Phase-field approximation of a vectorial, geometrically nonlinear cohesive fracture energy

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We consider a family of vectorial models for cohesive fracture, which may incorporate SO(n)-invariance. The deformation belongs to the space of generalized functions of bounded variation and the energy contains an (elastic) volume energy, an opening-dependent jump energy concentrated on the fractured surface, and a Cantor part representing diffuse damage. We show that this type of functional can be naturally obtained as Γ-limit of an appropriate phase-field model. The energy densities entering the limiting functional can be expressed, in a partially implicit way, in terms of those appearing in the phase-field approximation.

Contents

1 Introduction

In variational models of nonlinear elasticity a hyper-elastic body with reference configuration $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ undergoes a deformation $u : \Omega \to \mathbb{R}^m$, whose stored energy reads as

$$\int_\Omega \Psi(\nabla u)\mathrm{d}x. \tag{1.1}$$

External loads can be included, adding linear perturbations to this energy, and Dirichlet boundary conditions, restricting the set of admissible deformations u. The energy density $\Psi : \mathbb{R}^{m\times n} \to [0, +\infty)$, acting on the deformation gradient ∇u, is typically assumed to be minimized by matrices in the set of proper rotations $\mathrm{SO}(n)$ (with $m = n$) and to have p-growth at infinity, $p > 1$. Correspondingly, the natural space for the deformation u is a (subset of) the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$. There is an extensive literature on the theory of existence of minimizers of this type of functionals, and in particular the key property of weak lower semicontinuity of (1.1) is closely related to the quasiconvexity of the energy density Ψ.

Fracture phenomena, both brittle and cohesive, require a richer modeling framework. Physically, cohesive fracture is often understood as a gradual separation phenomenon: load-displacement curves usually exhibit an initial increase of the load up to a critical value, and a subsequent decrease to zero, which is the value indicating the complete separation [BFM08, Dug60, Bar62, FCO14]. See [dPT98, dPT01] for discussions on different load-displacement behaviours. Evolutionary models (prescribing the crack path) have been studied in [DMZ07, BFM08, Cag08, CT11, LS14, Alm17, ACFS17, NS17, TZ17, NV18, CLO18], see also references therein. See [DMG08, CCF20] for further results on the topic.

Variational models of fracture are typically formulated using the space $(G)BV$ of (generalised) functions of bounded variation [FM98, BFM08] and energy functionals of the form

$$\int_\Omega W(\nabla u)\mathrm{d}x + \int_\Omega l(\mathrm{d}D^c u) + \int_{J_u} g([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{1.2}$$

The deformation $u \in (G)BV(\Omega; \mathbb{R}^m)$ may exhibit discontinuities along a $(n-1)$-dimensional set J_u. We denote by $[u]$ and ν_u the opening of the crack and the normal vector to the crack set J_u, respectively, while $D^c u$ represents the Cantor derivative of u (see [AFP00] for the definition and the relevant properties of functions of bounded variation). Working within deformation theory, the functional (1.2) contains both energetic and dissipative terms, which are physically distinct but need not be separated for this variational modeling.

The densities W, l, and g entering (1.2) need to satisfy suitable growth conditions. Lower semicontinuity of the functional imposes several restrictions, as for example that l is positively one-homogeneous and quasiconvex, W quasiconvex, and g subadditive. Furthermore, l needs to match, after appropriate scaling, both the behavior of W at infinity and the behavior of g near zero. These properties will be discussed in more detail below (see, for example, Proposition 3.11).

The qualitative properties of W, l and g are selected according to the specific model of interest. For instance, the brittle regime is modelled by a constant surface density g and a superlinear bulk energy density W. These choices in turn imply that $l(\xi) = \infty$ for $\xi \neq 0$, so that $D^c u$ necessarily vanishes. The functional setting of the problem is then provided by the space of (generalised) special functions with bounded variation $(G)SBV(\Omega)$. In contrast, in cohesive models g is usually assumed to be approximately linear for small amplitudes and bounded.

The direct numerical simulation of functionals of the type (1.2) is highly problematic, due to the difficulty of finding good discretizations for $(G)BV$ functions and of differentiating the functional with respect to the coefficients entering the finite-dimensional approximation. Therefore a number of regularizations have been proposed, of which one of the most successful is given by phase-field functionals. These are energies depending on a pair of variables (u, v), having a Sobolev regularity, where u represents a regularization of a discontinuous displacement, while $v \in [0, 1]$ can be interpreted as a damage parameter, indicating the amount of damage at each point of the body (where $v = 1$ corresponds to the undamaged material and $v = 0$ to the completely damaged material). The basic structure of a phase-field model is

$$\mathcal{F}_\varepsilon(u, v) := \int_\Omega \left(f_\varepsilon^2(v)\Psi(\nabla u) + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 \right) \mathrm{d}x, \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter, f_ε is a damage coefficient acting on the damage variable v, increasing from 0 to 1, and Ψ is an elastic energy density, as in (1.1). The first term in (1.3) represents the stored elastic energy, the other two terms represent the stored energy and dissipation due to the damage.

Finding a variational approximation of the fracture model (1.2) by phase-field models means to construct f_ε and Ψ such that the functionals (1.3) converge, in the sense of Γ-convergence, to (1.2) as $\varepsilon \to 0$. This is not an easy task in general. The brittle case (g constant) in an antiplane shear, linear, framework ($m = 1$, Ψ quadratic) was the first outcome of this type [AT90, AT92]. It has been extended in several directions for different aims, giving rise to a very vast literature of both theoretical results [Sha96, AFM01, Cha04, Cha05, HMCX14, ALRC13, DMI13, Iur13, FI14, Iur14, BEZ15, CFZ21] and numerical simulations [BC94a, BSK06, Bou07, BFM08, BOS10, BOS13, BB21] (for other regularizations, see also [AFP00, BDMG99, Bra98, Fus03, BG06] and references therein). In particular, the extension of the results in [AT92] to the vector-valued (nonlinear) brittle case has been provided in [Foc01]. The variational approximation of cohesive models is considerably more involved. The antiplane shear, linear, case was obtained through a double Γ-limit of energies with 1-growth in [ABS99], then generalized to the vector-valued case in [AF02]. A drawback of these results is the 1-growth with respect to ∇u, which makes the approximants mechanically less meaningful and numerically less helpful.

To overcome these problems, in [CFI16] we proposed a different approximation of (1.2) in the antiplane shear case, with quadratic models of the form (1.3), based on a damage coefficient f_ε of the type

$$f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} \frac{\ell s}{1-s} \qquad s \in [0,1],\, \ell > 0\,, \tag{1.4}$$

and obtained Γ-convergence to a model of the type (1.2) in the scalar ($m = 1$) case. We remark that f_ε is equal to 1 when $v \sim 1$ (elastic response) and to 0 when $v \sim 0$ (brittle fracture response). Moreover, the first addend in the energy in (1.3) competes against the second term if v is less than but close to 1, and with all the terms of (1.3) otherwise (pre-fracture response). This phase-field approximation of this scalar cohesive fracture was investigated numerically in [FI17]. A 1D cohesive quasistatic evolution (not prescribing the crack path) is presented in [BCI21] and related to the phase-field models of [CFI16]. A different approximation of (1.2), still in the scalar-valued framework, is obtained in [DMOT16] using elasto-plastic models.

In this paper we study the approximation of vector-valued cohesive models of the type (1.2) via phase-field models of the type (1.3) with the damage coefficient (1.4), as proposed in [CFI16]. In particular, this permits to extend the results of [CFI16] to a geometrically nonlinear framework, we refer to (2.2)-(2.5) for the specific hypotheses on Ψ. The main result is given in Theorem 2.1, the precise assumptions are discussed in Section 2.1.

In order to illustrate our result, let us consider the simplest model for the energy density Ψ in finite kinematics and $m = n$,

$$\Psi_2(\xi) := \operatorname{dist}^2(\xi, \mathrm{SO}(n)) = \min_{R \in \mathrm{SO}(n)} |\xi - R|^2. \tag{1.5}$$

With this choice, our main result Theorem 2.1 states that the phase-field energies (1.3) Γ-converge in the L^1-topology as $\varepsilon \to 0$ to the energy (1.2), with

$$W(\xi) := (\operatorname{dist}^2(\cdot, \mathrm{SO}(n)) \wedge \ell \operatorname{dist}(\cdot, \mathrm{SO}(n)))^{\mathrm{qc}}(\xi)\,, \tag{1.6}$$

and

$$l(\xi) := \ell|\xi|, \quad g(z,\nu) := g_{\mathrm{scal}}(|z|)\,,$$

for every $\xi \in \mathbb{R}^{m\times n}$, $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$, where g_{scal} is the surface energy density appearing in the scalar model (cf. formula (4.4) for the definition of g_{scal}, item (iii) in Proposition 3.12 with $W = h^{\mathrm{qc}}$ and $l = h^{\mathrm{qc},\infty}$ to justify the second equality, and Corollary 3.5 for the third equality). As remarked above, g coincides with l asymptotically for infinitesimal amplitudes. Even in this simple case, the expression for W is somewhat implicit, as it involves a quasiconvex envelope, which in most cases can only be approximately computed numerically. We remark that even Ψ_2 itself as defined in (1.5) is not quasiconvex, we refer to [Š01, Example 4.2] for an explicit formula for its quasiconvex envelope Ψ_2^{qc} in the two-dimensional case.

We recall that in the scalar case several different choices for f_ε are possible without changing the overall effect of the approximation (cf. [CFI16, Section 4]).

A negative power-law divergence at 1 however leads to a corresponding power-law behaviour of g close to 0 (cf. [CFI16, Theorem 7.4]). We expect these findings to have a natural generalization to the current vectorial setting, this requires additional technical ingredients that will be the object of future work [CFI22].

Let us now briefly discuss some aspects of the proof of Theorem 2.1. One of the main difficulties is to identify the correct limit densities W, g, and l, given the density Ψ and the damage coefficient f_ε of the phase-field (1.3). We do not expect that the cohesive energies that arise in the limit of our approximation exhaust all possible energies of the form (1.2), with densities W, g, and l satisfying the growth conditions and matching properties specified above. Indeed, we prove that, even in the simplest case $\Psi(\xi) := |\xi|^2$, W is not convex (see Lemma 2.5 below). Thus, at least in this case, the limit energy is not given by the relaxation of a functional defined on $SBV(\Omega)$ (cf. [BC94b, Remark 2.2]). Convex functions may be obtained as densities of the bulk term of the energy under more specific choices of the damage variable (see for example [BIR21], where the damage variable is a characteristic function).

The effective surface energy density g of the Γ-limit of the family $(\mathcal{F}_\varepsilon)$ is defined in an abstract fashion by an asymptotic minimization formula as the Γ-limit of a simpler family of functionals computed on functions jumping on a hyperplane (cf. (2.12)). Alternative characterizations of g useful along the proofs are provided both in Propositions 3.1 and 3.2, in which we show that the test sequences in the very definition of g can be assumed to be periodic in $(n-1)$ mutually orthogonal directions and with L^2 integrability, and in Proposition 3.3, where g is represented in terms of an asymptotic homogenization formula. Finally, the energy density l of the Cantor part turns out to coincide with the recession function W^∞ of W. Furthermore, an explicit characterization of l in terms of Ψ is given in Proposition 3.10.

The proof of the lower bound in BV is based on the blow-up technique. Roughly, to get the local estimate for the diffuse part given $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in L^1, we analyze the asymptotic behaviour of the phase-field energies $\mathcal{F}_\varepsilon$ restricted on the δ-superlevel sets of v_ε, $\delta \in (0,1)$, and then let $\delta \uparrow 1$. More precisely, in Lemma 4.4 we bound from below $\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$ in (1.3) pointwise with a functional defined on $(G)SBV$, that is independent of v_ε and that is computed on a truncation of u_ε with the characteristic function of a suitable superlevel set of v_ε (depending on δ). This is actually true up to an error related to the measure of the corresponding sublevel set of v_ε, and up to prefactors depending on δ which are converging to 1 as $\delta \uparrow 1$ for the volume term and vanishing for the surface term. The lower semicontinuity in L^1 of the diffuse part of such a functional then implies the lower bound. In addition, a slight variation of this argument shows directly that $(GBV(\Omega))^m$ is the domain of the Γ-limit.

Instead, to prove the local estimate for the surface part we show that under a surface scaling assumption we may replace v_ε by its truncation at the threshold γ_ε, being γ_ε the smallest $z \in [0,1]$ satisfying $f_\varepsilon(z) = 1$. The mentioned asymptotic minimization formula defining g then provides a natural lower bound. The liminf inequality in GBV is finally obtained by a further truncation argument.

The upper bound in BV is proven through an integral representation argument. In particular, a direct computation provides a rough linear estimate from above, in fact optimal for the diffuse part. This allows to apply the representation result for linear functionals given in [BFM98]. The sharp estimate for the surface density is obtained using the aforementioned characterization of g involving periodic boundary conditions. The full upper bound in GBV follows by a truncation argument.

The paper is structured as follows. In Section 2.1 we present the model, introducing the main definitions and stating the Γ-convergence result in Theorem 2.1. In Section 2.2 we focus on a simplified model and we prove that in this case the limiting volume energy density W, obtained by quasiconvexification as in (1.6), is not convex (Lemma 2.5). In Section 3 several properties of the surface and Cantor densities are discussed. In particular, Propositions 3.1 and 3.2 deal with the change of boundary conditions within the minimum problem defining g. Proposition 3.3 provides an equivalent expression of g. Section 4 is devoted to the proof of the lower bound: Proposition 4.1 proves the surface estimate in BV. The lower bound in BV for the diffuse part is addressed in Proposition 4.2. Finally, in Theorem 4.9 the lower bound is extended to the full space GBV via a continuity argument (cf. Proposition 4.8). The proof of the upper bound is the object of Section 5, which concludes the proof of Theorem 2.1. Finally, Section 6 addresses the problems of compactness and convergence of minimizers.

2 Model

2.1 General definitions

In the entire paper $\Omega \subset \mathbb{R}^n$ is a bounded, open set with Lipschitz boundary, $\mathcal{A}(\Omega)$ denotes the family of open subsets of Ω and $|\cdot|$ denotes the Euclidean norm, $|\xi|^2 := \sum_{ij} \xi_{ij}^2 = \mathrm{Tr}\left(\xi^T \xi\right)$ for $\xi \in \mathbb{R}^{m\times n}$.

For all $\varepsilon > 0$ we consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^{m+1}) \times \mathcal{A}(\Omega) \to [0, \infty]$ given by

$$\mathcal{F}_\varepsilon(u, v; A) := \int_A \left(f_\varepsilon^2(v)\Psi(\nabla u) + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 \right) \mathrm{d}x \tag{2.1}$$

if $(u, v) \in W^{1,2}(\Omega; \mathbb{R}^m) \times W^{1,2}(\Omega; [0, 1])$ and ∞ otherwise, where for every $s \in [0, 1)$ we set

$$f(s) := \frac{\ell s}{1-s}, \qquad f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} f(s), \qquad f_\varepsilon(1) := 1\,; \tag{2.2}$$

and $\ell > 0$ is a parameter representing the critical yield stress. We write briefly $\mathcal{F}_\varepsilon(u, v) := \mathcal{F}_\varepsilon(u, v; \Omega)$, and analogously for all the functionals that shall be introduced in what follows.

We assume that $\Psi : \mathbb{R}^{m\times n} \to [0, \infty)$ is continuous and such that

$$\left(\frac{1}{c}|\xi|^2 - c\right) \vee 0 \le \Psi(\xi) \le c(|\xi|^2 + 1) \qquad \text{for all } \xi \in \mathbb{R}^{m\times n}. \tag{2.3}$$

We assume the ensuing limit to exist

$$\Psi_\infty(\xi) := \lim_{t\to\infty} \frac{\Psi(t\xi)}{t^2}\,, \tag{2.4}$$

and that it is uniform on the set of ξ with $|\xi| = 1$. This means that for every $\delta > 0$ there is $t_\delta > 0$ such that $|\Psi(t\xi)/t^2 - \Psi_\infty(\xi)| \le \delta$ for all $t \ge t_\delta$ and all ξ with $|\xi| = 1$, which is the same as

$$|\Psi(\xi) - \Psi_\infty(\xi)| \le \delta|\xi|^2 \qquad \text{for all } |\xi| \ge t_\delta. \tag{2.5}$$

By scaling, $\Psi_\infty(t\xi) = t^2\Psi_\infty(\xi)$ and in particular $\Psi_\infty(0) = 0$. Uniform convergence also implies $\Psi_\infty \in C^0(\mathbb{R}^{m\times n})$.

We define $h : \mathbb{R}^{m\times n} \to [0,\infty)$ by

$$h(\xi) := \Psi(\xi) \wedge \ell\Psi^{1/2}(\xi) \tag{2.6}$$

and denote by h^{qc} its quasiconvex envelope,

$$h^{\mathrm{qc}}(\xi) := \inf\Big\{ \int_{(0,1)^n} h(\xi + \nabla\varphi)\mathrm{d}x : \varphi \in C_c^\infty((0,1)^n;\mathbb{R}^m)\Big\}. \tag{2.7}$$

From (2.3) we infer that for every $\xi \in \mathbb{R}^{m\times n}$

$$\Big(\frac{1}{c}|\xi| - c\Big) \vee 0 \le h^{\mathrm{qc}}(\xi) \le h(\xi) \le c(|\xi| + 1). \tag{2.8}$$

Let $h^{\mathrm{qc},\infty}$ be its recession function,

$$h^{\mathrm{qc},\infty}(\xi) := \limsup_{t\to\infty} \frac{h^{\mathrm{qc}}(t\xi)}{t}. \tag{2.9}$$

We remark that the definitions of $h^{\mathrm{qc},\infty}$ and Ψ_∞ differ, to reflect the different growth of the two functions, quadratic for Ψ and linear for h. Recall that $h^{\mathrm{qc},\infty}$ is itself a quasiconvex function [FM93, Rem. 2.2 (ii)]. Therefore, it is locally Lipschitz continuous (cf. for instance [Dac08, Theorem 5.3 (ii)]). Moreover, in Proposition 3.10 below we shall prove that

$$h^{\mathrm{qc},\infty}(\xi) = \ell(\Psi^{1/2})^{\mathrm{qc},\infty}(\xi)\,, \tag{2.10}$$

where the latter quantity is defined as in (2.7)-(2.9). We remark that, at variance with the convex case, one cannot in general replace the lim sup in (2.9) by a limit [Mül92, Theorem 2].

For all open subsets $A \subseteq \mathbb{R}^n$, $u \in W^{1,2}(A;\mathbb{R}^m)$ and $v \in W^{1,2}(A;[0,1])$ it is convenient to introduce the functional

$$\mathcal{F}_\varepsilon^\infty(u,v;A) := \int_A \Big(\varepsilon f^2(v)\Psi_\infty(\nabla u) + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x. \tag{2.11}$$

The first term is interpreted to be zero whenever $\nabla u = 0$, even if $v = 1$. For any $\nu \in S^{n-1}$ we fix a cube Q^ν with side length 1, centered in the origin, and with one side parallel to ν. We write $Q^\nu_r := rQ^\nu$. We define $g : \mathbb{R}^m \times S^{n-1} \to [0,\infty)$ by

$$g(z,\nu) := \inf\{\liminf_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j, Q^\nu) : \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)} \to 0, \varepsilon_j \to 0\}. \tag{2.12}$$

Here $u_j \in W^{1,2}(Q^\nu;\mathbb{R}^m)$ and $v_j \in W^{1,2}(Q^\nu;[0,1])$; obviously one can restrict to sequences $v_j \to 1$ in $L^1(Q^\nu)$. We refer to Section 3 for the discussion of several properties of g.

We will prove the following result.

Theorem 2.1. *Let $\mathcal{F}_\varepsilon$ be the functional defined in* (2.1)*. Then for all $(u,v) \in L^1(\Omega;\mathbb{R}^{m+1})$ it holds*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) = \mathcal{F}_0(u,v)\,,$$

where

$$\mathcal{F}_0(u,v) := \int_\Omega h^{\rm qc}(\nabla u)\mathrm{d}x + \int_\Omega h^{{\rm qc},\infty}(\mathrm{d}D^c u) + \int_{J_u} g([u],\nu_u)d\mathcal{H}^{n-1}, \tag{2.13}$$

if $u \in (GBV \cap L^1(\Omega))^m$ and $v = 1$ $\mathcal{L}^n$-a.e., and $\mathcal{F}_0(u,v) := \infty$ otherwise.

Remark 2.2. *One can imagine several natural generalizations of Theorem 2.1. For example, one could allow Ψ to take negative values, replacing* (2.3) *by*

$$\frac{1}{c}|\xi|^2 - c \le \Psi(\xi) \le c(|\xi|^2+1).$$

Whereas in purely elastic models like (1.1) *one can add a constant to the energy density without any change in the analysis, the presence of the prefactor $f^2_\varepsilon(v)$ renders this modification nontrivial, and influences several steps in the proof. Indeed, the construction in Step 1 of the proof of Theorem 5.2 shows that the definition of h in* (2.6) *needs to be replaced by*

$$h(\xi) := \Psi(\xi) \wedge \ell\Psi^{1/2}_+(\xi)\,.$$

Alternatively, one could replace the quadratic growth of Ψ in (2.3) *by p-growth, $p > 1$. The requirement that the effective energy scales linearly for large strains leads to corresponding adaptations in the other parts of the functional.*

For simplicity we only address here the growth condition in (2.3)*.*

Notation. For A open we denote by $\mathcal{M}^+(A)$ the set of positive Radon measures on the set A, and by $\mathcal{M}^+_b(A)$ the subset of bounded measures. For $A \in \mathcal{A}(\Omega)$,

$$\Gamma(L^1)\text{-}\liminf \mathcal{F}_\varepsilon(u,v;A) := \inf\Big\{ \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon;A) : (u_\varepsilon,v_\varepsilon) \to (u,v) \text{ in } L^1(\Omega;\mathbb{R}^{m+1})\Big\}$$

and correspondingly for the Γ-$\limsup$. We drop the dependence on the reference set A if $A = \Omega$. We refer to Section 4.1 for the definition of the vector measure $D^c u$ if $u \in (GBV(\Omega))^m$.

2.2 Simplified model

In this Section we consider the simplified case $\Psi_{\text{simp}}(\xi) := |\xi|^2$, the corresponding unrelaxed energy density $h_{\text{simp}} : \mathbb{R}^{m\times n} \to [0,\infty)$,

$$h_{\text{simp}}(\xi) := |\xi|^2 \wedge \ell|\xi|, \tag{2.14}$$

its quasiconvex envelope $h^{\text{qc}}_{\text{simp}}$ as in (2.7), and its recession function $h^{\text{qc},\infty}_{\text{simp}}$ as in (2.9). These functions only depend on the space dimension and the single parameter $\ell > 0$, which could be eliminated by scaling.

In this case it is possible to obtain simple closed-form expressions for several of the quantities defined above. However, an explicit characterization of the quasiconvex envelope in (2.7) remains difficult. Indeed, we show in Lemma 2.5(iii) below that even in this simplified setting the result is not convex. Since it has linear growth, lower bounds with polyconvexity cannot be used, and an explicit determination of $h^{\text{qc}}_{\text{simp}}$ seems difficult. We believe this to be a strong indication that in most cases of interest the function h^{qc} can only be approximated numerically, and not computed explicitly. Lemma 2.5 and this observation are not used in the proof of Theorem 2.1.

Lemma 2.3. *For $n, m \geq 1$ let $h_{\text{simp}} : \mathbb{R}^{m\times n} \to [0,\infty)$ be defined as in* (2.14)*. Then:*

(i) its convex envelope is

$$h^{\text{conv}}_{\text{simp}}(\xi) = \begin{cases} |\xi|^2, & \text{if } |\xi| \leq \frac{\ell}{2}, \\ \ell|\xi| - \frac{\ell^2}{4}, & \text{if } |\xi| > \frac{\ell}{2}; \end{cases} \tag{2.15}$$

(ii) $\ell|\xi| - \frac{\ell^2}{4} \leq h^{\text{qc}}_{\text{simp}}(\xi) \leq \ell|\xi|$ *for all* $\xi \in \mathbb{R}^{m\times n}$;

(iii) $h^{\text{qc},\infty}_{\text{simp}}(\xi) = \ell|\xi|$ *and the* $\limsup$ *in* (2.9) *is a limit.*

Proof. (i): To prove (2.15) we consider $h_{\text{scal}} : [0,\infty) \to [0,\infty)$ defined by

$$h_{\text{scal}}(t) := t^2 \wedge \ell t \tag{2.16}$$

and compute its convex envelope

$$h^{\text{conv}}_{\text{scal}}(t) = \begin{cases} t^2, & \text{if } 0 \leq t \leq \frac{\ell}{2}, \\ \ell t - \frac{\ell^2}{4}, & \text{if } t > \frac{\ell}{2}. \end{cases} \tag{2.17}$$

Let $\eta \in \mathbb{R}^{m\times n}$ with $|\eta| = 1$. Then $h_{\text{simp}}(t\eta) = h_{\text{scal}}(t)$, hence $h^{\text{conv}}_{\text{simp}}(t\eta) \leq h^{\text{conv}}_{\text{scal}}(t)$. This proves one inequality in (2.15). At the same time, $h^{\text{conv}}_{\text{scal}}(|\xi|) \leq h_{\text{scal}}(|\xi|) = h_{\text{simp}}(\xi)$, and the function $\xi \mapsto h^{\text{conv}}_{\text{scal}}(|\xi|)$ is convex, since $h^{\text{conv}}_{\text{scal}}$ is convex and nondecreasing in $[0,\infty)$ and $\xi \mapsto |\xi|$ is convex. This proves the second inequality in (2.15).

