A SMECTIC LIQUID CRYSTAL MODEL IN THE PERIODIC SETTING

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Abstract. We consider the asymptotic behavior as ε goes to zero of the 2D smectics model in the periodic setting given by

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2}\int_{\mathbb{T}^2} \frac{1}{\varepsilon}\left(|\partial_1|^{-1}\left(\partial_2 w - \partial_1 \frac{1}{2} w^2\right)\right)^2 + \varepsilon\,(\partial_1 w)^2\,dx.$$

We show that the energy $\mathcal{E}_\varepsilon(w)$ controls suitable L^p and Besov norms of w and use this to demonstrate the existence of minimizers for $\mathcal{E}_\varepsilon(w)$, which has not been proved for this smectics model before, and compactness in L^p for an energy-bounded sequence. We also prove an asymptotic lower bound for $\mathcal{E}_\varepsilon(w)$ as $\varepsilon \to 0$ by means of an entropy argument.

1. Introduction. We consider the variational model

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2}\int_{\mathbb{T}^2} \frac{1}{\varepsilon}\left(|\partial_1|^{-1}\left(\partial_2 w - \partial_1 \frac{1}{2} w^2\right)\right)^2 + \varepsilon\,(\partial_1 w)^2\,dx\,, \tag{1.1}$$

where $w : \mathbb{T}^2 \to \mathbb{R}$ is a periodic function with vanishing mean in x_1, that is

$$\int_0^1 w(x_1, x_2)\,dx_1 = 0 \quad \text{for any } x_2 \in [0,1)\,.^1 \tag{1.2}$$

Here $|\partial_1|^{-1}$ is defined via its Fourier coefficients

$$\widehat{|\partial_1|^{-1} f}\,(k) = |k_1|^{-1}\,\widehat{f}\,(k) \quad \text{for } k \in (2\pi\mathbb{Z})^2\,,$$

and is well defined when (1.2) holds.

This model is motivated by a nonlinear approximate model of smectic liquid crystals. The following functional has been proposed as an approximate model for smectic liquid crystals [4, 14, 25, 30, 31] in two space dimensions:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \frac{1}{\varepsilon}\left(\partial_2 u - \frac{1}{2}(\partial_1 u)^2\right)^2 + \varepsilon(\partial_{11} u)^2\,dx, \tag{1.3}$$

where u is the Eulerian deviation from the ground state $\Phi(x) = x_2$ and ε is the characteristic length scale. The first term represents the compression energy and the second term represents the bending energy. For further background on the model, we refer to [25, 26] and the references contained therein. The 3D version of (1.3), which we analyzed in [26] but do not consider further here, is also used for example in the mathematical description of nuclear pasta in neutron stars [6]. Assuming that u is periodic on the torus $\mathbb{T}^2 = \Omega$ and setting $w = \partial_1 u$, (1.3) becomes

$$E_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{T}^2} \frac{1}{\varepsilon}\left(|\partial_1|^{-1}\left(\partial_2 w - \partial_1 \frac{1}{2} w^2\right)\right)^2 + \varepsilon\,(\partial_1 w)^2\,dx.$$

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[1] More generally, a periodic distribution f on $\mathbb{T}^2$ has "vanishing mean in x_1" if for all $(k_1, k_2) = k \in (2\pi\mathbb{Z})^2$ with $k_1 = 0$, $\widehat{f}(k) = 0$. If f corresponds to an L^p function, $p \in [1,\infty)$, this is equivalent to the existence of a sequence $\{\varphi_k\}$ of smooth, periodic functions satisfying (1.2) that converges in L^p to f.

Finally, a similar model to (1.1) with $|\partial_1|^{-1/2}$ replacing $|\partial_1|^{-1}$ has been derived in the context of micromagnetics [12]; see also [17].

The asymptotic behavior of (1.3) as ε goes to zero was studied in [25]. Given $\varepsilon_n \to 0$ and a sequence $\{u_n\}$ with bounded energies $E_{\varepsilon_n}(u_n)$, the authors proved pre-compactness of $\{\partial_1 u_n\}$ in L^q for any $1 \leq q < p$ and pre-compactness of $\{\partial_2 u_n\}$ in L^2 under the additional assumption $\|\partial_1 u_n\|_{L^p} \leq C$ for some $p > 6$. The compactness proof in [25] uses a compensated compactness argument based on entropies, following the work of Tartar [32, 33, 34] and Murat [22, 23, 24]. In addition, a lower bound on E_ε and a matching upper bound corresponding to a 1D ansatz was obtained as $\varepsilon \to 0$ under the assumption that the limiting function u satisfies $\nabla u \in (L^\infty \cap BV)(\Omega)$.

In this paper, we approach the compactness via a different argument in the periodic setting. Our proof is motivated by recent work on related variational models in the periodic setting [5, 10, 13, 27, 28, 36] where strong convergence of a weakly convergent L^2 sequence is proved via estimates on Fourier series. Given a sequence u_ε weakly converging in $L^2(\mathbb{T}^2)$, to prove strong convergence of u_ε in L^2, it is sufficient to show that there is no concentration in the high frequencies. The center piece of this approach relies on the estimates for solutions to Burgers equation

$$-\partial_1 \frac{1}{2} w^2 + \partial_2 w = \eta$$

in suitable Besov spaces. This type of compactness argument also applies to a sequence $\{w_n\}$ with $\mathcal{E}_\varepsilon(w_n) \leq C$ for any fixed ε. As a direct corollary, we obtain the existence of minimizers of E_ε in $W^{1,2}(\mathbb{T}^2)$ (see Corollary 2.12) for any fixed ε. We observe that to the best of our knowledge, the existence of minimizers of E_ε in any setting was not known due to the lack of compactness for sequence $\{u_n\}$ satisfying $E_\varepsilon(u_n) \leq C$ with fixed ε.

To further understand the minimization of $\mathcal{E}_\varepsilon$, we are also interested in a sharp lower bound for the asymptotic limit of $\mathcal{E}_\varepsilon$ as ε approaches zero. In the literature for such problems (see for example [1, 3, 11, 16]), one useful technique in achieving such a bound is an "entropy" argument, in which the entropy production $\int \operatorname{div} \Sigma(w)$ of a vector field $\Sigma(w)$ is used to bound the energy $\mathcal{E}_\varepsilon$ from below. For the 2D Aviles-Giga functional

$$\frac{1}{2} \int_\Omega \frac{1}{\varepsilon} (|\nabla u|^2 - 1)^2 + \varepsilon |\nabla^2 u|^2 \, dx \,, \tag{1.4}$$

such vector fields were introduced in [16, 8]. In [25, 26], the analogue for the smectic energy, in 2D and 3D respectively, of the Jin-Kohn entropies from [16] were used to prove a sharp lower bound which can be matched by a construction similar to [7, 29]. In this paper, we use the vector field

$$\Sigma(w) = \left(-\frac{1}{3} w^3, \frac{1}{2} w^2 \right) \tag{1.5}$$

which is $(-(\partial_1 u)^3/3, (\partial_1 u)^2/2)$ in terms of u, to prove a sharp lower bound. As $\varepsilon \to 0$, entropy production concentrates along curves and approximates the total variation of the distributional divergence of a BV vector field. An interesting open direction which motivates studying (1.5) is utilizing the correct version of (1.5) (or the entropies from [8, 9]) in 3D, for example in a compactness argument.

