A One-Dimensional Variational Problem for Cholesteric Liquid Crystals with Disparate Elastic Constants

Dmitry Golovaty[a], Michael Novack[b], Peter Sternberg[c]

[a]*Department of Mathematics, The University of Akron, Akron, OH 44325*
[b]*Department of Mathematics, The University of Texas at Austin 2515 Speedway, RLM 8.100, Austin, TX 78712*
[c]*Department of Mathematics, Indiana University, Rawles Hall 831 East 3rd St., Bloomington, IN 47405*

Abstract

We consider a one-dimensional variational problem arising in connection with a model for cholesteric liquid crystals. The principal feature of our study is the assumption that the twist deformation of the nematic director incurs much higher energy penalty than other modes of deformation. The appropriate ratio of the elastic constants then gives a small parameter ε entering an Allen-Cahn-type energy functional augmented by a twist term. We consider the behavior of the energy as ε tends to zero. We demonstrate existence of the local energy minimizers classified by their overall twist, find the Γ-limit of the relaxed energies and show that it consists of the twist and jump terms. Further, we extend our results to include the situation when the cholesteric pitch vanishes along with ε.

Keywords: Cholesteric liquid crystal, Gamma-convergence, local minimizer

1. Introduction

We seek an understanding of the energy landscape for the one-dimensional variational problem

$$\inf_{\mathcal{A}_\alpha} E_\varepsilon(u), \tag{1.1}$$

where $u : [0,1] \to \mathbb{R}^2$ so that $u = (u_1, u_2)$ with

$$E_\varepsilon(u_1, u_2) = \int_0^1 \frac{\varepsilon}{2} |u'|^2 + \frac{1}{4\varepsilon}(|u|^2 - 1)^2 + \frac{L}{2}(u_1\, u_2' - u_2\, u_1' - 2\pi N)^2\, dx, \tag{1.2}$$

and

$$\mathcal{A}_\alpha := \{u \in H^1((0,1);\mathbb{R}^2) :\ u(0) = 1,\ u(1) = e^{i\alpha}\}, \tag{1.3}$$

for some positive integer N and some $\alpha \in [0, 2\pi)$. When convenient, as above, we will view $u = (u_1, u_2)$ as a map into $\mathbb{C}$. We should note that the choice of N as an integer is made purely for convenience. One could work with any real number N but as we shall see, the key issue is simply how close the pure N twist state $e^{i2\pi Nx}$ is to satisfying the boundary conditions at $x = 1$.

Our purpose in this article is to continue the analysis of a family of models with disparate elastic constants arising in the mathematics of liquid crystals [5, 6, 7, 8]. In particular, the problem (1.1) can be viewed as a highly simplified, relaxed version of the Oseen-Frank model for *cholesteric liquid crystals*, [2, 12, 19, 20, 21, 22] based on the elastic deformations of an $\mathbb{S}^1$- or $\mathbb{S}^2$-valued director n, cf. [23]. Other models, of course, exist for nematic liquid crystals, including the Q-tensor based Landau-de Gennes model, whose energy density consists of a bulk potential favoring either a uniaxial nematic state, an isotropic state, or both, depending on temperature, cf. [15]. We refer the reader to the recent literature [5, 11] that establishes a precise asymptotic relationship between the Oseen-Frank and the Landau-de Gennes models.

We recall now the form of the Oseen-Frank energy,

$$\begin{aligned} F_{OF}(n) := \int_\Omega \bigg(&\frac{K_1}{2}(\operatorname{div} n)^2 + \frac{K_2}{2}((\operatorname{curl} n)\cdot n + q)^2 + \frac{K_3}{2}|(\operatorname{curl} n)\times n|^2 \\ &+ \frac{K_2+K_4}{2}(\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2)\bigg)\, dx, \end{aligned} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ represents the sample domain and the director n maps Ω to $\mathbb{S}^2$. The material constants K_1, K_2, K_3 and K_4 are the elastic coefficients associated with the deformations of splay, twist, bend and saddle-splay, respectively [23]. Most important for this article is the second term, the twist, where $q = \frac{2\pi}{p}$ with p being the pitch of the cholesteric helix. The distinction between nematic and cholesteric liquid crystals is manifested by the value of q. The liquid crystal is in a nematic state when $q = 0$ and, absent boundary conditions, a global minimizer of F_{OF} is a constant director field. On the other hand, a liquid crystal is in a cholesteric state whenever $q \neq 0$ and global minimizers of F_{OF} in $\mathbb{R}^3$ are rigid rotations of a uniformly twisted director field $n = (n_x, n_y, 0) = e^{\frac{2\pi i z}{p}}$.

In [8] we propose and analyze a model problem for nematic liquid crystals carrying a large energetic cost for splay. The model couples the Ginzburg-Landau potential to an elastic energy density with large elastic disparity, namely

$$\inf_{u\in H^1(\Omega;\mathbb{R}^2)} \frac{1}{2}\int_\Omega \left(\varepsilon|\nabla u|^2 + L(\operatorname{div} u)^2 + \frac{1}{\varepsilon}(1-|u|^2)^2\right)\,dx. \tag{1.5}$$

Here one should view L as playing a role analogous to K_1 in (1.4). The minimization is taken over competitors satisfying an $\mathbb{S}^1$-valued Dirichlet condition on $\partial\Omega$ so as to avoid a trivial minimizer. This choice of potential clearly favors $\mathbb{S}^1$-valued states, which are a stand-in in our models for uniaxial nematic states. Analysis of (1.5) in the $\varepsilon\to 0$ limit involves a 'wall energy' along a jump set J_u penalizing jumps of any $\mathbb{S}^1$-valued competitor u, and bulk elastic energy favoring low divergence. The conjectured Γ-limit of (1.5) is

$$\frac{L}{2}\int_\Omega (\operatorname{div} u)^2\,dx + \frac{1}{6}\int_{J_u\cap\Omega} |u_+ - u_-|^3\,d\mathcal{H}^1, \tag{1.6}$$

where u_+ and u_- are the one-sided traces of u along J_u which exhibit a jump discontinuity in their tangential components.

The model considered in this paper is a cholesteric analog of the problem in [8]. Just as the functional considered in [8] can be viewed as a Ginzburg-Landau-type relaxation of the splay K_1-term in (1.4), the problem (1.1) can be understood as a similar relaxation of the twist K_2-term in the same energy. For example, in 2D this relaxation may take the form

$$\inf_{\mathcal{A}} E_\varepsilon^{2D}(u), \tag{1.7}$$

where $u:\Omega\to\mathbb{R}^3$ with

$$E_\varepsilon^{2D}(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(|u|^2-1)^2 + \frac{L}{2}(u\cdot\operatorname{curl} u - 2\pi N)^2\,dx, \tag{1.8}$$

and

$$\mathcal{A} := \{u\in H^1(\Omega;\mathbb{R}^3):\ u|_{\partial\Omega} = u_0\}, \tag{1.9}$$

for some domain $\Omega\subset\mathbb{R}^2$, some positive integer N and boundary condition $u_0:\partial\Omega\to\mathbb{S}^2$. Results of simulations for the gradient flow dynamics associated with the problem (1.7) lead to intricate textures, such as that shown in Fig. 1, resembling cholesteric fingerprint textures observed in experiments [16].

While attempting to tackle the problem (1.7), we found that the energy landscape in (1.1) is already rich enough to merit a separate investigation in one dimension that we undertake in this paper. Even though the features of minimizers in one dimension will not be exactly the same as those in Fig. 1, they are motivated by a high penalty imposed on twist in both cases. We note that existence and stability of minimizers for the three-component cholesteric director within the framework of the Oseen-Frank model in one dimension was considered in [1] and [4] under the assumption that all elastic constants have comparable values. In addition, in [4], the energy functional included the effects of an electric field. However, these studies are not carried out in the present context of extreme disparity between the elastic constants.

Here we further assume that the component of u along the axis of the twist vanishes so that the target space for the director is two-dimensional. Thus, though we will write $u = (u_1(x), u_2(x))$ what we really have in mind is $u = (0, u_2(x), u_3(x))$. The thought experiment that allows us to impose this condition assumes that an electric field is applied along the axis of the twist and that the cholesteric has negative dielectric anisotropy that forces its molecules to orient perpendicular to the field, [10]. In the one-dimensional setting for highly disparate elastic constants, it turns out that if one includes a third x-dependent component, so that $u(x) = \big(u_1(x), u_2(x), u_3(x)\big)$, it leads to an energy where distinguishing textures are lost for $\varepsilon \ll 1$ and the energy landscape becomes highly degenerate, see Remark 3.3. On the other hand, we find that the one-dimensional, two-component model (1.2) leads to stable states more reminiscent of those described above for the two-dimensional problem.

The richness of the energy landscape is first revealed in Theorem 1.1 below, showing that local minimizers of E_ε exist for every positive integer value of twist–essentially for every winding number. More precisely, through a constrained minimization procedure keeping the modulus of competitors away from zero, we establish:

Theorem 1.1. *For every positive integer M and every $\alpha \in [0, 2\pi)$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there is an H^1-local minimizer $u_{\varepsilon,M} = \rho_{\varepsilon,M} e^{i\theta_{\varepsilon,M}}$ of E_ε within the class*

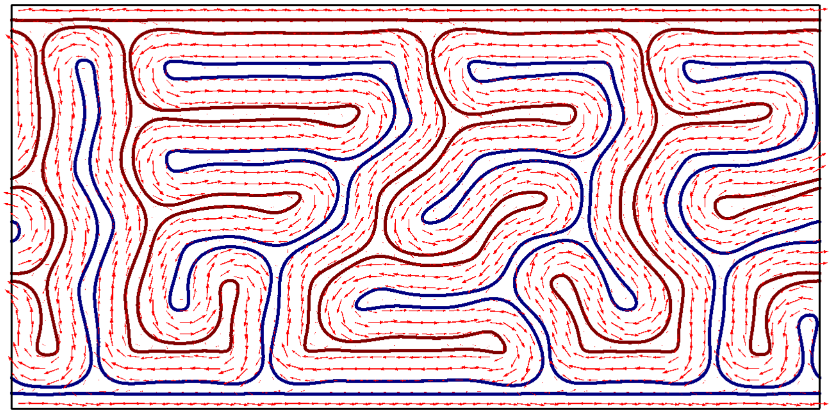

Figure 1: Numerical solution for the gradient flow associated with (1.7) obtained in COMSOL [3]. The arrows represent the director u, the blue and the red curves are level sets $u_3 = -0.92$ and $u_3 = 0.92$, respectively. The simulation was started from the initial condition $u = (\sin(7\pi y/2), 0, \cos(7\pi y/2))$ with the axis y of the twist oriented in a vertical direction and $y \in [-1, 1]$. The director is assumed to be oriented to the right and to the left on the top and the bottom boundaries, respectively. Periodic boundary conditions are imposed on vertical components of the boundary. Here $N = 10$, $L = 1$, and $\varepsilon = 0.005$.

$\mathcal{A}_\alpha$ *such that*

$$\limsup_{\varepsilon \to 0} \frac{\|\rho_{\varepsilon,M} - 1\|_{L^\infty(0,1)}}{\varepsilon} < \infty, \tag{1.10}$$

$$\lim_{\varepsilon \to 0} \theta'_{\varepsilon,M} = 2\pi M + \alpha \text{ uniformly in } x \in [0,1], \tag{1.11}$$

and

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_{\varepsilon,M}) = \frac{L}{2}\left(2\pi(M - N) + \alpha\right)^2. \tag{1.12}$$

This is proven in Section 2. Through Corollary 3.5 which readily follows from the Γ-convergence result below, we will ultimately find that in some parameter regimes, corresponding to α small and $M = N$, the local minimizers of Theorem 1.1 turn out in fact to be global minimizers. However, when $M \neq N$ or when $M = N$ but α exceeds a critical value, they will not.

Section 3 contains our principal result of this investigation, namely the identification of the Γ-limit of E_ε. The key feature of this analysis is that in the $\varepsilon \to 0$ limit, energy bounded sequences may exhibit a jump in phase. In order to gain or release twist in an energetically expedient manner, the modulus of such a sequence may plunge towards zero on a small set, effecting a Modica-Mortola type transition from modulus ≈ 1 down to 0 and then back. Over this small interval where the modulus ≈ 0, the phase is 'free' to jump any amount at minimal cost.

In light of this mechanism, our candidate for a limiting functional will be finite for $u \in H^1((0,1) \setminus J; S^1)$ where J is a jump set consisting of a finite collection of points, say $0 < x_1 < x_2 < \ldots < x_k < 1$ for some non-negative integer k, along with perhaps $x = 0$ and/or $x = 1$ depending on whether or not the traces of u satisfy the desired boundary conditions inherited from E_ε; that is, we include $x = 0$ in J only if $u(0^+) \neq 1$ and we include $x = 1$ in J only if $u(1^-) \neq e^{i\alpha}$. For such a u we will assume J is the minimal such set of points, meaning that if any point in $J \cap (0,1)$ were eliminated, the function u would no longer represent an H^1 function in the compliment of the smaller set of points. In particular, if $u \in H^1((0,1))$ and has the proper traces, then $J = \emptyset$.

Then we define $E_0 : L^2\big((0,1);\mathbb{R}^2\big) \to \mathbb{R}$ via

$$E_0(u) := \begin{cases} \dfrac{L}{2}\displaystyle\int_0^1 (u_1\, u_2' - u_2\, u_1' - 2\pi N)^2\, dx + \frac{2\sqrt{2}}{3}\mathcal{H}^0(J) & \text{if } u \in H^1((0,1) \setminus J; S^1) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{1.13}$$

Here $\mathcal{H}^0$ refers to zero-dimensional Hausdorff measure, i.e. counting measure.

We will establish:

Theorem 1.2. $\{E_\varepsilon\}$ Γ*-converges to* E_0 *in* $L^2\big((0,1);\mathbb{R}^2\big)$.

We also establish a compactness result for energy bounded sequences in Theorem 3.1.

Here we wish to emphasize the parallels between the present problem of high cost twist and the previous study [8] of high cost splay. Comparing the Γ-limits (1.6) and (1.13) we see that the former consists of bulk splay plus jump cost while the latter takes the form of bulk twist plus jump cost. One distinction, however, is that in the high twist model, the *size* of the phase jump does not affect the energetic cost.