(ii): This follows immediately from the fact that $\ell|\xi| - \frac{\ell^2}{4} \leq h^{\text{conv}}_{\text{simp}}(\xi) \leq h^{\text{qc}}_{\text{simp}}(\xi) \leq h_{\text{simp}}(\xi) \leq \ell|\xi|$ for any $\xi \in \mathbb{R}^{m\times n}$.

(iii): This follows immediately from the definition and (ii).

□

We next prove that the quasiconvex envelope $h_{\rm simp}^{\rm qc}$ is not convex. For this we need a linear algebra statement that we present first.

Lemma 2.4. *Let*

$$\mathbb{R}^{m\times n\times n}_{\rm sym} := \{\Gamma \in \mathbb{R}^{m\times n\times n} : \Gamma_{ijk} = \Gamma_{ikj}\} \tag{2.18}$$

and consider for $\xi \in \mathbb{R}^{m\times n}$ the linear map $T : \mathbb{R}^{m\times n\times n}_{\rm sym} \to \mathbb{R}^{m\times n\times n}$ of the form

$$(T\Gamma)_{ijk} := \Gamma_{ijk} - \xi_{ij} \sum_{a,b} \xi_{ab}\Gamma_{abk}\,. \tag{2.19}$$

If $\operatorname{rank}\xi \geq 2$, then T is injective. In particular, it has an inverse $S : T(\mathbb{R}^{m\times n\times n}_{\rm sym}) \to \mathbb{R}^{m\times n\times n}_{\rm sym}$.

Proof. It suffices to show that there is no $\Gamma \in \mathbb{R}^{m\times n\times n}_{\rm sym}$ with $T\Gamma = 0$ and $\Gamma \neq 0$. We assume it exists and define $v \in \mathbb{R}^n$ componentwise by

$$v_k := \sum_{a,b} \xi_{ab}\Gamma_{abk}. \tag{2.20}$$

Then $T\Gamma = 0$ is equivalent to

$$\Gamma_{ijk} - \xi_{ij} v_k = 0,$$

hence $\Gamma_{ijk} = \xi_{ij} v_k$, for all i, j, and k. Moreover, $\Gamma \neq 0$ in turn implies that $v \neq 0$. From $\Gamma \in \mathbb{R}^{m\times n\times n}_{\rm sym}$ we obtain

$$\xi_{ij} v_k = \xi_{ik} v_j.$$

As $\operatorname{rank}\xi \geq 2$ there is a vector $w \in \mathbb{R}^n$ with $v \cdot w = 0$ and $\xi w \neq 0$. We take the scalar product of the previous equation with w and obtain

$$\sum_k \xi_{ij} v_k w_k = \sum_k \xi_{ik} v_j w_k$$

which gives $0 = v_j(\xi w)_i$ for all i and j. As $v \neq 0$ and $\xi w \neq 0$, this is a contradiction. ☐

Lemma 2.5. *Let $\xi \in \mathbb{R}^{m\times n}$.*

(i) If $|\xi| \leq \frac{\ell}{2}$, then $h_{\rm simp}(\xi) = h_{\rm simp}^{\rm qc}(\xi) = h_{\rm simp}^{\rm conv}(\xi)$.

(ii) If $\operatorname{rank}\xi \leq 1$, then $h_{\rm simp}^{\rm qc}(\xi) = h_{\rm simp}^{\rm conv}(\xi)$.

(iii) If $\operatorname{rank}\xi \geq 2$ and $|\xi| > \frac{\ell}{2}$, then $h_{\rm simp}^{\rm conv}(\xi) < h_{\rm simp}^{\rm qc}(\xi)$.

Proof. We work for $\ell = 1$ (the general case can be reduced to this one by a rescaling), to shorten notation we write h for $h_{\rm simp}$.

(i): It is clear that $h^{\rm conv} \leq h^{\rm qc} \leq h$. If $|\xi| \leq \frac{1}{2}$ then $h^{\rm conv}(\xi) = h(\xi)$ (cf. (2.15)), and the assertion then follows.

(ii): If $\operatorname{rank}\xi = 1$ with $|\xi| > \frac{1}{2}$, then for any $t > |\xi|$ one has

$$\xi = \frac{t-|\xi|}{t-\frac{1}{2}}\frac{\xi}{2|\xi|} + \frac{|\xi|-\frac{1}{2}}{t-\frac{1}{2}}\frac{t\xi}{|\xi|}$$

and by rank-one convexity of h^{qc} we obtain

$$h^{\mathrm{qc}}(\xi) \le \frac{t-|\xi|}{t-\frac{1}{2}} h\Big(\frac{\xi}{2|\xi|}\Big) + \frac{|\xi|-\frac{1}{2}}{t-\frac{1}{2}} h\Big(t\frac{\xi}{|\xi|}\Big) \le \frac{t-|\xi|}{t-\frac{1}{2}}\frac{1}{4} + \frac{|\xi|-\frac{1}{2}}{t-\frac{1}{2}} t.$$

Taking $t \to \infty$ shows that $h^{\mathrm{qc}}(\xi) \le |\xi| - \frac{1}{4} = h^{\mathrm{conv}}(\xi)$. Recalling $h^{\mathrm{conv}} \le h^{\mathrm{qc}}$ concludes the proof.

(iii): We assume that $\operatorname{rank}\xi \ge 2$ and $|\xi| > \frac{1}{2}$, and show that $h^{\mathrm{conv}}(\xi) < h^{\mathrm{qc}}(\xi)$. From the explicit formulas given in Lemma 2.3(i) we know that $h^{\mathrm{conv}}(\xi) < h(\xi)$, from general theory $h^{\mathrm{conv}} \le h^{\mathrm{qc}}$.

Assume by contradiction that $h^{\mathrm{conv}}(\xi) = h^{\mathrm{qc}}(\xi)$. Then there is a sequence $\varphi_j \in C^\infty((0,1)^n;\mathbb{R}^m)$ such that $\varphi_j(x) = \xi x$ on $\partial(0,1)^n$ and

$$h^{\mathrm{conv}}(\xi) = \lim_{j\to\infty} \int_{(0,1)^n} h(\nabla\varphi_j)\mathrm{d}x. \tag{2.21}$$

We consider the affine function $L : \mathbb{R}^{m\times n} \to \mathbb{R}$,

$$L(\eta) := \frac{\eta\cdot\xi}{|\xi|} - \frac{1}{4}.$$

One easily checks that $h^{\mathrm{conv}}(t\xi) = L(t\xi) = t|\xi| - \frac{1}{4}$ for $t \ge \frac{1}{2|\xi|}$ (cf. (2.15)), and since $|\xi| > \frac{1}{2}$ this in particular holds for $t = 1$. Linearity and the boundary values of φ_j imply

$$\int_{(0,1)^n} L(\nabla\varphi_j)\mathrm{d}x = L\left(\int_{(0,1)^n} \nabla\varphi_j \,\mathrm{d}x\right) = L(\xi).$$

Subtracting from (2.21), and letting $g := h - L$, leads to

$$\lim_{j\to\infty} \int_{(0,1)^n} g(\nabla\varphi_j)\mathrm{d}x = 0. \tag{2.22}$$

We next show that $g(\eta)$ controls the distance of the matrix η from the set $\mathbb{R}\xi$. To do this, for $\eta \in \mathbb{R}^{m\times n}$ we define the orthogonal projections

$$\eta^{\parallel} := \frac{\eta\cdot\xi}{|\xi|} \in \mathbb{R} \qquad \text{and} \qquad \eta^{\perp} := \eta - \frac{\xi}{|\xi|}\eta^{\parallel} \in \mathbb{R}^{m\times n},$$

so that $|\eta|^2 = |\eta^{\parallel}|^2 + |\eta^{\perp}|^2$ and $L(\eta) = \eta^{\parallel} - \frac{1}{4}$.

We first consider the case $|\eta| \ge 1$, so that $h(\eta) = |\eta|$. Assume for a moment that both $\eta^{\parallel}$ and $\eta^{\perp}$ do not vanish. Letting $\gamma := |\eta^{\perp}|/|\eta^{\parallel}|$,

$$g(\eta) = |\eta| - L(\eta) \ge |\eta^{\parallel}|\sqrt{1+\gamma^2} - |\eta^{\parallel}| = \frac{\sqrt{1+\gamma^2}-1}{\gamma}|\eta^{\perp}|.$$

Let now $\varepsilon \in (0,1]$. If $\gamma \le \varepsilon$, then $|\eta^\perp| \le \varepsilon|\eta^\parallel|$. Otherwise, by monotonicity of $t \mapsto (\sqrt{1+t^2}-1)/t$ we have $g(\eta) \ge (\sqrt{1+\varepsilon^2}-1)|\eta^\perp|/\varepsilon$. Therefore

$$|\eta^\perp| \le \varepsilon|\eta^\parallel| + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}-1} g(\eta) \tag{2.23}$$

for all $\eta \in \mathbb{R}^{m\times n}$ with $|\eta| \ge 1$ (the two cases $\eta^\parallel = 0$ and $\eta^\perp = 0$ follow by continuity). If instead $|\eta| \le 1$,

$$g(\eta) = |\eta|^2 - L(\eta) = |\eta^\parallel|^2 + |\eta^\perp|^2 - \eta^\parallel + \frac{1}{4} \ge |\eta^\perp|^2.$$

Therefore for any $\varepsilon \in (0,1]$ we have for all $\eta \in \mathbb{R}^{m\times n}$ with $|\eta| \le 1$

$$|\eta^\perp| \le \varepsilon + \frac{1}{\varepsilon}|\eta^\perp|^2 \le \varepsilon + \frac{1}{\varepsilon} g(\eta)\,. \tag{2.24}$$

Combining (2.23) and (2.24) we see that for any $\varepsilon \in (0,1]$ there is $C_\varepsilon > 0$ such that for all $\eta \in \mathbb{R}^{m\times n}$

$$|\eta^\perp| \le \varepsilon(|\eta^\parallel| + 1) + C_\varepsilon g(\eta)\,.$$

In particular, for any j we have

$$|\nabla\varphi_j^\perp| \le \varepsilon(|\nabla\varphi_j^\parallel| + 1) + C_\varepsilon g(\nabla\varphi_j).$$

We integrate over $(0,1)^n$, take the limit $j \to \infty$ and recall that $g(\nabla\varphi_j) \to 0$ in L^1 by (2.22). We obtain

$$\limsup_{j\to\infty} \int_{(0,1)^n} |\nabla\varphi_j^\perp| \mathrm{d}x \le \varepsilon \limsup_{j\to\infty} \int_{(0,1)^n} (|\nabla\varphi_j^\parallel| + 1) \mathrm{d}x$$

for any $\varepsilon \in (0,1]$. By (2.21) and Lemma 2.3(ii) the sequence $\nabla\varphi_j$ is bounded in L^1, and since ε was arbitrary we conclude that

$$\limsup_{j\to\infty} \int_{(0,1)^n} |\nabla\varphi_j^\perp| \mathrm{d}x = 0. \tag{2.25}$$

We next prove that (2.25) implies that $\nabla\varphi_j$ converges to the constant ξ strongly in weak-L^1. To do this we show that standard singular integral estimates imply rigidity. To simplify notation, we write $u_j(x) := \varphi_j(x) - \xi x$ and $R_j := \nabla\varphi_j^\perp = \nabla u_j^\perp$, both extended by zero to the rest of $\mathbb{R}^n$, in the next steps. We observe that

$$R_j = \nabla u_j - \xi \frac{\xi \cdot \nabla u_j}{|\xi|^2} = \nabla u_j - \tilde\xi(\tilde\xi \cdot \nabla u_j)$$

where $\tilde\xi := \frac{\xi}{|\xi|}$. Taking a derivative, and writing components, we obtain

$$(\nabla R_j)_{cdk} = (\nabla^2 u_j)_{cdk} - \tilde\xi_{cd} \sum_{a,b} \tilde\xi_{ab} (\nabla^2 u_j)_{abk} = (T(\nabla^2 u_j))_{cdk},$$

with T obtained from $\tilde{\xi}$ as in Lemma 2.4. Let S be the inverse operator. Then

$$\nabla^2 u_j = S(\nabla R_j),$$

so that in particular Δu_j is given by a linear combination of the components of ∇R_j, with coefficients which depend only on ξ. As $u_j(x) = 0$ outside $(0,1)^n$, we obtain, denoting by N the fundamental solution of Laplace's equation in $\mathbb{R}^n$ (which solves $-\Delta N = \delta_0$),

$$-\partial_r u_j = \partial_r(N * \Delta u_j) = \partial_r(N * \operatorname{Tr} S(\nabla R_j)) = \operatorname{Tr} S(\Lambda_r(R_j)),$$

for every $r = 1, \dots, n$, where we have set $(\Lambda_r(R_j))_{cdk} := \partial_r \partial_k N * (R_j)_{cd}$ (recall that $R_j = 0$ outside of $(0,1)^n$), and $(\operatorname{Tr}\Gamma)_l := \sum_{i=1}^n \Gamma_{lii}$, for every $l = 1, \dots, m$ and $\Gamma \in \mathbb{R}^{m\times n\times n}$. By [Ste70, Theorem 4(b), page 42] we see that the operator $R \mapsto \Lambda_r(R)$ is of weak type $(1,1)$, so that

$$\|\nabla u_j\|_{w-L^1((0,1)^n)} \le c \|R_j\|_{L^1((0,1)^n)},$$

with c depending only on ξ. Recalling the definition of u_j and R_j as well as (2.25),

$$\lim_{j\to\infty} \|\nabla\varphi_j - \xi\|_{w-L^1((0,1)^n)} \le c \lim_{j\to\infty} \|\nabla\varphi_j^\perp\|_{L^1((0,1)^n)} = 0.$$

To conclude the proof we choose $z \in (h^{\mathrm{conv}}(\xi), h(\xi))$ (here we use again that $|\xi| > \frac{1}{2}$). By continuity of h, there is $\delta > 0$ such that $h(\eta) \ge z$ for all $\eta \in \mathbb{R}^{m\times n}$ with $|\eta - \xi| < \delta$. By definition of the weak-L^1 norm,

$$\limsup_{j\to\infty} \mathcal{L}^n(\{x \in (0,1)^n : |\nabla\varphi_j - \xi| \ge \delta\}) \le \limsup_{j\to\infty} \frac{\|\nabla\varphi_j - \xi\|_{w-L^1}}{\delta} = 0\,.$$

Therefore, recalling that $h \ge 0$ pointwise,

$$\begin{aligned}\liminf_{j\to\infty} \int_{(0,1)^n} h(\nabla\varphi_j)\mathrm{d}x \ge& \liminf_{j\to\infty} z\mathcal{L}^n(\{x \in (0,1)^n : |\nabla\varphi_j - \xi| < \delta\}) \\ =& z > h^{\mathrm{conv}}(\xi).\end{aligned}$$

This contradicts (2.21) and concludes the proof. □

3 Energy densities of the surface and Cantor part

In this section we discuss several properties of the energy densities g and $h^{\mathrm{qc},\infty}$. We warn the reader that while the results dealing with g contained in subsections 3.1 and 3.2 will be crucial in the proof of ¸Theorem 2.1, those in subsection 3.3 will not be employed in that proof. Actually, Proposition 3.9 and Corollary 3.11 take advantage of Theorem 2.1 itself (in particular of the lower semicontinuity of Γ-limits).

3.1 Equivalent characterizations of $g(z,\nu)$

We show below that we may reduce the test sequences in the definition of $g(z,\nu)$ in (2.12) to those converging in L^2 and satisfying periodic boundary conditions in $(n-1)$ directions orthogonal to ν and mutually orthogonal to each other. This is the content of the next two propositions, which will be crucial in the proof of the upper bound for the surface part (Theorem 5.2 Step 2). The proof draws inspiration from that of [BF94, Lemma 4.2]. We fix a mollifier $\varphi_1 \in C_c^\infty(B_1)$, with $\int_{B_1} \varphi_1 \mathrm{d}x = 1$, and set $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi_1(x/\varepsilon)$ in B_ε.

Proposition 3.1. *Assume an optimal sequence in* (2.12) *converges in* $L^2(Q^\nu;\mathbb{R}^{m+1})$. *Then there are* $\varepsilon_j \to 0$, $(u_j^*, v_j^*) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ *in* $L^2(Q^\nu;\mathbb{R}^{m+1})$, *with* $v_j^* \in [0,1]$ $\mathcal{L}^n$*-a.e. in* Ω, *such that*

$$\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}^\infty(u_j^*, v_j^*; Q^\nu) \le g(z,\nu)$$

and

$$u_j^* = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\varepsilon_j}\,, \qquad v_j^* = \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j\}} * \varphi_{\varepsilon_j} \qquad \text{on } \partial Q^\nu. \tag{3.1}$$

Proof. **Step 1. Construction of u_j^* and v_j^*.** Pick $\varepsilon_j \to 0$, v_j and $u_j \to z\chi_{\{x\cdot\nu>0\}}$ in $L^2(Q^\nu;\mathbb{R}^m)$ such that

$$g(z,\nu) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}^\infty(u_j, v_j; Q^\nu).$$

To simplify the notation we write

$$U_j := (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\varepsilon_j}\,, \qquad V_j := \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j\}} * \varphi_{\varepsilon_j}\,. \tag{3.2}$$

Obviously $\|U_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^2(Q^\nu)} \to 0$, so that $\|u_j - U_j\|_{L^2(Q^\nu)} \to 0$. Moreover, by construction $U_j = z\chi_{\{x\cdot\nu>0\}}$ if $|x\cdot\nu| \ge \varepsilon_j$, $V_j = 0$ if $|x\cdot\nu| \le \varepsilon_j$, and $V_j = 1$ if $|x\cdot\nu| \ge 3\varepsilon_j$. Therefore, by $\Psi_\infty(0) = 0$ and $f(0) = 0$, we have

$$\mathcal{F}_{\varepsilon_j}^\infty(U_j, V_j; Q^\nu) = \mathcal{F}_{\varepsilon_j}^\infty(0, V_j; Q^\nu) \le c + \varepsilon_j \int_{\{x\in Q^\nu : \varepsilon_j < |x\cdot\nu| < 3\varepsilon_j\}} |\nabla V_j|^2 \mathrm{d}x \le c\,,$$

as $\|\nabla V_j\|_{L^\infty(\mathbb{R}^m)} \le \frac{c}{\varepsilon_j}$, where c is a constant independent of $j \in \mathbb{N}$.

Next, we choose a sequence $\eta_j \to 0$ such that

$$\frac{\varepsilon_j + \|u_j - U_j\|_{L^2(Q^\nu)}^{2/3}}{\eta_j} \to 0 \tag{3.3}$$

and set $K_j := \lfloor \eta_j/\varepsilon_j \rfloor$, we can assume $K_j \geq 4$. We let $\hat{R}_k^j := Q^\nu_{1-k\varepsilon_j} \setminus Q^\nu_{1-(k+1)\varepsilon_j}$, where we write for brevity $Q^\nu_r := rQ^\nu$ for the scaled cube. We select $k_j \in \{K_j+1, \dots, 2K_j\}$ such that, writing $R_j := \hat{R}^j_{k_j}$,

$$\|u_j - U_j\|^2_{L^2(R_j)} \leq \frac{c}{K_j} \|u_j - U_j\|^2_{L^2(Q^\nu)} \tag{3.4}$$

and

$$\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; R_j) + \mathcal{F}^\infty_{\varepsilon_j}(U_j, V_j; R_j) \leq \frac{c}{K_j}. \tag{3.5}$$

We fix $\theta_j \in C^1_c(Q^\nu_{1-k_j\varepsilon_j})$ with $\theta_j = 1$ on $Q^\nu_{1-(k_j+1)\varepsilon_j}$ and $|\nabla \theta_j| \leq 3/\varepsilon_j$, and define

$$u_j^* := \theta_j u_j + (1-\theta_j) U_j.$$

The construction of v_j^* is more complex. In the interior part, it should match v_j. In the exterior, V_j. In the interpolation region, it should be not larger than v_j and V_j, but also not larger than $1-\eta_j$. Therefore we first define

$$\hat{v}_j(x) := \min\{1, 1-\eta_j + \frac{1}{\varepsilon_j}\mathrm{dist}(x, R_j)\}, \tag{3.6}$$

which coincides with $1-\eta_j$ in the interpolation region R_j, and with 1 at distance larger than $\eta_j\varepsilon_j$ from it, then

$$\hat{V}_j(x) := \min\{1, V_j(x) + \frac{1}{\varepsilon_j}\mathrm{dist}(x, Q^\nu \setminus Q^\nu_{1-(k_j+1)\varepsilon_j})\} \tag{3.7}$$

which coincides with V_j outside $Q^\nu_{1-(k_j+1)\varepsilon_j}$, and with 1 inside $Q^\nu_{1-(k_j+3)\varepsilon_j}$ as well as for $|x \cdot \nu| \geq 3\varepsilon_j$ (cf. the definition of V_j), and finally

$$\tilde{v}_j := \min\{1, v_j + \frac{2}{k_j\varepsilon_j}\mathrm{dist}(x, Q^\nu_{1-k_j\varepsilon_j})\}. \tag{3.8}$$

We then combine these three ingredients to obtain

$$v_j^* := \min\{\tilde{v}_j, \hat{V}_j, \hat{v}_j\}.$$

On ∂Q^ν the first and the last term are equal to 1, hence $v_j^* = \hat{V}_j = V_j$.