The paper is organized as follows. The pre-compactness of a sequence of functions with bounded energy is proved in Section 2, for both fixed ε and $\varepsilon \to 0$. The lower bound is established in Section 3.

2. Compactness of a sequence with bounded energy.

2.1. Preliminaries. Let $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ be unit vectors in $\mathbb{R}^2$. We recall some definitions from [13]. For $f : \mathbb{T}^2 \to \mathbb{R}$, we write

$$\partial_j^h f(x) = f(x + h\mathbf{e}_j) - f(x) \qquad x \in \mathbb{T}^2,\ h \in \mathbb{R}.$$

Definition 2.1. *Given $f : \mathbb{T}^2 \to \mathbb{R}$, $j \in \{1,2\}$, $s \in (0,1]$, and $p \in [1,\infty)$, the directional Besov seminorm is defined as*

$$\|f\|_{\dot{\mathcal{B}}^s_{p;j}} = \sup_{h\in(0,1]} \frac{1}{h^s}\left(\int_{\mathbb{T}^2} \left|\partial_j^h f\right|^p dx\right)^{\frac{1}{p}}$$

Remark 2.2. This is the $\mathcal{B}^{s;p,\infty}$ seminorm defined in each direction separately.

Remark 2.3. For $p = 2$ and $s \in (0,1)$, given $s' \in (s,1)$, the following inequality holds ([13, Equation (2.2)]):

$$\int_{\mathbb{T}^2} \left||\partial_j|^s f\right|^2 = \sum |k_j|^{2s}\left|\widehat{f}(k)\right|^2 = c_s \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^2} \left|\partial_j^h f\right|^2 dx \frac{dh}{|h|} \le C(s,s')\,\|f\|^2_{\dot{\mathcal{B}}^{s'}_{2;j}}\,.$$

We quote two results from [13].

Lemma 2.4. *[13, Proposition B.9] For every $p \in (1,\infty]$ and $q \in [1,p]$ with $(p,q) \neq (\infty,1)$, there exists a constant $C(p,q) > 0$ such that for every periodic function $f : [0,1) \to \mathbb{R}$ with vanishing mean,*

$$\left(\int_0^1 |f(z)|^p\,dz\right)^{\frac{1}{p}} \le C(p,q)\int_0^1 \frac{1}{h^{\frac{1}{q}-\frac{1}{p}}}\left(\int_0^1 \left|\partial_1^h f(z)\right|^q dz\right)^{\frac{1}{q}} \frac{dh}{h}, \tag{2.1}$$

with the usual interpretation for $p = \infty$ or $q = \infty$.

The following estimate was derived in the proof of Lemma B.10 in [13].

Lemma 2.5. *[13, In the proof of Lemma B.10] For every $p \in [1,\infty)$ and every periodic function $f : [0,1) \to \mathbb{R}$, $h \in (0,1]$, the following estimate holds.*

$$\left(\int_0^1 \left|\partial_1^h f(z)\right|^p dz\right)^{\frac{1}{p}} \le 2\left(\frac{1}{h}\int_0^h \int_0^1 \left|\partial_1^{h'} f(z)\right|^p dz\,dh'\right)^{\frac{1}{p}}. \tag{2.2}$$

We define $\eta_w = \partial_2 w - \partial_1 \frac{1}{2} w^2$, and thus (1.1) can be written as

$$\mathcal{E}_\varepsilon(w) = \frac{1}{2}\int_{\mathbb{T}^2} \frac{1}{\varepsilon}(|\partial_1|^{-1}\eta_w)^2 + \varepsilon(\partial_1 w)^2 dx. \tag{2.3}$$

Finally, we introduce the ε-independent energy

$$\mathcal{E}(w) = \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1}\eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2\,dx\right)^{\frac{1}{2}}, \tag{2.4}$$

and note that

$$\mathcal{E}(w) \le \mathcal{E}_\varepsilon(w) \quad \text{for all } \varepsilon > 0\,. \tag{2.5}$$

2.2. Besov and L^p estimates. We obtain the following estimates. The proofs follow closely those in [13, Propositions 2.3-2.4].

LEMMA 2.6. *There exists a universal constant $C_1 > 0$ such that if $w \in L^2\left(\mathbb{T}^2\right)$ and has vanishing mean in x_1 and $h \in (0,1]$, then*

$$\int_{\mathbb{T}^2} \left|\partial_1^h w\right|^3 dx \leq C_1 h \mathcal{E}(w) \tag{2.6}$$

and

$$\sup_{x_2 \in [0,1)} \int_0^h \int_0^1 \left|\partial_1^{h'} w\left(x_1, x_2\right)\right|^2 dx_1 dh' \leq C_1 \left(h\mathcal{E}(w) + h^{\frac{5}{3}} \mathcal{E}^{\frac{2}{3}}(w)\right). \tag{2.7}$$

Proof. Throughout the proof, we assume that w is smooth; once the estimates hold for smooth w, they hold in generality by approximation. The constant C_1 may change from line to line. Following [13, Equations (2.5)-(2.6)], we apply the modified Howarth-Kármán-Monin identities for the Burgers operator. For every $h' \in (0,1]$, we have

$$\partial_2 \frac{1}{2} \int_0^1 \left|\partial_1^{h'} w\right| \partial_1^{h'} w \, dx_1 - \frac{1}{6} \partial_{h'} \int_0^1 \left|\partial_1^{h'} w\right|^3 dx_1 = \int_0^1 \partial_1^{h'} \eta_w \left|\partial_1^{h'} w\right| dx_1, \tag{2.8}$$

$$\partial_2 \frac{1}{2} \int_0^1 \left(\partial_1^{h'} w\right)^2 dx_1 - \frac{1}{6} \partial_{h'} \int_0^1 \left(\partial_1^{h'} w\right)^3 dx_1 = \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w \, dx_1. \tag{2.9}$$

Integrating (2.8) over x_2 and using the periodicity of w yields

$$\begin{aligned} \partial_{h'} \int_{\mathbb{T}^2} \left|\partial_1^{h'} w\right|^3 dx &= -6 \int_{\mathbb{T}^2} \partial_1^{h'} \eta_w \left|\partial_1^{h'} w\right| dx \\ &= -6 \int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} \left|\partial_1^{h'} w\right| dx. \end{aligned} \tag{2.10}$$

Now

$$\begin{aligned} \left|\int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} \left|\partial_1^{h'} w\right| dx\right| &\leq \left(\int_{\mathbb{T}^2} \left(\left|\partial_1\right|^{-1} \eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 \partial_1^{-h'} \left|\partial_1^{h'} w\right|\right)^2 dx\right)^{\frac{1}{2}} \\ &\leq C_1 \left(\int_{\mathbb{T}^2} \left(\left|\partial_1\right|^{-1} \eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 w\right)^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

so that integrating (2.10) from 0 to h and using $\partial_1^0 w = 0$, we have

$$\int_{\mathbb{T}^2} \left|\partial_1^h w\right|^3 dx \leq C_1 \left(\int_{\mathbb{T}^2} \left(\left|\partial_1\right|^{-1} \eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 w\right)^2 dx\right)^{\frac{1}{2}} h \leq C_1 h \mathcal{E}(w).$$

