T_pM$ with basis $\{v,w\}$, is defined by

$$\mathrm{Sec}_p(\pi)=\mathrm{Sec}_p(v,w)=\frac{\mathrm{Riem}_p(v,w,w,v)}{\|v\|_p^2\|w\|_p^2-g_p^2(v,w)}\,.$$

This definition is well–posed since it is independent of the considered basis of π.
We then have other notions of curvature, one of them is the *Ricci tensor*, which is the symmetric $(0,2)$–tensor field obtained by tanking the "partial" trace (with respect to g) of the operator of Riemann, namely

$$\mathrm{Ric}(X,Y)=\mathrm{tr}\,(Z\mapsto \mathrm{R}(Z,X)\,Y)$$

for every $X,Y\in\Gamma(TM)$. In a local chart $\big(U,(x^1,\dots,x^n)\big)$, the smooth functions Ric_{ij} on U, defined as

$$\mathrm{Ric}_{ij}=\mathrm{Ric}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\,,$$

are given by

$$\mathrm{Ric}_{ij}=\mathrm{R}^k_{kij}=g^{kl}\mathrm{R}_{iklj}\,.$$

From the Ricci tensor we can obtain, by taking its trace, the *scalar curvature*,

$$\mathrm{R}(p)=\mathrm{tr}(\mathrm{Ric}_p)$$

which is then the smooth function on M, given in a local chart $\big(U,(x^1,\dots,x^n)\big)$ by

$$\mathrm{R}=\mathrm{Ric}^i_i=g^{ij}\mathrm{Ric}_{ij}=g^{ij}g^{kl}\mathrm{R}_{iklj}\,.$$

If $e_1,\dots,e_n\in T_p(M)$ is an orthonormal basis, we have

$$\begin{aligned}\mathrm{Ric}_p(v,w)&=\mathrm{R}_p(e_i,v,w,e_i)=\mathrm{R}_p(v,e_i,e_i,w)\,,\\ \mathrm{R}=\mathrm{Ric}_p(e_i,e_i)&=\mathrm{R}_p(e_i,e_j,e_j,e_i)=2\sum_{i<j}\mathrm{Sec}_p(e_i,e_j)\ .\end{aligned}$$

In particular, it follows that, when $n=2$, $\mathrm{R}=2K$, where K is the sectional curvature of (M,g) (also called *Gauss curvature*). A simple relationship between Gauss curvature and topology is provided by the famous Gauss–Bonnet theorem (see [71, Section 4.3]).

Theorem 1.1.2 (Gauss–Bonnet Theorem)**.** *For a closed surface M (compact and without boundary), we have*

$$\int_M K\, d\mu = 2\pi\chi(M)\,,$$

where K is the Gauss curvature and $\chi(M)$ is the Euler characteristic of M, which is a topological invariant.

Since every connected closed surface is diffeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes (see [9, 60]) and

$$\chi(M) = \begin{cases} 2 & \text{if } M \text{ is diffeomorphic to } \mathbb{S}^2 \\ 2-2n & \text{if } M \text{ is diffeomorphic to the connected sum of } n \text{ tori} \\ 2-n & \text{if } M \text{ is diffeomorphic to the connected sum of } n \text{ projective planes} \end{cases}$$

one can deduce $2\pi\chi(M) \leq 4\pi$, for every connected closed surface. This inequality will play a key role in the monotonicity formulas in Chapters 3 and 4 and the lack of an analogue of Gauss–Bonnet theorem is the reason why they do not hold in all dimensions.

1.2 The fundamental equations of submanifolds

Let Σ be an k–dimensional Riemannian submanifold of an n–dimensional Riemannian manifold (M, g). The *codimension of* Σ is the difference $\dim M - \dim \Sigma$, i.e. $n - k$ and the submanifolds of codimension 1 are called *hypersurfaces*. The Riemannian metric induced by (M, g) on Σ is denoted by g^Σ, though we use the notation $g(X, Y)$ for all vector fields X, Y along Σ as we identify $T_p\Sigma$ with its image in T_pM via the differential of the inclusion map in each point $p \in \Sigma$. Moreover, we denote covariant derivatives and curvatures associated with (M, g) in the usual way (∇, R, Riem, etc.) and write (∇^Σ, R^Σ, Riem^Σ, etc.) for those associated with (Σ, g^Σ). Let $T\Sigma$ and $N\Sigma$ be the tangent bundle and the normal bundle of Σ, respectively. At each point $p \in \Sigma$, the "ambient" tangent space T_pM splits as an orthogonal direct sum of $T_p\Sigma$ and $N_p\Sigma$, i.e. $T_pM = T_p\Sigma \oplus N_p\Sigma$. Therefore, we indicate with $v^\top \in T_p\Sigma$ and $v^\perp \in N_p\Sigma$ the tangential projection and the normal projection of every $v \in T_pM$, respectively. For all $X, Y \in \Gamma(T\Sigma)$, we recall that $\nabla_X Y$ is a well–defined vector field along Σ (as for every $p \in \Sigma$ the value of $\nabla_X Y$ at p depends only on the values of Y along some curve γ with $\gamma'(0) = X_p$) and consequently

$$\nabla_X Y = \big(\nabla_X Y\big)^\top + \big(\nabla_X Y\big)^\perp\,,$$

then, considering

$$\begin{aligned} \nabla^\top &: (X, Y) \in \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \big(\nabla_X Y\big)^\top \in \Gamma(T\Sigma) \\ \mathrm{II} &: (X, Y) \in \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to -\big(\nabla_X Y\big)^\perp \in \Gamma(N\Sigma) \end{aligned}$$

we get that $\nabla^\top$ is a linear connection, more precisely, it coincides with ∇^Σ (by the uniqueness of the Levi–Civita connection), thus

$$\nabla_X Y = \nabla^\Sigma_X Y - \mathrm{II}(X, Y)\,.$$

The bilinear map Π is called *second fundamental form* of Σ and it is symmetric, i.e $\Pi(X,Y) = \Pi(Y,X)$ for every $X, Y \in \Gamma(T\Sigma)$, since ∇ and ∇^Σ are torsion–free connections. Likewise, for every $X \in \Gamma(T\Sigma)$ and for each $\xi \in \Gamma(N\Sigma)$, there holds

$$\nabla_X \xi = \big(\nabla_X \xi\big)^\top + \big(\nabla_X \xi\big)^\perp ,$$

then, introducing

$$\begin{aligned} \nabla^\perp &: (X,\xi) \in \Gamma(T\Sigma) \times \Gamma(N\Sigma) \to \big(\nabla_X \xi\big)^\perp \in \Gamma(N\Sigma) \\ S_\xi &: X \in \Gamma(T\Sigma) \to \big(\nabla_X \xi\big)^\top \in \Gamma(T\Sigma) \end{aligned}$$

for every $\xi \in \Gamma(N\Sigma)$, $\nabla^\perp$ is a connection on $N\Sigma$, compatible with g in the sense that

$$Xg(\xi,\eta) = g(\nabla^\perp_X \xi, \eta) + g(\xi, \nabla^\perp_X \eta)$$

for every $X \in \Gamma(T\Sigma)$ and for all $\xi, \eta \in \Gamma(N\Sigma)$, then there holds

$$\nabla_X \xi = S_\xi X + \nabla^\perp_X \xi \,.$$

The operator $\nabla^\perp$ is called *normal connection*, while S_ξ is known as *shape operator of* Σ *in the direction* ξ. The second fundamental form and the shape operator are related as follows,

$$g(S_\xi X, Y) = g\big(\Pi(X,Y), \xi\big)$$

for all $X, Y \in \Gamma(T\Sigma)$ and for every $\xi \in \Gamma(N\Sigma)$, therefore S_ξ is self–adjoint.
We mention that the sign of the second fundamental form and of the shape operator can differ from our choice in the literature. In a way, they "measure" how a submanifold "curves inside" the ambient space.

Finally, we introduce the other fundamental equations for submanifolds in Riemannian geometry; the first one relates the Riemann tensors of M and Σ through the second fundamental form, the others describe the normal projection of the Riemannian operator on particular triples in terms of the second fundamental form and of its derivatives and of the shape operator.
For all vector fields $X, Y, Z, W \in \Gamma(T\Sigma)$ and $\xi, \eta \in \Gamma(N\Sigma)$, the following equations hold,

Gauss equation:

$$\mathrm{Riem}(X,Y,Z,W) = \mathrm{Riem}^\Sigma(X,Y,Z,W) + g\big(\Pi(X,Z), \Pi(Y,W)\big) - g\big(\Pi(X,W), \Pi(Y,Z)\big) ,$$

Codazzi equation:

$$\begin{aligned} \big(\mathrm{R}(X,Y)Z\big)^\perp = {} & \nabla^\perp_Y \Pi\big(X,Z\big) - \Pi(\nabla^\Sigma_Y X, Z) - \Pi(X, \nabla^\Sigma_Y Z) \\ & - \nabla^\perp_X \Pi\big(Y,Z\big) + \Pi(\nabla^\Sigma_X Y, Z) + \Pi(Y, \nabla^\Sigma_X Z) , \end{aligned}$$

Ricci equation:

$$\big(\mathrm{R}(X,Y)\xi\big)^\perp = \mathrm{R}^\perp(X,Y)\xi + \Pi(S_\xi X, Y) - \Pi(S_\xi Y, X) \,.$$

Here, $\mathrm{R}^\perp$ is the curvature tensor of the normal bundle of M defined by

$$\mathrm{R}^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp_Y \xi - \nabla^\perp_Y \nabla^\perp_X \xi - \nabla^\perp_{[X,Y]} \xi \,.$$

We notice that, being any immersed submanifold locally embedded, all that has been said so far can be extended to only–immersed (possibly not embedded) submanifolds.

1.3 Riemannian hypersurfaces

Let Σ be a Riemannian hypersurface of an n–dimensional Riemannian manifold (M, g). In this case, the normal bundle $N\Sigma$ of Σ is a vector bundle of rank 1 on Σ, therefore around each point of Σ we can define, up to a sign, a local unit normal vector field ν along Σ. If there exists a global choice of the unit normal vector field ν, i.e. Σ has a trivial normal bundle, Σ is called *two–sided*. If M is orientable, then Σ is two–sided if and only if Σ is orientable.
Unless we explicitly consider a neighborhood of a point with a unit normal vector field, we assume that Σ is two–sided with a global unit normal vector field ν. Then, $\nabla_X \nu$ is a vector field on Σ, namely

$$\nabla_X \nu = \big(\nabla_X \nu\big)^\top = S_\nu(X)$$

for every $X \in \Gamma(T\Sigma)$. Moreover, the symmetric $(0, 2)$–tensor h on Σ, called *(scalar) second fundamental form of* Σ *with respect to* ν, given by

$$\mathrm{h}(X, Y) = g(\nabla_X \nu, Y) = g(S_\nu(X), Y) = -g(\nu, \nabla_X Y)\,, \tag{1.3}$$

for every $X, Y \in \Gamma(T\Sigma)$, determines uniquely the second fundamental form II. Indeed,

$$\mathrm{II}(X, Y) = g(\mathrm{II}(X, Y), \nu)\nu = g(S_\nu(X), Y)\nu = \mathrm{h}(X, Y)\nu\,.$$

From now on we will denote S_ν by S for a chosen unit normal vector field ν and the *outward* unit normal vector field will be our default choice when Σ is the boundary of a bounded domain.
The *mean curvature* H *of* Σ is the trace (with the induced metric) of the second fundamental form h, or equivalently of the shape operator S. If $\{e_\alpha\}_{\alpha=1}^{n-1}$ is an orthonormal basis of $T_p\Sigma$, for $p \in \Sigma$, then

$$\mathrm{H}_p = \mathrm{h}_p(e_\alpha, e_\alpha) = g_p(\nabla_{e_\alpha} \nu, e_\alpha)\,.$$

In particular, H_p is equal to the sum of the *principal curvatures of* Σ *at* p, which are the eigenvalues of S_p.

If U is a neighborhood of p in M and f is a $C^\infty(U)$–function without critical points, such that $\Sigma \cap U = f^{-1}(c)$, then the second fundamental form and the mean curvature with respect to the unit normal vector field $\nu = \nabla f \,/\, |\nabla f|$ are given respectively by

$$\mathrm{h}(X, Y) = \frac{\nabla df(X, Y)}{|\nabla f|}\,, \tag{1.4}$$

$$\mathrm{H} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right) = \frac{\Delta f}{|\nabla f|} - \frac{\nabla df(\nabla f, \nabla f)}{|\nabla f|^3}\,, \tag{1.5}$$

for every $X, Y \in \Gamma(T\Sigma)$. It is useful to notice that in a small enough neighborhood of each point of a hypersurface, there always exists a function f as above.

If $\mathrm{H}_p = 0$ for all $p \in \Sigma$, then Σ is called *minimal*.

Concerning the basic equations of the submanifolds, in the codimension one case many of the previous formulas are simplified. In particular

$$\nabla_X Y = \nabla^\Sigma_X Y - \mathrm{h}(X,Y)\nu\,, \qquad \textit{(Gauss formula for a hypersurface)}$$
$$\nabla_X \nu = S(X)\,. \qquad \textit{(Weingarten formula for a hypersurface)}$$

The Gauss equation becomes

$$\mathrm{Riem}(X,Y,Z,W) = \mathrm{Riem}^\Sigma(X,Y,Z,W) + \mathrm{h}(X,Z)\mathrm{h}(Y,W) - \mathrm{h}(X,W)\mathrm{h}(Y,Z)\,,$$

from which we obtain, after taking traces,

$$\mathrm{R} = \mathrm{R}^\Sigma + 2\mathrm{Ric}(\nu,\nu) + |\mathrm{h}|^2 - \mathrm{H}^2\,, \tag{1.6}$$

known as *traced Gauss equation*. Indeed, fixed an orthonormal basis $\{e_\alpha\}_{\alpha=1}^{n-1}$ of $T_p\Sigma$ for $p \in \Sigma$, we have

$$\begin{aligned}
\mathrm{R} &= \mathrm{Riem}(e_\alpha, e_\beta, e_\beta, e_\alpha) + 2\mathrm{Riem}(e_\alpha, \nu, \nu, e_\alpha)\\
&= \mathrm{Riem}^\Sigma(e_\alpha, e_\beta, e_\beta, e_\alpha) + \mathrm{h}(e_\alpha, e_\beta)\mathrm{h}(e_\alpha, e_\beta) - \mathrm{h}(e_\alpha, e_\alpha)\mathrm{h}(e_\beta, e_\beta) + 2\mathrm{Ric}(\nu,\nu)\\
&= \mathrm{Riem}^\Sigma(e_\alpha, e_\beta, e_\beta, e_\alpha) + |\mathrm{h}|^2 - \mathrm{H}^2 + 2\mathrm{Ric}(\nu,\nu)\,,
\end{aligned}$$

where the Gauss equation implies the second equality.
The Codazzi equation gets the form

$$\mathrm{R}(X,Y,Z,\nu) = \big(\nabla^\Sigma_Y \mathrm{h}\big)(X,Z) - \big(\nabla^\Sigma_X \mathrm{h}\big)(Y,Z)\,,$$

while the Ricci equation is trivial.

Let (M,g) be a Riemannian manifold and Σ be a two-sided Riemannian hypersurface of M with unit normal ν. We call *tangential gradient* $\nabla^\top f(p)$ of a function of class C^1 defined in a neighborhood U in M of a point $p \in \Sigma$, the projection of $\nabla f(p)$ on $T_p\Sigma$, i.e. $(\nabla f(p))^\top$. It turns out that $\nabla^\top f$ depends only on the restriction of f to $\Sigma \cap U$ and coincides with $\nabla^\Sigma f|_\Sigma$.
We define the *tangential divergence* of a vector field X along Σ as

$$\mathrm{div}^\top X = g(\nabla_{E_\alpha} X, E_\alpha)$$

where $\{E_\alpha\}$ is a local orthonormal frame of Σ, which coincides with the divergence (relative to Σ) of X, if X is a vector field on the hypersurface. It follows

$$\begin{aligned}
\Delta^\Sigma f &= \mathrm{div}^\top \nabla^\top f = g(\nabla_{E_\alpha} \nabla^\top f, E_\alpha)\\
&= g(\nabla_{E_\alpha} \nabla f, E_\alpha) - g(\nabla f, \nu) g(\nabla_{E_\alpha}\nu, E_\alpha)\\
&= \Delta f - \nabla df(\nu,\nu) - g(\nabla f,\nu) g(S(E_\alpha), E_\alpha)\\
&= \Delta f - \nabla df(\nu,\nu) - g(\nabla f,\nu)\mathrm{H}\,,
\end{aligned}$$

where Δ^Σ is the Laplacian of Σ with the induced metric.
Moreover, arguing similarly,

$$\mathrm{div}^\top X = \mathrm{div}^\top X^\top + \mathrm{div}^\top X^\perp = \mathrm{div}^\top X^\top + g(X,\nu) g(\nabla_{E_\alpha}\nu, E_\alpha) = \mathrm{div}^\top X^\top + g(X,\nu)\mathrm{H}\,. \tag{1.7}$$

This last equality and the divergence theorem imply the *tangential divergence formula*

$$\int_\Sigma \operatorname{div}^\top X \, d\sigma = \int_\Sigma g(X,\nu)\mathrm{H}\, d\sigma\,,$$

for every vector field X along Σ with compact support.

We now discuss the first variation of the volume measure and mean curvature of a hypersurface. These computations will be used in the following chapters.
Let Σ be a two–sided Riemannian hypersurface of a Riemannian manifold (M, g) with unit normal ν. We call a *local variation of* Σ a smooth map

$$\Phi : I \times \Sigma \to M$$

such that I is an open interval around $0 \in \mathbb{R}$, the map $\phi_t = \Phi(t,\cdot) : \Sigma \to M$ is an embedding for each $t \in I$ and

$$\Phi(0,p) = p \quad \text{for every } p \in \Sigma\,,$$

finally, there exists a compact set $K \subseteq \Sigma$ such that $\Phi(t,\cdot) : \Sigma \to M$ is the identity outside K, for every $t \in I$. We then define $\Sigma_t = \Phi(\{t\} \times \Sigma) = \phi_t(\Sigma)$, for each fixed $t \in I$.
We assume that the vector field along Σ, called *infinitesimal generator* of the variation, given by

$$X(p) = d\Phi_{(0,p)}\Big(\frac{\partial}{\partial t}\Big) = \frac{\partial\Phi}{\partial t}(0,p)\,,$$

is normal, that is, $X = f\nu$, then we have,

$$\frac{d}{dt}\phi_t^*\sigma^{\Sigma_t}\Big|_{t=0} = f\,\mathrm{H}\sigma \qquad (\textit{Normal first variation of volume measure})$$

$$\frac{\partial}{\partial t}\mathrm{H}_t\Big|_{t=0} = -\Delta^\Sigma f - \big(|\mathrm{h}|^2 + \mathrm{Ric}(\nu,\nu)\big)f$$

(*Normal first variation of mean curvature*)

where $\mathrm{H}_t = \phi_t^*\mathrm{H}^{\Sigma_t}$ and h, H and σ are the second fundamental form, the mean value and volume measure of Σ, respectively.
For every coordinate chart $\big(W,(\theta^1,\dots,\theta^{n-1})\big)$ of Σ, the measures $\phi_t^*\sigma^{\Sigma_t}$ are determined by

$$\phi_t^*\sigma^{\Sigma_t}(A) = \int_A \sqrt{\det g^t_{\alpha\beta}}\, d\sigma$$

for all measurable sets $A \subseteq W$, with $g^t = \phi_t^* g^{\Sigma_t}$ and

$$g^t_{\alpha\beta} = g\bigg(\frac{\partial\phi_t}{\partial\theta^\alpha}, \frac{\partial\phi_t}{\partial\theta^\beta}\bigg)\,,$$

where we have set $\frac{\partial\phi_t}{\partial\theta^\alpha}(p) = d\phi_t\big(\frac{\partial}{\partial\theta^\alpha}\big|_p\big)$ for every $p \in W$. Then, we have,

$$\begin{aligned}
\frac{\partial}{\partial t}\, g^t_{\alpha\beta}\bigg|_{t=0} &= \frac{\partial}{\partial t}\, g\Big(\frac{\partial\phi_t}{\partial\theta^\alpha}, \frac{\partial\phi_t}{\partial\theta^\beta}\Big)\bigg|_{t=0}\\
&= g\Big(\nabla_t \frac{\partial\phi_t}{\partial\theta^\alpha}\bigg|_{t=0}, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_t \frac{\partial\phi_t}{\partial\theta^\beta}\bigg|_{t=0}, \frac{\partial}{\partial\theta^\alpha}\Big)\\
&= g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}} X, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_{\frac{\partial}{\partial\theta^\beta}} X, \frac{\partial}{\partial\theta^\alpha}\Big)\\
&= g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}} f\nu, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_{\frac{\partial}{\partial\theta^\beta}} f\nu, \frac{\partial}{\partial\theta^\alpha}\Big)\\
&= fg\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}} \nu, \frac{\partial}{\partial\theta^\beta}\Big) + fg\Big(\nabla_{\frac{\partial}{\partial\theta^\beta}} \nu, \frac{\partial}{\partial\theta^\alpha}\Big)\\
&= 2f\,\mathrm{h}\Big(\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\Big)\\
&= 2f\,\mathrm{h}_{\alpha\beta}\,,
\end{aligned}$$

as $X = f\nu$ is a normal vector field and we used equality (1.3).
Hence, since

$$\frac{d}{dt}\phi_t^*\sigma^{\Sigma_t}\bigg|_{t=0}(A) = \int_A \frac{\partial}{\partial t}\sqrt{\det g^t_{\alpha\beta}}\,\bigg|_{t=0} d\sigma$$

we compute, by means of the formula

$$\frac{d}{dt}\det A(t) = \det A(t)\,\mathrm{tr}\,[A^{-1}(t)A'(t)]\,,$$

holding for any invertible $n \times n$ squared matrix $A(t)$ dependent on t,

$$\frac{\partial}{\partial t}\sqrt{\det g^t_{\alpha\beta}}\,\bigg|_{t=0} = \frac{1}{2}\sqrt{\det g^t_{\alpha\beta}}\, g_t^{\lambda\mu}\frac{\partial g^t_{\lambda\mu}}{\partial t}\bigg|_{t=0} = f\,g^{\lambda\mu}\mathrm{h}_{\lambda\mu}\sqrt{\det g_{\alpha\beta}} = f\,\mathrm{H}\sqrt{\det g_{\alpha\beta}}\,,$$

which clearly gives the claimed normal variation of the volume measure.

Concerning the normal first variation of the mean curvature, for simplicity, we prove it in the case in which

$$\Phi : \big((-\delta,\delta)\times\Sigma, \widehat{f}^{\,2}(t,p)\,dt\otimes dt + g^\Sigma(t)\big) \longrightarrow (U, g)$$

is an isometry, where U is an open set of M and $\widehat{f}$ is a positive smooth function satisfying $\widehat{f}(0,p) = f(p)$ for every $p \in \Sigma$. In this case, we can consider a coordinate basis $\big\{\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta^1}, \dots, \frac{\partial}{\partial\theta^{n-1}}\big\}$ in a neighborhood V of an arbitrary point $p \in \Sigma$ in M such that $\big\{\frac{\partial}{\partial\theta^1}, \dots, \frac{\partial}{\partial\theta^{n-1}}\big\}$ is a coordinate basis of the tangent space to Σ around $p \in \Sigma$ and we observe that the vector field $\frac{\partial}{\partial t}$ (defined globally as $\nabla t/|\nabla t|^2$ for $t = \pi_2 \circ \Phi^{-1}$) is normal to each Σ_t. Then, with a slight abuse of notation, we denote by f also the function $\widehat{f} \circ \Phi^{-1}$ and by ν the vector field defined by the equality $f\nu = \frac{\partial}{\partial t}$. This latter coincides with ν_t along each Σ_t and in particular $X(p) = \frac{\partial}{\partial t}\big|_p$ for all $p \in \Sigma$. Moreover, we define the $(0,2)$-tensor field h and the smooth function H on V as

$$\begin{aligned}
\mathrm{h}(X,Y) &= g(\nabla_X\nu, Y) \quad \text{for every } X, Y \in \Gamma(TV),\\
\mathrm{H} &= g^{\alpha\beta}\mathrm{h}_{\alpha\beta}\,,
\end{aligned}$$

respectively. The restrictions of h and H to each Σ_t coincide with h^{Σ_t} and H^{Σ_t} (with restriction of h to Σ_t we mean $\iota^*_{\Sigma_t}\mathrm{h}$, being ι_{Σ_t} the inclusion map). With these notations, we have that

$$\frac{\partial \mathrm{H}}{\partial t} = \frac{\partial g^{\alpha\beta}}{\partial t}\mathrm{h}_{\alpha\beta} + g^{\alpha\beta}\frac{\partial \mathrm{h}_{\alpha\beta}}{\partial t}\,.$$

As we already know that

$$\frac{\partial g^{\alpha\beta}}{\partial t} = -2f\,g^{\alpha\lambda}g^{\beta\mu}\mathrm{h}_{\lambda\mu}\,,$$

the contribution of the first term on the right hand side is given by

$$\frac{\partial g^{\alpha\beta}}{\partial t}\mathrm{h}_{\alpha\beta} = -2f\,g^{\alpha\lambda}g^{\beta\mu}\mathrm{h}_{\alpha\beta}\mathrm{h}_{\lambda\mu} = -2f\,|\mathrm{h}|^2\,. \tag{1.8}$$

About the second term, we need to compute $\partial \mathrm{h}_{\alpha\beta}/\partial t$. We have

$$\begin{aligned}\frac{\partial \mathrm{h}_{\alpha\beta}}{\partial t} &= g\Big(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\frac{\partial}{\partial t}\Big)\\ &= g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nabla_{\frac{\partial}{\partial t}}\nu\,, \frac{\partial}{\partial\theta^\beta}\Big) + \mathrm{Riem}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta^\alpha}, \nu, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\frac{\partial}{\partial t}\Big)\,.\end{aligned}$$

Again with a small abuse of notation in the last term of the chain of equalities

$$\nabla_{\frac{\partial}{\partial t}}\nu = \nabla_{\frac{\partial}{\partial t}}\Big(\frac{1}{f}\frac{\partial}{\partial t}\Big) = -\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{\partial}{\partial t} + \frac{1}{f}\Big[\Gamma^t_{tt}\frac{\partial}{\partial t} + \Gamma^\alpha_{tt}\frac{\partial}{\partial\theta^\alpha}\Big] = -g^{\alpha\beta}\frac{\partial f}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha} = -\nabla^\top f\,,$$

where in the third equality we used the fact that

$$\begin{aligned}\Gamma^t_{tt} &= \frac{1}{f}\frac{\partial f}{\partial t}\,,\\ \Gamma^\alpha_{tt} &= -f g^{\alpha\beta}\frac{\partial f}{\partial\theta^\beta}\,.\end{aligned}$$

We then have

$$\frac{\partial \mathrm{h}_{\alpha\beta}}{\partial t} = -g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nabla^\top f\,, \frac{\partial}{\partial\theta^\beta}\Big) - f\,\mathrm{Riem}\Big(\frac{\partial}{\partial\theta^\alpha}, \nu, \nu, \frac{\partial}{\partial\theta^\beta}\Big) + g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\frac{\partial}{\partial t}\Big)\,.$$

Consequently,

$$g^{\alpha\beta}\frac{\partial \mathrm{h}_{\alpha\beta}}{\partial t} = -g^{\alpha\beta}g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nabla^\top f\,, \frac{\partial}{\partial\theta^\beta}\Big) - f\,\mathrm{Ric}(\nu,\nu) + f g^{\alpha\beta}g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\nu\Big)\,, \tag{1.9}$$

being $\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu$ tangential. Then, it follows by equations (1.8) and (1.9) that around any $p \in \Sigma$ there holds

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{H}_t\Big|_{t=0} &= -2f\,|\mathrm{h}|^2 - g^{\alpha\beta}g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nabla^\top f\,, \frac{\partial}{\partial\theta^\beta}\Big) - f\,\mathrm{Ric}(\nu,\nu) + f g^{\alpha\beta}g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\nu\Big)\\ &= -f\,|\mathrm{h}|^2 - \Delta^\Sigma f - f\,\mathrm{Ric}(\nu,\nu)\,,\end{aligned}$$

where we observed that

$$g^{\alpha\beta}g\Big(\nabla_{\frac{\partial}{\partial\theta^\alpha}}\nu\,, \nabla_{\frac{\partial}{\partial\theta^\beta}}\nu\Big) = \mathrm{tr}(S^2) = |\mathrm{h}|^2\,,$$

hence, we obtained the claimed normal variation of the mean curvature in our special case.

1.4 Asymptotically flat manifolds

We discussed in the introduction how asymptotically flat manifolds arise naturally in general relativity and their relevance for the theory, we now see their precise mathematical definition.

In all this thesis, we adopt the Landau big–O/little–o *convention, as follows.*
Let f be a function defined outside a compact set of $\mathbb{R}^n$ and let $\tau \in \mathbb{R}$. We will write:

$f = o(1)$ *when* $|f(x)| \to 0$ *as* $|x| \to +\infty$*;*
$f = O(1)$ *when there exists a constant* $C > 0$ *such that* $|f(x)| \leq C$ *on* $\{|x| \geq R\}$*, for* R *sufficiently large;*
$f = o_k(|x|^{-\tau})$ *if* $f \in C^k$ *and* $|x|^{|\alpha|+\tau}|\partial^\alpha f| = o(1)$ *for every multi–index* α *with* $0 \leq |\alpha| \leq k$*;*
$f = O_k(|x|^{-\tau})$ *if* $f \in C^k$ *and* $|x|^{|\alpha|+\tau}|\partial^\alpha f| = O(1)$ *for every multi–index* α *with* $0 \leq |\alpha| \leq k$*;*
$f = o_\infty(|x|^{-\tau})$ *if* $f \in C^\infty$ *and* $|x|^{|\alpha|+\tau}|\partial^\alpha f| = o(1)$ *for every multi–index* α*;*
$f = O_\infty(|x|^{-\tau})$ *if* $f \in C^\infty$ *and* $|x|^{|\alpha|+\tau}|\partial^\alpha f| = O(1)$ *for every multi–index* α*.*

Obviously, in a manifold these definitions naturally extend to functions defined on any coordinate domain diffeomorphic to $\mathbb{R}^n$ minus a compact set, via the coordinate chart.

Definition 1.4.1. An n–dimensional Riemannian manifold (M, g) (with or without boundary) is said to be *asymptotically flat* if there exists a closed and bounded subset K such that $M \setminus K$ is a finite union of pairwise disjoint open sets $M_1, \dots, M_k$, called *ends*, each of them diffeomorphic to $\mathbb{R}^n$ minus a closed ball $\overline{B}_r(0)$ by a coordinate chart $(x^1, \dots, x^n)$, called *asymptotically flat coordinate chart*, such that, setting $g = g_{ij}\, dx^i \otimes dx^j$, there holds

$$g_{ij} = \delta_{ij} + \sigma_{ij} \qquad \text{with} \qquad \sigma_{ij} = O_2(|x|^{-\tau}), \tag{1.10}$$

for some constant $\tau > \frac{n-2}{2}$ (the *order of decay* of g in the asymptotically flat coordinates chart $(x^1, \dots, x^n)$, briefly *the order*), where δ is the Kronecker delta function.

We will always require in the sequel, without mentioning it, that the scalar curvature of an asymptotically flat manifold (M, g) is nonnegative or belongs to $L^1(M)$. The reason for this choice will be clear in the last section of this chapter, in defining the ADM mass.

In all this thesis we will use the acronym AF for asymptotically flat. Moreover, given an AF coordinate chart $\big(E, (x^1, \dots, x^n)\big)$, we can consider on E the Riemannian metric g_e, defined as $g_e = \delta_{ij} dx^i \otimes dx^j$ and all the relative geometric quantities will be labeled with the letter e.

We remark that we can clearly always suppose, without loss of generality, that $|x| > 1$ for every AF coordinate chart on any end and that such charts can be always smoothly extended to the closure of the coordinate domain. Hence, all the quantities can be expressed in terms of the coordinates on a closed set diffeomorphic to $\mathbb{R}^n$ minus an open ball.

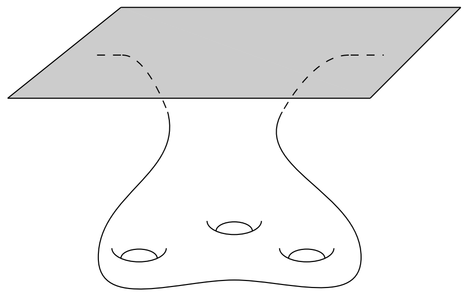

FIGURE 1.1: An asymptotically flat manifold with just one end. By the definition, such manifolds can possibly have a quite complicate topology but all "concentrated" in a bounded domain.

Example 1.4.2. Let m be a real number. The *(exterior spatial) Schwarzschild manifold of mass* m is defined as the Riemannian manifold $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ given by

$$(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)}) = \left(\mathbb{R}^n \setminus \overline{B}_{\left(\frac{|m|}{2}\right)^{\frac{1}{n-2}}}(0), \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathrm{eucl}}\right) \tag{1.11}$$

The Schwarzschild manifold $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ of mass m is clearly asymptotically flat, moreover, the metric g is spherically–symmetric (roughly speaking, it means that in polar coordinates all metric components are independent of the $\mathbb{S}^{n-1}$–factor and there are no mixed terms involving one–forms on $\mathbb{S}^{n-1}$), conformal to the Euclidean metric via a power of a harmonic function and the manifold $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ has zero scalar curvature. The map

$$\Phi : x \in \mathbb{R}^n \setminus \overline{B}_{\left(\frac{|m|}{2}\right)^{\frac{1}{n-2}}}(0) \mapsto \left(r = |x|\left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{2}{n-2}}, \vartheta = \frac{x}{|x|}\right) \in \mathrm{I}_m \times \mathbb{S}^{n-1}\,,$$

where I_m is the open interval $(0, +\infty)$, if $m \leq 0$ and $((2m)^{\frac{1}{n-2}}, +\infty)$ otherwise, is a diffeomorphism with inverse

$$\Phi^{-1} : (r, \vartheta) \in \mathrm{I}_m \times \mathbb{S}^{n-1} \mapsto x = \frac{r}{2^{\frac{2}{n-2}}}\left(1 + \sqrt{1 - \frac{2m}{r^{n-2}}}\right)^{\frac{2}{n-2}} \vartheta \in \mathbb{R}^n \setminus \overline{B}_{\left(\frac{|m|}{2}\right)^{\frac{1}{n-2}}}(0)\,, \tag{1.12}$$

more precisely it is an isometry between the Riemannian manifolds $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ and

$$\left(\mathrm{I}_m \times \mathbb{S}^{n-1}, \frac{dr \otimes dr}{1 - \frac{2m}{r^{n-2}}} + r^2 g_{\mathbb{S}^{n-1}}\right). \tag{1.13}$$

The geometry of $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ depends on the sign of m, indeed, in the case $m > 0$ the metric given in formula (1.11) is smooth on all of $\mathbb{R}^n \setminus \{0\}$, the manifold thus obtained is complete and has two asymptotically flat ends, with a reflection symmetry about the totally geodesic (hence, minimal) sphere at $\left\{|x| = (m/2)^{\frac{1}{n-2}}\right\}$. In the case $m = 0$, $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ can be smoothly extended across the origin and it is isometric to Euclidean space. Finally, if $m < 0$, the metric $g_{\mathrm{Sch}(m)}$ degenerates on ∂M.

From now on, for $m > 0$ with (exterior spatial) Schwarzschild manifold of mass m, *denoted with* $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$, *we mean the manifold*

$$\left(\mathbb{R}^n \setminus B_{\left(\frac{m}{2}\right)^{\frac{1}{n-2}}}(0), \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathrm{eucl}}\right).$$

Let M_i be an end of an AF manifold (M, g) and $(x^1, \dots, x^n)$ an AF coordinate chart for M_i, by definition, one has

$$\sum_{i,j=1}^{n} \left[|\sigma_{ij}(x)| + \sum_{k=1}^{n} |x| \left| \frac{\partial \sigma_{ij}}{\partial x^k} \right| + \sum_{k,l=1}^{n} |x|^2 \left| \frac{\partial^2 \sigma_{ij}}{\partial x^k \partial x^l} \right| \right] \leq \frac{C}{|x|^\tau}, \tag{1.14}$$

on $\overline{M}_i$. Being the metric g_{ij} and its inverse g^{ij} converging to δ_{ij}, as $|x| \to +\infty$, it is easy to see that there exists a constant C (possibly different by the one above), such that

$$C^{-1}\, \delta_{ij} v^i v^j \leq g_{ij}(p) v^i v^j \leq C\, \delta_{ij} v^i v^j\,, \tag{1.15}$$
$$C^{-1}\, \delta^{ij} v^i v^j \leq g^{ij}(p) v^i v^j \leq C\, \delta^{ij} v^i v^j \tag{1.16}$$

for every $p \in \overline{M}_i$ and for all $v = (v^1, \dots, v^n) \in \mathbb{R}^n$.
An easy consequence is the (metric) unboundedness of every end M_i (and also of every subset of M_i of which the image through $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ is unbounded).
This fact leads to *the uniqueness of number of ends of an AF manifold*. Indeed, we consider the sets K, M_1, $\dots$, M_k of Definition 1.4.1. Then

$$M = K \sqcup M_1 \sqcup \cdots \sqcup M_k$$

where K is a closed and bounded set, while M_1, $\dots$, M_k are unbounded, connected, open sets, each of them is diffeomorphic to $\mathbb{R}^n$ minus an open ball via an AF coordinate chart. Assume that

$$M = C \sqcup N_1 \sqcup \cdots \sqcup N_l$$

where C and N_1, $\dots$, N_l satisfy the same conditions of K and M_1, $\dots$, M_k, respectively. For each $i \in \{1, \dots, k\}$, we can consider an unbounded connected subset of M_i diffeomorphic, via an associated AF chart, to $\mathbb{R}^n$ minus a ball of radius sufficiently large, in order that it has empty intersection with C (this is possible since C is bounded), then we denote by $\widetilde{M}_i$ its connected component in $M_i \setminus C$. Notice that the other connected components of $M_i \setminus C$ are bounded. Similarly, for each $j \in \{1, \dots, l\}$, let $\widetilde{N}_j$ be the unbounded connected component of $N_j \setminus K$, obtained analogously. As each $\widetilde{M}_i$ is contained in one and only one $N_{j(i)}$, as $\widetilde{M}_i$ is connected, it is then well–defined a map $i \mapsto j(i)$ which is actually surjective, indeed each $\widetilde{N}_j$ is as well contained in one and only one M_i, by virtue of the fact that $\widetilde{N}_j \cap K = \emptyset$, along with its connectedness. It follows that $l \leq k$ and, by symmetry, the conclusion follows.

From now on we deal only with one–ended AF manifolds, that is, AF manifolds with only one end. Anyway, most of the results that we will present can be extended to each end of any AF manifold. We start with the behaviors in an AF coordinate chart at infinity of some relevant quantities.

Proposition 1.4.3. *Let (M^n, g) be a one–ended AF manifold and let $(x^1, \ldots, x^n)$ be an AF coordinate chart of order τ. It follows that*

$$\Gamma_{ij}^k = O(|x|^{-\tau-1}) \,, \tag{1.17}$$
$$\mathrm{Rm}_{ijk}^l = O(|x|^{-\tau-2}) \,, \tag{1.18}$$
$$\mathrm{Ric}_{ij} = O(|x|^{-\tau-2}) \,, \tag{1.19}$$
$$\mathrm{R} = O(|x|^{-\tau-2}) \,.$$

Proof. These decay orders are immediate consequence of the expressions of these quantities in terms of the derivatives of the metric, as seen in Section 1.1. □

Proposition 1.4.4. *Let (M^n, g) be a one–ended AF manifold and let $(x^1, \ldots, x^n)$ be an AF coordinate chart of order τ. We have*

$$\sqrt{\det g_{ij}} = 1 + \frac{1}{2}\mathrm{tr}_e(\sigma) + O(|x|^{-2\tau}) \,, \tag{1.20}$$
$$g^{ij} = \delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau}) \,, \tag{1.21}$$

where $\mathrm{tr}_e(\sigma) = \sigma_{ii}$.

Proof. We have, by using the Leibniz formula for the determinant,

$$\begin{aligned}
\det g_{ij} &= \sum_{\theta \in \mathrm{P}_n} \prod_{i=1}^n g_{i\theta(i)} = \prod_{i=1}^n g_{ii} + O(|x|^{-2\tau}) \\
&= \prod_{i=1}^n (1 + \sigma_{ii}) + O(|x|^{-2\tau}) = 1 + \mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})
\end{aligned}$$

where P_n is the set of all permutations of $\{1, \ldots, n\}$. Then, the asymptotic expansion in formula (1.20) follows by the Taylor expansion of the square root function. Concerning the second asymptotic expansion, we have

$$g^{ij} = \frac{(-1)^{i+j}}{\det g_{ij}} G_{ji} \,,$$

where G_{ji} is the determinant of the matrix obtained from (g_{ij}), by deleting the row of index j and the column of index i. If $i = j$, then

$$\begin{aligned}
g^{ii} &= \frac{g_{11} \cdots g_{i-1\,i-1}\, g_{i+1\,i+1} \cdots g_{nn} + O(|x|^{-2\tau})}{1 + \mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})} \\
&= \frac{1 + \mathrm{tr}_e(\sigma) - \sigma_{ii} + O(|x|^{-2\tau})}{1 + \mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})} \\
&= \left[1 + \mathrm{tr}_e(\sigma) - \sigma_{ii} + O(|x|^{-2\tau})\right]\left[1 - \mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})\right] \\
&= 1 - \sigma_{ii} + O(|x|^{-2\tau}) \,.
\end{aligned}$$

Instead, if $i \neq j$, for instance $i < j$,

$$
\begin{aligned}
g^{ij} &= \frac{(-1)^{i+j}}{\det g_{ij}} \mathrm{sign}(i, \dots, j-1) \big[g_{11} \cdots g_{i-1\, i-1}\, g_{i+1\, i+1} \cdots g_{j-1\, j-1}\, g_{ij}\, g_{j+1\, j+1} \cdots g_{nn} \big] \\
&\quad + O(|x|^{-2\tau}) \\
&= -\big[\sigma_{ij} + O(|x|^{-2\tau})\big] \big[1 - \mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})\big] \\
&= -\sigma_{ij} + O(|x|^{-2\tau}) \,,
\end{aligned}
$$

where $(i, \dots, j-1)$ is the permutation of $\{1, \dots, n-1\}$ which maps i in $i+1$, $i+1$ in $i+2$ and so on, up to $j-1$ which is mapped in i while the other elements are fixed. This permutation is the product of $j-i-1$ transpositions, therefore its sign is $(-1)^{j-i-1}$. $\square$

Proposition 1.4.5. *Let (M^n, g) be a one–ended AF manifold and $\big(E, (x^1, \dots, x^n)\big)$ an AF coordinate chart of order τ. If $\{\Sigma_l\}_{l \in \mathbb{R}^+}$ is a family of closed, two–sided Riemannian hypersurfaces such that each Σ_l is contained in E and $r_l = \inf\{\, |x(p)| : p \in \Sigma_l \,\} \to +\infty$, then*

$$
\nu^i = \nu_e^i + \frac{1}{2} \sigma_{ks}\, \nu_e^k\, \nu_e^s\, \nu_e^i - \sigma_{ij}\, \nu_e^j + O(|x|^{-2\tau}) \,, \tag{1.22}
$$

$$
\omega_i = \nu_i^\flat = \Omega_i + \frac{1}{2} \sigma_{ks}\, \nu_e^k\, \nu_e^s\, \Omega_i + O(|x|^{-2\tau}) \,,
$$

$$
d\sigma = \left[1 + \frac{1}{2} \varepsilon^{ij}\, \sigma_{ij} + O(|x|^{-2\tau}) \right] d\sigma_e \,, \tag{1.23}
$$

where $\Omega = \nu_e^{\flat_e}$ and $\varepsilon^{ij} = \delta^{ij} - \nu_e^i\, \nu_e^j$.