Step 2. Estimate of the elastic energy. By the definition of u_j^*,

$$|\nabla u_j^*| \leq |\nabla u_j| + |\nabla U_j| + \frac{3}{\varepsilon_j}|u_j - U_j|$$

therefore in R_j

$$\Psi_\infty(\nabla u_j^*) \le c\Psi_\infty(\nabla u_j) + c\Psi_\infty(\nabla U_j) + \frac{c}{\varepsilon_j^2}|u_j - U_j|^2.$$

We recall that $v_j^* \le \min\{v_j, V_j, 1-\eta_j\}$ in R_j and that $[0,1) \ni t \mapsto t/(1-t)$ is increasing. Since by construction $v_j^* = V_j = 0$ on $\{\nabla U_j \neq 0\} \cap R_j$ the term $\Psi_\infty(\nabla U_j)$ can be ignored. Therefore

$$\frac{\varepsilon_j (v_j^*)^2}{(1-v_j^*)^2}\Psi_\infty(\nabla u_j^*) \le c\frac{\varepsilon_j v_j^2}{(1-v_j)^2}\Psi_\infty(\nabla u_j) + c\frac{\varepsilon_j}{\eta_j^2}\frac{|u_j - U_j|^2}{\varepsilon_j^2}.$$

Integrating over R_j and using (3.5) in the first term, (3.4) in the second one,

$$\int_{R_j} \frac{\varepsilon_j (v_j^*)^2}{(1-v_j^*)^2}\Psi_\infty(\nabla u_j^*)\mathrm{d}x \le \frac{c}{K_j} + c\frac{\|u_j - U_j\|^2_{L^2(Q^\nu)}}{K_j \varepsilon_j \eta_j^2}.$$

Using first that the definition of K_j implies $\lim_{j\to\infty} K_j \varepsilon_j/\eta_j = 1$ and then (3.3),

$$\limsup_{j\to\infty} \frac{\|u_j - U_j\|^2_{L^2(Q^\nu)}}{K_j \varepsilon_j \eta_j^2} = \limsup_{j\to\infty} \frac{\|u_j - U_j\|^2_{L^2(Q^\nu)}}{\eta_j^3} = 0.$$

Therefore

$$\limsup_{j\to\infty} \int_{R_j} \frac{\varepsilon_j (v_j^*)^2}{(1-v_j^*)^2}\Psi_\infty(\nabla u_j^*)\mathrm{d}x = 0.$$

Using again that the supports of ∇U_j and V_j are disjoint, we have

$$\int_{Q^\nu \setminus Q^\nu_{1-k_j\varepsilon_j}} \frac{\varepsilon_j V_j^2}{(1-V_j)^2}\Psi_\infty(\nabla U_j)\mathrm{d}x = 0.$$

Therefore

$$\limsup_{j\to\infty} \int_{Q^\nu} \frac{\varepsilon_j (v_j^*)^2}{(1-v_j^*)^2}\Psi_\infty(\nabla u_j^*)\mathrm{d}x \le \limsup_{j\to\infty} \int_{Q^\nu} \frac{\varepsilon_j v_j^2}{(1-v_j)^2}\Psi_\infty(\nabla u_j)\mathrm{d}x. \quad (3.9)$$

Step 3. Estimate of the energy of the phase field. By the definition of v_j^*,

$$\mathcal{F}^\infty_{\varepsilon_j}(0, v_j^*; Q^\nu) \le \mathcal{F}^\infty_{\varepsilon_j}(0, \tilde{v}_j; Q^\nu) + \mathcal{F}^\infty_{\varepsilon_j}(0, \hat{V}_j; Q^\nu) + \mathcal{F}^\infty_{\varepsilon_j}(0, \hat{v}_j; Q^\nu). \qquad (3.10)$$

From (3.6) we have $|1-\hat{v}_j| \le \eta_j$ with $|\{\hat{v}_j \neq 1\}| \le c\varepsilon_j$ and $|\nabla \hat{v}_j| \le 1/\varepsilon_j$ with $|\{\nabla \hat{v}_j \neq 0\}| \le c\varepsilon_j\eta_j$, so that

$$\mathcal{F}^\infty_{\varepsilon_j}(0, \hat{v}_j; Q^\nu) = \int_{Q^\nu} \Big(\frac{(1-\hat{v}_j)^2}{4\varepsilon_j} + \varepsilon_j |\nabla \hat{v}_j|^2\Big)\mathrm{d}x \le c\eta_j.$$

From the definition of V_j and $\hat{V}_j$, we see that $|\{\hat{V}_j \neq 1\}| \le c\eta_j\varepsilon_j$ and $\varepsilon_j|\nabla \hat{V}_j| \le c$, so that

$$\mathcal{F}^\infty_{\varepsilon_j}(0, \hat{V}_j; Q^\nu) \le c\eta_j.$$

Similarly, $\tilde{v}_j = v_j$ in $Q^\nu_{1-k_j\varepsilon_j}$, $|\tilde{v}_j - 1| \le |v_j - 1|$, and $|\nabla\tilde{v}_j| \le |\nabla v_j| + 2/(k_j\varepsilon_j)$ in $Q^\nu \setminus Q^\nu_{1-k_j\varepsilon_j}$ lead to

$$\mathcal{F}^\infty_{\varepsilon_j}(0,\tilde{v}_j;Q^\nu) \le \mathcal{F}^\infty_{\varepsilon_j}(0,v_j;Q^\nu) + \frac{4\varepsilon_j\mathcal{L}^n(Q^\nu \setminus Q^\nu_{1-k_j\varepsilon_j})}{k_j^2\varepsilon_j^2}$$
$$+\frac{4}{k_j\varepsilon_j^{1/2}}\mathcal{F}^\infty_{\varepsilon_j}(0,v_j;Q^\nu)^{1/2}\mathcal{L}^n(Q^\nu \setminus Q^\nu_{1-k_j\varepsilon_j})^{1/2} \le \mathcal{F}^\infty_{\varepsilon_j}(0,v_j;Q^\nu) + \frac{c}{k_j^{1/2}}.$$

Recalling $k_j \ge K_j + 1 \to \infty$ and $\eta_j \to 0$, (3.10) leads to

$$\limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(0,v_j^*;Q^\nu) \le \limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(0,v_j;Q^\nu).$$

Combining this with (3.9) concludes the proof. □

We are now ready to perform the claimed reduction on the test sequences in the definition of $g(\cdot,\nu)$ in (2.12). To this aim we fix a sequence $(a_k)_k \subset (0,\infty)$ such that $a_k < a_{k+1}$, $a_k \uparrow \infty$, and such that there are functions $\mathcal{T}_k \in C^1_c(\mathbb{R}^m;\mathbb{R}^m)$ satisfying

$$\mathcal{T}_k(z) := \begin{cases} z, & \text{if } |z| \le a_k, \\ 0, & \text{if } |z| \ge a_{k+1} \end{cases} \tag{3.11}$$

and $\|\nabla\mathcal{T}_k\|_{L^\infty(\mathbb{R}^m)} \le 1$. Following De Giorgi's averaging/slicing procedure on the codomain, the family $\mathcal{T}_k$ will be used in several instances along the paper to obtain from a sequence converging in L^1 to a limit belonging to L^∞, a sequence with the same L^1 limit which is in addition equi-bounded in L^∞. Moreover, this substitution can be done up to paying an error in energy which can be made arbitrarily small.

Proposition 3.2. *For any $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ and any $\varepsilon_j^* \downarrow 0$ there is $(u_j^*,v_j^*) \to (z\chi_{\{x\cdot\nu>0\}},1)$ in $L^2(Q^\nu;\mathbb{R}^{m+1})$, with $v_j^* \in [0,1]$ $\mathcal{L}^n$-a.e. in Ω, such that*

$$\lim_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j^*}(u_j^*,v_j^*;Q^\nu)=g(z,\nu) \tag{3.12}$$

and

$$u_j^* = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\varepsilon_j^*}, \qquad v_j^* = \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j^*\}} * \varphi_{\varepsilon_j^*} \qquad \text{on } \partial Q^\nu.$$

Proof. **Step 1. Reduction to an optimal sequence in** (2.12) **converging in** $L^2(Q^\nu;\mathbb{R}^{m+1})$**.** Let $\varepsilon_j \to 0$, $(u_j,v_j) \to (z\chi_{\{x\cdot\nu>0\}},1)$ in $L^1(Q^\nu;\mathbb{R}^{m+1})$ be such that

$$g(z,\nu) = \lim_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu).$$

Recall that $v_j \in [0,1]$ $\mathcal{L}^n$-a.e. in Ω, therefore $v_j \to 1$ in $L^2(Q^\nu)$. We claim that for all j, $M \in \mathbb{N}$ there is $k_{M,j} \in \{M+1,\dots,2M\}$ such that

$$\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_{k_{M,j}}(u_j),v_j;Q^\nu) \le \Big(1+\frac{c}{M}\Big)\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)\,, \tag{3.13}$$

where $c > 0$ is a constant independent of M and j. If $a_M > 1 + |z| = 1 + \|z\chi_{\{x\cdot\nu>0\}}\|_{L^\infty(Q^\nu)}$ then $\mathcal{T}_{k_M,j}(u_j) \to z\chi_{\{x\cdot\nu>0\}}$ in $L^2(Q^\nu;\mathbb{R}^m)$, and (3.13) yields

$$\limsup_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_{k_M,j}(u_j), v_j; Q^\nu) \le \Big(1 + \frac{c}{M}\Big) g(z,\nu)\,,$$

in turn implying by the arbitrariness of $M \in \mathbb{N}$

$$g(z,\nu) = \inf\{\liminf_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu) : \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^2(Q^\nu)} \to 0, \varepsilon_j \to 0\}.$$

We are left with establishing (3.13). To this aim consider $\mathcal{T}_k(u_j)$ and note that

$$\begin{aligned}\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) &= \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; \{|u_j| \le a_k\}) \\ &\quad + \mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j), v_j; \{a_k < |u_j| < a_{k+1}\}) + \mathcal{F}^\infty_{\varepsilon_j}(0, v_j; \{|u_j| \ge a_{k+1}\})\,. \quad (3.14)\end{aligned}$$

We estimate the second term in (3.14). The growth conditions on Ψ (cf. (2.3)) and $\|\nabla\mathcal{T}_k\|_{L^\infty(\mathbb{R}^m)} \le 1$ yield for a constant $c > 0$

$$\begin{aligned}&\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j), v_j; \{a_k < |u_j| < a_{k+1}\}) \\ &\quad \le c\int_{\{a_k<|u_j|<a_{k+1}\}} \varepsilon_j f^2(v_j)\Psi_\infty(\nabla u_j)\mathrm{d}x + \mathcal{F}^\infty_{\varepsilon_j}(0, v_j; \{a_k < |u_j| < a_{k+1}\})\,. \end{aligned} \quad (3.15)$$

Collecting (3.14) and (3.15) and using $\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; A) + \mathcal{F}^\infty_{\varepsilon_j}(0, v_j; B) \le \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; A \cup B)$ for A and B disjoint we conclude that

$$\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) \le \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu) + c\int_{\{a_k<|u_j|<a_{k+1}\}} \varepsilon_j f^2(v_j)\Psi_\infty(\nabla u_j)\mathrm{d}x\,.$$

Let now $M \in \mathbb{N}$, by averaging there exists $k_{M,j} \in \{M+1, \dots, 2M\}$ such that

$$\begin{aligned}\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_{k_{M,j}}(u_j), v_j; Q^\nu) &\le \frac{1}{M}\sum_{k=M+1}^{2M} \mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) \\ &\le \Big(1 + \frac{c}{M}\Big)\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu)\,,\end{aligned}$$

i.e. (3.13).

Step 2. Conclusion. In view of Step 1 there is an optimal sequence for $g(z,\nu)$ in (2.12) converging in $L^2(Q^\nu;\mathbb{R}^{m+1})$. Let $(\varepsilon_k, u_k, v_k)$ be the sequence from Proposition 3.1. Since $\lim_{k\to\infty}\lim_{j\to 0}\varepsilon^*_j/\varepsilon_k = 0$, we can select a nondecreasing sequence $k(j) \to \infty$ such that $\lambda_j := \varepsilon^*_j/\varepsilon_{k(j)} \to 0$. We let $\tilde{Q}^\nu := (\mathrm{Id} - \nu\otimes\nu)Q^\nu \subset \nu^\perp \subset \mathbb{R}^n$ and select $x_1, \dots, x_{I_j} \in \tilde{Q}^\nu$, with $I_j := \lfloor 1/\lambda_j \rfloor^{n-1}$, such that $x_i + \tilde{Q}^\nu_{\lambda_j}$ are pairwise disjoint subsets of $\tilde{Q}^\nu$. We set

$$u^*_j(x) := \begin{cases} u_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x - x_i \in Q^\nu_{\lambda_j} \text{ for some } i, \\ U^*_j(x), & \text{otherwise in } Q^\nu \end{cases}$$

and

$$v_j^*(x) := \begin{cases} v_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x - x_i \in Q^\nu_{\lambda_j} \text{ for some } i, \\ V_j^*(x), & \text{otherwise in } Q^\nu, \end{cases}$$

where U_j^* and V_j^* are defined as in (3.2) using ε_j^*. One easily verifies that $U_j^*(x) = U_{k(j)}(\frac{x-y}{\lambda_j})$ for all $y \in \nu^\perp$, and the same for V. By the boundary conditions (3.1), these functions are continuous and therefore in $W^{1,2}(Q^\nu;\mathbb{R}^{m+1})$. We further estimate

$$\mathcal{F}^\infty_{\varepsilon_j^*}(u_j^*, v_j^*; Q^\nu) \le I_j \lambda_j^{n-1} \mathcal{F}^\infty_{\varepsilon_{k(j)}}(u_{k(j)}, v_{k(j)}; Q^\nu) + c\mathcal{H}^{n-1}(\tilde{Q}^\nu \setminus \cup_i (x_i + \tilde{Q}^\nu_{\lambda_j})).$$

Taking $j \to \infty$, and recalling that $\limsup_j \mathcal{F}^\infty_{\varepsilon_{k(j)}}(u_{k(j)}, v_{k(j)}; Q^\nu) \le g(z,\nu)$, concludes the proof. □

In what follows we provide an equivalent characterization for the surface energy g in the spirit of [CFI16, Proposition 4.3].

Proposition 3.3. *For any $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ one has*

$$g(z,\nu) = \lim_{T\to\infty} \inf_{(u,v)\in\mathcal{U}^T_{z,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u,v;Q^\nu_T)\,, \tag{3.16}$$

where

$$\mathcal{U}^T_{z,\nu} := \Big\{(u,v) \in W^{1,2}(Q^\nu_T;\mathbb{R}^{m+1})\colon 0 \le v \le 1,\ v = \chi_{\{|x\cdot\nu|\ge 2\}} * \varphi_1 \ \text{and} \\ u = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_1 \ \text{on } \partial Q^\nu_T\Big\}.$$

Proof. For every $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ and $T > 0$ set

$$g_T(z,\nu) := \inf_{(u,v)\in\mathcal{U}^T_{z,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u,v;Q^\nu_T)\,.$$

We first prove that

$$\limsup_{T\to\infty} g_T(z,\nu) \le g(z,\nu)\,. \tag{3.17}$$

Indeed, if $T_j \uparrow \infty$ is a sequence achieving the superior limit on the left-hand side above, thanks to Proposition 3.2 we may consider $(u_j, v_j) \in W^{1,2}(Q^\nu;\mathbb{R}^{m+1})$ with $0 \le v_j \le 1$, $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^2(Q^\nu;\mathbb{R}^{m+1})$,

$$u_j = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\frac{1}{T_j}}\,, \qquad v_j = \chi_{\{|x\cdot\nu|\ge\frac{2}{T_j}\}} * \varphi_{\frac{1}{T_j}} \text{ on } \partial Q^\nu, \tag{3.18}$$

and

$$\lim_{j\to\infty} \mathcal{F}^\infty_{\frac{1}{T_j}}(u_j, v_j; Q^\nu) = g(z,\nu). \tag{3.19}$$

Then, define $(\tilde{u}_j(y), \tilde{v}_j(y)) := \big(u_j(\frac{y}{T_j}), v_j(\frac{y}{T_j})\big)$ for $y \in Q^\nu_{T_j}$, and note that by a change of variable it is true that

$$\frac{1}{T_j^{n-1}} \mathcal{F}^\infty_1(\tilde{u}_j, \tilde{v}_j; Q^\nu_{T_j}) = \mathcal{F}^\infty_{\frac{1}{T_j}}(u_j, v_j; Q^\nu)\,,$$

and that $(\tilde{u}_j, \tilde{v}_j) \in \mathcal{U}^{T_j}_{z,\nu}$ in view of (3.18). Then, by (3.19), the choice of T_j and the definition of $g_T(z,\nu)$ we conclude straightforwardly (3.17).

In order to prove the converse inequality

$$\liminf_{T\to\infty} g_T(z,\nu) \geq g(z,\nu)\,, \tag{3.20}$$

we assume for the sake of notational simplicity $\nu = e_n$. We then fix $\rho > 0$ and take $T > 6$, depending on ρ, and $(u_T, v_T) \in \mathcal{U}^T_{z,e_n}$ such that

$$\frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(z,e_n) + \rho\,. \tag{3.21}$$

Let $\varepsilon_j \to 0$ and set

$$u_j(y) := \begin{cases} u_T\left(\dfrac{y}{\varepsilon_j} - d\right), & \text{if } y \in \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}, \\ (z\chi_{\{x\cdot e_n > 0\}} * \varphi_1)(\frac{y}{\varepsilon_j}), & \text{otherwise in } Q^{e_n}, \end{cases}$$

$$v_j(y) := \begin{cases} v_T\left(\dfrac{y}{\varepsilon_j} - d\right), & \text{if } y \in \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}, \\ (\chi_{\{|x\cdot e_n| > 2\}} * \varphi_1)(\frac{y}{\varepsilon_j}), & \text{otherwise in } Q^{e_n}, \end{cases}$$

with $d \in \mathbb{Z}^{n-1} \times \{0\}$. Then, $(u_j, v_j) \to (z\chi_{\{x\cdot e_n>0\}}, 1)$ in $L^1(Q^{e_n}; \mathbb{R}^{m+1})$, and letting $I_{\varepsilon_j} := \{d \in \mathbb{Z}^{n-1} \times \{0\} : \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}\}$, a change of variable yields (cf. also the discussion after (3.2))

$$\begin{aligned} g(z,e_n) &\leq \limsup_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^{e_n}) \\ &\leq \limsup_{j\to\infty} \Big(\sum_{d\in I_{\varepsilon_j}} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; \varepsilon_j(Q^{e_n}_T + d)) \\ &\quad + \frac{c}{\varepsilon_j}\mathcal{L}^n\Big(Q^{e_n} \cap \{\varepsilon_j \leq |x_n| \leq 3\varepsilon_j\} \setminus \bigcup_{d\in I_{\varepsilon_j}} \varepsilon_j(Q^{e_n}_T + d)\Big)\Big) \\ &= \limsup_{j\to\infty} \varepsilon_j^{n-1} \# I_{\varepsilon_j}\, \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \\ &\leq \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(z,e_n) + \rho\,, \end{aligned}$$

by the choice of (u_T, v_T) and T (cf. (3.21)). As $\rho \to 0$ we get (3.20).

Estimates (3.17) and (3.20) yield the existence of the limit of $g_T(z,\nu)$ as $T \uparrow \infty$ and equality (3.16), as well. □

With this representation of g at hand we can obtain a version of Proposition 3.2 which also accounts for a regularization term of the form $\eta_\varepsilon \int \Psi(\nabla u)\mathrm{d}x$.

Proposition 3.4. *For any $\varepsilon_j \downarrow 0$ and $\eta_j \downarrow 0$ with $\eta_j/\varepsilon_j \to 0$, and any $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ there is $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^2(Q^\nu; \mathbb{R}^{m+1})$, with $v_j \in [0,1]$ $\mathcal{L}^n$-a.e. in Q^ν, such that*

$$\lim_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu) = g(z,\nu)$$

$$\lim_{j\to\infty}\eta_j\int_{Q^\nu}|\nabla u_j|^2\mathrm{d}x=0\,,$$

and

$$u_j=(z\chi_{\{x\cdot\nu>0\}})*\varphi_{\varepsilon_j}\,,\qquad v_j=\chi_{\{|x\cdot\nu|\geq 2\varepsilon_j\}}*\varphi_{\varepsilon_j}\qquad \text{on } \partial Q^\nu.$$

Proof. We use the same construction as above (without loss of generality, explicitly written only for $\nu=e_n$), and compute similarly

$$\begin{aligned}\|\nabla u_j\|^2_{L^2(Q^{e_n})}&\leq\sum_{d\in I_{\varepsilon_j}}\|\nabla u_j\|^2_{L^2(\varepsilon_j(Q_T^{e_n}+d))}+\frac{c}{\varepsilon_j^2}\mathcal{L}^n\left(Q^{e_n}\cap\{|x_n|\leq\varepsilon_j\}\right)\\&=\varepsilon_j^{n-1}\#I_{\varepsilon_j}\,\|\nabla u_T\|^2_{L^2(Q_T)}+\frac{c}{\varepsilon_j}\leq\frac{C_T}{\varepsilon_j}.\end{aligned}$$

To conclude the proof it suffices to choose $T_j\to\infty$ so slow that $\eta_jC_{T_j}/\varepsilon_j\to 0$. □

Corollary 3.5. *If $\Psi_\infty(\xi)=|\xi|^2$, then $g(z,\nu)=g_{\mathrm{scal}}(|z|)$ for all $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$, where g_{scal} is defined as the right-hand side of equation* (3.16) *with $n=m=1$.*

For an equivalent definition of g_{scal} see equation (4.4) below and [CFI16, Proposition 4.3].

Proof. By [CFI16, Proposition 4.3] or by Proposition 3.3, the following characterization holds for g_{scal}:

$$g_{\mathrm{scal}}(s)=\lim_{T\uparrow\infty}\inf_{(\alpha,\beta)\in\mathcal{U}_s^T}\mathscr{F}_1^\infty(\alpha,\beta;(-{}^T\!/2,{}^T\!/2)),$$

with

$$\mathscr{F}_1^\infty(\alpha,\beta;(-{}^T\!/2,{}^T\!/2)):=\int_{-\frac{T}{2}}^{\frac{T}{2}}(f^2(\beta)|\alpha'|^2+\frac{(1-\beta)^2}{4}+|\beta'|^2)\mathrm{d}x$$

and

$$\begin{aligned}\mathcal{U}_s^T:=\{\alpha,\beta\in W^{1,2}((-{}^T\!/2,{}^T\!/2))\colon 0\leq\beta\leq1,\ \beta(\pm{}^T\!/2)=1\\ \alpha(-{}^T\!/2)=0\,,\ \alpha({}^T\!/2)=s\}.\end{aligned}$$

Let $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$, $z\neq0$. We first prove that

$$g(z,\nu)\geq g_{\mathrm{scal}}(|z|).\tag{3.22}$$

If $T>0$ and $(u,v)\in\mathcal{U}^T_{z,\nu}$ (see Proposition 3.3 for the definition of $\mathcal{U}^T_{z,\nu}$), then for $\mathcal{H}^{n-1}$-a.e. $y\in\tilde{Q}^\nu_T:=(\mathrm{Id}-\nu\otimes\nu)Q^\nu_T\subset\nu^\perp$ the slices

$$u_y^\nu(t):=\frac{z}{|z|}\cdot u(y+t\nu),\quad v_y^\nu(t):=v(y+t\nu)$$

belong to $\mathcal{U}^T_{|z|}$ and satisfy by Fubini's theorem

$$\frac{1}{T^{n-1}}\mathcal{F}_1^\infty(u,v;Q^\nu_T) \geq \frac{1}{T^{n-1}}\int_{\tilde{Q}^\nu_T} \mathscr{F}_1^\infty(u^\nu_y, v^\nu_y; (-T/2, T/2))\, \mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\geq \inf_{(\alpha,\beta)\in\mathcal{U}^T_{|z|}} \mathscr{F}_1^\infty(\alpha,\beta;(-T/2,T/2)).$$

Taking the infimum over $(u,v)\in\mathcal{U}^T_{z,\nu}$ and passing to the limit $T\to\infty$ we get (3.22).

Let us show now that

$$g(z,\nu) \leq g_{\mathrm{scal}}(|z|). \tag{3.23}$$

Let $T>0$ and $(\alpha,\beta)\in\mathcal{U}^T_{|z|}$. Fixed $\varepsilon_j\to 0$, we will construct a competitor (u_j,v_j) for the problem (2.12) defining g. We set

$$u_j(x) := \begin{cases} \alpha\Big(\dfrac{T}{\varepsilon_j}x\cdot\nu\Big)\dfrac{z}{|z|}, & \text{if } |x\cdot\nu|\leq\frac{\varepsilon_j}{2},\ x\in Q^\nu,\\ z\chi_{\{x\cdot\nu>0\}}, & \text{otherwise in } Q^\nu,\end{cases}$$

$$v_j(x) := \begin{cases} \beta\Big(\dfrac{T}{\varepsilon_j}x\cdot\nu\Big), & \text{if } |x\cdot\nu|\leq\frac{\varepsilon_j}{2},\ x\in Q^\nu,\\ 1, & \text{otherwise in } Q^\nu.\end{cases}$$

Hence by a change of variables we have $\|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)}\to 0$ and

$$\mathcal{F}^\infty_{\varepsilon_j/T}(u_j,v_j;Q^\nu) = \mathscr{F}_1^\infty(\alpha,\beta;(-T/2,T/2))\,.$$

Therefore, we conclude that

$$g(z,\nu)\leq \mathscr{F}_1^\infty(\alpha,\beta;(-T/2,T/2))\,.$$

As $(\alpha,\beta)\in\mathcal{U}^T_{|z|}$ varies, we obtain (3.23). □

Remark 3.6. *The same argument shows that if Ψ satisfies $\Psi_\infty(\xi)\geq\Psi_\infty(\xi\nu\otimes\nu)$ for every $\xi\in\mathbb{R}^{m\times n}$ and $\nu\in S^{n-1}$, then for all $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$*

$$g(z,\nu) = \lim_{T\uparrow\infty}\inf_{\widetilde{\mathcal{U}}^T_z}\int_{-T/2}^{T/2}\Big(f^2(\beta(t))\Psi_\infty\big(\alpha'(t)\otimes\nu\big) + \frac{(1-\beta(t))^2}{4} + |\beta'(t)|^2\Big)\mathrm{d}t$$

where

$$\widetilde{\mathcal{U}}^T_z := \{(\alpha,\beta)\in W^{1,2}((-T/2,T/2);\mathbb{R}^{m+1})\colon 0\leq\beta\leq 1,\ \beta(\pm T/2)=1$$
$$\alpha(-T/2)=0,\ \alpha(T/2)=z\}.$$

3.2 Structural properties of $g(z,\nu)$

We next deduce the coercivity properties of g.

Lemma 3.7. *There is $c>0$ such that, for all $z,\nu\in\mathbb{R}^m\times S^{n-1}$,*

$$\frac{1}{c}(|z|\wedge 1)\le g(z,\nu)\le c(|z|\wedge 1).$$

We provide here a direct proof of the lemma. Alternatively, these bounds may be derived estimating $\mathcal{F}_\varepsilon$ by its 1D counterpart (as in (4.2) below) and recalling the bounds holding for g_{scal}, see [CFI16, Prop. 4.1].