To prove (2.7), we integrate (2.9) from 0 to h and again utilize $\partial_1^0 w = 0$ to obtain

$$\partial_2 \frac{1}{2} \int_0^h \int_0^1 \left(\partial_1^{h'} w\right)^2 dx_1 dh' - \frac{1}{6} \int_0^1 \left(\partial_1^h w\right)^3 dx_1 = \int_0^h \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w \, dx_1 dh'. \tag{2.11}$$

We set

$$f\left(x_2\right) = \int_0^h \int_0^1 \left(\partial_1^{h'} w\right)^2 dx_1 dh',$$

and recall the Sobolev embedding inequality for $W^{1,1}(\mathbb{T}) \subset L^\infty(\mathbb{T})$:

$$\sup_{z\in\mathbb{T}} |f(z)| \le \int_{\mathbb{T}} |f(y)|\,dy + \int_{\mathbb{T}} |f'(y)|\,dy\,.$$

Then applying this to $f(x_2)$ and referring to (2.11), we have

$$\begin{aligned}(2.12)\qquad &\sup_{x_2\in[0,1)} \int_0^h \int_0^1 \left(\partial_1^{h'} w\right)^2 dx_1 dh' \\ &\le \int_0^h \int_{\mathbb{T}^2} \left(\partial_1^{h'} w\right)^2 dx\,dh' \\ &+ \frac{1}{3}\int_{\mathbb{T}^2} \left|\partial_1^h w\right|^3 dx + 2\int_0^h \int_0^1 \left| \int_0^1 \eta_w \partial_1^{-h'} \left|\partial_1^{h'} w\right| dx_1 \right| x_2\,dh'.\end{aligned}$$

Since

$$\int_{\mathbb{T}^2} \left(\partial_1^{h'} w\right)^2 dx \le \left(\int_{\mathbb{T}^2} \left|\partial_1^{h'} w\right|^3 dx\right)^{\frac{2}{3}} \le C_1\left(h'\mathcal{E}(w)\right)^{\frac{2}{3}},$$

and

$$\begin{aligned}&\int_0^1 \left| \int_0^1 \eta_w \partial_1^{-h'} \left|\partial_1^{h'} w\right| dx_1 \right| x_2 \\ &\le \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1}\eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(\partial_1 \partial_1^{-h'} \left|\partial_1^{h'} w\right|\right)^2 dx\right)^{\frac{1}{2}} \\ &\le C_1 \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{-1}\eta_w\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 dx\right)^{\frac{1}{2}},\end{aligned}$$

(2.12) therefore implies

$$\sup_{x_2\in[0,1)} \int_0^h \int_0^1 \left(\partial_1^{h'} w\right)^2 dx_1 dh' \le C_1\left(h^{\frac{5}{3}}\mathcal{E}^{\frac{2}{3}}(w) + h\mathcal{E}(w)\right),$$

which is (2.7). ∎

Lemma 2.7. *If $w \in L^2(\mathbb{T}^2)$ and has vanishing mean in x_1, then the following estimates hold:*

$$(2.13)\qquad \|w\|_{\dot{\mathcal{B}}^s_{3;1}} \le C_1 \mathcal{E}^{\frac{1}{3}}(w), \quad \text{for every } s\in\left(0,\frac{1}{3}\right],$$

where C_1 is as in Lemma 2.6;

$$(2.14)\qquad \|w\|_{L^p(\mathbb{T}^2)} \le C_2(p)\mathcal{E}^{\frac{2}{3\alpha}}(w)\left(\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w)\right)^{\frac{\alpha-2}{2\alpha}},$$

for every $1 \le p < \frac{10}{3}$, where $\alpha = \max\{2,p\}$; and

$$(2.15)\qquad \|w\|_{L^p(\mathbb{T}^2)} \le C_2(p)\varepsilon^{-\frac{1}{\alpha}}\mathcal{E}_\varepsilon^{\frac{1}{\alpha}}(w)\left(\mathcal{E}_\varepsilon(w) + \mathcal{E}_\varepsilon^{\frac{2}{3}}(w)\right)^{\frac{\alpha-2}{2\alpha}}$$

for every $\varepsilon > 0$ and $1 \le p < 6$, where again $\alpha = \max\{2,p\}$.

Proof. The estimate (2.13) follows from (2.6) and the definition of $\|\cdot\|_{\dot{\mathcal{B}}^s_{3;1}}$. Turning to (2.14)-(2.15), we first prove a preliminary estimate. We fix $x_2 \in [0,1)$ and apply Lemma 2.4 to $f(z) = w(z,x_2)$ with $q = 2$, $p > 2$ to deduce

$$\left(\int_0^1 |w(x_1,x_2)|^p dx_1\right)^{\frac{1}{p}} \le C_2(p)\int_0^1 \frac{1}{h^{\frac{1}{2}-\frac{1}{p}}}\left(\int_0^1 \left|\partial_1^h w(x_1,x_2)\right|^2 dx_1\right)^{\frac{1}{2}} \frac{dh}{h}.$$

Integrating over x_2, we thus have by Minkowski's integral inequality

$$
\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &= \left(\int_0^1\int_0^1 |w(x_1,x_2)|^p\,dx_1dx_2\right)^{\frac{1}{p}} \\
&\le C_2(p)\left(\int_0^1\left[\int_0^1 h^{\frac{1}{p}-\frac{3}{2}}\left(\int_0^1\left|\partial_1^h w(x_1,x_2)\right|^2dx_1\right)^{\frac{1}{2}}dh\right]^p dx_2\right)^{\frac{1}{p}} \\
&\le C_2(p)\int_0^1 h^{\frac{1}{p}-\frac{3}{2}}\left[\int_0^1\left(\int_0^1\left|\partial_1^h w(x_1,x_2)\right|^2dx_1\right)^{\frac{p}{2}}dx_2\right]^{\frac{1}{p}}dh \\
&\le C_2(p)\int_0^1 h^{\frac{1}{p}-\frac{3}{2}}\sup_{x_2\in[0,1)}\left(\int_0^1\left|\partial_1^h w(x_1,x_2)\right|^2dx_1\right)^{\frac{p-2}{2p}}\cdot\left(\int_{\mathbb{T}^2}\left|\partial_1^h w(x)\right|^2dx\right)^{\frac{1}{p}}dh\,.
\end{aligned}
$$

The first term in the integrand can be estimated using (2.2) and (2.7), which gives

$$
\begin{aligned}
\sup_{x_2\in[0,1)}\left(\int_0^1\left|\partial_1^h w(x_1,x_2)\right|^2dx_1\right)^{\frac{p-2}{2p}} &\le \sup_{x_2\in[0,1)}\left(\frac{4}{h}\int_0^h\int_0^1\left|\partial_1^{h'}w(x_1,x_2)\right|^2\,dx_1dh'\right)^{\frac{p-2}{2p}} \\
&\le C_1\left(\mathcal{E}(w)+h^{\frac{2}{3}}\mathcal{E}^{\frac{2}{3}}(w)\right)^{\frac{p-2}{2p}}\,,
\end{aligned}
$$

and therefore

$$
\|w\|_{L^p(\mathbb{T}^2)} \le C_2(p)\big(\mathcal{E}(w)+\mathcal{E}^{\frac{2}{3}}(w)\big)^{\frac{p-2}{2p}}\int_0^1 h^{\frac{1}{p}-\frac{3}{2}}\left(\int_{\mathbb{T}^2}\left|\partial_1^h w(x)\right|^2dx\right)^{\frac{1}{p}}dh\,. \tag{2.16}
$$