Proof. Let $p \in M$ be a point of Σ_l. Then, there exist a neighborhood U of p in M and a smooth function $f : U \to \mathbb{R}$ without critical points such that $U \cap \Sigma_l = f^{-1}(c)$ for some $c \in \mathbb{R}$ and $\nu = \nabla f \,/\, |\nabla f|$. Then

$$
\begin{aligned}
\nu^i &= \frac{\big[\delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau})\big] \partial_j f}{\sqrt{\big[\delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau})\big] \partial_i f\, \partial_j f}} = \frac{\big[\delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau})\big] \nu_e^j}{\sqrt{\big[\delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau})\big] \nu_e^i\, \nu_e^j}} \\
&= \Big[\nu_e^i - \sigma_{ij}\, \nu_e^j + O(|x|^{-2\tau}) \Big] \Big[1 + \frac{1}{2} \sigma_{ij}\, \nu_e^i\, \nu_e^j + O(|x|^{-2\tau}) \Big] \\
&= \nu_e^i + \frac{1}{2} \sigma_{rs}\, \nu_e^r\, \nu_e^s\, \nu_e^i - \sigma_{ij}\, \nu_e^j + O(|x|^{-2\tau}) \,,
\end{aligned}
$$

by the asymptotic expansion (1.21). Hence,

$$
\begin{aligned}
\omega_i = g_{ij}\, \nu^j &= (\delta_{ij} + \sigma_{ij}) \Big[\nu_e^j + \frac{1}{2} \sigma_{ks}\, \nu_e^k\, \nu_e^s\, \nu_e^j - \sigma_{js}\, \nu_e^s + O(|x|^{-2\tau}) \Big] \\
&= \Omega_i + \frac{1}{2} \sigma_{ks}\, \nu_e^k\, \nu_e^s\, \Omega_i + O(|x|^{-2\tau}) \,.
\end{aligned}
$$

Concerning the asymptotic expansion (1.23), we notice that

$$\begin{aligned} d\sigma &= \nu d\mu \llcorner \Sigma_l \\ &= \big[g_e(\nu,\nu_e)\nu_e + (\nu - g_e(\nu,\nu_e)\nu_e)\,\big]\sqrt{\det g_{ij}}\, d\mu_e \llcorner \Sigma_l \\ &= \sqrt{\det g_{ij}}\,\delta_{ij}\,\nu^i\,\nu_e^j\,d\sigma_e \\ &= \Big[1 + \frac{1}{2}\,\mathrm{tr}_e(\sigma) + O(|x|^{-2\tau})\Big]\Big[1 - \frac{1}{2}\,\sigma_{ks}\,\nu_e^k\,\nu_e^s + O(|x|^{-2\tau})\Big]\,d\sigma_e \\ &= \Big[1 + \frac{1}{2}\,\varepsilon^{ij}\,\sigma_{ij} + O(|x|^{-2\tau})\Big]\,d\sigma_e\,, \end{aligned}$$

by equations (1.20) and (1.22). □

Proposition 1.4.6 (see [37]). *Let (M^n,g) be a one–ended AF manifold and $\big(E,(x^1,\dots,x^n)\big)$ an AF coordinate chart of order τ. Let $\{\Sigma_l\}_{l\in\mathbb{R}^+}$ be a family of closed, two–sided Riemannian hypersurfaces that are the regular level sets $f^{-1}(l)$ of a smooth function $f : E \to \mathbb{R}$ and satisfy $r_l = \inf\{\,|x(p)| : p\in\Sigma_l\,\} \to +\infty$. Then,*

$$\begin{aligned} \mathrm{H} = \mathrm{H}_e - \varepsilon^{ik}\,\sigma_{ks}\,\varepsilon^{sj}\,\mathrm{h}^e_{ij} + \frac{1}{2}\,\mathrm{H}_e\,\sigma_{ks}\,\nu_e^k\,\nu_e^s - \varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk} + \frac{1}{2}\,\varepsilon^{ij}\,\nu_e^k\,\partial_k\sigma_{ij} + O(|x|^{-1-2\tau}) \\ + O(|x|^{-2\tau}|\mathrm{h}^e|_e)\,, \end{aligned}$$

where H and H_e are the mean curvatures with respect to the unit normals $\nabla f\,/\,|\nabla f|$ and $\nabla^e f\,/\,|\nabla^e f|_e$ associated to g and g_e, respectively, while h^e is defined as $\nabla^e df\,/\,|\nabla^e f|_e$.

Proof. We follow the notations of Proposition 1.4.5 and we define h as $\nabla df\,/\,|\nabla f|$. Then, we observe that

$$|\Omega|^2 = g^{ij}\,\Omega_i\Omega_j = \frac{|\nabla f|^2}{|\nabla^e f|_e^2}\,,$$

therefore, we obtain

$$|\Omega|\,\mathrm{h}_{ij} = \frac{\nabla df_{ij}}{|\nabla^e f|_e} = \mathrm{h}^e_{ij} - \Omega_k\Gamma^k_{ij}\,.$$

Since

$$|\Omega|^2 = g^{ij}\,\nu_e^i\,\nu_e^j = 1 - \sigma_{ij}\,\nu_e^i\,\nu_e^j + O(|x|^{-2\tau})\,,$$

and

$$\begin{aligned} g^{ij} - \nu^i\nu^j &= \delta^{ij} - \sigma_{ij} + O(|x|^{-2\tau}) - \Big[\nu_e^i + \frac{1}{2}\,\sigma_{ks}\,\nu_e^k\,\nu_e^s\,\nu_e^i - \sigma_{ir}\,\nu_e^r + O(|x|^{-2\tau})\Big] \\ &\quad\cdot\Big[\nu_e^j + \frac{1}{2}\,\sigma_{ks}\,\nu_e^k\,\nu_e^s\,\nu_e^j - \sigma_{js}\,\nu_e^s + O(|x|^{-2\tau})\Big] \\ &= \delta^{ij} - \sigma_{ij} - \nu_e^i\,\nu_e^j - \sigma_{ks}\,\nu_e^k\,\nu_e^s\,\nu_e^i\,\nu_e^j + \sigma_{js}\,\nu_e^s\,\nu_e^i + \sigma_{is}\,\nu_e^s\,\nu_e^j + O(|x|^{-2\tau}) \\ &= \varepsilon^{ij} - \varepsilon^{ik}\,\sigma_{ks}\,\varepsilon^{sj} + O(|x|^{-2\tau})\,, \end{aligned}$$

we have

$$\begin{aligned}
\mathrm{H} &= (g^{ij} - \nu^i\nu^j)\mathrm{h}_{ij} \\
&= \Big[\varepsilon^{ij} - \varepsilon^{ik}\sigma_{ks}\,\varepsilon^{sj} + O(|x|^{-2\tau})\Big]\Big[1 + \frac{1}{2}\,\sigma_{ks}\,\nu_e^k\,\nu_e^s + O(|x|^{-2\tau})\Big]\big(\mathrm{h}^e_{ij} - \Omega_k\Gamma^k_{ij}\big) \\
&= \Big[\varepsilon^{ij} - \varepsilon^{ik}\sigma_{ks}\,\varepsilon^{sj} + \frac{1}{2}\,\sigma_{ks}\,\nu_e^k\,\nu_e^s\,\varepsilon^{ij} + O(|x|^{-2\tau})\Big]\big(\mathrm{h}^e_{ij} - \Omega_k\Gamma^k_{ij}\big) \\
&= \mathrm{H}_e - \varepsilon^{ik}\sigma_{ks}\,\varepsilon^{sj}\,\mathrm{h}^e_{ij} + \frac{1}{2}\,\mathrm{H}_e\,\sigma_{ks}\,\nu_e^k\,\nu_e^s - \Omega_k\Gamma^k_{ij}\,\varepsilon^{ij} + O(|x|^{-1-2\tau}) + O(|x|^{-2\tau}|\mathrm{h}^e|_e) \\
&= \mathrm{H}_e - \varepsilon^{ik}\sigma_{ks}\,\varepsilon^{sj}\,\mathrm{h}^e_{ij} + \frac{1}{2}\,\mathrm{H}_e\,\sigma_{ks}\,\nu_e^k\,\nu_e^s - \varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk} + \frac{1}{2}\,\varepsilon^{ij}\,\nu_e^k\,\partial_k\sigma_{ij} + O(|x|^{-1-2\tau}) \\
&\quad + O(|x|^{-2\tau}|\mathrm{h}^e|_e)\,,
\end{aligned}$$

where the last equality follows by

$$\Gamma^k_{ij} = \frac{1}{2}\Big(\partial_i\sigma_{kj} + \partial_j\sigma_{ik} - \partial_k\sigma_{ij}\Big) + O(|x|^{-1-2\tau})\,.$$

□

In a complete one–ended AF manifold (M^n, g), given an AF coordinate chart $(x^1, \dots, x^n)$ and a function f defined outside a compact set of M, we will say that f converges to $l \in \mathbb{R} \cup \{\pm\infty\}$ at infinity, writing $f \to l$ *at* ∞, if for every neighborhood I of l there exists a compact set K such that $f(p) \in I$, for any $p \in M \setminus K$. This is equivalent to have $f(p) \to l$ as $|x(p)| \to +\infty$.

Proposition 1.4.7. *Let (M^n, g) be a complete one–ended AF manifold and $(x^1, \dots, x^n)$ an AF coordinate chart. Then, fixing any point $o \in M$, there holds*

$$\frac{d(p, o)}{|x(p)|} \to 1 \quad \textit{at } \infty\,.$$

Proof. Since $g_{ij} \to \delta_{ij}$ as $|x(p)| \to +\infty$, for every $\varepsilon > 0$ there exists $R > 0$ such that $o \notin \{q \in M \,:\, |x(q)| \geq R\}$ and

$$(1 - \varepsilon)(|x(p)| - R) \leq d(p, S) \leq (1 + \varepsilon)(|x(p)| - R)$$

for every $p \in M$ such that $|x(p)| > R$, with $S = \{q \in M \,:\, |x(q)| = R\}$.
As there holds

$$d(p, S) - d(o, S) \leq d(p, o) \leq d(p, S) + d(o, S) + \mathrm{diam}(S)$$

for all such points p, we have

$$\frac{(1 - \varepsilon)(|x(p)| - R) - d(o, S)}{|x(p)|} \leq \frac{d(p, o)}{|x(p)|} \leq \frac{(1 + \varepsilon)(|x(p)| - R) + d(o, S) + \mathrm{diam}(S)}{|x(p)|}\,,$$

the thesis then follows easily. □

In the rest of this section we present some results about the theory of weighted spaces on AF manifolds (for more details see [43, Appendix A] and the references therein).

Definition 1.4.8. Let (M^n, g) be a complete one–ended AF manifold and let $\big(E, \psi = (x^1, \dots, x^n)\big)$ be an AF coordinate chart (of order τ). We choose a smooth positive function on M which coincides on E with $|x| = \sqrt{\sum_{i=1}^{n}(x^i)^2}$ and with slightly abusing the notation this function will still be denoted with $|x|$.

(1) For every $1 \leq p < +\infty$ and $s \in \mathbb{R}$, the *weighted Lebesgue space* $L^p_s(M)$ is the set of functions f in $L^p_{loc}(M)$ such that

$$\|f\|_{L^p_s(M)} = \left(\int_M |f|^p \, |x|^{-sp-n} \, d\mu \right)^{\frac{1}{p}}$$

is finite.

(2) For every $k \in \mathbb{N}$, $1 \leq p < +\infty$ and $s \in \mathbb{R}$, the *weighted Sobolev space* $W^{k,p}_s(M)$ is the set of functions f in $W^{k,p}_{loc}(M)$ such that

$$\|f\|_{W^{k,p}_s(M)} = \sum_{i=0}^{k} \|\nabla^i f\|_{L^p_{s-i}(M)}$$

is finite.

(3) We let $C^k_s(M)$, with $k \in \mathbb{N}$ and $s \in \mathbb{R}$, to be the set of functions f in $C^k(M)$ such that

$$\|f\|_{C^k_s(M)} = \sum_{i=0}^{k} \sup_M \left(|x|^{i-s} \, |\nabla^i f| \right)$$

is finite.

(4) For every $k \in \mathbb{N}$, $s \in \mathbb{R}$ and $\alpha \in (0,1)$, the *weighted Hölder space* $C^{k,\alpha}_s(M)$ is the set of functions f in $C^{k,\alpha}_{loc}(M)$ such that

$$\begin{aligned} \|f\|_{C^{k,\alpha}_s(M)} = & \sum_{i=0}^{k} \sup_M \left(|x|^{i-s} \, |\nabla^i f| \right) \\ & + \sup_{p \in M} \sup_{\substack{q \in M \\ 0 < d(p,q) < \rho(p)}} \left[\left(\min\{|x(p)|, |x(q)|\} \right)^{k+\alpha-s} \frac{|\nabla^k f(p) - \nabla^k f(q)|}{d(p,q)^\alpha} \right] \end{aligned}$$

is finite, where $\rho(p)$ is the injectivity radius of p and $\nabla^k f(p) - \nabla^k f(q)$ denotes the difference of $\nabla^k f(p)$ with the parallel transport of $\nabla^k f(q)$ in p along the minimal geodesic joining p and q.

All these weighted spaces are Banach spaces and do not depend on the AF coordinate chart, by virtue of Proposition 1.4.7, in particular, every change of the AF coordinate chart produces equivalent norms.
Moreover, if $f \in C^2(M)$, then

$$\sum_{k=1}^{n} |x| \left| \frac{\partial f}{\partial x^k} \right| + \sum_{k,l=1}^{n} |x|^2 \left| \frac{\partial^2 f}{\partial x^k \partial x^l} \right| = O(|x|^{-s})$$

if and only if

$$|x| \, |\nabla f| + |x|^2 \, |\nabla df| = O(|x|^{-s}),$$

for every $s > 0$, by inequalities (1.16).

The importance of these spaces lies in the fact that they share analogues of many of the global elliptic regularity results for compact manifolds, which in general are not true on noncompact manifolds without considering weights. In the literature there are several ways of defining such spaces, we followed Bartnik [7] with the above definition.

The index s reflects the order of growth of the functions with respect to $|x|$, at infinity. This is stated in the following lemma, with some useful continuous embeddings.

Lemma 1.4.9 (Section 1 [17], Lemma 9.1 [48]). *With the notation and conventions of Definition 1.4.8,*

(1) $C^{k+1}_s(M) \subseteq C^{k,\alpha}_s(M)$ *and* $C^{k,\alpha}_{s_1}(M) \subseteq C^{k,\alpha}_{s_2}(M)$, *if* $s_1 \leq s_2$.

(2) *If* $l, k \in \mathbb{N}$, $p \in (1, +\infty)$ *and* $\alpha \in (0, 1)$ *satisfy the inequality* $l - k - \alpha > n/p$, *then for every* $\varepsilon > 0$ *there holds* $C^{l,\alpha}_{s-\varepsilon}(M) \subseteq W^{l,p}_s(M) \subseteq C^{k,\alpha}_s(M)$. *In particular, if* $f \in W^{k,p}_s(M)$ *with* $k > \frac{n}{p}$, *then* $f = O(|x|^s)$.

By working in these spaces, we satisfy the rough (unfortunately wrong) intuition that if a function on $\mathbb{R}^n$ decays at infinity with a certain order, then its Laplacian decays two orders faster, with respect to $|x|$. In this spirit, if we want to to solve the Poisson equation $\Delta v = f$, we look for a solution v that decays two orders slower than f.

Then, we state the following result about the Fredholm properties of the Laplacian of a complete one–ended AF manifold (M^n, g) on weighted spaces.

Theorem 1.4.10 (Theorem A.40 [43], Theorem 9.2 [48]). *Let* (M^n, g) *be a complete one–ended AF manifold and* Δ *denotes the Laplacian of* (M, g).

(1) *Let* $p > 1$ *and* $s \in \mathbb{R}$, *there exists a constant* C *such that*

$$\|u\|_{W^{2,p}_s(M)} \leq C \left(\|u\|_{L^p_s(M)} + \|\Delta u\|_{L^p_{s-2}(M)} \right)$$

for every $u \in W^{2,p}_s(M)$.

(2) *Let* s *be any real number not belonging to* exceptional set

$$\Lambda = \mathbb{Z} \setminus \{0, -1, \ldots, 3 - n\} .$$

Then, for any $p > 1$, *the map*

$$\Delta : W^{2,p}_s(M) \mapsto W^{0,p}_{s-2}(M)$$

is Fredholm (i.e. is a bounded linear operator between two Banach spaces with finite-dimensional kernel and cokernel). More precisely,

$$\dim\ker\Delta \;=\begin{cases}\sum_{k=0}^{[s]}\left[\binom{n+k-1}{n-1}-\binom{n+k-3}{n-1}\right] & \text{for } s\geq 0\\ 0 & \text{for } s<0\end{cases}$$

$$\dim\operatorname{coker}\Delta \;=\begin{cases}\sum_{k=0}^{[2-n-s]}\left[\binom{n+k-1}{n-1}-\binom{n+k-3}{n-1}\right] & \text{for } s\leq 2-n\\ 0 & \text{for } s>2-n\end{cases}$$

In particular, if $2-n<s<0$, the operator Δ is an isomorphism between $W^{2,p}_s(M)$ and $W^{0,p}_{s-2}(M)$.

(3) *If $u\in C^0_s(M)$ and $\Delta u\in C^{0,\alpha}_{s-2}(M)$, then $u\in C^{2,\alpha}_s(M)$ and*

$$\|u\|_{C^{2,\alpha}_s(M)}\leq C\big(\|u\|_{C^0_s(M)}+\|\Delta u\|_{C^{0,\alpha}_{s-2}(M)}\big)\,.$$

(4) *If $2-n<s<0$, $h\in C^{0,\alpha}_{s'}(M)$ for some $s'<-2$ and the operator $\Delta+h : C^{2,\alpha}_s(M)\to C^{0,\alpha}_{s-2}(M)$ is one–to–one, then it is an isomorphism.*

A key step in our line to obtain geometric inequalities is knowing the behavior at infinity, in an AF coordinate chart, of certain harmonic functions. We will need the following theorem.

Theorem 1.4.11. *Let Δ denote the Euclidean Laplacian of $\mathbb{R}^n$ and let $s\in\mathbb{N}$ and $s'\in\mathbb{R}$ be such that $s\geq s'\geq n-2$. If $f\in C^\infty(\mathbb{R}^n)$ is a real function with $f=O(|x|^{-s'})$ and $\Delta f=\rho=O_1(|x|^{-s-2-\alpha})$, for some $0<\alpha<1$, then, letting $\widehat{x}=x/|x|$ and $\widehat{y}=y/|y|$, we have*

$$f(x)=-\sum_{k=\lceil s'-n+2\rceil}^{s-n+2}\frac{1}{n(n-2)\omega_n}\,\frac{1}{|x|^{n-2+k}}\int_{\mathbb{R}^n}\rho(y)\,|y|^k\,P_k(\widehat{x}\cdot\widehat{y})\,dy+O_2(|x|^{-s-\alpha})\,,$$

outside a closed ball centered at the origin, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, $\lceil\cdot\rceil$ is the ceiling function ($\lceil x\rceil$ is the least integer greater than or equal to x) and $P_k=P_k^{(n-2)/2}$ are the ultraspherical *(or* Gegenbauer*) polynomials (see [1, Chap. 22] and [83, Chap. IV], for more details), given by*

$$P_k^{(n-2)/2}(t)=\sum_{l=0}^{[\frac{k}{2}]}(-1)^l\,\frac{\Gamma\left(k-l+\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\,l!(k-2l)!}\,(2t)^{k-2l}\,. \tag{1.24}$$

In particular, each function under the summation sign is a harmonic function.

Proof. By a classical representation formula and since $\rho=O(|x|^{-s-2-\alpha})$, the function

$$w(x)=-\frac{1}{n(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{\rho(y)}{|x-y|^{n-2}}\,dy\,, \tag{1.25}$$

is well–defined, of class C^2 and satisfies $\Delta w = \rho$ on $\mathbb{R}^n$. We assume $|x| \geq 2$ and we split $\mathbb{R}^n$ in the three separate pieces

$$B_{\frac{|x|}{2}}(0), \qquad B_{\frac{|x|}{2}}(x), \qquad \mathbb{R}^n \setminus \big(B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x)\big).$$

We recall that in the following computations C will denote a constant which may vary from line to line and it is independent of x. For any $y \in B_{\frac{|x|}{2}}(x)$, we have

$$\begin{aligned} \Bigg| \int\limits_{B_{\frac{|x|}{2}}(x)} \frac{\rho(y)}{|x-y|^{n-2}}\, dy \Bigg| &\leq \frac{C}{|x|^{s+2+\alpha}} \int\limits_{B_{\frac{|x|}{2}}(x)} \frac{1}{|x-y|^{n-2}}\, dy \\ &\leq C|x|^{-s-\alpha}. \end{aligned} \tag{1.26}$$

Since $|y-x| \geq \frac{|x|}{2}$ for $y \notin B_{\frac{|x|}{2}}(x)$, we have

$$\begin{aligned} \Bigg| \int\limits_{\mathbb{R}^n \setminus \left(B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x)\right)} \frac{\rho(y)}{|x-y|^{n-2}}\, dy \Bigg| &\leq C \frac{1}{|x|^{n-2}} \int\limits_{\mathbb{R}^n \setminus B_{\frac{|x|}{2}}(0)} |\rho(y)|\, dy \\ &\leq C|x|^{-s-\alpha}. \end{aligned} \tag{1.27}$$

In the region $B_{\frac{|x|}{2}}(0)$, where $\frac{|y|}{|x|} < \frac{1}{2}$, we can expand the fundamental solution $|x-y|^{2-n}$ as a power series in $\frac{|y|}{|x|}$ as follows,

$$|x-y|^{2-n} = \sum_{k=0}^{\infty} \frac{1}{|x|^{n-2}} \left(\frac{|y|}{|x|}\right)^k P_k(\widehat{x} \cdot \widehat{y})$$

for the polynomials P_k (see [43, Appendix A]), noticing that every function in the sum is harmonic in both variables x and y. We then consider

$$\sum_{k=0}^{s-n+2} \frac{1}{|x|^{n-2+k}} \int\limits_{\mathbb{R}^n} \rho(y)\, |y|^k\, P_k(\widehat{x} \cdot \widehat{y})\, dy$$

and we observe that

$$\Bigg| \sum_{k=0}^{s-n+2} \frac{1}{|x|^{n-2+k}} \int\limits_{\mathbb{R}^n \setminus B_{\frac{|x|}{2}}(0)} \rho(y)\, |y|^k\, P_k(\widehat{x} \cdot \widehat{y})\, dy \Bigg| \leq C|x|^{-s-\alpha}$$

$$\Bigg| \int\limits_{B_{\frac{|x|}{2}}(0)} \frac{\rho(y)}{|x-y|^{n-2}}\, dy - \sum_{k=0}^{s-n+2} \frac{1}{|x|^{n-2+k}} \int\limits_{B_{\frac{|x|}{2}}(0)} \rho(y)\, |y|^k\, P_k(\widehat{x} \cdot \widehat{y})\, dy \Bigg| \leq C|x|^{-s-\alpha},$$

thus, we can conclude that also

$$\Bigg| \int\limits_{\mathbb{R}^n} \frac{\rho(y)}{|x-y|^{n-2}}\, dy - \sum_{k=0}^{s-n+2} \frac{1}{|x|^{n-2+k}} \int\limits_{B_{\frac{|x|}{2}}(0)} \rho(y)\, |y|^k\, P_k(\widehat{x} \cdot \widehat{y})\, dy \Bigg| \leq C|x|^{-s-\alpha}, \tag{1.28}$$

by inequalities (1.26) and (1.27).
Indeed, we have

$$\begin{aligned}\Bigg|\sum_{k=0}^{s-n+2}\frac{1}{|x|^{n-2+k}}\int\limits_{\mathbb{R}^n\setminus B_{\frac{|x|}{2}}(0)}\rho(y)\,|y|^k\,P_k(\widehat{x}\cdot\widehat{y})\,dy\Bigg| &\leq C\sum_{k=0}^{s-n+2}\frac{1}{|x|^{n-2+k}}\int\limits_{\mathbb{R}^n\setminus B_{\frac{|x|}{2}}(0)}|\rho(y)|\,|y|^k\,dy\\ &\leq C\sum_{k=0}^{s-n+2}\frac{1}{|x|^{n-2+k}}\int\limits_{\mathbb{R}^n\setminus B_{\frac{|x|}{2}}(0)}\frac{1}{|y|^{s+2+\alpha-k}}\,dy\\ &\leq\frac{C}{|x|^{s+\alpha}}\,,\end{aligned}$$

as $P_k(\widehat{x}\cdot\widehat{y})$ are clearly bounded quantities.
About the second inequality, (following [62]) if $y\in B_{\frac{|x|}{2}}(0)$, there holds

$$\begin{aligned}\Bigg|\frac{1}{|x-y|^{n-2}}-\frac{1}{|x|^{n-2}}\sum_{k=0}^{s-n+2}\left(\frac{|y|}{|x|}\right)^k P_k(\widehat{x}\cdot\widehat{y})\Bigg| &= \Bigg|\frac{1}{|x|^{n-2}}\sum_{k=s-n+3}^{\infty}\left(\frac{|y|}{|x|}\right)^k P_k(\widehat{x}\cdot\widehat{y})\Bigg|\\ &\leq C\,\frac{1}{|x|^{n-2}}\left(\frac{|y|}{|x|}\right)^{s-n+3},\end{aligned}$$

since $\frac{|y|}{|x|}<\frac{1}{2}$, hence, we get

$$\begin{aligned}\Bigg|\int\limits_{B_{\frac{|x|}{2}}(0)}\frac{\rho(y)}{|x-y|^{n-2}}\,dy-\sum_{k=0}^{s-n+2}\frac{1}{|x|^{n-2+k}}\int\limits_{B_{\frac{|x|}{2}}(0)}\rho(y)\,|y|^k\,P_k(\widehat{x}\cdot\widehat{y})\,dy\Bigg|\\ \leq\frac{C}{|x|^{s+1}}\int\limits_{B_{\frac{|x|}{2}}(0)}|\rho(y)|\,|y|^{s-n+3}\,dy\\ \leq\frac{C}{|x|^{s+\alpha}}\,.\end{aligned}$$

Then, recalling equation (1.25), by inequality (1.28) we get

$$n(n-2)\omega_n\,w(x)=-\sum_{k=0}^{s-n+2}\frac{1}{|x|^{n-2+k}}\int\limits_{\mathbb{R}^n}\rho(y)\,|y|^k\,P_k(\widehat{x}\cdot\widehat{y})\,dy+h(x)\,,$$

where the first summand is a harmonic function and

$$h=O(|x|^{-s-\alpha})\,.$$

Since the function $f-w$ is harmonic and bounded on $\mathbb{R}^n$, then it is constant by Liouville's theorem and this constant must be zero by the behavior at infinity of f and w, then f and w coincide, hence,

$$\Delta h/\left(n(n-2)\omega_n\right)=\Delta w=\Delta f=\rho\,.$$

The higher order estimates of h follow by point (3) of Theorem 1.4.10, since $\rho = O_1(|x|^{-s-2-\alpha})$, along with point (1) of Lemma 1.4.9. Finally, the assumption about the behavior at infinity of f implies that in the sum remain those harmonic functions which tends to zero fast enough. □

A consequence is the following result.

Proposition 1.4.12. *With the notation and conventions of Definition 1.4.8, if v is a smooth positive harmonic function outside a compact subset of M such that $v \to 0$ at ∞, then there exists a constant C such that*

$$v = \frac{C}{|x|^{n-2}} + O_2(|x|^{2-n-\alpha})$$

for every $\alpha \in (0, \min\{1, \tau\})$.

Proof. Without loss of generality, we can assume that v is defined and harmonic on the domain E of the AF coordinate chart. We observe that

$$\Delta f = \delta^{ij}\partial_i\partial_j f + a^{ij}\partial_i\partial_j f + b^j\,\partial_j f\ , \tag{1.29}$$

for every $f \in C^\infty(E)$, where $a^{ij} = g^{ij} - \delta^{ij} = O(|x|^{-\tau})$ and $b^j = -g^{kl}\,\Gamma^j_{kl} = O_1(|x|^{-1-\tau})$, by estimates (1.16) and (1.17).
For a fixed $0 < \varepsilon < \tau$ and for an arbitrary $a > 0$ to be chosen later, we consider the function

$$\phi_a = a\left(\frac{1}{|x|^{n-2}} - \frac{1}{|x|^{n-2+\varepsilon}}\right).$$

By direct computations, one can check that

$$\begin{aligned}\partial_j\phi_a &= -a\left(\frac{n-2}{|x|^n} - \frac{n-2+\varepsilon}{|x|^{n+\varepsilon}}\right)x^j\\ \partial_i\partial_j\phi_a &= a\left[\frac{n\,(n-2)}{|x|^{n+2}} - \frac{(n+\varepsilon)(n-2+\varepsilon)}{|x|^{n+2+\varepsilon}}\right]x^i x^j - a\left[\frac{n-2}{|x|^n} - \frac{n-2+\varepsilon}{|x|^{n+\varepsilon}}\right]\delta_{ij}\,,\end{aligned}$$

in particular, $\phi_a = O_\infty(|x|^{2-n})$. Consequently, by equality (1.29) along with the behavior at infinity of the functions a^{ij} and b^j, we get

$$\Delta\phi_a = a\Big[-(n-2+\varepsilon)\varepsilon|x|^{-n-\varepsilon} + O(|x|^{-n-\tau})\Big]\,,$$

hence, there exists $R_2 > 1$, independent of a such that $\Delta\phi_a < 0$ in $\{|x| \geq R_2\}$, for every $a > 0$. We now choose $a > 0$ (since v is positive) such that

$$\phi_a > \max_{\{|x|=R_2\}} v \quad \text{on } \{|x| = R_2\}\,,$$

then by the maximum principle for elliptic equations [21, Section 6.4], we have

$$v \leq \phi_a \quad \text{in} \quad \{|x| \geq R_2\}\,. \tag{1.30}$$

Notice that from (1.30) one gets in particular that $v = O(|x|^{2-n})$. Then, point (3) of Theorem 1.4.10, applied to a smooth extension on all M of v, implies that $v = O_2(|x|^{2-n})$. Consequently, we also have $\Delta_e v = O(|x|^{-n-\tau})$. Hence, we can apply the first part of the proof of Theorem 1.4.11 to a smooth extension $\widehat{v}$ on all $\mathbb{R}^n$ of $\psi_* v$, to conclude that

$$\begin{aligned}\widehat{v} &= -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}\int\limits_{\mathbb{R}^n}\rho(y)\,P_0(\widehat{x}\cdot\widehat{y})\,dy+\widehat{h}\\ &= -\frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}\int\limits_{\mathbb{R}^n}\rho(y)\,dy+\widehat{h}\\ &= \frac{C}{|x|^{n-2}}+\widehat{h}\end{aligned}$$

outside a closed ball of $\mathbb{R}^n$ with center 0, with $\widehat{h}=O(|x|^{2-n-\alpha})$ for every $\alpha\in(0,\min\{1,\tau\})$. We underline that in this computation we used the fact that $P_0(t)=1$, by formula (1.24). It follows that

$$\Delta_{\psi_* g}\widehat{h}=O_1(|x|^{-n-\alpha})$$

as $\Delta_{\psi_* g}\widehat{v}=0$ outside a closed ball of $\mathbb{R}^n$ of radius large enough centered at the origin. By Shauder's interior estimates (see [28, Lemma 6.20]), we then have $\widehat{h}=O_2(|x|^{2-n-\alpha})$, thus, the statement of the proposition. □

A consequence of this proposition is the following result, which will be used in Chapters 2 and 4.

Corollary 1.4.13. *Let (M^n,g) be a complete one–ended AF manifold with compact boundary and let $\big(E,(x^1,\dots,x^n)\big)$ be an AF coordinate chart of order τ. If $v\in C^\infty(M)$ is the solution of the Dirichlet problem*

$$\begin{cases}\Delta v=0 & \text{in } M\\ v=1 & \text{on } \partial M\\ v\to 0 & \text{at } \infty\end{cases}$$

(the existence of such v can be obtained following [57], together with a "barrier" argument to ensure the convergence to zero at ∞, the uniqueness follows by the maximum principle, see [71] for instance) then,

$$v=\frac{\mathcal{C}}{|x|^{n-2}}+O_2(|x|^{2-n-\alpha}) \tag{1.31}$$

where

$$\mathcal{C}=\frac{1}{(n-2)\,|\mathbb{S}^{n-1}|}\int\limits_{\partial M}|\nabla v|\,d\sigma=\frac{1}{(n-2)\,|\mathbb{S}^{n-1}|}\int\limits_{M}|\nabla v|^2\,d\mu \tag{1.32}$$

and any $\alpha\in(0,\min\{1,\tau\})$.
The function v is called boundary capacity potential, *while the last integral in formula* (1.32) boundary capacity *of ∂M in (M,g) (see [56]).*

Proof. We notice that $0<v<1$ on $M\setminus\partial M$, by the strong maximum principle [71, Chapter 9], therefore, the behavior of v at infinity is given by the previous proposition. Furthermore, by the Hopf lemma [21, Section 6.4], $|\nabla v|>0$ on ∂M. In particular, 1 is a regular value of v, thus, the unit normal vector field along ∂M can be expressed in terms of v. Let K be the compact set which is the complement of E

in M, i.e. $M \setminus E$. Applying the divergence theorem, we obtain that

$$0 = \int\limits_{K\cup\{|x|<R\}} \Delta v\, d\mu = \int\limits_{\partial M} g\Big(\nabla v, \frac{\nabla v}{|\nabla v|}\Big)\, d\sigma + \int\limits_{\{|x|=R\}} g\Big(\nabla v, \frac{\nabla |x|}{|\nabla |x||}\Big)\, d\sigma\,,$$

hence,

$$\int\limits_{\partial M} |\nabla v|\, d\sigma = \int\limits_{\partial M} g\Big(\nabla v, \frac{\nabla v}{|\nabla v|}\Big)\, d\sigma = -\int\limits_{\{|x|=R\}} g\Big(\nabla v, \frac{\nabla |x|}{|\nabla |x||}\Big)\, d\sigma\,.$$

Now, thanks to formulas (1.14), (1.21), (1.22), (1.23) and (1.31), keeping in mind that $|\partial_i v| \leq C|x|^{1-n}$, we have

$$\begin{aligned}\int\limits_{\{|x|=R\}} g\Big(\nabla v, \frac{\nabla |x|}{|\nabla |x||}\Big)\, d\sigma &= \int\limits_{\{|x|=R\}} g_e\Big(\nabla^e v, \frac{x^i}{|x|}\,\partial_i\Big)\, d\sigma_e + O(R^{-\tau})\\ &= \int\limits_{\{|x|=R\}} g_e\Big(\nabla^e \frac{\mathcal{C}}{|x|^{n-2}}, \frac{x^i}{|x|}\,\partial_i\Big)\, d\sigma_e + O(R^{-\alpha})\\ &= -\mathcal{C}(n-2)|\mathbb{S}^{n-1}| + O(R^{-\alpha}).\end{aligned} \tag{1.33}$$

Then,

$$\int\limits_{\partial M} |\nabla v|\, d\sigma = -\lim_{R\to+\infty} \int\limits_{\{|x|=R\}} g\Big(\nabla v, \frac{\nabla |x|}{|\nabla |x||}\Big)\, d\sigma = \mathcal{C}(n-2)|\mathbb{S}^{n-1}|\,,$$

giving the first identity in formula (1.32).
Concerning the second identity, the divergence theorem implies

$$\begin{aligned}\int\limits_{K\cup\{|x|<R\}} |\nabla v|^2\, d\mu &= \int\limits_{K\cup\{|x|<R\}} \mathrm{div}(v\,\nabla v)\, d\mu\\ &= \int\limits_{\partial M} g\Big(\nabla v, \frac{\nabla v}{|\nabla v|}\Big)\, d\sigma + \int\limits_{\{|x|=R\}} v\, g\Big(\nabla v, \frac{\nabla |x|}{|\nabla |x||}\Big)\, d\sigma\,,\end{aligned}$$

since v is identically 1 on ∂M. Then, for $R \to +\infty$ we obtain the desired equality, as the last term goes to zero, as v. □

Finally, by means of Theorem 1.4.11, we describe the behavior of certain harmonic functions in the end (of a one–ended AF manifold) if it is isometric to the end of a Schwarzschild manifold (see Example 1.4.2) of arbitrary mass m (in particular, the parameter m is a real number).

Proposition 1.4.14. *Let (M^n, g) be a complete, one–ended AF manifold, possibly with boundary. Assume that there exists a distinguished AF coordinate chart $\big(E, \psi = (x^1, \ldots, x^n)\big)$ in which the metric g can be expressed as*

$$g = \Big(1 + \frac{m}{2|x|^{n-2}}\Big)^{\frac{4}{n-2}} \delta_{ij}\, dx^i \otimes dx^j\,.$$

If v is a positive harmonic function outside a compact subset of M with $v \to 0$ at ∞, then there exists a constant C such that

$$v = \frac{C}{|x|^{n-2}} + \sum_{k=1}^{n-2} \frac{\phi_k(x/|x|)}{|x|^{n-2+k}} - \left(1 + \frac{m}{2|x|^{n-2}}\right)^{-1} \frac{mC}{2|x|^{2(n-2)}} + O_2(|x|^{4-2n-\alpha})$$

for every $\alpha \in (0,1)$. Notice that each function under the summation sign is an Euclidean harmonic function.

Proof. Considering the push–forwards via ψ of the function v and of the metric g to $\mathbb{R}^n$ minus a ball, still denoted by v and g, respectively, we know that

$$v = \frac{C}{|x|^{n-2}} + h, \text{ and } h = O_2\big(|x|^{2-n-\alpha}\big)\,,$$

with $0 < \alpha < 1$. We set $\mathcal{U} = 1 + \frac{m}{2|x|^{n-2}}$. Then, we have

$$0 = \Delta v = \mathcal{U}^{-\frac{2(n-1)}{n-2}} \left[\Delta_e v + 2\mathcal{U}^{-1} g_e(\nabla^e \mathcal{U}, \nabla^e v)\right]\,,$$

hence,

$$\Delta_e v + 2\mathcal{U}^{-1} g_e(\nabla^e \mathcal{U}, \nabla^e v) = 0\,,$$

which implies

$$\Delta_e h - \mathcal{U}^{-1} \frac{(n-2)m}{|x|^{n-1}}\, g_e\left(\nabla^e h, \frac{x}{|x|}\right) = -C\mathcal{U}^{-1} \frac{m(n-2)^2}{|x|^{2(n-1)}}.$$

Then,

$$h = -\,\mathcal{U}^{-1} \frac{mC}{2|x|^{2(n-2)}} + f\,,$$

where the function f satisfies

$$\begin{aligned} \Delta_e f &= \mathcal{U}^{-1} \frac{(n-2)m}{|x|^{n-1}}\, g_e\left(\nabla^e f, \frac{x}{|x|}\right)\,, \\ f &= O_2\big(|x|^{2-n-\alpha}\big)\,. \end{aligned} \tag{1.34}$$

The claim then follows by applying Theorem 1.4.11 to a smooth extension of f on all $\mathbb{R}^n$, once noticed that for such an extension the right hand side of equation (1.34) is $O_1\big(|x|^{-2(n-1)-\alpha}\big)$. ☐

Corollary 1.4.15. *Under the assumptions of the previous proposition, in dimension 3, there exists a constant C such that*

$$v = \frac{C}{|x|} + \frac{1}{2|x|^2}\,\left(\phi(x/|x|) - mC\right) + O_2(|x|^{-2-\alpha})\,,$$

for every $\alpha \in (0,1)$, where ϕ satisfies $\Delta^{\mathbb{S}^2}\phi = -2\phi$.

Proof. We proceed as in the proof of Proposition 1.4.14. We first pass to the the push–forwards via ψ of the function v and of the metric g to $\mathbb{R}^3$ minus a ball, still denoted by v and g, respectively and we obtain with the same argument that

$$v = \frac{C}{|x|} - \frac{1}{4\pi|x|^2} \int_{\mathbb{R}^3} \rho(y)\,|y|^k\, P_1(\widehat{x}\cdot\widehat{y})\,dy - \left(1 + \frac{m}{2|x|}\right)^{-1} \frac{mC}{2|x|^2} + O_2(|x|^{-2-\alpha})$$

(outside a closed ball centered at the origin), where ρ is a suitable function. By formula (1.24) we have $P_1(t) = t$, therefore, we obtain

$$\frac{1}{|x|^2}\int_{\mathbb{R}^3} \rho(y)\,|y|^k\,P_1(\widehat{x}\cdot\widehat{y})\,dy = \frac{1}{|x|^2}\int_{\mathbb{R}^3} \rho(y)\,|y|^k\,\widehat{x}\cdot\widehat{y}\,dy = \frac{\widehat{x}\cdot a}{|x|^2}\,.$$

We now notice that $a\cdot\widehat{x}$ is the restriction to the unit sphere of the homogeneous harmonic polynomial $a\cdot x$, hence an eigenfunction of the $\mathbb{S}^2$–Laplacian with eigenvalue -2. The claim then follows, as

$$\left(1+\frac{m}{2|x|}\right)^{-1}\frac{mC}{2|x|^2} = \frac{mC}{2|x|^2} + O_2(|x|^{-3})\,.$$

□

1.5 The ADM mass

The asymptotically flat manifolds (explaining our general convention, just after Definition 1.4.1) have the remarkable property of having a well–defined notion of "total" mass, called *ADM mass*, after the names of R. Arnowitt, S. Deser and C. W. Misner, who introduced it in [3]. In this section we discuss such notion and see that it is a geometric invariant of a complete AF manifold.

First, we consider (M,g) to be a complete, one–ended AF manifold with scalar curvature in $L^1(M)$ and let $\big(E,(x^1,\dots,x^n)\big)$ be an AF coordinate chart of order τ, where $\tau > (n-2)/2$. We define on E the vector field

$$\mathbb{U} = \sqrt{\det g_{st}}\,g^{kl}g^{ij}\big(\partial_i g_{jl} - \partial_l g_{ij}\big)\partial_k$$

and we observe that

$$\mathbb{U}^k = \sqrt{\det g_{st}}\,\big(g^{ij}\Gamma^k_{ij} - g^{ki}\Gamma^j_{ij}\big)\,,$$

therefore, the divergence of $\mathbb{U}$ with respect to g_e can be written as

$$\begin{aligned}\mathrm{div}_e(\mathbb{U}) = \partial_k\mathbb{U}^k &= \sqrt{\det g_{st}}\,\big[g^{ij}\partial_k\Gamma^k_{ij} - g^{ij}\partial_i\Gamma^k_{jk} + 2g^{ij}\Gamma^k_{ij}\Gamma^l_{kl} - 2g^{ij}\Gamma^k_{il}\Gamma^l_{kj}\big]\\ &= \sqrt{\det g_{st}}\,\big[\mathrm{R} + g^{ij}\Gamma^k_{ij}\Gamma^l_{kl} - g^{ij}\Gamma^k_{il}\Gamma^l_{kj}\big]\\ &= \sqrt{\det g_{st}}\,\mathrm{R} + O\big(|x|^{-2(1+\tau)}\big)\end{aligned} \tag{1.35}$$

by assumption (1.10) and formula (1.17), where we used the following equalities,

$$\begin{aligned}\partial_k g_{ij} &= g_{jl}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj}\\ \partial_k\sqrt{\det g_{ij}} &= \sqrt{\det g_{ij}}\,\Gamma^l_{kl}\\ \partial_k g^{ij} &= -g^{il}g^{jt}\partial_k g_{lt} = -g^{il}\Gamma^j_{kl} - g^{jt}\Gamma^i_{kt}\\ \mathrm{R} &= g^{ij}\big[\partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{jk} + \Gamma^s_{ij}\Gamma^k_{sk} - \Gamma^s_{kj}\Gamma^k_{is}\big]\,.\end{aligned}$$

Equality (1.35) with the integrability of the scalar curvature R with respect to g, then implies the integrability of the divergence of $\mathbb{U}$ with respect to g_e on E by virtue of fact that $\tau > (n-2)/2$. This last integrability in turn implies, by the divergence theorem, the existence and the finiteness of the limit of

$$\widetilde{m}(r) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\{|x|=r\}} \sqrt{\det g_{st}}\, g^{kl} g^{ij} \left(\partial_i g_{jl} - \partial_l g_{ij}\right) \frac{x^k}{|x|}\, d\sigma_e\,,$$

as $r \to +\infty$. On the other side, we have

$$\sqrt{\det g_{st}}\, g^{kl} g^{ij} \left(\partial_i g_{jl} - \partial_l g_{ij}\right) \frac{x^k}{|x|} = \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \frac{x^i}{|x|} + O\big(|x|^{-1-2\tau}\big)\,, \tag{1.36}$$

by the formulas (1.20) and (1.21), as a consequence, the limit of

$$m(r) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\{|x|=r\}} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \frac{x^i}{|x|}\, d\sigma_e$$

as $r \to +\infty$ exists, it is finite and equal to $\lim\limits_{r\to+\infty} \widetilde{m}(r)$.

Remark 1.5.1. Analogously, if $\{\Sigma_l\}_{l\in\mathbb{R}^+}$ is a family of closed two–sided hypersurfaces such that

(1) for every $l \in \mathbb{R}^+$ there exists an open bounded domain D_l with $\partial D_l = \{|x| = r_0\} \sqcup \Sigma_l$ for some $r_0 > 1$;

(2) $r_l = \inf\{|x| : x \in \Sigma_l\} \to +\infty$ as $l \to +\infty$;

(3) there exists $L > 0$ such that $r_l^{1-n}\, \sigma_e(\Sigma_l) \leq L$ for every $l \in \mathbb{R}^+$;

then the limit

$$\frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{l\to+\infty} \int\limits_{\Sigma_l} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \nu_e^i\, d\sigma_e,$$

exists, it is finite and it is equal to $\lim\limits_{r\to+\infty} m(r)$, where ν_e is the ∞–pointing unit normal vector field along Σ_l, with respect to g_e. Moreover, we get directly by formulas (1.22) and (1.23),

$$\begin{aligned}\lim_{l\to+\infty} \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} &\int\limits_{\Sigma_l} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) dx^i(\nu)\, d\sigma \\ &= \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{l\to+\infty} \int\limits_{\Sigma_l} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \nu_e^i\, d\sigma_e.\end{aligned}$$

Hence, for every AF coordinate chart $\big(E, \psi = (x^1, \dots, x^n\big)$ of order τ of (M^n, g), with $\tau > (n-2)/2$, it is well–defined the limit

$$m(g,\psi) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{r\to+\infty} \int\limits_{\{|x|=r\}} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \frac{x^i}{|x|}\, d\sigma_e\,, \tag{1.37}$$

where $g = g_{ij} dx^i \otimes dx^j$. We then want to see that such limit is independent of the AF chart ψ, i.e. it is a *geometric invariant* of (M^n, g).