Proof. We start with the lower bound. Let $z\in\mathbb{R}^m$, $\nu\in S^{n-1}$, and fix sequences $\varepsilon_j\to 0$, v_j and $u_j\to z\chi_{\{x\cdot\nu>0\}}$ in $L^1(Q^\nu;\mathbb{R}^m)$ such that $\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)\to g(z,\nu)$. For every j and $y_j\in\nu^\perp\cap Q^\nu$ we define $v_j^*\in W^{1,2}((-\frac12,\frac12);[0,1])$ and $u_j^*\in W^{1,2}((-\frac12,\frac12);\mathbb{R}^m)$ by $v_j^*(t):=v_j(y_j+t\nu)$ and $u_j^*(t):=u_j(y_j+t\nu)$. The set of $y_j\in\nu^\perp\cap Q^\nu$ such that

$$\|u_j^*-z\chi_{\{t\ge 0\}}\|_{L^1((-\frac12,\frac12))}\le 3\|u_j-z\chi_{\{x\cdot\nu\ge 0\}}\|_{L^1(Q^\nu)}$$

has measure at least $\frac23$ and, using (2.3) to estimate $\frac1c|(u_j^*)'|^2(t)\le\Psi_\infty(\nabla u_j)(y_j+t\nu)$, the set of $y_j\in\nu^\perp\cap Q^\nu$ such that

$$\int_{(-\frac12,\frac12)}\Big(\frac{\varepsilon_j\ell^2(v_j^*)^2}{(1-v_j^*)^2}\frac{|(u_j^*)'|^2}{c}+\frac{(1-v_j^*)^2}{4\varepsilon_j}+\varepsilon_j|(v_j^*)'|^2\Big)\mathrm{d}t\le 3\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)$$

also has measure at least $\frac23$. Therefore we can fix y_j such that both inequalities hold. If $g(z,\nu)<\infty$, then necessarily $v_j^*\to 1$ in $L^2((-\frac12,\frac12))$, and it has a continuous representative. We can therefore assume that $\sup v_j^*\ge\frac34$ for large j. If $\inf v_j^*\le\frac12$ then

$$\begin{aligned}\frac12(1-v)^2\Big|_{1/2}^{3/4}&\le\int_{(-\frac12,\frac12)}|(1-v_j^*)(v_j^*)'|\mathrm{d}t\\&\le\int_{(-\frac12,\frac12)}\frac{(1-v_j^*)^2}{4\varepsilon_j}+\varepsilon_j|(v_j^*)'|^2\mathrm{d}t\le 3\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu).\end{aligned}$$

Otherwise, $v_j^*\ge\frac12$ pointwise and

$$\int_{(-\frac12,\frac12)}\Big(\frac{\varepsilon_j\ell^2(v_j^*)^2}{(1-v_j^*)^2}\frac{|(u_j^*)'|^2}{c}+\frac{(1-v_j^*)^2}{4\varepsilon_j}\Big)\mathrm{d}t\ge\frac{1}{2c^{1/2}}\ell\int_{(-\frac12,\frac12)}|(u_j^*)'|\mathrm{d}t.$$

Since $\|u_j^*-z\chi_{t\ge0}\|_{L^1((-\frac12,\frac12))}\to 0$, there are t_j,t_j' such that $u_j^*(t_j)\to 0$, $u_j^*(t_j')\to z$, and therefore $\liminf_{j\to\infty}\int_{(-\frac12,\frac12)}|(u_j^*)'|\mathrm{d}t\ge\liminf_{j\to\infty}|u_j^*(t_j)-u_j^*(t_j')|=|z|$. We conclude that $\liminf_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)\ge c(1\wedge\ell|z|)$.

We turn to the upper bound. We define $u_j(x) := u_j^*(x\cdot\nu)$, $v_j(x) := v_j^*(x\cdot\nu)$, where, denoting by AI the affine interpolation between the boundary data in the relevant segments,

$$u_j^*(t) := \begin{cases} 0, & \text{if } t \le -\varepsilon_j, \\ z, & \text{if } t \ge \varepsilon_j, \\ AI, & \text{if } -\varepsilon_j < t < \varepsilon_j, \end{cases} \qquad v_j^*(t) := \begin{cases} (1-(\ell|z|)^{1/2})_+, & \text{if } |t| \le \varepsilon_j, \\ 1, & \text{if } |t| \ge 2\varepsilon_j, \\ AI, & \text{if } |t| \in (\varepsilon_j, 2\varepsilon_j). \end{cases}$$

If $\ell|z| < 1$, then the upper bound in (2.3) leads to

$$\mathcal{F}_{\varepsilon_j}^\infty(u_j, v_j; Q^\nu) \le 2\varepsilon_j \frac{\varepsilon_j \ell^2 c(|z|/2\varepsilon_j)^2}{\ell|z|} + 4\varepsilon_j \frac{\ell|z|}{4\varepsilon_j} + 2\varepsilon_j\varepsilon_j \frac{\ell|z|}{\varepsilon_j^2} = (\frac{1}{2}c + 1 + 2)\ell|z|.$$

If instead $\ell|z| \ge 1$ the first term vanishes, and

$$\mathcal{F}_{\varepsilon_j}^\infty(u_j, v_j; Q^\nu) \le 0 + 4\varepsilon_j \frac{1}{4\varepsilon_j} + 2\varepsilon_j\varepsilon_j \frac{1}{\varepsilon_j^2} = 3. \qquad \square$$

We prove next the subadditivity and continuity of g.

Lemma 3.8. *(i) For any $\nu \in S^{n-1}$ and $z^1, z^2 \in \mathbb{R}^m$ one has*

$$g(z^1 + z^2, \nu) \le g(z^1, \nu) + g(z^2, \nu).$$

(ii) $g \in C^0(\mathbb{R}^m \times S^{n-1})$.

Proof. (i): Fix $z^1, z^2 \in \mathbb{R}^m$, $\nu \in S^{n-1}$. Let (u_j^i, v_j^i) be the sequences from Proposition 3.2 corresponding to $\varepsilon_j := 1/j$ and the pair (ν, z^i), for $i = 1, 2$. We implicitly extend both periodically in the directions of $\nu^\perp \cap Q^\nu$, and constant in the direction ν. In particular, for $\{x\cdot\nu \ge \frac{1}{2}\}$ we have $u_j^i = z^i$ and $v_j^i = 1$; for $\{x\cdot\nu \le -\frac{1}{2}\}$ we have $u_j^i = 0$ and $v_j^i = 1$ for $i \in \{1, 2\}$ and all j.

We use a rescaling similar to the one of Proposition 3.2. We fix a sequence $M_j \in \mathbb{N}$, $M_j \to \infty$, and define $(u_j, v_j) \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^m \times [0,1])$ by

$$u_j(x) := \begin{cases} u_j^1(M_j x + \frac{1}{2}\nu), & \text{if } x\cdot\nu < 0, \\ z^1 + u_j^2(M_j x - \frac{1}{2}\nu), & \text{if } x\cdot\nu \ge 0, \end{cases}$$

and, correspondingly,

$$v_j(x) := \begin{cases} v_j^1(M_j x + \frac{1}{2}\nu), & \text{if } x\cdot\nu < 0, \\ v_j^2(M_j x - \frac{1}{2}\nu), & \text{if } x\cdot\nu \ge 0. \end{cases}$$

By the periodicity of (u_j^i, v_j^i) in the directions of $\nu^\perp \cap Q^\nu$, these maps belong to $W^{1,2}(Q^\nu; \mathbb{R}^m)$. Furthermore, $u_j = 0$ and $v_j = 1$ if $x\cdot\nu \le -\frac{1}{M_j}$, $u_j = z^1 + z^2$

and $v_j = 1$ if $x \cdot \nu \geq \frac{1}{M_j}$, and (u_j, v_j) is $\frac{1}{M_j}$-periodic in the directions of $\nu^\perp \cap Q^\nu$. Therefore, by changing variables we find

$$\begin{aligned}
&\|u_j - (z^1 + z^2)\chi_{\{x\cdot\nu\geq 0\}}\|_{L^1(Q^\nu)} = \|u_j - (z^1 + z^2)\chi_{\{x\cdot\nu\geq 0\}}\|_{L^1(Q^\nu \cap \{|x\cdot\nu| \leq \frac{1}{M_j}\})} \\
&= \frac{1}{M_j^n}\|u_j^1\|_{L^1(M_j Q^\nu \cap \{|x\cdot\nu|\leq\frac{1}{2}\})} + \frac{1}{M_j^n}\|u_j^2 - z^2\|_{L^1(M_j Q^\nu \cap \{|x\cdot\nu|\leq\frac{1}{2}\})} \\
&= \frac{1}{M_j}\|u_j^1\|_{L^1(Q^\nu)} + \frac{1}{M_j}\|u_j^2 - z^2\|_{L^1(Q^\nu)} \\
&\leq \frac{1}{M_j}\|u_j^1 - z^1\chi_{\{x\cdot\nu\geq 0\}}\|_{L^1(Q^\nu)} + \frac{|z^1|}{2M_j} + \frac{1}{M_j}\|u_j^2 - z^2\chi_{\{x\cdot\nu\geq 0\}}\|_{L^1(Q^\nu)} + \frac{|z^2|}{2M_j},
\end{aligned}$$

so that $u_j \to (z^1 + z^2)\chi_{\{x\cdot\nu\geq 0\}}$ in $L^1(Q^\nu; \mathbb{R}^m)$. Arguing similarly, we infer

$$\mathcal{F}^\infty_{\varepsilon_j/M_j}(u_j, v_j; Q^\nu) = \mathcal{F}^\infty_{\varepsilon_j}(u_j^1, v_j^1; Q^\nu) + \mathcal{F}^\infty_{\varepsilon_j}(u_j^2, v_j^2; Q^\nu).$$

The conclusion follows taking the limit $j \to \infty$.

(ii): By (i) and Lemma 3.7 we have $g(z, \nu) \leq g(z', \nu) + c\ell|z - z'|$, which implies that for any $\nu \in S^{n-1}$ the function $g(\cdot, \nu)$ is $c\ell$-Lipschitz continuous. Therefore it suffices to prove continuity in ν at any fixed z.

Since Ψ_∞ is continuous and positive on the compact set $S^{nm-1} \subseteq \mathbb{R}^{m\times n}$, there is a monotone modulus of continuity $\omega : [0, \infty) \to [0, \infty)$, with $\omega_\rho \to 0$ as $\rho \to 0$, such that

$$\Psi_\infty(\xi) \leq (1 + \omega_{|\xi-\xi'|})\Psi_\infty(\xi') \text{ for } |\xi| = |\xi'| = 1.$$

This implies that

$$\Psi_\infty(\eta) \leq (1 + \omega_{|R-\mathrm{Id}|})\Psi_\infty(\eta R) \text{ for any } \eta \in \mathbb{R}^{m\times n}, R \in O(n) \tag{3.24}$$

(it suffices to insert $\eta/|\eta|$ and $\eta R/|\eta|$ in the above expression).

Fix $\nu \in S^{n-1}$, a sequence $\varepsilon_j \to 0$, and let (u_j, v_j) be as in Proposition 3.2, extended periodically in the directions of $\nu^\perp \cap Q^\nu$ and constant along ν, as in the proof of (i). Let $\tilde\nu \in S^{n-1}$, $\tilde\nu \neq \nu$, and choose $R \in O(n)$ such that $\nu = R\tilde\nu$ and $|R - \mathrm{Id}| \leq c|\nu - \tilde\nu|$ (for example, R can be the identity on vectors orthogonal to both ν and $\tilde\nu$, and map $(\tilde\nu, \tilde\nu^\perp)$ to $(\nu, \nu^\perp)$ in this two-dimensional subspace). We fix a sequence $M_j \to \infty$ (for example, $M_j := j$) and define

$$\tilde u_j(x) := u_j(M_j R x), \qquad \tilde v_j(x) := v_j(M_j R x).$$

From $u_j \to z\chi_{\{x\cdot\nu\geq 0\}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ we obtain $\tilde u_j \to z\chi_{\{x\cdot\tilde\nu\geq 0\}}$. Further, $\nabla\tilde u_j(x) = M_j \nabla u_j(M_j R x)R$, which implies, recalling (3.24),

$$\Psi_\infty(\nabla\tilde u_j)(x) = M_j^2 \Psi_\infty(\nabla u_j R)(M_j R x) \leq M_j^2(1 + \omega_{|R-\mathrm{Id}|})\Psi_\infty(\nabla u_j)(M_j R x).$$

Inserting in the definition of $\mathcal{F}^\infty_{\varepsilon_j}(\tilde u_j, \tilde v_j; Q^{\tilde\nu})$ and using a change of variables leads to

$$\mathcal{F}^\infty_{\varepsilon_j/M_j}(\tilde u_j, \tilde v_j; Q^{\tilde\nu}) \leq (1 + \omega_{|R-\mathrm{Id}|})M_j^{1-n}\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; M_j R Q^{\tilde\nu}).$$

We observe that, although $R\tilde{\nu} = \nu$, we cannot in general expect $RQ^{\tilde{\nu}} = Q^{\nu}$. However, as (u_j, v_j) are periodic in the directions orthogonal to ν, the $(n-1)$-dimensional square $\nu^{\perp} \cap M_j RQ^{\tilde{\nu}}$ can be covered by at most $M_j^{n-1} + cM_j^{n-2}$ disjoint translated copies of the $(n-1)$-dimensional unit square $\nu^{\perp} \cap Q^{\nu}$. Therefore

$$\begin{aligned} g(z,\tilde{\nu}) \leq & \limsup_{j\to\infty} \mathcal{F}^{\infty}_{\varepsilon_j/M_j}(\tilde{u}_j, \tilde{v}_j; Q^{\tilde{\nu}}) \\ \leq & (1+\omega_{|R-\mathrm{Id}|}) \limsup_{j\to\infty} (1+\frac{c}{M_j}) \mathcal{F}^{\infty}_{\varepsilon_j}(u_j, v_j; Q^{\nu}) \\ = & (1+\omega_{|R-\mathrm{Id}|}) g(z,\nu) \leq (1+\omega_{c|\nu-\tilde{\nu}|}) g(z,\nu). \end{aligned}$$

□

3.3 Density of the Cantor part

We study now the behaviour of the surface energy density g at small jump amplitudes. The next result is probably well known to experts. Despite this, we give a self-contained proof since we have not found a precise reference in the literature. Similar constructions are performed in [AFP00, Proposition 5.1] for isotropic functionals defined on vector-valued measures. The L^1 lower semicontinuity of $\mathcal{F}_0$ is assumed to hold in Proposition 3.9 below, as already mentioned at the beginning of Section 3. Such a property follows, for instance, from the validity of Theorem 2.1. We stress again that Proposition 3.9 is not used in the proof of Theorem 2.1, rather it provides a further piece of information on g showing its linear behavior at small amplitudes.

Proposition 3.9. *Assume that the functional $\mathcal{F}_0$ defined in* (2.13) *is $L^1(\Omega;\mathbb{R}^m)$ lower semicontinuous. Then, for all $\nu \in S^{n-1}$ we have*

$$\lim_{z\to 0} \frac{g(z,\nu)}{h^{\mathrm{qc},\infty}(z\otimes\nu)} = 1\,.$$

Proof. With fixed $\nu \in S^{n-1}$, let $x_0 \in \Omega$ and $\rho > 0$ be such that $Q^{\nu}_{\rho}(x_0) \subset \Omega$. Upon translating and scaling, it is not restrictive to assume $x_0 = 0$ and $\rho = 1$. For every $z \in \mathbb{R}^m$ consider the sequence

$$w_j(x) := \varphi(jx\cdot\nu)z\,, \qquad x \in Q^{\nu}, \tag{3.25}$$

where $\varphi(t) := (t\wedge 1)\vee 0$ for every $t \in \mathbb{R}$. Clearly, $w_j \to u_z(x) := z\chi_{\{x\cdot\nu\geq 0\}}$ in $L^1(Q^{\nu};\mathbb{R}^m)$, and thus by the $L^1(Q^{\nu};\mathbb{R}^m)$ lower semicontinuity of $\mathcal{F}_0$ we conclude that

$$\begin{aligned} g(z,\nu) = \mathcal{F}_0(u_z,1;Q^{\nu}) &\leq \liminf_{j\to\infty} \mathcal{F}_0(w_j,1;Q^{\nu}) = \liminf_{j\to\infty} \int_{Q^{\nu}} h^{\mathrm{qc}}(\nabla w_j)\mathrm{d}x \\ &= \liminf_{j\to\infty} \int_{\{x\in Q^{\nu}:\,0\leq x\cdot\nu\leq 1/j\}} h^{\mathrm{qc}}(jz\otimes\nu)\mathrm{d}x = \liminf_{j\to\infty} \frac{h^{\mathrm{qc}}(jz\otimes\nu)}{j} \\ &\leq h^{\mathrm{qc},\infty}(z\otimes\nu)\,. \end{aligned} \tag{3.26}$$

On the other hand, given $z \in \mathbb{R}^m$ and any couple of sequences $z_j \to z$ and $t_j \to 0^+$, denote by M_j the integer part of t_j^{-1} and define for every $k \in \mathbb{N}$, $k \geq 3$,

$$u_{j,k}(x) := \sum_{i=0}^{M_j-1} i t_j z_j \chi_{[\frac{i}{kM_j}, \frac{i+1}{kM_j})}(x \cdot \nu) + z \chi_{[\frac{1}{k}, \frac{1}{2}]}(x \cdot \nu) .$$

We show that $u_{j,k}$ converges, as $j \to \infty$, to w_k as defined in (3.25) for every $k \geq 3$. Indeed, for $s := x \cdot \nu \in [\frac{i}{kM_j}, \frac{i+1}{kM_j}) \subseteq [0, \frac{1}{k})$ we have

$$\begin{aligned} |it_j z_j - zks| &\leq |z - z_j| + |z_j|\,|it_j - ks| \leq |z - z_j| + |z_j| \left(\frac{i}{M_j} |M_j t_j - 1| + \frac{1}{M_j} \right) \\ &\leq |z - z_j| + |z_j| \left(|M_j t_j - 1| + \frac{1}{M_j} \right) \to 0 \end{aligned}$$

uniformly in i, hence $\|w_k - u_{j,k}\|_{L^\infty(Q^\nu;\mathbb{R}^m)} \to 0$ as $j \to \infty$. Further,

$$\begin{aligned} Du_{j,k} = D^j u_{j,k} = (t_j z_j \otimes \nu)\mathcal{H}^{n-1} \llcorner \cup_{i=1}^{M_j-1} \{x \in Q^\nu : x \cdot \nu = \tfrac{i}{kM_j}\} \\ + ((z - (M_j - 1)t_j z_j) \otimes \nu)\mathcal{H}^{n-1} \llcorner \{x \in Q^\nu : x \cdot \nu = \tfrac{1}{k}\} . \end{aligned}$$

Therefore, by the $L^1(Q^\nu;\mathbb{R}^m)$ lower semicontinuity of $\mathcal{F}_0$ we conclude that

$$\begin{aligned} \frac{1}{k} h^{\mathrm{qc}}(kz \otimes \nu) = \mathcal{F}_0(w_k, 1; Q^\nu) &\leq \liminf_{j\to\infty} \mathcal{F}_0(u_{j,k}, 1; Q^\nu) \\ &= \liminf_{j\to\infty} \int_{J_{u_{j,k}}} g([u_{j,k}](x), \nu) \, d\mathcal{H}^{n-1}(x) \\ &= \liminf_{j\to\infty} (M_j - 1) g(t_j z_j, \nu) = \liminf_{j\to\infty} \frac{g(t_j z_j, \nu)}{t_j} . \end{aligned}$$

As this holds for every sequence, this implies

$$h^{\mathrm{qc},\infty}(z \otimes \nu) \leq \liminf_{(t,z')\to(0,z)} \frac{g(tz', \nu)}{t} . \tag{3.27}$$

Indeed, the superior limit in the definition of $h^{\mathrm{qc},\infty}$ is actually a limit on rank-1 directions being $h^{\mathrm{qc},\infty}$ convex on those directions.

Let now $\widetilde{z}_j \to 0$ be a sequence for which

$$\liminf_{z\to 0} \frac{g(z, \nu)}{h^{\mathrm{qc},\infty}(z \otimes \nu)} = \lim_{j\to\infty} \frac{g(\widetilde{z}_j, \nu)}{h^{\mathrm{qc},\infty}(\widetilde{z}_j \otimes \nu)} .$$

Upon setting $z_j := \frac{\widetilde{z}_j}{|\widetilde{z}_j|}$, up to subsequences we may assume that $z_j \to z_\infty \in S^{n-1}$. In addition, $t_j := |\widetilde{z}_j| \to 0$. Therefore, being $h^{\mathrm{qc},\infty}$ one-homogeneous we have that

$$\frac{g(\widetilde{z}_j, \nu)}{h^{\mathrm{qc},\infty}(\widetilde{z}_j \otimes \nu)} = \frac{g(t_j z_j, \nu)}{t_j} \frac{1}{h^{\mathrm{qc},\infty}(z_j \otimes \nu)} .$$

By the latter equality, by (3.27) and by the continuity of $h^{\mathrm{qc},\infty}$ we infer

$$\liminf_{z\to 0}\frac{g(z,\nu)}{h^{\mathrm{qc},\infty}(z\otimes\nu)}\geq 1\,. \tag{3.28}$$

The conclusion follows at once from (3.26) and (3.28). □

We now identify $h^{\mathrm{qc},\infty}$ explicitly as stated in (2.10).

Proposition 3.10. *For all $\xi\in\mathbb{R}^{m\times n}$*

$$h^{\mathrm{qc},\infty}(\xi)=\ell(\Psi^{1/2})^{\mathrm{qc},\infty}(\xi)\,. \tag{3.29}$$

Proof. With fixed $\xi\in\mathbb{R}^{m\times n}$, the very definition of h in (2.6) and the growth condition (2.3) easily imply

$$h^{\mathrm{qc},\infty}(\xi)=\limsup_{t\to\infty}\frac{h^{\mathrm{qc}}(t\xi)}{t}\leq\limsup_{t\to\infty}\frac{\ell(\Psi^{1/2})^{\mathrm{qc}}(t\xi)}{t}=\ell(\Psi^{1/2})^{\mathrm{qc},\infty}(\xi)\,.$$

Let $\varepsilon>0$, then for every $t>0$ consider $\varphi_t\in C_c^\infty(Q_1;\mathbb{R}^m)$ such that

$$h^{\mathrm{qc}}(t\xi)\geq\int_{Q_1}h(t\xi+\nabla\varphi_t(x))\mathrm{d}x-\varepsilon\,. \tag{3.30}$$

Note that

$$\begin{aligned}E_t:=&\{x\in Q_1:\ h(t\xi+\nabla\varphi_t(x))=\Psi(t\xi+\nabla\varphi_t(x))\}\\ =&\{x\in Q_1:\ \Psi^{1/2}(t\xi+\nabla\varphi_t(x))\leq\ell\}\,,\end{aligned}$$

so that

$$\int_{E_t}\ell\Psi^{1/2}(t\xi+\nabla\varphi_t(x))\mathrm{d}x\leq\ell^2\,.$$

Therefore, being $h\geq 0$ (cf. again (2.3)) from (3.30) we infer that

$$h^{\mathrm{qc}}(t\xi)\geq\int_{Q_1}\ell\Psi^{1/2}(t\xi+\nabla\varphi_t(x))\mathrm{d}x-\ell^2-\varepsilon\geq\ell(\Psi^{1/2})^{\mathrm{qc}}(t\xi)-\ell^2-\varepsilon\,,$$

from which we conclude that

$$h^{\mathrm{qc},\infty}(\xi)\geq\limsup_{t\to\infty}\frac{\ell(\Psi^{1/2})^{\mathrm{qc}}(t\xi)}{t}=\ell(\Psi^{1/2})^{\mathrm{qc},\infty}(\xi)\,.$$ □

From Propositions 3.9 and 3.10 we deduce straightforwardly the ensuing statement.

Corollary 3.11. *For all $\nu\in S^{n-1}$ we have*

$$\lim_{z\to 0}\frac{g(z,\nu)}{\ell(\Psi^{1/2})^{\mathrm{qc},\infty}(z\otimes\nu)}=1\,.$$

We conclude this section by proving that, under our hypotheses, the superior limit in the definition of $\Psi^{1/2}$ is in fact a limit and that the operations of quasi-convexification and of recession for $\Psi^{1/2}$ commute.

Proposition 3.12. *We have that*

(i) $(\Psi^{1/2})^\infty(\xi) = (\Psi_\infty)^{1/2}(\xi) = \lim_{t\to\infty} \frac{\Psi^{1/2}(t\xi)}{t}$, *for all* $\xi \in \mathbb{R}^{m\times n}$*;*

(ii) $(\Psi^{1/2})^{\mathrm{qc},\infty} = (\Psi^{1/2})^{\infty,\mathrm{qc}}$.

(iii) In the special case $\Psi_2(\xi) := \mathrm{dist}^2(\xi, \mathrm{SO}(n))$ *one obtains* $h^{\mathrm{qc},\infty}(\xi) = \ell|\xi|$ *for all* $\xi \in \mathbb{R}^{m\times n}$.

Proof. The second equality in (i) follows immediately from (2.4). Then, the first is a consequence of the very definition of recession function. Alternatively, by (2.5) we infer that, for all $\delta > 0$, there is $C_\delta > 0$ satisfying

$$(\Psi^{1/2})^\infty(\xi) \le (1+\delta)\Psi^{1/2}(\xi) + C_\delta, \qquad \text{for all } \xi \in \mathbb{R}^{m\times n}. \tag{3.31}$$

This, together with the definition of recession function, implies (i).