To prove (2.14) and (2.15) we estimate the h-integrand in two different fashions before integrating. For (2.14), using Hölder's inequality and (2.6), we have the upper bound

$$
\left(\int_{\mathbb{T}^2}\left|\partial_1^h w(x)\right|^2dx\right)^{\frac{1}{p}} \le \left(\int_{\mathbb{T}^2}\left|\partial_1^h w(x)\right|^3dx\right)^{\frac{2}{3p}} \le C_1 h^{\frac{2}{3p}}\mathcal{E}^{\frac{2}{3p}}(w)\,.
$$

Inserting this into (2.16) and using $p\in(2,10/3)$ yields

$$
\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &\le C_2(p)\mathcal{E}^{\frac{2}{3p}}(w)\big(\mathcal{E}(w)+\mathcal{E}^{\frac{2}{3}}(w)\big)^{\frac{p-2}{2p}}\int_0^1 h^{\frac{5}{3p}-\frac{3}{2}}\,dh \\
&= C_2(p)\mathcal{E}^{\frac{2}{3p}}(w)\big(\mathcal{E}(w)+\mathcal{E}^{\frac{2}{3}}(w)\big)^{\frac{p-2}{2p}}\,,
\end{aligned}
$$

which is (2.14) when $p>2$. For $p\le 2$, we apply (2.14) with $p'>2$, use the fact that $\|w\|_{L^p}\le\|w\|_{L^{p'}}$, and let $p'\searrow 2$. Now for (2.15), we instead use the fundamental theorem of calculus and Jensen's inequality to estimate

$$
\begin{aligned}
\left(\int_{\mathbb{T}^2}\left|\partial_1^h w(x)\right|^2\,dx\right)^{\frac{1}{p}} &\le \left(h^2\int_{\mathbb{T}^2}(\partial_1 w(x))^2\,dx\right)^{\frac{1}{p}} \\
&\le h^{\frac{2}{p}}\varepsilon^{-\frac{1}{p}}\mathcal{E}_\varepsilon^{\frac{1}{p}}(w)\,.
\end{aligned}
$$

When plugged into (2.16) and combined with (2.5), this implies

$$
\begin{aligned}
\|w\|_{L^p(\mathbb{T}^2)} &\le C_2(p)\varepsilon^{-\frac{1}{p}}\mathcal{E}_\varepsilon^{\frac{1}{p}}(w)\big(\mathcal{E}_\varepsilon(w)+\mathcal{E}_\varepsilon^{\frac{2}{3}}(w)\big)^{\frac{p-2}{2p}}\int_0^1 h^{\frac{3}{p}-\frac{3}{2}}\,dh \\
&= C_2(p)\varepsilon^{-\frac{1}{p}}\mathcal{E}_\varepsilon^{\frac{1}{p}}(w)\big(\mathcal{E}_\varepsilon(w)+\mathcal{E}_\varepsilon^{\frac{2}{3}}(w)\big)^{\frac{p-2}{2p}}
\end{aligned}
$$

for $p\in(2,6)$. The case $p\in[1,2)$ is handled similarly as in (2.14). □

Remark 2.8. Generalizing the previous argument to the 3D smectics model from [26] is open. An intermediate step would be analyzing the Aviles-Giga model (which is a special case of the energy in [26]) on $\mathbb{T}^2$ using these type of ideas.

2.3. Compactness and existence. We prove compactness and existence theorems in this section. First we define the admissible sets

$$\mathcal{A}_\varepsilon = \left\{ w \in L^2\left(\mathbb{T}^2\right) : \int_0^1 w\left(x_1, x_2\right) dx_1 = 0 \text{ for each } x_2 \in [0,1) \text{ and } \mathcal{E}_\varepsilon(w) < \infty \right\}$$

and

$$\mathcal{A} = \left\{ w \in L^2\left(\mathbb{T}^2\right) : \int_0^1 w\left(x_1, x_2\right) dx_1 = 0 \text{ for each } x_2 \in [0,1) \text{ and } \mathcal{E}(w) < \infty \right\}.$$

Note that for any positive $\varepsilon > 0$, (2.5) implies that $\mathcal{A}_\varepsilon \subset \mathcal{A}$. We prove the following compactness result.

PROPOSITION 2.9. *If* $\{w_n\} \subset \mathcal{A}$ *satisfy* $\mathcal{E}_{\varepsilon_n}(w_n) \le C_3 < \infty$ *and* $\sup_n |\varepsilon_n| \le \varepsilon_0$, *then* $\{w_n\}$ *is precompact in* $L^2\left(\mathbb{T}^2\right)$.

Proof. By (2.14),

$$\|w_n\|_{L^2(\mathbb{T}^2)} \le C_2(p) \mathcal{E}^{\frac{2}{3\alpha}}(w) \left(\mathcal{E}(w) + \mathcal{E}^{\frac{2}{3}}(w) \right)^{\frac{\alpha-2}{2\alpha}},$$

and thus, by (2.5) (that is, $\mathcal{E}(w) \le \mathcal{E}_\varepsilon(w)$), $\|w_n\|_{L^2(\mathbb{T}^2)} \le C_4$ depending on p and C_3. As a consequence, we can find $w_0 \in L^2\left(\mathbb{T}^2\right)$ such that up to a subsequence, $w_n \rightharpoonup w_0$ weakly in $L^2\left(\mathbb{T}^2\right)$. Therefore, for each $k \in (2\pi\mathbb{Z})^2$,

$$\widehat{w_n}(k) \to \widehat{w_0}(k), \ |\widehat{w_n}(k)| \le \left(\int_{\mathbb{T}^2} w_n^2 \right)^{\frac{1}{2}} \le C_4, \text{ and } \left|\widehat{w_n^2}(k)\right| \le \int_{\mathbb{T}^2} w_n^2 \le C_4^2. \tag{2.17}$$

We therefore know that for any fixed $N \in \mathbb{N}$,

$$\sum_{\substack{|k_1| \le 2\pi N, \\ |k_2| \le 2\pi N}} \left|\widehat{w_n}(k) - \widehat{w_0}(k)\right|^2 \to 0 \text{ as } n \to \infty,$$

and so the strong convergence of $w_n \to w_0$ would follow if

$$\sum_{\substack{|k_1| > 2\pi N \\ \text{or} \\ |k_2| > 2\pi N}} \left|\widehat{w_n}(k)\right|^2 \to 0 \text{ uniformly in } n \text{ as } N \to \infty. \tag{2.18}$$

The rest of the proof is dedicated to showing (2.18).