Lemma 1.5.2 (Theorems 9.3 and 9.5 in [48]). *Let (M^n, g) be a complete, one–ended AF manifold. Let $\big(E_1, \psi_1 = (x^1, \dots, x^n)\big)$ and $\big(E_2, \psi_2 = (y^1, \dots, y^n)\big)$ be two AF coordinate charts with orders τ_1, τ_2, respectively, where $\tau_1,\ \tau_2 > (n-2)/2$, then there exists a* rigid motion $(A^j_i, a^i) \in O(n,\mathbb{R}) \times \mathbb{R}^n$ *of* $\mathbb{R}^n$*, such that*

$$x^i - (A^i_j\, y^j + a^i) = R^i$$
$$\sum_{i=1}^n \left[|R^i| + \sum_{j=1}^n |x| \left| \frac{\partial R^i}{\partial x^j} \right| + \sum_{j,\,k=1}^n |x|^2 \left| \frac{\partial^2 R^i}{\partial x^j \partial x^k} \right| \right] = O(|x|^{1-\tau})$$

outside some bounded open set of M *and for some* $\tau \in \left(\frac{n-2}{2}, \min\{\tau_1, \tau_2\}\right)$*.*

Proof. We can assume $\tau_1 < n-1$ and extend the functions x^i to smooth functions on M. First, we show that there exist functions $z^1, \dots, z^n \in C^\infty(M)$ such that each z^i is a harmonic function on M, $(z^1, \dots, z^n)$ is a coordinate system for M outside a compact subset and

$$\begin{cases} x^i - z^i \in C^{2,\alpha_1}_{-\tau_1+1+\varepsilon_1}(M) & \text{if } n = 3 \\ x^i - z^i \in C^{2,\alpha_1}_{-\tau_1+1}(M) & \text{if } n \geq 4 \end{cases}$$

for some $\alpha_1 \in (0,1)$ and $\varepsilon_1 > 0$. We observe that $\Delta x^i \in C^{0,\alpha}_{-\tau_1-1}(M)$ for every $\alpha \in (0,1)$, by point (1) of Lemma 1.4.9, as $\Delta x^i \in C^1_{-\tau_1-1}(M)$. If $n \geq 4$, there holds $2 - n < -\tau_1 + 1 < 0$, then point (4) of Theorem 1.4.10 guarantees the existence of $u^i \in C^{2,\alpha}_{-\tau_1+1}(M)$ such that $\Delta u^i = \Delta x^i$. If $n = 3$, then $\Delta x^i \in L^p_{-\tau_1-1+\varepsilon_1}(M)$ for every $p > 1, \varepsilon_1 > 0$, by point (2) of Lemma 1.4.9, hence, by choosing $\varepsilon_1 > 0$ in a way that $-\tau_1 + 1 + \varepsilon_1$ is not an integer and it is greater than $2 - n$, it follows by point (2) of Theorem 1.4.10, that there exist $u^i \in W^{2,p}_{-\tau_1+1+\varepsilon_1}(M)$ with $\Delta u^i = \Delta x^i$. If $p > n$, point (2) of Lemma 1.4.9 implies $u^i \in C^{0,\alpha'}_{-\tau_1+1+\varepsilon_1}(M)$ for every $\alpha' \in (0,1)$, thus $u^i \in C^{2,\alpha'}_{-\tau_1+1+\varepsilon_1}(M)$ by point (3) of Theorem 1.4.10.
Setting $z^i = x^i - u^i$, each form dz^i is a harmonic 1–form, by construction and since these 1–forms are "asymptotic" to dx^i, they form a (local) dual frame, near infinity. An application of the mean value theorem to the function $z \circ \psi^{-1}$ then implies that the map $z : M \to \mathbb{R}^n$ is one–to–one for $|z|$ large enough, thus $\{z^i\}$ form coordinates near infinity. Moreover, one can deduce

$$g\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\Big) - g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \begin{cases} O(|x|^{-\tau_1+\varepsilon_1}) & \text{if } n = 3 \\ O(|x|^{-\tau_1}) & \text{if } n \geq 4 \end{cases} \tag{1.38}$$

from the formula of change of coefficients of the metric g. Analogously, assuming $\tau_2 < n-1$ and extending the functions y^i to smooth functions on M, we find functions $w^1, \dots, w^n \in C^\infty(M)$ such that each w^i is a harmonic function on M, $(w^1, \dots, w^n)$ is a coordinate system near infinity and

$$\begin{cases} y^i - w^i \in C^{2,\alpha_2}_{-\tau_2+1+\varepsilon_2}(M) & \text{if } n = 3 \\ y^i - w^i \in C^{2,\alpha_2}_{-\tau_2+1}(M) & \text{if } n \geq 4 \end{cases}$$

for some $\alpha_2 \in (0,1), \varepsilon_2 > 0$ and

$$g\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) - g\Big(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j}\Big) = \begin{cases} O(|x|^{-\tau_2+\varepsilon_2}) & \text{if } n = 3 \\ O(|x|^{-\tau_2}) & \text{if } n \geq 4 \end{cases} \tag{1.39}$$

in particular, ε_1 and ε_2 can be chosen in such a way that $\tau_1 - \varepsilon_1, \tau_2 - \varepsilon_2 > (n-2)/2$. Observe now that $z^i, w^i \in W^{2,p}_s(M)$, for every $1 < s < 2$ and recall that, by point (2) of Theorem 1.4.10, the dimension of $\ker \Delta$ on $W^{2,p}_s(M)$ is $n+1$, then $\{w^1, \dots, w^n, 1\}$

form a basis of such kernel. Thus, there exist a matrix A^i_j and constants a^i such that

$$z^i = A^i_j w^j + a^i \, .$$

Hence, formulas (1.38) and (1.39), sending $|x| \to +\infty$, imply that $A \in O(n, \mathbb{R})$. The statement of the lemma then follows by setting $\tau = \min\{\tau_1 - \varepsilon_1, \tau_2 - \varepsilon_2\}$, if $n = 3$ and $\tau = \min\{\tau_1, \tau_2\}$, if $n \geq 4$. □

Theorem 1.5.3 (Invariance of the ADM mass – I). *Let (M^n, g) be a complete, one–ended AF manifold with scalar curvature in $L^1(M)$. Given any AF coordinate chart $\big(E, \psi = (x^1, \ldots, x^n)\big)$, the limit*

$$m(g, \psi) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{r \to +\infty} \int\limits_{\{|x|=r\}} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{|x|} \, d\sigma_e \, ,$$

exists finite and it is independent of the AF chart.

Proof. We only need to show that the limit is independent of the chart ψ.
We first prove that, given $(A^j_i, a^i) \in O(n, \mathbb{R}) \times \mathbb{R}^n$ and set

$$y^i = A^i_j \, x^j + a^i \, ,$$

then, possibly choosing a smaller set E, the chart $\big(E, \widetilde{\psi} = (y^1, \ldots, y^n)\big)$ is AF and the associated limits coincide. In E we have

$$dy^i = A^i_k \, dx^k \tag{1.40}$$

$$\frac{\partial}{\partial y^i} = A^k_i \frac{\partial}{\partial x^k} \tag{1.41}$$

$$C_1 \, |x| \leq |y| \leq C_2 \, |x| \tag{1.42}$$

where the second equality follows from the orthogonality of the matrix A and the last inequalities are clearly satisfied, possibly choosing a smaller set E. Then, there holds

$$\begin{aligned}
\sigma^{(y)}_{ij} \, dy^i \otimes dy^j &= \Big[g\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) - \delta_{ij} \Big] \, dy^i \otimes dy^j \\
&= \Big[g\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) - A^i_k A^i_l \Big] \, dx^k \otimes dx^l \\
&= \Big[g\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) - \delta_{kl} \Big] \, dx^k \otimes dx^l = \sigma^{(x)}_{kl} \, dx^k \otimes dx^l \, ,
\end{aligned}$$

implying the equality

$$\sigma^{(y)}_{ij} = A^k_i \sigma^{(x)}_{kl} A^l_j \, . \tag{1.43}$$

It follows that also $\big(E, \widetilde{\psi} = (y^1, \ldots, y^n)\big)$ is an AF coordinate chart and, by Remark 1.5.1 and inequalities (1.42), we conclude

$$m(g, \widetilde{\psi}) = \int\limits_{\{|x|=r\}} \Big(\frac{\partial \sigma^{(y)}_{ij}}{\partial y^j} - \frac{\partial \sigma^{(y)}_{jj}}{\partial y^i} \Big) dy^i(\nu) \, d\sigma = \int\limits_{\{|x|=r\}} \Big(\frac{\partial \sigma^{(x)}_{ij}}{\partial x^j} - \frac{\partial \sigma^{(x)}_{jj}}{\partial x^i} \Big) dx^i(\nu) \, d\sigma = m(g, \psi)$$

where the second equality is a direct consequence of formulas (1.40), (1.41) and (1.43).

If now, we consider any two AF coordinate charts $\big(E_1, \psi_1 = (x^1, \dots, x^n)\big)$ and $\big(E_2, \psi_2 = (y^1, \dots, y^n)\big)$ of orders τ_1 and τ_2, respectively, by Lemma 1.5.2 and the previous step, we can assume

$$y^i(p) - x^i(p) = R^i(p) \tag{1.44}$$

$$\sum_{i=1}^n \left[|R^i| + \sum_{j=1}^n |x| \left| \frac{\partial R^i}{\partial x^j} \right| + \sum_{j,\,k=1}^n |x|^2 \left| \frac{\partial^2 R^i}{\partial x^j \partial x^k} \right| \right] = O(|x|^{1-\tau}) \tag{1.45}$$

outside some bounded open set of M, for some $\tau \in \big(\frac{n-2}{2}, \min\{\tau_1, \tau_2\}\big)$. As consequences of formulas (1.44) and (1.45), we obtain that

$$C_1 |x| \le |y| \le C_2 |x| \,, \tag{1.46}$$

$$\frac{\partial}{\partial x^i} = \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \frac{\partial}{\partial y^k} \,, \tag{1.47}$$

for some $C_1, C_2 > 0$ and outside some bounded open set containing the previous one. Moreover, we have

$$\begin{aligned} g_{ij}^{(x)} = g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \Big) &= \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) g\Big(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l} \Big) \\ &= \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) g_{kl}^{(y)} \,, \end{aligned}$$

which implies

$$\begin{aligned} \frac{\partial g_{ij}^{(x)}}{\partial x^s} &= \frac{\partial^2 R^k}{\partial x^i \partial x^s} \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) g_{kl}^{(y)} + \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \frac{\partial^2 R^l}{\partial x^j \partial x^s} g_{kl}^{(y)} + \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) \frac{\partial g_{kl}^{(y)}}{\partial x^s} \\ &= \frac{\partial^2 R^k}{\partial x^i \partial x^s} \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) (\delta_{kl} + \sigma_{kl}^{(y)}) + \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \frac{\partial^2 R^l}{\partial x^j \partial x^s} (\delta_{kl} + \sigma_{kl}^{(y)}) \\ &\quad + \left(\delta_i^k + \frac{\partial R^k}{\partial x^i} \right) \left(\delta_j^l + \frac{\partial R^l}{\partial x^j} \right) \left(\delta_s^t + \frac{\partial R^t}{\partial x^s} \right) \frac{\partial g_{kl}^{(y)}}{\partial y^t} \\ &= \frac{\partial g_{ij}^{(y)}}{\partial y^s} + \frac{\partial^2 R^j}{\partial x^i \partial x^s} + \frac{\partial^2 R^i}{\partial x^j \partial x^s} + O(|x|^{-1-2\tau}) \,, \end{aligned}$$

where we used equality (1.47) to obtain the second equality, while for the last one we took advantage of formulas (1.45) and (1.46). In particular, there holds

$$\frac{\partial g_{ij}^{(y)}}{\partial y^s} = \frac{\partial g_{ij}^{(x)}}{\partial x^s} - \frac{\partial^2 R^j}{\partial x^i \partial x^s} - \frac{\partial^2 R^i}{\partial x^j \partial x^s} + O(|x|^{-1-2\tau}) \,. \tag{1.48}$$

By equality (1.48), we then obtain

$$\begin{aligned}
&\left(\frac{\partial g^{(y)}_{ij}}{\partial y^j} - \frac{\partial g^{(y)}_{jj}}{\partial y^i}\right) dy^i(\nu) \\
&= \left(\frac{\partial g^{(x)}_{ij}}{\partial x^j} - \frac{\partial^2 R^j}{\partial x^i \partial x^j} - \frac{\partial^2 R^i}{\partial x^j \partial x^j} - \frac{\partial g^{(x)}_{jj}}{\partial x^i} + 2\frac{\partial^2 R^j}{\partial x^j \partial x^i} + O(|x|^{-1-2\tau})\right)\left(\delta^i_k + \frac{\partial R^i}{\partial x^k}\right) dx^k(\nu) \\
&= \left(\frac{\partial g^{(x)}_{ij}}{\partial x^j} - \frac{\partial g^{(x)}_{jj}}{\partial x^i}\right) dx^i(\nu) + \frac{\partial}{\partial x^j}\left(\frac{\partial R^j}{\partial x^i} - \frac{\partial R^i}{\partial x^j}\right) dx^i(\nu) + O(|x|^{-1-2\tau})\,.
\end{aligned}$$

Thus, by formulas (1.22) and (1.23), there holds

$$\begin{aligned}
&\int\limits_{\{|x|=r\}} \left(\frac{\partial g^{(y)}_{ij}}{\partial y^j} - \frac{\partial g^{(y)}_{jj}}{\partial y^i}\right) dy^i(\nu)\, d\sigma \\
&= \int\limits_{\{|x|=r\}} \left(\frac{\partial g^{(x)}_{ij}}{\partial x^j} - \frac{\partial g^{(x)}_{jj}}{\partial x^i}\right) dx^i(\nu)\, d\sigma + \int\limits_{\{|x|=r\}} \frac{\partial}{\partial x^j}\left(\frac{\partial R^i}{\partial x^j} - \frac{\partial R^j}{\partial x^i}\right) dx^i(\nu)\, d\sigma + O(r^{n-2-2\tau}) \\
&= \int\limits_{\{|x|=r\}} \left(\frac{\partial g^{(x)}_{ij}}{\partial x^j} - \frac{\partial g^{(x)}_{jj}}{\partial x^i}\right) \frac{x^i}{r}\, d\sigma^{(x)}_e + \int\limits_{\{|x|=r\}} \frac{\partial}{\partial x^j}\left(\frac{\partial R^i}{\partial x^j} - \frac{\partial R^j}{\partial x^i}\right) \frac{x^i}{r}\, d\sigma^{(x)}_e + O(r^{n-2-2\tau}).
\end{aligned} \tag{1.49}$$

We then notice that, by the divergence theorem,

$$\int\limits_{\{|x|=r\}} \frac{\partial}{\partial x^j}\left(\frac{\partial R^i}{\partial x^j} - \frac{\partial R^j}{\partial x^i}\right) \frac{x^i}{r}\, d\sigma^{(x)}_e = -\int\limits_{\{|x|\geq r\}} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial R^i_r}{\partial x^j} - \frac{\partial R^j_r}{\partial x^i}\right) d\mu^{(x)}_e = 0$$

where R^i_r is a smooth function with compact support, coinciding with R^i on an open set which contains $\{|x| = r\}$, for each $i \in \mathbb{N}$. Being $\tau \geq \frac{n-2}{2}$, the last equality in formula (1.49) combined with this last result implies that

$$m(g, \psi_1) = m(g, \psi_2)\,.$$

$\square$

Concerning the asymptotically flat manifolds with nonnegative and not summable scalar curvature, we have the following more immediate result.

Theorem 1.5.4 (Invariance of the ADM mass – II). *Let (M^n, g) be a complete, one–ended AF manifold with nonnegative scalar curvature* R*, not in $L^1(M)$. Given any AF coordinate chart $\big(E, \psi = (x^1, \dots, x^n)\big)$ of order $\tau > \frac{n-2}{2}$, then*

$$m(g, \psi) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{r\to+\infty} \int\limits_{\{|x|=r\}} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{|x|}\, d\sigma_e = +\infty\,,$$

In particular, the limit above is independent of the AF chart.

Proof. From formulas (1.35) and (1.36), it follows

$$\int_{\{|x|=r\}} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{|x|}\, d\sigma_e = \int_{\{r_0<|x|<r\}} \mathrm{R}\, d\mu + \int_{\{r_0<|x|<r\}} q\, d\mu_e$$
$$+ \int_{\{|x|=r_0\}} \sqrt{\det g_{st}}\, g^{kl} g^{ij} (\partial_i g_{jl} - \partial_l g_{ij}) \frac{x^k}{|x|}\, d\sigma_e + O(r^{n-2-2\tau})$$

for $r_0 > 1$ sufficiently large and for every $r > r_0$, where the function q, which is $O\big(|x|^{-2(1+\tau)}\big)$, is then in $L^1(E)$. Now, passing to the limit for $r \to +\infty$, at the right hand side of the above equality, the first term diverges positively by the monotone convergence theorem, the second one converges by the dominate convergence theorem and the last one tends to zero as $\tau > (n-2)/2$. Consequently, we have the thesis. $\square$

Definition 1.5.5 (ADM mass)**.** Let (M^n, g) be a one–ended AF manifold and let $\big(E, (x^1, \dots, x^n)\big)$ be an AF coordinate chart. The limit

$$m_{\mathrm{ADM}} = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{r\to+\infty} \int_{\{|x|=r\}} (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{|x|}\, d\sigma_e\,,$$

where $g = g_{ij} dx^i \otimes dx^j$, exists and it is independent of the AF coordinate chart (proved first by Bartnik [7] and then independently by Chruściel [18]). This geometric invariant is called *ADM mass* of (M, g), named after the physicists Arnowitt, Deser and Misner [3].

Example 1.5.6. We consider outside an open ball in $\mathbb{R}^n$ spherically–symmetric metrics given in polar coordinates as

$$g = \phi(r) dr \otimes dr + \chi(r) r^2 g_{\mathbb{S}^{n-1}}$$

and we observe that

$$\begin{aligned}\phi(r) dr \otimes dr + \chi(r) r^2 g_{\mathbb{S}^{n-1}} &= g_{\mathrm{eucl}} + \big(\chi(r) - 1\big) g_{\mathrm{eucl}} + \big(\phi(r) - \chi(r)\big) dr \otimes dr \\ &= \left[\delta_{ij} + \big(\chi(|x|) - 1\big)\delta_{ij} + \big(\phi(|x|) - \chi(|x|)\big) \frac{x^i x^j}{|x|^2}\right] dx^i \otimes dx^j\,.\end{aligned}$$

Then, if

$$\sigma_{ij} = \big(\chi(|x|) - 1\big)\delta_{ij} + \big(\phi(|x|) - \chi(|x|)\big) \frac{x^i x^j}{|x|^2} = O_2(|x|^{-\tau})\,,$$

for $\tau > \frac{n-2}{2}$ and if the scalar curvature of g is integrable or nonnegative, we have a one–ended AF manifold with

$$\begin{aligned} m_{\mathrm{ADM}} &= \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{r\to+\infty} \int_{\{|x|=r\}} (n-1) \left[\frac{\phi(r) - \chi(r)}{r} - \chi'(r)\right] d\sigma_e \\ &= \frac{1}{2}\Big[r^{n-2}\big(\phi(r) - \chi(r)\big) - r^{n-1}\chi'(r)\Big]\,, \end{aligned} \tag{1.50}$$

since

$$\begin{aligned}\partial_k\sigma_{ij} &= \partial_k\Big[\big(\chi(|x|)-1\big)\delta_{ij}+\big(\phi(|x|)-\chi(|x|)\big)\frac{x^ix^j}{|x|^2}\Big]\\ &= \chi'(|x|)\frac{x^k}{|x|}\delta_{ij}+\big(\phi'(|x|)-\chi'(|x|)\big)\frac{x^ix^jx^k}{|x|^3}\\ &\quad+\frac{\phi(|x|)-\chi(|x|)}{|x|}\Big[\delta_{ik}\frac{x^j}{|x|}+\delta_{jk}\frac{x^i}{|x|}-\frac{2x^ix^jx^k}{|x|^3}\Big].\end{aligned}$$

For the Schwarzschild manifold of mass m, introduced in Example 1.4.2 and given by formula (1.13) up to an isometry, one has $\chi\equiv 1$ and $\phi = 1/(1-2mr^{2-n})$, therefore the parameter m coincides with the ADM mass by equality (1.50).

Several other formulas that "produce" the ADM mass are known, such as the $r\to+\infty$ limits of the *Brown–York mass* or of the *Hawking mass* of $\{|x|=r\}$ (see [22] and the references therein). In a different spirit, the ADM mass can also be recovered through an expression involving the Ricci curvature at infinity (see [65]).

Remark 1.5.7. In 1960, Arnowitt, Deser and Misner in [4–6] studied in detail the isolated gravitational systems. They adopted a Hamiltonian viewpoint, namely, they chose a spacelike hypersurface as an "initial hypersurface" and wrote the Einstein's equations as evolution equations of this initial data. Then, they discovered a conserved quantity, the ADM mass and concluded that it represented the "total" mass of the isolated system. A disadvantage of the ADM mass is that it is defined only globally, hence, since the 1970s, physicists and mathematicians are looking for a suitable *quasi–local* notion which describes the mass of an isolated system, using only metric–related quantities contained in a bounded region of space.

We conclude this section presenting an example due to Denisov and Solov'ev [19] showing that, in general, a change of coordinate chart could destroy the asymptotic properties of functions and tensors, in particular, the geometric invariance of the limit (1.37) if one considers also AF coordinate charts with orders $\tau\in(0,\frac{n-2}{2}]$ (this explains the condition on the order in Definition 1.4.1).

Example 1.5.8. We define the map

$$\mathbb{R}^3\setminus\{0\}\ni x\mapsto y=\big(1+|x|^{-\alpha}\big)x\in\mathbb{R}^3\setminus\{0\}\,,$$

where $\alpha>0$, which is clearly a diffeomorphism outside a closed ball centered at the origin with a large enough radius. Then, there holds

$$\begin{aligned}g_{\mathrm{eucl}} &= \delta_{ij}dy^i\otimes dy^j\\ &= \Big[(1+|x|^{-\alpha})^2\delta_{ij}-\alpha(2+2|x|^{-\alpha}-\alpha|x|^{-\alpha})\,\frac{x^ix^j}{|x|^{2+\alpha}}\Big]dx^i\otimes dx^j\\ &= \big[\delta_{ij}+O_2(|x|^{-\alpha})\big]dx^i\otimes dx^j\end{aligned}$$

and

$$\frac{1}{16\pi}\lim_{r\to+\infty}\int\limits_{\{|x|=r\}}(\partial_jg_{ij}-\partial_ig_{jj})\frac{x^i}{r}\,d\sigma_e=\lim_{r\to+\infty}\frac{\alpha^2}{2\,r^{2\alpha-1}}=\begin{cases}+\infty & \text{for } 0<\alpha<1/2\\ 1/8 & \text{for } \alpha=1/2\\ 0 & \text{for } \alpha>1/2\end{cases}$$

where $g_{\mathrm{eucl}}=g_{ij}dx^i\otimes dx^j$.

Chapter 2

Sub–static manifolds with harmonic potential

In this chapter, the object under investigation is a triple (M, g, u) satisfying the following two conditions:

(a) (M, g) is a complete, one–ended AF manifold of dimension $n \geq 3$, with compact boundary ∂M (which could have several connected components).

(b) $u \in C^\infty(M)$ satisfies the system

$$\begin{cases} u\mathrm{Ric} - \nabla du \geq 0 & \text{in } M \\ \Delta u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases} \tag{2.1}$$

We will refer to such a triple (M, g, u) as a *sub–static harmonic triple* and to u as *the potential* of (M, g).
If the equality holds everywhere in the first equation of system (2.1), the triple (M, g, u) is called *static*.

In dimension 3, if assumptions (a) and (b) hold, then the asymptotically flat spacetime $(\mathcal{M}, \mathbf{g})$, given by $\mathcal{M} := \mathbb{R} \times \big(M \setminus \{u = 0\}\big)$ with the Lorentzian metric $\mathbf{g} := -u^2 dt \otimes dt + g$, satisfies the so called *null convergence condition* [85], i.e. $\mathbf{Ric}(\mathbf{V}, \mathbf{V}) \geq 0$ for every $\mathbf{V} \in \Gamma(T\mathcal{M})$ such that $\mathbf{g}(\mathbf{V}, \mathbf{V}) = 0$. This is exactly the curvature assumption made in Penrose's celebrated *singularity theorem* [33, p. 263, Theorem 1], since typically a singularity theorem has three ingredients: an energy condition on the matter; a condition on the global structure of spacetime; gravity strong enough (somewhere) to trap a region. Recall that, in general relativity, a singularity is a place that objects or light rays can reach in finite time but where the curvature becomes infinite, or the spacetime stops being a manifold. Before Penrose, it was conceivable that, for example, in the collapse of a star inside its Schwarzschild radius ($r = 2m$), if the star is spinning and thus possesses some angular momentum, maybe the centrifugal force could partly counteract the gravity and avoid the formation of a singularity. The singularity theorem shows that this cannot happen, hence a singularity will always develop.

The triples (M, g, u) satisfying assumptions (a) and (b) with equality in the first equation of system (2.1) are of great relevance. The asymptotically flat spacetime $(\mathcal{M}, \mathbf{g})$, defined as before, then solves the vacuum Einstein equation (i.e. the Einstein equation with $\mathbf{T} = 0$). In general, a spacetime $(\mathcal{M}, \mathbf{g})$ is called *static* if there exists a timelike Killing vector field X that is irrotational. Examples of static spacetimes are the *static standard spacetimes* $(\mathcal{M}, \mathbf{g})$ which can be globally decomposed as $\mathcal{M} = \mathbb{R} \times M^3$ and $\mathbf{g} = -N^2 dt \otimes dt + g$ (the function $N : M^3 \to (0, +\infty)$ is called *static lapse function*). It is then justified to interpret the static lapse function N of a static system as telling us how fast the time t flows at different points in the space M^3, where t is the time measured by a static observer on ∂_t.

By the strong maximum principle, one clearly has that $u \in (0,1)$ in $\mathrm{Int}(M) = M \setminus \partial M$, hence, taking the trace of the first inequality, we see that the scalar curvature R is nonnegative everywhere (and zero in the static case). By the last condition, $u : M \to [0,1)$ is proper, consequently, each level set of u is compact and it follows that it has finite $(n-1)$–Hausdorff measure,by applying [32, Theorem 1.7] via coordinate charts. Then, for every regular value $t \in [0,1)$ of u there exists $\varepsilon_t > 0$ such that $(t - \varepsilon_t, t + \varepsilon_t) \cap [0,1)$ does not contain any critical value, hence, the set of the critical values of u is a closed set of zero Lebesgue measure, by Sard's theorem.
Moreover, $|\nabla u| > 0$ on ∂M by the Hopf lemma, hence zero is a regular value of u. More precisely, the function $|\nabla u|$ attains a positive constant value on each connected component of ∂M and the boundary ∂M is a totally geodesic hypersurface in M (in particular, each of its connected components is a minimal hypersurface), since $\nabla du \equiv 0$ on ∂M as a consequence of the first two conditions in system (2.1), restricted to ∂M and the second fundamental form of the boundary is proportional to ∇du, by formula (1.4).
By Corollary 1.4.13, in an AF coordinate chart $(x^1, \dots, x^n)$, being $1-u$ the *boundary capacity potential* there mentioned, we also know that

$$u = 1 - \frac{\mathcal{C}}{|x|^{n-2}} + o_2(|x|^{2-n}) \quad \text{as} \quad |x| \to +\infty\,, \tag{2.2}$$

with

$$\mathcal{C} = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial M} |\nabla u| \, d\sigma\,, \tag{2.3}$$

which is the *boundary capacity* of ∂M in (M,g) by formula (1.32).
Then

$$\partial_i u = (n-2)\, \mathcal{C}\, |x|^{-n} x^i \, + \, o(|x|^{1-n})\,, \tag{2.4}$$

$$\partial_i \partial_j u = -(n-2)\, \mathcal{C}\, |x|^{-n-2}(n x^i x^j - |x|^2 \delta_{ij}) \, + \, o(|x|^{-n})\,. \tag{2.5}$$

Consequently, the set of critical points $\mathrm{Crit}(u)$ is compact, therefore it has finite $(n-2)$–dimensional Hausdorff measure, by applying [31, Theorem 1.1]. More precisely, $\mathrm{Crit}(u)$ is a countably $(n-2)$–*rectifiable subset* (see Federer [24] or [82]).
In general, it is convenient to notice that

$$\int_{\{u=t\}} |\nabla u| \, d\sigma = (n-2)|\mathbb{S}^{n-1}|\, \mathcal{C}\,, \tag{2.6}$$

for every $t \in [0,1)$ regular value of u. Indeed, by applying the divergence theorem to the vector field ∇u on $\{0 < u < t\}$ (see Remark 1.1.1), one has

$$0 = \int_{\{0<u<t\}} \Delta u \, d\mu = \int_{\{u=t\}} |\nabla u| d\sigma - \int_{\partial M} |\nabla u| \, d\sigma \, = \int_{\{u=t\}} |\nabla u| d\sigma - (n-2)|\mathbb{S}^{n-1}|\, \mathcal{C}\,,$$

where the first equality follows by the fact that u is a harmonic function and the last one by formula (2.3).

Remark 2.0.1. It is also useful to observe that:

(1) for every $t \in (0,1)$ sufficiently close to 1, the level set $\{u = t\}$ is regular and diffeomorphic to the $(n-1)$–sphere $\mathbb{S}^{n-1}$;

(2) there hold $\{u \geq t\} = \overline{\{u > t\}}$ and $\{0 \leq u \leq t\} = \overline{\{0 < u < t\}}$ for every $t \in (0,1)$. Moreover, one has $\{t \leq u \leq T\} = \overline{\{t < u < T\}}$ for every $t, T \in (0,1)$ such that $t < T$;

(3) the sets $\{u \geq t\}$ are all connected, for every $t \in (0,1)$.

We check (1) first. We start by observing that due to formula (2.4) there holds $|\nabla u| \neq 0$ in $\{u \geq t_0\}$ for some $0 < t_0 < 1$. This fact establishes a diffeomorphism between $\{u \geq t_0\}$ and $\{u = t_0\} \times [t_0, 1)$ and tells us at the same time that the level sets $\{u = t\}$ are pairwise diffeomorphic for every $t \geq t_0$. It is thus sufficient to show that $\{u = t_0\}$ is connected. Suppose by contradiction that this is not the case. Without loss of generality we can assume that $\{u = t_0\}$ can be decomposed into the disjoint union of two connected sets C_1 and C_2, indeed the same argument works *a fortiori* if the connected components are more than two. Now, by definition of asymptotically flat manifold, there exists a compact set $K \subseteq M$ such that $M \setminus K^\circ$ is diffeomorphic to $\mathbb{R}^n \setminus B$ via an AF chart ψ, where B is a suitable open ball with center at 0 and radius greater than 1. Here, for each subset E of M, E° and $\overline{E}$ denote the interior and the closure of E in M, respectively. Also, we can suppose, up to choosing a larger t_0, that $\{u \geq t_0\} \subseteq M \setminus K^\circ$. Now, in view of the asymptotic expansion of u, there exist two positive constants $A < B$ such that

$$\frac{A}{|x|^{n-2}} \leq 1 - u \leq \frac{B}{|x|^{n-2}}\,.$$

In particular, setting $R_0 = [B/(1-t_0)]^{1/(n-2)}$, we have

$$\{|x| > R_0\} \subseteq \{u \geq t_0\} \approx \big\{C_1 \times [t_0, 1)\big\} \sqcup \big\{C_2 \times [t_0, 1)\big\},$$

where the symbol $\approx$ indicates that the manifolds involved are diffeomorphic. At the same time, $\{|x| > R_0\} \subseteq C_i \times [t_0, 1)$ for some $i \in \{1,2\}$, since $\{|x| > R_0\}$ is connected and each $C_i \times [t_0, 1)$ is a closed set of M. Then, we have

$$\begin{aligned}\big\{C_1 \times [t_0,1)\big\} \sqcup \big\{C_2 \times [t_0,1)\big\} &= \{u \geq t_0\} \subseteq M \setminus K^\circ \\ &= \big[(M \setminus K^\circ) \cap \{|x| \leq R_0\}\big] \sqcup \big[(M \setminus K^\circ) \cap \{|x| > R_0\}\big] \\ &\subseteq \big[(M \setminus K^\circ) \cap \{|x| \leq R_0\}\big] \sqcup \big\{C_i \times [t_0,1)\big\},\end{aligned}$$

which gives the contradiction that the noncompact set $C_j \times [t_0, 1)$, where $j \in \{1,2\} \setminus \{i\}$, is contained into the compact one $(M \setminus K^\circ) \cap \{|x| \leq R_0\}$. Therefore, $\{u = t_0\}$ is connected. Now, $\{\widetilde{u} = t_0\}$, with $\widetilde{u} := \psi_* u$, is a closed and connected hypersurface of $\mathbb{R}^n$, having strictly positive sectional curvature as Riemannian submanifold of $(\mathbb{R}^n, g_{\mathrm{eucl}})$, up to a larger t_0, due to formula (2.5). Consequently, $\{\widetilde{u} = t_0\}$ is diffeomorphic to $\mathbb{S}^{n-1}$ by the Gauss map (see [25, Section 5.B] for more details). Statement (1) thus follows, being $\{u = t_0\}$ and $\{\widetilde{u} = t_0\}$ diffeomorphic.

Concerning point (2), the first claims are obvious if $t \in (0,1)$ is a regular value of the function u, while the last one is clear for every $0 < t < T < +\infty$ regular values of u, we will now show these statements in general. We consider $t \in (0,1)$. One has immediately that $\overline{\{0 < u < t\}} \subseteq \{0 \leq u \leq t\}$. We suppose by contradiction that $\overline{\{0 < u < t\}} \not\subseteq \{0 \leq u \leq t\}$, then

$$\{u > t\} = \complement\left(\{0 \leq u \leq t\}\right) \not\subseteq \complement\left(\overline{\{0 < u < t\}}\right) = \complement\left(\overline{\complement(\{u \geq t\})}\right) = \{u \geq t\}^\circ \subseteq \{u \geq t\}\,,$$

where in the second equality we have used that $\overline{\{0 \leq u < t\}} = \overline{\{0 < u < t\}}$ (which is a consequence of the fact that 0 is a regular value of u). Thus, there exists a point p in $\{u \geq t\}^\circ$ such that $u(p) = t$, in particular p is a point of local minimum in this open set, which is impossible by the strong maximum principle. Similarly, one can check that $\overline{\{u > t\}} = \{u \geq t\}$ as $\{0 \leq u < t\} = \{0 \leq u \leq t\}^\circ$. Then, if $t, T \in (0,1)$ such that $t < T$, one gets $\{t \leq u \leq T\} = \overline{\{t < u < T\}}$, by observing that the following chain of inclusions,

$$\begin{aligned}
\{0 \leq u < t\} \cup \{u > T\} &= \complement\left(\{t \leq u \leq T\}\right) \subseteq \complement\left(\overline{\{t < u < T\}}\right) \\
&= \complement\left(\overline{\complement\left(\{0 \leq u \leq t\} \cup \{u \geq T\}\right)}\right) = \left(\{0 \leq u \leq t\} \cup \{u \geq T\}\right)^\circ \\
&= \{0 \leq u \leq t\}^\circ \cup \{u \geq T\}^\circ = \{0 \leq u < t\} \cup \{u > T\}
\end{aligned}$$

is a chain of equalities. Here, the fourth equality follows since $\{0 \leq u \leq t\}$ and $\{u \geq T\}$ are disjoint closed sets (we refer to [41] for some properties of interior and closure).

Finally, to show (3), we suppose, by contradiction, that $\{u \geq t\}$ is disconnected, for some $t \in (0,1)$. Since $u \to 1$ at ∞, we know that only one connected component of $\{u \geq t\}$ can be unbounded with an argument similar to the first part of point (1). Then, any other connected component K is compact and at same time, its interior must be nonempty and contain points where $u > t$, since $\partial K \subseteq \{u = t\}$ and $\{u \geq t\} = \overline{\{u > t\}}$. Therefore, K attains a local maximum in its interior and we obtain a contradiction as before.

A fundamental sub–static harmonic triple is the so called *Schwarzschild solution of mass* $m > 0$, denoted with $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)}, u_{\mathrm{Sch}(m)})$, where $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ is the Schwarzschild manifold of mass m defined in Example 1.4.2 and the function $u_{\mathrm{Sch}(m)}$ is given in the following way:

$$u_{\mathrm{Sch}(m)} = \frac{1 - \frac{m}{2|x|^{n-2}}}{1 + \frac{m}{2|x|^{n-2}}}\,.$$

Up to an isometry (see Example 1.4.2), the triple $\left(\mathrm{Int}\left(M_{\mathrm{Sch}(m)}\right), g_{\mathrm{Sch}(m)}, u_{\mathrm{Sch}(m)}\right)$ is equal to

$$\left(\left((2m)^{\frac{1}{n-2}}, +\infty\right) \times \mathbb{S}^{n-1}, \frac{dr \otimes dr}{1 - 2mr^{2-n}} + r^2 g_{\mathbb{S}^{n-1}}, \sqrt{1 - 2mr^{2-n}}\right),$$

we will use this last triple in the rest of this chapter. We recall that the parameter $m > 0$ coincides with the ADM mass of $(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)})$ as showed in Example 1.5.6.

2.1 Monotonicity and outer rigidity

We state now a *monotonicity and outer rigidity theorem*, which will be used later to prove the *capacitary Riemannian Penrose inequality* (2.55). The expression "outer rigidity" here means "rigidity" (that is, existence of an isometry with a model space) of a piece of type $\{u \geq t_0\}$ of a Riemannian manifold, where $t_0 \in [0, 1)$ is a regular value of a smooth proper function $u : M \to [0, 1)$, converging to 1 at ∞.

In all the sequel, we will denote for simplicity with σ the $(n-1)$–Hausdorff measure of (M, g).

Theorem 2.1.1. *Let (M, g, u) be a sub–static harmonic triple and let $F_\beta : [1, +\infty) \to [0, +\infty)$ be the function defined by*

$$F_\beta(\tau) := (1+\tau)^{\beta\,\frac{n-1}{n-2}} \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta+1}\, d\sigma\,,$$

for every $\beta > \frac{n-2}{n-1}$. Each function F_β is continuous and convex on $[1, +\infty)$ and it is continuously differentiable with nonpositive derivative in $(1, +\infty)$. Moreover, if there exists $\tau_0 \in (1, +\infty)$ such that $F'_\beta(\tau_0) = 0$ for some $\beta > \frac{n-2}{n-1}$, then $t_0 = \sqrt{\frac{\tau_0-1}{\tau_0+1}}$ is a regular value of u and $(\,\{u \geq t_0\}, g\,)$ is isometric to the following end of the Schwarzschild manifold

$$\left([r_0, +\infty) \times \mathbb{S}^{n-1}\,, \frac{dr \otimes dr}{1 - 2\mathcal{C} r^{2-n}} + r^2 g_{\mathbb{S}^{n-1}}\right), \qquad \textit{with } r_0 = [\mathcal{C}(1+\tau_0)]^{\frac{1}{n-2}}\,,$$

of mass equal to the boundary capacity

$$\mathcal{C} = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int\limits_{\partial M} |\nabla u|\, d\sigma\,.$$

Notice that the functions F_β are all well–defined, in view of the properties of the set of the critical points of u, discussed in the previous section and since the integrand function is bounded on the level sets of u (which are all compact).
Once Theorem 2.1.1 has been proved, the monotonicity and convexity of F_β also extend to the case $\beta = \frac{n-2}{n-1}$ by the dominated convergence theorem.
Moreover, at every value τ such that $\left\{u = \sqrt{(\tau-1)/(\tau+1)}\,\right\}$ is a regular level set, thus for a.e. $\tau > 1$, each function F_β is twice differentiable, with first and second

derivative given by

$$F'_\beta(\tau) = -\frac{\beta(\tau+1)^{\beta\frac{n-1}{n-2}-\frac{3}{2}}}{\sqrt{\tau-1}} \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^\beta \left[\mathrm{H} - \frac{n-1}{n-2}\,\frac{2u}{1-u^2}\,|\nabla u|\right] d\sigma\,,$$

$$\begin{aligned}
F''_\beta(\tau) = \frac{\beta(\tau+1)^{\beta\frac{n-1}{n-2}-3}}{\tau-1}\Bigg\{ \left(\beta - \frac{n-2}{n-1}\right) &\int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1} \left[\mathrm{H} - \frac{n-1}{n-2}\,\frac{2u}{1-u^2}\,|\nabla u|\right]^2 d\sigma \\
&+ \beta \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-3}\,|\nabla^\top|\nabla u||^2\, d\sigma + \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1}|\mathring{\mathrm{h}}|^2\, d\sigma \\
&- \frac{1}{\beta-1} \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} \Delta^\Sigma\big(|\nabla u|^{\beta-1}\big)\, d\sigma + \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1}\left[\mathrm{Ric}(\nu,\nu) - \frac{\nabla du(\nu,\nu)}{u}\right] d\sigma \Bigg\} \\
= \frac{\beta(\tau+1)^{\beta\frac{n-1}{n-2}-3}}{\tau-1}\Bigg\{ \left(\beta - \frac{n-2}{n-1}\right) &\int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1} \left[\mathrm{H} - \frac{n-1}{n-2}\,\frac{2u}{1-u^2}\,|\nabla u|\right]^2 d\sigma \\
&+ \beta \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-3}\,|\nabla^\top|\nabla u||^2\, d\sigma + \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1}|\mathring{\mathrm{h}}|^2\, d\sigma \\
&+ \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta-1}\left[\mathrm{Ric}(\nu,\nu) - \frac{\nabla du(\nu,\nu)}{u}\right] d\sigma \Bigg\}\,, \qquad (2.7)
\end{aligned}$$

where Δ^Σ is the Laplacian relative to the hypersurface/level set $\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}$. Here, we have used the normal first variation of volume measure and of mean curvature (see the end of Section 1.3) and the divergence theorem to obtain the expression of F''_β. The symbols H and h stand respectively for the mean curvature and the second fundamental form of $\left\{u = \sqrt{(\tau-1)/(\tau+1)}\right\}$ with respect to the ∞–pointing unit normal vector field $\nu = \nabla u/|\nabla u|$ and are given respectively by

$$\begin{aligned}
\mathrm{H} &= -\frac{\nabla du(\nabla u, \nabla u)}{|\nabla u|^3}\,, \qquad (2.8)\\
\mathrm{h}(X,Y) &= \frac{\nabla du\,(X,Y)}{|\nabla u|}\,,
\end{aligned}$$

for every pair of vector fields X, Y tangent to $\left\{u = \sqrt{(\tau-1)/(\tau+1)}\right\}$, see formulas (1.4) and (1.5). Also, $\nabla^\top$ denotes the tangential part of the gradient, that is,

$$\nabla^\top f = \nabla f - g(\nabla f, \nu)\nu,$$

for every $f \in C^1(M)$.
Then, in absence of critical points of u, the function F_β is twice differentiable in $(1, +\infty)$ with nonnegative second derivative by formula (2.7) together with the first

condition of system (2.1), therefore the first derivative F'_β is nondecreasing in $(1,+\infty)$. Then, being the limit of $F'_\beta(\tau)$ zero for $\tau\to+\infty$, we also conclude that F'_β is always nonpositive.
We now show the claim that $\lim_{\tau\to+\infty}F'_\beta(\tau)=0$. We first rewrite the function F'_β in the following way

$$F'_\beta(\tau)=-\beta\sqrt{\frac{\tau+1}{\tau-1}}(\tau+1)^{-\frac{n-3}{n-2}}\int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}}\left(\frac{|\nabla u|}{(1-u^2)^{\frac{n-1}{n-2}}}\right)^{\beta-1}\left[\mathrm{H}-\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|\right]|\nabla u|\,d\sigma\,.$$

Then, let $(x^1,\dots,x^n)$ be an AF coordinate chart of order τ. In these coordinates, $g=g_{ij}\,dx^i\otimes dx^j$. Then, we have, by formulas (1.21), (2.2) and (2.4),

$$\begin{aligned}\frac{|\nabla u|^2}{(1-u^2)^{2\frac{n-1}{n-2}}}&=\frac{g^{ij}\,\partial_i u\,\partial_j u}{(1-u^2)^{2\frac{n-1}{n-2}}}\\&=\frac{\big[\delta^{ij}+O\big(|x|^{-\tau}\big)\big]\,\big[(n-2)^2\,\mathcal{C}^2\,|x|^{-2n}\,x^ix^j+o(|x|^{2-2n})\big]}{\big[2\,\mathcal{C}\,|x|^{2-n}+o(|x|^{2-n})\big]^{2\,(\frac{n-1}{n-2})}}\\&=\frac{(n-2)^2\,\mathcal{C}^2}{(2\,\mathcal{C})^{2\,(\frac{n-1}{n-2})}}(1+o(1))\longrightarrow\frac{(n-2)^2\,\mathcal{C}^2}{(2\,\mathcal{C})^{2\,(\frac{n-1}{n-2})}}\qquad\text{at }\infty\,.\end{aligned}\tag{2.9}$$

with $\mathcal{C}=\frac{1}{(n-2)|\mathbb{S}^{n-1}|}\int_{\partial M}|\nabla u|\,d\sigma$. Concerning the term $\mathrm{H}-\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|$, by formula (2.8) it is always equal to

$$-\left[\frac{\nabla du(\nabla u,\nabla u)}{|\nabla u|^3}+\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|\right],$$

which is $o(|x|^{-1})$, arguing as before. This fact and limit (2.9) imply

$$\left(\frac{|\nabla u|}{(1-u^2)^{\frac{n-1}{n-2}}}\right)^{\beta-1}\left[\mathrm{H}-\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|\right]\longrightarrow 0\text{ at }\infty\,.$$

Therefore, for every $\varepsilon>0$ there exists $\tau_\varepsilon>1$ such that, whenever $p\in\{u\geq\sqrt{\frac{\tau_\varepsilon-1}{\tau_\varepsilon+1}}\}$, one has

$$-\varepsilon<\left(\frac{|\nabla u|}{(1-u^2)^{\frac{n-1}{n-2}}}\right)^{\beta-1}\left[\mathrm{H}-\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|\right](p)<\varepsilon$$

and consequently,

$$\int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}}\left(\frac{|\nabla u|}{(1-u^2)^{\frac{n-1}{n-2}}}\right)^{\beta-1}\left[\mathrm{H}-\frac{n-1}{n-2}\frac{2u}{1-u^2}|\nabla u|\right]|\nabla u|\,d\sigma\in\big[-a\varepsilon,\,a\varepsilon\big]\,,$$

for every $\tau\geq\tau_\varepsilon$, where $a:=(n-2)|\mathbb{S}^{n-1}|\,\mathcal{C}$, by virtue of property (2.6).