(ii) Since $(\Psi^{1/2})^{\mathrm{qc}} \le \Psi^{1/2}$, we immediately deduce $(\Psi^{1/2})^{\mathrm{qc},\infty} \le (\Psi^{1/2})^\infty$. By [FM93, Rem. 2.2(ii)], $(\Psi^{1/2})^{\mathrm{qc},\infty}$ is quasiconvex, hence $(\Psi^{1/2})^{\mathrm{qc},\infty} \le (\Psi^{1/2})^{\infty,\mathrm{qc}}$.

Let us check the converse inequality. Let $\xi \in \mathbb{R}^{m\times n}$. By definition of quasi-convexification and (3.31) we have

$$(\Psi^{1/2})^{\infty,\mathrm{qc}}(\xi) \le \int_{(0,1)^n} (\Psi^{1/2})^\infty(\xi + \nabla\varphi)\mathrm{d}x \le (1+\delta)\int_{(0,1)^n} \Psi^{1/2}(\xi+\nabla\varphi)\mathrm{d}x + C_\delta,$$

for all $\varphi \in C_c^\infty((0,1)^n;\mathbb{R}^m)$. Hence, taking the infimum over φ gives

$$(\Psi^{1/2})^{\infty,\mathrm{qc}}(\xi) \le (1+\delta)(\Psi^{1/2})^{\mathrm{qc}}(\xi) + C_\delta.$$

Since $(\Psi^{1/2})^\infty$ and therefore $(\Psi^{1/2})^{\infty,\mathrm{qc}}$ are positively one-homogeneous, we obtain

$$(\Psi^{1/2})^{\infty,\mathrm{qc}} \le (\Psi^{1/2})^{\mathrm{qc},\infty},$$

which yields the thesis.

(iii) From the definition of Ψ_2 one easily obtains $(\Psi_2^{1/2})^\infty(\xi) = |\xi|$. As this function is quasiconvex, it coincides with $(\Psi_2^{1/2})^{\infty,\mathrm{qc}}$, the assertion follows then from (ii) and Proposition 3.10 . □

4 Lower bound

4.1 Domain of the limits

In order to characterize the compactness properties and the space in which the limit is finite it is useful to consider the scalar simplification of functional, $\mathcal{F}_\varepsilon^{\mathrm{scal}} : W^{1,2}(A;\mathbb{R}\times[0,1]) \to [0,\infty]$,

$$\mathcal{F}_\varepsilon^{\mathrm{scal}}(u,v;A) := \int_A \Big(f_\varepsilon^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x. \tag{4.1}$$

From (2.3), one immediately obtains that for any $(u,v)\in W^{1,2}(A;\mathbb{R}^m\times[0,1])$

$$\frac{1}{c}\max_{i=1,\dots,m}\mathcal{F}_\varepsilon^{\mathrm{scal}}(u_i,v;A)-c\mathcal{L}^n(A) \le \mathcal{F}_\varepsilon(u,v;A) \le c\sum_{i=1}^m \mathcal{F}_\varepsilon^{\mathrm{scal}}(u_i,v;A)+c\mathcal{L}^n(A) \tag{4.2}$$

with the same constant $c\ge 1$ as in (2.3). In particular, [CFI16, Prop. 6.1] implies that if $(u_\varepsilon,v_\varepsilon)\to(u,v)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ with

$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)<\infty$$

then $u\in(GBV(\Omega))^m$ and $v=1$ $\mathcal{L}^n$-a.e. Ω (for a different proof see Remark 4.7). In addition, for every $i\in\{1,\dots,m\}$

$$\int_\Omega h_{\mathrm{scal}}^{\mathrm{conv}}(|\nabla u_i|)\mathrm{d}x + \int_{J_{u_i}} g_{\mathrm{scal}}(|[u_i](x)|)\mathrm{d}\mathcal{H}^{n-1} + \ell|D^c u_i|(\Omega) < \infty\,. \tag{4.3}$$

Here $h_{\mathrm{scal}}^{\mathrm{conv}} : [0,\infty)\to[0,\infty)$ is the convex function explicitly defined by

$$h_{\mathrm{scal}}^{\mathrm{conv}}(t) := (\ell t\wedge t^2)^{\mathrm{conv}} = \begin{cases} t^2, & \text{if } t\in[0,\frac{\ell}{2}],\\ \ell t-\frac{\ell^2}{4}, & \text{otherwise,}\end{cases}$$

(cf. (2.16)-(2.17)). We remark that it coincides with the simplified model h_{simp} for $m=1$ (cf. Lemma 2.3). Further, $g_{\mathrm{scal}} : [0,\infty)\to[0,1]$ is the function implicitly defined by

$$g_{\mathrm{scal}}(t) := \inf_{\mathcal{U}_t}\int_0^1 |1-\beta|\sqrt{f^2(\beta)|\alpha'|^2+|\beta'|}\mathrm{d}s \tag{4.4}$$

where $\mathcal{U}_t := \{\alpha,\,\beta\in W^{1,2}((0,1)) : \alpha(0)=0,\, \alpha(1)=t,\, 0\le\beta\le 1,\, \beta(0)=\beta(1)=1\}$. In particular, g_{scal} satisfies

(i) g_{scal} is subadditive: $g_{\mathrm{scal}}(t_1+t_2)\le g_{\mathrm{scal}}(t_1)+g_{\mathrm{scal}}(t_2)$ for every $t_1,\, t_2\in[0,\infty)$,

(ii) $0\le g_{\mathrm{scal}}(t)\le 1\wedge\ell t$,

(iii) $\frac{g_{\mathrm{scal}}(t)}{t}\to\ell$ as $t\to 0^+$

(cf. formula (1.6) in [CFI16, Theorem 1.1] for the definition of g_{scal}, and [CFI16, Section 4] for further properties).

In formula (4.3) the total variation of the Cantor part of the scalar function $u_i \in GBV(\Omega)$, $|D^c u_i|(\Omega)$, is defined as the least upper bound of the family of measures $|D^c\big((u_i \wedge k) \vee (-k)\big)|(\Omega)$, for $k > 0$ (cf. [AFP00, Definition 4.33, Theorem 4.34]). A similar construction can be performed for every $u \in (GBV(\Omega))^m$.

Precisely, [AF02, Lemma 2.10] implies that for every $u \in (GBV(\Omega))^m$ for which $|D^c u|$ is a finite measure on Ω, one can construct a vector measure on Ω with total variation coinciding exactly with $|D^c u|(B)$ for every Borel subset B of Ω. For this reason such a vector measure, is denoted by $D^c u$. Let us briefly recall the construction of $D^c u$. To this aim, the family of truncations $\mathcal{T}_k$ defined in (3.11) is employed. Indeed, for every $u \in (GBV(\Omega))^m$ such that $|D^c u|$ is a finite measure on Ω, it is possible to show that the following limit exists for every Borel subset B of Ω

$$\lambda(B) := \lim_{k\to\infty} D^c(\mathcal{T}_k(u))(B)\,. \tag{4.5}$$

In addition, λ is actually independent from the chosen family of truncations. The set function λ turns out to be a vector Radon measure on Ω, and moreover equality $|\lambda|(B) = |D^c u|(B)$ is true for every B as above.

Finally, for functions $u \in (GBV(\Omega))^m$ satisfying estimate (4.3) it is also true that

$$\mathcal{H}^{n-1}(\{x \in J_u : u^+(x) = \infty \text{ or } u^-(x) = \infty\}) = 0 \tag{4.6}$$

(cf. [AF02, Proposition 2.12, Remark 2.13]), here one works with the one-point compactification of $\mathbb{R}^m$. We remark that we deal with $(GBV(\Omega))^m$ and not with the strictly larger space $GBV(\Omega;\mathbb{R}^m)$, which is not even a vector space, see [AFP00, Remark 4.27].

4.2 Surface energy in BV

We prove below the lower bound in BV for the surface term. We recall that the definition of the surface energy density g has been given in (2.12).

Proposition 4.1. *Let $u \in BV(\Omega;\mathbb{R}^m)$, and $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$. Then for all $A \in \mathcal{A}(\Omega)$*

$$\int_{J_u \cap A} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \tag{4.7}$$

where g has been defined in (2.12).

Proof. Let $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ be such that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) < \infty.$$

Up to subsequences and with a small abuse of notation, we can assume that the previous lower limit is in fact a limit. Let us define the measures $\mu_\varepsilon \in \mathcal{M}_b^+(A)$

$$\mu_\varepsilon := \left(f_\varepsilon^2(v_\varepsilon)\Psi(\nabla u_\varepsilon) + \frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathcal{L}^n \llcorner A.$$

Extracting a further subsequence, we can assume that

$$\mu_\varepsilon \rightharpoonup \mu \quad \textit{weakly}^* \text{ in } \mathcal{M}(A) = (C^0_c(A))' \tag{4.8}$$

as $\varepsilon \to 0$, for some $\mu \in \mathcal{M}^+_b(A)$, so that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \geq \mu(A).$$

Equation (4.7) will follow once we have proved that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \geq g([u](x_0), \nu_u(x_0)), \quad \mathcal{H}^{n-1}\text{-a.e. } x_0 \in J_u \cap A\,. \tag{4.9}$$

We will prove the last inequality for points $x_0 \in J_u \cap A$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q^\nu_\rho(x_0))}{\mathcal{H}^{n-1}(J_u \cap Q^\nu_\rho(x_0))} \quad \text{exists finite,}$$

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q^\nu_\rho(x_0))}{\rho^{n-1}} = 1,$$

where $\nu := \nu_u(x_0)$ and $Q^\nu_\rho(x_0) := x_0 + \rho Q^\nu$ is the cube centred in x_0, with side length ρ, and one face orthogonal to ν. We remark that such conditions define a set of full measure in $J_u \cap A$.

For $x_0 \in J_u \cap A$ as above, we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q^\nu_\rho(x_0))}{\rho^{n-1}} = \lim_{\substack{\rho\in I\\ \rho\to 0}} \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q^\nu_\rho(x_0))}{\rho^{n-1}}$$

where we used (4.8) and

$$I := \Big\{\rho \in (0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0, \partial A)) : \mu(\partial Q^\nu_\rho(x_0)) = 0\Big\}.$$

We introduce

$$\gamma_\varepsilon := \inf\{z \in [0,1] : f(z) \geq \varepsilon^{-1/2}\} = \frac{1}{1 + \ell\varepsilon^{1/2}},$$

$$\tilde{v}_\varepsilon := \min\{v_\varepsilon, \gamma_\varepsilon\}$$

and compute

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon; Q^\nu_\rho(x_0)) &= \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; Q^\nu_\rho(x_0) \setminus \{v_\varepsilon > \gamma_\varepsilon\}) \\ &\quad + \int_{Q^\nu_\rho(x_0)\cap\{v_\varepsilon>\gamma_\varepsilon\}} \Psi(\nabla u_\varepsilon)\mathrm{d}x + \frac{(1-\gamma_\varepsilon)^2}{4\varepsilon}\mathcal{L}^n(Q^\nu_\rho(x_0) \cap \{v_\varepsilon > \gamma_\varepsilon\}) \\ &\leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; Q^\nu_\rho(x_0)) + \frac{\ell^2}{4}\rho^n\,,\end{aligned}$$

where in the last step we used that the definition of γ_ε implies $1-\gamma_\varepsilon = \ell\gamma_\varepsilon\varepsilon^{1/2} \le \ell\varepsilon^{1/2}$. Therefore

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \ge \limsup_{\substack{\rho\in I\\ \rho\to 0}} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon, \tilde{v}_\varepsilon; Q_\rho^\nu(x_0))}{\rho^{n-1}}. \tag{4.10}$$

By (2.5), for every $\delta \in (0,1)$ one has $\Psi(\xi) \ge (1-\delta)\Psi_\infty(\xi)$ for ξ sufficiently large. As Ψ_∞ is continuous, there is $C(\delta) > 0$ such that

$$\Psi(\xi) + C(\delta) \ge (1-\delta)\Psi_\infty(\xi) \qquad \text{for all } \xi.$$

We choose $\delta_\rho \to 0$ such that $\rho C(\delta_\rho) \to 0$. As $\varepsilon f^2(\tilde{v}_\varepsilon) \le 1$, we have

$$\varepsilon f^2(\tilde{v}_\varepsilon)\Psi(\nabla u_\varepsilon) \ge (1-\delta_\rho)\varepsilon f^2(\tilde{v}_\varepsilon)\Psi_\infty(\nabla u_\varepsilon) - C(\delta_\rho)$$

with $\rho^{1-n}\mathcal{L}^n(Q_\rho^\nu)C(\delta_\rho) = \rho C(\delta_\rho) \to 0$ as $\rho \to 0$. We conclude by (4.10) that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \ge \limsup_{\substack{\rho\in I\\ \rho\to 0}} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon^\infty(u_\varepsilon, \tilde{v}_\varepsilon; Q_\rho^\nu(x_0))}{\rho^{n-1}}, \tag{4.11}$$

where $\mathcal{F}_\varepsilon^\infty$ has been defined in (2.11). Setting $y := (x-x_0)/\rho \in Q^\nu$, and changing variable in the previous expression we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \ge \limsup_{\substack{\rho\in I\\ \rho\to 0}} \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^\infty(u_\varepsilon^\rho, \tilde{v}_\varepsilon^\rho; Q^\nu),$$

where $w^\rho(y) := w(\rho y + x_0)$ for $y \in Q^\nu$. Recalling that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^m)$, by diagonalization we can find subsequences $\{\rho_k\}_k$ and $\{\varepsilon(\rho_k)\}_k$ such that $u_{\varepsilon(\rho_k)}^{\rho_k} \to [u](x_0)\chi_{\{y\cdot\nu>0\}} + u^-(x_0)$ in $L^1(Q^\nu;\mathbb{R}^m)$ and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \ge \lim_{k\to\infty} \mathcal{F}_{\varepsilon(\rho_k)}^\infty(u_{\varepsilon(\rho_k)}^{\rho_k}, \tilde{v}_{\varepsilon(\rho_k)}^{\rho_k}; Q^\nu).$$

Being $\mathcal{F}_\varepsilon^\infty$ invariant for translations of the first argument, we find

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) \ge \liminf_{k\to\infty} \mathcal{F}_{\varepsilon(\rho_k)}^\infty(u_{\varepsilon(\rho_k)}^{\rho_k}, \tilde{v}_{\varepsilon(\rho_k)}^{\rho_k}; Q^\nu) \ge g([u](x_0), \nu_u(x_0)),$$

that is (4.9), and this concludes the proof. □

4.3 Diffuse part in BV

Proposition 4.2. *Let $u \in BV(\Omega;\mathbb{R}^m)$, $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$, $A \in \mathcal{A}(\Omega)$. Then*

$$\int_A h^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_A h^{\mathrm{qc},\infty}(\mathrm{d}D^c u) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \tag{4.12}$$

where h^{qc} and $h^{\mathrm{qc},\infty}$ have been defined in (2.6)-(2.9).

We remark that this statement can be proven using the lower-semicontinuity result by Fonseca and Leoni [FL01, Th. 1.8], following an argument similar to that used in [AF02, Subsection 4.1]. Instead, our proof is based on the following result from [ADM92, Theorem 4.1], see also [AFP00, Theorem 5.47].

Theorem 4.3 (Ambrosio-Dal Maso). *Let $\phi : \mathbb{R}^{m\times n} \to [0,\infty)$ be quasiconvex and such that*

$$0 \le \phi(\xi) \le c(1+|\xi|) \quad \text{for all } \xi \in \mathbb{R}^{m\times n},$$

and define $F : L^1(\Omega;\mathbb{R}^m) \to \mathbb{R}$ by

$$F(u) := \begin{cases} \int_\Omega \phi(\nabla u)\mathrm{d}x, & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^m), \\ \infty, & \text{otherwise in } L^1(\Omega;\mathbb{R}^m). \end{cases}$$

Then for any $u \in BV(\Omega;\mathbb{R}^m)$ we have

$$sc^-(L^1)\text{-}F(u) = \int_\Omega \phi(\nabla u)\mathrm{d}x + \int_\Omega \phi^\infty(\mathrm{d}D^s u),$$

where $\phi^\infty(\xi) := \limsup_{t\to\infty} \phi(t\xi)/t$. In particular the latter functional is lower semicontinuous with respect to the strong $L^1(\Omega;\mathbb{R}^m)$ convergence.

We start with a truncation result.

Lemma 4.4. *There are two functions $\alpha,\beta : (0,1) \to (0,1)$, with $\lim_{\delta\uparrow 1} \alpha_\delta = 1$ and $\lim_{\delta\uparrow 1} \beta_\delta = 0$, such that for any $\varepsilon > 0$, $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^m \times [0,1])$, $\delta \in (0,1)$ and $A \in \mathcal{A}(\Omega)$ there is $\tilde u_\varepsilon^\delta \in GSBV(A;\mathbb{R}^m)$ such that*

$$H_\delta(\tilde u_\varepsilon^\delta; A) \le \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) + h(0)\mathcal{L}^n(A \cap \{v_\varepsilon \le \delta\}),$$

where H_δ is defined for $A \in \mathcal{A}(\Omega)$ and $w \in L^1(A;\mathbb{R}^m)$ by

$$H_\delta(w;A) := \begin{cases} \alpha_\delta \int_A h^{\mathrm{qc}}(\nabla w)\mathrm{d}x + \beta_\delta \mathcal{H}^{n-1}(A \cap J_w), & \text{if } w \in GSBV(A;\mathbb{R}^m), \\ \infty, & \text{otherwise.} \end{cases} \tag{4.13}$$

If one has $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ as $\varepsilon \to 0$, then $\tilde u_\varepsilon^\delta \to u$ in $L^1(A;\mathbb{R}^m)$ as $\varepsilon \to 0$, for any fixed $\delta \in (0,1)$.

We stress that, for the sake of notational simplicity, we will omit here and below the explicit dependence of $\tilde u_\varepsilon^\delta$ on the set A.

Proof. We fix $\delta \in (0,1)$ and $\varepsilon > 0$. We compute, for any pair $(u,v) \in$

$W^{1,2}(\Omega;\mathbb{R}^m\times[0,1])$,

$$
\begin{aligned}
\mathcal{F}_\varepsilon(u,v;A) \geq & \int_{\{\varepsilon f^2(v)>1\}\cap A} \Psi(\nabla u)\mathrm{d}x + \int_{\{\varepsilon f^2(v)\leq 1\}\cap A} \Big(\varepsilon f^2(v)\Psi(\nabla u) + \delta^2 \frac{(1-v)^2}{4\varepsilon}\Big)\mathrm{d}x \\
& + \int_A \Big((1-\delta^2)\frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x \\
& \geq \int_{\{\varepsilon f^2(v)>1\}\cap A} \Psi(\nabla u)\mathrm{d}x + \delta \int_{\{\varepsilon f^2(v)\leq 1\}\cap A} v\ell\Psi^{1/2}(\nabla u)\mathrm{d}x \\
& + \sqrt{1-\delta^2}\int_A |\nabla(\Phi(v))|\mathrm{d}x \\
& \geq \delta \int_A \Big(\Psi(\nabla u)\wedge v\ell\Psi^{1/2}(\nabla u)\Big)\mathrm{d}x + \sqrt{1-\delta^2}\int_A |\nabla(\Phi(v))|\mathrm{d}x \\
& \geq \delta \int_A vh(\nabla u)\mathrm{d}x + \sqrt{1-\delta^2}\int_A |\nabla(\Phi(v))|\mathrm{d}x,
\end{aligned} \tag{4.14}
$$

where h has been introduced in (2.6) and $\Phi:[0,1]\to[0,\frac{1}{2}]$ is defined by

$$
\Phi(t) := \int_0^t (1-s)\mathrm{d}s = t - \frac{1}{2}t^2\,. \tag{4.15}
$$

We observe that Φ is strictly increasing, $\Phi(1)=\frac{1}{2}$ and in particular Φ is bijective. By the coarea formula,

$$
\int_A |\nabla(\Phi(v))|\mathrm{d}x = \int_0^{1/2} \mathcal{H}^{n-1}(A\cap\partial^*\{\Phi(v)>t\})\mathrm{d}t.
$$

Therefore there is $\bar t\in(\Phi(\delta^2),\Phi(\delta))$ such that

$$
(\Phi(\delta)-\Phi(\delta^2))\mathcal{H}^{n-1}(A\cap\partial^*\{\Phi(v)>\bar t\}) \leq \int_A |\nabla(\Phi(v))|\mathrm{d}x.
$$

We define

$$
\tilde u := u\chi_{\{\Phi(v)>\bar t\}\cap A} \in GSBV(A;\mathbb{R}^m)
$$

(dropping the dependence on both ε and δ from $\tilde u$) and obtain from (4.14),

$$
\begin{aligned}
\mathcal{F}_\varepsilon(u,v;A) \geq & \delta\Phi^{-1}(\bar t)\int_A h(\nabla\tilde u)\mathrm{d}x + \sqrt{1-\delta^2}(\Phi(\delta)-\Phi(\delta^2))\mathcal{H}^{n-1}(A\cap J_{\tilde u}) \\
& - h(0)\mathcal{L}^n(\{\Phi(v)\leq\bar t\}\cap A).
\end{aligned}
$$

We recall that $\bar t \geq \Phi(\delta^2)$ and that Φ is increasing, define $\alpha_\delta := \delta^3$, $\beta_\delta := \sqrt{1-\delta^2}(\Phi(\delta)-\Phi(\delta^2))$, and conclude

$$
\mathcal{F}_\varepsilon(u,v;A) \geq \alpha_\delta \int_A h(\nabla\tilde u)\mathrm{d}x + \beta_\delta\mathcal{H}^{n-1}(A\cap J_{\tilde u}) - h(0)\mathcal{L}^n(\{v\leq\delta\}\cap A).
$$

We also remark that $\|\tilde u - u\|_{L^1(A)} \leq \|u\|_{L^1(\{v\leq\Phi^{-1}(\bar t)\})}$, hence, if the sequence u_ε is equiintegrable and $v_\varepsilon\to 1$ in $L^1(A)$, we obtain that $u_\varepsilon - \tilde u_\varepsilon \to 0$ in $L^1(A;\mathbb{R}^m)$. □

The next lemma is a minor reformulation of [Lar98, Lemma 5.1]. The latter improves the statement of [AFP00, Theorem 3.95] on the convergence of the blow-ups of a BV-function in a Cantor point. A more general version of this result can be found in [Rin18, Lemma 10.6].

Lemma 4.5. *Let $u \in BV(\Omega;\mathbb{R}^m)$ and let $\eta : \Omega \to S^{m-1}$, $\xi : \Omega \to S^{n-1}$ be Borel maps such that $D^c u = \eta \otimes \xi |D^c u|$. Then, for $|D^c u|$-a.e. $x \in \Omega$ and for all given $\mu \in \mathcal{M}^+(\Omega)$, there exists a sequence $\rho_i \to 0$, as $i \to \infty$, such that*

$$\mu(\partial Q^{\xi(x)}_{\rho_i}(x)) = 0, \qquad \text{for all } i \geq 1, \tag{4.16}$$

$$t_{\rho_i} := \frac{|Du|(Q^{\xi(x)}_{\rho_i}(x))}{\rho_i^n} \to \infty, \qquad t_{\rho_i}\rho_i \to 0, \tag{4.17}$$

$$\frac{u(x+\rho_i y) - u_{Q^{\xi(x)}_{\rho_i}(x)}}{t_{\rho_i}\rho_i} \to \eta(x)\chi(y\cdot\xi(x)) \qquad \text{strictly-}BV(Q^{\xi(x)};\mathbb{R}^m), \tag{4.18}$$

as $i \to \infty$, for some nondecreasing function $\chi : (-1/2, 1/2) \to \mathbb{R}$ with

$$|D\chi|((-1/2, 1/2)) = 1, \tag{4.19}$$

where $u_{Q^{\xi(x)}_{\rho_i}(x)}$ denotes the average of u over $Q^{\xi(x)}_{\rho_i}(x)$.

Proof. For simplicity we will denote $Q_1 := Q^{\xi(x)}$, $Q_\rho(x) := x + \rho Q_1$, and

$$u^\rho_x(y) := \frac{u(x+\rho y) - u_{Q^{\xi(x)}_{\rho}(x)}}{t_\rho \rho}, \qquad \text{for } y \in Q_1.$$

By general properties of BV functions (4.17) holds for the entire family $\rho \to 0$ and by Radon-Nikodym differentiation

$$\lim_{\rho\to 0} \frac{D^c u^\rho_x(Q_1)}{|D^c u^\rho_x|(Q_1)} = \eta(x) \otimes \xi(x), \tag{4.20}$$

$|D^c u|$-a.e. $x \in \Omega$. Up to a further $|D^c u|$-negligible set, [AFP00, Theorem 3.95] and [Lar98, Lemma 5.1] provide a sequence $\rho_i \to 0$ such that

$$|Du^{\rho_i}_x| \rightharpoonup \gamma \qquad \text{weakly*-}\mathcal{M}(Q_1), \tag{4.21}$$

$$u^{\rho_i}_x(y) \to u_x(y) := \eta(x)\chi(y\cdot\xi(x)) \qquad \text{weakly*-}BV(Q_1;\mathbb{R}^m), \tag{4.22}$$

as $i \to \infty$, for some $\gamma \in \mathcal{M}^+(Q_1)$ with $\gamma(Q_1) = 1$ and some nondecreasing function $\chi : (-1/2, 1/2) \to \mathbb{R}$ with $|D\chi|((-1/2, 1/2)) \leq 1$.