We fix $0 < s < 1/3$ and appeal to Remark 2.3 and (2.13) to calculate

$$\begin{aligned} \int_{\mathbb{T}^2} \left| |\partial_1|^s w_n \right|^2 &= \sum |k_1|^{2s} \left|\widehat{w_n}(k)\right|^2 \le C(s, 1/3) \|w_n\|^2_{\dot{\mathcal{B}}^{1/3}_{2;1}} \\ &\le C(s, 1/3) \|w_n\|^2_{\dot{\mathcal{B}}^{1/3}_{3;1}} \le C(s, 1/3) C_1 \mathcal{E}^{\frac{2}{3}}(w_n) \le C_5, \end{aligned} \tag{2.19}$$

for suitable C_5. We recall the formula

$$\eta_w = \partial_2 w - \partial_1 \frac{1}{2} w^2,$$

which, in terms of Fourier coefficients, reads

$$\widehat{\eta_w}(k) = -ik_2\widehat{w}(k) + \frac{1}{2}ik_1\widehat{w^2}(k).$$

For M_1, $M_2 \in \mathbb{N}$ to be chosen momentarily, we combine this with (2.17) and then (2.19) to find

$$\begin{aligned}
&\sum_{\substack{|k_1|>2\pi M_1 \\ \text{or} \\ |k_2|>2\pi M_2}} |\widehat{w_n}(k)|^2 \\
&\le \sum_{|k_1|>2\pi M_1} |\widehat{w_n}(k)|^2 + \sum_{\substack{|k_1|\le 2\pi M_1 \\ |k_2|>2\pi M_2}} |\widehat{w_n}(k)|^2 \\
&\le (2\pi M_1)^{-2s} \sum_{|k_1|>2\pi M_1} |k_1|^{2s} |\widehat{w_n}(k)|^2 + 2 \sum_{\substack{|k_1|\le 2\pi M_1 \\ |k_2|>2\pi M_2}} \frac{1}{|k_2|^2} |\widehat{\eta_{w_n}}(k)|^2 \\
&\qquad + \frac{1}{2} \sum_{\substack{|k_1|\le 2\pi M_1 \\ |k_2|>2\pi M_2}} \frac{|k_1|^2}{|k_2|^2} \left|\widehat{w_n^2}(k)\right| \\
&\le (2\pi M_1)^{-2s} \sum_{|k_1|>2\pi M_1} |k_1|^{2s} |\widehat{w_n}(k)|^2 + \frac{2M_1^2}{M_2^2} \sum_{\substack{|k_1|\le 2\pi M_1 \\ |k_2|>2\pi M_2}} \frac{1}{|k_1|^2} |\widehat{\eta_{w_n}}(k)|^2 + \frac{C_4^2}{2} \sum_{\substack{|k_1|\le 2\pi M_1 \\ |k_2|>2\pi M_2}} \frac{|k_1|^2}{|k_2|^2} \\
&\le (2\pi M_1)^{-2s} C_5 + \frac{2M_1^2}{M_2^2} \times \varepsilon_0 \mathcal{E}_{\varepsilon_n}(w_n) + \frac{C_4^2}{2} \times 2(2\pi M_1)^3 \times \frac{1}{\pi M_2}.
\end{aligned}$$

Taking $M_1 = M \in \mathbb{N}$ and $M_2 = M^4$, we find that

$$\sum_{\substack{|k_1|>2\pi M \\ \text{or} \\ |k_2|>2\pi M^4}} |\widehat{w_n}(k)|^2 \to 0 \text{ uniformly in } n \text{ as } M \to \infty,$$

which concludes the proof of (2.18). ∎

COROLLARY 2.10. *If $\{w_n\} \subset \mathcal{A}$ satisfy $\mathcal{E}_{\varepsilon_n}(w_n) \le C < \infty$ and $\sup_n |\varepsilon_n| \le \varepsilon_0$, then $\{w_n\}$ is precompact in $L^p(\mathbb{T}^2)$ for any $p \in [1, \frac{10}{3})$.*

Proof. The conclusion follows from the precompactness of $\{w_n\}$ in $L^2(\mathbb{T}^2)$, the bound (2.14) from Lemma 2.7, and interpolation. ∎

COROLLARY 2.11. *If $\{w_n\} \subset \mathcal{A}$ satisfy $\mathcal{E}_\varepsilon(w_n) \le C < \infty$ for a fixed ε, then $\{w_n\}$ is precompact in $L^p(\mathbb{T}^2)$ for any $p \in [1, 6)$.*

Proof. We again appeal to the precompactness of w_n in $L^2(\mathbb{T}^2)$ (taking $\varepsilon_n = \varepsilon$ in Proposition 2.9), but instead use the bound (2.15) from Lemma 2.7 before interpolating. ∎

As a direct application of Corollary 2.11, we can prove an existence theorem for the original smectic energy E_ε defined in (1.3). For any periodic $g : \mathbb{T}^1 \to \mathbb{R}$, we define

$$\widetilde{\mathcal{A}}_{\varepsilon,g} = \left\{ u \in W^{1,2}(\mathbb{T}^2) : E_\varepsilon(u) < \infty,\ \int_0^1 u(x_1, x_2)\,dx_1 = g(x_2) \text{ for a.e. } x_2 \in [0,1) \right\}.$$

We note that $\widetilde{\mathcal{A}}_{\varepsilon,g}$ is non-empty for example when g is smooth.

COROLLARY 2.12. *For fixed $\varepsilon > 0$, if $\widetilde{\mathcal{A}}_{\varepsilon,g}$ is non-empty, then there exists $u_\varepsilon \in \widetilde{\mathcal{A}}_{\varepsilon,g}$ such that $E_\varepsilon(u_\varepsilon) = \inf_{u\in\widetilde{\mathcal{A}}_{\varepsilon,g}} E_\varepsilon(u)$.*

Proof. Since admissible class is non-empty, we can let u_n be a minimizing sequence for

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \frac{1}{\varepsilon}\left(\partial_2 u - \frac{1}{2}(\partial_1 u)^2\right)^2 + \varepsilon(\partial_{11}u)^2\,dx;$$

in particular, the energies are uniformly bounded. By Corollary 2.11, we have, up to a subsequence that we do not relabel,

$$\partial_1 u_n \to \partial_1 u_0 \quad \text{in } L^4\left(\mathbb{T}^2\right) \tag{2.20}$$

for some u_0. Since u_n is a minimizing sequence, the first term in E_ε combined with the L^4-convergence of $\partial_1 u_n$ implies that $\{\partial_2 u_n\}$ are uniformly bounded in $L^2(\mathbb{T}^2)$. Thus, up to a further subsequence which we do not notate, there exists $v_0 \in L^2$ such that $\partial_2 u_n \rightharpoonup v_0$ weakly in $L^2(\mathbb{T}^2)$. Furthermore, by the uniqueness of weak limits, it must be that $v_0 = \partial_2 u_0$, so $u_0 \in W^{1,2}(\mathbb{T}^2)$. Expanding

$$\int_\Omega \left(\partial_2 u_n - \frac{(\partial_1 u_n)^2}{2}\right)^2 dx = \int_\Omega \left[(\partial_2 u_n)^2 - (\partial_1 u_n)^2\partial_2 u_n + \frac{1}{4}(\partial_1 u_n)^4\right] dx\,,$$

we see that by (2.20), the lower semicontinuity of the L^2-norm under weak convergence, and the fact that

$$\lim_{n\to\infty}\int_{\mathbb{T}^2}(\partial_1 u_n)^2\partial_2 u_n\,dx = \int_{\mathbb{T}^2}(\partial_1 u_0)^2\partial_2 u_0\,dx,$$

we have

$$\liminf_{n\to\infty}\int_{\mathbb{T}^2}\left(\partial_2 u_n - \frac{(\partial_1 u_n)^2}{2}\right)^2 dx \ge \int_{\mathbb{T}^2}\left(\partial_2 u - \frac{(\partial_1 u)^2}{2}\right)^2 dx. \tag{2.21}$$