We underline that, by direct computation, one can check that the all functions F_β are constant, respectively identically equal to

$$(n-2)^{\beta+1}|\mathbb{S}^{n-1}|\,m^{1-\frac{\beta}{n-2}}$$

in $[1,+\infty)$, for every *Schwarzschild solution of mass* $m>0$.

Strategy of the proof. To give an idea, we focus our attention on the rigidity statement and for simplicity, we let $\beta = 2$. At the same time, we provide some heuristics for the monotonicity statement. The method is based on a conformal splitting technique which consists of two main steps. The first step is the construction of the so called *cylindrical ansatz* and amounts to finding an appropriate conformal metric $\overline{g}$ of g in terms of the potential u. In the case under consideration, the natural deformation is given by

$$\overline{g} = (1-u^2)^{\frac{2}{n-2}} g \,. \tag{2.10}$$

Indeed, when (M, g, u) is the Schwarzschild solution, the metric $\overline{g}$ obtained through the above formula is cylindrical, more precisely $(M, \overline{g})$ is a round cylinder (with boundary). In general, the cylindrical ansatz leads to a conformal reformulation of system (2.1), in which the metric $\overline{g}$ satisfies

$$Q := \overline{\mathrm{Ric}} - \coth(\varphi)\, \overline{\nabla} d\varphi + \frac{1}{n-2}\, d\varphi \otimes d\varphi - \frac{1}{n-2}\, \|\overline{\nabla}\varphi\|^2\, \overline{g} \geq 0 \qquad \text{in} \quad \mathrm{Int}(M)\,,$$

where ($\overline{\nabla}$, $\overline{\mathrm{R}}$, $\overline{\mathrm{Riem}}$, etc.) denote the Levi–Civita connection and the curvatures associated with $(M, \overline{g})$, $\|\cdot\|$ indicates the norm correlated with $\overline{g}$ and finally φ is the $\overline{g}$–harmonic function given by $\varphi = \log\left(\frac{1+u}{1-u}\right)$. The importance of the function φ lies in the fact that if (M, g, u) is the Schwarzschild solution, then φ is a non–trivial *affine* function, i.e. φ is smooth and its Hessian $\overline{\nabla} d\varphi$ vanishes everywhere (or equivalently, φ is smooth and its gradient vector field $\overline{\nabla}\varphi$ is parallel). Viceversa, it is known that a complete Riemannian manifold admitting a nonconstant affine function, splits as a Riemannian product, thus, if φ is an affine function with respect to $\overline{g}$, then $(M, \overline{g})$ splits as a Riemannian product in the direction $\overline{\nabla}\varphi$ as well as (M, g) in the direction ∇u and, being this latter asymptotically flat, it must be a piece of a Schwarzschild manifold, up to isometry. With this in mind, the second step of our strategy consists in finding a nonnegative or nonpositive quantity, whose vanishing guarantees that the function φ is affine. More precisely, we use the previous conformal reformulation of the original system together with the Bochner formula to deduce the inequality

$$\overline{\Delta}\, \|\overline{\nabla}\varphi\|^2 - \overline{g}\left(\overline{\nabla}\|\overline{\nabla}\varphi\|^2, \overline{\nabla}\log(\sinh\varphi)\right) = \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi) \geq 0\,,$$

in a way that

$$\overline{\mathrm{div}}\left(\frac{\|\overline{\nabla}\varphi\|^2}{\sinh\varphi}\right) = \frac{\|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)}{\sinh\varphi} \geq 0\,.$$

This will imply the monotonicity statement once we obtain the equality

$$F_2'(\tau) = -2^{\frac{2}{n-2}} \int\limits_{\{\varphi > s(\tau)\}} \frac{\|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)}{\sinh\varphi}\, d\overline{\mu} \leq 0\,.$$

A delicate point is justifying such identity in a region where critical points of the potential are present. Also, if the left hand side of the above identity vanishes, then the Hessian of φ must be zero in an open unbounded region of M. Consequently, the function φ is affine on this region of M with respect to $\overline{g}$ and the partial isometry with a Schwarzschild manifold follows.

Given a sub–static harmonic triple (M, g, u), let us consider the conformal change defined by

$$\overline{g} = (1-u^2)^{\frac{2}{n-2}} g\,, \qquad \varphi = \log\left(\frac{1+u}{1-u}\right). \tag{2.11}$$

The conformal metric $\overline{g}$ of g is well–defined, being $0 \leq u < 1$ in M. Moreover, one obtains that $(M, \overline{g})$ is complete by using the Hopf–Rinow theorem [73], together with the completeness of (M, g) and formulas (1.15) and (2.2) (we refer also to [20]). Clearly,

$$g = \cosh^{\frac{4}{n-2}}(\varphi/2)\,\overline{g}\,, \qquad u = \tanh(\varphi/2)\,.$$

From now on in this chapter, a bar over a symbol will denote the relative geometric object associated to the metric $\overline{g}$ on M, for instance,$(\overline{\nabla}, \overline{\mathrm{R}}, \overline{\mathrm{Riem}}$, etc.) will denote the Levi–Civita connection and the curvatures associated with $(M, \overline{g})$ and $\|\cdot\|$ will be the relative norm.

Then, it follows by the formulas in [10, Theorem 1.159] that

$$\nabla u = \frac{1}{2}\cosh^{-\frac{2n}{n-2}}(\varphi/2)\,\overline{\nabla}\varphi\,, \tag{2.12}$$

$$\nabla du = \frac{1}{2}\frac{\overline{\nabla} d\varphi}{\cosh^2(\varphi/2)} - \frac{1}{2(n-2)}\frac{\sinh(\varphi/2)}{\cosh^3(\varphi/2)}\left[n\, d\varphi\otimes d\varphi - \|\overline{\nabla}\varphi\|^2\,\overline{g}\right], \tag{2.13}$$

$$\Delta u = \frac{1}{2}\cosh^{-\frac{2n}{n-2}}(\varphi/2)\,\overline{\Delta}\varphi\,,$$

and

$$\begin{aligned}\mathrm{Ric} = \overline{\mathrm{Ric}} - \tanh(\varphi/2)\,\overline{\nabla} d\varphi + \left[\frac{\tanh^2(\varphi/2)}{n-2} - \frac{1}{2}\frac{1}{\cosh^2(\varphi/2)}\right] d\varphi\otimes d\varphi \\ - \frac{1}{(n-2)}\left[\frac{1}{2}\frac{1}{\cosh^2(\varphi/2)} + \tanh^2(\varphi/2)\right]\|\overline{\nabla}\varphi\|^2\,\overline{g}\,.\end{aligned}$$

Consequently, rewriting system (2.1) in terms of $\overline{g}$ and φ, we get that the triple $(M, \overline{g}, \varphi)$ satisfies

$$\begin{cases} \overline{\mathrm{Ric}} - \coth(\varphi)\,\overline{\nabla} d\varphi + \dfrac{1}{n-2}\, d\varphi\otimes d\varphi - \dfrac{1}{n-2}\,\|\overline{\nabla}\varphi\|^2\,\overline{g} \geq 0 & \text{in } \mathrm{Int}(M) \\ \overline{\Delta}\varphi = 0 & \text{in } M \\ \varphi = 0 & \text{on } \partial M \\ \varphi \to +\infty & \text{at } \infty \end{cases} \tag{2.14}$$

Remark 2.1.2. Since $\{|\nabla u| = 0\} = \mathrm{Crit}(u) = \mathrm{Crit}(\varphi) := \{\|\overline{\nabla}\varphi\| = 0\}$ and $\{\varphi = s\} = \{u = \tanh(s/2)\}$, it follows from the results at the beginning of this chapter that: $\mathrm{Crit}(\varphi)$ has zero $\overline{\mu}$–measure and zero $(n-1)$–Hausdorff measure in $(M, \overline{g})$; the level sets of φ have finite $(n-1)$–Hausdorff measure in $(M, \overline{g})$; there exists $s_0 \geq 0$ such that $\{\varphi = s\}$ is regular and diffeomorphic to $\mathbb{S}^{n-1}$ for every $s \geq s_0$; $\{s \leq \varphi \leq S\} = \overline{\{s < \varphi < S\}}$ and it is a compact set, for every $0 < s < S < +\infty$; $\{\varphi \geq s\}$ is connected for every $s \geq 0$. Finally, we will denote for simplicity by $\overline{\sigma}$ the $(n-1)$–Hausdorff measure of $(M, \overline{g})$.

Along $\{\varphi = s\} \setminus \mathrm{Crit}(\varphi)$, we consider the ∞–pointing normal unit vector field

$$\overline{\nu} = \frac{\overline{\nabla}\varphi}{\|\overline{\nabla}\varphi\|}\,,$$

with associated mean curvature

$$\overline{\mathrm{H}} = -\frac{\overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi)}{\|\overline{\nabla}\varphi\|^3}\,, \tag{2.15}$$

and second fundamental form

$$\overline{\mathrm{h}}(X, Y) = \frac{\overline{\nabla} d\varphi\,(X, Y)}{\|\overline{\nabla}\varphi\|}\,, \tag{2.16}$$

for every pair of vector fields X, Y tangent to $\{\varphi = s\} \setminus \mathrm{Crit}(\varphi)$, see formulas (1.4) and (1.5).

By formulas (2.12) and (2.13), we get

$$\overline{\nabla}\varphi = \frac{2}{(1-u^2)^{\frac{n}{n-2}}}\,\nabla u\,, \tag{2.17}$$
$$\overline{\nabla} d\varphi = \frac{2}{1-u^2}\left[\nabla du + \frac{n}{n-2}\,\frac{2u}{1-u^2}\,du\otimes du - \frac{1}{n-2}\,\frac{2u}{1-u^2}\,|\nabla u|^2\,g\right],$$

hence,

$$\|\overline{\nabla} d\varphi\|^2 = \frac{4}{(1-u^2)^{\frac{2n}{n-2}}}\,|\nabla du|^2 + \frac{16n}{n-2}\,\frac{u}{(1-u^2)^{\frac{3n-2}{n-2}}}\,\nabla du(\nabla u, \nabla u) + \frac{16n(n-1)}{(n-2)^2}\,\frac{u^2}{(1-u^2)^{4\left(\frac{n-1}{n-2}\right)}}\,|\nabla u|^4\,. \tag{2.18}$$

These equalities, together with the asymptotic flatness of (M, g) and the behavior at infinity of u, described at the beginning of this chapter, allow us to obtain an upper bound for the functions $\|\overline{\nabla}\varphi\|$ and $\|\overline{\nabla} d\varphi\|$ and for the $\overline{\sigma}$–measure of the level sets of φ sufficiently "close" to infinity. This is the content of the following lemma.

Lemma 2.1.3. *There exists $0 \leq s_0 < +\infty$ such that*

$$\sup_M \|\overline{\nabla}\varphi\| + \sup_M \|\overline{\nabla} d\varphi\| + \sup_{s\geq s_0} \int\limits_{\{\varphi=s\}} d\overline{\sigma} < +\infty\,.$$

Proof. Equality (2.17) and limit (2.9) imply

$$\|\overline{\nabla}\varphi\|^2 \longrightarrow \frac{4\,(n-2)^2\,\mathcal{C}^2}{(2\,\mathcal{C})^{2\left(\frac{n-1}{n-2}\right)}} = (2\,\mathcal{C})^{-\frac{2}{n-2}}(n-2)^2 \text{ at } \infty\,. \tag{2.19}$$

Now, computing the limit at ∞ of the right hand side of equality (2.18), as for the limit (2.9), thanks to formulas (1.17), (1.21), (2.2), (2.4) and (2.5), we have

$$\|\overline{\nabla} d\varphi\|^2 \longrightarrow (n-1)\,(15n-1)\,(n-2)^2\,(2\,\mathcal{C})^{-\frac{4}{n-2}} \quad \text{at } \infty\,. \tag{2.20}$$

In particular, from limits (2.19) and (2.20), it follows

$$\sup_M \|\overline{\nabla}\varphi\| + \sup_M \|\overline{\nabla} d\varphi\| < +\infty\,,$$

since φ is a smooth function. Moreover, as a consequence of limit (2.9), there exist a constant $L > 0$ and a value $s_0 > 0$ such that

$$(1-u^2)^{\frac{n-1}{n-2}} \leq L|\nabla u|$$

on $\{\varphi \geq s_0\}$. Then, we have that every $s \geq s_0$ is a regular value of φ and

$$\int\limits_{\{\varphi=s\}} d\overline{\sigma} = \int\limits_{\{u=\tanh(s/2)\}} (1-u^2)^{\frac{n-1}{n-2}}\, d\sigma \leq L \int\limits_{\{u=\tanh(s/2)\}} |\nabla u|\, d\sigma = L \int\limits_{\partial M} |\nabla u|\, d\sigma\,,$$

where the last equality follows by virtue of property (2.6) (as the set of the regular values of u coincides with the set of the regular values of φ, since there holds the equality $\{|\nabla u| = 0\} = \{\|\overline{\nabla}\varphi\| = 0\}$). Thus,

$$\sup_{s \geq s_0} \int\limits_{\{\varphi=s\}} d\overline{\sigma} < +\infty\,.$$

$\square$

A key point of our argument is to exhibit a suitable vector field with nonnegative divergence, relative to the conformal metric $\overline{g}$. To do this, let us focus on the set $\mathrm{Int}(M) \setminus \mathrm{Crit}(\varphi)$ and notice first that the classical Bochner formula, applied to the $\overline{g}$–harmonic function φ, becomes

$$\begin{aligned} \frac{1}{2}\,\overline{\Delta}\,\|\overline{\nabla}\varphi\|^2 &= \|\overline{\nabla} d\varphi\|^2 + \overline{\mathrm{Ric}}(\overline{\nabla}\varphi, \overline{\nabla}\varphi) + \overline{g}(\overline{\nabla}\,\overline{\Delta}\varphi, \overline{\nabla}\varphi) \\ &= \|\overline{\nabla} d\varphi\|^2 + \overline{\mathrm{Ric}}(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\,. \end{aligned} \tag{2.21}$$

Then, we obtain

$$\begin{aligned} \overline{\Delta}\|\overline{\nabla}\varphi\|^\beta &= \overline{\mathrm{div}}\Big(\overline{\nabla}\|\overline{\nabla}\varphi\|^\beta\Big) = \frac{\beta}{2}\,\overline{\mathrm{div}}\Big(\|\overline{\nabla}\varphi\|^{\beta-2}\,\overline{\nabla}\|\overline{\nabla}\varphi\|^2\Big) \\ &= \frac{\beta}{2}\left[\overline{g}\left(\overline{\nabla}\|\overline{\nabla}\varphi\|^{\beta-2}, \overline{\nabla}\|\overline{\nabla}\varphi\|^2\right) + \|\overline{\nabla}\varphi\|^{\beta-2}\overline{\Delta}\|\overline{\nabla}\varphi\|^2\right] \\ &= \beta\,\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + \overline{\mathrm{Ric}}(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right]\,, \end{aligned} \tag{2.22}$$

where in the third equality we used formula (2.21). Now, we observe from the non-negativity of the tensor

$$Q := \overline{\mathrm{Ric}} - \coth(\varphi)\,\overline{\nabla} d\varphi + \frac{1}{n-2}\,d\varphi \otimes d\varphi - \frac{1}{n-2}\,\|\overline{\nabla}\varphi\|^2\,\overline{g} \tag{2.23}$$

(see system (2.14)) that

$$Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi) = \overline{\mathrm{Ric}}(\overline{\nabla}\varphi, \overline{\nabla}\varphi) - \coth(\varphi)\,\overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi) \geq 0\,. \tag{2.24}$$

Therefore, by adding and subtracting the term $\beta\|\overline{\nabla}\varphi\|^{\beta-2}\coth(\varphi)\overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi)$ on the right–hand side of equality (2.22), we get

$$\begin{aligned} &\overline{\Delta}\|\overline{\nabla}\varphi\|^\beta - \beta\|\overline{\nabla}\varphi\|^{\beta-2}\coth(\varphi)\overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi) \\ &\qquad = \beta\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right]\,. \end{aligned} \tag{2.25}$$

Since

$$\beta\,\|\overline{\nabla}\varphi\|^{\beta-2}\coth(\varphi)\,\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)=\coth(\varphi)\,\overline{g}\Big(\overline{\nabla}\|\overline{\nabla}\varphi\|^{\beta},\overline{\nabla}\varphi\Big)\,,$$

setting

$$Y_\beta:=\frac{\overline{\nabla}\|\overline{\nabla}\varphi\|^{\beta}}{\sinh\varphi}\,, \tag{2.26}$$

there holds

$$\overline{\operatorname{div}}\,Y_\beta=\frac{\overline{\Delta}\|\overline{\nabla}\varphi\|^{\beta}}{\sinh\varphi}-\frac{\cosh\varphi}{\sinh^2\varphi}\,\overline{g}\Big(\overline{\nabla}\|\overline{\nabla}\varphi\|^{\beta},\overline{\nabla}\varphi\Big)$$

and from equality (2.25), we get

$$\sinh(\varphi)\,\overline{\operatorname{div}}\,Y_\beta=\beta\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\|\overline{\nabla}d\varphi\|^2+Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right].$$

Now, the *refined Kato inequality* for harmonic functions,

$$\frac{n}{n-1}\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2\leq\|\overline{\nabla}d\varphi\|^2\,, \tag{2.27}$$

which is a consequence of

$$\begin{aligned}\|\overline{\nabla}d\varphi\|^2&=\|\overline{\nabla}\varphi\|^2\|\overline{\mathrm{h}}\|^2+\|\overline{\nabla}\|\overline{\nabla}\varphi\|\|^2+\|\nabla^\top\|\overline{\nabla}\varphi\|\|^2\\&=\|\overline{\nabla}\varphi\|^2\|\mathring{\overline{\mathrm{h}}}\|^2+\frac{n}{n-1}\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\frac{n-2}{n-1}\|\nabla^\top\|\overline{\nabla}\varphi\|\|^2\,,\end{aligned}$$

where the first identity is simply the definition of norm of a tensor (see [23, Proposition 18]), while the second one follows from equality (2.15), implies

$$\begin{aligned}&(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\|\overline{\nabla}d\varphi\|^2+Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\\&\quad=\Big(\beta-\frac{n-2}{n-1}\Big)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\Big[\|\overline{\nabla}d\varphi\|^2-\frac{n}{n-1}\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2\Big]+Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\geq0\,,\end{aligned} \tag{2.28}$$

whenever $\beta\geq\frac{n-2}{n-1}$. Hence,

$$\overline{\operatorname{div}}\,Y_\beta\geq0\qquad\text{for every }\beta\geq\frac{n-2}{n-1}\,. \tag{2.29}$$

We now show some fundamental integral identities.

Proposition 2.1.4. *Let (M,g,u) be a sub–static harmonic triple, let $\overline{g}$ and φ be the metric and the function defined by formulas (2.11). Then, the following integral identities hold.*

(1) *For every $\beta>0$ and for every $0<s<S<+\infty$,*

$$\begin{aligned}&\int\limits_{\{\varphi=S\}}\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma}-\int\limits_{\{\varphi=s\}}\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma}=\\&\qquad=\int\limits_{\{s<\varphi<S\}}\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)-\coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu}\,.\end{aligned} \tag{2.30}$$

(2) *For every* $\beta > 0$ *and for every* $0 < s < +\infty$,

$$\int_{\{\varphi=s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\, d\overline{\sigma} = \int_{\{\varphi>s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\coth(\varphi)\|\overline{\nabla}\varphi\|^4 - \beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right]}{\sinh\varphi}\, d\overline{\mu}\,. \tag{2.31}$$

(3) *For every* $\beta > \frac{n-2}{n-1}$ *and for every* $0 < s < S < +\infty$ *regular values of the function* φ,

$$\int_{\{\varphi=s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} - \int_{\{\varphi=S\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} = \tag{2.32}$$

$$= \int_{\{s<\varphi<S\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 + \|\overline{\nabla}d\varphi\|^2 + Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right]}{\sinh\varphi}\, d\overline{\mu} \geq 0,$$

where the tensor Q *is defined by formula* (2.23).

(4) *For every* $\beta > \frac{n-2}{n-1}$ *and for every* $0 < s < +\infty$ *regular value of the function* φ,

$$\int_{\{\varphi=s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} \tag{2.33}$$

$$= \int_{\{\varphi>s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 + \|\overline{\nabla}d\varphi\|^2 + Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right]}{\sinh\varphi}\, d\overline{\mu} \geq 0\,.$$

We underline that in this proposition and in all the sequel, in all the integrals Remark 2.1.2 is kept into account tacitly.

Before to proceed, we remark that the integrals in the right hand side of equalities (2.30) and (2.31) are well–defined. Indeed,

$$\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\coth(\varphi)\|\overline{\nabla}\varphi\|^4 - \beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right]}{\sinh\varphi} \in L^1\big(\{\varphi \geq s\};\overline{\mu}\big) \tag{2.34}$$

for every $s \in (0,+\infty)$, as there holds

$$\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left|\coth(\varphi)\|\overline{\nabla}\varphi\|^4 - \beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right|}{\sinh\varphi} \leq \left[\coth(\varphi)\,\|\overline{\nabla}\varphi\|^2 + \beta\|\overline{\nabla}d\varphi\|\right]\frac{\|\overline{\nabla}\varphi\|^{\beta}}{\sinh\varphi}$$

$\overline{\mu}$–a.e. in $\{\varphi \geq s\}$ and the function at the right hand side of this inequality belongs $L^1\big(\{\varphi \geq S\},\overline{\mu}\big)$ for S sufficiently large, by the coarea formula (1.1), in connection with limit (2.19) and Lemma 2.1.3. Notice that the well–definition of the integrals in the right hand side of equalities (2.32) and (2.33) is instead immediate since the integrand function is well–defined and nonnegative $\overline{\mu}$–a.e. in M, by formula (2.28).

Finally, we remark that the first two points can be proved as in [2, Proposition 4.1]. Here, we provide an alternate proof which is self–contained and does not make use of any fine measure–theoretic property of $\mathrm{Crit}(\varphi)$.

Proof of Proposition 2.1.4 (1). For every $\beta > 0$, we consider on the open set $\mathrm{Int}(M) \setminus \mathrm{Crit}(\varphi)$ the vector field

$$X_\beta = \frac{\|\overline{\nabla}\varphi\|^\beta \, \overline{\nabla}\varphi}{\sinh\varphi}\,,$$

which satisfies

$$\overline{\mathrm{div}}\, X_\beta = \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi) - \coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,.$$

If $\{s \le \varphi \le S\} \cap \mathrm{Crit}(\varphi) = \emptyset$, then the statement is a straightforward application of the divergence theorem to X_β on $\{s < \varphi < S\}$ (see Remark 1.1.1). Now, suppose that $\{s \le \varphi \le S\}\cap\mathrm{Crit}(\varphi) \neq \emptyset$. Since there always exists $\overline{s} \in (s, S)$ regular value of φ, up to splitting the right–hand side of equality (2.30) into two integrals by virtue of the fact that its integrand belongs to $L^1\big(\{s \le \varphi \le S\}; \overline{\mu}\big)$, we can suppose without loss of generality that one among s and S is a regular value of φ. To fix the ideas, suppose that S is the regular value. We are going to change the function φ in a neighborhood of the set $\mathrm{Crit}(\varphi)$. To do this, for every $\varepsilon > 0$ sufficiently small, by Sard's theorem, we can fix a positive real number $\delta(\varepsilon)$ such that $s + \delta(\varepsilon) < S$ is a regular value of φ and $\delta(\varepsilon) < d\varepsilon$, where $d > 0$ will be specified later. Then, considering a smooth and nondecreasing cut–off function $\xi_\varepsilon : [0, +\infty) \to [0, 1]$ satisfying the conditions

$$\xi_\varepsilon(\tau) = 0 \quad \text{in } \left[0, \frac{1}{2}\varepsilon\right], \quad 0 \le \xi'_\varepsilon(\tau) \le \frac{c}{\varepsilon} \quad \text{in } \left[\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon\right], \quad \xi_\varepsilon(\tau) = 1 \quad \text{in } \left[\frac{3}{2}\varepsilon, +\infty\right), \tag{2.35}$$

where c is a positive real constant independent of ε, we define on M the following smooth functions

$$\Xi_\varepsilon := \xi_\varepsilon \circ \|\overline{\nabla}\varphi\|^2\,, \tag{2.36}$$

and

$$\varphi_\varepsilon := \varphi - (1 - \Xi_\varepsilon)\,\delta(\varepsilon)\,.$$

Clearly,

$$\nabla\varphi_\varepsilon = \nabla\varphi + \delta(\varepsilon)\,\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\,\overline{\nabla}\|\overline{\nabla}\varphi\|^2 \tag{2.37}$$

and

$$\varphi_\varepsilon = \varphi \text{ in } \left\{\|\overline{\nabla}\varphi\|^2 \ge \frac{3}{2}\varepsilon\right\}. \tag{2.38}$$

Notice that s is a regular value for the function φ_ε. To see this, let p be a point of $\{\varphi_\varepsilon = s\}$ and distinguish two cases

$$\|\overline{\nabla}\varphi\|^2(p) \le \frac{1}{2}\varepsilon\,, \qquad \|\overline{\nabla}\varphi\|^2(p) > \frac{1}{2}\varepsilon\,.$$

In the first case, $s = \varphi_\varepsilon(p) = \varphi(p) - \delta(\varepsilon)$ and $\nabla\varphi_\varepsilon(p) = \nabla\varphi(p)$. Since $s + \delta(\varepsilon)$ is a regular value for φ, it follows then $\nabla\varphi_\varepsilon(p) \neq 0$.
In the second case, observing that $s \le \varphi(p) \le s + \delta(\varepsilon)$, there holds $p \in \{s \le \varphi \le S\}$.

Consequently, we have from formula (2.37) at the point p

$$\begin{aligned}\|\overline{\nabla}\varphi_\varepsilon\| &\geq \|\overline{\nabla}\varphi\| - \delta(\varepsilon)\,\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\right\| \\ &= \|\overline{\nabla}\varphi\|\Big(1 - 2\delta(\varepsilon)\,\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|\Big) \\ &\geq \|\overline{\nabla}\varphi\|\Big(1 - 2d\,\varepsilon\,\frac{c}{\varepsilon}\max_{\{s\leq\varphi\leq S\}}\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|\Big)\,,\end{aligned}$$

where c is given by conditions (2.35). Now, notice that $\max_{\{s\leq\varphi\leq S\}}\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\| > 0$, otherwise, due to the presence of critical points in $\{s \leq \varphi \leq S\}$, there should be a connected component of $\{s \leq \varphi \leq S\}$ where $\nabla\varphi \equiv 0$, but this is impossible because $\{s \leq \varphi \leq S\} = \overline{\{s < \varphi < S\}}$ and by the size of $\mathrm{Crit}(\varphi)$. Hence, choosing

$$d \leq \frac{1}{4\,c\max\limits_{\{s\leq\varphi\leq S\}}\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|}\,,$$

we obtain $\|\overline{\nabla}\varphi_\varepsilon\|(p) \geq \|\overline{\nabla}\varphi\|(p)/2 > \sqrt{\varepsilon}/(2\sqrt{2})$.
Moreover, for $\varepsilon > 0$ sufficiently small, S is also a regular value of φ_ε and $\{\varphi_\varepsilon = S\} = \{\varphi = S\}$, as there exists $0 < \varepsilon < S - s$ such that the interval $[S-\varepsilon, S+\varepsilon]$ does not contain critical values of φ and $\{S - \varepsilon \leq \varphi \leq S + \varepsilon\}$ is compact.
By virtue of fact that s, S are regular values of φ_ε, we apply the divergence theorem to the vector field $\Xi_{4\varepsilon}X_\beta$ in $\{s < \varphi_\varepsilon < S\}$. Then,

$$\begin{aligned}&\int\limits_{\{\varphi_\varepsilon=S\}} \overline{g}\left(\Xi_{4\varepsilon}X_\beta, \frac{\overline{\nabla}\varphi_\varepsilon}{\|\overline{\nabla}\varphi_\varepsilon\|}\right) d\overline{\sigma} - \int\limits_{\{\varphi_\varepsilon=s\}} \overline{g}\left(\Xi_{4\varepsilon}X_\beta, \frac{\overline{\nabla}\varphi_\varepsilon}{\|\overline{\nabla}\varphi_\varepsilon\|}\right) d\overline{\sigma} \\ &\qquad = \int\limits_{\{s<\varphi_\varepsilon<S\}} \Xi_{4\varepsilon}\,\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi) - \coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu} \\ &\qquad\quad + 2\int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi_\varepsilon<S\}} \frac{\xi'_{4\varepsilon}\left(\|\overline{\nabla}\varphi\|^2\right)\|\overline{\nabla}\varphi\|^\beta\,\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)}{\sinh\varphi}\,d\overline{\mu}\,,\end{aligned}$$

where

$$U_\mu := \{\|\overline{\nabla}\varphi\|^2 < \mu\} \quad \text{for every } \mu > 0\,. \tag{2.39}$$

By construction $\Xi_{4\varepsilon} \equiv 0$ in $\{\|\overline{\nabla}\varphi\|^2 \leq 2\varepsilon\} \supseteq \{\|\overline{\nabla}\varphi\|^2 \leq (3/2)\varepsilon\}$, therefore from formula (2.38) it follows

$$\begin{aligned}&\int\limits_{\{\varphi=S\}} \Xi_{4\varepsilon}\,\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma} - \int\limits_{\{\varphi=s\}} \Xi_{4\varepsilon}\,\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma} \\ &\qquad = \int\limits_{\{s<\varphi<S\}} \Xi_{4\varepsilon}\,\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi) - \coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu} \\ &\qquad\quad + 2\int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\xi'_{4\varepsilon}\left(\|\overline{\nabla}\varphi\|^2\right)\|\overline{\nabla}\varphi\|^\beta\,\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)}{\sinh\varphi}\,d\overline{\mu}\,.\end{aligned} \tag{2.40}$$

Looking at the right–hand side of equality (2.40), we notice

$$\left| \int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\xi'_{4\varepsilon}(\|\overline{\nabla}\varphi\|^2)\,\|\overline{\nabla}\varphi\|^\beta\,\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)}{\sinh\varphi}\,d\overline{\mu}\right| \leq \frac{C}{\varepsilon}\int\limits_{U_{6\varepsilon}\cap\{s<\varphi<S\}} \|\overline{\nabla}\varphi\|^{\beta+2}\|\overline{\nabla}d\varphi\|\,d\overline{\mu}$$
$$\leq C\frac{\varepsilon^{\frac{\beta}{2}+1}}{\varepsilon}\overline{\mu}(\{s\leq\varphi\leq S\})\to 0\,,$$

where we have used the properties (2.35) of ξ_ε to obtain the first inequality and Lemma 2.1.3 for the second inequality, while, by the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0^+}\int\limits_{\{s<\varphi<S\}} \Xi_{4\varepsilon}\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)-\coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu}$$
$$=\int\limits_{\{s<\varphi<S\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)-\coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu}\,.$$

Finally, looking at the left–hand side of equality (2.40), we notice that $\{\varphi=S\}$ is a compact set contained in $\{\|\overline{\nabla}\varphi\|^2>6\varepsilon\}$ for $\varepsilon>0$ sufficiently small and we observe that ξ_ε can always be chosen to be nonincreasing in ε in a way that, in turn, Ξ_ε is nonincreasing, therefore by the monotone convergence theorem it follows

$$\lim_{\varepsilon\to 0^+}\int\limits_{\{\varphi=s\}}\Xi_{4\varepsilon}\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma}=\int\limits_{\{\varphi=s\}}\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma}\,.$$

Passing to the limit as $\varepsilon\to 0^+$ in equality (2.40), we obtain the desired identity. □

Proof of Proposition 2.1.4 (2). Lemma 2.1.3 implies

$$\lim_{S\to+\infty}\int\limits_{\{\varphi=S\}}\frac{\|\overline{\nabla}\varphi\|^{\beta+1}}{\sinh\varphi}\,d\overline{\sigma}=0\,.$$

Therefore, passing to the limit as $S\to+\infty$ in equality (2.30) and using the dominated convergence theorem, by virtue of formula (2.34), the statement follows. □

In order to prove Proposition 2.1.4 (3), it is useful to have a precise estimate of $\int_{\{\|\overline{\nabla}\varphi\|^2=\delta\}}\frac{\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\|}{\sinh\varphi}\,d\overline{\sigma}$ in terms of a suitable power of δ, for every δ regular value of $\|\overline{\nabla}\varphi\|^2$ sufficiently close to zero (notice that the set of the critical values of $\|\overline{\nabla}\varphi\|^2$ has zero Lebesgue measure by Sard's theorem being $\|\overline{\nabla}\varphi\|^2$ a smooth function). This is the content of the following lemma.

Lemma 2.1.5. *There exists $\delta_0>0$ such that*

$$\sup\left\{\frac{1}{\delta^{\frac{1}{2}\frac{n}{n-1}}}\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\delta\}}\frac{\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\|}{\sinh\varphi}\,d\overline{\sigma}\ :\ 0<\delta<\delta_0 \textit{ is a regular value of } \|\overline{\nabla}\varphi\|^2\right\}<+\infty\,.$$

Proof. Applying Sard's theorem to the smooth function $\|\overline{\nabla}\varphi\|^2$, there exists a regular value ε_0 of $\|\overline{\nabla}\varphi\|^2$ such that

$$0<\varepsilon_0<\min_{\partial M}\|\overline{\nabla}\varphi\|^2,\ \text{the limit of } \|\overline{\nabla}\varphi\|^2 \text{ at } \infty\,.$$

Here, the limit exists and is equal to $(2\mathcal{C})^{-\frac{1}{n-2}}(n-2)$ by formula (2.19). In particular, $\{\|\overline{\nabla}\varphi\|^2\leq\varepsilon_0\}$ is a compact set contained in $\mathrm{Int}(M)$. Now, let us consider on $\{0<\|\overline{\nabla}\varphi\|^2\leq\varepsilon_0\}$ the vector field

$$Z:=2\,\frac{n-1}{n-2}\,Y_{\frac{n-2}{n-1}}=\frac{1}{\sinh\varphi}\,\frac{\overline{\nabla}\|\overline{\nabla}\varphi\|^2}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}$$

having $\overline{\mathrm{div}}Z\geq 0$, by definition (2.26) and formula (2.29). Therefore, for every regular value $0<\varepsilon<\varepsilon_0$ of $\|\overline{\nabla}\varphi\|^2$, applying the divergence theorem to Z on $U_{\varepsilon_0}\setminus\overline{U_\varepsilon}$ (see Remark 1.1.1), where U_μ is given by formula (2.39), we get

$$\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon_0\}}\frac{1}{\sinh\varphi}\,\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}\,d\overline{\sigma}-\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon\}}\frac{1}{\sinh\varphi}\,\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}\,d\overline{\sigma}=\int\limits_{U_{\varepsilon_0}\setminus\overline{U_\varepsilon}}\overline{\mathrm{div}}Z\,d\overline{\mu}\geq 0\,,$$

then, it follows

$$\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon_0\}}\frac{1}{\sinh\varphi}\,\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}\,d\overline{\sigma}\geq\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon\}}\frac{1}{\sinh\varphi}\,\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}\,d\overline{\sigma}\,.$$

Consequently, setting

$$c_1:=\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon_0\}}\frac{1}{\sinh\varphi}\,\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\|\overline{\nabla}\varphi\|^{\frac{n}{n-1}}}\,d\overline{\sigma}>0\,,$$

we obtain

$$\frac{1}{\varepsilon^{\frac{1}{2}\frac{n}{n-1}}}\int\limits_{\{\|\overline{\nabla}\varphi\|^2=\varepsilon\}}\frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\sinh\varphi}\,d\overline{\sigma}\leq c_1\,.$$

Thus, the thesis follows. □

Proof of Proposition 2.1.4 (3). In $\mathrm{Int}(M)\setminus\mathrm{Crit}(\varphi)$ we consider the vector field Y_β, defined by formula (2.26), satisfying

$$0\leq\overline{\mathrm{div}}\,Y_\beta=\frac{\beta\|\overline{\nabla}\varphi\|^{\beta-2}\Big[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\|\overline{\nabla}d\varphi\|^2+Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\Big]}{\sinh\varphi}\,.$$

When $\{s<\varphi<S\}\cap\mathrm{Crit}(\varphi)=\emptyset$, then the statement is a straightforward application of the divergence theorem to Y_β on $\{s<\varphi<S\}$ keeping into account both Remark 1.1.1 and expression (2.15) of mean curvature along regular level sets of φ. Now, suppose that $\{s<\varphi<S\}\cap\mathrm{Crit}(\varphi)\neq\emptyset$. In this case, we consider for every $\varepsilon>0$ sufficiently small the smooth nondecreasing cut–off function $\xi_\varepsilon:[0,+\infty)\to[0,1]$ satisfying conditions (2.35) and the smooth function $\Xi_\varepsilon:M\to[0,1]$, given by formula (2.36) and we apply the divergence theorem (in the same way, as before) to the vector field $\Xi_\varepsilon\, Y_\beta$ in $\{s < \varphi < S\}$. It follows

$$\begin{aligned}
&\int\limits_{\{\varphi=s\}} \frac{\|\overline{\nabla}\varphi\|^\beta\, \overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} - \int\limits_{\{\varphi=S\}} \frac{\|\overline{\nabla}\varphi\|^\beta\, \overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} \\
&\qquad = \int\limits_{\{s<\varphi<S\}} \Xi_\varepsilon\, \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\Big[(\beta-2)\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\Big]}{\sinh\varphi}\, d\overline{\mu} \\
&\qquad\quad + \int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\,\|\overline{\nabla}\varphi\|^{\beta-2}\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|^2}{2\sinh\varphi}\, d\overline{\mu}\,,
\end{aligned}$$

where U_μ is defined by formula (2.39). Notice that ξ_ε can always be chosen to be nonincreasing in ε, hence, also Ξ_ε is nonincreasing. Therefore, applying the monotone convergence theorem, as $\varepsilon \to 0^+$, the first term on the right of the equality tends to

$$\int\limits_{\{s<\varphi<S\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\Big[(\beta-2)\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\Big]}{\sinh\varphi}\, d\overline{\mu}\,.$$

To conclude, we finally need to show that

$$\lim_{\varepsilon\to 0^+} \int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\,\|\overline{\nabla}\varphi\|^{\beta-2}\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|^2}{2\sinh\varphi}\, d\overline{\mu} = 0\,. \tag{2.41}$$

First we observe that

$$\begin{aligned}
&\int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\,\|\overline{\nabla}\varphi\|^{\beta-2}\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|^2}{2\sinh\varphi}\, d\overline{\mu} \\
&\qquad\qquad \le \int\limits_{U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}} \frac{\xi'_\varepsilon\left(\|\overline{\nabla}\varphi\|^2\right)\,\|\overline{\nabla}\varphi\|^{\beta-2}\,\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|^2}{2\sinh\varphi}\, d\overline{\mu} \\
&\qquad\qquad \le \frac{c}{2\varepsilon}\int_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2}}\, ds \int\limits_{\{\|\overline{\nabla}\varphi\|^2=s\}} \frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\sinh\varphi}\, d\overline{\sigma}\,,
\end{aligned}$$

where, keeping in mind the properties satisfied by ξ_ε, in the first inequality we have used the nonnegativity of the integrand and in the last one the coarea formula (1.1). Now, the Sard's theorem applied to the smooth function $\|\overline{\nabla}\varphi\|^2$ and Lemma 2.1.5 imply the existence of $\varepsilon_0,\, c_1 > 0$ such that the inequality

$$\frac{1}{s^{\frac{1}{2}\frac{n}{n-1}}} \int\limits_{\{\|\overline{\nabla}\varphi\|^2=s\}} \frac{\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\big\|}{\sinh\varphi}\, d\overline{\sigma} \le c_1$$

holds for a.e. $s \in \left[\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon\right]$ for every $0 < \varepsilon < \frac{2}{3}\varepsilon_0$, therefore we get

$$\int\limits_{\left(U_{\frac{3}{2}\varepsilon} \setminus \overline{U_{\frac{1}{2}\varepsilon}}\right) \cap \{s<\varphi<S\}} \frac{\xi_\varepsilon'\left(\|\overline{\nabla}\varphi\|^2\right)\, \|\overline{\nabla}\varphi\|^{\beta-2} \left\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\right\|^2}{2\sinh\varphi}\, d\overline{\mu}$$
$$\leq \frac{c}{2\varepsilon} \int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2}}\, ds \int\limits_{\{\|\overline{\nabla}\varphi\|^2 = s\}} \frac{\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|^2\right\|}{\sinh\varphi}\, d\overline{\sigma} \leq \frac{c\, c_1}{2\varepsilon} \int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2} + \frac{1}{2}\frac{n}{n-1}}\, ds \leq C\, \varepsilon^{\frac{1}{2}\left(\beta - \frac{n-2}{n-1}\right)},$$

which clearly implies limit (2.41), as $\beta > \frac{n-2}{n-1}$. □

Proof of Proposition 2.1.4 (4). For every S large enough, S is a regular value of φ and Lemma 2.1.3 together with limit (2.19) imply

$$\left| \int\limits_{\{\varphi = S\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma} \right| \leq \int\limits_{\{\varphi=S\}} \|\overline{\nabla}\varphi\|^{\beta-1}\, \|\overline{\nabla} d\varphi\|\, d\overline{\sigma} \leq C\,.$$

In particular,

$$\lim_{S \to +\infty} \frac{1}{\sinh S} \int\limits_{\{\varphi = S\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma} = 0\,.$$

Therefore, the desired identity can be obtained by the monotone convergence theorem, by passing to the limit as $S \to +\infty$, in equality (2.32). □

Remark 2.1.6. For every $\beta > \frac{n-2}{n-1}$, as a consequence of integral identity (2.32), we have

$$\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right] \in L^1_{loc}\big(\mathrm{Int}(M), \overline{\mu}\big)\,. \tag{2.42}$$

Consequently, there holds

$$\|\overline{\nabla}\varphi\|^{\beta-3}\; \overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi) \;\in L^1_{loc}\big(\mathrm{Int}(M), \overline{\mu}\big)\,, \tag{2.43}$$

since

$$\int\limits_K \|\overline{\nabla}\varphi\|^{\beta-3}\, |\overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi)|\, d\overline{\mu} \leq \int\limits_K \|\overline{\nabla}\varphi\|^{\beta-1}\, \left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|\, d\overline{\mu}$$
$$= \int\limits_K \|\overline{\nabla}\varphi\|^{\frac{\beta}{2}}\, \|\overline{\nabla}\varphi\|^{\frac{\beta-2}{2}} \left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|\, d\overline{\mu}$$

for every compact set $K \subseteq \mathrm{Int}(M)$, then applying Hölder inequality, keeping into account formulas (2.28) and (2.42).
Moreover, for every $\beta > \frac{n-2}{n-1}$, the integral identity (2.33) implies that

$$\frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\left\|\overline{\nabla}\|\overline{\nabla}\varphi\|\right\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right]}{\sinh\varphi} \in L^1\big(\{\varphi \geq s\}; \overline{\mu}\big)$$

for all $s \in (0, +\infty)$, by the Sard's theorem and being the integrand function well–defined and nonnegative $\overline{\mu}$–a.e. in M.

By means of these integral identities, we are able to show the following proposition of monotony and partial rigidity which, reread in terms of the original data, will allow us to obtain Theorem 2.1.1.

Proposition 2.1.7. *Let (M, g, u) be a sub–static harmonic triple, let $\overline{g}$ and φ be the metric and the function defined by formulas (2.11) and let $\Phi_\beta : [0, \infty) \to \mathbb{R}$ be the function*

$$\Phi_\beta(s) := \int\limits_{\{\varphi = s\}} \|\overline{\nabla}\varphi\|^{\beta+1}\, d\overline{\sigma} \tag{2.44}$$

for every $\beta > \frac{n-2}{n-1}$. Then, Φ_β is continuously differentiable with nonpositive derivative given by

$$\Phi'_\beta(s) = -\beta \sinh(s) \int\limits_{\{\varphi > s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\Big[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\Big]}{\sinh\varphi}\, d\overline{\mu} \leq 0\,, \tag{2.45}$$

for every $s > 0$.
Moreover, if there exists $s_0 > 0$ such that $\Phi'_\beta(s_0) = 0$ for some $\beta > \frac{n-2}{n-1}$, then s_0 is a regular value of φ and $(\{\varphi \geq s_0\}, \overline{g})$ is isometric to $\big([0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + \overline{g}_{\{\varphi = s_0\}}\big)$, where ρ is the $\overline{g}$–distance function from $\{\varphi = s_0\}$ and φ is an affine function of ρ in $\{\varphi \geq s_0\}$, i.e. there exist $a, b \in \mathbb{R}$ such that $\varphi = a\rho + b$.

Notice that we now know the integrability of the integrand function in formula (2.45) on the unbounded set of integration, by Remark 2.1.6.