As a consequence of Theorem 1.2, we demonstrate in Theorem 3.4 and Corollary 3.5 that in certain parameter regimes depending on L and α, global energy minimizers with jumps are energetically favorable. Indeed, this is the most dramatic effect of the assumption of disparate elastic constants present in our model. The relatively expensive cost of twist leads the global minimizer of (1.1), which of course is necessarily smooth, to rapidly change its phase, a process that can only be achieved with finite energetic cost by having the modulus simultaneously plunge towards zero.

In Section 4 we establish an energy barrier between the local minimizers of different winding numbers exposed in Theorem 1.1, cf. Theorem 4.1. This readily leads to the existence of saddle points in Theorem 4.2 via the Mountain Pass Theorem, thus filling out the energy landscape for E_ε.

Finally, in Section 5 we investigate the energy (5.1) motivated by studies of so-called twist bend nematics, where twisting of the director occurs at much shorter scales than in cholesterics [17]. Here we model this situation by tying the pitch (or the period of the twist) $1/N$ to the Ginzburg-Landau parameter ε so that twisting "averages out" in the limit $\varepsilon \to 0$. We show in Theorem 5.2 that, in fact, the weak limit of uniformly energy bounded director fields is equal to zero but we are nonetheless able to recover some information about fine scale behavior of these fields. Then in Theorems 5.3 and 5.4 we establish Γ-convergence in this setting.

2. Global and local minimizers that stay bounded away from zero

We begin with the observation for problem (1.2)-(1.3) that a global minimizer exists for fixed $\varepsilon > 0$.

Theorem 2.1. *For each fixed $\varepsilon > 0$ there exists a minimizer of E_ε within the class $\mathcal{A}_\alpha$.*

Proof. Existence follows readily from the direct method as follows. Suppressing the ε-dependence, let $\{u^j\} = \{(u_1^j, u_2^j)\}$ denote a minimizing sequence:

$$E_\varepsilon(u_1^j, u_2^j) \to m := \{\inf E_\varepsilon(u) : \ u \in \mathcal{A}_\alpha\}.$$

Compactness of a minimizing sequence follows from the immediate energy bounds

$$\int_0^1 \left|u^{j\,\prime}\right|^2 dx < C, \quad \int_0^1 \left|u^j\right|^4 dx < C, \quad \int_0^1 \left(u_1^j u_2^{j\,\prime} - u_2^j u_1^{j\,\prime}\right)^2 dx < C.$$

So, in particular we have a uniform H^1-bound on $\{u^j\}$. Thus, up to subsequences, we get uniform (in fact Holder) convergence of $u^j \to \bar{u} = (\bar{u}_1, \bar{u}_2)$, and $u^{j\,\prime} \rightharpoonup \bar{u}'$ weakly in $L^2((0,1))$ for some $\bar{u} \in \mathcal{A}_\alpha$.

Turning to the issue of lower-semicontinuity, we note that verification for the first two terms in E_ε is standard. For the third term we observe that

$$u_1^j u_2^{j\,\prime} - u_2^j u_1^{j\,\prime} \rightharpoonup \bar{u}_1 \bar{u}_2{}' - \bar{u}_2 \bar{u}_1{}' \text{ weakly in } L^2,$$

through the pairing of weak L^2 and uniform convergence.

Then we have

$$\int_0^1 (u_1^j\, u_2^{j\,\prime} - u_2^j\, u_1^{j\,\prime} - 2\pi N)^2\, dx =$$
$$\int_0^1 (u_1^j\, u_2^{j\,\prime} - u_2^j\, u_1^{j\,\prime})^2\, dx - 4\pi N \int_0^1 \left(u_1^j\, u_2^{j\,\prime} - u_2^j\, u_1^{j\,\prime}\right) dx + 4\pi^2 N^2.$$

The middle term is continuous given the strong convergence of u^j to $\bar{u}$. For the first term, we appeal to the lower-semicontinuity of the L^2 norm under weak L^2 convergence. Thus, $E_\varepsilon(\bar{u}) = m$. □

It turns out that characterization of the global minimizer in the case where $\alpha = 0$, so that the boundary conditions are simply $u(0) = u(1) = 1$, is much simpler than when $\alpha \in (0, 2\pi)$. In particular, we have the following result.

Theorem 2.2. *Let u_ε denote a global minimizer of E_ε within the admissible class $\mathcal{A}_0$. Then $\rho_\varepsilon(x) := |u_\varepsilon(x)|$ converges to 1 uniformly on $[0,1]$ as $\varepsilon \to 0$.*

Proof. We proceed by contradiction and assume that for some $\delta > 0$ there exists a sequence $\varepsilon_j \to 0$ and values $x_j \in [0,1]$ such that

$$\rho_{\varepsilon_j}(x_j) \le 1 - \delta.$$

The case where $\rho_{\varepsilon_j}(x_j) \ge 1 + \delta$ is handled similarly.

We begin with the observation that

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(e^{i2\pi N x}) = 2(\pi N)^2 \varepsilon. \tag{2.1}$$

It then follows that for some $C_0 > 0$ independent of ε one has

$$\int_0^1 (\rho_\varepsilon')^2 \ + \rho_\varepsilon^4 \, dx < C_0,$$

which in turn implies a bound of the form

$$\|\rho_\varepsilon\|_{H^1(0,1)} < C_1 = C_1(C_0). \quad \text{Hence,} \quad \|\rho_\varepsilon\|_{C^{0,1/2}(0,1)} < C_1.$$

Then invoking the Hölder bound above, we have

$$|\rho_\varepsilon(x) - \rho_\varepsilon(x_j)| \le C_1 \, |x - x_j|^{1/2}$$

and so for $|x - x_j| \le \left(\frac{\delta}{2C_1}\right)^2$ one would have

$$\rho_\varepsilon(x) \le \rho_\varepsilon(x_j) + C_1 \, |x - x_j|^{1/2} \le 1 - \frac{\delta}{2}.$$

This in turn would imply

$$\begin{aligned} E_\varepsilon(u_\varepsilon) \ge \frac{1}{4\varepsilon} \int_0^1 (\rho_\varepsilon^2 - 1)^2 \, dx \qquad &\ge \frac{1}{4\varepsilon} \int_{\left\{x:\, |x - x_j| \le \left(\frac{\delta}{2C_1}\right)^2\right\}} (\rho_\varepsilon^2 - 1)^2 \, dx \\ &\ge \frac{\delta^4}{64 C_1^2 \varepsilon}. \end{aligned}$$

This cannot hold in light of (2.1) for $\varepsilon < \varepsilon_0$ where

$$\varepsilon_0 = \frac{\delta^2}{8\sqrt{2} C_1 \pi N}.$$

□

Next we turn to the construction of local minimizers of E_ε within the class $\mathcal{A}_\alpha$ for $\alpha \in [0, 2\pi)$, namely the proof of Theorem 1.1. Like the global minimizers constructed for the case $\alpha = 0$ in Theorem 2.2, the modulus of these local minimizers will converge uniformly to 1 as $\varepsilon \to 0$.

Proof of Theorem 1.1: To capture these local minimizers we will rephrase our problem by switching to polar coordinates via the substitution

$$u_1 = \rho \cos \theta, \quad u_2 = \rho \sin \theta.$$

The boundary conditions corresponding to (1.3) are

$$\rho(0) = 1 = \rho(1), \quad \theta(0) = 0,\ \theta(1) = 2\pi M + \alpha \quad \text{for some integer } M > 0. \tag{2.2}$$

We find that in these variables,

$$E_\varepsilon = E_\varepsilon(\rho,\theta) = \int_0^1 \frac{\varepsilon}{2}\big((\rho')^2 + \rho^2(\theta')^2\big) + \frac{1}{4\varepsilon}(\rho^2-1)^2 + \frac{L}{2}(\rho^2\theta' - 2\pi N)^2\,dx.$$

We will minimize $E_\varepsilon(\rho,\theta)$ subject to (2.2) via a constrained minimization procedure. To this end, for any number $\rho_0 \in (0,1)$ we introduce the admissible class

$$\mathcal{H}_{\rho_0} := \{\rho \in H^1(0,1):\ \rho(0) = 1 = \rho(1),\ \rho(x) \geq \rho_0 \text{ on } [0,1]\} \tag{2.3}$$

and for any positive integer M and any $\alpha \in [0,2\pi)$ we denote

$$\mathcal{H}_{M,\alpha} := \{\theta \in H^1(0,1):\ \theta(0) = 0,\ \theta(1) = 2\pi M + \alpha\}. \tag{2.4}$$

We note that for each fixed $\varepsilon > 0$ and $\rho_0 \in (0,1)$, the direct method provides for a minimizing pair $(\rho_{\varepsilon,M}, \theta_{\varepsilon,M})$ to the constrained problem:

$$\mu_{\varepsilon,M} := \inf_{\rho\in\mathcal{H}_{\rho_0},\,\theta\in\mathcal{H}_{M,\alpha}} E_\varepsilon(\rho,\theta). \tag{2.5}$$

The only point to be made here is that the lower bound $\rho_j \geq \rho_0$ on a minimizing sequence $\{\rho_j, \theta_j\}$ allows for H^1 control of $\{\theta_j\}$. Also the H^1 control on $\{\rho_j\}$ yields uniform convergence of a subsequence so that the constraint is satisfied by the limiting $\rho_{\varepsilon,M}$.

We remark for later use that $\mu_{\varepsilon,M}$ is bounded independent of ε since

$$\mu_{\varepsilon,M} \leq E_\varepsilon(1, (2\pi M + \alpha)x) = \frac{L}{2}\left(2\pi(M-N) + \alpha\right)^2 + O(\varepsilon) \tag{2.6}$$

We will now argue that for any integer $M > 0$ and any $\rho_0 \in (0,1)$, these solutions to the constrained problem in fact satisfy $\rho_{\varepsilon,M}(x) > \rho_0$ for all $x \in [0,1]$ when ε is sufficiently small. Hence, they correspond to H^1-local minimizers of $E_\varepsilon(u)$ subject to the boundary conditions (1.3) since the representation $u_{\varepsilon,M} = \rho_{\varepsilon,M}e^{i\theta_{\varepsilon,M}}$ is global.

CLAIM: For any positive integer M, any $\alpha \in [0,2\pi)$, and any $\rho_0 \in (0,1)$ we have

$$\rho_{\varepsilon,M}(x) > \rho_0 \text{ for all } x \in [0,1] \quad \text{provided } \varepsilon \text{ is sufficiently small.} \tag{2.7}$$

To pursue this claim, we first observe that since the constraint falls only on $\rho_{\varepsilon,M}$, this minimizing pair $(\rho_{\varepsilon,M},\theta_{\varepsilon,M})$ must satisfy

$$\lim_{t\to 0^+}\frac{E_\varepsilon\big(\rho_{\varepsilon,M}+tf,\theta_{\varepsilon,M}\big)-E_\varepsilon\big(\rho_{\varepsilon,M},\theta_{\varepsilon,M}\big)}{t}\geq 0, \tag{2.8}$$

for all $f\in H_0^1(0,1)$ such that $f(x)\geq 0$ on $[0,1]$, and

$$\frac{d}{dt}_{t=0}E_\varepsilon\left(\rho_{\varepsilon,M},\theta_{\varepsilon,M}+t\psi\right)=0\quad\text{for all }\psi\in H_0^1(0,1). \tag{2.9}$$

Computing these quantities we find that (2.8) takes the form

$$\int_0^1 \varepsilon\rho'_{\varepsilon,M}f'+\Big(\varepsilon\rho_{\varepsilon,M}\big(\theta'_{\varepsilon,M}\big)^2+\frac{1}{\varepsilon}\left(\rho_{\varepsilon,M}^2-1\right)\rho_{\varepsilon,M}$$
$$-2L\left(2\pi N-\rho_{\varepsilon,M}^2\theta'_{\varepsilon,M}\right)\rho_{\varepsilon,M}\theta'_{\varepsilon,M}\Big)\,f\,dx\geq 0 \tag{2.10}$$

for all nonnegative $f\in H_0^1(0,1)$, and (2.9) takes the form

$$\left[\left(\varepsilon\theta'_{\varepsilon,M}-L\left(2\pi N-\rho_{\varepsilon,M}^2\theta'_{\varepsilon,M}\right)\right)\rho_{\varepsilon,M}^2\right]'=0. \tag{2.11}$$

Thus,

$$\big(\varepsilon\theta'_{\varepsilon,M}-L(2\pi N-\rho_{\varepsilon,M}^2\theta'_{\varepsilon,M})\big)\rho_{\varepsilon,M}^2=C_\varepsilon\quad\text{for some constant }C_\varepsilon, \tag{2.12}$$

allowing us to solve for $\theta'_{\varepsilon,M}$ to find

$$\theta'_{\varepsilon,M}=\frac{2\pi NL\rho_{\varepsilon,M}^2+C_\varepsilon}{L\rho_{\varepsilon,M}^4+\varepsilon\rho_{\varepsilon,M}^2}. \tag{2.13}$$

Integrating (2.13) over the interval $[0,1]$ and using the boundary conditions on $\theta_{\varepsilon,M}$ we obtain a formula for C_ε:

$$C_\varepsilon=\frac{2\pi M+\alpha-2\pi LN\int_0^1(L\rho_{\varepsilon,M}^2+\varepsilon)^{-1}\,dx}{\int_0^1(L\rho_{\varepsilon,M}^4+\varepsilon\rho_{\varepsilon,M}^2)^{-1}\,dx}. \tag{2.14}$$

Now by (2.6),

$$\int_0^1(\rho_{\varepsilon,M}^2-1)\left|\rho'_{\varepsilon,M}\right|\,dx\leq\sqrt{2}\int_0^1\frac{\varepsilon}{2}(\rho'_{\varepsilon,M})^2+\frac{1}{4\varepsilon}(\rho_{\varepsilon,M}^2-1)^2\,dx\leq\sqrt{2}\,\mu_{\varepsilon,M}.$$

Since $\rho_{\varepsilon,M}(0)=1$, it then follows from (2.6) and this total variation bound that $\rho_{\varepsilon,M}$ is bounded above uniformly in ε. Thus, by (2.14), the same is true of $|C_\varepsilon|$.

