# THE BOUNDARY HARNACK PRINCIPLE ON OPTIMAL DOMAINS

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ABSTRACT. We give a short and self-contained proof of the Boundary Harnack inequality for a class of domains satisfying some geometric conditions given in terms of a state function that behaves as the distance function to the boundary, is subharmonic inside the domain and satisfies some suitable estimates on the measure of its level sets. We also discuss the applications of this result to some shape optimization and free boundary problems.

## 1. INTRODUCTION

In this paper, we prove a Boundary Harnack inequality for domains satisfying some geometric conditions, which naturally arise in shape optimization and free boundary problems. One consequence of our analysis is that if a domain $\Omega \subset B_1$ admits a function, which is harmonic in $\Omega$, vanishes on $\partial\Omega$, behaves as the distance function to the boundary and the measure of its level sets decays linearly (see Theorem 1.2 for the complete list of hypotheses), then the Boundary Harnack inequality holds on $\Omega$. This general principle is well-known in the free boundary community and was used for instance in [4], [6] and [18, 19]. In all these cases the strategy of the proof is to show that the optimal domain $\Omega$ is NTA and then to obtain the Boundary Harnack inequality by applying the well-known result of Jerison and Kënig [10].

In this paper we give a direct proof of the Boudary Harnack inequality, without passing through the result for NTA domains ([10]). Our proof is completely self-contained and essentially uses only the mean value formula for harmonic functions and the classical Alt-Caffarelli-Friedman monotonicity formula for subharmonic functions. In Section 2 we prove interior Harnack inequalities, which are the key part of the proof and encode the geometric properties of the domains. In Section 3, we prove the Boundary Harnack inequality (Theorem 1.2); we follow step-by-step the strategy from the recent paper of De Silva and Savin [8] and the results from Section 2. In Section 4, for the sake of completeness, we show how to deduce the Boundary Harnack Principle (Definition 1.1) from the Boundary Harnack Inequality (Theorem 1.3).

The Boundary Harnack principle is a key tool in proving the $C^{1,\alpha}$ regularity of free boundaries arising in vectorial free boundary and shape optimization problems. We discuss some applications in Section 5.

Throughout this paper $\Omega$ will be an open subset of the unit ball $B_1 \subset \mathbb{R}^d$.

**Definition 1.1** (Boundary Harnack Principle)**.** We say that the Boundary Harnack Principle holds in $\Omega$, if there is a constant $\alpha > 0$ such that, for every

$$u : B_1 \to \mathbb{R} \qquad \text{and} \qquad v : B_1 \to \mathbb{R}$$

which are:

- continuous on $B_1$,
- positive and harmonic in $\Omega \cap B_1$,
- vanishing identically on $B_1 \setminus \Omega$,

the ratio $\dfrac{u}{v} : \Omega \to \mathbb{R}$ can be extended to a $C^{0,\alpha}$-regular function on $B_{1/2} \cap \overline{\Omega}$.

The aim of this paper is to prove the following theorem.

**Theorem 1.2.** *Let $\Omega \subset B_1$ be an open set with $0 \in \partial\Omega$ and $\phi : B_1 \to \mathbb{R}$ a continuous function such that:*

(a) *$\phi > 0$ on $\Omega$ and $\phi \equiv 0$ on $B_1 \setminus \Omega$;*
(b) *$\phi$ is L-Lipschitz continuous on $B_1$, where $L > 0$ is a given constant;*
(c) *$\phi$ behaves as the distance function to the set $B_1 \setminus \Omega$; precisely, there is a constant $\kappa > 0$ such that*

$$\phi \geq \kappa \, \mathrm{dist}_{B_1 \setminus \Omega} \quad \textit{in} \quad B_{1/2} \ ;$$

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(d) *we have the inequality*

$$\Delta\phi \geq 0 \quad \text{in sense of distributions in } B_1;$$

(e) *there is a constant $\mu > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$, we have*

$$|B_r(x_0) \setminus \Omega| \geq \mu |B_r(x_0)| \quad \text{for every} \quad r \in (0, 1 - |x_0|);$$

(f) *there is a constant $\Lambda > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, we have*

$$\big|\{0 < \phi < rt\} \cap B_r(x_0)\big| \leq \Lambda t |B_r| \quad \text{for every} \quad t > 0.$$

(g) *there is a constant $\eta > 0$ such that for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, we have*

$$\sup_{B_r(x_0)} \phi \geq \eta r.$$

*Then the Boundary Harnack Principle holds in $\Omega$ in the sense of Definition 1.1.*

In Section 4, we will deduce Theorem 1.2 from the following theorem.