Proof. We divide the proof in three steps.
Step 1 – Continuity. The boundary ∂M is a regular level set of φ, therefore there exists $\varepsilon > 0$ such that the interval $[0, \varepsilon]$ does not contain critical values of φ. Consequently, we can consider on $\{0 \leq \varphi \leq \varepsilon\}$ the vector field $\|\overline{\nabla}\varphi\|^\beta\, \overline{\nabla}\varphi$ and apply to such field the divergence theorem in $\{0 < \varphi < \varepsilon\}$. Then, by Remark 1.1.1, we obtain

$$\begin{aligned}\Phi_\beta(\varepsilon) - \Phi_\beta(0) &= \int\limits_{\{\varphi=\varepsilon\}} \|\overline{\nabla}\varphi\|^{\beta+1}\, d\overline{\sigma} - \int\limits_{\{\varphi=0\}} \|\overline{\nabla}\varphi\|^{\beta+1}\, d\overline{\sigma} = \int\limits_{\{0<\varphi<\varepsilon\}} \overline{\mathrm{div}}\big(\|\overline{\nabla}\varphi\|^\beta\, \overline{\nabla}\varphi\big)\, d\overline{\mu} \\ &= \int\limits_{\{0<\varphi<\varepsilon\}} \overline{g}\big(\nabla\|\overline{\nabla}\varphi\|^\beta, \overline{\nabla}\varphi\big)\, d\overline{\mu} = \int\limits_{\{0<\varphi<\varepsilon\}} \beta\, \|\overline{\nabla}\varphi\|^{\beta-2}\, \overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\, d\overline{\mu}\,,\end{aligned} \tag{2.46}$$

where the third equality follows as φ is a $\overline{g}$–harmonic function. Then, the dominate convergence theorem implies the continuity of Φ_β at 0. Now, a straightforward application of Proposition 2.1.4 (1) and of the dominate convergence theorem gives the right and left continuity of the function

$$\Upsilon_\beta : s \in (0, +\infty) \to \frac{\Phi_\beta(s)}{\sinh s} \in \mathbb{R}\,, \tag{2.47}$$

then Φ_β is also continuous in $(0, +\infty)$.
Step 2 – Continuous differentiability and monotonicity. By applying the coarea formula (1.1) in equality (2.46) and using identity (2.15), we obtain

$$\Phi_\beta(\varepsilon) = \Phi_\beta(0) + \beta \int_0^\varepsilon ds \int_{\{\varphi=s\}} \|\overline{\nabla}\varphi\|^{\beta-3}\, \overline{\nabla} d\varphi(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\, d\overline{\sigma} = \Phi_\beta(0) - \beta \int_0^\varepsilon ds \int_{\{\varphi=s\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma}\,.$$

Now, being

$$\begin{aligned}
&\int_{\{\varphi=s_1\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma} - \int_{\{\varphi=s_2\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma} \\
&\quad = \int_{\{\varphi=s_2\}} \|\overline{\nabla}\varphi\|^{\beta-1}\, \overline{g}\Big(\nabla \|\overline{\nabla}\varphi\|, \frac{\overline{\nabla}\varphi}{\|\overline{\nabla}\varphi\|}\Big)\, d\overline{\sigma} - \int_{\{\varphi=s_1\}} \|\overline{\nabla}\varphi\|^{\beta-1}\, \overline{g}\Big(\nabla \|\overline{\nabla}\varphi\|, \frac{\overline{\nabla}\varphi}{\|\overline{\nabla}\varphi\|}\Big)\, d\overline{\sigma} \\
&\quad = \beta^{-1} \int_{\{s_1\le\varphi\le s_2\}} \overline{\mathrm{div}}\big(\nabla \|\overline{\nabla}\varphi\|^{\beta}\big)\, d\overline{\mu}
\end{aligned}$$

for every $0 \le s_1 < s_2 \le \varepsilon$ (by Remark 1.1.1 and expression (2.15) of the mean curvature along regular level sets of φ), by the dominated convergence theorem the function

$$s \in [0, \varepsilon] \to \int_{\{\varphi=s\}} \|\overline{\nabla}\varphi\|^{\beta}\, \overline{\mathrm{H}}\, d\overline{\sigma} \in \mathbb{R}$$

is continuous, therefore the fundamental theorem of integral calculus implies the continuous differentiability of Φ_β on the closed interval $[0, \varepsilon]$.
Let s_0 be a regular value of the function φ. By Remark 2.1.6 we can define the function $\Psi_\beta : (0, +\infty) \to \mathbb{R}$ as follows

$$\Psi_\beta(s) = \begin{cases} \displaystyle\int_{\{\varphi=s_0\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} + \int_{\{s<\varphi<s_0\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\big[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\big]}{\sinh\varphi}\, d\overline{\mu} & \text{if } s \le s_0 \\[2ex] \displaystyle\int_{\{\varphi=s_0\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma} - \int_{\{s_0<\varphi<s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\big[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\big]}{\sinh\varphi}\, d\overline{\mu} & \text{if } s > s_0 \end{cases}$$

Then, $\Psi_\beta(s) = \int_{\{\varphi=s\}} \frac{\|\overline{\nabla}\varphi\|^{\beta}\,\overline{\mathrm{H}}}{\sinh\varphi}\, d\overline{\sigma}$ for every $s > 0$ regular value of φ, by Proposition 2.1.4 (3). Moreover, Ψ_β is a continuous function on the open interval $(0, +\infty)$, as

$$\Psi_\beta(s) - \Psi_\beta(\overline{s}) = \int_{\{s<\varphi<\overline{s}\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\big[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 + \|\overline{\nabla} d\varphi\|^2 + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\big]}{\sinh\varphi}\, d\overline{\mu} \tag{2.48}$$

for every couple $0 < s < \overline{s} < +\infty$, therefore Remark 2.1.6 and the dominated convergence theorem imply the right and left continuity of Ψ_β on the interval $(0, +\infty)$.
Now, considering the function Υ_β given by formula (2.47), for every $0 < s < \overline{s} < +\infty$

we have

$$\begin{aligned}\frac{\Upsilon_\beta(\bar{s})-\Upsilon_\beta(s)}{\bar{s}-s} &= \frac{1}{\bar{s}-s}\int\limits_{\{s<\varphi<\bar{s}\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)-\coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\mu}\\ &= \frac{1}{\bar{s}-s}\int_s^{\bar{s}} d\tau \int\limits_{\{\varphi=\tau\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-3}\left[\beta\overline{\nabla}d\varphi(\overline{\nabla}\varphi,\overline{\nabla}\varphi)-\coth(\varphi)\|\overline{\nabla}\varphi\|^4\right]}{\sinh\varphi}\,d\overline{\sigma}\\ &= -\frac{\beta}{\bar{s}-s}\int\limits_s^{\bar{s}}\Psi_\beta(\tau)\,d\tau - \frac{1}{\bar{s}-s}\int\limits_s^{\bar{s}}\coth(\tau)\Upsilon_\beta(\tau)\,d\tau\,,\end{aligned}$$

where the first equality follows from Proposition 2.1.4 (1), the second one from the coarea formula (1.1), keeping in mind formula (2.43) and the last one is a consequence of the properties of Ψ_β, together with Sard's theorem. Then, from the continuity of the functions Υ_β and Ψ_β it follows that the function Υ_β is C^1 and

$$\Upsilon'_\beta(\,\cdot\,) = -\beta\,\Psi_\beta(\,\cdot\,) - \coth(\,\cdot\,)\,\Upsilon_\beta(\,\cdot\,)\,.$$

In turn, this implies that $\Phi_\beta \in C^1(0,+\infty)$ and $\Phi'_\beta(s) = -\beta\,\sinh(s)\Psi_\beta(s)$, as $\Phi_\beta(s) = \sinh(s)\Upsilon_\beta(s)$, for every $s>0$. Moreover, by equality (2.48),

$$\begin{aligned}\frac{\Phi'_\beta(S)}{\sinh(S)}-\frac{\Phi'_\beta(s)}{\sinh(s)} &= -\beta\,\Psi_\beta(S)+\beta\,\Psi_\beta(s)\\ &= \beta\int\limits_{\{s<\varphi<S\}} \frac{\|\overline{\nabla}\varphi\|^{\beta-2}\left[(\beta-2)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2+\|\overline{\nabla}d\varphi\|^2+Q(\overline{\nabla}\varphi,\overline{\nabla}\varphi)\right]}{\sinh\varphi}\,d\overline{\mu} \geq 0\end{aligned} \tag{2.49}$$

for every $0<s<S<+\infty$. Therefore, the integral representation (2.45) follows from the passage to the limit, as $S\to+\infty$, in formula (2.49), by using the monotone convergence theorem and by the fact that

$$\lim_{S\to+\infty}\frac{\Phi'_\beta(S)}{\sinh(S)} = -\beta\lim_{S\to+\infty}\Psi_\beta(S) = -\beta\lim_{S\to+\infty}\int\limits_{\{\varphi=S\}}\frac{\|\overline{\nabla}\varphi\|^\beta\,\overline{\mathrm{H}}}{\sinh\varphi}\,d\overline{\sigma} = 0\,,$$

where the last equality is explained in the proof of Proposition 2.1.4 (4).

Before to proceed with the last point, the outer rigidity, we need to recall briefly some known definitions and results involving the *normal exponential map* (see [75]). Let $(\widehat{M},\widehat{g})$ be a complete Riemannian manifold with compact boundary $\partial\widehat{M}$ and we denote by $\widehat{\nu}$ the inner–pointing unit normal vector field along $\partial\widehat{M}$. Moreover, for $p\in\partial\widehat{M}$, we denote by $\widehat{\gamma}_p : I_p \to M$ the (maximal) geodesic, in the usual sense in Riemannian geometry, with initial conditions $\widehat{\gamma}_p(0)=p$ and $\widehat{\gamma}'_p(0)=\widehat{\nu}_p$, where I_p is an open or closed interval starting at 0. Now, on the set $\mathcal{O}$ of $v\in N\partial\widehat{M}$ such that $v = t\nu_p$ for every $t\in I_p$ and for all $p\in\partial\widehat{M}$, it is well-defined the map $\exp^\perp : v\in\mathcal{O}\to\widehat{\gamma}_{\pi(v)}(|v|_{\widehat{g}})$, called *normal exponential map of $\partial\widehat{M}$*, which is smooth. By using the compactness of $\partial\widehat{M}$ there exists $\varepsilon>0$ such that $[0,\varepsilon)\subseteq I_p$, the restriction $\widehat{\gamma}_p|_{[0,\varepsilon)}$ is a minimal geodesic in M, i.e. $d_{\widehat{M}}\big(\widehat{\gamma}_p(s),\widehat{\gamma}_p(t)\big) = |t-s|$ for every $s,t\in[0,\varepsilon)$ and p is the unique point of $\partial\widehat{M}$ that realizes the distance of every point $q\in\widehat{\gamma}_p\big([0,\varepsilon)\big)$ from

$\partial\widehat{M}$. Consequently, the function $c : \partial\widehat{M} \to (0, +\infty]$, defined as

$$c(p) := \sup \left\{ t \in I_p \ : \ d_{\widehat{M}}(\widehat{\gamma}_p(t), \partial\widehat{M}) = t \right\},$$

is well–defined and actually, it is a continuous function. Take $T > 0$ and $p \in \partial\widehat{M}$ with $c(p) < +\infty$. Then $T = c(p)$ if and only if $T = d_{\widehat{M}}(\widehat{\gamma}_p(T), \partial\widehat{M})$ and at least one of the following holds: or $\widehat{\gamma}_p(T)$ is the first conjugate point of $\partial\widehat{M}$ along $\widehat{\gamma}_p$ (which means that there exists a Jacobi filed Y along $\widehat{\gamma}_p$ satisfying the initial conditions $Y(0) \in T_p\partial\widehat{M}$ and $\nabla_0 Y - S_\nu Y(0) \in N_p\partial\widehat{M}$) or there exists a *foot point* $\widehat{p} \in \partial\widehat{M} \setminus \{p\}$ on $\partial\widehat{M}$ of $\widehat{\gamma}_p(T)$ (where for a point $q \in \widehat{M}$, a point $\widetilde{p} \in \partial\widehat{M}$ is called *a foot point on* $\partial\widehat{M}$ *of* q if $d_{\widehat{M}}(q, \widetilde{p}) = d_{\widehat{M}}(q, \partial\widehat{M})$). We put

$$\begin{aligned}
\mathrm{D}_{\partial\widehat{M}} &:= \exp^{\perp}\Big(\big\{t\nu_p \ : \ t \in [0, c(p)) \text{ and } p \in \partial\widehat{M}\big\}\Big),\\
\mathrm{Cut}\,\partial\widehat{M} &:= \exp^{\perp}\Big(\big\{c(p)\nu_p \ : \ p \in \partial\widehat{M} \text{ such that } c(p) < +\infty\big\}\Big),
\end{aligned}$$

where $\mathrm{Cut}\,\partial\widehat{M}$ is called *the cut locus of the boundary* $\partial\widehat{M}$. Then, it follows that $\mathrm{Int}(M) = \big(\mathrm{D}_{\partial\widehat{M}} \setminus \partial\widehat{M}\big) \sqcup \mathrm{Cut}\,\partial\widehat{M}$ and $\big\{t\nu_p \ : \ t \in [0, c(p)) \text{ and } p \in \partial\widehat{M}\big\}$ is the maximal domain where $\exp^{\perp}$ is a diffeomorphism.

Finally, if $p \in \partial\widehat{M}$ be a foot point on $\partial\widehat{M}$ of a point $q \in \mathrm{Int}(\widehat{M})$, then there exists a unique curve $\sigma : [0, l] \to M$ parametrized in arclength from p to q such that $d_{\widehat{M}}\big(\sigma(s), \sigma(t)\big) = |t - s|$ for every $s, t \in [0, l]$ and coincide with $\widehat{\gamma}_p$, where $l = d_{\widehat{M}}(q, \partial\widehat{M})$.

Step 3 – Outer rigidity. Suppose that $\Phi'_\beta(s_0) = 0$, for some $s_0 > 0$. Then, it follows from identity (2.45) together with formula (2.28), that

$$\Big(\beta - \frac{n-2}{n-1}\Big)\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 \equiv 0\,, \qquad \|\overline{\nabla}d\varphi\|^2 - \frac{n}{n-1}\big\|\overline{\nabla}\|\overline{\nabla}\varphi\|\big\|^2 \equiv 0$$

in $\{\varphi \geq s_0\} \setminus \mathrm{Crit}(\varphi)$. These equalities imply $\overline{\nabla}d\varphi \equiv 0$ in $\{\varphi \geq s_0\}$, as $\overline{\nabla}d\varphi$ is a smooth function and $\overline{\mu}\big(\mathrm{Crit}(\varphi)\big) = 0$, consequently $\|\overline{\nabla}\varphi\|^2 \equiv a^2$ with $a > 0$, since $\{\varphi \geq s_0\}$ is connected. Then, s_0 is a regular value of φ and $\{\varphi \geq s_0\}$, with the induced Riemannian metric, is a noncompact, connected and complete Riemannian manifold, with compact and totally geodesic boundary (since $\overline{\nabla}d\varphi \equiv 0$ in $\{\varphi = s_0\}$ and its second fundamental form is related to the $\overline{g}$–Hessian of φ by formula (2.16)) and having $\overline{\mathrm{Ric}} \geq 0$ (by the first line of system (2.14), applying the Cauchy–Schwarz inequality). Therefore, by [42, Theorem C], we obtain that the level set $\{\varphi = s_0\}$ is connected and that $\{\varphi \geq s_0\}$ is isometric to the product $[0, +\infty) \times \{\varphi = s_0\}$. Moreover, this isometry from the product $[0, +\infty) \times \{\varphi = s_0\}$ to $\{\varphi \geq s_0\}$ is determined by the normal exponential map of the boundary $\{\varphi = s_0\}$, which in this case it is a diffeomorphism (see [42]). Finally, we prove that φ is an affine function of ρ on $\{\varphi \geq s_0\}$, i.e. there exist $a, b \in \mathbb{R}$ such that $\varphi = a\rho + b$. First, we notice that every integral curve γ_p of $\overline{\nabla}\varphi$ outgoing from a point p of $\{\varphi = s_0\}$ is defined on the interval $[0, +\infty)$ and it is contained in $\{\varphi \geq s_0\}$, by the completeness and $\|\overline{\nabla}\varphi\| > 0$. Then, $\varphi \circ \gamma_p(t) = a^2 t + s_0$ for every $t \in [0, +\infty)$ and all the curves γ_p realize the distance between the hypersurfaces $\{\varphi = s_0\}$ and $\{\varphi = s_1\}$ with $s_1 > s_0$. Indeed, for every C^1–curve $\sigma : [0, l] \to \{\varphi \geq s_0\}$ parametrized by arclength and joining a point of $\{\varphi = s_0\}$ to a point of $\{\varphi = s_1\}$, we have

$$L_{\overline{g}}(\sigma) = \int_0^l \|\dot\sigma(\tau)\|\,d\tau \geq \left|\int_0^l \frac{1}{a}\,\overline{g}\big(\dot\sigma(\tau), \overline{\nabla}\varphi\,(\sigma(\tau))\big)\,d\tau\right| = \frac{1}{a}\,|\varphi\circ\sigma(l) - \varphi\circ\sigma(0)|$$
$$= at = L_{\overline{g}}\left(\gamma_{\sigma(0)}|_{[0,t]}\right)\,,$$

where s_1, s_0 and t satisfy $s_1 = a^2 t + s_0$.
Since $\overline{\nu} = (1/a)\overline{\nabla}\varphi$ is the (inward–pointing) unit normal vector field along the boundary $\{\varphi = s_0\}$ and we know that the normal exponential map is a diffeomorphism, then the point $\exp^\perp(t\overline{\nu}_p)$ has distance from $\{\varphi = s_0\}$ equal to t and p is the unique point of $\{\varphi = s_0\}$ that realizes such distance, as said before, therefore $\exp^\perp(t\overline{\nu}_p)$ coincides with $\gamma_p(t/a)$ (recall the last part of the above discussion on the properties of the normal exponential map), by the properties of the integral curves $\gamma_{\widehat{p}}$ for $\widehat{p} \in \{\varphi = s_0\}$. Hence,

$$\varphi\big(\exp^\perp(t\overline{\nu}_p)\big) = \varphi\circ\gamma_p(t/a) = at + s_0 = a\,\rho\big(\exp^\perp(t\overline{\nu}_p)\big) + s_0\,.$$

This shows that φ is an affine function of ρ on $\{\varphi \geq s_0\}$. $\square$

While the previous proposition gives an outer rigidity result, the following corollary provides a "global" rigidity result.

Corollary 2.1.8. *Let (M, g, u) be a sub–static harmonic triple, let $\overline{g}$ and φ be the metric and the function defined by formulas (2.11) and let $\Phi_\beta : [0,\infty) \to \mathbb{R}$ be the function defined by formula (2.44), for every $\beta > \frac{n-2}{n-1}$. If Φ_β is constant for some $\beta > \frac{n-2}{n-1}$, then ∂M is connected and $(M, \overline{g})$ is isometric to $\big([0,+\infty)\times\partial M, d\rho\otimes d\rho + \overline{g}_{\partial M}\big)$, where ρ is the $\overline{g}$–distance function to ∂M and φ is an affine function of ρ.*

Proof. As $\Phi'_\beta(s) = 0$, for every $s > 0$, by identity (2.45) and formula (2.28), the following integral

$$\int_{\{\varphi>s\}} \|\overline{\nabla}\varphi\|^{\beta-2}\left\{\Big(\beta - \frac{n-2}{n-1}\Big)\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 + \Big[\|\overline{\nabla}d\varphi\|^2 - \frac{n}{n-1}\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2\Big] + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right\}d\overline{\mu}$$

is zero for every $s > 0$. In turn, the monotone convergence theorem implies that

$$\int_M \|\overline{\nabla}\varphi\|^{\beta-2}\left\{\Big(\beta - \frac{n-2}{n-1}\Big)\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 + \Big[\|\overline{\nabla}d\varphi\|^2 - \frac{n}{n-1}\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2\Big] + Q(\overline{\nabla}\varphi, \overline{\nabla}\varphi)\right\}d\overline{\mu}$$

is zero, too. Then, one has

$$\Big(\beta - \frac{n-2}{n-1}\Big)\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 \equiv 0\,, \qquad \|\overline{\nabla}d\varphi\|^2 - \frac{n}{n-1}\big|\overline{\nabla}\|\overline{\nabla}\varphi\|\big|^2 \equiv 0$$

in $M \setminus \mathrm{Crit}(\varphi)$, by the Kato inequality for harmonic functions (2.27) and by (2.24). Consequently, $\overline{\nabla}d\varphi \equiv 0$ in M, therefore $\|\overline{\nabla}\varphi\|^2 \equiv a^2$ with $a > 0$ and $\overline{\mathrm{Ric}} \geq 0$ by the continuity of the Ricci tensor. The same argument of the proof of the outer rigidity in Proposition 2.1.7 implies that ∂M is connected and $(M, \overline{g})$ is isometric to

$$\big([0,+\infty)\times\partial M, d\rho\otimes d\rho + \overline{g}_{\partial M}\big)$$

where ρ is the $\overline{g}$–distance to ∂M and φ is an affine function of ρ. $\square$

We are now ready to prove Theorem 2.1.1. Also in this case we proceed by steps.
Step 1 – Continuity, differentiability, monotonicity and convexity. Let $\overline{g}$ and φ be the metric and the function defined by formulas (2.11) and let $\Phi_\beta : [0,\infty) \to \mathbb{R}$ be the function defined by formula (2.44), for every $\beta > \frac{n-2}{n-1}$. Notice that

$$F_\beta(\tau) = 2^{\frac{\beta}{n-2}-1}\,\Phi_\beta\left(\log\left(\frac{\sqrt{\tau+1}+\sqrt{\tau-1}}{\sqrt{\tau+1}-\sqrt{\tau-1}}\right)\right),$$

for every $\tau \in [1,+\infty)$. Then, by Theorem 2.1.7, F_β is continuous in $[1,+\infty)$ and continuously differentiable in $(1,+\infty)$ with

$$F'_\beta(\tau) = \frac{2^{\frac{\beta}{n-2}-1}}{\sqrt{\tau^2-1}}\,\Phi'_\beta\left(\log\left(\frac{\sqrt{\tau+1}+\sqrt{\tau-1}}{\sqrt{\tau+1}-\sqrt{\tau-1}}\right)\right) \le 0\,. \tag{2.50}$$

The convexity of F_β is a consequence of its continuity and of the fact that F'_β is nondecreasing in $(1,+\infty)$, which follows from formula (2.49) after observing that

$$\sinh\left(\log\left(\frac{\sqrt{\tau+1}+\sqrt{\tau-1}}{\sqrt{\tau+1}-\sqrt{\tau-1}}\right)\right) = \sqrt{\tau^2-1}$$

and that the function $\log\left(\frac{\sqrt{\tau+1}+\sqrt{\tau-1}}{\sqrt{\tau+1}-\sqrt{\tau-1}}\right)$ is nondecreasing.
Step 2 – Outer rigidity. Let us assume that there exists $\tau_0 \in (1,\infty)$ such that $F'_\beta(\tau_0) = 0$, for some $\beta > \frac{n-2}{n-1}$. Then, equality (2.50) implies

$$\Phi'_\beta(s_0) = 0 \quad \text{with } s_0 = \log\left(\frac{\sqrt{\tau_0+1}+\sqrt{\tau_0-1}}{\sqrt{\tau_0+1}-\sqrt{\tau_0-1}}\right).$$

Therefore, it follows from Theorem 2.1.7 that

$$(\{\varphi \ge s_0\}, \overline{g}) \cong \Big([0,+\infty) \times \{\varphi = s_0\},\, d\rho \otimes d\rho + \overline{g}_{\{\varphi = s_0\}}\Big),$$

where the symbol $\cong$ means that the Riemannian manifolds are isometric, ρ is the $\overline{g}$–distance function from $\{\varphi = s_0\}$ and

$$\varphi = (n-2)(2\mathcal{C})^{-\frac{1}{n-2}}\rho + s_0\,,$$

by virtue of the fact that $\|\overline{\nabla}\rho\| \equiv 1$, together with limit (2.19). Then, we have the following isometries

$$
\begin{gathered}
p \in \big(\{\varphi \geq s_0\},\, \overline{g}\big) \\
\big\downarrow \\
(\rho, q) \in \Big([0, +\infty) \times \{\varphi = s_0\},\, d\rho \otimes d\rho + \overline{g}_{\{\varphi = s_0\}}\Big) \\
\big\downarrow \\
\Big(\varphi = \frac{n-2}{(2\mathcal{C})^{\frac{1}{n-2}}}\,\rho + s_0, q\Big) \in \Big([s_0, +\infty) \times \{\varphi = s_0\},\, \frac{(2\mathcal{C})^{\frac{2}{n-2}}}{(n-2)^2}\, d\varphi \otimes d\varphi + \overline{g}_{\{\varphi = s_0\}}\Big) \\
\big\downarrow \\
\big(u = \tanh(\varphi/2), q\big) \in \Big([t_0, 1) \times \{u = t_0\},\, \frac{2^{2\frac{n-1}{n-2}}\,\mathcal{C}^{\frac{2}{n-2}}}{(n-2)^2(1-u^2)^2}\, du \otimes du + \overline{g}_{\{u = t_0\}}\Big)\,.
\end{gathered}
$$

where $t_0 = \tanh(s_0/2)$.
Here, the map $p \to (\rho, q)$ associates to every point p of $\{\varphi \geq s_0\}$ the pair having as first coordinate the $\overline{g}$–distance of p from $\{\varphi = s_0\}$ and as second coordinate the point q of $\{\varphi = s_0\}$ that realizes such distance, or equivalently, the map associating to every point p of $\{u \geq t_0\}$ the pair having as first coordinate the $\overline{g}$–distance of p from $\{u = t_0\}$ and as second coordinate the point q of $\{u = t_0\}$ that realizes such distance (by virtue of the fact that $\{\varphi \geq s_0\} = \{u \geq t_0\}$ and $\{\varphi = s_0\} = \{u = t_0\}$). Then, in view of equality (2.10) and with the same notation as above, we have the isometries

$$
\begin{gathered}
p \in \big(\{u \geq t_0\},\, g\big) \\
\big\downarrow \\
(u, q) \in \Big([t_0, 1) \times \{u = t_0\},\, \frac{2^{2\frac{n-1}{n-2}}\,\mathcal{C}^{\frac{2}{n-2}}}{(n-2)^2(1-u^2)^{2\frac{n-1}{n-2}}}\, du \otimes du + (1-u^2)^{-\frac{2}{n-2}}\, \overline{g}_{\{u = t_0\}}\Big) \\
\big\downarrow \\
\bigg(r = \Big(\frac{2\mathcal{C}}{1-u^2}\Big)^{\frac{1}{n-2}}, q\bigg) \in \Big([r_0, +\infty) \times \{r = r_0\},\, \frac{dr \otimes dr}{1 - 2\mathcal{C} r^{2-n}} + (2\mathcal{C})^{-\frac{2}{n-2}}\, r^2 \overline{g}_{\{r = r_0\}}\Big)\,,
\end{gathered}
\tag{2.51}
$$

where $r_0 = \big(\frac{2\mathcal{C}}{1-t_0^2}\big)^{\frac{1}{n-2}}$. Then, by a straightforward computation, it follows

$$
|\mathrm{Riem}|^2(p) = \frac{(2\mathcal{C})^{\frac{4}{n-2}}}{r^4(p)} \left| \mathrm{Riem}_{\overline{g}_{\{r=r_0\}}} + \frac{1 - 2\mathcal{C} r^{2-n}}{2^{\frac{n}{n-2}}\,\mathcal{C}^{\frac{2}{n-2}}}\, \overline{g}_{\{r=r_0\}} \owedge \overline{g}_{\{r=r_0\}} \right|^2_{\overline{g}_{\{r=r_0\}}} (q) + \frac{c_2}{r^{2n}(p)}\,,
\tag{2.52}
$$

where c_2 is a suitable positive constant, q is the point of $\{r = r_0\}$ that realizes the $\overline{g}$–distance of p from $\{r = r_0\}$ and $\overline{g}_{\{r=r_0\}} \owedge \overline{g}_{\{r=r_0\}}$ is the $(0,4)$–tensor field on $\{r = r_0\}$

given by the Kulkarni–Nomizu product

$$\overline{g}_{\{r=r_0\}} \owedge \overline{g}_{\{r=r_0\}}\,(X,Y,Z,W) = 2\overline{g}_{\{r=r_0\}}\,(X,Z)\,\overline{g}_{\{r=r_0\}}\,(Y,W) - 2\overline{g}_{\{r=r_0\}}\,(X,W)\,\overline{g}_{\{r=r_0\}}\,(Y,Z)$$

for every X, Y, Z, W vector fields on $\{r = r_0\}$. Now, denoting by Θ the isometry from $\{u \geq t_0\}$ to $[r_0, +\infty) \times \{r = r_0\}$ introduced by formula (2.51), for every $q_0 \in \{r = r_0\}$ it follows from equality (2.52) that

$$\lim_{r\to+\infty} (2\mathcal{C})^{-\frac{4}{n-2}} r^4 |\mathrm{Riem}|^2 \big(\Theta^{-1}(r, q_0)\big) = \left| \mathrm{Riem}_{\overline{g}_{\{r=r_0\}}} + \frac{(2\mathcal{C}\,)^{-\frac{2}{n-2}}}{2} \overline{g}_{\{r=r_0\}} \owedge \overline{g}_{\{r=r_0\}} \right|^2_{\overline{g}_{\{r=r_0\}}} (q_0)\,. \tag{2.53}$$

At the same time, we have

$$\lim_{r\to+\infty} r^4 |\mathrm{Riem}|^2 \big(\Theta^{-1}(r, q_0)\big) = 0\,, \tag{2.54}$$

by observing that

$$|\mathrm{Riem}| = O(|x|^{-\tau-2}) \qquad \text{and} \qquad \frac{r}{|x|} \to 1 \quad \text{at } \infty$$

for any AF coordinate chart $(x^1, \ldots, x^n)$ of order $\tau > \frac{n-2}{2}$, which are consequences of formulas (1.18) and (2.2), respectively.
Combining limits (2.53) and (2.54), the arbitrariness of the point q_0 in $\{r = r_0\}$ gives

$$\mathrm{Riem}_{\overline{g}_{\{r=r_0\}}} = -\frac{(2\mathcal{C}\,)^{-\frac{2}{n-2}}}{2} \overline{g}_{\{r=r_0\}} \owedge \overline{g}_{\{r=r_0\}}\,.$$

Hence, $\big(\{r = r_0\}, \overline{g}_{\{r=r_0\}}\big)$ is a complete $(n-1)$–dimensional Riemannian manifold with constant (sectional) curvature $(2\mathcal{C}\,)^{-\frac{2}{n-2}}$. Consequently, being all the level sets $\{u = t\}$ diffeomorphic to $\{u = t_0\}$ for $t > t_0$ and to $\mathbb{S}^{n-1}$ for t sufficiently to close to 1, the set $\{r = r_0\} = \{u = t_0\}$ is simply connected. Therefore, from [25, Section 3.F] it follows

$$\big(\{r = r_0\}, \overline{g}_{\{r=r_0\}}\big) \cong (\mathbb{S}^{n-1},\, (2\mathcal{C}\,)^{\frac{2}{n-2}} g_{\mathbb{S}^{n-1}})\,,$$

hence, we conclude

$$(\{u \geq t_0\}, g) \cong \left([r_0, +\infty) \times \mathbb{S}^{n-1},\, \frac{dr \otimes dr}{1 - 2\mathcal{C} r^{2-n}} + r^2 g_{\mathbb{S}^{n-1}} \right)\,.$$

2.2 A capacitary Riemannian Penrose inequality

A straightforward application of the monotonicity of F_β leads to the following geometric inequality.

Theorem 2.2.1 (Capacitary Riemannian Penrose inequality). *Let (M, g, u) be a sub–static harmonic triple with connected boundary having associated boundary capacity*

$$\mathcal{C} = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial M} |\nabla u|\, d\sigma\,,$$

as defined in formula (2.3). Then

$$\mathcal{C} \geq \frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} . \tag{2.55}$$

Moreover, the equality holds if and only if (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$.

Proof. The monotonicity of F_β in $(1, +\infty)$ together with its continuity in $[1, +\infty)$, given by Theorem 2.1.1, imply

$$F_\beta(1) \geq \lim_{\tau\to+\infty} F_\beta(\tau)\,, \tag{2.56}$$

for every $\beta > \frac{n-2}{n-1}$. As explained at the beginning of this chapter, just before formula (2.2), we have the $|\nabla u|$ attains a positive constant value on ∂M, which is connected. Then, from equality (2.3), it follows

$$|\nabla u| \equiv \frac{(n-2)\mathcal{C}\,|\mathbb{S}^{n-1}|}{|\partial M|} \quad \text{ on } \partial M\,,$$

hence,

$$F_\beta(1) = \frac{2^{\beta\,\frac{n-1}{n-2}}\left[(n-2)\mathcal{C}\,|\mathbb{S}^{n-1}|\right]^{\beta+1}}{|\partial M|^\beta} . \tag{2.57}$$

Now, we know by limit 2.9 that

$$\frac{|\nabla u|}{(1-u^2)^{\frac{n-1}{n-2}}} \longrightarrow 2^{-\frac{n-1}{n-2}}(n-2)\mathcal{C}^{-\frac{1}{n-2}} \quad \text{ at } \infty\,.$$

Therefore, fixed $\varepsilon > 0$, there exists $\tau_0 > 1$ such that

$$|\nabla u| \geq (1-u^2)^{\frac{n-1}{n-2}}\left(2^{-\frac{n-1}{n-2}}(n-2)\mathcal{C}^{-\frac{1}{n-2}} - \varepsilon\right)$$

in $\left\{u \geq \sqrt{\frac{\tau_0-1}{\tau_0+1}}\right\}$ and the level sets $\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}$ are regular for all $\tau \geq \tau_0$. Consequently, for every $\tau \geq \tau_0$ we have

$$\begin{aligned}
F_\beta(\tau) &= (1+\tau)^{\beta\,\frac{n-1}{n-2}} \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\nabla u|^{\beta+1}\,d\sigma \\
&\geq (1+\tau)^{\beta\,\frac{n-1}{n-2}} \int\limits_{\left\{u=\sqrt{\frac{\tau-1}{\tau+1}}\right\}} (1-u^2)^{\beta\,\frac{n-1}{n-2}}\left(2^{-\frac{n-1}{n-2}}(n-2)\mathcal{C}^{-\frac{1}{n-2}} - \varepsilon\right)^\beta |\nabla u|\,d\sigma \\
&= 2^{\beta\,\frac{n-1}{n-2}}\left(2^{-\frac{n-1}{n-2}}(n-2)\mathcal{C}^{-\frac{1}{n-2}} - \varepsilon\right)^\beta \int\limits_{\partial M} |\nabla u|\,d\sigma \\
&= 2^{\beta\,\frac{n-1}{n-2}}(n-2)\mathcal{C}\left(2^{-\frac{n-1}{n-2}}(n-2)\,\mathcal{C}^{-\frac{1}{n-2}} - \varepsilon\right)^\beta |\mathbb{S}^{n-1}|\,,
\end{aligned}$$

where in the second equality we used property (2.6). In particular,

$$\lim_{\tau\to+\infty} F_\beta(\tau) \geq 2^{\beta\,\frac{n-1}{n-2}}(n-2)\mathcal{C}\left(2^{-\frac{n-1}{n-2}}(n-2)\,\mathcal{C}^{-\frac{1}{n-2}} - \varepsilon\right)^\beta |\mathbb{S}^{n-1}|\,,$$

thus, the arbitrariness of $\varepsilon > 0$ implies

$$\lim_{\tau \to +\infty} F_\beta(\tau) \geq (n-2)^{\beta+1} \mathcal{C}^{1-\frac{\beta}{n-2}} |\mathbb{S}^{n-1}| .$$

In a similar way we can obtain the reverse inequality, then we conclude

$$\lim_{\tau \to +\infty} F_\beta(\tau) = (n-2)^{\beta+1} \mathcal{C}^{1-\frac{\beta}{n-2}} |\mathbb{S}^{n-1}| . \tag{2.58}$$

Putting together formulas (2.56), (2.57) and (2.58), we obtain inequality (2.55).
Concerning the rigidity statement, first we observe that if (M, g) is isometric to the Schwarzschild manifold of mass $m > 0$, then the right–hand and left–hand sides of inequality (2.55) are both equal to m, by straightforward computations.
Suppose now that the equality holds in (2.55). Then, for every $\beta > \frac{n-2}{n-1}$, the function F_β is constant. In turn, Φ_β is also constant, being

$$\Phi_\beta(s) = 2^{1-\frac{\beta}{n-2}} F_\beta\left(\frac{1+\tanh^2\left(\frac{s}{2}\right)}{1-\tanh^2\left(\frac{s}{2}\right)}\right) .$$

Hence, from Corollary 2.1.8, it follows

$$(M, \overline{g}) \cong \big([0,+\infty) \times \partial M,\, d\rho \otimes d\rho + \overline{g}_{\partial M}\big) ,$$

where ρ is the $\overline{g}$–distance to ∂M and φ is an affine function of ρ. Thus, (M, g) is isometric to the Schwarzschild manifold with mass $\mathcal{C}$, with a slight refinement of the argument in the proof of the outer rigidity in Theorem 2.1.1 (notice that the argument for the outer rigidity in Theorem 2.1.1 give us a diffeomorphism between the manifolds without boundary, composing it with the diffeomorphism (1.12), we obtain a map that can be extended also on the boundaries, which is the wanted isometry). $\square$

2.3 A uniqueness theorem for sub–static manifolds

Using the positive mass theorem [54, 79] for every dimension $n \geq 3$, more precisely a consequence of it contained in [35, Theorem 1.5], one can prove the following uniqueness statement. We remark that then, the capacitary Riemannian Penrose inequality (2.55) is an obvious consequence, thus, such inequality is actually relevant in the other cases, where the hypotheses of the following theorem are not satisfied.

Theorem 2.3.1 (Uniqueness theorem for sub–static harmonic triples)**.** *Let (M, g, u) be a sub–static harmonic triple with associated boundary capacity $\mathcal{C}$, given by formula (2.3). Assume that there exists a distinguished AF coordinate chart $\big(E, (x^1, \dots, x^n)\big)$ with order of decay τ_1, with $\tau_1 > \frac{n-2}{2}$, such that the scalar curvature satisfies*

$$\mathrm{R} = O(|x|^{-\tau_2}) , \tag{2.59}$$

for some $\tau_2 > n$. Then (M, g) is the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$.

Proof. By condition (2.59) and by the fact that $\nabla du \equiv 0$ on ∂M, which in turn implies the minimality of the boundary ∂M, we have that the hypothesis of [35, Theorem 1.5] are fulfilled, therefore

$$m_{\mathrm{ADM}} \geq \mathcal{C} .$$

Now, we want to show that the reverse inequality $m_{\mathrm{ADM}} \leq \mathcal{C}$ holds, this would imply that we are in the "rigidity case" $m_{\mathrm{ADM}} = \mathcal{C}$ of [35, Theorem 1.5], therefore

(M, g) is the Schwarzschild manifold of mass $\mathcal{C}$.
To this aim, we introduce an equivalent expression for the ADM mass involving the Ricci tensor (see [65] and references therein), given by

$$m_{\mathrm{ADM}} = \lim_{r\to+\infty} -\frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{1}{2}\mathrm{R}g\Big)(X,\nu)\, d\sigma \ , \tag{2.60}$$

where ν is the ∞–pointing unit normal vector field along $\{|x| = r\}$ with respect to g and X is the "Euclidean" conformal Killing vector field $x^i \frac{\partial}{\partial x^i}$. We then rewrite characterization (2.60) as

$$\begin{aligned} m_{\mathrm{ADM}} = \lim_{r\to+\infty} -\frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \Bigg\{ &\int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{\nabla du}{u}\Big)(X,\nu_e)\, d\sigma \\ + \int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{\nabla du}{u}\Big)(X,\nu-\nu_e)\, d\sigma &+ \int\limits_{\{|x|=r\}} \frac{\nabla du}{u}(X,\nu)\, d\sigma - \frac{1}{2}\int\limits_{\{|x|=r\}} \mathrm{R}g(X,\nu)\, d\sigma \Bigg\}, \end{aligned}$$

where $\nu_e = \frac{x^i}{|x|}\frac{\partial}{\partial x^i} = \frac{1}{|x|}X$. As $u\mathrm{Ric} - \nabla du \geq 0$ in system (2.1), we notice that

$$\int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{\nabla du}{u}\Big)(X,\nu_e)\, d\sigma = \frac{1}{r}\int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{\nabla du}{u}\Big)(X,X)\, d\sigma \geq 0\,. \tag{2.61}$$

Then, recalling that $(\nabla du)_{ij} = \partial_i\partial_j u - \Gamma_{ij}^k\partial_k u$, where Γ_{ij}^k are the the Christoffel symbols related to g, by formulas (1.17) (2.4) and (2.5) we get

$$\begin{aligned} |(\nabla du)_{ij} - (\nabla^e du)_{ij}| = |\Gamma_{ij}^k\partial_k u| = O\big(|x|^{-n-\tau_1}\big) \\ (\nabla du)_{ij} = O(|x|^{-n})\,. \end{aligned}$$

These decays, coupled with formulas (1.19) (1.22) and (1.23) yield

$$\begin{aligned} \Bigg|\int\limits_{\{|x|=r\}} \Big(\mathrm{Ric} - \frac{\nabla du}{u}\Big)(X,\nu-\nu_e)\, d\sigma\Bigg| &\leq C \int\limits_{\{|x|=r\}} \frac{1}{|x|^{\tau_1+\min\{\tau_1+2,n\}-1}}\, d\sigma_e \\ &= \frac{C}{r^{\tau_1+\min\{\tau_1+2,n\}-n}} \to 0\,, \end{aligned} \tag{2.62}$$

being $\tau_1 > \frac{n-2}{2}$. We now observe that

$$\int\limits_{\{|x|=r\}} \frac{\nabla du}{u}(X,\nu)\, d\sigma \to -(n-1)(n-2)\mathcal{C}|\mathbb{S}^{n-1}| \quad \text{as } r\to+\infty\,, \tag{2.63}$$

indeed,

$$\begin{aligned} \int\limits_{\{|x|=r\}} \frac{\nabla du}{u}(X,\nu)\, d\sigma = \int\limits_{\{|x|=r\}} \frac{\nabla du}{u}(X,\nu-\nu_e)\, d\sigma &+ \int\limits_{\{|x|=r\}} \frac{\nabla du - \nabla^e du}{u}(X,\nu_e)\, d\sigma \\ + \int\limits_{\{|x|=r\}} \frac{\nabla^e du}{u}(X,\nu_e)\, d\sigma\,, \end{aligned}$$

and one can show that the first two terms of this sum tend to 0 for $r \to +\infty$, with similar estimates as before, while

$$\int\limits_{\{|x|=r\}} \frac{\nabla^e du}{u}(X, \nu_e)\, d\sigma \to -(n-1)(n-2)\mathcal{C}|\mathbb{S}^{n-1}| \quad \text{as } r \to +\infty\,,$$

using formulas (2.5) and (1.23). Hence, limit (2.63) is proven.
Gathering together formulas (2.61), (2.62) and (2.63), there holds

$$m_{\mathrm{ADM}} \leq \mathcal{C} + \limsup_{r\to+\infty} \frac{1}{2(n-2)(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\{|x|=r\}} \mathrm{R}\, g(X,\nu)\, d\sigma\,. \tag{2.64}$$

Now, since

$$\begin{aligned} g(X,\nu) = g_{ij}X^i\nu^j &= \Big(\delta_{ij} + O(|x|^{-\tau_1})\Big)\, X^i \left(\nu_e^j + \nu^j - \nu_e^j\right) \\ &= g_e(X,\nu_e) + O(|x|^{-\tau_1+1}) = |x| + O(|x|^{-\tau_1+1})\,, \end{aligned}$$

by formulas (1.23) and (2.59), we obtain

$$\left|\,\int\limits_{\{|x|=r\}} \mathrm{R}\, g(X,\nu)\, d\sigma\,\right| \leq C \int\limits_{\{|x|=r\}} \frac{1}{|x|^{\tau_2-1}}\, d\sigma_e \leq C r^{-\tau_2+n} \to 0 \quad \text{as } r \to +\infty\,.$$

Then, from inequality (2.64), it follows $m_{\mathrm{ADM}} \leq \mathcal{C}$ and we are done. □

It remains an open question to see whether it is possible to remove the assumption on the decay of R, at least in dimension $n = 3$.

Chapter 3

Positive mass inequality via linear potential theory

In this chapter, a new proof of the positive mass theorem is established through a newly discovered monotonicity formula, holding along the level sets of an appropriate harmonic function, related to the minimal positive Green's function with a pole of a complete one–ended asymptotically flat 3–manifold with nonnegative scalar curvature and sufficiently simple topology.

3.1 The positive mass theorem

Arnowitt, Deser and Misner in [5] conjectured that the ADM mass, measured along a spacelike hypersurface in a physical spacetime modeling an isolated gravitational system, is nonnegative (and zero only if the spacetime is "empty" of matter). The metric of any physical spacetime is a solution of the Einstein's equation

$$\mathbf{Ric} - \frac{1}{2}\mathbf{R}\mathbf{g} = \frac{8\pi G}{c^4}\mathbf{T}$$

where $\mathbf{T}$ is the *energy–momentum tensor* which, in realistic physical models, satisfies a certain positivity condition, called *dominant energy condition* (for instance, it implies that the matter cannot travel faster than light, see [33, Section 4.3]). In particular, totally geodesic spacelike slices inside a spacetime modeling an isolated gravitational system such that the dominant energy condition holds, have nonnegative scalar curvature. Then, one case of the ADM conjecture is the following.

Theorem 3.1.1 (Positive mass theorem)**.** *Let (M, g) be a 3-dimensional, complete, one–ended asymptotically flat manifold with nonnegative scalar curvature. Then, the ADM mass of (M, g) is nonnegative*

$$m_{\mathrm{ADM}} \geq 0.$$

Moreover, $m_{\mathrm{ADM}} = 0$ if and only if (M, g) is isometric to $(\mathbb{R}^3, g_{\mathrm{eucl}})$.

Before proceeding with our proof of this theorem, we mention that since the original work of Schoen and Yau, in which they used minimal surfaces techniques, several other approaches have been proposed to prove this relevant result. Far from being complete and referring the reader to [43] for a comprehensive survey on this topic, we just mention that the first alternate proof was found by Witten [87] (see also [69]), using harmonic spinors. Another route to the positive mass theorem was subsequently provided by the Huisken and Ilmanen via the weak inverse mean curvature flow [37]. Yet another proof of the positive mass theorem has been recently proposed by Li [52], using the Ricci flow. Finally, in a very recent preprint, Bray, Kazaras, Kuhri and Stern [12] were able to provide a new argument, based on the study of the level sets of linearly growing harmonic functions. This latter approach turned out to be flexible enough to also allow for the treatment of the "spacetime case" (see [11,34]) and together with some of the computations carried out in [39,40], it is possibly the method closest to ours.