Next we use (2.13) to find that

$$\theta'_{\varepsilon,M} - \left(\frac{2\pi NL + C_\varepsilon}{L+\varepsilon}\right) = \frac{2\pi NL\rho^2_{\varepsilon,M} + C_\varepsilon}{L\rho^4_{\varepsilon,M} + \varepsilon\rho^2_{\varepsilon,M}} - \left(\frac{2\pi NL + C_\varepsilon}{L+\varepsilon}\right)$$
$$= \left(\frac{2\pi NL^2\rho^2_{\varepsilon,M} + C_\varepsilon\left[L(1+\rho^2_{\varepsilon,M}) + \varepsilon\right]}{\rho^2_{\varepsilon,M}(L\rho^2_{\varepsilon,M}+\varepsilon)(L+\varepsilon)}\right)(1-\rho^2_{\varepsilon,M}) =: \Lambda_\varepsilon(1-\rho^2_{\varepsilon,M})$$

where $|\Lambda_\varepsilon| \leq C = C(N,M,L)$ independent of ε by the uniform bounds on C_ε and $\rho_{\varepsilon,M}$. Hence,

$$\int_0^1 \left|\theta'_{\varepsilon,M} - \left(\frac{2\pi NL + C_\varepsilon}{L+\varepsilon}\right)\right| \leq C\int_0^1 (1-\rho^2_{\varepsilon,M})\,dx$$
$$\leq 2C\sqrt{\varepsilon}\left(\int_0^1 \frac{1}{4\varepsilon}(1-\rho^2_{\varepsilon,M})^2\,dx\right)^{1/2} \leq 2C\sqrt{\mu_{\varepsilon,M}}\sqrt{\varepsilon}. \tag{2.15}$$

Since

$$2\pi M + \alpha = \int_0^1 \left(\theta'_{\varepsilon,M} - \left(\frac{2\pi NL + C_\varepsilon}{L+\varepsilon}\right)\right)\,dx + \frac{2\pi NL + C_\varepsilon}{L+\varepsilon}$$

we can then invoke (2.15) to conclude that

$$C_\varepsilon = 2\pi L(M-N) + L\alpha + O(\sqrt{\varepsilon}). \tag{2.16}$$

Substituting this back into (2.13) we find

$$\theta'_{\varepsilon,M} = \frac{2\pi LM + L\alpha + 2\pi LN(\rho^2_{\varepsilon,M}-1)}{L\rho^4_{\varepsilon,M} + \varepsilon\rho^2_{\varepsilon,M}} + O(\sqrt{\varepsilon}). \tag{2.17}$$

With these estimates we can now establish Claim (2.7).

In light of the boundary conditions, we need only consider $x \in (0,1)$. First, suppose by contradiction, that $\{x : \rho_{\varepsilon,M} = \rho_0\}$ contains an isolated point $x_0 \in (0,1)$. Since the obstacle in (2.5) is smooth, it follows from standard regularity theory of obstacle problems (see e.g. [18]) that $\rho_{\varepsilon,M}$ makes $C^{1,1}$ contact with the obstacle $y(x) \equiv 1$. However, we also have that $\rho_{\varepsilon,M}$ satisfies the Euler-Lagrange equation on either side of x_0, that is,

$$\varepsilon\rho''_{\varepsilon,M} = \varepsilon\rho_{\varepsilon,M}(\theta'_{\varepsilon,M})^2 + \frac{1}{\varepsilon}(\rho^2_{\varepsilon,M}-1)\rho_{\varepsilon,M} - 2L(2\pi N - \rho^2_{\varepsilon,M}\theta'_{\varepsilon,M})\rho_{\varepsilon,M}\theta'_{\varepsilon,M} \tag{2.18}$$

cf. (2.10). Consequently the limits $x \to x_0^+$ and $x \to x_0^-$ agree for $\rho''_{\varepsilon,M}(x)$ so we find that in fact $\rho_{\varepsilon,M} \in C^2$ in a neighborhood of x_0 with

$$\rho''_{\varepsilon,M}(x_0) = \varepsilon(\theta'_{\varepsilon,M}(x_0))^2 + \frac{1}{\varepsilon}(\rho_0^2 - 1)\rho_0 - 2L(2\pi N - \theta'_{\varepsilon,M}(x_0))\theta'_{\varepsilon,M}(x_0).$$

Invoking (2.17) evaluated at $x = x_0$, we see

$$\theta'_{\varepsilon,M} \sim \frac{2\pi M + \alpha + 2\pi N(\rho_0^2 - 1)}{\rho_0^4} + O(\sqrt{\varepsilon}) \tag{2.19}$$

so that

$$\rho''_{\varepsilon,M}(x_0) \sim \frac{1}{\varepsilon}(\rho_0^2 - 1)\rho_0 + O(1) \tag{2.20}$$

But since $\rho_{\varepsilon,M}$ has a minimum at x_0, this contradicts the requirement that $\rho''_{\varepsilon,M}(x_0) \geq 0$ when ε is sufficiently small.

Next we suppose by way of contradiction that $\{x : \rho_{\varepsilon,M} = \rho_0\}$ contains an interval $I \subset [0,1]$. Fix a smooth non-negative function f compactly supported in I. Then by (2.10) we must have

$$\int_I \left(\varepsilon(\theta'_{\varepsilon,M})^2 + \frac{1}{\varepsilon}(\rho_0^2 - 1)\rho_0 - 2L(2\pi N - \theta'_{\varepsilon,M})\theta'_{\varepsilon,M} \right) f\,dx \geq 0,$$

again leading to a contradiction for ε small. Claim (2.7) is established and the local minimality of $u_{\varepsilon,M}$ follows.

We remark in passing that for the case $M < N$, one can establish the stronger statement that in fact $\rho_{\varepsilon,M}(x) > 1$ for all $x \in (0,1)$ by choosing $\rho_0 = 1$ in the definition of the constrained set (2.3). Then the same contradiction argument works with (2.19) replaced by

$$\theta'_{\varepsilon,M} \sim 2\pi M + \alpha + O(\sqrt{\varepsilon})$$

and (2.20) replaced by

$$\rho''_{\varepsilon,M}(x_0) \sim -2L\big(2\pi(N - M) - \alpha\big)\big(2\pi M + \alpha\big) + O(\sqrt{\varepsilon}).$$

Finally, in light of the uniform in ε bound on $\theta'_{\varepsilon,M}$ provided by (2.17), we observe that for any fixed values of M and N, the minimizing $\rho_{\varepsilon,M}$ must satisfy (1.10), since otherwise, a presumed maximum of $\rho_{\varepsilon,M}$ at x_0 that is bigger than 1 or a presumed minimum that is less than 1 would violate (2.18). Then applying (1.10) to (2.17), we obtain (1.11) as well. We then may conclude that

$$\begin{aligned} \liminf_{\varepsilon\to 0} E_\varepsilon(\rho_{\varepsilon,M}, \theta_{\varepsilon,M}) &\geq \liminf_{\varepsilon\to 0} \frac{L}{2}\int_0^1 (\rho_{\varepsilon,M}^2\theta'_{\varepsilon,M} - 2\pi N)^2\,dx \\ &= \frac{L}{2}\left(2\pi(M - N) + \alpha\right)^2, \end{aligned}$$

and so (1.12) follows, in view of (2.6). □

3. Γ-convergence of E_ε

As we shall see, the local minimizers described in Theorem 1.1 and established in the previous section are also global minimizers only in certain parameter regimes. In order to fill out the characterization of global minimizers in all parameter regimes, we will turn to the machinery of Γ-convergence and establish Theorem 1.2.

In this section we will also prove the following compactness result.

Theorem 3.1. *If $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies*

$$E_\varepsilon(u_\varepsilon) \leq C_0 < \infty, \tag{3.1}$$

then there exists a function $u \in H^1((0,1)\setminus J'; S^1)$ where J' is a finite, perhaps empty, set of points in $(0,1)$ such that along a subsequence $\varepsilon_\ell \to 0$ one has

$$u_{\varepsilon_\ell} \to u \text{ in } L^2\big((0,1);\mathbb{R}^2\big). \tag{3.2}$$

Furthermore, writing $u(x) = e^{i\theta(x)}$ for $\theta \in H^1((0,1)\setminus J')$, we have that for every compact set $K \subset\subset (0,1)\setminus J'$, there exists an $\varepsilon_0(K) > 0$ such that for every $\varepsilon_\ell < \varepsilon_0$ one has $|u_{\varepsilon_\ell}| > 0$ on K and there is a lifting whereby $u_{\varepsilon_\ell}(x) = \rho_{\varepsilon_\ell}(x)e^{i\theta_{\varepsilon_\ell}(x)}$ on K, with

$$\theta_{\varepsilon_\ell} \rightharpoonup \theta \text{ weakly in } H^1_{loc}\big((0,1)\setminus J'\big). \tag{3.3}$$

Remark 3.2. *It is not necessarily the case that J' is minimal for u; that is, it can happen that $u \in H^1((0,1)\setminus J; S^1)$ for some proper subset $J \subset J'$ and in that case it is the minimal such set J which one uses to evaluate the Γ-limit E_0 at u. However, one cannot guarantee the validity of (3.3) with J' replaced by such a minimal J. For example, in a neighborhood of, say, $x = 1/2$ whose size shrinks with ε, an energy-bounded sequence $\{u_\varepsilon\}$ could undergo a rapid jump in phase by 2π while the modulus of u_ε plunges to zero–or even stays positive but very small– in this neighborhood. Then the limiting u could have well-behaved lifting across $x = 1/2$ while for all $\varepsilon > 0$, the function u_ε would not.*

Remark 3.3. *The appearance of a jump set contribution to the Γ-limit E_0 is associated with the cost of a Modica-Mortola type transition layer for the modulus from value 1 down to 0 and* *back, accompanied by a rapid shift in the phase. If one instead considers a three-component model for $u = (u_1(x), u_2(x), u_3(x))$ then such a phase shift can be achieved with asymptotically vanishing cost by plunging $u_2(x)^2 + u_3(x)^2$ to zero while compensating with $u_1(x)$ to keep $|u| \approx 1$. In this way, twist can be smoothly added or subtracted while decreasing energy–in, particular, without the cost of a Modica-Mortola type transition. In the absence of such an energy barrier, we believe this leads to an absence of local minimizers with states eventually 'melting' under a gradient flow to global minimizers given asymptotically by* (3.31) *of Theorem 3.4 below. In fact, the degeneracy in such a three-component model is worse than just this: If one introduces cylindrical coordinates so that $(u_1, u_2, u_3) = (\rho\cos\theta, \rho\sin\theta, u_3)$ and then one writes $\rho = \cos\phi$ and $u_3 = \sin\phi$ for some angle $\phi(x)$, a three-component version of E_ε would take the form*

$$\frac{1}{2}\int_0^1 \varepsilon\phi'(x)^2 + \varepsilon\theta'(x)^2 + L\big(\cos^2\phi(x)\theta'(x) - 2\pi N\big)^2\,dx.$$

Note then that for ε small there is no control on ϕ', nor is there control on θ' when $\phi \approx \pi/2$.

We now present the proofs of Theorem 1.2 and Theorem 3.1. We will begin with the proof of Theorem 3.1 since elements of it will be called upon in the proof of Theorem 1.2.

Proof of Theorem 3.1. We fix an integer $q \geq 2$ and consider a sequence satisfying (3.1). Denoting $\rho_\varepsilon := |u_\varepsilon|$, since u_ε is H^1, we have that ρ_ε is continuous and we may define the open sets

$$\mathcal{I}_\varepsilon := \{y \in [0,1] : \rho_\varepsilon(y) > 1 - 2^{-q}\}.$$

As open sets on the real line, each is a countable disjoint union of open intervals

$$\mathcal{I}_\varepsilon = \cup_{m=1}^{\infty}\mathcal{I}_m^\varepsilon = \cup_{m=1}^{\infty}(a_m^\varepsilon, b_m^\varepsilon),$$

with

$$\rho_\varepsilon(a_m^\varepsilon) = \rho_\varepsilon(b_m^\varepsilon) = 1 - 2^{-q}.$$

Note that by the energy bound (3.1),

$$\mathbb{1}_{\mathcal{I}_\varepsilon} \to \mathbb{1}_{(0,1)} \text{ in } L^1((0,1)). \tag{3.4}$$

Now we consider the open sets

$$(0,1) \setminus \overline{\mathcal{I}}_\varepsilon = \mathring{\mathcal{I}}_\varepsilon^c.$$

and similarly decompose $\mathring{\mathcal{I}}_\varepsilon^c$ into a countable union of intervals

$$\cup_{m=1}^\infty (b_m^\varepsilon, a_{m+1}^\varepsilon).$$

Now some of the intervals $(b_m^\varepsilon, a_{m+1}^\varepsilon)$ could contain a point c_m^ε such that

$$\rho(c_m^\varepsilon) = 2^{-q},$$

and we collect those intervals and label them $(b_{m_j}^\varepsilon, a_{m_j+1}^\varepsilon)$, where j belongs to an index set S_ε. A priori S_ε could be finite or infinite. Let B_ε be the union of these "bad intervals." These are the intervals over which it is possible that a limit of u_ε exhibits a jump discontinuity. We first prove that the number of these intervals is finite and bounded uniformly in ε. We observe that

$$\begin{aligned} C_0 &\geq \int_{B_\varepsilon} \frac{\varepsilon}{2} |u_\varepsilon'|^2 + \frac{1}{4\varepsilon}(|u_\varepsilon|^2 - 1)^2 \, dx \\ &\geq \sum_{j \in S_\varepsilon} \int_{b_{m_j}^\varepsilon}^{c_{m_j}^\varepsilon} \frac{\varepsilon}{2}(\rho_\varepsilon')^2 + \frac{1}{4\varepsilon}(\rho_\varepsilon^2 - 1)^2 \, dy + \int_{c_{m_j}^\varepsilon}^{a_{m_j+1}^\varepsilon} \frac{\varepsilon}{2}(\rho_\varepsilon')^2 + \frac{1}{4\varepsilon}(\rho_\varepsilon^2 - 1)^2 \, dy \\ &\geq \sum_{j \in S_\varepsilon} \int_{b_{m_j}^\varepsilon}^{c_{m_j}^\varepsilon} \frac{|\rho_\varepsilon'||\rho_\varepsilon^2 - 1|}{\sqrt{2}} \, dy + \int_{c_{m_j}^\varepsilon}^{a_{m_j+1}^\varepsilon} \frac{|\rho_\varepsilon'||\rho_\varepsilon^2 - 1|}{\sqrt{2}} \, dy \\ &\geq \sum_{j \in S_\varepsilon} \sqrt{2} \int_{2^{-q}}^{1-2^{-q}} |z^2 - 1| \, dz. \end{aligned} \tag{3.5}$$

Rearranging (3.5) yields an estimate on the size of S_ε:

$$\mathcal{H}^0(S_\varepsilon) \leq \left(\sqrt{2} \int_{2^{-q}}^{1-2^{-q}} |z^2 - 1| \, dz \right)^{-1} C_0. \tag{3.6}$$

Next, on $(0,1) \setminus B_\varepsilon$, we observe that $\rho_\varepsilon \geq 2^{-q}$, which allows us define a lifting of u_ε as $\rho_\varepsilon e^{i\theta_\varepsilon}$ and to find a positive constant C_1 such that

$$\begin{aligned} \int_{(0,1)\setminus B_\varepsilon} (\theta_\varepsilon')^2 \, dy &\leq C_1 + C_1 \int_{(0,1)\setminus B_\varepsilon} \frac{L}{2}(\rho_\varepsilon^2 \theta_\varepsilon' - 2\pi N)^2 \, dy \\ &\leq C_1 + C_1 E_\varepsilon(u_\varepsilon) \leq C_1 + C_1 C_0 < \infty. \end{aligned} \tag{3.7}$$

On each of the (finitely many) intervals comprising $(0,1) \setminus B_\varepsilon$ we may choose our lifting such that the value of θ_ε at, say, the left endpoint of the interval lies in $[0, 2\pi)$ and from the fundamental theorem of calculus and Cauchy-Schwarz it then follows from (3.7) that $\|\theta_\varepsilon\|_{L^\infty((0,1)\setminus B_\varepsilon)}$ is bounded uniformly in ε by a constant depending on C_0 and C_1. Consequently, we have a bound of the form

$$\|\theta_\varepsilon\|_{H^1((0,1)\setminus B_\varepsilon)} < C_2, \tag{3.8}$$

for some constant C_2 independent of ε.