Let us check that the sequence $\rho_i \to 0$ can be chosen such that (4.16) holds. Indeed, fixed $i \in \mathbb{N} \setminus \{0\}$, we have $\mu(\partial Q_{s\rho_i}(x_0)) = 0$ for $\mathcal{L}^1$-a.e. $s \in (0, 1/\rho_i)$. Moreover, the maps

$$s \in (0, 1/\rho_i) \mapsto u^{s\rho_i}_x \in L^1(Q_1, \mathbb{R}^m),$$
$$s \in (0, 1/\rho_i) \mapsto |Du^{s\rho_i}_x| \in \mathcal{M}^+(Q_1)$$

are continuous as $s \to 1^-$, respectively for the convergences $L^1(Q_1;\mathbb{R}^m)$ and weak*-$\mathcal{M}(Q_1)$, by definition of u_x^ρ and t_ρ. Hence, we can find $s_i \in (0,1)$ such that (4.16), (4.17) and the $L^1(Q_1,\mathbb{R}^m)$ convergence in (4.18) hold for $s_i\rho_i$ in place of ρ_i.

We next check (4.19). By (4.21) and (4.22) we have that $|Du_x| \le \gamma$. Hence, for $t \in (0,1)$ such that $\gamma(\partial Q_t) = 0$, recalling that $|Du_x^{\rho_i}|(Q_1) = \gamma(Q_1) = 1$, we obtain

$$|Du_x^{\rho_i}|(Q_t) \to \gamma(Q_t), \qquad |Du_x^{\rho_i}|(Q_1 \setminus Q_t) \to \gamma(Q_1 \setminus Q_t),$$
$$Du_x^{\rho_i}(Q_t) \to Du_x(Q_t).$$

We infer that

$$\limsup_{i\to\infty} |Du_x^{\rho_i}(Q_1) - Du_x(Q_1)| \le 2\gamma(Q_1 \setminus Q_t),$$

and letting $t \to 1^-$ gives $Du_x^{\rho_i}(Q_1) \to Du_x(Q_1)$ as $i \to \infty$. In conclusion

$$Du_x(Q_1) = \lim_{i\to\infty} Du_x^{\rho_i}(Q_1) = \lim_{i\to\infty} \frac{Du_x^{\rho_i}(Q_1)}{|Du_x^{\rho_i}|(Q_1)} = \eta(x) \otimes \xi(x),$$

and then $D\chi(-1/2, 1/2) = 1$. This gives (4.19) by monotonicity of χ. Finally, $|Du_x|(Q_1) = 1$ provides the strict-$BV(Q_1;\mathbb{R}^m)$ convergence in (4.18). □

Proof of Proposition 4.2. Step 0: Preparation. We assume $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ for some $u \in BV(\Omega;\mathbb{R}^m)$. Let $A \subseteq \mathcal{A}(\Omega)$, $\delta \in (0,1)$ and let $\tilde u_\varepsilon^\delta$ be as in Lemma 4.4. We define the measure

$$\mu_\varepsilon^\delta := \alpha_\delta h^{\mathrm{qc}}(\nabla \tilde u_\varepsilon^\delta)\mathcal{L}^n \llcorner A + \beta_\delta \mathcal{H}^{n-1} \llcorner (A \cap J_{\tilde u_\varepsilon^\delta}),$$

so that $\mu_\varepsilon^\delta(A) = H_\delta(\tilde u_\varepsilon^\delta; A) \le \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) + h(0)\mathcal{L}^n(A \cap \{v_\varepsilon \le \delta\})$. Passing to a subsequence we can assume that $\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A)$ exists finite and that $\mu_\varepsilon^\delta \rightharpoonup \mu^\delta$ weakly* in the sense of measures on A as $\varepsilon \to 0$, for some $\mu^\delta \in \mathcal{M}_b^+(A)$. If we can show that

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \ge \alpha_\delta h^{\mathrm{qc}}(\nabla u(x_0)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x_0 \in A \tag{4.23}$$

and

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0) \ge \alpha_\delta h^{\mathrm{qc},\infty}\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x_0)\right) \quad \text{for } |D^c u|\text{-a.e. } x_0 \in A \tag{4.24}$$

for all $\delta \in (0,1)$, then the conclusion follows.

Step 1: Absolutely continuous part. We prove (4.23). We can assume that the left-hand side is finite. First we observe that for $\mathcal{L}^n$-a.e. $x_0 \in A$ one has

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho(x_0))}{\rho^n} = \lim_{\substack{\rho\to 0\\ \rho\in I}} \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon^\delta(Q_\rho(x_0))}{\rho^n}$$

where $Q_\rho(x_0) := x_0 + (-\frac{1}{2}\rho, \frac{1}{2}\rho)^n$ and $I := \{\rho \in (0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0, \partial A)) : \mu^\delta(\partial Q_\rho(x_0)) = 0\}$. We define $u^\rho : Q_1 \to \mathbb{R}^m$ by

$$u^\rho(y) := \frac{u(x_0 + \rho y) - u(x_0)}{\rho}.$$

By the properties of BV, for $\mathcal{L}^n$-a.e. $x_0 \in A$, after possibly extracting a further subsequence, $u^\rho(y) \to \nabla u(x_0)y$ in $L^1(Q_1; \mathbb{R}^m)$ as $\rho \to 0$. We further define

$$u^\rho_\varepsilon(y) := \frac{\tilde{u}^\delta_\varepsilon(x_0 + \rho y) - u(x_0)}{\rho}$$

so that $u^\rho_\varepsilon \to u^\rho$ in $L^1(Q_1; \mathbb{R}^m)$ as $\varepsilon \to 0$ for any fixed $\rho > 0$ (and $\delta \in (0,1)$). We take a diagonal subsequence so that $w_i(y) := u^{\rho_i}_{\varepsilon_i}(y) \to \nabla u(x_0)y$ in $L^1(Q_1; \mathbb{R}^m)$ and

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{i\to\infty} \left[\int_{Q_1} \alpha_\delta h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x + \frac{\beta_\delta}{\rho_i}\mathcal{H}^{n-1}(J_{w_i} \cap Q_1) \right]. \tag{4.25}$$

We fix $M \in \mathbb{N}$ and for every i, by averaging we choose $k_i \in \{M+1, \ldots, 2M\}$ such that

$$\int_{\{a_{k_i} < |w_i| < a_{k_i+1}\}} h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x \le \frac{1}{M} \int_{Q_1} h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x\,, \tag{4.26}$$

which implies that $\hat{w}_i := \mathcal{T}_{k_i}(w_i)$, with $\mathcal{T}_{k_i}$ defined in (3.11), obeys

$$\int_{Q_1} h^{\mathrm{qc}}(\nabla \hat{w}_i)\mathrm{d}x \le (1 + \frac{C}{M}) \int_{Q_1} h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x + C\mathcal{L}^n(\{|w_i| \ge a_{k_i}\}). \tag{4.27}$$

Indeed, in view of (2.8) and $\|\nabla \mathcal{T}_{k_i}\|_{L^\infty(\mathbb{R}^m)} \le 1$ we have

$$\begin{aligned}
\int_{Q_1} h^{\mathrm{qc}}(\nabla \hat{w}_i)\mathrm{d}x \le& \int_{\{|w_i| \le a_{k_i}\}} h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x \\
&+ \int_{\{a_{k_i} < |w_i| < a_{k_i+1}\}} h^{\mathrm{qc}}(\nabla \hat{w}_i)\mathrm{d}x + h(0)\mathcal{L}^n(\{|w_i| \ge a_{k_i+1}\}) \\
&\le \int_{Q_1} h^{\mathrm{qc}}(\nabla w_i)\mathrm{d}x + C\int_{\{a_{k_i} < |w_i| < a_{k_i+1}\}} h^{\mathrm{qc}}(\nabla w_i) + C\mathcal{L}^n(\{|w_i| \ge a_{k_i}\})\,.
\end{aligned}$$

The inequality in (4.27) then follows from (4.26).

Moreover, note that if $a_M > \|\nabla u(x_0)y\|_{L^\infty(Q_1)} + 1$ then $w_i \to \nabla u(x_0)y$ implies $\hat{w}_i \to \nabla u(x_0)y$ in $L^1(Q_1; \mathbb{R}^m)$.

We recall that $\mathcal{T}_{k_i} \in C^1$ implies $\mathcal{H}^{n-1}(J_{\hat{w}_i} \cap Q_1) \le \mathcal{H}^{n-1}(J_{w_i} \cap Q_1)$. From (4.25) and $\rho_i \to 0$ we deduce $\mathcal{H}^{n-1}(J_{w_i} \cap Q_1) \to 0$ and, with $|\hat{w}_i| \le a_{M+1}$ pointwise,

$$|D^s \hat{w}_i|(Q_1) = \int_{J_{\hat{w}_i} \cap Q_1} |[\hat{w}_i]|\mathrm{d}\mathcal{H}^{n-1} \le 2a_{M+1}\mathcal{H}^{n-1}(J_{w_i} \cap Q_1) \to 0$$

and therefore

$$\int_{Q_1} h^{\mathrm{qc},\infty}(\mathrm{d}D^s\hat{w}_i) \le c|D^s\hat{w}_i|(Q_1) \to 0.$$

With (4.25) and (4.27), using that $w_i \to \nabla u(x_0)y$ in measure, we get

$$\alpha_\delta \lim_{i\to\infty}\left[\int_{Q_1} h^{\mathrm{qc}}(\nabla\hat{w}_i)\mathrm{d}x + \int_{Q_1} h^{\mathrm{qc},\infty}(\mathrm{d}D^s\hat{w}_i)\right] \le (1+\frac{C}{M})\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0).$$

By the lower semicontinuity of the functional in the left-hand side (Theorem 4.3) and $\hat{w}_i \to \nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$ we deduce

$$\alpha_\delta h^{\mathrm{qc}}(\nabla u(x_0)) \le (1+\frac{C}{M})\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)$$

for $\mathcal{L}^n$-a.e. x_0, every M, and every δ. This proves (4.23).

Step 2: Cantor part. We prove (4.24). By Alberti's rank-one theorem we can assume without loss of generality that

$$\frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x_0) = \eta(x_0)\otimes\xi(x_0) \tag{4.28}$$

with $\eta(x_0) \in S^{m-1}$, $\xi(x_0) \in S^{n-1}$ for $|D^cu|$-a.e. $x_0 \in A$. We fix a unit cube $Q_1 := Q^{\xi(x_0)}$ with one face orthogonal to $\xi(x_0)$, write $Q_\rho(x_0) := x_0 + \rho Q_1$, and select a sequence $\rho_i \to 0$ as in Lemma 4.5, applied for the given $u \in BV(\Omega,\mathbb{R}^m)$ and $\mu := \mu^\delta$.

As above, for $|D^cu|$-a.e. x_0 one has

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0) = \lim_{\rho\to0}\frac{\mu^\delta(Q_\rho(x_0))}{|Du|(Q_\rho(x_0))} = \lim_{i\to\infty}\lim_{\varepsilon\to0}\frac{\mu^\delta_\varepsilon(Q_{\rho_i}(x_0))}{|Du|(Q_{\rho_i}(x_0))}.$$

We define

$$u^\rho_\varepsilon(y) := \frac{\tilde{u}^\delta_\varepsilon(x_0+\rho y) - u_{Q_\rho(x_0)}}{t_\rho\rho},$$

so that, defining $u_{x_0}(y) := \eta(x)\chi_{x_0}(y\cdot\xi(x_0))$, $\lim_{\rho\to0}\lim_{\varepsilon\to0}u^\rho_\varepsilon = u_{x_0}$ in $L^1(Q_1;\mathbb{R}^m)$ (for every $\delta\in(0,1)$) and

$$\begin{aligned}\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0) &= \lim_{i\to\infty}\lim_{\varepsilon\to0}\left[\frac{\alpha_\delta}{\rho_i^n t_{\rho_i}}\int_{Q_{\rho_i}(x_0)} h^{\mathrm{qc}}(\nabla\tilde{u}^\delta_\varepsilon)\mathrm{d}x + \frac{\beta_\delta}{\rho_i^n t_{\rho_i}}\mathcal{H}^{n-1}(J_{\tilde{u}^\delta_\varepsilon}\cap Q_{\rho_i}(x_0))\right]\\ &= \lim_{i\to\infty}\lim_{\varepsilon\to0}\left[\frac{\alpha_\delta}{t_{\rho_i}}\int_{Q_1} h^{\mathrm{qc}}(t_{\rho_i}\nabla u^{\rho_i}_\varepsilon)\mathrm{d}y + \frac{\beta_\delta}{\rho_i t_{\rho_i}}\mathcal{H}^{n-1}(J_{u^{\rho_i}_\varepsilon}\cap Q_1)\right].\end{aligned}$$

Taking a diagonal subsequence we see that there is $\varepsilon_i \to 0$ such that

$$w_i := u^{\rho_i}_{\varepsilon_i} \to u_{x_0} \text{ in } L^1(Q_1;\mathbb{R}^m)$$

with $|Du_{x_0}|(Q_1) = 1$, and setting $t_i := t_{\rho_i} \to \infty$,

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0) = \lim_{i\to\infty}\left[\frac{\alpha_\delta}{t_i}\int_{Q_1} h^{\mathrm{qc}}(t_i\nabla w_i)\mathrm{d}y + \frac{\beta_\delta}{\rho_i t_i}\mathcal{H}^{n-1}(J_{w_i}\cap Q_1)\right].$$

We fix $M>0$ and, by averaging, for every i choose $k_i\in\{M+1,\dots,2M\}$ such that

$$\int_{Q_1\cap\{a_{k_i}<|w_i|<a_{k_i+1}\}} h^{\rm qc}(t_i\nabla w_i)\mathrm{d}x\le\frac{1}{M}\int_{Q_1}h^{\rm qc}(t_i\nabla w_i)\mathrm{d}x,$$

which implies that $\hat w_i:=\mathcal{T}_{k_i}(w_i)\in SBV\cap L^\infty(Q_1;\mathbb{R}^m)$ obeys, arguing as in Step 1 above and by taking into account that $t_i\to\infty$,

$$\begin{aligned}
&\limsup_{i\to\infty}\int_{Q_1}\frac{\alpha_\delta}{t_i}h^{\rm qc}(t_i\nabla\hat w_i)\mathrm{d}y+\frac{\beta_\delta}{\rho_i t_i}\mathcal{H}^{n-1}(J_{\hat w_i}\cap Q_1)\\
&\le(1+\frac{C}{M})\lim_{i\to\infty}\int_{Q_1}\frac{\alpha_\delta}{t_i}h^{\rm qc}(t_i\nabla w_i)\mathrm{d}y+\frac{\beta_\delta}{\rho_i t_i}\mathcal{H}^{n-1}(J_{w_i}\cap Q_1)+\frac{C}{t_i}\mathcal{L}^n(\{|w_i|>a_{k_i}\})\\
&=(1+\frac{C}{M})\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0).
\end{aligned}\tag{4.29}$$

Further, since χ is bounded, for M sufficiently large we have $r_i:=\|\hat w_i-u_{x_0}\|_{L^1(Q_1)}\to0$. For every i we select $q_i\in(1-r_i^{1/2},1)$ such that

$$\int_{\partial Q_{q_i}}|\hat w_i^- - u_{x_0}^+|\mathrm{d}\mathcal{H}^{n-1}\le\frac{1}{r_i^{1/2}}\|\hat w_i-u_{x_0}\|_{L^1(Q_1)}=r_i^{1/2}\to0,$$

where $\hat w_i^-$ and $u_{x_0}^+$ denote the inner and outer trace, respectively, and define

$$w_i^*:=\begin{cases}\hat w_i, & \text{in } Q_{q_i},\\ u_{x_0}, & \text{in } Q_1\setminus Q_{q_i}.\end{cases}$$

Then, the choice of q_i, (4.29), and $\rho_i t_i\to0$ yield

$$\int_{Q_1}h^{\rm qc,\infty}(\mathrm{d}D^s w_i^*)\le c r_i^{1/2}+c_M\mathcal{H}^{n-1}(J_{\hat w_i}\cap Q_1)+\int_{Q_1\setminus Q_{q_i}}h^{\rm qc,\infty}(\mathrm{d}D^s u_{x_0})\to0.\tag{4.30}$$

In addition, we get from (2.8) and $t_i\to\infty$

$$\lim_{i\to\infty}\int_{Q_1\setminus Q_{q_i}}\frac{1}{t_i}h^{\rm qc}\left(t_i\nabla u_{x_0}\right)\mathrm{d}y\le\lim_{i\to\infty}c\int_{Q_1\setminus Q_{q_i}}\mathrm{d}|Du_{x_0}|=0\,.\tag{4.31}$$

Further, $w_i^*\in BV(Q_1;\mathbb{R}^m)$ and $\mathrm{supp}(w_i^*-u_{x_0})\subset\subset Q_1$. By [AFP00, Lemma 5.50] and Theorem 4.3

$$\int_{Q_1}\frac{1}{t_i}h^{\rm qc}\left(t_i\nabla w_i^*\right)\mathrm{d}y+\int_{Q_1}h^{\rm qc,\infty}(\mathrm{d}D^s w_i^*)\ge\frac{1}{t_i}h^{\rm qc}\left(t_i Du_{x_0}(Q_1)\right).$$

Therefore, being $Du_{x_0}(Q_1)=\eta(x_0)\otimes\xi(x_0)D\chi((-1/2,1/2))$ a rank-one matrix, the latter estimate together with (4.30) and (4.31) yield that

$$\begin{aligned}
h^{\rm qc,\infty}(Du_{x_0}(Q_1))&=\lim_{i\to\infty}\frac{1}{t_i}h^{\rm qc}(t_i Du_{x_0}(Q_1))\\
&\le\liminf_{i\to\infty}\left[\int_{Q_1}\frac{1}{t_i}h^{\rm qc}(t_i\nabla w_i^*)\mathrm{d}y+\int_{Q_1}h^{\rm qc,\infty}(\mathrm{d}D^s w_i^*)\right]\\
&\le\liminf_{i\to\infty}\int_{Q_1}\frac{1}{t_i}h^{\rm qc}(t_i\nabla\hat w_i)\mathrm{d}y.
\end{aligned}$$

Recalling (4.29), we infer that

$$\alpha_\delta h^{\mathrm{qc},\infty}(Du_{x_0}(Q_1)) \leq (1+\frac{C}{M})\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0),$$

for every M sufficiently large. Therefore, by letting $M\to\infty$ we conclude that

$$\alpha_\delta h^{\mathrm{qc},\infty}(Du_{x_0}(Q_1)) \leq \frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x_0).$$

As $Du_{x_0}(Q_1) = \eta(x_0)\otimes\xi(x_0)D\chi((-1/2,1/2)) = \eta(x_0)\otimes\xi(x_0)$, this and (4.28) conclude the proof of (4.24). □

The lower bound in BV follows at once from the lower bounds for the surface and the diffuse parts.

Theorem 4.6. *Let $u\in BV(\Omega;\mathbb{R}^m)$. Then, for all $A\in\mathcal{A}(\Omega)$*

$$\mathcal{F}_0(u,1;A) \leq \Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u,1;A), \tag{4.32}$$

where $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ have been defined in (2.1) *and* (2.13).

Proof. For simplicity, we will prove the statement for $A=\Omega$. We argue by localization. Assume that $(u_\varepsilon,v_\varepsilon)\to(u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$, with $u\in BV(\Omega;\mathbb{R}^m)$, and that

$$\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)<\infty.$$

Set

$$\mu(A) := \liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon;A), \qquad \text{for all } A\in\mathcal{A}(\Omega),$$
$$\lambda := \mathcal{L}^n \llcorner \Omega + \mathcal{H}^{n-1}\llcorner J_u + |D^c u|$$
$$\psi_1 := g([u],\nu_u), \qquad \psi_2 := h^{\mathrm{qc}}(\nabla u) + h^{\mathrm{qc},\infty}(\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}),$$

and notice that μ is a monotone set function which is superadditive on disjoint open sets, λ is a positive Borel measure and ψ_i are positive Borel functions satisfying

$$\mu(A)\geq\int_A\psi_i\mathrm{d}\lambda, \qquad \text{for } i=1,2 \text{ and } A\in\mathcal{A}(\Omega)$$

thanks to Propositions 4.1 and 4.2. By [Bra98, Proposition 1.16] we conclude

$$\mu(\Omega)\geq\int_\Omega(\psi_1\vee\psi_2)\mathrm{d}\lambda,$$

which gives the thesis. □

Remark 4.7. *From the argument in Lemma 4.4 one can also deduce directly that $u\in(GBV(\Omega))^m$. Indeed, consider $(u_\varepsilon,v_\varepsilon)\to(u,v)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ with $\sup_\varepsilon\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)<\infty$. Necessarily $v=1$ $\mathcal{L}^n$-a.e. on Ω. Moreover, with fixed*

$\delta \in (0,1)$, keeping the notation introduced in Lemma 4.4, using the growth conditions on h (see (2.8)) we get

$$\int_\Omega |\nabla \tilde{u}^\delta_\varepsilon| \mathrm{d}x + \mathcal{H}^{n-1}(J_{\tilde{u}^\delta_\varepsilon}) \leq c(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + 1)\,,$$

for some positive constant c depending on δ and on $\mathcal{L}^n(\Omega)$. In particular, for each component $(\tilde{u}^\delta_\varepsilon)_i$ of $\tilde{u}^\delta_\varepsilon$, $i \in \{1, \dots, n\}$, we have $(\tilde{u}^\delta_\varepsilon)_i \in GSBV(\Omega)$ and

$$\int_\Omega |\nabla (\tilde{u}^\delta_\varepsilon)_i| \mathrm{d}x + \mathcal{H}^{n-1}(J_{(\tilde{u}^\delta_\varepsilon)_i}) \leq c(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + 1)\,.$$

Then, if $k > 0$ and $\tau_k(s) := (s \vee k) \wedge (-k)$, from the estimate above we infer that $|D(\tau_k((\tilde{u}^\delta_\varepsilon)_i))|(\Omega) \leq C_k$, with $C_k > 0$ depending on k and on the sequence, but not on ε. Therefore, there is a subsequence that converges weakly in $BV(\Omega)$. This implies, recalling that $\tilde{u}^\delta_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^m)$ as $\varepsilon \to 0$ for all $\delta \in (0,1)$, that $\tau_k(u_i) \in BV(\Omega)$ for all k. In conclusion, we deduce that $u_i \in GBV(\Omega)$, for all $i \in \{1, \dots, n\}$, and thus $u \in (GBV(\Omega))^m$.

4.4 Lower bound in GBV

In this section we extend the validity of the lower bound Theorem 4.6 to every $u \in (GBV(\Omega))^m$. We first prove that the functional $\mathcal{F}_0$ is continuous under truncations.

Proposition 4.8. *Let $\mathcal{F}_0$ and $\mathcal{T}_k$ be defined as in (2.13) and (3.11), respectively. Then, for all $u \in (GBV(\Omega))^m$ with $\mathcal{F}_0(u,1) < \infty$ we have*

$$\lim_{k\to\infty} \mathcal{F}_0(\mathcal{T}_k(u), 1) = \mathcal{F}_0(u,1)\,.$$

Proof. We prove the convergence of the volume, Cantor and surface terms separately. It is useful to recall for the rest of the proof that $\|\nabla \mathcal{T}_k\|_{L^\infty(\mathbb{R}^m)} \leq 1$.

For the volume part, we observe that (2.8) implies $|\nabla u| \in L^1(\Omega)$. We have $\nabla(\mathcal{T}_k(u)) = \nabla u$ for $\mathcal{L}^n$-a.e. $x \in \Omega_k := \{|u| \leq a_k\}$, therefore in view of (2.8) we get

$$\Big| \int_\Omega h^{\mathrm{qc}}\big(\nabla(\mathcal{T}_k(u))\big)\mathrm{d}x - \int_\Omega h^{\mathrm{qc}}(\nabla u)\mathrm{d}x \Big| \leq c \int_{\Omega\setminus\Omega_k} (1 + |\nabla u|)\mathrm{d}x\,,$$

so that, as $a_k \to \infty$ as $k \uparrow \infty$, we conclude

$$\lim_{k\to\infty} \int_\Omega h^{\mathrm{qc}}\big(\nabla(\mathcal{T}_k(u))\big)\mathrm{d}x = \int_\Omega h^{\mathrm{qc}}(\nabla u)\mathrm{d}x\,.$$

For the surface term we recall that $J_{\mathcal{T}_k(u)} \subseteq J_u$ for every $k \in \mathbb{N}$ with $\nu_{\mathcal{T}_k(u)} = \nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_{\mathcal{T}_k(u)}$. Then, thanks to (4.6) we infer that $(\mathcal{T}_k(u))^\pm \to u^\pm$, $\chi_{J_{\mathcal{T}_k(u)}} \to \chi_{J_u}$ and $|[\mathcal{T}_k(u)]| \leq |[u]|$ $\mathcal{H}^{n-1}$-a.e. in J_u, and then we conclude

$$\begin{aligned}\lim_{k\to\infty} \int_{J_{\mathcal{T}_k(u)}} g([\mathcal{T}_k(u)], \nu_{\mathcal{T}_k(u)})\mathrm{d}\mathcal{H}^{n-1} &= \lim_{k\to\infty} \int_{J_u} g([\mathcal{T}_k(u)], \nu_u)\chi_{J_{\mathcal{T}_k(u)}}\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{J_u} g([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

thanks to Lemmata 3.7 and 3.8 (ii) and to the Dominated Convergence Theorem.