Our proof of the positive mass theorem follows from a monotonicity result holding along the level sets of an appropriate function, related to the positive minimal Green's function $\mathcal{G}_o$ with pole o in a 3–dimensional complete one–ended AF manifold (M, g), for an arbitrary point $o \in M$, under certain conditions on (M, g). We start discussing its existence, noticing that the condition of minimality directly implies the uniqueness.

Since (M, g) is a one–ended AF manifold, there exist a bounded open subset Ω containing o with smooth boundary and a positive smooth function ϕ, defined an open set containing $M \setminus \Omega$, that is superharmonic in $M \setminus \Omega$, identically 1 on $\partial\Omega$ and 0 at infinity (see the proof of Proposition 1.4.12). Then, we consider a cover $\{\Omega_k\}_{k\in\mathbb{N}}$ of M with bounded open sets, having smooth boundary and satisfying the following two conditions: $\Omega \subset\subset \Omega_0$ and $\Omega_k \subset\subset \Omega_{k+1}$ for every $k \in \mathbb{N}$. Afterwards, we construct a nondecreasing sequence of functions, $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$, where each $\mathcal{G}_k$ is a positive Green kernel of Ω_k with pole at o, i.e. $\mathcal{G}_k$ is a positive function in $\Omega_k \setminus \{o\}$ satisfying the following conditions

$$\Delta\mathcal{G}_k = -\delta_o \text{ in } \Omega_k \quad \text{and} \quad \lim_{p\to q} \mathcal{G}_k(p) = 0 \text{ for all } q \in \partial\Omega_k,$$

(we refer to [36, Definition 3.9, Lemma 3.15, Theorem 3.19, Theorem 3.25] for the existence of such a sequence $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$). By Harnack principle (see [36, Section 2] and references therein), $\mathcal{G}_o := \lim_{k\to+\infty} \mathcal{G}_k$ is either harmonic in $M \setminus \{o\}$ or identically $+\infty$. To see that $\mathcal{G}_o$ is not identically $+\infty$, it is sufficient to show that $L_k := \max_{\partial\Omega} \mathcal{G}_k$ (greater than zero, as $\mathcal{G}_k$ is a smooth, positive function in $\Omega_k \setminus \{o\}$) does not go to $+\infty$. Now, if by contradiction $L_k \to +\infty$ (or if an arbitrary subsequence tends to $+\infty$), then there should exist a subsequence of $\{L_k^{-1}\mathcal{G}_k\}_{k\in\mathbb{N}}$ that converges pointwise to 1 in $M \setminus \{o\}$. Indeed, one has

$$0 \leq L_k^{-1}\mathcal{G}_k \leq L_k^{-1}[\sup_{\Omega(r)} \mathcal{G}_k - \inf_{\Omega(r)} \mathcal{G}_k] + L_k^{-1} \inf_{\Omega(r)} \mathcal{G}_k \leq L_k^{-1}[\sup_{\Omega(r)} \mathcal{G}_k - \inf_{\Omega(r)} \mathcal{G}_k] + 1\,, \tag{3.1}$$

where $\Omega(r) := \Omega \setminus \overline{B}_r(o)$ for $r > 0$ sufficiently small, by noticing that $\min_{\Omega(r)} \mathcal{G}_k = \min_{\partial\Omega} \mathcal{G}_k$, due to the fact that

$$\inf_{\Omega\setminus\{o\}} \mathcal{G}_k = \min_{\partial\Omega} \mathcal{G}_k\,,$$

from the maximum principle (recall that $\mathcal{G}_k(p) \to +\infty$ for $p \to o$). Then, since the sequence $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$ has uniformly bounded oscillations in any compact subset K of $M \setminus \{o\}$ for sufficiently large k such that $K \subseteq \Omega_k$, [51, Lemma 1] and $L_k \to +\infty$, the sequence $\{L_k^{-1}\mathcal{G}_k\}_{k\in\mathbb{N}}$ is locally uniformly bounded in $\Omega \setminus \{o\}$ and as a consequence of elliptic Hölder estimates in [43, Theorem A.6], it is also locally equicontinuous in $\Omega\setminus\{o\}$, therefore, up to a subsequence, it converges uniformly on compact subsets of $\Omega\setminus\{o\}$ to a harmonic function $\mathcal{G}$ which is bounded between 0 and 1, by formula (3.1). Consequently, $\mathcal{G}$ has a removable singularity at o, see [27] and can be extended to a bounded harmonic function on all Ω, still denoted by $\mathcal{G}$. Now, the function $\mathcal{G}$ admits a maximum in o, by virtue of the fact that each $L_k^{-1}\mathcal{G}_k$ assumes maximum in $\overline{\Omega}(r)$ on the interior boundary $\partial B_r(o)$ as

$$\max_{\overline{\Omega}_k \setminus B_r(o)} L_k^{-1}\mathcal{G}_k = \max_{\partial B_r(o)} L_k^{-1}\mathcal{G}_k\,,$$

by the maximum principle (recall that $L_k^{-1}\mathcal{G}_k$ is a positive function in $\Omega_k \setminus \{o\}$ and $L_k^{-1}\mathcal{G}_k \equiv 0$ on $\partial\Omega_k$). Then, $\mathcal{G}$ in Ω is equal to a constant $\gamma \in [0,1]$, again by the strong maximum principle. Notice that $\{L_k^{-1}\mathcal{G}_k\}_{k\in\mathbb{N}}$ is locally uniformly bounded in $M \setminus \overline{B}_r(o)$, since $0 \leq L_k^{-1}\mathcal{G}_k \leq L_k^{-1} \max_{\partial B_r(o)} \mathcal{G}_k$ and $L_k^{-1}\mathcal{G}_k$ converges uniformly to γ on $\partial B_r(o)$. Similarly as before, up to a subsequence, $\{L_k^{-1}\mathcal{G}_k\}_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $M\setminus\overline{B}_r(o)$, therefore it converges uniformly on compact subsets of $M \setminus \{o\}$ to a harmonic function on all M, which we will still denote by $\mathcal{G}$ (being its harmonic extension). Now, since $\mathcal{G}$ is a constant in Ω, one obtains that $\mathcal{G}$ is constant and equal to γ on all M. Let us show that $\gamma = 1$. Being $\partial\Omega$ compact, there exist a sequence $\{p_k\} \subseteq \partial\Omega$ and a point $p \in \partial\Omega$, such that $L_k^{-1}\mathcal{G}_k(p_k) = 1$ and $p_k \to p$. Consequently, $\gamma = \mathcal{G}(p) = 1$, by the uniform convergence of $\{L_k^{-1}\mathcal{G}_k\}_{k\in\mathbb{N}}$ on $\partial\Omega$, together with the continuity of $\mathcal{G}$.
We are then ready to show the contradiction, indeed, observing first that $\phi \geq L_k^{-1}\mathcal{G}_k$ in $\overline{\Omega}_k\setminus\Omega$ by the maximum principle (since the inequality is trivially true on its boundary) and later passing pointwise to the limit, we have $\phi \geq 1$ in $M \setminus \Omega$, which is not possible.
Thus, $\mathcal{G}_o$ is a positive harmonic function in $M \setminus \{o\}$, more precisely, it satisfies $\Delta\mathcal{G}_o = -\delta_o$ in M (arguing as [50, p. 198–199]) and tends to 0 at ∞, since $\mathcal{G}_o \leq L\phi$ in $M \setminus \Omega$, where L is the limit of a converging subsequence of the bounded sequence $\{L_k\}_{k\in\mathbb{N}}$. Moreover, there hold

$$\Big|\mathcal{G}_o - \frac{1}{4\pi r}\Big| = o(r^{-1}) \tag{3.2}$$

$$\Big|\nabla\mathcal{G}_o + \frac{1}{4\pi r^2}\,\nabla r\Big| = o(r^{-2}) \tag{3.3}$$

$$\Big|\nabla d\mathcal{G}_o - \frac{1}{4\pi r^2}\Big(\frac{2}{r}\,dr\otimes dr - \nabla dr\Big)\Big| = o(r^{-3}) \tag{3.4}$$

where r stands for the distance function from o in (M,g), by [57, Appendix] (there, these formulas are proven for every $\mathcal{G}$ distributional solution of $\Delta\mathcal{G} = -\delta_o$ in an open set $U \subseteq M$ containing o).
Since a comparison theorem holds for Green kernels (see [58, Corollary 2.6]), $\mathcal{G}_o$ is unique, in the sense that it is independent by the particular exhaustion and minimal among all positive distributional solutions of the equation $\Delta\mathcal{G} = -\delta_o$ in M (as a direct consequence of the fact that $(1+\varepsilon)\mathcal{G} \geq \mathcal{G}_k$ on $\overline{\Omega}_k \setminus \{o\}$, for every $\mathcal{G}$ positive distributional solution of $\Delta\mathcal{G} = -\delta_o$ on M and for every $\varepsilon > 0$, by the maximum principle and the behavior near o of these functions).
Finally, we recall the property

$$\int\limits_{\{\mathcal{G}_o=\tau\}} |\nabla\mathcal{G}_o|\,d\sigma = 1 \tag{3.5}$$

for every regular value τ of $\mathcal{G}_o$, by arguing as in [58, Section 2] (with $\psi \equiv 1$) and keeping into account the convergence of $\mathcal{G}_o$ to 0 at ∞ (together with the density of $C_c^\infty(M)$ in $\mathrm{Lip}_c(M)$).

We now state the monotonicity result that will imply the positive mass inequality.

Theorem 3.1.2. *Let (M,g) be a 3-dimensional, complete, one–ended AF manifold with nonnegative scalar curvature and satisfying $H_2(M;\mathbb{Z})=\{0\}$. Let u be the distributional solution of*

$$\begin{cases} \Delta u = 4\pi\delta_o & \text{in } M \\ u \to 1 & \text{at } \infty \end{cases} \tag{3.6}$$

for some $o \in M$, given by $u = 1-4\pi\mathcal{G}_o$, where $\mathcal{G}_o$ is the minimal positive Green's function $\mathcal{G}_o$ with pole at o which tends to 0 at ∞.
Let $F:(0,+\infty)\to\mathbb{R}$ be the function defined as

$$F(t) = 4\pi t - t^2 \int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|\,\mathrm{H}\,d\sigma + t^3 \int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|^2\,d\sigma\,, \tag{3.7}$$

where H is the mean curvature of the surface $\Sigma_t=\{u=1-1/t\}\setminus\mathrm{Crit}(u)$, computed with respect to the ∞–pointing unit normal vector field $\nu=\nabla u/|\nabla u\,|$ and σ is the 2–Hausdorff measure of (M,g). Then F coincides a.e. with a nondecreasing locally absolutely continuous function on $(0,+\infty)$, still denoted by F, such that

$$F'(t) = 4\pi + \int\limits_{\{u=1-\frac{1}{t}\}} \left[-\frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2\,|\nabla u|}{1-u} - \mathrm{H}\right)^2\right] d\sigma \tag{3.8}$$

a.e. in $(0,+\infty)$, in particular, at all values of t such that $1-1/t$ is a regular value of u.

Before to proceed with the proof of this theorem, we list some useful properties of the function u and we discuss the well–definition of the function F.

The function u is smooth on $M\setminus\{o\}$, tends $-\infty$ as $p\to o$ and is proper (as a function from $M\setminus\{o\}$ to $(-\infty,1)$). In particular, its level sets are compact. It follows then from [32, Theorem 1.7] that they also have finite 2–dimensional Hausdorff measure. At same time, the set $\mathrm{Crit}(u)$ of the critical points of u has locally finite 1–dimensional Hausdorff measure (see for instance [31, Theorem 1.1]) and the set $\widehat{\mathcal{N}}$ of the critical values of u has zero Lebesgue measure by Sard's theorem, whereas the set of regular values of u is open.
A key fact in the proof of the monotonicity is that *the regular level sets of u are connected.* Here is where the assumption $H_2(M;\mathbb{Z})=\{0\}$ comes into play. To see this, suppose by contradiction that for some $\tau\in(-\infty,1)\setminus\widehat{\mathcal{N}}$ the (regular) level set $\Sigma=\{u=\tau\}$ is given by the disjoint union of at least two connected components, each one being a connected closed surface. Considering two of such connected closed surfaces Σ' and Σ'', by the triviality of $H_2(M;\mathbb{Z})$, we have that each connected closed surface in (M,g) is the boundary of a bounded open domain. In particular, there exist two bounded connected open subsets $\Omega',\Omega''\subseteq M$ such that $\partial\Omega'=\Sigma'$ and $\partial\Omega''=\Sigma''$. If o doesn't belong to Ω', then $\overline{\Omega'}$ is contained in $M\setminus\{o\}$ and by the maximum principle u must then be constant in $\overline{\Omega'}$, but this is no possible since the level sets of u have finite 2–dimensional Hausdorff measure. Therefore, o belongs to Ω' and to Ω'', in particular Ω' and Ω'' have nonempty intersection. Consequently, $\Omega'\subseteq\Omega''$ or $\Omega''\subseteq\Omega'$, since $\partial\Omega'=\Sigma'$ and $\partial\Omega''=\Sigma''$ are disjoint and by the maximum principle, u must be constant in $\overline{\Omega''}\setminus\Omega'$ or in $\overline{\Omega'}\setminus\Omega''$ respectively, which is a contradiction.

Concerning the well–definition of the function F given by formula (3.7), it is sufficient to observe that the integrand functions are σ–a.e. bounded on each level set of the function u, by virtue of the fact that the level sets of u have finite σ–measure. To justify this sentence, one only needs to check that $\big||\nabla u|\mathrm{H}\big|$ is bounded on $\{u=\tau\}\setminus\mathrm{Crit}(u)$, for every $\tau\in(-\infty,1)$, being $\sigma\big(\mathrm{Crit}(u)\big)=0$. Since u is harmonic, H can be expressed as

$$\mathrm{H}=-\frac{\nabla du(\nabla u,\nabla u)}{|\nabla u|^3}=-\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|^2},\tag{3.9}$$

on the open set $M_o\setminus\mathrm{Crit}(u)$, setting

$$M_o:=M\setminus\{o\}.$$

Consequently, one has

$$\big||\nabla u|\mathrm{H}\big|\leq|\nabla du(\nu,\nu)|\leq|\nabla du|\,,\tag{3.10}$$

where $|\nabla u|\neq 0$.

We are now ready to prove Theorem 3.1.2. For the sake of clarity, we first give a proof under the favorable assumption that $\mathrm{Crit}(u)=\emptyset$, then we proceed with the proof in the general case.

Proof of Theorem 3.1.2 in the absence of critical points. In this case, all of the level sets of u are regular and diffeomorphic between them, in turn the function F is everywhere continuously differentiable in its interval of definition. We claim that $F'(t)\geq 0$ for every $t\in(0,+\infty)$. We start observing that

$$\begin{aligned}\frac{d}{dt}\int\limits_{\{u=1-\frac{1}{t}\}}|\nabla u|^2\,d\sigma&=-\frac{1}{t^2}\int\limits_{\{u=1-\frac{1}{t}\}}|\nabla u|\,\mathrm{H}\,d\sigma\,,\\ \frac{d}{dt}\int\limits_{\{u=1-\frac{1}{t}\}}|\nabla u|\,\mathrm{H}\,d\sigma&=-\frac{1}{t^2}\int\limits_{\{u=1-\frac{1}{t}\}}|\nabla u|\left[\Delta^{\Sigma_t}\left(\frac{1}{|\nabla u|}\right)+\frac{|\mathrm{h}|^2+\mathrm{Ric}(\nu,\nu)}{|\nabla u|}\right]d\sigma\,,\end{aligned}\tag{3.11}$$

where Δ^{Σ_t} is the Laplace–Beltrami operator of the induced metric g^{Σ_t} on $\Sigma_t=\{u=1-\frac{1}{t}\}$ and h denotes the second fundamental form of Σ_t, computed with respect to $\nu=\nabla u/|\nabla u|$. Here, we have used the normal first variation of volume measure and of mean curvature (see the end of Section 1.3). Now, with the help of the traced Gauss equation (1.6), the integrand on the right hand side of equality (3.11) can be expressed as

$$\begin{aligned}|\nabla u|\left[\Delta^{\Sigma_t}\left(\frac{1}{|\nabla u|}\right)+\frac{|\mathrm{h}|^2+\mathrm{Ric}(\nu,\nu)}{|\nabla u|}\right]&=\\ &=-\Delta^{\Sigma_t}(\log|\nabla u|)+\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2}+\frac{\mathrm{R}}{2}-\frac{\mathrm{R}^{\Sigma_t}}{2}+\frac{|\mathring{\mathrm{h}}|^2}{2}+\frac{3}{4}\mathrm{H}^2,\end{aligned}$$

where R^{Σ_t} and $\mathring{\mathrm{h}}$ denote the scalar curvature and the traceless second fundamental form of Σ_t, respectively, whereas ∇^{Σ_t} is the Levi–Civita connection of g^{Σ_t}. Substituting the latter expression into formula (3.11) and using standard manipulations, one arrives at

$$F'(t) = 4\pi - \int_{\Sigma_t} \frac{\mathrm{R}^{\Sigma_t}}{2}\, d\sigma + \int_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2\,|\nabla u|}{1-u} - \mathrm{H}\right)^2 \right] d\sigma. \tag{3.12}$$

Now, we notice that the last summand of the right hand side is always nonnegative, as the scalar curvature of (M, g) is nonnegative by assumption. The first two summands also give a nonnegative contribution, by virtue of Gauss–Bonnet theorem 1.1.2 and the discussion thereafter, as Σ_t is a connected closed surface for every $t \in (0, +\infty)$, hence

$$\int_{\Sigma_t} \frac{\mathrm{R}^{\Sigma_t}}{2}\, d\sigma = \int_{\Sigma_t} K^{\Sigma_t}\, d\sigma = 2\pi\chi(\Sigma_t) \leq 4\pi\,,$$

where $K^{\Sigma_t} = \mathrm{R}^{\Sigma_t}/2$ is the Gauss curvature and $\chi(\Sigma_t)$ is the Euler characteristic of Σ_t. □

Proof of Theorem 3.1.2 in the general case. Let us consider the vector field Y, defined as

$$Y = \frac{\nabla|\nabla u|}{(1-u)^2} + \frac{|\nabla u|}{(1-u)^3}\nabla u\,, \tag{3.13}$$

where u is a solution of problem (3.6), then the vector field Y is well–defined on the open set $M_o \setminus \mathrm{Crit}(u)$. With the help of Bochner formula,

$$\frac{1}{2}\,\Delta\,|\nabla f|^2 = |\nabla df|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla\,\Delta f, \nabla f)$$

for every $f \in C^\infty(M)$, the divergence of Y on $M_o \setminus \mathrm{Crit}(u)$ can be expressed as

$$\mathrm{div}(Y) = \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,|\nabla u|^2}{(1-u)^2} + \frac{3\,g(\nabla|\nabla u|, \nabla u)}{(1-u)\,|\nabla u|} + \frac{|\nabla du|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u)}{|\nabla u|^2}\right],$$

where in the computation we used the fact that u is harmonic. By the traced Gauss equation (1.6) and the identity

$$|\nabla du|^2 = |\nabla u|^2|\mathrm{h}|^2 + |\nabla|\nabla \mathrm{u}||^2 + |\nabla^\Sigma|\nabla \mathrm{u}||^2\,,$$

one can work out an equivalent expression for $\mathrm{div}(Y)$, adapted to the (regular parts of the) level sets of u, namely

$$\mathrm{div}(Y) = \frac{|\nabla u|}{(1-u)^2}\left[-\frac{\mathrm{R}^\Sigma}{2} + \frac{|\nabla^\Sigma|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2\,|\nabla u|}{1-u} - \mathrm{H}\right)^2\right]. \tag{3.14}$$

Here, $\mathrm{h}, \mathrm{H}, \mathrm{R}^\Sigma$ and ∇^Σ are all referred to the regular level set of u that passes for the point where $\mathrm{div}(Y)$ is computed.
First of all, we show that $\mathrm{div}(Y) \in L^1_{loc}(M_o)$, keeping into account that $\mathrm{div}(Y)$ is μ–a.e. well–defined and smooth since $\mu\big(\mathrm{Crit}(u)\big) = 0$. Let $K \subseteq M_o$ be a compact set, then, by Sard's theorem, K is contained in $E^t_s := \{1 - \frac{1}{s} < u < 1 - \frac{1}{t}\}$ for some $t > s > 0$ such that $1 - 1/s$ and $1 - 1/t$ are regular values of u. We consider the non–trivial case in which the interval $(1 - 1/s, 1 - 1/t)$ contains some critical value of u, in particular the open subset $\{1 - 1/s < u < 1 - 1/t\}$ contains critical points, hence, the vector field Y is not well–defined on $\{1 - 1/s \leq u \leq 1 - 1/t\}$. To overcome

this difficulty, we consider a pointwise nondecreasing sequence of cut–off functions $\{\eta_k\}_{k\in\mathbb{N}^+}$ such that, for every $k \in \mathbb{N}^+$, the function $\eta_k : [0,+\infty) \to [0,1]$ is smooth, nondecreasing and satisfies

$$\eta_k(\tau) \equiv 0 \quad \text{in} \left[0\,,\frac{1}{2k}\right], \qquad 0 \le \eta_k'(\tau) \le 2k \quad \text{in} \left[\frac{1}{2k}\,,\frac{3}{2k}\right], \qquad \eta_k(\tau) \equiv 1 \quad \text{in} \left[\frac{3}{2k}\,,+\infty\right). \tag{3.15}$$

Using these cut–off functions, we define for every $k \in \mathbb{N}^+$ the vector field

$$Y_k \;=\; \eta_k\Big(\frac{|\nabla u|}{1-u}\Big) Y\,.$$

It is immediate to see that all the vector fields Y_k are well–defined on M_o and they coincide with the vector field Y, defined by formula (3.13), whenever restricted to a compact set sitting inside $M_o \setminus \mathrm{Crit}(u)$, for k large enough. Their divergence can be computed as follows,

$$\begin{aligned}\mathrm{div}(Y_k) = {} & \frac{|\nabla u|}{(1-u)^2}\left\{\eta_k\Big(\frac{|\nabla u|}{1-u}\Big)\left[\frac{3\,|\nabla u|^2}{(1-u)^2} + \frac{|\nabla du|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}\right]\right\} \\ & + \frac{|\nabla u|}{(1-u)^2}\left\{\eta_k\Big(\frac{|\nabla u|}{1-u}\Big)\left[\frac{3\,g(\nabla|\nabla u|,\nabla u)}{(1-u)|\nabla u|} + \frac{\mathrm{Ric}(\nabla u,\nabla u)}{|\nabla u|^2}\right]\right\} \\ & + \frac{|\nabla u|^2}{(1-u)^3}\;\eta_k'\Big(\frac{|\nabla u|}{1-u}\Big)\left|\frac{\nabla u}{1-u} + \frac{\nabla|\nabla u|}{|\nabla u|}\right|^2.\end{aligned}$$

A remarkable feature of the above expression is that the last summand is nonnegative. Thus, considering the function $\Phi : (0,+\infty) \to \mathbb{R}$, defined by

$$\Phi(t) = -\;t^2\int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|\,\mathrm{H}\;d\sigma \;+\; t^3\int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|^2\;d\sigma$$

which is well-defined, as $\Phi(t) = F(t) - 4\pi t$ in $(0,+\infty)$ and since, for large enough k, the vector field Y_k coincides with Y at the boundary of $\{1-1/s < u < 1-1/t\}$, the divergence theorem, applied to Y_k on $\{1-1/s < u < 1-1/t\}$ (see Remark 1.1.1), implies

$$\Phi(t) - \Phi(s) \;=\; \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} \mathrm{div}(Y_k)\,d\mu \;\ge\; \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} P_k\,d\mu \;+\; \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} D_k\,d\mu\,, \tag{3.16}$$

where we set

$$\begin{aligned} P_k &:= \eta_k\Big(\frac{|\nabla u|}{1-u}\Big)\,P \quad \text{with} \quad P := \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,|\nabla u|^2}{(1-u)^2} + \frac{|\nabla du|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2}\right], \\ D_k &:= \eta_k\Big(\frac{|\nabla u|}{1-u}\Big)\,D \quad \text{with} \quad D := \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,g(\nabla|\nabla u|,\nabla u)}{(1-u)|\nabla u|} + \frac{\mathrm{Ric}(\nabla u,\nabla u)}{|\nabla u|^2}\right]. \end{aligned}$$

Notice that the functions D and P are μ–a.e. well–defined and smooth as well as $\mathrm{div}(Y)$. Now, the functions P_k are clearly nonnegative and they pointwise converge monotonically to the function $P\,\mathbb{I}_{M_o\setminus \mathrm{Crit}(u)}$ in M_o, where $\mathbb{I}_{M_o\setminus \mathrm{Crit}(u)}$ denotes the characteristic function of $M_o \setminus \mathrm{Crit}(u)$. The monotone convergence theorem thus yields

$$\lim_{k\to+\infty} \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} P_k \, d\mu = \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} P \, d\mu \,, \tag{3.17}$$

by virtue of the fact that $\mu\big(\mathrm{Crit}(u)\big) = 0$. Concerning the functions D_k, they are not necessarily positive. Indeed, in $M_o \setminus \mathrm{Crit}(u)$ the terms of D in the square brackets are related both to the mean curvature of the regular level sets of u and to the Ricci tensor of M and we do not have information about them. However, we know that all the functions D_k belong to $L^1_{loc}(M_o)$, since

$$|D_k| \;\leq\; |D| \;\leq\; \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,|\nabla du|}{1-u} + |\mathrm{Ric}|\right]$$

in $M_o \setminus \mathrm{Crit}(u)$ and converge pointwise to the function $D\,\mathbb{I}_{M_o\setminus \mathrm{Crit}(u)}$ in M_o. Then, $D \in L^1(E_s^t)$ and the dominated convergence theorem implies

$$\lim_{k\to+\infty} \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} D_k \, d\mu = \int\limits_{\{1-\frac{1}{s}<u<1-\frac{1}{t}\}} D \, d\mu \,, \tag{3.18}$$

being $\mu\big(\mathrm{Crit}(u)\big) = 0$. As a consequence of inequality (3.16) together with limits (3.17) and (3.18), we have that the sequence given by the integrals of the functions P_k is bounded, in particular $P \in L^1(E_s^t)$. Being $\mathrm{div}(Y) = P + D$ in $M_o \setminus \mathrm{Crit}(u)$, it follows then that $\mathrm{div}(Y) \in L^1_{loc}(M_o)$. More precisely, we obtained

(1) $0 \leq P_k \uparrow P\,\mathbb{I}_{M_o\setminus \mathrm{Crit}(u)}$ in M_o and $P \in L^1_{loc}(M_o)$;

(2) $|D_k| \leq |D| \leq \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,|\nabla du|}{1-u} + |\mathrm{Ric}|\right]$ in $M_o \setminus \mathrm{Crit}(u)$, in particular, $|D|/|\nabla u| \in L^1_{loc}(M_o)$ and $D_k \to D\,\mathbb{I}_{M_o\setminus \mathrm{Crit}(u)}$ in M_o;

(3) $\mathrm{div}(Y) = P + D$ in $M_o \setminus \mathrm{Crit}(u)$.

Now, we show that $\Phi \in W^{1,1}_{loc}(0,+\infty)$ with weak derivative given by

$$\Phi'(t) = \int\limits_{\{u=1-\frac{1}{t}\}} \left[-\,\frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2|\nabla u|}{1-u} - \mathrm{H}\right)^2 \right] d\sigma$$

a.e. in $(0,+\infty)$ with an argument inspired by [8]. We consider a function $\chi \in C^\infty_c\big((0,+\infty)\big)$, then, we have

$$\begin{aligned}\int_0^{+\infty} \chi'(s)\,\Phi(s)\,ds &= \int_0^{+\infty} ds \int_{\{u=1-\frac{1}{s}\}} \chi'\Big(\frac{1}{1-u}\Big)\,\frac{g(Y,\nabla u)}{|\nabla u|}\,d\sigma \\ &= \int_{-\infty}^{1} d\tau \int_{\{u=\tau\}} \frac{1}{(1-u)^2}\,\chi'\Big(\frac{1}{1-u}\Big)\,\frac{g(Y,\nabla u)}{|\nabla u|}\,d\sigma \\ &= \int_{M_o} g\Big(Y,\nabla\Big[\chi\Big(\frac{1}{1-u}\Big)\Big]\Big)\,d\mu \\ &= \lim_{k\to+\infty}\int_{M_o} g\Big(Y_k,\nabla\Big[\chi\Big(\frac{1}{1-u}\Big)\Big]\Big)\,d\mu \\ &= -\lim_{k\to+\infty}\int_{M_o} \chi\Big(\frac{1}{1-u}\Big)\,\mathrm{div}(Y_k)\,d\mu\,. \end{aligned} \tag{3.19}$$

Here, the third equality follows by the coarea formula (1.1), by observing that in $M_o \setminus \mathrm{Crit}(u)$ there holds

$$\left|\frac{1}{(1-u)^2}\,\chi'\Big(\frac{1}{1-u}\Big)\,\frac{g(Y,\nabla u)}{|\nabla u|}\right| \leq \frac{1}{(1-u)^4}\,\|\chi'\|_{L^\infty(0,+\infty)}\left[|\nabla du|+\frac{|\nabla u|^2}{1-u}\right] \in L^1_{loc}(M_o)\,. \tag{3.20}$$

The fourth equality is a consequence of the dominated convergence theorem, since $Y_k \to Y\mathbb{I}_{M_o\setminus\mathrm{Crit}(u)}$ pointwise in M_o, the function $\chi\big(\frac{1}{1-u}\big)$ has compact support in M_o and keeping into account that in $M_o \setminus \mathrm{Crit}(u)$ one has

$$\left|g\Big(Y_k,\nabla\Big[\chi\Big(\frac{1}{1-u}\Big)\Big]\Big)\right| \leq \left|g\Big(Y,\nabla\Big[\chi\Big(\frac{1}{1-u}\Big)\Big]\Big)\right| = \left|\frac{|\nabla u|}{(1-u)^2}\,\chi'\Big(\frac{1}{1-u}\Big)\,\frac{g(Y,\nabla u)}{|\nabla u|}\right|$$

together with formula (3.20). Finally, the last equality follows by the properties of the divergence operator combined with the divergence theorem applied to $\chi\big(\frac{1}{1-u}\big)\,Y_k$ on $E^b_a := \{1-\frac{1}{a} < u < 1-\frac{1}{b}\}$, for $a,b\in(0,+\infty)$ such that $1-1/a,\ 1-1/b$ are regular values of u and $\mathrm{supp}\chi \subseteq (a,b)$. In order to compute the last limit in formula (3.19), we recall that

$$\int_{M_o} \chi\Big(\frac{1}{1-u}\Big)\,\mathrm{div}(Y_k)\,d\mu = \int_{E^b_a} \chi\Big(\frac{1}{1-u}\Big)\,\Big\{P_k + D_k + \frac{|\nabla u|}{1-u}\,\eta'_k\Big(\frac{|\nabla u|}{1-u}\Big)\,Q\Big\}\,d\mu\,, \tag{3.21}$$

where

$$Q := \frac{|\nabla u|}{(1-u)^2}\left|\frac{\nabla u}{1-u}+\frac{\nabla|\nabla u|}{|\nabla u|}\right|^2.$$

Observe that the function Q is μ–a.e. well–defined and smooth, moreover, $Q \in L^1_{loc}(M_o)$. Indeed, in $M_o \setminus \mathrm{Crit}(u)$, since there hold

$$\begin{aligned}
Q &= \frac{|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|^2}{(1-u)^2} - \frac{2\,|\nabla u|\,\mathrm{H}}{1-u} + \frac{|\nabla^\Sigma |\nabla u||^2}{|\nabla u|^2} + \mathrm{H}^2\right] \geq 0\,,\\
P &= \frac{|\nabla u|}{(1-u)^2}\left[\frac{3\,|\nabla u|^2}{(1-u)^2} + \frac{|\nabla^\Sigma |\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{H}^2}{2} + |\mathring{\mathrm{h}}|^2\right] \in L^1_{loc}(M_o)\,,
\end{aligned}$$

then, we obtain

$$Q \leq 4P + \frac{2|\nabla u|}{(1-u)^3}|\nabla du|$$

from formula (3.10), consequently $Q \in L^1_{loc}(M_o)$. This property of Q along with the dominated convergence theorem yields

$$\lim_{k\to+\infty} \int\limits_{E_a^b} \chi\Big(\frac{1}{1-u}\Big)\,\frac{|\nabla u|}{(1-u)}\,\eta_k'\Big(\frac{|\nabla u|}{1-u}\Big)\,Q\,d\mu \;=\; 0\,, \tag{3.22}$$

indeed, $\chi\big(\frac{1}{1-u}\big)\,\frac{|\nabla u|}{(1-u)}\,\eta_k'\big(\frac{|\nabla u|}{1-u}\big)\,Q \to 0$ pointwise in M_o and

$$\begin{aligned}
\Big|\chi\Big(\frac{1}{1-u}\Big)\Big|\,\frac{|\nabla u|}{1-u}\,\eta_k'\Big(\frac{|\nabla u|}{1-u}\Big)\,Q &\leq \|\chi\|_{L^\infty(0,+\infty)}\,\frac{|\nabla u|}{1-u}\,\eta_k'\Big(\frac{|\nabla u|}{1-u}\Big)\,Q\,\mathbb{I}_{\left\{\frac{1}{2k}\leq\frac{|\nabla u|}{1-u}\leq\frac{3}{2k}\right\}}\\
&\leq \|\chi\|_{L^\infty(0,+\infty)}\,\frac{3}{2k}\,2k\,Q\,\mathbb{I}_{\left\{\frac{1}{2k}\leq\frac{|\nabla u|}{1-u}\leq\frac{3}{2k}\right\}}\\
&\leq\,3\,\|\chi\|_{L^\infty(0,+\infty)}\,Q \in L^1(E_a^b)\,,
\end{aligned}$$

by virtue of properties (3.15) of the cut–off functions η_k. At the same time, from the dominated convergence theorem it also follows

$$\int\limits_{E_a^b}\chi\Big(\frac{1}{1-u}\Big)\,(P_k\,+\,D_k)d\mu \;\to\; \int\limits_{E_a^b}\chi\Big(\frac{1}{1-u}\Big)\,(P\,+\,D)d\mu\,, \tag{3.23}$$

by points (1) and (2) above and being χ bounded. Then, by formulas (3.19), (3.21) and limits (3.22) and (3.23), we have

$$\begin{aligned}
&\int\limits_0^{+\infty}\chi'(s)\,\Phi(s)\,ds = -\lim_{k\to+\infty}\int\limits_{M_o}\chi\Big(\frac{1}{1-u}\Big)\,\mathrm{div}(Y_k)\,d\mu = -\int\limits_{M_o}\chi\Big(\frac{1}{1-u}\Big)\,(P\,+\,D)\,d\mu\\
&= -\int\limits_0^{+\infty} ds\,\Bigg\{\chi(s)\int\limits_{\{u=1-\frac{1}{s}\}}\Bigg[-\frac{\mathrm{R}^{\Sigma_s}}{2} + \frac{|\nabla^{\Sigma_s}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\Big(\frac{2\,|\nabla u|}{1-u} - \mathrm{H}\Big)^2\Bigg]d\sigma\Bigg\}\,,
\end{aligned}$$

where the third equality follows by the coarea formula (1.1) by virtue of points (1), (2), (3) above and equality (3.14).
Thus, $\Phi \in W^{1,1}_{loc}(0,+\infty)$, hence $F \in W^{1,1}_{loc}(0,+\infty)$ with weak derivative given a.e. by the expression in formula (3.8), as $F(t) = 4\pi t + \Phi(t)$. Consequently, F coincides a.e. with a locally absolutely continuous function on $(0,+\infty)$, still denoted by F. The weak derivative of F coincides with the classical derivative almost everywhere,

thus, for any pair of positive real numbers $s < t$, we have

$$
\begin{aligned}
&F(t) - F(s)\\
&= \int_s^t d\tau \left\{ 4\pi + \int\limits_{\{u=1-\frac{1}{\tau}\}} \left[-\frac{\mathrm{R}^{\Sigma_\tau}}{2} + \frac{|\nabla^{\Sigma_\tau}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2\,|\nabla u|}{1-u} - \mathrm{H} \right)^2 \right] d\sigma \right\}\\
&= \int\limits_{[s,t]\setminus\mathcal{N}} d\tau \left\{ 4\pi - \int\limits_{\{u=1-\frac{1}{\tau}\}} \frac{\mathrm{R}^{\Sigma_\tau}}{2}\, d\sigma + \int\limits_{\{u=1-\frac{1}{\tau}\}} \left[\frac{|\nabla^{\Sigma_\tau}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{2|\nabla u|}{1-u} - \mathrm{H} \right)^2 \right] d\sigma \right\},
\end{aligned}
$$

where

$$\mathcal{N} := \{\tau \in (0, +\infty) : 1 - 1/\tau \text{ is a critical value of } u\}\,.$$

Notice that in the last identity we used the fact that $\mathcal{N}$ is negligible, by Sard's theorem. Then, the monotonicity of F follows by the very same considerations we made after formula (3.12). □

Combining the above theorem with some standard facts about the asymptotic behavior of the minimal positive Green's functions near the pole, one gets the following corollary.

Corollary 3.1.3. *Under the assumptions of Theorem 3.1.2, we have*

$$0 \le \lim_{t\to+\infty} F(t)\,. \tag{3.24}$$

Moreover, if $\lim_{t\to+\infty} F(t) = 0$*, then* (M, g) *is isometric to* $(\mathbb{R}^3, g_{\mathrm{eucl}})$*.*

Proof. We first claim that $\lim_{t\to0^+} F(t) = 0$. To see this fact, we recall that u is related to the minimal positive Green's function $\mathcal{G}_o$ of (M, g) with pole at o, by the equality $u = 1 - 4\pi\mathcal{G}_o$. Hence, there holds

$$\int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|\, d\sigma \;\equiv\; 4\pi \tag{3.25}$$

for every $t \in (0, +\infty) \setminus \mathcal{N}$, by identity (3.5). As a consequence of the asymptotic behavior near the pole of $\mathcal{G}_o$, see formulas (3.2) (3.3) and (3.4), in a sufficiently small neighborhood of $o \in M$, the function u satisfies the bounds

$$\frac{C_1}{r} \le 1 - u \le \frac{C_2}{r}, \qquad \frac{C_3}{r^2} \le |\nabla u| \le \frac{C_4}{r^2}, \qquad |\nabla du| \le \frac{C_5}{r^3},$$

for some positive constants $C_i > 0$, $i = 1, \ldots, 5$. In particular, the function u has no critical points near o, hence $1 - 1/t$ is a regular value of u for $t > 0$ sufficiently close to 0. Combining these bounds with formulas (3.10) and (3.25), we conclude

$$
\begin{aligned}
t^2 \int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|^2\, d\sigma \;&\le \int\limits_{\{u=1-\frac{1}{t}\}} \frac{C_4}{r^2(1-u)^2}\, |\nabla u|\, d\sigma \;\le\; \frac{C_4}{C_1^2} \int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|\, d\sigma \;\le\; \frac{4\pi C_4}{C_1^2}\,,\\
t \int\limits_{\{u=1-\frac{1}{t}\}} |\mathrm{H}|\,|\nabla u|\; d\sigma \;&\le \int\limits_{\{u=1-\frac{1}{t}\}} \frac{|\nabla du|}{1-u}\, d\sigma \;\le\; \frac{C_5}{C_1 C_3} \int\limits_{\{u=1-\frac{1}{t}\}} |\nabla u|\, d\sigma \;\le\; \frac{4\pi C_5}{C_1 C_3}\,.
\end{aligned}
$$

From these estimates, the fact that $F(t) \to 0$, as $t \to 0^+$, easily follows. Then, the monotonicity of F yields inequality (3.24).

Let us now focus our attention on the rigidity statement. By the above discussion, we have that u behaves like $1 - 1/r$ and ∇u behaves like $\nabla r / r^2$ in a sufficiently small neighborhood of the pole o. In particular, there exists a maximal time T such that $\nabla u \neq 0$ in $\{u < 1 - 1/T\}$, where F is continuously differentiable. Notice that the open set $\{u < 1 - 1/T\}$ is connected, then, arguing by contradiction and using inequality (3.24), one easily gets that $F' \equiv 0$ in $(0, T)$. In particular, all the positive summands in formula (3.12) are forced to vanish for every $t \in (0, T)$. Then, $\nabla^{\Sigma_t} |\nabla u| \equiv 0$ implies that $|\nabla u| = f(u)$, for some positive function $f : (0, T) \to (0, +\infty)$ which can actually be made explicit. Indeed, from formula (3.12) one also has that $\mathrm{H} = 2f(u)/(1-u)$ and it follows from equalities (3.9) that $\mathrm{H} = -g(\nabla |\nabla u|, \nabla u)/|\nabla u|^2 = -f'(u)$. Hence, we have that f satisfies the ODE

$$f'(u) = -\frac{2f(u)}{1-u}\,.$$

Now, the only solution to this ODE which is compatible with the asymptotic behavior of u and $|\nabla u|$, as $u \to -\infty$, is given by $f(u) = (1-u)^2$. Since $u < 1$ on the whole manifold, f never vanishes, hence $T = +\infty$ and $|\nabla u| \neq 0$ everywhere. In particular, all the level sets of u are regular and diffeomorphic to each other. So, up to an isometry, we have that $M_o = (-\infty, 1) \times \{u = 0\}$, every slice $\{t\} \times \{u = 0\}$ is the level set $\{u = t\}$ and the metric g can be written on M_o as

$$g \;=\; \frac{du \otimes du}{(1-u)^4} \;+\; g_{\alpha\beta}(u,\vartheta)\, d\vartheta^\alpha \otimes d\vartheta^\beta\,,$$

where $g_{\alpha\beta}(u,\vartheta)\, d\vartheta^\alpha \otimes d\vartheta^\beta$ represents the metric induced by g on the level sets of u. Exploiting the vanishing of the traceless second fundamental form of the level sets in formula (3.12), i.e. $\mathrm{h}_{\alpha\beta} = (\mathrm{H}/2)\, g_{\alpha\beta}$, in combination with equality $\mathrm{h}_{\alpha\beta} = \nabla du_{\alpha\beta}/|\nabla u|$ by equality (1.4), it turns out that the coefficients $g_{\alpha\beta}(u,\vartheta)$ satisfy the following first order system of PDE's

$$\frac{\partial g_{\alpha\beta}}{\partial u} \;=\; \frac{2\, g_{\alpha\beta}}{1-u}\,,$$

from which one can deduce

$$g_{\alpha\beta}(u,\vartheta)\, d\vartheta^\alpha \otimes d\vartheta^\beta = (1-u)^{-2} c_{\alpha\beta}(\vartheta)\, d\vartheta^\alpha \otimes d\vartheta^\beta\,.$$

At the same time, the traced Gauss equation (1.6), together with Bochner formula imply

$$
\begin{aligned}
\mathrm{R}^{\{u=u_0\}} &= \mathrm{R} - 2\mathrm{Ric}(\nu,\nu) - |\mathrm{h}|^2 + \mathrm{H}^2 \\
&= -2\,|\nabla u|^{-2}\,\mathrm{Ric}(\nabla u,\nabla u) + \frac{\mathrm{H}^2}{2} \\
&= |\nabla u|^{-2}\,\big[-\Delta|\nabla u|^2 + 2\,|\nabla du|^2\big] + \frac{\mathrm{H}^2}{2} \\
&= |\nabla u|^{-2}\,\Big\{-\Delta|\nabla u|^2 + 2\,\Big[|\mathrm{h}|^2|\nabla u|^2 + |\nabla^\perp|\nabla u||^2\Big]\Big\} + \frac{\mathrm{H}^2}{2} \\
&= |\nabla u|^{-2}\,\big\{-\Delta|\nabla u|^2 + 2\,\big[|\mathrm{h}|^2|\nabla u|^2 + \mathrm{H}^2|\nabla u|^2\big]\big\} + \frac{\mathrm{H}^2}{2} \\
&= |\nabla u|^{-2}\,\big\{-\Delta|\nabla u|^2 + 3\,\mathrm{H}^2|\nabla u|^2\big\} + \frac{\mathrm{H}^2}{2} \\
&= -\frac{\Delta|\nabla u|^2}{|\nabla u|^2} + \frac{7\,\mathrm{H}^2}{2} \\
&= 2(1-u_0)^2\,,
\end{aligned}
$$

where we used again the fact that all the positive summands in formula (3.12) are forced to vanish on each level set of u and equalities $|\nabla u| = (1-u)^2$ and $\mathrm{H} = 2\,(1-u)$. Then, $\{u = u_0\}$ with the Riemannian metric induced by (M,g) has constant sectional curvature (equal to $(1-u_0)^2$) and, by the vanishing of the Gauss–Bonnet term in formula (3.12), it is diffeomorphic to a 2–sphere. Consequently, $(\{u = u_0\}, g_{\{u=u_0\}})$ is isometric to $(\mathbb{S}^2, (1-u_0)^{-2} g_{\mathbb{S}^2})$ from [25, Section 3.F], thus, up to an isometry, one has $M_o = (-\infty, 1)\times\mathbb{S}^2$ and

$$
g = \frac{du\otimes du}{(1-u)^4} + \frac{g_{\mathbb{S}^2}}{(1-u)^2}\,.
$$

Then, (M_o, g) is isometric to $(\mathbb{R}^3\setminus\{O\}, g_{\mathrm{eucl}})$, being the map

$$
(u,\vartheta) \in \big((-\infty,1)\times\mathbb{S}^2, g\big) \mapsto \left(\frac{1}{1-u}, \vartheta\right) \in \big((0,+\infty)\times\mathbb{S}^2, dr\otimes dr + r^2 g_{\mathbb{S}^2}\big)
$$

an isometry. However, this isometry can be extended to a homeomorphism from M to $\mathbb{R}^3$ and consequently, M is simply connected. Finally, the rest of the claim follows from the fact the manifold (M,g) is complete, simply connected and flat (that is, it has constant zero curvature), as $|\mathrm{Riem}|$ is a continuous function, then it is isometric to $\mathbb{R}^3$ with its standard metric (see [25, Section 3.F], for instance). □

By means of Theorem 3.1.2, we present now a new proof of the positive mass theorem 3.1.1. As we will see, it also exploits the recent result [12, Proposition 2.1], that for every Riemannian manifold (M,g), satisfying the assumptions of the positive mass theorem 3.1.1 and having scalar curvature in $L^1(M)$, for every $\varepsilon > 0$, there exists a Riemannian manifold $(\overline{M},\overline{g})$ satisfying the same assumptions of (M,g) and also the following properties: $\overline{M}$ is diffeomorphic to $\mathbb{R}^3$; there holds $|m - \overline{m}| < \varepsilon$ where m and $\overline{m}$ are the ADM masses of (M,g) and $(\overline{M},\overline{g})$ respectively; there exists an AF coordinate chart (x^1,x^2,x^3) such that $\overline{g} = (1 + \frac{\overline{m}}{2|x|})^4\,\delta_{ij}dx^i\otimes dx^j$.