Now we are going to obtain a subsequence of ε approaching zero along which the bad intervals converge to a finite set of points. To this end, we start with the sequence of all the endpoints of the left-most subinterval in B_ε and extract a subsequential limit, calling it x_1. Then, along this subsequence of $\varepsilon's$, we move on to the left endpoints of the second subinterval of B_ε, and passing to a further subsequence, arrive at a limit point x_2, etc. In light of (3.6), this procedure generates a finite number of points $x_1 < x_2 \ldots < x_k$ in $[0,1]$. (If this procedure ever yields $x_j = x_{j+1}$ then we drop x_{j+1} from this list.) In this manner, we arrive at a subsequence, $\varepsilon_\ell \to 0$ such that:

$$\mathcal{H}^0(S_{\varepsilon_\ell}) \text{ is independent of } \ell \text{ and equal to some fixed } k \in \mathbb{N},$$

and, in light of (3.4), the subintervals of B_{ε_ℓ} collapse to these k points as $\varepsilon_\ell \to 0$; that is

$$B_{\varepsilon_\ell} \to J' := \{x_1, x_2, \ldots, x_k\} \text{ as } \varepsilon_\ell \to 0. \tag{3.9}$$

.

If we then fix any finite union of closed intervals $K_1 \subset\subset [0,1] \setminus J'$, it follows from (3.9) that

$$K_1 \cap B_{\varepsilon_\ell} = \emptyset \tag{3.10}$$

for $\varepsilon < \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(K_1)$ small enough. Therefore, u_{ε_ℓ} has a lifting on the various intervals comprising $K_1 \cap B_{\varepsilon_\ell}$ and invoking (3.8), we have, after passing to a further subsequence, (with notation suppressed) that

$$\theta_{\varepsilon_\ell} \rightharpoonup \theta \quad \text{in } H^1(K_1), \; \theta_{\varepsilon_\ell} \to \theta \quad \text{in } L^2(K_1) \tag{3.11}$$

for some $\theta \in H^1(K_1)$ such that

$$\|\theta\|_{H^1(K_1)} \leq C_2. \tag{3.12}$$

Repeating this procedure on a nested sequence of sets

$$K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset K_p \subset\subset \cdots [0,1] \setminus J' \tag{3.13}$$

which exhaust $[0,1] \setminus J'$, and passing to further subsequences via a diagonalization procedure we arrive at a subsequence (still denoted here by $\varepsilon_\ell \to 0$) such that (3.3) holds for some $\theta \in H^1\big((0,1) \setminus J'\big)$.

Finally, we define $u \in H^1\big((0,1) \setminus J'; S^1\big)$ via $u(x) := e^{i\theta(x)}$ and verify (3.2). The uniform bound (3.1) implies that $\rho_\varepsilon \to 1$ in $L^2((0,1))$ and also that

$$C_0 \geq \int_0^1 \left|1 - |\rho_\varepsilon|^2\right| |\rho_\varepsilon'| \, dx \geq \left| \int_{x_\varepsilon}^y (1-\rho_\varepsilon^2)\rho_\varepsilon' \, dx \right| \tag{3.14}$$

for any $y \in (0,1)$ where $x_\varepsilon \in (0,1)$ is any point selected such that, say, $\rho_\varepsilon(x_\varepsilon) \leq 2$. It follows that $\|\rho_\varepsilon\|_{L^\infty(0,1)} < M$ for some $M = M(C_0)$ independent of ε. Hence, for any $\eta > 0$ if we select a compact set $K \subset [0,1] \setminus J'$ such that $|[0,1] \setminus K| < \eta$, we can appeal to (3.11) to conclude (3.2) since

$$\limsup_{l\to\infty} \int_0^1 |u_{\varepsilon_l} - u|^2 \, dx \leq \limsup_{l\to\infty} \int_K |u_{\varepsilon_l} - u|^2 \, dx + \limsup_{l\to\infty} \int_{K^c} |u_{\varepsilon_l} - u|^2 \, dx$$
$$\leq \limsup_{l\to\infty} \int_{K^c} |u_{\varepsilon_l} - u|^2 \, dx \leq 2\int_{K^c} (M^2+1) \, dx < 2(M^2+1)\eta.$$

□

Proof of Theorem 1.2. We will first assume that $u_\varepsilon \to u$ in $L^2\big((0,1);\mathbb{R}^2\big)$ and establish the inequality

$$\liminf E_\varepsilon(u_\varepsilon) \geq E_0(u). \tag{3.15}$$

To this end, we may certainly assume that

$$\liminf E_\varepsilon(u_\varepsilon) \leq C_0 < \infty \quad \text{for some } C_0 > 0,$$

since otherwise (3.15) is immediate. Let $\{u_{\varepsilon_\ell}\}$ be a subsequence which achieves the limit inferior. As in (3.5) in the proof of Theorem 3.1, we can then assert that for any integer $q \geq 2$

and up to a further subsequence for which we suppress the notation, one has the lower bound

$$\liminf_{\ell\to\infty}\int_{B^q_{\varepsilon_\ell}}\frac{\varepsilon_\ell}{2}\left|{u_{\varepsilon_\ell}}'\right|^2+\frac{1}{4\varepsilon_\ell}(|u_{\varepsilon_\ell}|^2-1)^2\,dx\geq\left(\sqrt{2}\int_{2^{-q}}^{1-2^{-q}}|z^2-1|\,dz\right)\mathcal{H}^0(J^q) \tag{3.16}$$

along with

$$\theta_{\varepsilon_\ell}\rightharpoonup\theta\quad\text{in } H^1_{\rm loc}\big((0,1)\setminus J^q\big). \tag{3.17}$$

Here we have emphasized the q dependence to write J^q for the finite set of points in $[0,1]$ and $B^q_{\varepsilon_\ell}$ for the set of 'bad intervals' collapsing to J^q over which $|u_{\varepsilon_\ell}|$ dips from values of $1-2^{-q}$ to 2^{-q}. Next, we note that for any two positive integers $q_1<q_2$ one has the containment $B^{q_2}_\varepsilon\subset B^{q_1}_\varepsilon$ and so, for any sequence $\varepsilon_\ell\to0$, the finite set of points arising as the limit of $B^{q_2}_{\varepsilon_\ell}$ must be a subset of the corresponding limit of the finite collection of collapsing intervals comprising $B^{q_1}_{\varepsilon_\ell}$. Also, since the limiting phase θ of u will be in H^1_{loc} of the complement of any such limit of bad intervals, and since J is assumed to be the minimal one, we have

$$\mathcal{H}^0(J)\leq\mathcal{H}^0(J^q)<C_1\quad\text{for any } q<\infty,$$

for some $C_1=C_1(C_0)$ in light of (3.6). Thus, passing to the limit $q\to\infty$ in (3.16) gives

$$\lim_{\ell\to\infty}\int_0^1\frac{\varepsilon_\ell}{2}\left|{u_{\varepsilon_\ell}}'\right|^2+\frac{1}{4\varepsilon_\ell}(|u_{\varepsilon_\ell}|^2-1)^2\,dx\geq\frac{2\sqrt{2}}{3}\mathcal{H}^0(J). \tag{3.18}$$

Turning to the lower-semi-continuity of the twist term, we can repeat the argument of Theorem 3.1 to obtain that, again up to a further subsequence which we do not notate,

$$\theta_{\varepsilon_\ell}\rightharpoonup\theta\quad\text{in } H^1_{\rm loc}\big((0,1)\setminus\tilde{J}^q\big), \tag{3.19}$$

where $\tilde{J}_q$ is the finite set of points in $[0,1]$ which is the limit of bad intervals $\tilde{B}^q_{\varepsilon_\ell}$ where $|u_{\varepsilon_\ell}|\leq1-2^{-q-1}$ and dips from $1-2^{-q-1}$ to $1-2^{-q}$. Of course, it could turn out that $\tilde{J}^q=\emptyset$, in which case the convergence of $\theta_{\varepsilon_\ell}$ to θ occurs weakly in $H^1_{\rm loc}\big((0,1)\big)$. We also note that

$$\rho^2_\varepsilon\to1 \text{ in } L^2((0,1)), \tag{3.20}$$

which combined with (3.19) implies that for any $K\subset\subset(0,1)\setminus J^q$

$$\lim_{\ell\to\infty}\int_K\rho^2_{\varepsilon_\ell}\theta'_{\varepsilon_\ell}\,dx=\int_K\theta'\,dx. \tag{3.21}$$

For convenience, we now introduce the following notation for the twist term:

$$\mathcal{T}(u) := u_1\, u_2' - u_2\, u_1'.$$

Then, using (3.21), the weak convergence of $\theta'_{\varepsilon_\ell}$ to θ', and the fact that $\rho_{\varepsilon_\ell} \geq 1 - 2^{-q}$ on K for large ℓ, we can estimate

$$\begin{aligned}
\liminf_{\ell\to\infty} \int_0^1 (\mathcal{T}(u_{\varepsilon_\ell}) - 2\pi N)^2\, dx &\geq \liminf_{\ell\to\infty} \int_K \rho_{\varepsilon_\ell}^4 (\theta'_{\varepsilon_\ell})^2 - 4\pi N \rho_{\varepsilon_\ell}^2 \theta'_{\varepsilon_\ell} + 4\pi^2 N^2\, dx \\
&\geq \liminf_{\ell\to\infty} \int_K (1-2^{-q})^4 (\theta'_{\varepsilon_\ell})^2 - 4\pi N \rho_{\varepsilon_\ell}^2 \theta'_{\varepsilon_\ell} + 4\pi^2 N^2\, dx \\
&\geq \int_K (1-2^{-q})^4 (\theta')^2 - 4\pi N \theta' + 4\pi^2 N^2\, dx. \qquad (3.22)
\end{aligned}$$

Choosing larger and larger K and using that $\mathcal{H}^0(J^q) < \infty$, we find

$$\liminf_{\ell\to\infty} \int_0^1 (\mathcal{T}(u_{\varepsilon_\ell}) - 2\pi N)^2\, dx \geq \int_0^1 (1-2^{-q})^4 (\theta')^2 - 4\pi N \theta' + 4\pi^2 N^2\, dx.$$

Finally, sending $q \to \infty$ yields

$$\liminf_{\ell\to\infty} \int_0^1 (\mathcal{T}(u_{\varepsilon_\ell}) - 2\pi N)^2\, dx \geq \int_0^1 (\mathcal{T}(u) - 2\pi N)^2\, dx. \qquad (3.23)$$

Combining (3.18) with (3.23) completes the proof of lower semi-continuity.

Moving on now to the construction of the recovery sequence for any $u \in L^2\big((0,1);\mathbb{R}^2\big)$, if $u \notin H^1((0,1)\setminus J; S^1)$ for any finite set J, then $E_0(u) = \infty$ and taking the trivial recovery sequence $v_\varepsilon \equiv u$ will suffice.

Thus we may assume $u \in H^1((0,1)\setminus J; S^1)$ for a finite set J and our task is to construct a sequence $\{v_\varepsilon\} \subset H^1\big((0,1);\mathbb{R}^2\big)$ such that

$$v_\varepsilon \to u \text{ in } L^2\big((0,1);\mathbb{R}^2\big) \quad \text{and} \quad \lim_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon) = E_0(u). \qquad (3.24)$$

In case the traces of u satisfy the desired boundary conditions for admissibility in E_ε, that is, in case $u(0^+) = 1$ and $u(1^-) = e^{i\alpha}$ so that $x = 0$ and $x = 1$ do not lie in J, our construction will take the form $v_\varepsilon = \rho_\varepsilon u$ for a sequence $\{\rho_\varepsilon\} \subset H^1\big((0,1);[0,1]\big)$ to be described below. We first describe the construction for this case and then discuss how it is slightly altered in case 0 or 1 lie in J. Denoting J by $\{x_1, x_2, \ldots, x_k\}$ with then $J \subset (0,1)$ by assumption, we then take ρ_ε to satisfy the following conditions:

(i) ρ_ε is smooth on $[0,1]$.

(ii) $\rho_\varepsilon \equiv 0$ on $(x_j - \varepsilon^2, x_j + \varepsilon^2)$.

(iii) ρ_ε makes a standard Modica-Mortola style transition from 1 to 0 on I_j^1, an interval of size say $O(\sqrt{\varepsilon})$ with right endpoint $x_j - \varepsilon^2$, and makes a transition from 0 back to 1 on an interval of size $O(\sqrt{\varepsilon})$ with left endpoint $x_j + \varepsilon^2$ that we denote by I_j^2, cf. [14].

(iv) $\rho_\varepsilon \equiv 1$ on $(0,1) \setminus \cup_j (I_j^1 \cup (x_j - \varepsilon^2, x_j + \varepsilon^2) \cup I_j^2)$.