Also, the uniform L^2-bound on $\partial_{11}u$ and the uniqueness of limits implies that, up to a subsequence, $\partial_{11}u_n \rightharpoonup \partial_{11}u_0$ weakly in $L^2(\mathbb{T}^2)$, and thus

$$\liminf_{n\to\infty}\int_\Omega(\partial_{11}u_n)^2\,dx \ge \int_{\mathbb{T}^2}(\partial_{11}u_0)^2\,dx\,. \tag{2.22}$$

Putting together (2.21)-(2.22), we conclude

$$\inf_{\mathcal{A}_{\varepsilon,g}} E_\varepsilon = \liminf_{n\to\infty} E_\varepsilon(u_n) \ge E_\varepsilon(u_0).$$

Finally, by Poincare's inequality and the weak convergence of ∇u_n to ∇u_0 in $L^2(\mathbb{T}^2)$, we conclude that u_n converges to u_0 strongly in $L^2(\mathbb{T}^2)$. Hence

$$\int_0^1 u(x_1,x_2)dx_1 = \lim_{n\to\infty}\int_0^1 u_n(x_1,x_2)dx_1 = g(x_2) \text{ for a.e. } x_2 \in [0,1),$$

therefore u_0 belongs to $\mathcal{A}_{\varepsilon,g}$ and is a minimizer. $\square$

3. Lower bound. We consider the question of finding a limiting functional as a lower bound for E_ε as ε goes to zero. Given a sequence $\{w_\varepsilon\}$ with $E_\varepsilon(w_\varepsilon) \le C$ and $\varepsilon \to 0$, then

$$\int_{\mathbb{T}^2}(|\partial_1|^{-1}\eta_{w_\varepsilon})^2 dx \to 0. \tag{3.1}$$

Therefore $\eta_{w_\varepsilon} \to 0$ distributionally and the natural function space for the limiting problem is

$$\mathcal{A}_0 = \{w \in L^2(\mathbb{T}^2) : \eta_w = -\partial_1\frac{1}{2}w^2 + \partial_2 w = 0 \text{ in } \mathcal{D}'\}.$$

3.1. Properties of BV functions. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We first recall the BV structure theorem. For $v \in [BV(\Omega)]^2$, the Radon measure Dv can be decomposed as

$$Dv = D^a v + D^c v + D^j v$$

where $D^a v$ is the absolutely continuous part of Dv with respect to Lebesgue measure $\mathcal{L}^2$ and $D^c v$, $D^j v$ are the Cantor part and the jump part, respectively. All three measures are mutually singular. Furthermore, $D^a v = \nabla v \mathcal{L}^2 \llcorner \Omega$ where ∇v is the approximate differential of v; $D^c v = D_s v \llcorner (\Omega \backslash S_v)$ and $D^j v = D_s v \llcorner J_v$, where $D_s v$ is the singular part of Dv with respect to $\mathcal{L}^2$, S_v is the set of approximate discontinuity points of v, and J_v is the jump set of v. Since J_v is countably $\mathcal{H}^1$-rectifiable, $D^j v$ can be expressed as

$$(v^+ - v^-) \otimes \nu \, \mathcal{H}^1 \llcorner J_v,$$

where ν is orthogonal to the approximate tangent space at each point of J_v and v^+, v^- are the traces of v from either side of J_v.

Next we quote the following general chain rule formula for BV functions.

Theorem 3.1. *([2, Theorem 3.96]) Let $w \in [BV(\Omega)]^2$, $\Omega \subset \mathbb{R}^2$, and $f \in [C^1(\mathbb{R}^2)]^2$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. Then $v = f \circ w$ belongs to $[BV(\Omega)]^2$ and*

$$Dv = \nabla f(w) \nabla w \mathcal{L}^2 \llcorner \Omega + \nabla f(\tilde{w}) D^c w + (f(w^+) - f(w^-)) \otimes \nu_w \mathcal{H}^1 \llcorner J_w. \tag{3.2}$$

Here $\tilde{w}(x)$ is the approximate limit of w at x and is defined on $\Omega \backslash J_w$.

In what follows, we will use Theorem 3.1 to compute the distributional divergence of such $f \circ w$ as the trace of the measure (3.2), that is

$$\operatorname{div}(f \circ w) = \operatorname{tr}(\nabla f(w) \nabla w) \mathcal{L}^2 \llcorner \Omega + \operatorname{tr}(\nabla f(\tilde{w}) D^c w) + (f(w^+) - f(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w \tag{3.3}$$

as measures.

Lemma 3.2. *If $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, then denoting by $D_i^a w$ and $D_i^c w$ the i-th components of the measures $D^a w$ and $D^c w$, we have*

$$(-w D_1^a w + D_2^a w) = 0 \quad \text{and} \quad (-\tilde{w} D_1^c w + D_2^c w) = 0$$

as measures, and, setting $\sigma(w) = (-w^2/2, w)$,

$$[\sigma(w^+) - \sigma(w^-)] \cdot \nu_w = 0 \quad \mathcal{H}^1\text{-a.e. on } J_w\,. \tag{3.4}$$

Proof. Let $\sigma(w) = (-w^2/2, w)$. By virtue of $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$ and (3.3), we know that, in the sense of distributions,

$$\begin{aligned} 0 &= -\partial_1 \frac{1}{2} w^2 + \partial_2 w \\ &= \operatorname{div} \sigma(w) \\ &= (-w D_1^a w + D_2^a w) + (-\tilde{w} D_1^c w + D_2^c w) + (\sigma(w^+) - \sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w\,. \end{aligned} \tag{3.5}$$

But the measures $D^a w$, $D^c w$, and $D^j w$ are mutually singular, which implies that each individual term in (3.5) is the zero measure. The lemma immediately follows. ∎

3.2. Limiting functional and the proof of the lower bound. Let

$$\Sigma(w) = \left(-\frac{1}{3}w^3, \frac{1}{2}w^2\right).$$

If $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, we can apply the chain rule (3.2) and Lemma 3.2 to $\Sigma(w)$, yielding

$$\begin{aligned}\operatorname{div}\Sigma(w) &= w(-wD_1^a w + D_2^a w)\mathcal{L}^2 + \tilde{w}(-\tilde{w}D_1^c w + D_2^c w) \\ &\quad + (\Sigma(w^+) - \Sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w \\ &= (\Sigma(w^+) - \Sigma(w^-)) \cdot \nu_w \mathcal{H}^1 \llcorner J_w \,. \end{aligned} \tag{3.6}$$

Remark 3.3. Observe if $w = u_x$ and $u_z = \frac{1}{2}u_x^2$, the entropy $\Sigma(w)$ here is exactly the entropy $\tilde{\Sigma}(\nabla u) = -(u_x u_z - \frac{1}{6}u_x^3, \frac{1}{2}u_x^2)$, which we used in the lower bound estimates in [25]. In fact, the argument below also gives a proof of the lower bound on any domain $\Omega \subset \mathbb{R}^2$; the only necessary modification of the proof presented above is that one does not use $|\partial_1|^{-1}\eta_w$ to represent the compression energy, but rather the original expression from (1.3).