Proof of the positive mass theorem 3.1.1. We set $m = m_{\mathrm{ADM}}$ and we deal the first part of the positive mass statement, i.e., $m \geq 0$. If the scalar curvature R is not in $L^1(M)$, then $m = +\infty$ and the inequality is obvious, hence we assume $\mathrm{R} \in L^1(M)$. By [12, Proposition 2.1], the analysis can be reduced to the special case where the underlying manifold M is diffeomorphic to $\mathbb{R}^3$ and there exists a distinguished AF coordinate chart $x = (x^1, x^2, x^3)$ – called *Schwarzschildian coordinate chart* – in which the metric g can be expressed as

$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\, dx^i \otimes dx^j .$$

By virtue of the monotonicity of the function F, given by formula (3.7), established in Theorem 3.1.2, we have

$$\lim_{t\to 0^+} F(t) \;\leq\; \lim_{t\to +\infty} F(t)\,. \tag{3.26}$$

We now claim that $\lim_{t\to+\infty} F(t) = 8\pi m$. It is clear that combining this claim and $\lim_{t\to 0^+} F(t) = 0$, given by Corollary 3.1.3, by inequality (3.26), one easily gets $m \geq 0$. In order to compute the limit of $F(t)$ as $t \to +\infty$, we need to understand the behavior at infinity of u, in particular, we have

$$u \;=\; 1 - \frac{1}{|x|} + \frac{1}{2|x|^2}\left(m + \phi(x/|x|)\right) + O_2\big(|x|^{-2-\alpha}\big)\,, \tag{3.27}$$

where $x = (x^1, x^2, x^3)$ is a Schwarzschildian coordinate chart, ϕ fulfills $\Delta^{\mathbb{S}^2}\phi = -2\phi$ and finally $0 < \alpha < 1$. This formula follows by Corollary 1.4.15 (whose hypothesis are trivially fulfilled by $1-u$), in view of identity (3.25), indeed, fixing a regular value $\tau \in (-\infty, 1)$ of u and taken $R_0 > 0$ sufficiently large in a way that $\{u \geq \tau,\ |x| \leq R\}$ is a Riemannian submanifold with boundary, given by $\{u = \tau\} \sqcup \{|x| \leq R\}$, for every $R \geq R_0$, we first apply the divergence theorem to ∇u on $\{u \geq \tau,\ |x| \leq R\}$ and then we take the limit as $R \to +\infty$. We then obtain

$$4\pi = \int\limits_{\{u=\tau\}} |\nabla u|\, d\sigma = \lim_{R\to+\infty} \int\limits_{\{|x|=R\}} g\left(\nabla u, \frac{\nabla|x|}{|\nabla|x||}\right) d\sigma = 4\pi C\,,$$

where C is the constant in the asymptotic expansion

$$u = 1 - \frac{C}{|x|} + \frac{1}{2|x|^2}\left(mC + \phi(x/|x|)\right) + O_2(|x|^{-2-\alpha})\,.$$

Here, the limit follows by the equalities analogous to those of formula (1.33). Thus, C is equal to 1.

Notice that, as a first consequence of expansion (3.27), the function u has no critical points near infinity, hence there exists $t_0 \in (0, +\infty)$ such that $1 - 1/t$ is a regular value of u for every $t \geq t_0$. Then, to compute the limit on the right hand side of inequality (3.26), it is convenient to rewrite F as

$$F(t) \;=\; \int\limits_{\{u=1-\frac{1}{t}\}} \frac{1}{1-u}\left[1 + \frac{g(\nabla|\nabla u|, \nabla u)}{(1-u)\,|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2}\right] |\nabla u|\, d\sigma$$

and, by formula (3.27), we get

$$\begin{aligned}
|\nabla u| \;&=\; \frac{1}{|x|^2}\left[1 - \frac{1}{|x|}\left(2m + \phi(x/|x|)\right) + O\big(|x|^{-2-\alpha}\big)\right],\\
\frac{g(\nabla|\nabla u|, \nabla u)}{|\nabla u|^2} \;&=\; -\frac{2}{|x|}\left[1 - \frac{1}{2|x|}\left(4m + \phi(x/|x|)\right) + O\big(|x|^{-2-\alpha}\big)\right],
\end{aligned}$$

therefore

$$\lim_{|x|\to+\infty} \frac{1}{1-u}\left[1 + \frac{g(\nabla|\nabla u|,\nabla u)}{(1-u)\,|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2}\right] \,=\, 2m\,.$$

In particular, for every $\varepsilon > 0$, there exists $t_\varepsilon > t_0$ such that whenever $u(p) \geq 1 - 1/t_\varepsilon$ one has

$$2m - \varepsilon \,\leq\, \frac{1}{1-u}\left[1 + \frac{g(\nabla|\nabla u|,\nabla u)}{(1-u)\,|\nabla u|^2} + \frac{|\nabla u|}{(1-u)^2}\right](p) \,\leq\, 2m + \varepsilon\,.$$

Using this fact in combination with equality (3.25), we deduce that, for every $t \geq t_\varepsilon$, there holds

$$4\pi(2m-\varepsilon) \,\leq\, F(t) \,\leq\, 4\pi(2m+\varepsilon).$$

Therefore, we have that $\lim_{t\to+\infty} F(t) = 8\pi m$, hence $m \geq 0$.
Concerning the rigidity statement, one of the implications is obvious, while the claim that (M, g) and $(\mathbb{R}^3, g_{\mathrm{eucl}})$ are isometric if $m = 0$ follows by the argument in the original Schoen–Yau's paper [76] (see also [43, p. 95–97 and p. 102]). $\square$

Chapter 4

ADM mass, area and boundary capacity

In this chapter, we show two sharp comparison results for three–dimensional, complete, one–ended asymptotically flat manifolds (M, g) with minimal, compact and connected boundary and with nonnegative scalar curvature, by means of two monotonicity formulas holding along regular level sets of a suitable harmonic function associated to the boundary of M, under the assumption that the first Betti number of M vanishes.

Let (M, g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Let $u \in C^\infty(M)$ be the solution of the following Dirichlet problem,

$$\begin{cases} \Delta u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases} \tag{4.1}$$

and let $\mathcal{C} > 0$ be the boundary capacity of ∂M in (M, g), given by

$$\mathcal{C} = \frac{1}{4\pi} \int\limits_{\partial M} |\nabla u| d\sigma = \frac{1}{4\pi} \int\limits_{M} |\nabla u|^2 d\mu \,, \tag{4.2}$$

as $1 - u$ is the boundary capacity potential, see Corollary 1.4.13. Since u satisfies system (4.1), by the strong maximum principle, we have

$$\mathrm{Int}(M) = M \setminus \partial M = \{0 < u < 1\} \,.$$

Then, from the Hopf lemma, it follows $|\nabla u| > 0$ on ∂M, in particular, zero is a regular value of u. Moreover, by the last condition in system (4.1), $u : M \to [0, 1)$ is proper. Consequently, each level set of u is compact, therefore, it has finite 2–Hausdorff measure of (M, g), see [32, Theorem 1.7]. Another consequence of the fact that u is proper is that for every regular value $t \in [0, 1)$ of u, there exists $\varepsilon_t > 0$ such that $(t - \varepsilon_t, t + \varepsilon_t) \cap [0, 1)$ does not contain any critical value (the set of the critical values of u has zero Lebesgue measure, by Sard's theorem). By Corollary 1.4.13 we also know that in a fixed AF coordinate chart (x^1, x^2, x^3) of order τ, $\tau > 1/2$, one has

$$u = 1 - \frac{\mathcal{C}}{|x|} + O_2(|x|^{-1-\alpha}) \tag{4.3}$$

for some $1/2 < \alpha < \min\{\tau, 1\}$ (where the choice $\alpha > 1/2$ will be clear in Section 4.1, due to the computations of some limits). A consequence of formula (4.3) is the compactness of $\mathrm{Crit}(u)$ which implies that $\mathrm{Crit}(u)$ has finite 1–dimensional Hausdorff measure, see [31, Theorem 1.1]. The divergence theorem (see Remark 1.1.1), together with Sard's theorem imply

$$\int_{\{u=s\}} |\nabla u|\, d\sigma = 4\pi\mathcal{C} \tag{4.4}$$

for a.e. $s \in [0, 1)$, in particular, any s regular value for u.

4.1 Monotonicity formulas and rigidity statements

Proposition 4.1.1. *Let (M, g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Let $u \in C^\infty(M)$ be the solution of Dirichlet problem (4.1) and let $\mathcal{C} > 0$ be the boundary capacity of ∂M in (M, g) given by formula (4.2). Moreover, let $\widehat{G} : [\mathcal{C}/2, +\infty) \to \mathbb{R}$ be the function defined as*

$$\widehat{G}(t) = 4\pi t + \frac{t^3}{\mathcal{C}^2}\left(1 + \frac{\mathcal{C}}{2t}\right)^3\left(1 - \frac{3\mathcal{C}}{2t}\right)\int_{\Sigma_t} |\nabla u|^2\, d\sigma - \frac{t^2}{\mathcal{C}}\left(1 + \frac{\mathcal{C}}{2t}\right)^2\int_{\Sigma_t} |\nabla u|\,\mathrm{H}\, d\sigma\,, \tag{4.5}$$

where Σ_t is the level set of u, given by

$$\Sigma_t := \left\{u = \left(1 - \frac{\mathcal{C}}{2t}\right)\Big/\left(1 + \frac{\mathcal{C}}{2t}\right)\right\},$$

H is the mean curvature of $\Sigma_t \setminus \mathrm{Crit}(u)$ with respect to the ∞–pointing unit normal vector field $\nu = \nabla u/|\nabla u|$ and σ is 2–Hausdorff measure of (M, g). Then, there hold

$$\widehat{G}\,(\mathcal{C}/2) = 2\mathcal{C}\left[\pi - \int_{\partial M} |\nabla u|^2\, d\sigma\right], \tag{4.6}$$

$$\limsup_{t\to+\infty} \widehat{G}(t) \leq 8\pi\,(m_{\mathrm{ADM}} - \mathcal{C})\,. \tag{4.7}$$

Finally, if all regular level sets of u are connected, then $\widehat{G}$ is nondecreasing on the set $\widehat{\mathcal{T}}$, given by

$$\widehat{\mathcal{T}} := \left\{t \in [\mathcal{C}/2, +\infty) \,:\, \left(1 - \frac{\mathcal{C}}{2t}\right)\Big/\left(1 + \frac{\mathcal{C}}{2t}\right) \text{ is a regular value of } u\right\}. \tag{4.8}$$

We observe that the function $\widehat{G}$ is well–defined, indeed, it is sufficient to notice that the integrand functions are σ–a.e. bounded on each level set of u, since such level sets have finite σ–measure. To justify this claim, one only needs to check that $\big||\nabla u|\mathrm{H}\big|$ is bounded on $\{u = s\} \setminus \mathrm{Crit}(u)$, where $\mathrm{Crit}(u)$ is the set of the critical points of u, for every $s \in [0, 1)$, being $\sigma\big(\mathrm{Crit}(u)\big) = 0$. As u is harmonic, H can be expressed by

$$\mathrm{H} = -\frac{\nabla du(\nabla u, \nabla u)}{|\nabla u|^3} = -\frac{g(\nabla|\nabla u|, \nabla u)}{|\nabla u|^2}\,. \tag{4.9}$$

Consequently, one has

$$\big||\nabla u|\mathrm{H}\big| \leq |\nabla du(\nu, \nu)| \leq |\nabla du|\,, \tag{4.10}$$

wherever $|\nabla u| \neq 0$.

Finally, we notice that $\widehat{\mathcal{T}}$ differs from $[\mathcal{C}/2, +\infty)$ only for a negligible set and it is a disjoint countable union of open intervals and of only one interval of type $[a, b)$, with $a = \mathcal{C}/2$, as it is an open set of $[\mathcal{C}/2, +\infty)$.

Proof. The function $\widehat{G}$ is easily seen to satisfy equality (4.6), now, we check formula (4.7). By virtue of the compactness of $\mathrm{Crit}(u)$, there exists $t_0 \in [\mathcal{C}/2, +\infty)$ such that every $t \in [t_0, +\infty)$ belongs to $\widehat{\mathcal{T}}$, therefore, we reduce ourselves to work on the interval $[t_0, +\infty)$. We break $\widehat{G}$ in two pieces, namely

$$\widehat{G}_1(t) = 4\pi t + \frac{t^3}{\mathcal{C}^2}\left(1 + \frac{\mathcal{C}}{2t}\right)^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma - \frac{t^2}{\mathcal{C}}\left(1 + \frac{\mathcal{C}}{2t}\right)^2 \int_{\Sigma_t} |\nabla u| \, \mathrm{H} \, d\sigma \,, \tag{4.11}$$

$$\widehat{G}_2(t) = -\frac{3t^2}{2\mathcal{C}}\left(1 + \frac{\mathcal{C}}{2t}\right)^3 \int_{\Sigma_t} |\nabla u|^2 \, d\sigma \,.$$

It is convenient, in order to compute the limit of $\widehat{G}_2(t)$, as $t \to +\infty$, to rewrite the function as

$$\widehat{G}_2(t) = -\frac{3\mathcal{C}}{2}\left(1 + \frac{\mathcal{C}}{2t}\right) \int_{\Sigma_t} \frac{|\nabla u|^2}{(1-u)^2} \, d\sigma \,. \tag{4.12}$$

By the expansion (4.3) of u, we have

$$|\nabla u| = \frac{\mathcal{C}}{|x|^2}\left[1 + O\big(|x|^{-\alpha}\big)\right] , \tag{4.13}$$

therefore,

$$\lim_{|x| \to +\infty} \frac{|\nabla u|}{(1-u)^2} = \mathcal{C}^{-1} \,. \tag{4.14}$$

Thus,

$$\lim_{t \to +\infty} \widehat{G}_2(t) = -6\pi\mathcal{C} \,. \tag{4.15}$$

Indeed, by virtue of limit (4.14), for every $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$\mathcal{C}^{-1} - \varepsilon \leq \frac{|\nabla u|}{(1-u)^2} \leq \mathcal{C}^{-1} + \varepsilon$$

in $\{u \geq (1 - \frac{\mathcal{C}}{2t_\varepsilon})/(1 + \frac{\mathcal{C}}{2t_\varepsilon})\}$. Using this fact in combination with formula (4.4), for every $t \geq t_\varepsilon$, there holds

$$4\pi - 4\pi\mathcal{C}\varepsilon \leq \int_{\Sigma_t} \frac{|\nabla u|^2}{(1-u)^2} \, d\sigma \leq 4\pi + 4\pi\mathcal{C}\varepsilon \,.$$

Then, the integral term in expression (4.12) of $\widehat{G}_2(t)$ converges to 4π, as $t \to +\infty$, consequently the limit (4.15) holds. In order to compute the upper limit of $\widehat{G}_1$, given by formula (4.11), we introduce an auxiliary function $\rho : M \to [\mathcal{C}/2, +\infty)$,

$$\rho = \frac{\mathcal{C}}{2}\,\frac{1+u}{1-u}$$

called *Euclidean fake distance*. This name is justified by the fact that

$$\rho = |x| + O_2(|x|^{1-\alpha}). \tag{4.16}$$

Then, the expression of $\widehat{G}_1$ becomes

$$\widehat{G}_1(t) = \frac{t}{4}\left(1+\frac{\mathcal{C}}{2t}\right)\left\{16\pi\left(1+\frac{\mathcal{C}}{2t}\right)^{-1} - \int\limits_{\{\rho=t\}} \mathrm{H}^2\,d\sigma + \int\limits_{\{\rho=t\}} \left[\frac{g(\nabla|\nabla\rho|,\nabla\rho)}{|\nabla\rho|^2}\right]^2 d\sigma\right\}.$$

We break $\widehat{G}_1$ in three pieces,

$$\begin{aligned}
\widehat{G}_{11}(t) &= -2\pi\mathcal{C}\,,\\
\widehat{G}_{12}(t) &= \frac{t}{4}\left(1+\frac{\mathcal{C}}{2t}\right)\left[16\pi - \int\limits_{\{\rho=t\}} \mathrm{H}^2\,d\sigma\right], \qquad (4.17)\\
\widehat{G}_{13}(t) &= \frac{t}{4}\left(1+\frac{\mathcal{C}}{2t}\right)\int\limits_{\{\rho=t\}} \left[\frac{g(\nabla|\nabla\rho|,\nabla\rho)}{|\nabla\rho|^2}\right]^2 d\sigma = \frac{1}{4}\left(1+\frac{\mathcal{C}}{2t}\right)\int\limits_{\{\rho=t\}} \rho\left[\frac{\nabla d\rho(\nabla\rho,\nabla\rho)}{|\nabla\rho|^3}\right]^2 d\sigma\,.
\end{aligned} \tag{4.18}$$

For simplicity, we start with the computation of the limit of $\widehat{G}_{13}$. By expansion (4.3) of u, there exist some constants $0 < A_1 < A_2$ and $0 < B_1 < B_2$ such that

$$\frac{A_1}{|x|} \leq 1-u \leq \frac{A_2}{|x|}\,, \tag{4.19}$$

$$\frac{B_1}{|x|^2} \leq |\nabla u| \leq \frac{B_2}{|x|^2}\,. \tag{4.20}$$

on $\{|x| \geq R_0\}$ for $R_0 > 1$ sufficiently large. Since $\Sigma_t \subseteq \{|x| \geq R_0\}$ for every $t \geq t_1$, with t_1 large enough, from formula (4.19) it follows

$$\frac{A_1 t}{\mathcal{C}} \leq r_t \leq |x(p)| \leq R_t \leq \frac{2A_2 t}{\mathcal{C}} \tag{4.21}$$

for every $p \in \Sigma_t$ and for all $t \geq t_1$, where we set

$$\begin{aligned}
r_t &:= \min\{|x(p)| : p \in \Sigma_t\}\,,\\
R_t &:= \max\{|x(p)| : p \in \Sigma_t\}\,.
\end{aligned} \tag{4.22}$$

In particular, as a consequence of inequalities (4.21), one has

$$\frac{1}{R_t} \geq \frac{\mathcal{C}}{2A_2 t} = \frac{A_1}{2A_2}\frac{\mathcal{C}}{A_1 t} \geq \frac{A_1}{2A_2}\frac{1}{r_t}\,, \tag{4.23}$$

then, there holds

$$4\pi\mathcal{C} = \int\limits_{\Sigma_t} |\nabla u|\,d\sigma \geq \frac{B_1}{R_t^2}\,\mathrm{Area}(\Sigma_t) \geq \frac{B_1 A_1^2}{4A_2^2}\,\frac{\mathrm{Area}(\Sigma_t)}{r_t^2}\,, \tag{4.24}$$

where the first inequality follows by formulas (4.20) and (4.21), while the second one by inequality (4.23). Now, by formulas (4.21) and (4.24), we have

$$
\begin{aligned}
&r_t \to +\infty && \text{for } t \to +\infty\,, && (4.25)\\
&\mathrm{Area}(\Sigma_t) \leq C r_t^2 && \text{for every } t \geq t_1\,, && (4.26)
\end{aligned}
$$

respectively. Therefore, up to choosing a larger t_1, from formula (1.23) of Proposition 1.4.6, it follows

$$
\mathrm{Area}_e(\Sigma_t) \leq C r_t^2 \quad \text{for every } t \geq t_1\,. \tag{4.27}
$$

At the same time, the behavior near infinity of ρ, described by formula (4.16), implies

$$
\frac{\nabla d\rho(\nabla\rho,\nabla\rho)}{|\nabla\rho|^3} = O(|x|^{-1-\alpha})\,,
$$

then,

$$
\rho\left[\frac{\nabla d\rho(\nabla\rho,\nabla\rho)}{|\nabla\rho|^3}\right]^2 = O(|x|^{-1-2\alpha})\,. \tag{4.28}
$$

Hence, we have

$$
\begin{aligned}
0 \leq \int\limits_{\{\rho=t\}} \rho\left[\frac{\nabla d\rho(\nabla\rho,\nabla\rho)}{|\nabla\rho|^3}\right]^2 d\sigma &\leq \frac{C}{r_t^{1+2\alpha}}\,\mathrm{Area}(\Sigma_t)\\
&\leq C r_t^{1-2\alpha}\,,
\end{aligned}
$$

for every $t \geq t_1$, where the second inequality follows by formula (4.28), together with definition (4.22) and the third one by formula (4.26). Consequently, being $\alpha > 1/2$, one gets the convergence of $\widehat{G}_{13}(t)$ to zero, as $t \to +\infty$, where $\widehat{G}_{13}$ is given by equalities (4.18), by limit (4.25).

We remark that a key point is the knowledge that the error term in formula (4.3) is $O_2(|x|^{-1-\alpha})$, with $\alpha > 1/2$. Indeed, if this error term were only $o_2(|x|^{-1})$, then the limit of $\widehat{G}_{13}(t)$ would be a indeterminate form.

Concerning the upper limit of $\widehat{G}_{12}(t)$ for $t \to +\infty$, where $\widehat{G}_{12}$ is given by equality (4.17), this is known to be less or equal than $8\pi m_{\mathrm{ADM}}$ from the celebrated work of Huisken and Ilmanen [37], but for completeness, we show its computation.

In the fixed AF coordinate chart (x^1, x^2, x^3) of order $\tau > 1/2$, by Proposition 1.4.6, it follows

$$
\begin{aligned}
\mathrm{H} = \mathrm{H}_e - \varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \frac{1}{2}\mathrm{H}_e\sigma_{ks}\nu_e^k\nu_e^s - \varepsilon^{ij}\nu_e^k\partial_i\sigma_{jk} + \frac{1}{2}\varepsilon^{ij}\nu_e^k\partial_k\sigma_{ij} + O(|x|^{-1-2\tau})&\\
+ O(|x|^{-2\tau}|\mathrm{h}^e|_e)\,,& \qquad (4.29)
\end{aligned}
$$

where $\sigma_{ij} = g_{ij} - \delta_{ij}$ (being $g = g_{ij}\,dx^i \otimes dx^j$), $\nu_e = \nabla^e\rho/|\nabla^e\rho|_e\,, \varepsilon^{ij} = \delta^{ij} - \nu_e^i\nu_e^j$ and finally,

$$
\begin{aligned}
\mathrm{h}^e &= \frac{\nabla^e d\rho}{|\nabla^e\rho|_e}\,, && (4.30)\\
\mathrm{H}_e &= \frac{\Delta_e\rho}{|\nabla^e\rho|_e} - \frac{\nabla^e d\rho(\nabla^e\rho,\nabla^e\rho)}{|\nabla^e\rho|_e^3}\,. && (4.31)
\end{aligned}
$$

Recall that $\mathrm{h}^e_p|_{T_p\Sigma_t\times T_p\Sigma_t}$ is the second fundamental form of Σ_t at every $p \in \Sigma_t$ with respect to ν_e, associated to the metric g_e, while H_e is the mean curvature of Σ_t with

respect to ν_e associated to g_e, by formulas (1.4) and (1.5).
By virtue of behavior of ρ near infinity in equation (4.16), together with the equalities (4.30) and (4.31), we know that

$$\mathrm{h}^e = \left[\frac{\varepsilon_{ij}}{|x|} + O(|x|^{-1-\alpha})\right] dx^i \otimes dx^j \,, \tag{4.32}$$

$$\mathrm{H}_e = \frac{2}{|x|} + O(|x|^{-1-\alpha})\,, \tag{4.33}$$

where $\varepsilon_{ij} = \delta_{ij} - \nu^e_i \nu^e_j$ and $\nu^e_i = \partial_i \rho / |\nabla^e \rho|_e$. Then, being $\mathrm{h}^e_{ij} = O(|x|^{-1})$, by equality (4.32), from formula (4.29) we get

$$\mathrm{H} = \mathrm{H}_e - \varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \frac{1}{2}\mathrm{H}_e\sigma_{ks}\nu^k_e\nu^s_e - \varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk} + \frac{1}{2}\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij} + O(|x|^{-1-2\alpha})\,,$$
$$\mathrm{H}^2 = \mathrm{H}^2_e - 2\mathrm{H}_e\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \mathrm{H}^2_e\sigma_{ks}\nu^k_e\nu^s_e - 2\mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk} + \mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij} + O(|x|^{-2-2\alpha}).$$

This last equality, along with formula (1.23) of Proposition 1.4.6, implies

$$\begin{aligned}\mathrm{H}^2\, d\sigma = \Big[&\mathrm{H}^2_e + \frac{1}{2}\mathrm{H}^2_e\varepsilon^{ij}\sigma_{ij} - 2\mathrm{H}_e\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \mathrm{H}^2_e\sigma_{ks}\nu^k_e\nu^s_e - 2\mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk}\\ &+ \mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij} + O(|x|^{-2-2\alpha})\Big]\, d\sigma_e\,.\end{aligned} \tag{4.34}$$

Then, we obtain

$$\begin{aligned}\int\limits_{\{\rho=t\}}\mathrm{H}^2\, d\sigma &= \int\limits_{\{\rho=t\}}\Big[\mathrm{H}^2_e + \frac{1}{2}\mathrm{H}^2_e\varepsilon^{ij}\sigma_{ij} - 2\mathrm{H}_e\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \mathrm{H}^2_e\sigma_{ks}\nu^k_e\nu^s_e\\ &\qquad - 2\mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk} + \mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij}\Big]\, d\sigma_e + O(r_t^{-2\alpha})\\ &\geq 16\pi + \int\limits_{\{\rho=t\}}\Big[\frac{1}{2}\mathrm{H}^2_e\varepsilon^{ij}\sigma_{ij} - 2\mathrm{H}_e\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \mathrm{H}^2_e\sigma_{ks}\nu^k_e\nu^s_e\\ &\qquad - 2\mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk} + \mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij}\Big]\, d\sigma_e + O(r_t^{-2\alpha})\,,\end{aligned}$$

where the equality is a consequence of formula (4.34) together with inequalities (4.21) and (4.27), while the inequality follows from *Willmore inequality*

$$\int\limits_{\Sigma}\mathrm{H}^2_e\, d\sigma_e \geq 16\pi$$

for every orientable, immersed, closed surface $\Sigma \subseteq \mathbb{R}^3$ (see [86]). Consequently, by expansion (4.16) of ρ near infinity, there holds

$$\begin{aligned}t\Big(16\pi - \int\limits_{\{\rho=t\}}\mathrm{H}^2\, d\sigma\Big) \leq -\int\limits_{\{\rho=t\}}|x|\Big[&\frac{1}{2}\mathrm{H}^2_e\varepsilon^{ij}\sigma_{ij} - 2\mathrm{H}_e\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij} + \mathrm{H}^2_e\sigma_{ks}\nu^k_e\nu^s_e\\ &- 2\mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_i\sigma_{jk} + \mathrm{H}_e\varepsilon^{ij}\nu^k_e\partial_k\sigma_{ij}\Big]\, d\sigma_e + O(r_t^{1-2\alpha})\,.\end{aligned}$$

Then, thanks to formula (4.33), it follows

$$t\left(16\pi-\int\limits_{\{\rho=t\}}\mathrm{H}^2\,d\sigma\right)\leq\int\limits_{\{\rho=t\}}\Big[-\frac{2}{|x|}\varepsilon^{ij}\sigma_{ij}+4\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij}-\frac{4}{|x|}\sigma_{ks}\nu_e^k\nu_e^s$$
$$+4\varepsilon^{ij}\nu_e^k\partial_i\sigma_{jk}-2\varepsilon^{ij}\nu_e^k\partial_k\sigma_{ij}\Big]\,d\sigma_e+O(r_t^{1-2\alpha})\,.$$

Since equality (4.32) with the observation that

$$\varepsilon^{ik}\varepsilon_{ij}\varepsilon^{sj}=\left(\delta^{ik}-\nu_e^i\nu_e^k\right)\left(\delta_{ij}-\nu^e_i\nu^e_j\right)\left(\delta^{sj}-\nu_e^s\nu_e^j\right)=\varepsilon^{sk}$$

implies

$$\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\mathrm{h}^e_{ij}=\frac{1}{|x|}\varepsilon^{ik}\sigma_{ks}\varepsilon^{sj}\varepsilon_{ij}+O(|x|^{-1-2\alpha})=\frac{1}{|x|}\varepsilon^{sk}\sigma_{ks}+O(|x|^{-1-2\alpha})\,, \tag{4.35}$$

we obtain

$$t\left(16\pi-\int\limits_{\{\rho=t\}}\mathrm{H}^2\,d\sigma\right)\leq\int\limits_{\{\rho=t\}}\Big[-\frac{2}{|x|}\varepsilon^{ij}\sigma_{ij}+\frac{4}{|x|}\varepsilon^{ks}\sigma_{ks}-\frac{4}{|x|}\sigma_{ks}\nu_e^k\nu_e^s$$
$$+4\varepsilon^{ij}\nu_e^k\partial_i\sigma_{jk}-2\varepsilon^{ij}\nu_e^k\partial_k\sigma_{ij}\Big]\,d\sigma_e+O(r_t^{1-2\alpha})$$
$$=\int\limits_{\{\rho=t\}}\Big[\frac{2}{|x|}\varepsilon^{ks}\sigma_{ks}-\frac{4}{|x|}\sigma_{ks}\nu_e^k\nu_e^s+4\varepsilon^{ij}\nu_e^k\partial_i\sigma_{jk}-2\varepsilon^{ij}\nu_e^k\partial_k\sigma_{ij}\Big]\,d\sigma_e$$
$$+O(r_t^{1-2\alpha})\,. \tag{4.36}$$

Then, we define $X=\sigma(\nu_e,\cdot)^{\sharp_e}$, i.e. $X=X^i\partial_i$ with $X^i=\sigma_{ij}\nu_e^j$ and we observe

$$\begin{aligned}\varepsilon^{ij}\partial_i\big(\nu_e^k\sigma_{jk}\big)&=\varepsilon^{ij}\partial_iX^j=\big[\delta^{ij}-\nu_e^i\nu_e^j\big]\,\partial_iX^j=\partial_iX^i-g_e\left(\nabla^e_{\nu_e}X,\nu_e\right)\\&=\mathrm{div}_eX-g_e\left(\nabla^e_{\nu_e}X,\nu_e\right)=g_e\left(\nabla^e_{E_\alpha}X,E_\alpha\right)\\&=\mathrm{div}_e^{\{\rho=t\}}(X^\top)+\mathrm{H}_e\sigma_{ij}\nu_e^i\nu_e^j\,,\end{aligned}$$

where $\{E_\alpha\}$ is a local orthonormal frame on $\{\rho=t\}$, with respect to the metric g_e, keeping into account equality (1.7), moreover, we have

$$\partial_i\nu_e^k=\partial_i\Big(\frac{\partial_k\rho}{|\nabla^e\rho|_e}\Big)=\frac{\partial_i\partial_k\rho}{|\nabla^e\rho|_e}-\frac{(\partial_j\rho)\,(\partial_k\rho)\,(\partial_i\partial_j\rho)}{|\nabla^e\rho|_e^3}=\varepsilon^{kj}\mathrm{h}^e_{ij}\,.$$

Hence, by the divergence theorem on the closed surface $\{\rho = t\}$, we compute

$$\begin{aligned}
\int\limits_{\{\rho=t\}} \varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk}\,d\sigma_e &= \int\limits_{\{\rho=t\}} \varepsilon^{ij}\left[\partial_i\big(\nu_e^k\,\sigma_{jk}\big) - \sigma_{jk}\,\partial_i\nu_e^k\right] d\sigma_e \\
&= \int\limits_{\{\rho=t\}} \left[\mathrm{H}_e\,\nu_e^j\,\nu_e^i\sigma_{ij} - \varepsilon^{ij}\,\sigma_{jk}\,\varepsilon^{ks}\,\mathrm{h}_{is}^e\right] d\sigma_e \\
&= \int\limits_{\{\rho=t\}} \left[\frac{2}{|x|}\nu_e^j\,\nu_e^i\sigma_{ij} - \frac{1}{|x|}\varepsilon^{ij}\,\sigma_{ij}\right] d\sigma_e + O(r_t^{1-2\alpha})\,,
\end{aligned}$$

by means of formulas (4.33), (4.35), (4.21) and (4.27), thus there holds

$$\int\limits_{\{\rho=t\}} \left[\frac{2}{|x|}\varepsilon^{ij}\,\sigma_{ij} - \frac{4}{|x|}\nu_e^j\,\nu_e^i\sigma_{ij}\right] d\sigma_e = -2\int\limits_{\{\rho=t\}} \varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk}\,d\sigma_e + O(r_t^{1-2\alpha})\,.$$

Now, by using this last equality in formula (4.36), we obtain

$$\begin{aligned}
t\left(16\pi - \int\limits_{\{\rho=t\}} \mathrm{H}^2\,d\sigma\right) &\leq \int\limits_{\{\rho=t\}} \left[-2\varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk} + 4\varepsilon^{ij}\,\nu_e^k\,\partial_i\sigma_{jk} - 2\varepsilon^{ij}\,\nu_e^k\partial_k\sigma_{ij}\right] d\sigma_e \\
&\quad + O(r_t^{1-2\alpha}) \\
&= 2\int\limits_{\{\rho=t\}} \varepsilon^{ij}\big(\partial_i\sigma_{jk} - \partial_k\sigma_{ij}\big)\nu_e^k\,d\sigma_e + O(r_t^{1-2\alpha}) \\
&= 2\int\limits_{\{\rho=t\}} \left[\delta^{ij}\big(\partial_i\sigma_{jk} - \partial_k\sigma_{ij}\big)\nu_e^k - \nu_e^i\nu_e^j\nu_e^k\partial_i\sigma_{jk} + \nu_e^i\nu_e^j\nu_e^k\partial_k\sigma_{ij}\right] d\sigma_e \\
&\quad + O(r_t^{1-2\alpha}) \\
&= 2\int\limits_{\{\rho=t\}} \big(\partial_i\sigma_{ij} - \partial_j\sigma_{ii}\big)\nu_e^j\,d\sigma_e + O(r_t^{1-2\alpha}) \;\to 32\pi m_{\mathrm{ADM}} \;\text{ as } t\to+\infty.
\end{aligned}$$

where at the end we used Remark 1.5.1 and the fact that $\alpha > 1/2$, keeping into account formula (4.27). In conclusion

$$\widehat{G} = \widehat{G}_{11}(t) + \widehat{G}_{12}(t) + \widehat{G}_{13}(t) + \widehat{G}_2(t)\,,$$

where

$$\begin{aligned}
\widehat{G}_{11}(t) &= -2\pi\mathcal{C}\,, \\
\limsup_{t\to+\infty} \widehat{G}_{12}(t) &\leq 8\pi m_{\mathrm{ADM}}\,, \\
\lim_{t\to+\infty} \widehat{G}_{13}(t) &= 0\,, \\
\lim_{t\to+\infty} \widehat{G}_2(t) &= -6\pi\mathcal{C}\,,
\end{aligned}$$

hence formula (4.7) is proved.

Now we show the monotonicity in absence of critical points. In this case, the function $\widehat{G}$ is everywhere continuously differentiable in its domain of definition, with first derivative given by

$$\widehat{G}'(t) = 4\pi - \int\limits_{\Sigma_t} \frac{\mathrm{R}^{\Sigma_t}}{2}\, d\sigma + \int\limits_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{4u}{1-u^2}\, |\nabla u| - \mathrm{H} \right)^2 \right] d\sigma\,. \tag{4.37}$$

Indeed,

$$\frac{d}{dt}\int\limits_{\Sigma_t} |\nabla u|^2\, d\sigma = -\frac{\mathcal{C}}{t^2}\left(1+\frac{\mathcal{C}}{2t}\right)^{-2} \int\limits_{\Sigma_t} |\nabla u|\, \mathrm{H}\, d\sigma\,, \tag{4.38}$$

$$\begin{aligned}
\frac{d}{dt}\int\limits_{\Sigma_t} |\nabla u|\, \mathrm{H}\, d\sigma &= -\frac{\mathcal{C}}{t^2}\left(1+\frac{\mathcal{C}}{2t}\right)^{-2} \int\limits_{\Sigma_t} |\nabla u| \left[\Delta^{\Sigma_t}\left(\frac{1}{|\nabla u|}\right) + \frac{|\mathrm{h}|^2 + \mathrm{Ric}(\nu,\nu)}{|\nabla u|} \right] d\sigma \\
&= -\frac{\mathcal{C}}{t^2}\left(1+\frac{\mathcal{C}}{2t}\right)^{-2} \int\limits_{\Sigma_t} \Bigg[-\Delta^{\Sigma_t}(\log|\nabla u|) + \frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \\
&\qquad\qquad - \frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3\,\mathrm{H}^2}{4} \Bigg] d\sigma \\
&= -\frac{\mathcal{C}}{t^2}\left(1+\frac{\mathcal{C}}{2t}\right)^{-2} \int\limits_{\Sigma_t} \left[\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} - \frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3\,\mathrm{H}^2}{4} \right] d\sigma\,,
\end{aligned}$$

where ∇^{Σ_t}, Δ^{Σ_t} are the Levi–Civita connection and the Laplace–Beltrami operator of the induced metric g^{Σ_t}, respectively, R^{Σ_t} is the scalar curvature of Σ_t and finally h, $\mathring{\mathrm{h}}$ denote the second fundamental form of Σ_t and its traceless version, with respect to $\nu = \nabla u/|\nabla u|$. Here, the first and the second equality are consequences of the normal first variation of volume measure and of mean curvature (see the end of Section 1.3), whereas the third one and the last one follow with the help of the traced Gauss equation (1.6) and of the divergence theorem. Now, we notice that the last summand of the right hand side of equality (4.37) is always nonnegative, as the scalar curvature of (M, g) is nonnegative, by assumption. At same time, the absence of critical points implies that all the level sets of u are diffeomorphic, in particular they are connected as $\partial M = \{u = 0\}$ is connected, by hypothesis. Consequently, the first two summands also give a nonnegative contribution, by virtue of Gauss–Bonnet theorem, as each Σ_t is a connected closed surface (see the precise explanation at the end of the proof of Theorem 3.1.2 – in absence of critical points). Thus, $\widehat{G}'(t) \geq 0$ for every $t \in [\mathcal{C}/2, +\infty)$, then $\widehat{G}$ is nondecreasing therein.
Now, we show the monotonicity also in presence of critical points. We consider the vector field X given by

$$X := \frac{1+u}{2(1-u)}\nabla u + \frac{\mathcal{C}}{(1-u)^2}\nabla|\nabla u| + \frac{2\mathcal{C}(2u-1)}{(1+u)(1-u)^3}|\nabla u|\nabla u\,, \tag{4.39}$$

on the open set $M \setminus \mathrm{Crit}(u)$, noticing that it is well–defined. With the help of Bochner formula,

$$\frac{1}{2}\,\Delta\,|\nabla f|^2 = |\nabla df|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla\,\Delta f, \nabla f)\,, \tag{4.40}$$

for every $f \in C^\infty(M)$ and by virtue of the fact that u is harmonic, the divergence of X can be expressed as

$$\operatorname{div}(X) = \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}} + \frac{12u^2}{(1-u^2)^2}|\nabla u|^2 + \frac{6u}{1-u^2}\frac{g(\nabla|\nabla u|, \nabla u)}{|\nabla u|} \right. \\ \left. + \frac{|\nabla du|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u)}{|\nabla u|^2}\right].$$

Using the traced Gauss equation (1.6), together with the identity

$$|\nabla du|^2 = |\nabla u|^2|\mathrm{h}|^2 + |\nabla|\nabla \mathrm{u}||^2 + |\nabla^\Sigma|\nabla \mathrm{u}||^2,$$

one can obtain an equivalent expression for $\operatorname{div}(X)$, adapted to the (regular portions of the) level sets of u, namely

$$\operatorname{div}(X) = \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}} - \frac{\mathrm{R}^\Sigma}{2} + \frac{|\nabla^\Sigma|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{4u}{1-u^2}|\nabla u| - \mathrm{H}\right)^2\right]. \tag{4.41}$$

Here, $\mathrm{h}, \mathrm{H}, \mathrm{R}^\Sigma$ and ∇^Σ are all the ones associated to the (regular portions of the) level set of u that passes for the point where $\operatorname{div}(X)$ is computed.
Let $t, T \in \widehat{\mathcal{T}}$ such that $t < T$, where $\widehat{\mathcal{T}}$ is given by equality (4.8). We want to show that $\widehat{G}(t) \leq \widehat{G}(T)$. To simplify the exposition, we introduce the diffeomorphism $f : [\mathcal{C}/2, +\infty) \to [0, 1)$, defined by

$$f(t) := \frac{1 - \frac{\mathcal{C}}{2t}}{1 + \frac{\mathcal{C}}{2t}}. \tag{4.42}$$

We treat the non–trivial case in which the open interval $\big(f(t), f(T)\big)$ contains critical values of u. In this case, the vector field X is no longer well–defined in $\{f(t) \leq u \leq f(T)\}$ and to overcome this difficulty, we consider the same pointwise nondecreasing sequence of cut–off functions $\{\eta_k\}_{k\in\mathbb{N}^+}$ introduced in proof of Theorem 3.1.2, in the general case, namely, for every $k \in \mathbb{N}^+$, the functions $\eta_k : [0, +\infty) \to [0, 1]$ are smooth, nondecreasing and satisfy

$$\eta_k(\tau) \equiv 0 \quad \text{in} \left[0\,, \frac{1}{2k}\right], \qquad 0 \leq \eta_k'(\tau) \leq 2k \quad \text{in} \left[\frac{1}{2k}\,, \frac{3}{2k}\right], \qquad \eta_k(\tau) \equiv 1 \quad \text{in} \left[\frac{3}{2k}\,, +\infty\right).$$

Using these cut–off functions, we define for every $k \in \mathbb{N}^+$, the vector fields

$$X_k := \frac{1+u}{2(1-u)}\nabla u + \eta_k\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\left[\frac{\mathcal{C}}{(1-u)^2}\nabla|\nabla u| + \frac{2\mathcal{C}(2u-1)}{(1+u)(1-u)^3}|\nabla u|\nabla u\right].$$

Notice that the vector fields X_k are well–defined in M and they coincide with the vector field X in formula (4.39), whenever restricted to a compact set sitting inside

$M \setminus \mathrm{Crit}(u)$, for k large enough. Moreover, they have divergence given by the following formula,

$$\mathrm{div}(X_k) = \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left\{\eta_k\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\left[\frac{6u}{1-u^2}\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|}+\frac{\mathrm{Ric}(\nabla u,\nabla u)}{|\nabla u|^2}\right]\right.$$
$$\left.+\eta_k\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\left[\frac{12u^2}{(1-u^2)^2}|\nabla u|^2+\frac{|\nabla du|^2-|\nabla|\nabla u||^2}{|\nabla u|^2}\right]+\frac{|\nabla u|}{\mathcal{C}}\right\}$$
$$+\frac{\mathcal{C}}{(1-u^2)^3}\,\eta_k'\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\left|\frac{2(2u-1)}{1-u^2}|\nabla u|\nabla u+\nabla|\nabla u|\right|^2.$$

Since in the above expression the last summand is nonnegative and since, for large enough k, the vector field X_k coincides with X at the boundary of $\{f(t)<u<f(T)\}$, the divergence theorem, applied to X_k on $\{f(t)<u<f(T)\}$ (see Remark 1.1.1), gives

$$\widehat{G}(T)-\widehat{G}(t) = \int\limits_{\{f(t)<u<f(T)\}}\mathrm{div}(X_k)\,d\mu \geq \int\limits_{\{f(t)<u<f(T)\}}\widehat{P}_k\,d\mu+\int\limits_{\{f(t)<u<f(T)\}}\widehat{D}_k\,d\mu\,, \quad (4.43)$$

where we set

$$\widehat{P}_k := \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}}+\eta_k\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\widehat{P}\right],$$
$$\widehat{D}_k := \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,\eta_k\left(\frac{|\nabla u|}{(1-u)(1+u)^3}\right)\widehat{D},$$

with

$$\widehat{P} := \frac{12u^2}{(1-u^2)^2}|\nabla u|^2+\frac{|\nabla du|^2-|\nabla|\nabla u||^2}{|\nabla u|^2},$$
$$\widehat{D} := \frac{6u}{1-u^2}\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|}+\frac{\mathrm{Ric}(\nabla u,\nabla u)}{|\nabla u|^2}.$$

We notice that the functions $\widehat{P}$ and $\widehat{D}$ are μ–a.e. well–defined and smooth as well as $\mathrm{div}(X)$, being $\mu\big(\mathrm{Crit}(u)\big)=0$ and furthermore, we observe that the following facts hold,

(1)
$$0 \leq P_k \nearrow \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}}+\widehat{P}\,\mathbb{I}_{M\setminus\mathrm{Crit}(u)}\right]$$
in M, where $\mathbb{I}_{M\setminus\mathrm{Crit}(u)}$ denotes the characteristic function of $M\setminus\mathrm{Crit}(u)$,

(2)
$$|\widehat{D}_k| \leq \frac{\mathcal{C}|\nabla u|}{(1-u)^2}|\widehat{D}| \qquad \text{and} \qquad |\widehat{D}| \leq \left[\frac{6u}{1-u^2}|\nabla du|+|\mathrm{Ric}|\right] \in L^1_{loc}(M)$$
in $M\setminus\mathrm{Crit}(u)$, keeping into account formula (4.9) and inequality (4.10), moreover,
$$\widehat{D}_k \to \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,\widehat{D}\,\mathbb{I}_{M\setminus\mathrm{Crit}(u)}$$

in M,

(3)
$$\mathrm{div}(X) = \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}} + \widehat{P} + \widehat{D}\right]$$
in $M \setminus \mathrm{Crit}(u)$.