In case either $u(0^+) \neq 1$ or $u(1^-) \neq e^{i\alpha}$ so that 0 and/or 1 lies in J, this procedure must be slightly altered near the endpoints. For example, if $u(0^+) \neq 1$ then one requires ρ_ε to make a Modica-Mortola style transition from 1 down to 0 on the interval $[0, \sqrt{\varepsilon}]$, $\rho_\varepsilon \equiv 0$ on $[\sqrt{\varepsilon}, \sqrt{\varepsilon} + \varepsilon^2]$ and a Modica-Mortola transition from 0 back up to 1 on $[\sqrt{\varepsilon} + \varepsilon^2, 2\sqrt{\varepsilon} + \varepsilon^2]$. Then we define

$$\theta_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in [0, \sqrt{\varepsilon} + \varepsilon^2/2) \\ \theta & \text{if } x > \sqrt{\varepsilon} + \varepsilon^2/2, \end{cases}$$

where $u = e^{i\theta}$, and take $v_\varepsilon = \rho_\varepsilon e^{i\theta_\varepsilon}$. A similar recipe is taken in a neighborhood of $x = 1$ in case $u(1^-) \neq e^{i\alpha}$.

Computing the transition energy of such a construction is classical and can be found in e.g. [13, 14]. One finds from conditions (ii)-(iv), that

$$\int_0^1 \frac{\varepsilon}{2}(\rho_\varepsilon')^2 + \frac{1}{4\varepsilon}(\rho_\varepsilon^2 - 1)^2\,dx \to \frac{2\sqrt{2}}{3}\mathcal{H}^0(J).$$

Furthermore, since $\theta \in H^1\big((0,1) \setminus J\big)$, and $\rho_\varepsilon \to 1$ in $L^2\big((0,1)\big)$ it is easily seen that

$$\lim_{\varepsilon \to 0} \int_0^1 \frac{\varepsilon}{2}\rho_\varepsilon^2(\theta_\varepsilon')^2\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{L}{2}\int_0^1 \mathcal{T}(v_\varepsilon)\,dx = \frac{L}{2}\mathcal{T}(u)\,dx.$$

The proof of (3.24) is complete. □

We observe that for $u \in H^1((0,1) \setminus J; S^1)$, one has

$$E_0(u) = \frac{L}{2}\int_0^1 (\theta' - 2\pi N)^2\,dx + \frac{2\sqrt{2}}{3}\mathcal{H}^0(J), \tag{3.25}$$

where $u = e^{i\theta}$ for $\theta \in H^1((0,1) \setminus J)$. Using this formulation, it is then straight-forward to identify the global minimizers of the Γ-limit, and consequently the limits of global minimizers of E_ε as well:

Theorem 3.4.

(i) When

$$L\alpha^2 < \frac{4\sqrt{2}}{3} \text{ and } \alpha \in [0, \pi] \tag{3.26}$$

the global minimizer of E_0 is given by the function

$$u(x) = e^{i(2\pi N + \alpha)x} \tag{3.27}$$

which has constant twist and no jumps.

(ii) When

$$L(2\pi - \alpha)^2 < \frac{4\sqrt{2}}{3} \text{ and } \alpha \in [\pi, 2\pi) \tag{3.28}$$

the global minimizer of E_0 is given by the function

$$u(x) = e^{i(2\pi(N-1)+\alpha)x} \tag{3.29}$$

which again has constant twist and no jumps.

(iii) When

$$L\alpha^2 > \frac{4\sqrt{2}}{3} \text{ and } L(2\pi - \alpha)^2 > \frac{4\sqrt{2}}{3} \tag{3.30}$$

the global minimizers of E_0 are given by the one-parameter set of functions

$$u(x) = \begin{cases} e^{i2\pi N x} & \text{if } x < x_0, \\ e^{i(2\pi N x + \alpha)} & \text{if } x > x_0, \end{cases} \tag{3.31}$$

for any $x_0 \in (0,1)$. These have one jump and twist $2\pi N$ away from the jump.

Since any limit of global minimizers of a Γ-converging sequence must itself be a global minimizer of the Γ-limit, one immediately concludes the following result based on Theorem 3.4 and the compactness result Theorem 3.1:

Corollary 3.5. *Let* $\{u_\varepsilon\}$ *denote a family of minimizers of* E_ε *subject to the boundary conditions* (1.3)*. Then if* (3.30) *holds, we have* $u_\varepsilon \to u$ *in* L^2 *for some* u *in the one-parameter family given by* (3.31)*, while if* (3.26) *or* (3.28) *holds, there will be a subsequence* $u_{\varepsilon_j} \to u$ *in* L^2 *with* $u = e^{i(2\pi N+\alpha)x}$ *or* $u = e^{i(2\pi(N-1)+\alpha)x}$*, respectively.*

Remark 3.6. *It is in the case where* $L\alpha^2 > \frac{4\sqrt{2}}{3}$ *and* $L(2\pi-\alpha)^2 > \frac{4\sqrt{2}}{3}$ *that one really sees the most dramatic effect of the assumption of disparate elastic constants present in our model. The relatively expensive cost of twist leads the global minimizer of* E_ε*, which of course is necessarily smooth, to rapidly change its phase, a process that can only be achieved with small energetic cost by having the modulus simultaneously plunge towards zero.*

Remark 3.7. *We have not attempted to determine the optimal location of the jump location* x_0 *for minimizers of* E_ε *in scenario* (3.31)*. We suspect this might entail much higher order energetic considerations–perhaps even at an exponentially small order–but we are not sure.*

Proof of Theorem 3.4. When $\alpha = 0$ then clearly the global minimizer is uniquely given by $u = e^{i2\pi Nx}$ since it has zero energy. Consider then the case $\alpha \in (0, 2\pi)$. By selecting any point $x_0 \in (0, 1)$, and taking u to be given by (3.31), we see that there is always a competitor with one jump having energy given simply by $\frac{2\sqrt{2}}{3}$. Any competitor jumping more than once has energy no lower than twice that value. On the other hand, minimization of E_0 among competitors with $J = \emptyset$ is standard, since criticality implies θ' is constant. Given the boundary conditions, this requires $u = e^{i(2\pi M+\alpha)x}$ for some $M \in \mathbb{Z}$ to be determined. The energy of such a u is $\frac{L}{2}(2\pi(M-N)+\alpha)^2$. Since $\alpha \in (0, 2\pi)$, the minimum over M is $\frac{L}{2}(2\pi(N-N)+\alpha)^2 = \frac{L}{2}\alpha^2$ if $\alpha < 2\pi - \alpha$ and $\frac{L}{2}(2\pi(N-1-N)+\alpha)^2 = \frac{L}{2}(2\pi-\alpha)^2$ if $2\pi - \alpha < \alpha$. Comparing these two energies to that of the one-jump competitors in (3.31), the theorem follows. We note that if $\alpha = \pi$ in this regime, there are two global minimizers. □

Next we state a result on local minimizers of the Γ-limit. These functions are the $\varepsilon \to 0$ limit of the non-vanishing local minimizers captured in Theorem 1.1.

Theorem 3.8. *For any positive integer* M *the function* $u = e^{i(2\pi M+\alpha)x}$ *is an isolated* L^2*-local minimizer of* E_0*.*

By invoking Theorem 4.1 of [9], one can conclude from Theorem 3.8 and Theorem 1.2 that there exist local minimizers of E_ε for ε small that converge to this isolated local minimizer of E_0. This provides for an alternative proof of existence for these local minimizers to the one given in Proposition 1.1. However, the approach in Theorem 1.1 yields much more detailed information on the structure of these functions via (1.10), (1.11) and (1.12).

Proof of Theorem 3.8. We fix a positive integer M and a number $\alpha \in [0, 2\pi)$. We will consider the case $M < N$. The case $M \geq N$ is similar. Of course, in case $M = N$ and (3.26) holds, then in fact $u = e^{i(2\pi N+\alpha)x}$ is the global minimizer, as was already addressed in Theorem 3.4. Let us denote $\theta_M := 2\pi Mx + \alpha x$. In light of (3.25), our goal is to show that for some $\delta > 0$, one has $E_0(\theta) > E_0(\theta_M)$ whenever $\theta \in H^1\big((0,1) \setminus J; S^1\big)$ for some finite set J provided $0 < \|\theta - \theta_M\|_{L^2(0,1)} < \delta$.

We begin with the easiest case where $J = \emptyset$ and where $\theta(0) = \theta_M(0)$, $\theta(1) = \theta_M(1)$. Writing $v := \theta - \theta_M$, we calculate

$$\begin{aligned} E_0(\theta) - E_0(\theta_M) &= \frac{L}{2}\int_0^1 \big(\theta_M' + v' - 2\pi N\big)^2\,dx - \frac{L}{2}\int_0^1 \big(\theta_M' - 2\pi N\big)^2\,dx \\ &= 2\pi L(M - N + \alpha/2\pi)\int_0^1 v'\,dx + \frac{L}{2}\int_0^1 (v')^2\,dx = \frac{L}{2}\int_0^1 (v')^2\,dx > 0, \end{aligned}$$

since in the case under consideration, $v(0) = 0 = v(1)$.

Now we turn to the general case where $J \neq \emptyset$. To this end, consider a competitor $\theta \in H^1\big(\cup_{j=1}^{\ell}(a_j, b_j)\big)$ where then $J = (0,1) \setminus \cup_{j=1}^{\ell}(a_j, b_j)$, along with perhaps $x = 0$ and/or $x = 1$, depending upon whether a competitor satisfies the boundary conditions. Thus, depending upon the boundary conditions of a competitor, we note that

$$\mathcal{H}^0(J) \in \{\ell - 1, \ell, \ell + 1\}. \tag{3.32}$$

Again we introduce $v := \theta - \theta_M$ and after a rearrangement of the indices, we suppose that for $j = 1, 2 \ldots, \ell'$, one has the condition

$$k_j := v(b_j) - v(a_j) < \frac{\sqrt{2}}{6\pi N} := k_0, \tag{3.33}$$

while for $j = \ell' + 1, \ldots, \ell$, the opposite inequality holds. We allow for the possibility that either $\ell' = 0$ or $\ell' = \ell$.

Then we again calculate the energy difference $E_0(\theta) - E_0(\theta_M)$ by splitting up the sum as follows:

$$E_0(\theta) - E_0(\theta_M) = \frac{2\sqrt{2}}{3}\mathcal{H}^0(J) + 2\pi L(M - N + \alpha/2\pi)\sum_{j=1}^{\ell'}\int_{a_j}^{b_j} v'\,dx$$

$$+2\pi L(M - N + \alpha/2\pi)\sum_{j=\ell'+1}^{\ell}\int_{a_j}^{b_j} v'\,dx + \frac{L}{2}\sum_{j=1}^{\ell}\int_{a_j}^{b_j}(v')^2\,dx$$

$$> \frac{2\sqrt{2}}{3}\mathcal{H}^0(J) - 2\pi LNk_0\ell'$$

$$-2\pi LN\sum_{j=\ell'+1}^{\ell} k_j + \frac{L}{2}\sum_{j=\ell'+1}^{\ell}\int_{a_j}^{b_j}(v')^2\,dx$$

$$> \sum_{j=\ell'+1}^{\ell}\big(- 2\pi LNk_j + \frac{L}{2}\int_{a_j}^{b_j}(v')^2\,dx\big), \tag{3.34}$$

in light of (3.33) and (3.32).

If $\ell' = \ell$ then the last sum is vacuous and the proof is complete. If not, then we now fix any $j \in \{\ell' + 1, \ldots, \ell\}$ for which the reverse inequality to (3.33) holds, and observe that

$$\delta_j^2 := \int_{a_j}^{b_j} v^2\,dx \geq \int_{(a_j,b_j)\cap\{|v|>k_j/4\}} v^2\,dx \geq \frac{k_j^2}{16}\text{meas}\,\big((a_j, b_j) \cap \{|v| > k_j/4\}\big). \tag{3.35}$$

Also,

$$\frac{k_j}{4} < \int_{(a_j,b_j)\cap\{|v|>k_j/4\}} |v'|\,dx$$

$$\leq \text{meas}\,\big((a_j, b_j) \cap \{|v| > k_j/4\}\big)^{1/2}\big(\int_{(a_j,b_j)\cap\{|v|>k_j/4\}}(v')^2\,dx\big)^{1/2}.$$

Combining this with (3.35) yields the inequality

$$\int_{a_j}^{b_j}(v')^2\,dx \geq \frac{k_j^4}{256\delta_j^2}$$

which we now substitute into (3.34) to conclude that

$$E_0(\theta) - E_0(\theta_M) > \sum_{j=\ell'+1}^{\ell}\big(\frac{Lk_j^4}{512\delta_j^2} - 2\pi LNk_j\big). \tag{3.36}$$

Choosing δ (which we recall denotes $(\|v\|_{L^2(0,1)})$ such that

$$\delta^2 < \frac{k_0^3}{1024\pi N},$$

and using that $\delta_j \le \delta$ while $k_j \ge k_0$ for all j, we obtain positivity of the right-hand side of (3.36).

□

4. An energy barrier leading to saddle points

The local minimizers provided by Theorem 1.1 can be viewed as the least energy critical points of E_ε within a given degree or winding number class given by the amount of twist. One might anticipate then that to pass continuously from one of these classes to another requires both the emergence of a zero in the order parameter and the expenditure of a certain amount of energy. What is more, one might expect the presence of saddle points in some sense interspersed between the distinct degree classes. That is the content of the two results in this section.

In the first theorem we demonstrate that the energy barrier between any two local minimizers u_{ε,M_1} and u_{ε,M_2} with $M_1 \neq M_2$ is at least $\frac{2\sqrt{2}}{3}$ when ε is sufficiently small. To this end, given a $\Lambda > 0$, we define the energy sublevel set

$$E_\varepsilon^\Lambda := \{u \in \mathcal{A}_\alpha : E_\varepsilon(u) < \Lambda\}.$$

We have the following:

Theorem 4.1. *Let $M_1, M_2 \in \mathbb{N}$ be such that $M_1 \neq M_2$ and assume that u_{ε,M_1} and u_{ε,M_2} are local minimizers of E_ε as obtained in Theorem 1.1. Suppose that*

$$\gamma^\varepsilon : [0,1] \to \mathcal{A}_\alpha \quad \textit{with } \gamma^\varepsilon(0) = u_{\varepsilon,M_1} \textit{ and } \gamma^\varepsilon(1) = u_{\varepsilon,M_2} \tag{4.1}$$

is a continuous path in $\mathcal{A}_\alpha$ that connects u_{ε,M_1} and u_{ε,M_2}. Fix an $h > 0$ and set $\Lambda_h := 2\pi^2 L(N - M_1 - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} - h$. There exists an $\varepsilon_h > 0$ such that the curve γ^ε leaves the set $E_\varepsilon^{\Lambda_h}$ whenever $\varepsilon < \varepsilon_h$.