THEOREM 3.4. *Let $\varepsilon_n \searrow 0$, $\{w_n\} \subset L^2(\mathbb{T}^2)$ with $\partial_1 w_n \in L^2(\mathbb{T}^2)$ such that*

$$w_n \to w \ \text{ in } L^3(\mathbb{T}^2), \tag{3.7}$$

for some $w \in (BV \cap L^\infty)(\mathbb{T}^2)$. Then

$$\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(w_n) \geq \int_{J_w} \frac{|w^+ - w^-|^3}{12\sqrt{1 + \frac{1}{4}(w^+ + w^-)^2}} d\mathcal{H}^1. \tag{3.8}$$

Remark 3.5. Due to recent progress on the rectifiability for the defect set to certain solutions of Burgers equation [21], the lower bound should in fact be valid among a larger class of limiting functions. Specifically, if $w \in \mathcal{A}_0 \cap L^\infty(\mathbb{T}^2)$ and for every smooth convex entropy $\Phi : \mathbb{R} \to \mathbb{R}$ and corresponding entropy flux $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi'(v) = -\Phi'(v)v$,

$$\partial_1 \Psi(w) + \partial_2 \Phi(w) \text{ is a finite Radon measure,} \tag{3.9}$$

then there exists an H^1-rectifiable set J_w with strong traces on either side such that

$$|\operatorname{div}\Sigma(w)| = \frac{|w^+ - w^-|^3}{12\sqrt{1 + \frac{1}{4}(w^+ + w^-)^2}} \mathcal{H}^1 \llcorner J_w. \tag{3.10}$$

In particular, by substituting any entropy/entropy flux pair for Σ in the the argument below, one finds that for an energy bounded sequence, any limiting function w satisfies (3.9) and thus (3.10). Technically, applying the results of [21] to deduce (3.10) would require extending the arguments there from $[0,T] \times \mathbb{R}$ to the bounded domain $\mathbb{T}^2$ as in [18, 19] and proving that (3.9) implies that $w \in C^0([0,1]; L^1(\mathbb{T}^1))$ (the continuous in time dependence being a technical assumption in [21, Definition 1.1]). Regarding the regularity assumption, it is known (see e.g. [20, Remark 5.2], [15, pg. 191]) that the argument of Vasseur [35] applies in this context and gives a representative of w belonging to $C^0([0,1]; L^1(\mathbb{T}^1))$. The extension of [21] to a bounded domain should not present serious difficulties, although we have not pursued the details further. The concentration of the entropy measures on an $\mathcal{H}^1$-rectifiable jump set should be a key step in obtaining the full Γ-convergence of E_ε in (1.3) to the limiting energy (3.10). The remaining obstacles to such a result are the construction of a recovery sequence for functions that with gradients that do not belong to $BV \cap L^\infty$ (as the existing technology from [7, 29] uses both those assumptions) and the strengthening of the results of [21] to include functions which do not belong to L^∞.

Proof of Theorem 3.4. Without loss of generality, we assume $\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(w_n) < \infty$, so that $w \in \mathcal{A}_0$ by (3.1). Now for any smooth v, direct calculation shows

$$\begin{aligned}\operatorname{div}\Sigma(v) &= \partial_1(-\frac{1}{3}v^3) + \partial_2(\frac{1}{2}v^2)\\ &= v(\partial_2 v - v\partial_1 v) = v\eta_v.\end{aligned} \tag{3.11}$$

On the other hand, we can bound $\mathcal{E}_\varepsilon$ from below as follows:

$$\begin{aligned}\mathcal{E}_\varepsilon(v) &= \frac{1}{2}\int_{\mathbb{T}^2}\frac{1}{\varepsilon}\left(|\partial_1|^{-1}\left(\partial_2 v - \partial_1\frac{1}{2}v^2\right)\right)^2 + \varepsilon(\partial_1 v)^2 dx\\ &= \frac{1}{2\varepsilon}\left\||\partial_1|^{-1}\eta_v\right\|^2_{L^2(\mathbb{T}^2)} + \frac{\varepsilon}{2}\left\|\partial_1 v\right\|^2_{L^2(\mathbb{T}^2)}\\ &\geq \left\||\partial_1|^{-1}\eta_v\right\|_{L^2(\mathbb{T}^2)}\left\|\partial_1 v\right\|_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), given any smooth periodic function ϕ, for any smooth v, we have

$$\begin{aligned}\left|-\int_{\mathbb{T}^2}\Sigma(v)\cdot\nabla\phi\,dx\right| &= \left|\int_{\mathbb{T}^2}\operatorname{div}\Sigma(v)\phi\,dx\right|\\ &\leq \left(\int_{\mathbb{T}^2}||\partial_1|^{-1}\eta_v|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{T}^2}|\partial_1(v\phi)|^2dx\right)^{\frac{1}{2}}\\ &\leq \left\||\partial_1|^{-1}\eta_v\right\|_{L^2(\mathbb{T}^2)}\left\|\partial_1 v\right\|_{L^2(\mathbb{T}^2)}\left\|\phi\right\|_{L^\infty(\mathbb{T}^2)}\\ &\quad + \left\||\partial_1|^{-1}\eta_v\right\|_{L^2(\mathbb{T}^2)}\left\|v\right\|_{L^2(\mathbb{T}^2)}\left\|\partial_1\phi\right\|_{L^\infty(\mathbb{T}^2)}\\ &\leq \mathcal{E}_\varepsilon(v)\left\|\phi\right\|_{L^\infty(\mathbb{T}^2)} + C\sqrt{\varepsilon}\mathcal{E}_\varepsilon(v)^{\frac{1}{2}}\left\|v\right\|_{L^2(\mathbb{T}^2)}\left\|\partial_1\phi\right\|_{L^\infty(\mathbb{T}^2)}.\end{aligned} \tag{3.13}$$

By the density of smooth functions in $L^2(\mathbb{T}^2)$, (3.13) holds for any $v \in L^2(\mathbb{T}^2)$ with $|\partial_1|^{-1}\eta_v, \partial_1 v \in L^2(\mathbb{T}^2)$. Thus

$$\begin{aligned}&\left|-\int_{\mathbb{T}^2}\Sigma(w_n)\cdot\nabla\phi\,dx\right|\\ &\leq \mathcal{E}_{\varepsilon_n}(w_n)\left\|\phi\right\|_{L^\infty(\mathbb{T}^2)} + C\sqrt{\varepsilon_n}\mathcal{E}_{\varepsilon_n}(w_n)^{\frac{1}{2}}\left\|w_n\right\|_{L^2(\mathbb{T}^2)}\left\|\partial_1\phi\right\|_{L^\infty(\mathbb{T}^2)}.\end{aligned} \tag{3.14}$$

Letting $n \to \infty$, by the strong convergence of w_n in $L^3(\mathbb{T}^2)$, we have $\Sigma(w_n) \to \Sigma(w)$ in $L^1(\mathbb{T}^2)$, so that

$$\begin{aligned}-\int_{\mathbb{T}^2}\Sigma(w)\cdot\nabla\phi\,dx &= -\lim_{n\to\infty}\int_{\mathbb{T}^2}\Sigma(w_n)\cdot\nabla\phi\,dx\\ &\leq \liminf_{n\to\infty}\mathcal{E}_{\varepsilon_n}(w_n)\left\|\phi\right\|_{L^\infty(\mathbb{T}^2)}.\end{aligned} \tag{3.15}$$