By point (2), the dominated convergence theorem implies

$$\lim_{k\to+\infty} \int\limits_{\{f(t)<u<f(T)\}} \widehat{D}_k \, d\mu = \int\limits_{\{f(t)<u<f(T)\}} \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\, \widehat{D}\, d\mu\,, \tag{4.44}$$

whereas, by point (1) and the monotone convergence theorem, it follows

$$\lim_{k\to+\infty} \int\limits_{\{f(t)<u<f(T)\}} \widehat{P}_k \, d\mu = \int\limits_{\{f(t)<u<f(T)\}} \frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}} + \widehat{P}\right] d\mu\,. \tag{4.45}$$

As a consequence of inequality (4.43) with the existence of limit (4.44) finite, the sequence of nonnegative real numbers given by the integrals of the functions P_k is bounded from above, then

$$\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{|\nabla u|}{\mathcal{C}} + \widehat{P}\right] \in L^1\big(\{f(t) < u < f(T)\}\big)\,.$$

Then, passing to the limit, as $k \to +\infty$, in inequality (4.43), by limits (4.44) and (4.45) together with point (3) above, we get

$$\begin{aligned}
\widehat{G}(T) - \widehat{G}(t) &\geq \int\limits_{\{f(t)<u<f(T)\}} \mathrm{div}(X)\, d\mu \\
&= \int\limits_{[f(t),f(T)]\setminus\mathcal{N}} d\tau\, \frac{\mathcal{C}}{(1-\tau)^2}\Bigg\{\int\limits_{\{u=\tau\}}\left[\frac{|\nabla^\Sigma|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2}\right] d\sigma \\
&\qquad + \frac{3}{4}\int\limits_{\{u=\tau\}}\left(\frac{4u}{1-u^2}|\nabla u| - \mathrm{H}\right)^2 d\sigma \\
&\qquad + 4\pi - \int\limits_{\{u=\tau\}} \frac{\mathrm{R}^\Sigma}{2}\, d\sigma\Bigg\}\,,
\end{aligned}$$

where $\mathcal{N}$ is the set of the critical values of u. Here, the equality follows first by using the coarea formula (1.1), then by applying the equality (4.41) for the divergence of X and finally by Sard's theorem. Since we are integrating only along the regular level sets of u and since every regular level set is a connected (by assumption) closed surface, we can invoke Gauss–Bonnet theorem to deduce that the last two summands also give a nonnegative contribute (see the precise explanation at the end of the proof of Theorem 3.1.2 – in absence of critical points), while the other summands are always nonnegative, as $\mathrm{R} \geq 0$ (by assumption). The claimed monotonicity of $\widehat{G}$ hence follows. □

Under the assumptions of above proposition, we conjecture that it is possible to prove (by an argument similar to the one of Section 3.1) that the function $\widehat{G}$ defined by formula (4.5), coincides a.e. with a nondecreasing locally absolutely continuous function on $[\mathcal{C}/2, +\infty)$, still denoted by $\widehat{G}$, such that

$$\widehat{G}'(t) = 4\pi + \int\limits_{\Sigma_t} \left[-\frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\left(\frac{4u}{1-u^2}|\nabla u| - \mathrm{H}\right)^2\right] d\sigma$$

a.e. in $[\mathcal{C}/2, +\infty)$, in particular, at all values $t \in \widehat{\mathcal{T}}$.

Proposition 4.1.2. *Let (M, g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Let $u \in C^\infty(M)$ be the solution of Dirichlet problem (4.1) and let $\mathcal{C} > 0$ be the boundary capacity of ∂M in (M, g), given by formula (4.2). Let $G : [\mathcal{C}/2, +\infty) \to \mathbb{R}$ be the function defined by*

$$G(t) = -\frac{\pi\mathcal{C}^2}{t} + \frac{t}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^4 \int\limits_{\Sigma_t} |\nabla u|^2 \, d\sigma\,, \tag{4.46}$$

where Σ_t is the level set of u, given by

$$\Sigma_t := \left\{ u = \left(1 - \frac{\mathcal{C}}{2t}\right) \Big/ \left(1 + \frac{\mathcal{C}}{2t}\right) \right\},$$

and σ is 2–Hausdorff measure of (M, g). Then, the function G satisfies

$$G(\mathcal{C}/2) = -2\mathcal{C}\left[\pi - \int\limits_{\partial M} |\nabla u|^2 \, d\sigma\right], \tag{4.47}$$

$$\lim_{t \to +\infty} G(t) = 0\,, \tag{4.48}$$

and admits a locally absolutely continuous representative in $[\mathcal{C}/2, +\infty)$, still denoted by G, such that

$$G'(t) = \frac{\pi\mathcal{C}^2}{t^2} + \frac{1}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^3 \left(1 - \frac{3\mathcal{C}}{2t}\right) \int\limits_{\Sigma_t} |\nabla u|^2 \, d\sigma - \frac{\mathcal{C}}{4t}\left(1 + \frac{\mathcal{C}}{2t}\right)^2 \int\limits_{\Sigma_t} |\nabla u| \, \mathrm{H} \, d\sigma \tag{4.49}$$

a.e. in $[\mathcal{C}/2, +\infty)$, in particular, at all the values $t \in \widehat{\mathcal{T}}$, where the set $\widehat{\mathcal{T}}$ is defined by equality (4.8). Finally, if all the regular level sets of u are connected, then G is nondecreasing in $\widehat{\mathcal{T}}$.

Notice that the function G is well–defined, as the integrand function is bounded on each level set of u and each level set of u has finite σ–measure.

Proof. The function G is easily seen to satisfy equality (4.47). Concerning limit (4.48), it is convenient to rewrite the second summand in the definition (4.46) of function G as

$$\frac{t}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^4 \int\limits_{\{u = f(t)\}} |\nabla u|^2 \, d\sigma = \frac{\mathcal{C}}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^3 \int\limits_{\{u = f(t)\}} \frac{|\nabla u|}{1-u} |\nabla u| \, d\sigma\,,$$

where $f : [\mathcal{C}/2, +\infty) \to [0, 1)$ is the diffeomorphism defined by formula (4.42). By formulas (4.3) and (4.13), we have

$$\lim_{|x|\to+\infty} \frac{|\nabla u|}{1-u} = 0\,,$$

therefore, the second summand of G tends to zero for $t \to +\infty$, by applying an argument similar to the one leading to limit (4.15), thus limit (4.48) follows.
In absence of critical points, the function G is everywhere continuously differentiable in its interval of definition, with first derivative given by

$$G'(t) = \frac{\pi\mathcal{C}^2}{t^2} + \frac{1}{4}\left(1+\frac{\mathcal{C}}{2t}\right)^3\left(1-\frac{3\mathcal{C}}{2t}\right)\int\limits_{\Sigma_t}|\nabla u|^2\,d\sigma - \frac{\mathcal{C}}{4t}\left(1+\frac{\mathcal{C}}{2t}\right)^2\int\limits_{\Sigma_t}|\nabla u|\,\mathrm{H}\,d\sigma\,,$$

keeping into account formula (4.38).
In presence of critical points, G is continuously differentiable only on $\widehat{\mathcal{T}}$, with first derivative given as above. In order to obtain the rest of statement, let us consider the function $G_1 : [\mathcal{C}/2, +\infty) \to (0, +\infty)$ defined by

$$G_1(t) := \frac{t}{4}\left(1+\frac{\mathcal{C}}{2t}\right)^4 \int\limits_{\{u=f(t)\}} |\nabla u|^2\,d\sigma\,.$$

Obviously it is well–defined and we want to show that $G_1 \in W^{1,1}_{loc}(\mathcal{C}/2, +\infty)$. Notice that

$$\int\limits_a^b G_1(t)\,dt = \int\limits_{\{f(a)\le u\le f(b)\}} \frac{2\mathcal{C}^2}{(1-u^2)^3}|\nabla u|^3\,d\mu < +\infty$$

for every $a, b \in (\mathcal{C}/2, +\infty)$ such that $a < b$, where the equality follows by a change of variable together with the coarea formula (1.1). Thus, $G_1 \in L^1_{loc}(\mathcal{C}/2, +\infty)$. Now, let $\chi \in C^\infty_c\big((\mathcal{C}/2, +\infty)\big)$, we have

$$\begin{aligned}
\int\limits_{\mathcal{C}/2}^{+\infty} \chi'(t)G_1(t)dt &= \int\limits_{\mathcal{C}/2}^{+\infty} dt\left[\chi'(t)\frac{t}{4}\left(1+\frac{\mathcal{C}}{2t}\right)^4\int\limits_{\{u=f(t)\}}|\nabla u|^2\,d\sigma\right] \\
&= \int\limits_0^1 ds \int\limits_{\{u=s\}} \chi'\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\frac{2\mathcal{C}^2}{(1-u^2)^3}|\nabla u|^2\,d\sigma \\
&= \int\limits_M \chi'\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\frac{2\mathcal{C}^2}{(1-u^2)^3}|\nabla u|^3\,d\mu \\
&= \lim_{k\to+\infty}\int\limits_M \chi'\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\frac{2\mathcal{C}^2}{(1-u^2)^3}\eta_k(|\nabla u|^2)|\nabla u|^3\,d\mu \\
&= \lim_{k\to+\infty}\int\limits_M g\left(\nabla\left[\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\right], \frac{2\mathcal{C}|\nabla u|}{(1-u)(1+u)^3}\eta_k(|\nabla u|^2)\nabla u\right)d\mu \\
&= -\lim_{k\to+\infty}\int\limits_M \chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\mathrm{div}\left(\frac{2\mathcal{C}|\nabla u|}{(1-u)(1+u)^3}\eta_k(|\nabla u|^2)\nabla u\right)d\mu\,.
\end{aligned} \tag{4.50}$$

Here, the third equality is a consequence of the coarea formula (1.1), the fourth one follows by the dominate convergence theorem, since

$$\chi'\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\frac{2\mathcal{C}^2}{(1-u^2)^3}\,\eta_k(|\nabla u|^2)|\nabla u|^3 \to \chi'\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\frac{2\mathcal{C}^2}{(1-u^2)^3}\,|\nabla u|^3\,\mathbb{I}_{M\setminus\mathrm{Crit}(u)}$$

pointwise in M and

$$\left|\chi'\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\frac{2\mathcal{C}^2}{(1-u^2)^3}\,\eta_k(|\nabla u|^2)|\nabla u|^3\right| \leq \|\chi'\|_{L^\infty(\mathcal{C}/2,+\infty)}\,\frac{2\mathcal{C}^2}{(1-u^2)^3}\,|\nabla u|^3 \in L^1_{loc}(M)\,,$$

finally, the last equality is obtained by the properties of the divergence operator combined with the divergence theorem applied to

$$\chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\frac{2\mathcal{C}|\nabla u|}{(1-u)(1+u)^3}\,\eta_k(|\nabla u|^2)\,\nabla u$$

on $E_a^b := \{f(a) < u < f(b)\}$, for $a, b \in \widehat{\mathcal{T}}$ such that $\mathrm{supp}\chi \subseteq (a,b)$. We observe that

$$\begin{aligned}
&\int_M \chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\mathrm{div}\Big(\frac{2\mathcal{C}|\nabla u|}{(1-u)(1+u)^3}\,\eta_k(|\nabla u|^2)\nabla u\Big)d\mu \\
&\quad= \int_{E_a^b} \chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\mathrm{div}\Big(\frac{2\mathcal{C}|\nabla u|}{(1-u)(1+u)^3}\,\eta_k(|\nabla u|^2)\nabla u\Big)d\mu \\
&\quad= \int_{E_a^b} \chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\eta_k(|\nabla u|^2)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,\Big[\frac{4(2u-1)}{(1+u)^4}|\nabla u|^2 + \frac{2(1-u)}{(1+u)^3}\,\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|}\Big]d\mu \\
&\qquad+ \int_{E_a^b} \chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\eta_k'(|\nabla u|^2)\,\frac{4\mathcal{C}|\nabla u|^2}{(1-u)(1+u)^3}\,g(\nabla|\nabla u|,\nabla u)d\mu\,.
\end{aligned} \tag{4.51}$$

Now, keeping into account that

$$\begin{aligned}
&\left|\chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\eta_k'(|\nabla u|^2)\,\frac{4\mathcal{C}|\nabla u|^2}{(1-u)(1+u)^3}\,g(\nabla|\nabla u|,\nabla u)\right| \\
&\qquad\leq \frac{4\mathcal{C}\,\|\chi\|_{L^\infty(\mathcal{C}/2,+\infty)}}{(1-u)(1+u)^3}\,\eta_k'(|\nabla u|^2)\,|\nabla u|^3\,|\nabla du|\,\mathbb{I}_{\{\frac{1}{2k}\leq|\nabla u|^2\leq\frac{3}{2k}\}} \\
&\qquad\leq \frac{4\mathcal{C}\,\|\chi\|_{L^\infty(\mathcal{C}/2,+\infty)}}{(1-u)(1+u)^3}\,\frac{3^{3/2}}{\sqrt{2k}}\,|\nabla du|\,,
\end{aligned}$$

we have

$$\left|\int_{E_a^b} \chi\Big(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\Big)\,\eta_k'(|\nabla u|^2)\,\frac{4\mathcal{C}|\nabla u|^2}{(1-u)(1+u)^3}\,g(\nabla|\nabla u|,\nabla u)d\mu\right| \leq \frac{C}{\sqrt{2k}} \longrightarrow 0\,. \tag{4.52}$$

At the same time, setting

$$Q := \frac{4(2u-1)}{(1+u)^4}|\nabla u|^2 + \frac{2(1-u)}{(1+u)^3}\,\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|}\,,$$

we get

$$\lim_{k\to+\infty}\int\limits_{E_a^b}\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\eta_k(|\nabla u|^2)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\,d\mu=\int\limits_{E_a^b}\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\,d\mu$$
$$=\int\limits_{M}\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\,d\mu\,, \tag{4.53}$$

as a consequence of the dominate convergence theorem, since

$$\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\eta_k(|\nabla u|^2)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\to\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\,\mathbb{I}_{M\setminus\mathrm{Crit}(u)}$$

pointwise in M and

$$|Q|\leq\frac{4\,|2u-1|}{(1+u)^4}|\nabla u|^2+\frac{2(1-u)}{(1+u)^3}|\nabla du|\,\in L^1(E_a^b)\,, \tag{4.54}$$

$$\left|\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\,\eta_k(|\nabla u|^2)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,Q\right|\leq\|\chi\|_{L^\infty(\mathcal{C}/2,+\infty)}\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\,|Q|\,\in L^1(E_a^b)\,,$$

where these inequalities hold in $M\setminus\mathrm{Crit}(u)$. Then, from formula (4.50), by virtue of equality (4.51) together with limits (4.52) and (4.53), it follows

$$\int\limits_{\mathcal{C}/2}^{+\infty}\chi'(t)G_1(t)\,dt$$
$$=-\int\limits_{M}\chi\left(\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\right)\,\frac{\mathcal{C}|\nabla u|}{(1-u)^2}\left[\frac{4(2u-1)}{(1+u)^4}|\nabla u|^2+\frac{2(1-u)}{(1+u)^3}\,\frac{g(\nabla|\nabla u|,\nabla u)}{|\nabla u|}\right]\,d\mu$$
$$=-\int\limits_{\mathcal{C}/2}^{+\infty}dt\,\chi(t)\left[\frac{1}{4}\left(1+\frac{\mathcal{C}}{2t}\right)^3\left(1-\frac{3\mathcal{C}}{2t}\right)\int\limits_{\Sigma_t}|\nabla u|^2\,d\sigma-\frac{\mathcal{C}}{4t}\left(1+\frac{\mathcal{C}}{2t}\right)^2\int\limits_{\Sigma_t}|\nabla u|\,\mathrm{H}\,d\sigma\right],$$

where the second equality is obtained by the coarea formula (1.1), along with a change of variable. In this way, we conclude that G_1 has a weak derivative in the open interval $(\mathcal{C}/2,+\infty)$, now we want to see that it is in $L^1_{loc}(\mathcal{C}/2,+\infty)$. Notice that each summand of $\mathcal{C}\,Q/(1-u)^2$ is in $L^1_{loc}(M)$, keeping into account formula (4.54), therefore the functions

$$t\mapsto\frac{1}{4}\left(1+\frac{\mathcal{C}}{2t}\right)^3\left(1-\frac{3\mathcal{C}}{2t}\right)\int\limits_{\Sigma_t}|\nabla u|^2\,d\sigma$$
$$t\mapsto\frac{\mathcal{C}}{4t}\left(1+\frac{\mathcal{C}}{2t}\right)^2\int\limits_{\Sigma_t}|\nabla u|\,\mathrm{H}\,d\sigma\,,$$

defined a.e., belongs to $L^1_{loc}(\mathcal{C}/2,+\infty)$. Consequently, $G_1\in W^{1,1}_{loc}(\mathcal{C}/2,+\infty)$ and this conclusion, along with the fact G_1 is also of class C^1 in an interval of type $[a,b)$, with $a=\mathcal{C}/2$, implies that G_1 admits a locally absolutely continuous representative in $[\mathcal{C}/2,+\infty)$ (see [49], for instance, about the relation between Sobolev spaces and the

locally absolutely continuous functions). Thus, G admits a locally absolutely continuous representative in $[\mathcal{C}/2,+\infty)$, as $G(t)=-\frac{\pi\mathcal{C}^2}{t}+G_1(t)$ in $[\mathcal{C}/2,+\infty)$, coinciding with G on $\widehat{\mathcal{T}}$.
Being the function G continuously differentiable on $\widehat{\mathcal{T}}$, with first derivative given by formula (4.49), it follows easily that the equality

$$\widehat{G}(t)=\frac{4t^3}{\mathcal{C}^2}\,G'(t) \tag{4.55}$$

holds, for every $t\in\widehat{\mathcal{T}}$. We set

$$\mathcal{A}:=2\mathcal{C}\left[\pi-\int\limits_{\partial M}|\nabla u|^2\,d\sigma\right],$$

therefore,

$$\begin{aligned}G\,(\mathcal{C}/2)&=-\mathcal{A}\,,\\ \widehat{G}\,(\mathcal{C}/2)&=\mathcal{A}\,,\end{aligned}$$

by formulas (4.47) and (4.6), respectively. Then, by virtue of the fact that $\mathcal{C}/2\in\widehat{\mathcal{T}}$, the monotonicity of $\widehat{G}$ proved in Proposition 4.1.1, under the assumption of connectedness of all regular level sets of u, implies

$$\frac{4t^3}{\mathcal{C}^2}\,G'(t)-\mathcal{A}=\widehat{G}(t)-\widehat{G}\,(\mathcal{C}/2)\geq 0\,,$$

for every $t\in\widehat{\mathcal{T}}$, in particular

$$G'(t)\geq\frac{\mathcal{C}^2}{4t^3}\,\mathcal{A}\,,$$

for every $t\in\widehat{\mathcal{T}}$. Notice that this inequality is true a.e. in $[\mathcal{C}/2,+\infty)$, as $\widehat{\mathcal{T}}$ differs from $[\mathcal{C}/2,+\infty)$ only for a negligible set, by Sard's theorem. Consequently, integrating between $\mathcal{C}/2$ and $t\in\widehat{\mathcal{T}}$ and since G admits a locally absolutely continuous representative in $[\mathcal{C}/2,+\infty)$, coinciding with it on $\widehat{\mathcal{T}}$, it follows

$$\begin{aligned}G(t)-G\,(\mathcal{C}/2)&\geq-\frac{\mathcal{C}^2\mathcal{A}}{8t^2}+\frac{\mathcal{A}}{2}\,,\\ G(t)+\mathcal{A}&\geq-\frac{\mathcal{C}^2\mathcal{A}}{8t^2}+\frac{\mathcal{A}}{2}\,,\\ G(t)&\geq-\frac{\mathcal{C}^2\mathcal{A}}{8t^2}-\frac{\mathcal{A}}{2}\,,\end{aligned} \tag{4.56}$$

for every $t\in\widehat{\mathcal{T}}$. By the compactness of $\mathrm{Crit}(u)$, there exists $t_0\in[\mathcal{C}/2,+\infty)$ such that $[t_0,+\infty)\subseteq\widehat{\mathcal{T}}$, thus, passing in inequality (4.56) to the limit, as $t\to+\infty$, we get $\mathcal{A}\geq 0$, by limit (4.48). Now, the nonnegativity of $\mathcal{A}$ implies that $\widehat{G}(t)\geq 0$ and in turn that $G'(t)\geq 0$, for every $t\in\widehat{\mathcal{T}}$. This implies that G is nondecreasing in $\widehat{\mathcal{T}}$, since $\widehat{\mathcal{T}}$ differs from $[\mathcal{C}/2,+\infty)$ for a negligible set and G admits a locally absolutely continuous representative in $[\mathcal{C}/2,+\infty)$, coinciding with it on $\widehat{\mathcal{T}}$. $\square$

Under the assumptions of the above proposition, we conjecture (as before) that it is possible to prove that the function G is of class C^1 in $[\mathcal{C}/2, +\infty)$ with

$$G'(t) = \frac{\pi \mathcal{C}^2}{t^2} + \frac{1}{4}\left(1 + \frac{\mathcal{C}}{2t}\right)^3 \left(1 - \frac{3\mathcal{C}}{2t}\right) \int\limits_{\Sigma_t} |\nabla u|^2 \, d\sigma - \frac{\mathcal{C}}{4t}\left(1 + \frac{\mathcal{C}}{2t}\right)^2 \int\limits_{\Sigma_t} |\nabla u| \, \mathrm{H} \, d\sigma$$

for a.e $t \in (\mathcal{C}/2, +\infty)$.

Until now, we defined two functions G and $\widehat{G}$ which are monotone on a set that differs from their interval of definition only for a negligible set, under the assumption of connectedness of all the regular level sets of u. Whereas the monotonicity of function G can be interpreted as a version, in presence of boundary, of the monotonicity obtained by Munteanu and Wang in [66], the monotonicity of the function $\widehat{G}$ is new. Another fundamental property of these functions is that their being constant characterizes the (exterior spatial) Schwarzschild manifolds of mass $m > 0$, as we are going to see in the next two propositions. A key point of these results lies in the fact that when they are constant, there are no critical points of u, which in turn implies, in both cases, that $\widehat{G}' =\equiv 0$, where $\widehat{G}'$ is sum of nonnegative terms. From a careful analysis of the consequences of the vanishing of these nonnegative terms, one can deduce that the Riemannian manifold is isometric to a (exterior spatial) Schwarzschild manifold of positive mass.

Proposition 4.1.3 (Rigidity – I)**.** *Let (M, g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Let $u \in C^\infty(M)$ be the solution of Dirichlet problem (4.1) and let $\mathcal{C} > 0$ be the boundary capacity of ∂M in (M, g), given by formula (4.2). Consider the function $\widehat{G} : [\mathcal{C}/2, +\infty) \to \mathbb{R}$ defined by formula (4.5). Then, $\widehat{G}$ is constant in $[\mathcal{C}/2, +\infty)$ if and only if (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$ in Example (1.4.2).*

Proof. If (M, g) is the (exterior spatial) Schwarzschild manifold with mass $m > 0$,

$$u = \frac{1 - \frac{m}{2|x|}}{1 + \frac{m}{2|x|}}, \qquad |\nabla u| = \left(1 + \frac{m}{2|x|}\right)^{-4} \frac{m}{|x|^2}, \qquad \mathrm{H} = \frac{2}{|x|} \frac{1 - \frac{m}{2|x|}}{\left(1 + \frac{m}{2|x|}\right)^3}. \tag{4.57}$$

Notice that u has no critical points. By a straightforward computation, one has

$$\mathcal{C} := \frac{1}{4\pi} \int\limits_{\partial M} |\nabla u| \, d\sigma = m \qquad \text{and} \qquad \widehat{G} \equiv 0 \,.$$

Now, we assume that $\widehat{G}$ is constant in $[\mathcal{C}/2, +\infty)$. We know that there exists a maximal time T such that $\nabla u \neq 0$ in $\big\{0 \leq u < (1 - \frac{\mathcal{C}}{2T})/(1 + \frac{\mathcal{C}}{2T})\big\}$, since $\partial M = \{u = 0\}$ is a regular level set of u and $u : M \to [0, 1)$ is proper. Then, $\widehat{G}$ is continuously differentiable in $\big[\frac{\mathcal{C}}{2}, T\big)$ with $\widehat{G}'$ given by formula (4.37) and nonnegative, since each level set $\{u = (1 - \frac{\mathcal{C}}{2t})/(1 + \frac{\mathcal{C}}{2t})\}$ is a connected closed surface, as it is diffeomorphic to $\partial M = \{u = 0\}$, which is connected by assumption, for every $t \in \big[\frac{\mathcal{C}}{2}, T\big)$. At the same time, $\widehat{G}'(t) = 0$ in the same interval, as $\widehat{G}$ is constant. Therefore, all the nonnegative summands in formula (4.37) are forced to vanish for every $t \in [\mathcal{C}/2, T)$. This fact gives $\nabla^{\Sigma_t}|\nabla u| = \nabla^\top|\nabla u| = 0$ and $\mathrm{H} = \frac{4u}{1-u^2}|\nabla u|$, which in turn imply

$$\nabla|\nabla u| = \nabla^\top|\nabla u| + \nabla^\perp|\nabla u| = g\left(\nabla|\nabla u|, \frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|} = -\mathrm{H}\,\nabla u = -\frac{4u}{1-u^2}\,|\nabla u|\,\nabla u\,. \tag{4.58}$$

Consequently,

$$\nabla\left(\log|\nabla u|\right) = \nabla\left(2\log\left(1-u^2\right)\right)$$
$$\nabla\left[\log\left(\frac{|\nabla u|}{\left(1-u^2\right)^2}\right)\right] = 0\,.$$

Thus, the function $|\nabla u|/\left(1-u^2\right)^2$ is constant on every connected component of $\left\{0 \le u < (1-\frac{\mathcal{C}}{2T})/(1+\frac{\mathcal{C}}{2T})\right\}$, but this latter set is connected since it is diffeomorphic to $\left[0,\,(1-\frac{\mathcal{C}}{2T})/(1+\frac{\mathcal{C}}{2T})\right)\times\partial M$ and ∂M is connected. In conclusion, $|\nabla u| = a(1-u^2)^2$, where $a \in \mathbb{R}$ is a positive constant, therefore, being $0 \le u < 1$ on the whole manifold, $T = +\infty$ and $|\nabla u| \neq 0$ everywhere. In particular, all the level sets of u are regular and diffeomorphic to each other, which clearly implies that they are all connected, hence $\widehat{G}'(t)$ can be written as sum of nonnegative terms, which are forced to vanish, as $\widehat{G}$ is constant, as before. Concerning the constant a, from formulas (4.3) and (4.13), it follows

$$\mathcal{C} = \lim_{|x|\to+\infty}|x|^2|\nabla u| = a\lim_{|x|\to+\infty}|x|^2(1-u^2)^2 = 4a\,\mathcal{C}^2\,,$$

therefore $a = (4\mathcal{C})^{-1}$. Now, up to an isometry, we have that $M = [0,1)\times\partial M$, every slice $\{t\}\times\partial M$ is the level set $\{u = t\}$ and the metric g can be written as

$$g = \frac{(4\mathcal{C})^2}{(1-u^2)^4}\,du\otimes du + g_{\alpha\beta}(u,\vartheta)\,d\vartheta^\alpha\otimes d\vartheta^\beta\,,$$

where $g_{\alpha\beta}(u,\vartheta)\,d\vartheta^\alpha\otimes d\vartheta^\beta$ represents the metric induced by g on the level sets of u. By the vanishing of the traceless second fundamental form of the level sets in formula (4.37), i.e. $\mathrm{h}_{\alpha\beta} = (\mathrm{H}/2)g_{\alpha\beta}$, in combination with equality $\mathrm{h}_{\alpha\beta} = \nabla du_{\alpha\beta}/|\nabla u|$, by equality (1.4), it turns out that the coefficients $g_{\alpha\beta}(u,\vartheta)$ satisfy the following first order system of PDE's

$$\frac{\partial g_{\alpha\beta}}{\partial u} = \frac{4u}{1-u^2}\,g_{\alpha\beta}\,,$$

from which we can deduce

$$g_{\alpha\beta}(u,\vartheta)\,d\vartheta^\alpha\otimes d\vartheta^\beta = (1-u^2)^{-2}c_{\alpha\beta}(\vartheta)\,d\vartheta^\alpha\otimes d\vartheta^\beta\,.$$

At the same time, for every $u_0 \in [0,1)$, we also have

$$\frac{1}{2}\mathrm{R}^{\{u=u_0\}} = \frac{(1-u_0^2)^2}{4\mathcal{C}^2}\,,$$

indeed, from the traced Gauss equation (1.6) together with Bochner formula (4.40) (coupled with the fact that u is harmonic), it follows

$$\begin{aligned}\mathrm{R}^{\{u=u_0\}} &= \mathrm{R} - 2\mathrm{Ric}(\nu,\nu) - |\mathrm{h}|^2 + \mathrm{H}^2 \\ &= -2\,|\nabla u|^{-2}\,\mathrm{Ric}(\nabla u,\nabla u) + \frac{\mathrm{H}^2}{2} \\ &= |\nabla u|^{-2}\,\big[-\Delta|\nabla u|^2 + 2\,|\nabla du|^2\big] + \frac{\mathrm{H}^2}{2}\,.\end{aligned}$$

Here, the second equality is a consequence of the vanishing of the scalar curvature of M and of the traceless second fundamental form of the level sets, in equality (4.37), in particular, the vanishing of this latter also implies

$$|\nabla du|^2 = |\nabla u|^2|\mathrm{h}|^2 + 2\big|\nabla^\top|\nabla u|\big|^2 + \big|\nabla^\perp|\nabla u|\big|^2 = \frac{3}{2}|\nabla u|^2\mathrm{H}^2\,,$$

keeping into account formula (4.58), consequently,

$$\mathrm{R}^{\{u=u_0\}} = -|\nabla u|^{-2}\,\Delta|\nabla u|^2 + \frac{7\,\mathrm{H}^2}{2}\,,$$

but, being

$$\begin{aligned}|\nabla u| &= a(1-u^2)^2 \\ \mathrm{H} &= \frac{4u}{1-u^2}|\nabla u| = 4au(1-u^2)\,,\end{aligned}$$

with $a = (4\mathcal{C})^{-1}$, as already explained, one obtains

$$\mathrm{R}^{\{u=u_0\}} = 8(4\mathcal{C})^{-2}(1-u_0^2)^2\,.$$

Then, $\{u = u_0\}$ with the Riemannian metric induced by (M,g), has constant sectional curvature (equal to $(1-u_0^2)^2/(4\mathcal{C}^2)$) and by the vanishing of the Gauss–Bonnet term in formula (4.37), it is diffeomorphic to a 2–sphere. Consequently, $(\{u = u_0\}, g_{\{u=u_0\}})$ is isometric to $(\mathbb{S}^2, \frac{4\mathcal{C}^2}{(1-u_0^2)^2}\, g_{\mathbb{S}^2})$, by [25, Section 3.F], in particular, $(\partial M, g_{\partial M})$ is isometric to $(\mathbb{S}^2, 4\mathcal{C}^2\, g_{\mathbb{S}^2})$. Thus, up to an isometry, one has $M = [0,1)\times\mathbb{S}^2$ and

$$g = \frac{(4\mathcal{C})^2}{(1-u^2)^4}\,du\otimes du + \frac{4\mathcal{C}^2}{(1-u^2)^2}\,g_{\mathbb{S}^2}\,.$$

Then, the map

$$(u,\vartheta)\in(M,\,g)\mapsto\frac{\mathcal{C}}{2}\frac{1+u}{1-u}\,\vartheta\in\big(M_{\mathsf{Sch}(\mathcal{C})},\,g_{\mathsf{Sch}(\mathcal{C})}\big)$$

is an isometry. □

Proposition 4.1.4 (Rigidity – II). *Let (M,g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Let $u\in C^\infty(M)$ be the solution of Dirichlet problem (4.1) and let $\mathcal{C}>0$ be the boundary capacity of ∂M in (M,g), given by formula (4.2). Consider the function $G : [\mathcal{C}/2,+\infty)\to\mathbb{R}$ defined by formula (4.46). Then, G is constant in $[\mathcal{C}/2,+\infty)$*

if and only if (M, g) *isometric to the (exterior spatial) Schwarzschild manifold of mass* $\mathcal{C}$ *in Example* (1.4.2)*.*

Proof. If (M, g) is the (exterior spatial) Schwarzschild manifold with mass $m > 0$, then, by the equalities (4.57) together with the observation $m = \mathcal{C}$, one obtains directly that $G \equiv 0$. Now, we assume that G is constant in $[\mathcal{C}/2, +\infty)$, then, as before, there exists a maximal time T such that $\nabla u \neq 0$ in $\{0 \leq u < (1 - \frac{\mathcal{C}}{2T})/(1 + \frac{\mathcal{C}}{2T})\}$, since $\partial M = \{u = 0\}$ is a regular level set of u and $u : M \to [0, 1)$ is proper. Hence, in $\big[\frac{\mathcal{C}}{2}, T\big)$ the function G is of class C^2, with $G'(t)$ given by formula (4.49) and at same time with $G'(t) = 0$, while $\widehat{G}$ is of class C^1 in the same interval, with $\widehat{G}'$ given by formula (4.37) and at the same time $\widehat{G}'(t) = 0$, as $\widehat{G} \equiv 0$, due to equality (4.55). Consequently, arguing as in the proof of Proposition 4.1.3, one obtains that $T = +\infty$ and $|\nabla u| \neq 0$ everywhere. In particular, $\widehat{G}$ is constant on its interval of definition and the conclusion follows again by Proposition 4.1.3. $\square$

4.2 Some sharp comparison results

In this section, two sharp inequalities are derived by means of Proposition 4.1.1 and Proposition 4.1.2, under a suitable topological assumption in order to guarantee the connectedness of all the regular level sets of u. One of these results is an improvement of the mass–capacity inequality obtained by Bray in [13] and the other one is a sharp upper bound of the boundary capacity of ∂M in terms of its area, proven by Bray and Miao in [15], anyway, our assumptions are different from the ones of these authors.

Lemma 4.2.1. *Let* (M, g) *be a* 3*–dimensional, complete, one–ended asymptotically flat manifold with compact and connected boundary. Let* $u \in C^\infty(M)$ *be the solution of Dirichlet problem* (4.1)*. Assume that the first Betti number of* M *is zero, then, all regular level sets of* u *are connected.*

Proof. Let $t \in (0, 1)$ be a regular value of u. We know that $\{u \geq t\} = \overline{\{u > t\}}$ and $\{0 \leq u \leq t\} = \overline{\{0 < u < t\}}$, as explained after Remark 2.0.1, we want to see that they are connected. First, we show the connectedness of $\{0 \leq u \leq t\}$. Supposing it is not connected, it must have a connected component K disjoint from ∂M (as this latter is connected and compact). Then, $\partial K \subseteq \{u = t\}$ and, since $\{0 \leq u \leq t\} = \overline{\{0 < u < t\}}$, the interior of K must be nonempty and contain some points where $0 < u < t$, which is not possible, by the maximum principle. On the other side, we already know that $\{u \geq t\}$ is connected, as explained after Remark 2.0.1.
Let now $\varepsilon > 0$ such that $[t - \varepsilon, t + \varepsilon]$ doesn't contain critical values of u, we consider the reduced Mayer–Vietoris exact sequence of the pair $\{0 \leq u \leq t + \varepsilon\}$ and $\{u \geq t\}$,

$$\widetilde{H}_1(M; \mathbb{Z}) \to \widetilde{H}_0\left(\{t \leq u \leq t + \varepsilon\}; \mathbb{Z}\right) \to \widetilde{H}_0\left(\{0 \leq u \leq t + \varepsilon\}; \mathbb{Z}\right) \oplus \widetilde{H}_0\left(\{u \geq t\}; \mathbb{Z}\right),$$

We recall that $\widetilde{H}_n(X; \mathbb{Z}) \simeq H_n(X; \mathbb{Z})$, for all positive integers $n \in \mathbb{N}$ and $H_0(X; \mathbb{Z}) \simeq \widetilde{H}_0(X; \mathbb{Z}) \oplus \mathbb{Z}$, for any topological space X. Then, from the connectedness of the sets $\{0 \leq u \leq t + \varepsilon\}$ and $\{u \geq t\}$, it follows that the last space, $\widetilde{H}_0\left(\{0 \leq u \leq t + \varepsilon\}; \mathbb{Z}\right) \oplus \widetilde{H}_0\left(\{u \geq t\}; \mathbb{Z}\right)$, is trivial, therefore, $\widetilde{H}_0\left(\{t \leq u \leq t + \varepsilon\}; \mathbb{Z}\right)$ is the image of $\widetilde{H}_1(M; \mathbb{Z}) \simeq H_1(M; \mathbb{Z})$, but this image is trivial. Indeed, the assumption that the first Betti number of M is zero implies that $H_1(M; \mathbb{Z})$ coincides with its torsion subgroup (i.e. the subgroup of all its elements with finite order), but at same time $\widetilde{H}_0\left(\{t \leq u \leq t + \varepsilon\}; \mathbb{Z}\right)$ is torsion–free (since $H_0(X; \mathbb{Z})$ is isomorphic to a direct sum of $\mathbb{Z}$'s, one for each path–connected component of any topological space X). Thus, $\widetilde{H}_0\left(\{t \leq u \leq t + \varepsilon\}; \mathbb{Z}\right) = 0$

and, consequently, $\{t \leq u \leq t+\varepsilon\}$ is connected, but, being $\{t \leq u \leq t+\varepsilon\}$ diffeomorphic to $\{u = t\} \times [t, t+\varepsilon]$, the number of the connected components of $\{t \leq u \leq t+\varepsilon\}$ and $\{u = t\}$ is the same. $\square$

Theorem 4.2.2. *Let (M, g) be a 3–dimensional, complete, one–ended asymptotically flat manifold with minimal, compact and connected boundary and with nonnegative scalar curvature. Assume that the first Betti number of M is zero. Let $u \in C^\infty(M)$ be the solution of Dirichlet problem (4.1) and let $\mathcal{C} > 0$ be the boundary capacity of ∂M in (M, g), given by formula (4.2). Then, the following statements hold.*

(1)

$$\mathrm{Area}\left(\left\{u = \frac{1 - \frac{\mathcal{C}}{2t}}{1 + \frac{\mathcal{C}}{2t}}\right\}\right) \geq 4\pi\, t^2 \left(1 + \frac{\mathcal{C}}{2t}\right)^4$$

for every $t \in \widehat{\mathcal{T}}$, where the set $\widehat{\mathcal{T}}$ is given by equality (4.8). In particular,

$$\mathcal{C} \leq \sqrt{\frac{\mathrm{Area}(\partial M)}{16\pi}}\,, \tag{4.59}$$

with equality if and only if (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$ in Example (1.4.2).

(2)

$$m_{\mathrm{ADM}} - \mathcal{C} \geq \frac{\mathcal{C}}{4\pi}\left[\pi - \int\limits_{\partial M} |\nabla u|^2\, d\sigma\right] \geq 0\,, \tag{4.60}$$

with equality in the first or second inequality if and only if (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$.
In particular,

$$m_{\mathrm{ADM}} \geq \mathcal{C}\,, \tag{4.61}$$

with equality if and only if (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$.

Proof. By Lemma (4.2.1), all the regular level sets of u are connected, in turn this implies that the functions G and $\widehat{G}$, given respectively by formulas (4.46) and (4.5), are nondecreasing on $\widehat{\mathcal{T}}$, by Propositions 4.1.2 and 4.1.1, respectively. Thus, for every $t \in \widehat{\mathcal{T}}$,

$$G(\mathcal{C}/2) \leq G(t) \leq \lim_{t \to +\infty} G(t) = 0\,, \tag{4.62}$$

$$0 \leq -G(\mathcal{C}/2) = \widehat{G}(\mathcal{C}/2) \leq \widehat{G}(t) \leq \lim_{t \to +\infty} \widehat{G}(t) \leq 8\pi\,(m_{\mathrm{ADM}} - \mathcal{C})\,, \tag{4.63}$$

by formulas (4.47), (4.48), (4.6), (4.7). Now, dividing by 8π in formula (4.63), the inequalities (4.60) follow. If the equality holds in the first inequality of formula (4.60), then $\widehat{G}$ is constant on $\widehat{\mathcal{T}}$ and this is sufficient to guarantee that $\mathrm{Crit}(u) = \emptyset$, as $T = +\infty$, by the same argument of the proof of Proposition 4.1.3, where $T = \sup\{t : [0, f(t))$ does not contain critical values of $u\}$ for $f(t) := (1 - \frac{\mathcal{C}}{2t})/(1 + \frac{\mathcal{C}}{2t})$. Consequently, $\widehat{G}$ is constant on $[\mathcal{C}/2), +\infty)$ and the rigidity statement that (M, g) is isometric to the (exterior spatial) Schwarzschild manifold of mass $\mathcal{C}$, follows by Proposition 4.1.3. Analogously, if the equality holds in the second inequality of formula (4.60), then G is constant on $\widehat{\mathcal{T}}$ and this is sufficient to guarantee that $\mathrm{Crit}(u) =$

$\emptyset$, as $T = +\infty$, by the same argument of the proof of Proposition 4.1.4, where T is defined as before. Thus, G is constant on $[\mathcal{C}/2), +\infty)$ and the rigidity statement is a consequence of Proposition 4.1.4.
Inequality (4.61) follows immediately from formula (4.60) and the rigidity statement, in the case that the equality holds in inequality (4.61), is a simple consequence of the fact that equality holds in the first and the second inequality of formula (4.60). On the other hand, in a (exterior spatial) Schwarzschild manifold with mass $m > 0$, one has that $m = \mathcal{C}$ and the functions G and $\widehat{G}$ are identically zero, as showed in the proof of Propositions 4.1.4 and 4.1.3, hence, the equalities in formula (4.60) also hold.
By the last inequality in formula (4.62) and recalling the definition (4.46) of the function G, for every $t \in \widehat{\mathcal{T}}$, there hold

$$\int_{\Sigma_t} |\nabla u|^2 \, d\sigma \leq \frac{4\pi\mathcal{C}^2}{t^2} \left(1 + \frac{\mathcal{C}}{2t}\right)^{-4} .$$

Consequently, we have

$$\begin{aligned} 4\pi\mathcal{C} = \int_{\Sigma_t} |\nabla u| \, d\sigma &\leq \left[\int_{\Sigma_t} |\nabla u|^2 \, d\sigma\right]^{1/2} [\mathrm{Area}(\Sigma_t)]^{1/2} \\ &\leq (4\pi)^{1/2} \, \frac{\mathcal{C}}{t} \left(1 + \frac{\mathcal{C}}{2t}\right)^{-2} [\mathrm{Area}(\Sigma_t)]^{1/2} , \end{aligned}$$

where the equality comes from formula (4.4) and the first inequality is a consequence of Hölder inequality. Thus,

$$\mathrm{Area}(\Sigma_t) \geq 4\pi \, t^2 \left(1 + \frac{\mathcal{C}}{2t}\right)^4 ,$$

in particular,

$$\mathrm{Area}(\partial M) \geq 16 \, \pi \, \mathcal{C}^2 ,$$

from which we obtain

$$\mathcal{C} \leq \sqrt{\frac{\mathrm{Area}(\partial M)}{16\pi}} .$$

Finally, if we assume that the equality holds, then G is constant on $\widehat{\mathcal{T}}$ and the rigidity statement follows as before. On other side, in a (exterior spatial) Schwarzschild manifold with mass $m > 0$, the equality in formula (4.59) can be checked directly, keeping into account that $m = \mathcal{C}$. $\square$

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Ringraziamenti

Vorrei ringraziare il Prof. Carlo Mantegazza per la sua guida, supporto e disponibilità nel darmi chiarimenti e suggerimenti. La passione, profondità e chiarezza con cui svolge l'attività di ricerca sarà sempre un obiettivo da cercare di eguagliare il più possibile.

Vorrei ringraziare la Prof.ssa Virginia Agostiniani e il Prof. Lorenzo Mazzieri per avermi fatto conoscere la bellezza di questo campo di ricerca e aver condiviso con me alcune intuizioni, competenze e conoscenze su di esso, senza le quali l'attività di ricerca portata avanti in questi anni e questa tesi non sarebbero state possibili.

Ringrazio Luca Benatti, il Prof. Riccardo Benedetti e Stefano Borghini per i loro preziosi chiarimenti durante questo percorso di studio. Sono grata anche a Giacomo Ascione, Mattia Fogagnolo, Marco Pozzetta e ai miei colleghi di dottorato per la loro disponibiltà.

Ringrazio la mia famiglia per il loro affetto e l'avermi supportata/sopportata nei momenti difficili del dottorato.