Proof. Fix any $h \in (0,1)$ and any curve γ^ε satisfying (4.1). Denote

$$\gamma^\varepsilon(t) := u_t^\varepsilon \,(= u_t^\varepsilon(x)\,) \text{ and } |u_t^\varepsilon| := \rho_t^\varepsilon$$

for every $t \in [0,1]$. The non-vanishing functions $e^{-i\alpha x}u_{\varepsilon,M_1}$ and $e^{-i\alpha x}u_{\varepsilon,M_2}$ have winding numbers M_1 and M_2 respectively on $[0,1]$ and so $u_t^\varepsilon(x)$ has to vanish for some $x \in (0,1)$ and $t \in (0,1)$. Since γ^ε is continuous and $u_t^\varepsilon(\cdot)$ is a continuous function for every $t \in [0,1]$, it follows that, given any $\delta \in (0,1/2)$, we can find $t_\delta^\varepsilon \in (0,1)$ such that $\min_{x\in(0,1)} \rho_{t_\delta^\varepsilon}^\varepsilon(x) = \delta$ and the winding number for $e^{-i\alpha x}u_{t_\delta^\varepsilon}^\varepsilon$ is still equal to M_1.

Now suppose by way of contradiction that $\gamma^\varepsilon([0,1]) \subset E_\varepsilon^{\Lambda_h}$. We would like to estimate $E_\varepsilon(u_{t_\delta^\varepsilon}^\varepsilon)$. First, by minimizing $E_\varepsilon(\rho_{t_\delta^\varepsilon}^\varepsilon e^{i\theta})$ over $\theta \in \mathcal{H}_{M_1,\alpha}$, (cf. (2.4)), note that the same approach that led to (2.17) can be followed to show that there exists a $\bar{\theta}_\varepsilon \in \mathcal{H}_{M_1,\alpha}$ such that

$$\bar{\theta}_\varepsilon' = \frac{2\pi L M_1 + L\alpha + 2\pi L N((\rho_{t_\delta^\varepsilon}^\varepsilon)^2 - 1)}{L(\rho_{t_\delta^\varepsilon}^\varepsilon)^4 + \varepsilon(\rho_{t_\delta^\varepsilon}^\varepsilon)^2} + O(\sqrt{\varepsilon}) \tag{4.2}$$

on $(0,1)$, and necessarily

$$E_\varepsilon\left(\rho_{t_\delta}^\varepsilon e^{i\bar{\theta}_\varepsilon}\right) \le E_\varepsilon\left(u_{t_\delta}^\varepsilon\right). \tag{4.3}$$

Using the standard Modica-Mortola arguments, we now have

$$\int_0^1 \frac{\varepsilon}{2}((\rho_{t_\delta^\varepsilon}^\varepsilon)')^2 + \frac{1}{4\varepsilon}\left((\rho_{t_\delta^\varepsilon}^\varepsilon)^2 - 1\right)^2 dx \ge c(\delta),$$

where $\lim_{\delta\to 0} c(\delta) = \frac{2\sqrt{2}}{3}$. Further, we can appeal to (4.2)-(4.3) and the assumption that $\gamma^\varepsilon(t_\delta^\varepsilon) \in E_\varepsilon^{\Lambda_h}$ to show that

$$\begin{aligned}
\frac{L}{2}\int_0^1 \left(2\pi N - (\rho_{t_\delta^\varepsilon}^\varepsilon)^2\bar{\theta}_\varepsilon'\right)^2 dx &= \frac{L}{2}\int_0^1 \left(2\pi N - \frac{2\pi L M_1 + L\alpha + 2\pi L N\left((\rho_{t_\delta^\varepsilon}^\varepsilon)^2 - 1\right)}{L(\rho_{t_\delta^\varepsilon}^\varepsilon)^2 + \varepsilon}\right)^2 dx + O(\sqrt{\varepsilon}) \\
&= 2\pi^2 L \int_0^1 \left(\frac{N - M_1 - \alpha/2\pi + (\varepsilon/L)N}{(\rho_{t_\delta^\varepsilon}^\varepsilon)^2 + \varepsilon/L}\right)^2 dx + O(\sqrt{\varepsilon}) \\
&= 2\pi^2 L(N - M_1 - \alpha/2\pi)^2 + O(\sqrt{\varepsilon}).
\end{aligned} \tag{4.4}$$

It then follows from (4.3) that

$$E_\varepsilon(u_{t_\delta^\varepsilon}^\varepsilon) \ge 2\pi^2 L(N - M_1 - \alpha/2\pi)^2 + c(\delta) + O(\sqrt{\varepsilon}).$$

It is clear, however, that one can select a positive δ sufficiently small, and then an $\varepsilon_h > 0$ such that the last expression exceeds Λ_h whenever $\varepsilon < \varepsilon_h$. □

The energy threshold provided by Theorem 4.1 leads to a straight-forward application of the Mountain Pass Theorem to establish saddle points for E_ε.

Theorem 4.2. *For every positive integer M and $\alpha \in [0, 2\pi)$ there exists a critical point v_ε of E_ε within the class $\mathcal{A}_\alpha$. Furthermore, the corresponding critical value $E_\varepsilon(v_\varepsilon)$ satisfies the asymptotic condition*

$$E_\varepsilon(v_\varepsilon) \to 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} \quad \text{as } \varepsilon \to 0. \tag{4.5}$$

Proof. First, we note that the arguments in Theorem 4.1 can easily be adapted with the same energy threshold to a curve that connects the states $U_M := e^{i(2\pi M + \alpha)x}$ and $U_{M_1} := e^{i(2\pi M_1 + \alpha)x}$ for any two positive integers M and M_1. Fixing $\varepsilon > 0$, one defines the potential critical value c_ε via

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} E_\varepsilon(\gamma(t)),$$

where Γ_ε is the set of continuous curves γ such that

$$\gamma : [0, 1] \to \mathcal{A}_\alpha \quad \text{with } \gamma(0) = U_M \text{ and } \gamma(1) = U_{M+1}. \tag{4.6}$$

Beginning with the case $M < N$ we have that

$$E_\varepsilon(U_M) \sim 2\pi^2 L(N - M - \alpha/2\pi)^2$$

while

$$E_\varepsilon(U_{M+1}) \sim 2\pi^2 L(N - M - 1 - \alpha/2\pi)^2,$$

so that, in particular, $E_\varepsilon(U_{M+1}) < E_\varepsilon(U_M)$. Then the implication of Theorem 4.1 is that E_ε exhibits the requisite mountain pass structure since for any $h > 0$ one has

$$\max_{t \in [0,1]} E_\varepsilon(\gamma(t)) \geq 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} - h > E_\varepsilon(U_M) > E_\varepsilon(U_{M+1}) \tag{4.7}$$

for any $\gamma \in \Gamma_\varepsilon$, provided ε is sufficiently small.

Subtracting off the boundary conditions by writing any competitor $u \in \mathcal{A}_\alpha$ as $u = \tilde{u} + \ell(x)$ where $\ell(x) := 1 + x\big(e^{i\alpha} - 1\big)$, we can work in the space $H_0^1\big((0, 1)\big)$. It remains to verify the Palais-Smale condition. Under assumptions

$$E_\varepsilon(\tilde{u}_k + \ell) < C_0 \quad \text{and} \quad \|\delta E_\varepsilon(\tilde{u}_k + \ell)\| \to 0 \quad \text{as } k \to 0, \tag{4.8}$$

for $\{\tilde{u}_k\} \subset H_0^1\big((0,1)\big)$, it immediately follows from the uniform energy bound that after passing to a subsequence (with notation suppressed), one has

$$\tilde{u}_k \rightharpoonup \tilde{u}_{\varepsilon,M} \text{ weakly in } H^1 \quad \text{and} \quad \tilde{u}_k \to \tilde{u}_{\varepsilon,M} \text{ uniformly} \quad \text{as } k \to \infty, \tag{4.9}$$

for some $\tilde{u}_{\varepsilon,M} \in H_0^1$. Then one writes E_ε as the sum of the Allen-Cahn energy and the twist energy, say $E_\varepsilon = I_1 + I_2$ with

$$I_1(\tilde{u}) = \int_0^1 \frac{\varepsilon}{2} |\tilde{u}' + \ell'|^2 + \frac{1}{4\varepsilon}(|\tilde{u} + \ell|^2 - 1)^2 \quad \text{and} \quad I_2(\tilde{u}) := \frac{L}{2}\int_0^1 \mathcal{T}(\tilde{u} + \ell)\, dx.$$

In light of (4.8), we know, in particular, that

$$\delta E_\varepsilon(\tilde{u}_k + \ell; \tilde{u}_k) = \delta I_1(\tilde{u}_k; \tilde{u}_k) + \delta I_2(\tilde{u}_k; \tilde{u}_k) \to 0$$

and

$$\delta E_\varepsilon(\tilde{u}_k + \ell; \tilde{u}_{\varepsilon,M}) = \delta I_1(\tilde{u}_k; \tilde{u}_{\varepsilon,M}) + \delta I_2(\tilde{u}_k; \tilde{u}_{\varepsilon,M}) \to 0$$

as $k \to \infty$. Since for any $v = (v^{(1)}, v^{(2)}) \in H_0^1(0,1)$ we can compute that

$$\delta I_2(\tilde{u}_k; v) = L \int_0^1 (\tilde{u}_k^{(2)\prime} + \ell^{(2)\prime})v^{(1)} - (\tilde{u}_k^{(1)\prime} + \ell^{(1)\prime})v^{(2)}\, dx,$$

it follows from (4.9) that as $k \to \infty$ one has

$$\begin{aligned} &\delta I_2(\tilde{u}_k; \tilde{u}_k) - \delta I_2(\tilde{u}_k; \tilde{u}_{\varepsilon,M}) = \\ &L \int_0^1 \big(\tilde{u}_k^{(2)\prime} + \ell^{(2)\prime}\big)\big(\tilde{u}_k^{(1)} - \tilde{u}_{\varepsilon,M}^{(1)}\big) - \big(\tilde{u}_k^{(1)\prime} + \ell^{(1)\prime}\big)\big(\tilde{u}_k^{(2)} - \tilde{u}_{\varepsilon,M}^{(2)}\big)\, dx \to 0. \end{aligned}$$

Thus,

$$\begin{aligned} &0 = \lim_{k\to\infty} \delta E_\varepsilon(\tilde{u}_k + \ell; \tilde{u}_k) - \delta E_\varepsilon(\tilde{u}_k + \ell; \tilde{u}_{\varepsilon,M}) = \\ &\lim_{k\to\infty} \delta I_1(\tilde{u}_k + \ell; \tilde{u}_k) - \delta I_1(\tilde{u}_k + \ell; \tilde{u}_{\varepsilon,M}) = \\ &\lim_{k\to\infty} \int_0^1 \varepsilon\big(\tilde{u}_k' + \ell'\big)\big(\tilde{u}_k' - \tilde{u}_{\varepsilon,M}'\big) + \frac{1}{\varepsilon}\big(|\tilde{u}_k + \ell|^2 - 1\big)\big(\tilde{u}_k + \ell\big)\big(\tilde{u}_k - \tilde{u}_{\varepsilon,M}\big)\, dx \end{aligned}$$

Since the second term in the last integral vanishes in the limit due to (4.9), the first must as well, from which it follows that $\int_0^1 |\tilde{u}_k'|^2\, dx \to \int_0^1 \left|\tilde{u}_{\varepsilon,M}'\right|^2\, dx$. Hence, the weak H^1-convergence of $\{\tilde{u}_k\}$ has been upgraded to strong convergence, completing the verification of the Palais-Smale condition.

We conclude from the Mountain Pass Theorem that a critical point $v_{\varepsilon,M} := \tilde{u}_{\varepsilon,M} + \ell$ exists with $E_\varepsilon(v_{\varepsilon,M}) = c_\varepsilon$.

Now we turn to the proof of condition (4.5). Again, we know from Theorem 4.1 that for any $h > 0$, one has the inequality (4.7) for ε small enough, so that

$$\liminf c_\varepsilon \geq 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} \tag{4.10}$$

On the other hand, we can build a continuous path $\gamma^\varepsilon : [0,1] \to \mathcal{A}_\alpha$ as follows:

(1) $0 \leq t \leq 1/3$. We write $U_M = e^{i\theta_M}$ with $\theta_M(x) := \big(2\pi M + \alpha\big)x$. Then as t varies between 0 and say 1/3, the modulus gradually depresses towards 0 in a small interval of x-values about $x = 1/2$ via the standard Modica-Mortola construction, so that $\gamma^\varepsilon(1/3) \equiv 0$ for say $1/2 - \varepsilon^2 \leq x \leq 1/2 + \varepsilon^2$. For this interval of t-values one leaves the phase θ_M unchanged. Following the approach used in the proof of Theorem 1.2 for the recovery sequence construction, such a procedure can be executed with

$$E_\varepsilon(\gamma^\varepsilon(t)) \leq 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} + O(\varepsilon) \text{ for each } t \in [0, 1/3).$$

(2) $1/3 \leq t \leq 2/3$. Beginning at $t = 1/3$ we introduce a discontinuity in the phase θ_M at $x = 1/2$. Since $x = 1/2$ lies inside the x-interval where the modulus vanishes for the t-interval $[1/3, 2/3]$, the map γ^ε remains smooth. The process in this interval is that as t increases from $t = 1/3$ to $t = 2/3$, the phase gradually converges to θ_{M+1} and $\theta_{M+1} - 2\pi$ for $x \in [0, 1/2)$ and $x \in (1/2, 1]$, respectively, while leaving the modulus unchanged. Since $M < N$, the $O(1)$ energy contribution of the twist will decrease under this process of increasing phase. As t approaches 2/3, $\gamma^\varepsilon(t)$ will converge to U_{M+1} except for the small interval about $x = 1/2$ where the modulus is depressed. Explicitly, we take the phase, say $\theta^t(x)$, to be given by

$$\theta^t(x) = \begin{cases} \Big(2\pi[M + 3(t - 1/3)] + \alpha\Big)x & \text{for } 0 \leq x < 1/2 \\ \Big(2\pi[M + 3(t - 1/3)] + \alpha\Big)x - 6\pi(t - 1/3) & \text{for } 1/2 < x \leq 1. \end{cases}$$

for $t \in [1/3, 2/3]$. We note that the term $6\pi(t - 1/3)$ is needed to maintain the boundary condition that $\gamma^\varepsilon(t) = e^{i\alpha}$ when $x = 1$ throughout the t-interval $[1/3, 2/3]$. We observe also that the jump discontinuity in the phase closes up once $t = 2/3$.