**Theorem 1.3** (Boundary Harnack Inequality). *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (c), (d), (e), (f) *and* (g) *of Theorem 1.2. Then, there are constants $M > 0$, $\delta \in (0, \eta]$ and $0 < \rho < R \leq 1$, depending on the dimension $d$ and the constants from* (b), (c), (e), (f) *and* (g), *such that the following Boundary Harnack inequality holds. Suppose that*

$$u, v : B_1 \to \mathbb{R}$$

*are nonnegative continuous functions satisfying*

$$\begin{cases} \Delta u = \Delta v = 0 \quad \text{in} \quad \Omega \cap B_1; \\ u = v = 0 \quad \text{on} \quad B_1 \setminus \Omega; \\ u(P) = v(P) \quad \text{for some point} \quad P \in B_R \cap \{\phi > \delta R\}. \end{cases} \tag{1}$$

*Then*

$$\frac{1}{M} v \leq u \leq M v \quad \text{in} \quad B_\rho \ .$$

In general, The Boundary Harnack Principle (Definition 1.1) on a domain $\Omega$ is a consequence of the validity of the Boundary Harnack Inequality at any scale and for any couple of nonnegative functions $u, v$ satisfying (1) on a rescaling of $\Omega$. This implication is well-known (see for instance [10]) and in Section 4 we give a short proof of this fact in our context. In order to do so, we need that the Boundary Harnack Inequality holds at any scale. This follows from the fact that the assumptions of Theorem 1.2 and Theorem 1.3 are scale-invariant:

**Remark 1.4** (Scale invariance). Let $\Omega$ and $\phi$ be as in Theorem 1.2. Then, for every $x_0 \in \partial\Omega \cap B_1$ and every $r \in (0, 1 - |x_0|)$, the rescalings $\Omega_{r,x_0} \subset B_1$ and $\phi_{r,x_0} : B_1 \to \mathbb{R}$ defined as

$$\Omega_{r,x_0} := \frac{1}{r}(-x_0 + \Omega) \qquad \text{and} \qquad \phi_{r,x_0}(x) := \frac{\phi(x_0 + rx)}{r} \ ,$$

satisfy the properties (a), (b), (c), (d), (e) and (f) of Theorem 1.2 with the same constants.

**Remark 1.5** (On the assumption (g)). We also notice that the assumption (g) is only needed to assure that, for $\delta$ small enough, the set $B_R \cap \{\phi > \delta R\}$ from (1) is non-empty. Indeed, if $\Omega$ is an open set in $B_1$ and $\phi : B_1 \to \mathbb{R}$ is a function satisfying (a) and (g) of Theorem 1.2, then

$$B_r(x_0) \cap \{\phi > r\delta\} \neq \emptyset \quad \text{for every} \quad x_0 \in \partial\Omega \cap B_1, \quad r \in (0, 1 - |x_0|) \quad \text{and} \quad \delta \in (0, \eta).$$

**Remark 1.6.** Several versions of the Boundary Harnack Inequality (B.H.I.) appeared recently in the literature. See for instance [17], where the authors established a B.H.I. on the class of nodal domains of solutions to uniformly elliptic equations in divergence form; [1] where B.H.I. was proved for solutions with right-hand side on sufficiently flat Lipschitz domains; we also refer to [9] for a higher order Boundary Harnack Principle.

## 2. Harnack chains and interior Harnack inequalities

In thise Section we prove the existence of Harnack chains, and consequently the validity of Harnack-type inequalities, by differentiating between those points that are close to the boundary and those that are away.

2.1. **Harnack chains and Harnack inequality close to the boundary.** In this subsection we show how to construct short Harnack chains starting from a point close to the boundary of a domain $\Omega$ satisfying the hypotheses of Theorem 1.2. This is done in the following simple lemma, which is essential for the proof of Theorem 1.3 (see Section 4, Lemma 3.1).

**Lemma 2.1** (Short Harnack chains close to the boundary)**.** *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (c) *and* (d) *of Theorem 1.2. Let*

$$x_0 \in \{\phi > 0\} \cap B_1 \qquad \text{be such that} \qquad 3\,\mathrm{dist}(x_0, \partial\Omega) < 1 - |x_0|,$$

*let $r$ be the distance from $x_0$ to $\partial\Omega$ and $z_0$ be a projection of $x_0$ on $\partial\Omega$ (thus, $z_0 \in \partial\Omega \cap B_1$ and $|x_0 - z_0| = r$).*
*Then, there is $y_0 \in \partial B_r(x_0)$ such that the following holds:*

(i) *$\phi(y_0) \geq (1+\sigma)\phi(x_0)$, where the constant $\sigma > 0$, depends only on the dimension $d$, and the constants $L$ from* (b) *and $\kappa$ from* (c)*;*
(ii) *there is a constant $\mathcal{H} > 1$, depending only on the dimension $d$, the constants $L$ from* (b) *and $\kappa$ from* (c)*, such that for every positive harmonic function $w : \Omega \to \mathbb{R}$,*

$$\frac{1}{\mathcal{H}} w(y_0) \leq w(x_0) \leq \mathcal{H} w(y_0).$$

*Proof.* We fix a parameter $\varepsilon > 0$ that we will choose later on. First, we notice that by the condition (d), we have that

$$\fint_{\partial B_r(x_0)} \phi - \phi(x_0) = \frac{1}{d\omega_d} \int_0^r s^{1-d} \Delta\phi(B_s(x_0))\, ds \geq 0.$$