For what the Cantor part of the energy is concerned, by (2.8) we have that $0 \le h^{\mathrm{qc},\infty}(\xi) \le c|\xi|$. Further, the definitions of $\mathcal{T}_k$ and of $D^c u$ outlined in (4.5) yield in particular

$$D^c(\mathcal{T}_k(u)) \llcorner \Omega_k = D^c u \llcorner \Omega_k$$

$$|D^c(\mathcal{T}_k(u))| \ll |D^c u|, \qquad \frac{\mathrm{d}|D^c(\mathcal{T}_k(u))|}{\mathrm{d}|D^c u|} \le 1\,.$$

Thus,

$$\Big| \int_\Omega h^{\mathrm{qc},\infty}(\mathrm{d}D^c(\mathcal{T}_k(u))) - \int_{\Omega_k} h^{\mathrm{qc},\infty}(\mathrm{d}D^c u) \Big| \le c \int_{\Omega\setminus\Omega_k} \mathrm{d}|D^c u| = c|D^c u|(\Omega \setminus \Omega_k),$$

and therefore

$$\lim_{k\to\infty} \int_\Omega h^{\mathrm{qc},\infty}(\mathrm{d}D^c(\mathcal{T}_k(u))) = \int_\Omega h^{\mathrm{qc},\infty}(\mathrm{d}D^c u),$$

which concludes the proof. □

We are ready to prove the lower bound for generalized functions of bounded variations.

Theorem 4.9. *Let $u \in (GBV(\Omega))^m$. Then*

$$\mathcal{F}_0(u,1) \le \Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,1), \tag{4.33}$$

where $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ have been defined in (2.1) *and* (2.13).

Proof. Let $u \in (GBV(\Omega))^m$ and let $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$ be such that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$, with $v_\varepsilon \in [0,1]$ $\mathcal{L}^n$-a.e. in Ω. Without loss of generality we can suppose that $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$, that the latter is actually a limit (up to a subsequence not relabeled), and that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ $\mathcal{L}^n$-a.e. in Ω. In particular, from Section 4.1 we infer that $u \in (GBV(\Omega))^m$, with $|\nabla u| \in L^1(\Omega)$ and satisfying (4.6) and (4.3), so that $\mathcal{F}_0(u,1) < \infty$.

Recalling the definition of the truncation $\mathcal{T}_k$ in (3.11), we have that $\mathcal{T}_k(u_\varepsilon) \to \mathcal{T}_k(u)$ in $L^1(\Omega;\mathbb{R}^m)$ for any k and that $\mathcal{T}_k(u) \in BV(\Omega;\mathbb{R}^m)$, being $\mathcal{F}_0(u,1) < \infty$. Hence, we can apply Theorem 4.6 to say that

$$\mathcal{F}_0(\mathcal{T}_{k_M}(u),1) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon). \tag{4.34}$$

We claim that for all $M \in \mathbb{N}$ there is $k_M \in \{M+1,\dots,2M\}$ independent of ε such that after extracting a further subsequence

$$\mathcal{F}_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon) \le \Big(1 + \frac{c}{M}\Big)\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + c\mathcal{L}^n(\{|u_\varepsilon| > a_M\}), \tag{4.35}$$

for some $c > 0$ independent of ε and of M. Given this for granted, we get by (4.34), (4.35) and by the convergence $u_\varepsilon \to u$ in measure

$$\limsup_{M\to\infty} \mathcal{F}_0(\mathcal{T}_{k_M}(u), 1) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Finally, using the continuity under truncations for $\mathcal{F}_0$ established in Proposition 4.8, we obtain

$$\mathcal{F}_0(u, 1) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$$

and hence (4.33).

It remains to prove (4.35). To this aim we argue as in Proposition 3.2 using De Giorgi's averaging-slicing method on the range. First, for all $k \in \mathbb{N}$ we split the energy contributions

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon) &= \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \{|u_\varepsilon| \le a_k\}) \\ &\quad + \mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}) + \mathcal{F}_\varepsilon(0, v_\varepsilon; \{|u_\varepsilon| \ge a_{k+1}\}) . \qquad (4.36)\end{aligned}$$

By (2.3) and the definition of $\mathcal{T}_k$, the last but one term in the previous expression can be estimated as

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}) &\le c \int_{\{a_k<|u_\varepsilon|<a_{k+1}\}} f_\varepsilon^2(v_\varepsilon)\Psi(\nabla u_\varepsilon)\mathrm{d}x \\ &\quad + c\mathcal{L}^n(\{a_k < |u_\varepsilon| < a_{k+1}\}) + \mathcal{F}_\varepsilon(0, v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}) , \qquad (4.37)\end{aligned}$$

for some $c > 0$. Summing (4.36) and (4.37) and averaging, we conclude that there exists $k_{M,\varepsilon} \in \{M + 1, \dots, 2M\}$ such that

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_{k_{M,\varepsilon}}(u_\varepsilon), v_\varepsilon) &\le \frac{1}{M} \sum_{k=M+1}^{2M} \mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon) \\ &\le \Big(1 + \frac{c}{M}\Big)\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + c\mathcal{L}^n(\{|u_\varepsilon| > a_M\}) ,\end{aligned}$$

for some $c > 0$. As $\varepsilon \to 0$, there exists a subsequence of $\{k_{M,\varepsilon}\}$ that is independent of ε. This yields (4.35) and concludes the proof. □

5 Upper bound

In this Section we prove the $\Gamma - \limsup$ inequality in Theorem 2.1. In order to be able to obtain existence of minimizers for the perturbed functionals (see Section 6), we consider a perturbed version of the functional which includes an additional uniformly coercive term, and prove the upper bound directly for the modified functional. We fix a function $\eta : (0,1] \to [0,1]$ such that

$$\lim_{\varepsilon\to 0} \frac{\eta_\varepsilon}{\varepsilon} = 0 \tag{5.1}$$

and define

$$\mathcal{F}^\eta_\varepsilon(u,v;A) := \mathcal{F}_\varepsilon(u,v;A) + \eta_\varepsilon \int_A \Psi(\nabla u)\mathrm{d}x, \tag{5.2}$$

where $\mathcal{F}_\varepsilon$ has been defined in (2.1).

One key ingredient in the proof of the upper bound is that the $\overline{\Gamma}$-limit of $\mathcal{F}^\eta_\varepsilon$ satisfies the hypotheses of [BFM98, Theorem 3.12], so that it can be represented as an integral functional. Its diffuse and surface densities will be identified by a direct computation.

In order to prove that $\overline{\Gamma}\text{-}\lim_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u,1;\cdot)$ is a Borel measure, we first check the weak subadditivity of the Γ-upper limit of $\mathcal{F}^\eta_\varepsilon$.

Lemma 5.1. *Let $u \in L^1(\Omega;\mathbb{R}^m)$, let $A', A, B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$, then*

$$\begin{aligned}\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u,1;A'\cup B) \le \\ \Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u,1;A) + \Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u,1;B),\end{aligned} \tag{5.3}$$

where $\mathcal{F}^\eta_\varepsilon$ has been defined in (5.2)*.*

Proof. To simplify the notation let us set $\mathcal{F}'' := \Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon$. It is not restrictive to assume that the right-hand side of (5.3) is finite, so that $u \in (GBV \cap L^1(A\cup B))^m$. Let $(u^A_\varepsilon, v^A_\varepsilon)$, $(u^B_\varepsilon, v^B_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$ be such that

$$(u^J_\varepsilon, v^J_\varepsilon) \to (u,1) \text{ in } L^1(\Omega;\mathbb{R}^m)\times L^1(\Omega)\,, \tag{5.4}$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u^J_\varepsilon, v^J_\varepsilon; J) = \mathcal{F}''(u,1;J), \tag{5.5}$$

for $J \in \{A,B\}$.

Step 1. Estimate (5.3) **is valid if** $u \in BV\cap L^2(A\cup B;\mathbb{R}^m)$ **and** (5.5) **holds for two sequences converging to** u **in** $L^2(\Omega;\mathbb{R}^m)$**.** For $\delta := \mathrm{dist}(A', \partial A) > 0$ and some $M \in \mathbb{N}$, we set for all $i \in \{1,\dots,M\}$

$$A_i := \left\{ x\in\Omega :\ \mathrm{dist}(x,A') < \frac{\delta}{M} i \right\},$$

and $A_0 := A'$, so that $A_{i-1} \subset\subset A_i \subset A$. Let $\varphi_i \in C_c^1(\Omega)$ be a cut-off function between A_{i-1} and A_i, i.e., $\varphi_i|_{A_{i-1}} = 1$, $\varphi_i|_{A_i^c} = 0$, and $\|\nabla\varphi_i\|_{L^\infty(\Omega)} \leq \frac{2M}{\delta}$. Then, we define

$$u_\varepsilon^i := \varphi_i\, u_\varepsilon^A + (1-\varphi_i)u_\varepsilon^B, \tag{5.6}$$

and

$$v_\varepsilon^i := \begin{cases} \varphi_{i-1}\, v_\varepsilon^A + (1-\varphi_{i-1})(v_\varepsilon^A \wedge v_\varepsilon^B), & \text{on } A_{i-1}, \\ v_\varepsilon^A \wedge v_\varepsilon^B, & \text{on } A_i \setminus A_{i-1}, \\ \varphi_{i+1}(v_\varepsilon^A \wedge v_\varepsilon^B) + (1-\varphi_{i+1})\, v_\varepsilon^B, & \text{on } \Omega \setminus A_i. \end{cases} \tag{5.7}$$

For $i \in \{2, \ldots, M-1\}$, $(u_\varepsilon^i, v_\varepsilon^i) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$. Arguing exactly as in [CFI16, Lemma 6.2], for all $\varepsilon > 0$ we can find an index $i_\varepsilon \in \{2, \ldots, M-1\}$ such that

$$\begin{aligned} &\mathcal{F}_\varepsilon^\eta(u_\varepsilon^{i_\varepsilon}, v_\varepsilon^{i_\varepsilon}; A' \cup B) \leq \mathcal{F}_\varepsilon^\eta(u_\varepsilon^A, v_\varepsilon^A; A) + \mathcal{F}_\varepsilon^\eta(u_\varepsilon^B, v_\varepsilon^B; B) \\ &+ \frac{c}{M}\Big(\mathcal{F}_\varepsilon^\eta(u_\varepsilon^A, v_\varepsilon^A; B\cap(A\setminus A')) + \mathcal{F}_\varepsilon^\eta(u_\varepsilon^B, v_\varepsilon^B; B\cap(A\setminus A')) + \mathcal{L}^n(B\cap(A\setminus A'))\Big) \\ &\qquad + \frac{c\,M}{\delta^2}\int_{B\cap(A\setminus A')} |u_\varepsilon^A - u_\varepsilon^B|^2\,\mathrm{d}x + \frac{c\,M\varepsilon}{\delta^2}\int_{B\cap(A\setminus A')} |v_\varepsilon^A - v_\varepsilon^B|^2\,\mathrm{d}x. \end{aligned}$$

Passing first to the limit as $\varepsilon \to 0$ and then as $M \to \infty$ we obtain (5.3) having assumed that $u_\varepsilon^J \to u$ in $L^2(\Omega;\mathbb{R}^m)$, $J \in \{A, B\}$.

Step 2. Estimate (5.3) **is valid if** $u \in (GBV \cap L^1(A \cup B))^m$**.** We use De Giorgi's slicing/averaging techniques on the co-domain by employing the truncation functions introduced in (3.11). The argument is analogous to that developed in Step 1 of Proposition 3.2.

Note that if $u \in (GBV(A \cup B))^m$ then $\mathcal{T}_k(u) \in BV \cap L^\infty(A \cup B;\mathbb{R}^m)$. In addition, for all $k \in \mathbb{N}$ and $J \in \{A, B\}$ it is easy to check that $\mathcal{T}_k(u_\varepsilon^J) \in W^{1,2}(\Omega;\mathbb{R}^m)$, that $\mathcal{T}_k(u_\varepsilon^J) \to \mathcal{T}_k(u)$ as $\varepsilon \to 0$ in $L^2(\Omega;\mathbb{R}^m)$, and that

$$\begin{aligned} \mathcal{F}_\varepsilon^\eta(\mathcal{T}_k(u_\varepsilon^J), v_\varepsilon^J; J) = {}& \mathcal{F}_\varepsilon^\eta(u_\varepsilon^J, v_\varepsilon^J; \{|u_\varepsilon^J| \leq a_k\}) \\ &+ \mathcal{F}_\varepsilon^\eta(\mathcal{T}_k(u_\varepsilon^J), v_\varepsilon^J; \{a_k < |u_\varepsilon^J| < a_{k+1}\}) \\ &+ \mathcal{F}_\varepsilon^\eta(0, v_\varepsilon^J; \{|u_\varepsilon^J| \geq a_{k+1}\}). \end{aligned} \tag{5.8}$$

We estimate the last but one term. The growth conditions on Ψ (cf. (2.3)) and $\|\nabla\mathcal{T}_k\|_{L^\infty(\mathbb{R}^m)} \leq 1$ yield for a constant $c > 0$

$$\begin{aligned} \mathcal{F}_\varepsilon^\eta(\mathcal{T}_k(u_\varepsilon^J), v_\varepsilon^J; \{a_k < |u_\varepsilon^J| < a_{k+1}\}) &\leq c\int_{\{a_k < |u_\varepsilon^J| < a_{k+1}\}} (\eta_\varepsilon + f_\varepsilon^2(v_\varepsilon^J))\Psi(\nabla u_\varepsilon^J)\mathrm{d}x \\ + c\mathcal{L}^n(\{a_k < |u_\varepsilon^J| < a_{k+1}\}) &+ \mathcal{F}_\varepsilon(0, v_\varepsilon^J; \{a_k < |u_\varepsilon^J| < a_{k+1}\}). \end{aligned} \tag{5.9}$$

Collecting (5.8) and (5.9) we conclude that

$$\begin{aligned} &\mathcal{F}_\varepsilon^\eta(\mathcal{T}_k(u_\varepsilon^J), v_\varepsilon^J; J) \leq \mathcal{F}_\varepsilon^\eta(u_\varepsilon^J, v_\varepsilon^J; J) \\ &\quad + c\int_{\{a_k < |u_\varepsilon^J| < a_{k+1}\}} (\eta_\varepsilon + f_\varepsilon^2(v_\varepsilon^J))\Psi(\nabla u_\varepsilon^J)\mathrm{d}x + c\mathcal{L}^n(\{|u_\varepsilon^J| > a_k\}). \end{aligned}$$

Let now $M \in \mathbb{N}$, by summing up the latter inequality for both A and B and by averaging, there exists $k_{\varepsilon,M} \in \{M+1, \dots, 2M\}$ such that

$$\begin{aligned} &\mathcal{F}^\eta_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u^A_\varepsilon), v^A_\varepsilon; A) + \mathcal{F}^\eta_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u^B_\varepsilon), v^B_\varepsilon; B) \\ &\leq \frac{1}{M} \sum_{k=M+1}^{2M} \Big(\mathcal{F}^\eta_\varepsilon(\mathcal{T}_k(u^A_\varepsilon), v^A_\varepsilon; A) + \mathcal{F}^\eta_\varepsilon(\mathcal{T}_k(u^B_\varepsilon), v^B_\varepsilon; B) \Big) \\ &\leq \Big(1 + \frac{c}{M}\Big) \Big(\mathcal{F}^\eta_\varepsilon(u^A_\varepsilon, v^A_\varepsilon; A) + \mathcal{F}^\eta_\varepsilon(u^B_\varepsilon, v^B_\varepsilon; B) \Big) \\ &+ c\mathcal{L}^n(\{|u^A_\varepsilon| \geq a_{M+1}\}) + c\mathcal{L}^n(\{|u^B_\varepsilon| \geq a_{M+1}\}). \end{aligned} \tag{5.10}$$

Up to a subsequence, we may take the index $k_{\varepsilon,M} = k_M$, i.e. to be independent of ε. Therefore, passing to the limit as $\varepsilon \to 0$, the convergence $u^J_\varepsilon \to u$ in measure for $J \in \{A, B\}$, (5.4), (5.5), (5.10) and Step 1 yield

$$\mathcal{F}''(\mathcal{T}_{k_M}(u), 1; A' \cup B) \leq \mathcal{F}''(\mathcal{T}_{k_M}(u), 1; A) + \mathcal{F}''(\mathcal{T}_{k_M}(u), 1; B)$$
$$\leq \Big(1 + \frac{c}{M}\Big) \Big(\mathcal{F}''(u, 1; A) + \mathcal{F}''(u, 1; B) \Big) \tag{5.11}$$
$$+ c\mathcal{L}^n(\{|u| \geq a_{M+1}\}) \tag{5.12}$$

Eventually, since $\mathcal{T}_{k_M}(u) \to u$ in $L^1(\Omega; \mathbb{R}^m)$ as $M \uparrow \infty$, by the lower semicontinuity of $\mathcal{F}''$ for the $L^1(\Omega; \mathbb{R}^m)$ convergence we conclude (5.3). □

We are now ready to prove the upper bound inequality.

Theorem 5.2. *Let $\mathcal{F}^\eta_\varepsilon$ and $\mathcal{F}_0$ be defined in* (5.2) *and* (2.13)*, respectively. For every $(u, v) \in L^1(\Omega; \mathbb{R}^{m+1})$ it holds*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}^\eta_\varepsilon(u, v) \leq \mathcal{F}_0(u, v). \tag{5.13}$$

Proof. Given a subsequence $(\mathcal{F}^\eta_{\varepsilon_k})$ of $(\mathcal{F}^\eta_\varepsilon)$, there exists a further subsequence, not relabeled, which $\overline{\Gamma}$-converges to some functional $\widehat{\mathcal{F}}$, that is,

$$\widehat{\mathcal{F}} = (\mathcal{F}')_- = (\mathcal{F}'')_-, \tag{5.14}$$

where $\mathcal{F}'$ and $\mathcal{F}''$ denote here the $\Gamma(L^1)$-lower and upper limits of $\mathcal{F}^\eta_{\varepsilon_k}$ and where the subscript $_-$ denotes the inner regular envelope of the relevant functional ([Dal93, Definition 16.2 and Theorem 16.9]).

We remark that $\widehat{\mathcal{F}}(u, v; \cdot)$ is the restriction of a Borel measure to open sets by [Dal93, Theorem 14.23]. Indeed, $\widehat{\mathcal{F}}(u, v; \cdot)$ is increasing and inner regular by definition; additivity follows from (5.14), once one checks that $(\mathcal{F}')_-$ is superadditive and $(\mathcal{F}'')_-$ is subadditive. The former condition is a direct consequence of the additivity of $\mathcal{F}_\varepsilon(u, v; \cdot)$ and [Dal93, Proposition 16.12]. The latter follows from Lemma 5.1 along the lines of [Dal93, Proposition 18.4], using Lemma 5.1 instead of [Dal93, (18.6)].

We divide the proof of (5.13) into several steps. First note that it is sufficient to prove it for $v = 1$ $\mathcal{L}^n$-a.e. on Ω.

Step 1. Estimate on the diffuse part for $u \in BV(\Omega;\mathbb{R}^m)$. We first prove a global rough estimate for $\mathcal{F}''$ which actually turns out to be sharp for the diffuse part if $u \in BV(\Omega;\mathbb{R}^m)$. To this aim we set $H : L^1(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega)\to[0,\infty]$ as

$$H(u;A) := \int_A h(\nabla u)\,\mathrm{d}x \tag{5.15}$$

if $u \in W^{1,1}(\Omega;\mathbb{R}^m)$, and ∞ otherwise, where h has been defined in (2.6). We next prove the bound

$$\mathcal{F}''(u,1;A) \le H(u;A) \tag{5.16}$$

for $u \in W^{1,1}(\Omega;\mathbb{R}^m)$. Given this estimate for granted, on setting $H^* : L^1(\Omega;\mathbb{R}^m)\times\mathcal{A}(\Omega)\to[0,\infty]$

$$H^*(u;A) := \int_A h^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x \tag{5.17}$$

if $u \in W^{1,1}(\Omega;\mathbb{R}^m)$, and ∞ otherwise, the lower semicontinuity of $\mathcal{F}''$ with respect to the $L^1(\Omega;\mathbb{R}^m)$ topology and the relaxation result with respect to the sequential weak topology in $W^{1,1}(\Omega;\mathbb{R}^m)$ in [AF84, Statement III.7] (or [Dac08, Theorem 9.1]) imply then that

$$\mathcal{F}''(u,1;A) \le H^*(u;A)\,.$$

In turn, from the estimate above, Theorem 4.3 finally yields

$$\mathcal{H}(u;A) := \mathrm{sc}^-(L^1)\text{-}H^*(u;A) = \int_A h^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x + \int_A h^{\mathrm{qc},\infty}\big(\mathrm{d}D^s u\big)\,, \tag{5.18}$$

for every $u \in BV(\Omega;\mathbb{R}^m)$. Therefore, the bound

$$\mathcal{F}''(u,1;A) \le \mathcal{H}(u;A) \tag{5.19}$$

follows for every $u \in BV(\Omega;\mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

To prove (5.16), assume first that u is an affine function, say $u(x) = \xi x + b$, with $\xi \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Then, the pair

$$u_k := u, \qquad v_k := 1 - \sqrt{2\ell\varepsilon_k}\,\Psi^{1/4}(\xi),$$

is such that $(u_k, v_k) \to (u,1)$ in $L^2(\Omega;\mathbb{R}^m)\times L^1(\Omega)$ and recalling $\eta_{\varepsilon_k}\to 0$

$$\limsup_{k\to\infty} \mathcal{F}^\eta_{\varepsilon_k}(u_k,v_k;A) \le \mathcal{L}^n(A)\ell\Psi^{1/2}(\xi).$$

Instead, if

$$\bar{u}_k := u, \qquad \bar{v}_k := 1$$

we get

$$\limsup_{k\to\infty} \mathcal{F}^\eta_{\varepsilon_k}(u_k,v_k;A) = \mathcal{L}^n(A)\Psi(\xi).$$

Therefore, we conclude (5.16) for every affine function u in view of the last two estimates.

Assume now that $u \in C^0(\Omega;\mathbb{R}^m)$ is a piecewise affine function, say $u(x) = \sum_{i=1}^N (\xi_i x + b_i)\chi_{\Omega_i}(x)$, with $\xi_i \in \mathbb{R}^{m\times n}$, $b_i \in \mathbb{R}^m$, and $\Omega_i \in \mathcal{A}(\Omega)$ disjoint and with Lipschitz boundary, and such that $\mathcal{L}^n(\Omega \setminus \cup_{i=1}^N \Omega_i) = 0$. Then, set

$$u_k := u\,, \qquad v_k := \sum_{i=1}^N \varphi_i v_k^i$$

where for each $i \in \{1,\dots,N\}$

$$v_k^i := \begin{cases} 1 - \sqrt{2\ell\varepsilon_k}\Psi^{1/4}(\xi_i), & \text{if } \Psi^{1/2}(\xi_i) > \ell, \\ 1, & \text{if } \Psi^{1/2}(\xi_i) \le \ell, \end{cases}$$

and $\{\varphi_i\}_{1\le i\le N}$ is a partition of unity subordinated to the covering $\{\Omega_i^\delta\}_{1\le i\le N}$ of Ω, Ω_i^δ an open δ-neighborhood of Ω_i for $\delta > 0$, i.e. $\varphi_i \in C_c^\infty(\Omega_i^\delta)$, $0 \le \varphi_i \le 1$, $\varphi_i = 1$ on $\Omega_i^{-\delta}$, $\sum_{i=1}^N \varphi_i = 1$ in Ω (we write $\Omega_i^{-\delta} := \{x : B_\delta(x) \subseteq \Omega_i\}$). Then, a straightforward computation shows that

$$\limsup_{k\to\infty} \mathcal{F}^\eta_{\varepsilon_k}(u_k, v_k; A) \le \sum_{i=1}^N \mathcal{L}^n(\Omega_i^\delta \cap A) h(\xi_i) + c\sum_{i=1}^N \mathcal{L}^n(\Omega_i^\delta \setminus \Omega_i^{-\delta}),$$

where c depends on ℓ, Ψ, and $\xi_1,\dots,\xi_N$. Therefore we conclude (5.16) when u is piecewise affine, namely as $\delta \to 0$ in the latter inequality we have

$$\mathcal{F}''(u,1;A) \le \sum_{i=1}^N \mathcal{L}^n(\Omega_i \cap A) h(\xi_i) = H(u;A)\,.$$

If $u \in W^{1,1}(\Omega;\mathbb{R}^m)$, we consider an extension of u itself (still denoted by u for convenience) to $W_0^{1,1}(\Omega';\mathbb{R}^m)$, for some open and bounded $\Omega' \supset\supset \Omega$ (recall that Ω is assumed to be Lipschitz regular). Then, we use a classical density result [ET99, Proposition 2.1 in Chapter X] to find $u_k \in W_0^{1,1}(\Omega';\mathbb{R}^m)$ piecewise affine such that $u_k \to u$ in $W^{1,1}(\Omega';\mathbb{R}^m)$. The continuity of H for the $W^{1,1}(\Omega;\mathbb{R}^m)$ convergence, and the lower semicontinuity of $\mathcal{F}''$ for the $L^1(\Omega;\mathbb{R}^{m+1})$ convergence finally imply (5.16).