(3) $2/3 \le t \le 1$. In the time interval $t \in [2/3, 1]$ one smoothly raises the modulus back up to 1 on $1/2 - \varepsilon^2 \le x \le 1/2 + \varepsilon^2$ while leaving the phase unchanged, so that at $t = 1$ one indeed has $\gamma^\varepsilon(1) = U_{M+1}$, as desired. Again, this process decreases energy so that throughout the interval $0 \le t \le 1$ one maintains the estimate

$$E_\varepsilon(\gamma^\varepsilon(t)) \le 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3} + O(\varepsilon).$$

Hence, we conclude that

$$\limsup c_\varepsilon \le \limsup E_\varepsilon(\gamma^\varepsilon) \le 2\pi^2 L(N - M - \alpha/2\pi)^2 + \frac{2\sqrt{2}}{3}$$

and together with (4.10) we arrive at (4.5). □

5. The case of unbounded twist

Finally, we consider the situation of an energy that encourages more and more twist in the $\varepsilon \to 0$ limit. To this end, we replace N in (1.2) by $N_\varepsilon := 1/\varepsilon^\beta$ where β is a positive number chosen less than 1/2 in order to retain an energy bound that is uniform in ε. Thus, we study global and local minimizers of an energy $\tilde{E}_\varepsilon$ given by

$$\tilde{E}_\varepsilon(u) = \int_0^1 \frac{\varepsilon}{2}\,|u'|^2 + \frac{1}{4\varepsilon}(|u|^2 - 1)^2 + \frac{L}{2}(u_1\, u_2' - u_2\, u_1' - 2\pi\varepsilon^{-\beta})^2\, dx, \tag{5.1}$$

again subject to the boundary conditions $u(0) = 1,\ u(1) = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$.

Of course existence of global minimizers for each $\varepsilon > 0$ follows as in Theorem 2.1. One also can establish a version of the local minimizer result Theorem 1.1:

Theorem 5.1. *Fix any positive integer m and any $\alpha \in [0, 2\pi)$. Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exist non-vanishing local minimizers $u_{\varepsilon,\pm} = \rho_{\varepsilon,\pm} e^{i\theta_{\varepsilon,\pm}}$ of $\tilde{E}_\varepsilon$ within the class $\mathcal{A}_\alpha$ such that*

$$\limsup \frac{\|\rho_{\varepsilon,\pm} - 1\|_{L^\infty(0,1)}}{\varepsilon} < \infty \ \textit{as}\ \varepsilon \to 0 \tag{5.2}$$

and (5.3)

$$\theta'_{\varepsilon,\pm} \to 2\pi\left(\lfloor \varepsilon^{-\beta} \rfloor \pm m\right) + \alpha \ \textit{as}\ \varepsilon \to 0 \ \textit{uniformly in}\ x \in [0, 1]. \tag{5.4}$$

Proof. The proof follows along similar lines as the proof of Theorem 1.1. First define $M_\varepsilon^\pm = \lfloor \varepsilon^{-\beta} \rfloor \pm m$. Then one writes competitors for constrained minimization of $\tilde{E}_\varepsilon = \tilde{E}_\varepsilon(\rho,\theta)$ in polar form (ρ,θ) where ρ satisfies (2.3) and $\theta(0) = 0,\ \theta(1) = 2\pi M_\varepsilon^\pm + \alpha$. The requirement $\beta < 1/2$ assures that a version of the uniform energy bound (2.6) still holds. Similarly, a uniform bound on the constant of integration C_ε is achievable as in (2.16), with the bound now depending on m. The rest of the argument is unchanged. $\square$

Next we consider the asymptotic behavior as $\varepsilon \to 0$ of $\tilde{E}_\varepsilon$. Due to the fact that $\varepsilon^{-\beta} \to \infty$ as $\varepsilon \to 0$, we expect that the elements of an energy bounded sequence will oscillate more and more rapidly as $\varepsilon \to 0$.

Theorem 5.2. *Suppose that for some* $0 < \beta < 1/2$, $\{u_\varepsilon\} \subset \mathcal{A}_\alpha$ *satisfies the uniform energy bound*

$$\tilde{E}_\varepsilon(u_\varepsilon) \le C_0 < \infty. \tag{5.5}$$

Then $|u_\varepsilon|^2 \to 1$ *in* $L^2(0,1)$ *and there exists a finite set* $J' \subset (0,1)$ *and a subsequence* $\{u_{\varepsilon_\ell}\}$ *such that for every compact set* $K \subset\subset (0,1) \setminus J'$, *there exists an* $\varepsilon_0(K) > 0$ *such that for every* $\varepsilon_\ell < \varepsilon_0$, *one has* $|u_{\varepsilon_\ell}| > 0$ *on* K *and there is a lifting whereby* $u_{\varepsilon_\ell} = \rho_{\varepsilon_\ell} e^{2\pi i v_{\varepsilon_\ell}/\varepsilon_\ell^\beta}$, *with*

$$v_{\varepsilon_\ell} \to x \text{ strongly in } H^1_{loc}((0,1) \setminus J'). \tag{5.6}$$

In addition, we have

$$u_\varepsilon \rightharpoonup 0 \text{ weakly in } L^2((0,1);\mathbb{C}), \tag{5.7}$$

so that the entire sequence converges weakly to 0.

Proof of Theorem 5.2. By the same argument as the one leading up to (3.6), we can identify finite unions of open intervals B_ε such that on $(0,1) \setminus B_\varepsilon$, $\rho_\varepsilon \ge 1/4$. Also, by restricting to a subsequence $\{\varepsilon_\ell\}$, we can assume that the sets B_{ε_ℓ} collapse to a finite set of points J'. We may therefore define liftings $\theta_{\varepsilon_\ell} : (0,1) \setminus B_{\varepsilon_\ell} \to \mathbb{R}$ such that on each of the finitely many intervals comprising $(0,1) \setminus B_{\varepsilon_\ell}$, the value of $\theta_{\varepsilon_\ell}$ at the left endpoint of an interval is greater than the value of $\theta_{\varepsilon_\ell}$ at the right endpoint of the previous interval, with a difference of no more than

2π. Also, we can without loss of generality suppose that 0 is in the domain of $\theta_{\varepsilon_\ell}$ and set $\theta_{\varepsilon_\ell}(0) = 0$. If we define

$$v_{\varepsilon_\ell} := \frac{\varepsilon_\ell^\beta \theta_{\varepsilon_\ell}}{2\pi}, \tag{5.8}$$

then due to the choice of $\theta_{\varepsilon_\ell}$ on each subinterval of $(0,1) \setminus B_{\varepsilon_\ell}$, we see that from the right endpoint of one subinterval to the left endpoint of the subsequent one,

$$\textit{the value of } v_{\varepsilon_\ell} \textit{ differs by no more than } \varepsilon_\ell^\beta. \tag{5.9}$$

Furthermore, we may rewrite the twist term in terms of v_{ε_ℓ} and employ the uniform energy bound to find that

$$\frac{1}{2}\int_{(0,1)\setminus B_{\varepsilon_\ell}} \frac{L}{\varepsilon_\ell^{2\beta}} \left(\rho_{\varepsilon_\ell}^2 v_{\varepsilon_\ell}' - 1\right)^2 \leq C_0. \tag{5.10}$$

Using (5.10), the fact that $\rho_{\varepsilon_\ell} \geq 1/4$ on $(0,1) \setminus B_{\varepsilon_\ell}$, and the energy bound, we estimate

$$\begin{aligned}
\int_{(0,1)\setminus B_{\varepsilon_\ell}} (v_{\varepsilon_\ell}' - 1)^2\, dx &\leq 2\int_{(0,1)\setminus B_{\varepsilon_\ell}} \left(v_{\varepsilon_\ell}' - \frac{1}{\rho_{\varepsilon_\ell}^2}\right)^2 + \left(\frac{1}{\rho_{\varepsilon_\ell}^2} - 1\right)^2 dx \\
&\leq 2\int_{(0,1)\setminus B_{\varepsilon_\ell}} 4^4\rho_{\varepsilon_\ell}^4 \left(v_{\varepsilon_\ell}' - \frac{1}{\rho_{\varepsilon_\ell}^2}\right)^2 + 4^4\rho_{\varepsilon_\ell}^4 \left(\frac{1}{\rho_{\varepsilon_\ell}^2} - 1\right)^2 dx \\
&\leq C\int_{(0,1)\setminus B_{\varepsilon_\ell}} \left(\rho_{\varepsilon_\ell}^2 v_{\varepsilon_\ell}' - 1\right)^2 + \left(\rho_{\varepsilon_\ell}^2 - 1\right)^2 dx \\
&\leq C\varepsilon^{2\beta}
\end{aligned} \tag{5.11}$$

We conclude from (5.11) that for any $K \subset\subset [0,1] \setminus J'$,

$$v_{\varepsilon_\ell}' \to 1 \text{ in } L^2(K). \tag{5.12}$$

From (5.9), (5.12), and the condition $v_{\varepsilon_\ell}(0) = 0$, we deduce that

$$v_{\varepsilon_\ell} \to x \text{ in } L^\infty(K), \tag{5.13}$$

which ends the proof of (5.6).

To prove (5.7), we must demonstrate that for any $w \in L^2((0,1);\mathbb{C})$,

$$\int_0^1 u_\varepsilon \overline{w}\, dx \to 0, \tag{5.14}$$

where the bar denotes complex conjugation. By the density of step functions in L^2, it is enough to show that for any $\eta > 0$ and interval I,

$$\limsup_{\varepsilon\to 0}\left|\int_I u_\varepsilon\,dx\right| \le \eta. \tag{5.15}$$

We first choose a subsequence u_{ε_ℓ} such that

$$\limsup_{\varepsilon\to 0}\left|\int_I u_\varepsilon\,dx\right| = \lim_{\ell\to\infty}\left|\int_I u_{\varepsilon_\ell}\,dx\right|. \tag{5.16}$$

By restricting to a further subsequence, there exists a finite set J' such that $u_{\varepsilon_\ell} = e^{2\pi i v_{\varepsilon_\ell}}$ off of finite unions of open intervals B_{ε_ℓ} which collapse to J' as described in (5.8)-(5.13). Since $|u_{\varepsilon_\ell}| \to 1$ in L^2, $\{|u_{\varepsilon_\ell}|\}$ is uniformly integrable. Let $\delta > 0$ be such that $|M| < \delta$ implies that $\int_M |u_{\varepsilon_\ell}|\,dx < \eta$. Thus there exist finitely many open intervals $\{B_j\}$ whose union has measure less than δ and contains J' such that for all ℓ,

$$\left|\int_{\cup_j B_j} u_{\varepsilon_\ell}\,dx\right| < \eta. \tag{5.17}$$

Since B_{ε_ℓ} collapse to J', we may safely assume that $B_{\varepsilon_\ell} \subset \cup B_j$ for all ℓ. Now

$$\begin{aligned}\int_{I\setminus\cup_j B_j} u_{\varepsilon_\ell}\,dx &= \int_{I\setminus\cup_j B_j} e^{2\pi i v_{\varepsilon_\ell}\varepsilon_\ell^{-\beta}}\,dx\\ &= \int_{I\setminus\cup_j B_j} e^{2\pi i x\varepsilon_\ell^{-\beta}} e^{2\pi i(v_{\varepsilon_\ell}-x)\varepsilon_\ell^{-\beta}}\,dx\end{aligned} \tag{5.18}$$

We next show that

$$\|e^{2\pi i(v_{\varepsilon_\ell}-x)\varepsilon_\ell^{-\beta}}\|_{W^{1,1}((0,1)\setminus B_{\varepsilon_\ell})} \le C < \infty, \tag{5.19}$$

so that there is a subsequence converging strongly in $L^2(I\setminus \cup B_j)$. The pairing of weak and strong convergence in (5.18) will then allow us to conclude the proof. Using (5.11), we have

$$\int_{(0,1)\setminus B_{\varepsilon_\ell}} |v'_{\varepsilon_\ell} - 1|\varepsilon_\ell^{-\beta}\,dx \le \varepsilon_\ell^{-\beta}|(0,1)\setminus B_{\varepsilon_\ell}|^{1/2}\|v'_{\varepsilon_\ell}-1\|_{L^2} \le C|(0,1)\setminus B_{\varepsilon_\ell}|^{1/2}.$$

In light of (5.9) and $v_{\varepsilon_\ell}(0) = 0$, the uniform L^1 bound on $(v'_{\varepsilon_\ell}-1)\varepsilon_\ell^{-\beta}$ implies that

$$\|(v_{\varepsilon_\ell}-x)\varepsilon_\ell^{-\beta}\|_{W^{1,1}((0,1)\setminus B_{\varepsilon_\ell})} \le C,$$

so that $(v_{\varepsilon_\ell}-x)\varepsilon_\ell^{-\beta}$ are uniformly bounded in $W^{1,1}(I\backslash\cup B_j)$. By the compact embedding of $W^{1,1}$ into L^2, we may extract a subsequence $\{v_{\varepsilon_m}\}$ and L^2 function v_0 such that $(v_{\varepsilon_m}-x)\varepsilon_m^{-\beta}\to v_0$ in $L^2(I\setminus\cup B_j)$. Due to the fact that $e^{2\pi ix}$ is Lipschitz, we have

$$e^{2\pi i(v_{\varepsilon_m}-x)\varepsilon_m^{-\beta}}\to e^{2\pi iv_0}\quad\text{in } L^2(I\setminus\cup B_j). \tag{5.20}$$

Together with the weak L^2 convergence of $e^{2\pi ix\varepsilon_m^{-\beta}}$ to 0 and the strong L^2 convergence of $e^{2\pi i(v_{\varepsilon_m}-x)\varepsilon_m^{-\beta}}$, (5.16)-(5.18) give

$$\begin{aligned}\limsup_{\varepsilon\to0}\left|\int_I u_\varepsilon\,dx\right| &= \lim_{m\to\infty}\left|\int_I u_{\varepsilon_m}\,dx\right|\\ &\le \limsup_{m\to\infty}\left|\int_{I\backslash\cup_j B_j} u_{\varepsilon_m}\,dx\right|+\left|\int_{\cup_j B_j} u_{\varepsilon_m}\,dx\right|\\ &\le \limsup_{m\to\infty}\left|\int_{I\backslash\cup_j B_j} e^{2\pi ix\varepsilon_m^{-\beta}}e^{2\pi i(v_{\varepsilon_m}-x)\varepsilon_m^{-\beta}}\,dx\right|+\eta\\ &=\eta.\end{aligned}$$

This is exactly (5.15), so the proof is complete.