By taking the supremum over all smooth test functions ϕ with $\|\phi\|_{L^\infty} \leq 1$ in (3.15), we see that $|\operatorname{div}\Sigma(w)|(\mathbb{T}^2)$ is a lower bound for the energies. To derive the explicit expression for this measure, we note that since $w \in \mathcal{A}_0 \cap (BV \cap L^\infty)(\mathbb{T}^2)$, (3.4) and (3.6) apply, so that

$$|\operatorname{div}\Sigma(w)|(\mathbb{T}^2) = \left|[\Sigma(w^+) - \Sigma(w^-)] \cdot \frac{(\sigma(w^+) - \sigma(w^-))^\perp}{|\sigma(w^+) - \sigma(w^-)|}\right| \mathcal{H}^1 \llcorner J_w \,.$$

The right hand side of this equation can be calculated directly from the formulas for $\sigma(w)$ and $\Sigma(w)$ and simplifies to (3.8) (see [25, Proof of Lemma 4.1, Equation (6.3)]). $\square$

Remark 3.6. When comparing with the lower bound proof from [25], this proof requires an extra integration by parts, as it does not rely on a pointwise lower bound on the energy density (see e.g. [25, Equation (4.11)]). The relationship between these two entropies and the structure of the corresponding arguments is exactly mirrored in the entropies devised in [16, 8] for the Aviles-Giga problem - they are equal on the zero set of the potential term, and both give lower bounds, with only one of them ([16]) bounding the energy density from below pointwise.

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REFERENCES

[1] L. Ambrosio, C. De Lellis, and C. Mantegazza, *Line energies for gradient vector fields in the plane*, Calc. Var. Partial Differential Equations, 9 (1999), pp. 327–255.

[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] P. Aviles and Y. Giga, *On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields*, Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), pp. 1–17.

[4] E. A. Brener and V. I. Marchenko, *Nonlinear theory of dislocations in smectic crystals: An exact solution*, Phys. Rev. E, 59 (1999), pp. R4752–R4753.

[5] R. Cantero-Álvarez, F. Otto, and J. Steiner, *The concertina pattern: A bifurcation in ferromagnetic thin films*, J Nonlinear Sci, 17 (2007), pp. 221–281.

[6] M. E. Caplan, A. S. Schneider, and C. J. Horowitz, *Elasticity of nuclear pasta*, Phys. Rev. Lett., 121 (2018), p. 132701.

[7] S. Conti and C. De Lellis, *Sharp upper bounds for a variational problem with singular perturbation*, Math. Ann., 338 (2007), pp. 119–146.

[8] A. DeSimone, S. Müller, R. V. Kohn, and F. Otto, *A compactness result in the gradient theory of phase transitions*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 131 (2001), p. 833–844.

[9] F. Ghiraldin and X. Lamy, *Optimal Besov differentiability for entropy solutions of the eikonal equation*, Comm. Pure Appl. Math., 73 (2020), pp. 317–349.

[10] M. Goldman, M. Josien, and F. Otto, *New bounds for the inhomogenous Burgers and the Kuramoto-Sivashinsky equations*, Communications in Partial Differential Equations, 40 (2015), pp. 2237–2265.

[11] R. Ignat and B. Merlet, *Entropy method for line-energies*, Calc. Var. Partial Differential Equations, 44 (2012), pp. 375–418.

[12] R. Ignat and F. Otto, *The magnetization ripple: a nonlocal stochastic PDE perspective*, J. Math. Pures Appl. (9), 130 (2019), pp. 157–199.

[13] R. Ignat, F. Otto, T. Reid, and P. Tsatsoulis, *Variational methods for a singular SPDE yielding the universality of the magnetization ripple*, arxiv.org/pdf/2010.13123.

[14] T. Ishikawa and O. D. Lavrentovich, *Dislocation profile in cholesteric finger texture*, Phys. Rev. E, 60 (1999), pp. R5037–R5039.

[15] P.-E. Jabin, F. Otto, and B. Perthame, *Line-energy Ginzburg-Landau models: zero-energy states*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 1 (2002), pp. 187–202.
[16] W. Jin and R. V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci., 10 (2000), pp. 355–390.
[17] Jutta Steiner, *The Formation of the Concertina Pattern : Experiments, Analysis, and Numerical Simulations*, PhD thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, Sept. 2011.
[18] E. Marconi, *Characterization of minimizers of Aviles-Giga functionals in special domains*, Arch. Ration. Mech. Anal., 242 (2021), pp. 1289–1316.
[19] E. Marconi, *Rectifiability of entropy defect measures in a micromagnetics model*, Advances in Calculus of Variations, (2021), p. 000010151520210012.
[20] E. Marconi, *On the structure of weak solutions to scalar conservation laws with finite entropy production*, Cal. Var. PDE, 61 (2022), pp. aper No. 32, 30 pp.
[21] ———, *The rectifiability of the entropy defect measure for Burgers equation*, J. Funct. Anal., 283 (2022), pp. Paper No. 109568, 19.
[22] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 5 (1978), pp. 489–507.
[23] ———, *Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 8 (1981), pp. 69–102.
[24] ———, *L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$*, J. Math. Pures Appl. (9), 60 (1981), pp. 309–322.
[25] M. Novack and X. Yan, *Compactness and sharp lower bound for a 2D smectics model*, J. Nonlinear Sci., 31 (2021), pp. Paper No. 60, 26.
[26] ———, *Nonlinear approximation of 3D smectic liquid crystals: sharp lower bound and compactness*, Calc. Var. Partial Differential Equations, 61 (2022), pp. Paper No. 157, 29.
[27] F. Otto, *Optimal bounds on the Kuramoto–Sivashinsky equation*, Journal of Functional Analysis, 257 (2009), pp. 2188–2245.
[28] F. Otto and J. Steiner, *The concertina pattern*, Calc. Var. Partial Differential Equations, 39 (2010), pp. 139–181.
[29] A. Poliakovsky, *Upper bounds for singular perturbation problems involving gradient fields*, J. Eur. Math. Soc. (JEMS), 9 (2007), pp. 1–43.
[30] C. D. Santangelo, *Geometry and the nonlinear elasticity of defects in smectic liquid crystals*, Liquid Crystals Today, 15 (2006), pp. 11–18.
[31] C. D. Santangelo and R. D. Kamien, *Bogomol'nyi, Prasad, and Sommerfield configurations in smectics*, Phys. Rev. Lett., 91 (2003), p. 045506.
[32] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, vol. 39 of Res. Notes in Math., Pitman, Boston, Mass.-London, 1979, pp. 136–212.
[33] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, in Systems of nonlinear partial differential equations (Oxford, 1982), vol. 111 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Reidel, Dordrecht, 1983, pp. 263–285.
[34] ———, *Compensation effects in partial differential equations*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5), 29 (2005), pp. 395–453.
[35] A. Vasseur, *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal., 160 (2001), pp. 181–193.
[36] R. Venkatraman, V. Dabade, and R. D. James, *Bounds on the energy of a soft cubic ferromagnet with large magnetostriction*, J. Nonlinear Sci., 30 (2020), pp. 3367–3388.