Step 2. Inner regularity of $\mathcal{F}''(u,1;\cdot)$ and existence of the $\Gamma(L^1)$-limit in $A \in \mathcal{A}(\Omega)$ for $u \in BV(\Omega;\mathbb{R}^m)$. First we show that if $u \in BV(\Omega;\mathbb{R}^m)$ then

$$\mathcal{F}''(u,1;\cdot) = (\mathcal{F}'')_-(u,1;\cdot). \tag{5.20}$$

Given an open set A and $\delta > 0$, we can find open sets A', A'', and C, with $A' \subset\subset A'' \subset\subset A$ and $A \setminus A' \subset C$, such that $\mathcal{H}(u;C) \le \delta$, where $\mathcal{H}$ is defined in (5.18). Then, by Lemma 5.1 and (5.19) we get

$$\mathcal{F}''(u,1;A) \le \mathcal{F}''(u,1;A' \cup C) \le \mathcal{F}''(u,1;A'') + \mathcal{H}(u;C) \le \mathcal{F}''(u,1;A'') + \delta.$$

Hence, (5.20) holds true and in turn by (5.14) we have

$$\widehat{\mathcal{F}}(u,1;\cdot) \le \mathcal{F}'(u,1;\cdot) \le \mathcal{F}''(u,1;\cdot) = \widehat{\mathcal{F}}(u,1;\cdot),$$

so that the Γ-limit of $\mathcal{F}^\eta_{\varepsilon_k}(u,1;\cdot)$ exists and coincides with $\widehat{\mathcal{F}}(u,1;\cdot)$ for all $u \in BV(\Omega;\mathbb{R}^m)$.

Step 3. Integral representation of the $\Gamma(L^1)$-limit on $BV(\Omega;\mathbb{R}^m)\times\{1\}$. We now would like to represent $\widehat{\mathcal{F}}$ as an integral functional through [BFM98, Theorem 3.12] and to estimate its diffuse and surface densities. In order to satisfy the coercivity hypothesis [BFM98, Eq. (2.3')], we introduce an auxiliary functional

$$\widehat{\mathcal{F}}_\lambda(u,1) := \widehat{\mathcal{F}}(u,1) + \lambda|Du|(\Omega)$$

for all $u \in BV(\Omega;\mathbb{R}^m)$, where $\lambda \in (0,1]$ is a small parameter. Indeed, (4.2), (4.3), (2.16) and (5.19) yield

$$\lambda|Du|(\Omega) - c\mathcal{L}^n(\Omega) \le \widehat{\mathcal{F}}_\lambda(u,1) \le c(|Du|(\Omega) + \mathcal{L}^n(\Omega))\,,$$

for all $u \in BV(\Omega;\mathbb{R}^m)$ and for some $c>0$. Note that $\widehat{\mathcal{F}}_\lambda$ also satisfies the continuity hypothesis [BFM98, Eq. (2.4)], since

$$\mathcal{F}^\eta_{\varepsilon_k}(u(\cdot-z),v(\cdot-z);z+A) = \mathcal{F}^\eta_{\varepsilon_k}(u,v;A),$$

$$\mathcal{F}^\eta_{\varepsilon_k}(u+b,v;A) = \mathcal{F}^\eta_{\varepsilon_k}(u,v;A),$$

for all $(u,v)\in W^{1,2}(\Omega;\mathbb{R}^{m+1})$, $z,b\in\mathbb{R}^m$, $A\in\mathcal{A}(\Omega)$, and analogous properties then hold for $\widehat{\mathcal{F}}$.

The integral representation result [BFM98, Theorem 3.12] then applies to $\widehat{\mathcal{F}}_\lambda + c\mathcal{L}^n$ and gives, for $u\in BV(\Omega;\mathbb{R}^m)$ and $A\in\mathcal{A}(\Omega)$, taking also into account the aforementioned translation invariance,

$$\widehat{\mathcal{F}}_\lambda(u,1;A) = \int_A h_\lambda(\nabla u)\mathrm{d}x + \int_{J_u\cap A} g_\lambda([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} + \int_A h_\lambda^\infty(\mathrm{d}D^c u),$$

where

$$h_\lambda(\xi) := \limsup_{\delta\downarrow 0}\frac{1}{\delta^n}\inf\Big\{\widehat{\mathcal{F}}_\lambda(w,1;\delta Q) :\, w\in BV\big(\delta Q;\mathbb{R}^m\big),\\ w(x)=\xi x \text{ on } \partial(\delta Q)\Big\}, \quad (5.21)$$

for $\xi\in\mathbb{R}^{m\times n}$, Q being a cube with side length 1 centered in the origin;

$$g_\lambda(z,\nu) := \limsup_{\delta\downarrow 0}\frac{1}{\delta^{n-1}}\inf\Big\{\widehat{\mathcal{F}}_\lambda(w,1;\delta Q^\nu) :\, w\in BV\big(\delta Q^\nu;\mathbb{R}^m\big),\\ w=w_z \text{ on } \partial(\delta Q^\nu)\Big\}, \quad (5.22)$$

for $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$, Q^ν being a cube with side length 1 and a face orthogonal to ν and $w_z := z\chi_{\{x\cdot\nu>0\}}$;

$$h_\lambda^\infty(\xi) := \limsup_{t\to\infty} \frac{h_\lambda(t\xi)}{t},$$

for $\xi \in \mathbb{R}^{m\times n}$. Let us estimate separately the three densities above. First, observe that by (5.19) we have

$$h_\lambda(\xi) \le \frac{1}{\delta^n}\widehat{\mathcal{F}}_\lambda(\xi x, 1; \delta Q) \le h^{\mathrm{qc}}(\xi)+\lambda|\xi|, \tag{5.23}$$

so that

$$h_\lambda^\infty(\xi) \le h^{\mathrm{qc},\infty}(\xi)+\lambda|\xi|, \tag{5.24}$$

for all $\xi \in \mathbb{R}^{m\times n}$. We next show that

$$g_\lambda(z,\nu) \le g(z,\nu) + \lambda|z|, \tag{5.25}$$

for $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$. From (5.22) we have

$$\begin{aligned} g_\lambda(z,\nu) &\le \limsup_{\delta\downarrow 0} \frac{1}{\delta^{n-1}}\widehat{\mathcal{F}}_\lambda(w_z, 1; \delta\, Q^\nu) \\ &= \limsup_{\delta\downarrow 0} \frac{1}{\delta^{n-1}}\widehat{\mathcal{F}}(w_z, 1; \delta\, Q^\nu) + \lambda|z|. \end{aligned} \tag{5.26}$$

In turn, by definition of $\widehat{\mathcal{F}}$ for every sequence $(\tilde u_k, \tilde v_k) \to (w_z, 1)$ in $L^1(\delta\, Q^\nu; \mathbb{R}^{m+1})$ we have

$$\widehat{\mathcal{F}}(w_z, 1; \delta\, Q^\nu) \le \limsup_{k\to\infty} \mathcal{F}^{\eta}_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu). \tag{5.27}$$

The proof of (5.25) therefore reduces to the construction of a suitable sequence $(\tilde u_k, \tilde v_k)$, which depends implicitly on $\delta \in (0,1)$, z and ν. By Proposition 3.4 applied with the sequences $\varepsilon_k^* := \varepsilon_k/\delta$ and $\eta_k^* := \eta_{\varepsilon_k}$, there are $(u_k^*, v_k^*) \to (w_z, 1)$ in $L^2(Q^\nu; \mathbb{R}^{m+1})$, such that

$$\lim_{k\to\infty} \mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu) = g(z,\nu) \tag{5.28}$$

and

$$\lim_{k\to\infty} \eta_k^* \|\nabla u_k^*\|^2_{L^2(Q^\nu)} = 0. \tag{5.29}$$

We define $(\tilde u_k, \tilde v_k) \in L^2(\delta\, Q^\nu; \mathbb{R}^{m+1})$ by

$$\tilde u_k(y) := u_k^*\left(\frac{y}{\delta}\right), \qquad \tilde v_k(y) := v_k^*\left(\frac{y}{\delta}\right).$$

Obviously $(\tilde u_k, \tilde v_k) \to (w_z, 1)$ in $L^2(\delta\, Q^\nu; \mathbb{R}^{m+1})$. A change of variable and a straightforward computation using $\varepsilon_k = \delta\varepsilon_k^*$ yield

$$\begin{gathered} \mathcal{F}^\infty_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu) = \delta^{n-1}\mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu), \\ \|\nabla \tilde u_k\|^2_{L^2(\delta\, Q^\nu)} = \delta^{n-2}\|\nabla u_k^*\|^2_{L^2(Q^\nu)}. \end{gathered} \tag{5.30}$$

Fixed $\rho > 0$, by (2.5) we have

$$\Psi(\xi) \le (1+\rho)\Psi_\infty(\xi),$$

for $|\xi|$ large, and then

$$\Psi(\xi) \le (1+\rho)\Psi_\infty(\xi) + C(\rho),$$

for some $C(\rho) > 0$ and all $\xi \in \mathbb{R}^{m\times n}$. Then, with (5.30)

$$\begin{aligned}\mathcal{F}_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu) &\le (1+\rho)\mathcal{F}^\infty_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu) + C(\rho)\mathcal{L}^n(\delta\, Q^\nu)\\ &= (1+\rho)\delta^{n-1}\mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu) + C(\rho)\delta^n.\end{aligned}$$

Similarly, from the growth conditions in (2.3) and (5.30),

$$\eta_{\varepsilon_k}\int_{\delta\, Q^\nu}\psi(\nabla\tilde u_k)\mathrm{d}x \le c\eta_{\varepsilon_k}(\|\nabla\tilde u_k\|^2_{L^2(\delta\, Q^\nu)} + \delta^n) = c\eta_{\varepsilon_k}\delta^{n-2}\|\nabla u_k^*\|^2_{L^2(Q^\nu)} + c\eta_{\varepsilon_k}\delta^n.$$

Summing these two estimates,

$$\mathcal{F}^\eta_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu) \le (1+\rho)\delta^{n-1}\mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu) + C(\rho)\delta^n + c\eta_{\varepsilon_k}\delta^{n-2}\|\nabla u_k^*\|^2_{L^2(Q^\nu)} + c\eta_{\varepsilon_k}\delta^n,$$

and taking the limit $k\to\infty$, by (5.27), (5.28) and (5.29),

$$\widehat{\mathcal{F}}(w_z, 1; \delta\, Q^\nu) \le \limsup_{k\to\infty}\mathcal{F}^\eta_{\varepsilon_k}(\tilde u_k, \tilde v_k; \delta\, Q^\nu) \le (1+\rho)\delta^{n-1}g(z,\nu) + C(\rho)\delta^n. \tag{5.31}$$

We divide by δ^{n-1} and take the limit $\delta\to 0$. Comparing with (5.26),

$$g_\lambda(z,\nu) \le (1+\rho)g(z,\nu) + \lambda|z|, \tag{5.32}$$

and since ρ was arbitrary (5.25) follows.

In conclusion, as $\lambda \to 0$, estimates (5.23), (5.24) and (5.25) imply that for all $u \in BV(\Omega;\mathbb{R}^m)$

$$\widehat{\mathcal{F}}(u,1) \le \mathcal{F}_0(u,1).$$

This, together with the lower bound Theorem 4.6 allows to identify uniquely the Γ-limit of the subsequence $\mathcal{F}^\eta_{\varepsilon_k}$. Finally, Urysohn's property ([Dal93, Proposition 8.3]) extends the result to the whole family $\mathcal{F}^\eta_\varepsilon$.

Step 4. Representation of the $\Gamma(L^1)$-limit on $(GBV(\Omega))^m \times \{1\}$. To extend the validity of (5.13) to $u \in (GBV(\Omega))^m$ we argue by truncation. Indeed, if $k \in \mathbb{N}$ and $\mathcal{T}_k$ is the truncation operator defined in (3.11), then by Steps 1-3 we infer that

$$\mathcal{F}''(\mathcal{T}_k(u), 1) \le \mathcal{F}_0(\mathcal{T}_k(u), 1)\,.$$

The conclusion then follows by the L^1-lower semicontinuity of $\mathcal{F}''$ and by Proposition 4.8. □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. The lower bound has been proven in Theorem 4.9. The upper bound follows by Theorem 5.2 with $\eta_\varepsilon = 0$. □

6 Compactness and convergence of minimizers

Next theorem establishes the compactness of sequences equibounded in energy and in L^1.

Theorem 6.1. *Let $\mathcal{F}_\varepsilon$ be defined in* (2.1)*. If $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$ is such that*

$$\sup_\varepsilon \big(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_{L^1(\Omega)}\big) < \infty,$$

then there exists a subsequence (u_j, v_j) of $(u_\varepsilon, v_\varepsilon)$ and a function $u \in (GBV \cap L^1(\Omega))^m$ such that $u_j \to u$ $\mathcal{L}^n$-a.e. and $v_j \to 1$ in $L^1(\Omega)$.

Proof. This follows arguing componentwise, that is, estimating $\mathcal{F}_\varepsilon$ with its one-dimensional counterpart evaluated in a component, and applying the one-dimensional compactness result obtained in [CFI16, Theorem 3.3] as done in subsection 4.1 (see also the argument in Remark 4.7). □

Convergence of minimizers and of minimum values follow now in a standard way by Theorems 2.1 and 6.1. Let $\eta_\varepsilon > 0$ be as in (5.1), i.e. such that ${}^{\eta_\varepsilon}/_{\varepsilon} \to 0$ as $\varepsilon \to 0$, consider the corresponding family $\mathcal{F}^\eta_\varepsilon$ defined in (5.2) and let $w \in L^q(\Omega;\mathbb{R}^m)$, with $q > 1$. Let now $\mathcal{G}_\varepsilon$, $\mathcal{G}_0 : L^q(\Omega;\mathbb{R}^m) \times L^1(\Omega) \to [0,\infty]$ be defined as

$$\mathcal{G}_\varepsilon(u,v) := \begin{cases} \mathcal{F}^\eta_\varepsilon(u,v) + \displaystyle\int_\Omega |u-w|^q \mathrm{d}x\,, & \text{if } (u,v) \in W^{1,2}(\Omega;\mathbb{R}^m \times [0,1]), \\ \infty, & \text{otherwise} \end{cases}$$

and

$$\mathcal{G}_0(u,v) := \mathcal{F}_0(u,v) + \int_\Omega |u-w|^q \mathrm{d}x\,,$$

where $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ have been defined in (2.1) and (2.13), respectively.

The assumption on the asymptotic ratio ${}^{\eta_\varepsilon}/_{\varepsilon} \to 0$ as $\varepsilon \to 0$ is needed to avoid that the term $\eta_\varepsilon \Psi(\nabla u)$ competes with the term $(1-v)^2/\varepsilon$, overall influencing the limit behaviour. Indeed, if $\eta_\varepsilon \sim \varepsilon$, we expect to gain a control on $|[u]|$, so loosing the limit cohesive effect (compare with [FI14]).

Instead, the addition of the term $\eta_\varepsilon \Psi(\nabla w)$ is instrumental to guarantee the existence of a minimizer for $\mathcal{G}_\varepsilon$, provided that Ψ is quasiconvex. In general, the coercivity of $\mathcal{G}_\varepsilon$ only ensures existence of minimizing sequences $(u^j_\varepsilon)_j$ converging weakly in $W^{1,2}(\Omega;\mathbb{R}^m)$ to some $\bar{u}_\varepsilon$ minimizing the relaxation of $\mathcal{G}_\varepsilon$. Since existence at fixed ε does not interact with the Γ-convergence, we state our result for asymptotically minimizing sequences.

Corollary 6.2. *Let $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$ be such that*

$$\limsup_{\varepsilon\to 0} \big(\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) - m_\varepsilon\big) = 0,$$

where $m_\varepsilon := \inf_{(u,v)\in W^{1,2}(\Omega;\mathbb{R}^{m+1})} \mathcal{G}_\varepsilon(u,v)$. Then $v_\varepsilon \to 1$ in $L^1(\Omega)$ and a subsequence of u_ε converges in $L^q(\Omega;\mathbb{R}^m)$ to a solution of

$$\min_{u\in(GBV(\Omega))^m} \mathcal{G}_0(u,1).$$

Moreover, m_ε tends to the minimum value of $\mathcal{G}_0$.

Proof. The proof of the corollary will be divided in three steps.

Step 1. Γ-limit of $\mathcal{F}^\eta_\varepsilon$ in $L^q \times L^1$. We check that passing from the $L^1 \times L^1$ to the $L^q \times L^1$ topology, the expression of the Γ-limit of $\mathcal{F}^\eta_\varepsilon$ remains the same

$$\Gamma(L^q \times L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u,v) = \mathcal{F}_0(u,v).$$

The lower bound is an immediate consequence of that in $L^1 \times L^1$ (Theorem 4.9, being the L^q convergence stronger than the L^1 convergence).

As for the upper bound, we argue by truncation. First take a subsequence of $\mathcal{F}^\eta_\varepsilon$ (not relabelled for convenience) and fix $u \in BV \cap L^\infty(\Omega;\mathbb{R}^m)$ with $\mathcal{F}_0(u,1) < \infty$. Then Theorem 5.2 yields the existence of a sequence $(u_\varepsilon, v_\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^{m+1})$, such that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ and

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u_\varepsilon, v_\varepsilon) \le \mathcal{F}_0(u,1).$$

Fix $M \in \mathbb{N}$ large enough such that $a_M > \|u\|_\infty$ (see (3.11) for the definition of a_M) and, for every $\varepsilon > 0$, choose $k_{\varepsilon,M} \in \{M+1,\dots,2M\}$ such that

$$\int_{\{a_{k_{\varepsilon,M}} < |u_\varepsilon| < a_{k_{\varepsilon,M}+1}\}} (\eta_\varepsilon + f^2_\varepsilon(v_\varepsilon))\Psi(\nabla u_\varepsilon)\mathrm{d}x \le \frac{1}{M}\int_\Omega (\eta_\varepsilon + f^2_\varepsilon(v_\varepsilon))\Psi(\nabla u_\varepsilon)\mathrm{d}x.$$

This implies

$$\mathcal{F}^\eta_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u_\varepsilon), v_\varepsilon) \le (1+\frac{C}{M})\mathcal{F}^\eta_\varepsilon(u_\varepsilon, v_\varepsilon) + C\mathcal{L}^n(\{a_{M+1} < |u_\varepsilon|\}),$$

with $\mathcal{T}_{k_{\varepsilon,M}}(u_\varepsilon)$ uniformly bounded in L^∞, $\mathcal{T}_{k_{\varepsilon,M}}$ being defined in (3.11). This argument has been used several times throughout the paper, see for example Theorem 4.9. Passing to a further subsequence in ε, we can take $k_{\varepsilon,M} = k_M$ independent of ε. Since $(\mathcal{T}_{k_M}(u_\varepsilon))_\varepsilon$ is uniformly bounded in L^∞ and M is large, we get $\mathcal{T}_{k_M}(u_\varepsilon) \to \mathcal{T}_{k_M}(u) = u$ in $L^q(\Omega;\mathbb{R}^m)$ and in particular $\mathcal{L}^n(\{a_{M+1} < |u_\varepsilon|\}) \to 0$ as $\varepsilon \to 0$, hence

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon) \le \Big(1+\frac{C}{M}\Big)\mathcal{F}_0(u,1).$$

Diagonalizing with respect to M and recalling the lower estimate, we conclude that every subsequence of $\{\mathcal{F}^\eta_\varepsilon\}_\varepsilon$ has a subsequence that $\Gamma(L^q \times L^1)$-converges to $\mathcal{F}_0$ in $L^\infty(\Omega;\mathbb{R}^m) \times L^1(\Omega)$. Finally Urysohn's lemma gives the convergence of the entire sequence in the same space.

Let us consider now the general case $u \in (GBV \cap L^q(\Omega))^m$. Then $\mathcal{T}_k(u) \in (BV \cap L^\infty(\Omega))^m$, with $\mathcal{T}_k$ again defined by (3.11), and

$$\Gamma(L^q \times L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(\mathcal{T}_k(u), 1) \le \mathcal{F}_0(\mathcal{T}_k(u), 1),$$

by the first part of the proof. As $k \to \infty$ we have $\mathcal{T}_k(u) \to u$ in $L^q(\Omega;\mathbb{R}^m)$ and we conclude by the lower semicontinuity of the Γ-limsup and the continuity of $\mathcal{F}_0$ (see Proposition 4.8).

Step 2. Γ-limit of $\mathcal{G}_\varepsilon$ in $L^1 \times L^1$. We check now that

$$\Gamma(L^1 \times L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u,v) = \mathcal{G}_0(u,v).$$

The lower bound simply follows by Theorem 4.9 using $\eta_\varepsilon \geq 0$ and the lower semicontinuity of $\int_\Omega |w-u|^q \mathrm{d}x$ with respect to the convergence in L^1. In particular, if $\Gamma(L^1 \times L^1)\text{-}\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u,v) < \infty$, then $u \in (GBV(\Omega) \cap L^q)^m$ and $v = 1$ $\mathcal{L}^n$-a.e. on Ω.

As for the upper bound, from Step 1 we know that for all $u \in (GBV(\Omega) \cap L^q)^m$ there exists a recovery sequence for $\mathcal{F}^\eta_\varepsilon$ in $L^q \times L^1$. This is in particular a recovery sequence for $\mathcal{G}_\varepsilon$ in $L^1 \times L^1$, which gives the conclusion.

Step 3. Convergence of minimizers. Let now $(u_\varepsilon, v_\varepsilon) \in W^{1,2} \cap L^q(\Omega;\mathbb{R}^{m+1})$ be a minimizing sequence for $\mathcal{G}_\varepsilon$. Being

$$\sup_{\varepsilon>0} \left(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_{L^q(\Omega)}\right) < \infty,$$

Theorem 6.1 gives the existence of a function $u \in (GBV(\Omega) \cap L^q)^m$ and of a subsequence, not relabelled, such that $u_\varepsilon \to u$ $\mathcal{L}^n$-a.e. on Ω and $v_\varepsilon \to 1$ in $L^1(\Omega;\mathbb{R}^m)$. In addition, by Hölder inequality

$$\begin{aligned}\int_{\{|u_\varepsilon-u|>1\}} |u_\varepsilon - u| \mathrm{d}x &\leq \|u_\varepsilon - u\|_{L^q(\Omega)} \big(\mathcal{L}^n(\{|u_\varepsilon - u| > 1\})\big)^{1-1/q} \\ &\leq c\big(\mathcal{L}^n\big(\{|u_\varepsilon - u| > 1\}\big)^{1-1/q},\end{aligned}$$

and the right-hand side tends to 0 since $u_\varepsilon \to u$ in measure on Ω. Also, $(u_\varepsilon - u)\chi_{\{|u_\varepsilon - u|\leq 1\}} \to 0$ in $L^1(\Omega;\mathbb{R}^m)$ by dominated convergence, hence we conclude that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^m)$.

By Step 2 and a general property of Γ-convergence [Dal93, Corollary 7.20], we conclude that $(u,1)$ is a minimizer of $\mathcal{G}_0$ and that $\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathcal{G}_0(u,1)$. Finally, we check that in fact $u_\varepsilon \to u$ in $L^q(\Omega;\mathbb{R}^m)$. From the previous steps we have

$$\begin{gathered}\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathcal{G}_0(u,1), \\ \int_\Omega |u-w|^q \mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon - w|^q \mathrm{d}x, \\ \mathcal{F}_0(u,1) \leq \liminf_{\varepsilon\to 0} \mathcal{F}^\eta_\varepsilon(u_\varepsilon, v_\varepsilon),\end{gathered}$$

so that

$$\int_\Omega |u_\varepsilon - w|^q \mathrm{d}x \to \int_\Omega |u-w|^q \mathrm{d}x.$$

Together with the pointwise convergence, this implies $u_\varepsilon \to u$ in $L^q(\Omega;\mathbb{R}^m)$ by generalized dominated convergence. □

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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