□

We would also like to describe the asymptotic behavior of minimizers in this regime by identifying a limiting problem. As demonstrated in the previous theorem, no meaningful limit can be extracted from simply looking at the sequence $\{u_\varepsilon\}$. Instead, we examine the "microscale" behavior of u_ε by eliminating the excess twist in the limit $\varepsilon\to0$, in the sense that we obtain a limiting asymptotic problem for the rescaled functions

$$w(x):=u(x)e^{-2\pi i\lfloor\varepsilon^{-\beta}\rfloor x}.$$

Here $\lfloor\varepsilon^{-\beta}\rfloor$ denotes the integer part of $\varepsilon^{-\beta}$.

In terms of w, the energy $\tilde{E}_\varepsilon(u)$ is given by

$$\begin{aligned}\tilde{E}_\varepsilon(u)=F_\varepsilon(w):=\int_0^1\frac{\varepsilon}{2}\left|(we^{2\pi i\lfloor\varepsilon^{-\beta}\rfloor x})'\right|^2+\frac{1}{4\varepsilon}(|w|^2-1)^2\\ +\frac{L}{2}(w_1\,w_2'-w_2\,w_1'+|w|^2 2\pi\lfloor\varepsilon^{-\beta}\rfloor-2\pi\varepsilon^{-\beta})^2\,dx.\end{aligned}$$

The boundary conditions imposed on competitors for F_ε are the same as those for $\tilde{E}_\varepsilon$. The asymptotic behavior of minimizers of $\tilde{E}_\varepsilon$ can therefore be completely understood in terms of F_ε, so we pursue an asymptotic limit for F_ε. Let us define the limiting functional as in Section 3, with slightly altered notation to emphasize the dependence on preferred twist:

$$E_{0,A}(w) := \begin{cases} \frac{L}{2}\int_0^1 (w_1\, w_2' - w_2\, w_1' - 2\pi A)^2\, dx + \frac{2\sqrt{2}}{3}\mathcal{H}^0(J) & \text{if } w \in H^1((0,1)\setminus J; S^1) \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that 0 and/or 1 belongs to J depending on whether or not the traces of u satisfy the desired boundary conditions inherited from E_ε; that is, we include $x = 0$ in J only if $u(0^+) \neq 1$ and we include $x = 1$ in J only if $u(1^-) \neq e^{i\alpha}$.

Theorem 5.3. *Let $0 < \beta < 1/2$ and suppose that for a subsequence $\{\varepsilon_\ell\} \to 0$ and some $A \in [0,1]$ we have*

$$\varepsilon_\ell^{-\beta} - \lfloor \varepsilon_\ell^{-\beta} \rfloor \to A.$$

Then $\{F_{\varepsilon_\ell}\}$ Γ-converges to $E_{0,A}$ in $L^2\left((0,1);\mathbb{R}^2\right)$.

We also have the compactness result

Theorem 5.4. *If $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies*

$$\tilde{E}_\varepsilon(u_\varepsilon) = F_\varepsilon(w_\varepsilon) \leq C_0 < \infty, \tag{5.21}$$

and

$$\varepsilon_\ell^{-\beta} - \lfloor \varepsilon_\ell^{-\beta} \rfloor \to A \tag{5.22}$$

for some $0 < \beta < 1/2$, then there exists a function $w \in H^1((0,1)\setminus J'; S^1)$ where J' is a finite, perhaps empty, set of points in $(0,1)$ such that along a subsequence $\varepsilon_\ell \to 0$ one has

$$u_{\varepsilon_\ell} e^{-2\pi i \lfloor \varepsilon_\ell^{-\beta} \rfloor x} = w_{\varepsilon_\ell} \to w \text{ in } L^2\big((0,1);\mathbb{C}\big). \tag{5.23}$$

Furthermore, writing $w(x) = e^{i\theta(x)}$ for $\theta \in H^1((0,1)\setminus J')$, we have that for every compact set $K \subset\subset (0,1)\setminus J'$, there exists an $\varepsilon_0(K) > 0$ such that for every $\varepsilon_\ell < \varepsilon_0$ one has $|u_{\varepsilon_\ell}| = |w_{\varepsilon_\ell}| > 0$ on K and there is a lifting whereby $u_{\varepsilon_\ell}(x)e^{-2\pi i \lfloor \varepsilon_\ell^{-\beta} \rfloor x} = w_{\varepsilon_\ell}(x) = \rho_{\varepsilon_\ell}(x)e^{i\theta_{\varepsilon_\ell}(x)}$ on K, with

$$\theta_{\varepsilon_\ell} \rightharpoonup \theta \text{ weakly in } H^1_{loc}\big((0,1)\setminus J'\big). \tag{5.24}$$

Proof of Theorem 5.4. The proof is based on the proof of Theorem 3.1. First, we estimate that

$$\int_0^1 \frac{\varepsilon}{2} \left|(w_\varepsilon e^{2\pi i \lfloor \varepsilon^{-\beta} \rfloor x})'\right|^2 \, dx = \int_0^1 \frac{\varepsilon}{2} \left|w_\varepsilon' + 2\pi i w \lfloor \varepsilon^{-\beta} \rfloor\right|^2 \, dx$$

$$= \int_0^1 \frac{\varepsilon}{2} \left|w_\varepsilon'\right|^2 \, dx + O(\varepsilon^{1/2-\beta}) \, dx \tag{5.25}$$

for an energy bounded sequence $\{w_\varepsilon\}$. Therefore,

$$F_\varepsilon(w_\varepsilon) = \int_0^1 \frac{\varepsilon}{2} |w_\varepsilon'|^2 + \frac{1}{4\varepsilon}(|w_\varepsilon|^2 - 1)^2 + \frac{L}{2}(\mathcal{T}(w_\varepsilon) + |w_\varepsilon|^2 2\pi \lfloor \varepsilon^{-\beta} \rfloor - 2\pi \varepsilon^{-\beta})^2 \, dx + O(\varepsilon^{1/2-\beta}). \tag{5.26}$$

The rest of the proof follows almost exactly as in Theorem 3.1. Indeed, the only difference between E_ε in that theorem and the right hand side of (5.26) here is the preferred twist $2\pi N$ versus $|w_\varepsilon|^2 2\pi \lfloor \varepsilon^{-\beta} \rfloor - 2\pi \varepsilon^{-\beta}$, respectively. For the purpose of showing compactness, this distinction is immaterial, since it is only the uniform boundedness of the preferred twist $2\pi N$ in L^2 that was used in (3.7) to obtain compactness. Using $\beta < 1/2$, we can estimate

$$\begin{aligned}
\left\| |w_\varepsilon|^2 2\pi \lfloor \varepsilon^{-\beta} \rfloor - 2\pi \varepsilon^{-\beta} \right\|_{L^2} &\le \left\| (|w_\varepsilon|^2 - 1) 2\pi \lfloor \varepsilon^{-\beta} \rfloor \right\|_{L^2} + \left\| 2\pi \lfloor \varepsilon^{-\beta} \rfloor - 2\pi \varepsilon^{-\beta} \right\|_{L^2}, \\
&\le 2\pi \left(\left\| (|w_\varepsilon|^2 - 1) \varepsilon^{-1/2} \right\|_{L^2} + 1 \right) \\
&\le 2\pi \left(2\sqrt{C_0} + 1 \right),
\end{aligned}$$

so we are done. □

Proof of Theorem 5.3. We begin with the lower-semicontinuity condition. Let $w_\varepsilon \to w$ in L^2. We can assume that

$$\liminf_{\varepsilon \to 0} F_\varepsilon(w_\varepsilon) \le C_0 < \infty, \tag{5.27}$$

otherwise the lower-semicontinuity is trivial. The proof is similar to the proof of (3.15) in Theorem 1.2. Also, due to (5.26), it is enough to show that

$$\liminf_{\varepsilon \to 0} \int_0^1 \frac{\varepsilon}{2} |w_\varepsilon'|^2 + \frac{1}{4\varepsilon}(|w_\varepsilon|^2 - 1)^2 + \frac{L}{2}(\mathcal{T}(w_\varepsilon) + |w_\varepsilon|^2 2\pi \lfloor \varepsilon^{-\beta} \rfloor - 2\pi \varepsilon^{-\beta})^2 \, dx \ge E_{0,A}(w). \tag{5.28}$$

First, for the twist term, it must be verified that under the assumption (5.27),

$$\liminf_{\varepsilon\to 0}\int_0^1 \frac{L}{2}(\mathcal{T}(w_\varepsilon)+|w_\varepsilon|^2 2\pi\lfloor\varepsilon^{-\beta}\rfloor - 2\pi\varepsilon^{-\beta})^2\,dx \\ \geq \frac{L}{2}\int_0^1 (\mathcal{T}(w)-2\pi A)^2\,dx. \tag{5.29}$$

In Theorem 1.2, after (3.19), we proved the inequality

$$\int_K (1-2^{-q})^4(\theta'_{\varepsilon_\ell})^2 - 4\pi N(\rho_{\varepsilon_\ell})^2\theta'_{\varepsilon_\ell} + 4\pi^2 N^2\,dx \geq \int_K (1-2^{-q})^4(\theta')^2 - 4\pi N\theta' + 4\pi^2 N^2\,dx,$$

where K is a compact set on which $\theta'_{\varepsilon_\ell} \rightharpoonup \theta'$ and $\rho_{\varepsilon_\ell} \geq 1-2^{-q}$, followed by an exhaustion argument in K and q to prove lower-semicontinuity of the twist in (3.23). The corresponding inequality to be verified in this case is

$$\int_K (1-2^{-q})^4(\theta'_{\varepsilon_\ell})^2 + 4\pi\left(|w_{e_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor - \varepsilon_\ell^{-\beta}\right)|w_{\varepsilon_\ell}|^2\theta'_{\varepsilon_\ell} + 4\pi^2\left(|w_{e_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor - \varepsilon_\ell^{-\beta}\right)^2\,dx \\ \geq \int_K (1-2^{-q})^4(\theta')^2 - 4\pi A\theta' + 4\pi^2 A^2\,dx, \tag{5.30}$$

which is the left-hand side of (5.29) expanded out and estimated using $|w_{\varepsilon_\ell}| \geq 1-2^{-q}$ on K, on which $\theta'_{\varepsilon_\ell} \rightharpoonup \theta'$. The desired inequality (5.30) would follow immediately from the weak convergence of $\theta'_{\varepsilon_\ell}$ and the two conditions

$$\varepsilon_\ell^{-\beta} - |w_{\varepsilon_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor \to A \text{ in } L^2 \tag{5.31}$$

and

$$|w_{\varepsilon_\ell}|^2(\varepsilon_\ell^{-\beta} - |w_{\varepsilon_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor) \to A \text{ in } L^2, \tag{5.32}$$

which we check in turn. First for (5.31), we estimate

$$\left\|\varepsilon_\ell^{-\beta} - |w_{\varepsilon_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor - A\right\|_{L^2} \leq \left\|\varepsilon_\ell^{-\beta} - \lfloor\varepsilon_\ell^{-\beta}\rfloor - A\right\|_{L^2} + \left\|(1-|w_{\varepsilon_\ell}|^2)\lfloor\varepsilon_\ell^{-\beta}\rfloor\right\|_{L^2}.$$

The first term goes to zero as $\varepsilon \to 0$ due to (5.22), and the second vanishes due to the uniform energy bound (5.27), since $\beta < 1/2$. Moving on to (5.32), we can repeat the argument (3.14) to find that

$$\|w_{\varepsilon_\ell}\|_{L^\infty} \leq M(C_0).$$

The second condition (5.32) can be shown as consequence of this L^∞ bound, (5.31), and (5.27) after writing

$$|w_{\varepsilon_\ell}|^2(\varepsilon_\ell^{-\beta} - |w_{\varepsilon_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor) - A = |w_{\varepsilon_\ell}|^2(\varepsilon_\ell^{-\beta} - |w_{\varepsilon_\ell}|^2\lfloor\varepsilon_\ell^{-\beta}\rfloor - A) + (|w_{\varepsilon_\ell}|^2 - 1)A.$$

Choosing larger and larger K which exhaust $(0,1)$ and letting $q \to \infty$ as in Theorem 1.2, the proof of (5.29) is finished. The remainder of the lower-semicontinuity proof follows from the proof of Theorem 1.2 and (5.26). The recovery sequence is very similar to the proof of Theorem 1.2, which is evident due to the similarity of (5.26) with E_ε, so we omit the details. We only mention that on the set of size $O(\varepsilon)$ where $|w_\varepsilon| \neq 1$, the assumption $\beta < 1/2$ is needed to make sure the twist term vanishes in the limit $\varepsilon \to 0$. □

Finally, we identify the minimizers of $E_{0,A}$. As in Corollary 3.5, this provides a description of all subsequential limits of a family of minimizers $\{u_\varepsilon\}$ for F_ε and thus $\tilde{E}_\varepsilon$. We omit the proof since it follows the same strategy as the proof of Corollary 3.5.

Theorem 5.5. *Let $N = N(A,\alpha)$ be the closest integer to $A - \frac{\alpha}{2\pi}$, so that $N \in \{-1,0,1\}$. Then the global minimizer(s) of $E_{0,A}$ are given by*

(i) the function

$$u(x) = e^{i(2\pi N+\alpha)x} \tag{5.33}$$

having constant twist and no jumps when

$$L(2\pi(N-A)+\alpha)^2 < \frac{4\sqrt{2}}{3}. \tag{5.34}$$

(ii) the one-parameter set of functions given by

$$u(x) = \begin{cases} e^{i2\pi Ax} & \text{if } x < x_0, \\ e^{i(2\pi Ax+\alpha-2\pi A)} & \text{if } x > x_0, \end{cases} \tag{5.35}$$

for any $x_0 \in (0,1)$, that have one jump and twist $2\pi A$ away from the jump, when

$$L(2\pi(N-A)+\alpha)^2 > \frac{4\sqrt{2}}{3}. \tag{5.36}$$

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