Let now the radius $\rho > 0$ be such that

$$\mathcal{H}^{d-1}\big(\partial B_r(x_0) \cap B_\rho(z_0)\big) = \varepsilon^{d-1}\mathcal{H}^{d-1}\big(\partial B_r(x_0)\big).$$

Now, for $\varepsilon$ small enough $\rho$ is comparable to $\varepsilon r$. In particular, by choosing $\varepsilon$ small enough (depending only on the dimension), we have $\rho \leq 2\varepsilon r$, so the Lipschitz continuity of $\phi$ gives that

$$\phi(x) \leq 2L\varepsilon r \quad \text{for} \quad x \in \partial B_r(x_0) \cap B_\rho(z_0),$$

where $L$ is the Lipschitz constant from (b). Thus, setting

$$M := \max\big\{\phi(x) \ : \ x \in \partial B_r(x_0)\big\},$$

we get that

$$\begin{aligned}
\phi(x_0) \leq \fint_{\partial B_r(x_0)} \phi &\leq \frac{1}{r^{d-1}}\Big((2\varepsilon r)^{d-1} 2L\varepsilon r + M\big(r^{d-1} - (2\varepsilon r)^{d-1}\big)\Big) \\
&\leq \frac{1}{r^{d-1}}\Big((2\varepsilon r)^{d-1} \frac{2L\varepsilon}{\kappa}\phi(x_0) + M\big(r^{d-1} - (2\varepsilon r)^{d-1}\big)\Big) \\
&\leq (2\varepsilon)^{d-1}\frac{2L\varepsilon}{\kappa}\phi(x_0) + M\big(1 - (2\varepsilon)^{d-1}\big),
\end{aligned}$$

which implies that

$$\Big(1 - \varepsilon^{d-1}\frac{2L\varepsilon}{\kappa}\Big)\,\phi(x_0) \leq \big(1 - \varepsilon^{d-1}\big)M.$$

We now choose $\varepsilon$ such that

$$\frac{2L\varepsilon}{\kappa} \leq \frac{1}{2^{d-1}}.$$

Thus, there is a point $y_0 \in \partial B_r(x_0)$ such that

$$(1+\sigma)\phi(x_0) \leq \phi(y_0),$$

where

$$1 + \sigma := \frac{1}{1 - (2\varepsilon)^{d-1}}\Big(1 - \varepsilon^{d-1}\Big).$$

In order to prove (ii), we notice that by the Lipschitz continuity of $\phi$, we have

$$\mathrm{dist}_{B_1\setminus\Omega}(y_0) \geq \frac{1}{L}\phi(y_0) \geq \frac{1}{L}\phi(x_0) \geq \frac{\kappa}{L} r.$$

Thus,

$$B_r(x_0) \cap B_{r\kappa/L}(y_0) \subset \Omega,$$

and the claim (ii) follows by the classical Harnack inequality. $\square$

As a consequence, by iterating this result, we obtain the following Harnack-type inequality close the boundary.

**Lemma 2.2** (Interior Harnack inequality close to the boundary)**.** *As in Lemma 2.1, we suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a)*,* (b)*,* (c) *and* (d) *of Theorem 1.2. Then, there are constants $A > 0$ and $\delta_0 > 0$, depending only on $d$ and the constants $L$ from* (b) *and $\kappa$ from* (c)*, such that for every positive harmonic function $w : \Omega \to \mathbb{R}$, we have*

$$\sup_{B_{1/2}\cap\{\phi>\frac{\delta}{2}\}} w \le A \sup_{B_1\cap\{\phi>\delta\}} w \qquad \textit{and} \qquad \inf_{B_{1/2}\cap\{\phi>\frac{\delta}{2}\}} w \ge \frac{1}{A} \inf_{B_1\cap\{\phi>\delta\}} w.$$

*for every $\delta \in (0, \delta_0]$.*

*Proof.* Let $x_0 \in B_{1/2} \cap \{\phi > \frac{\delta}{2}\}$. If $\phi(x_0) > \delta$, then for any $A \ge 1$, we clearly have the inequalities

$$\frac{1}{A} \inf_{B_1\cap\{\phi>\delta\}} w \le w(x_0) \quad \text{and} \quad w(x_0) \le A \sup_{B_1\cap\{\phi>\delta\}} w,$$

Thus, we consider the case $x_0 \in B_{1/2} \cap \{\frac{\delta}{2} < \phi \le \delta\}$. Let $x_1$ be the point $y_0$ from Lemma 2.1. Then,

$$\phi(x_1) \ge (1+\sigma)\phi(x_0) \ge (1+\sigma)\frac{\delta}{2}.$$

Now, by construction $x_1 \in B_r(x_0)$ and $r = \text{dist}_{B_1\setminus\Omega}(x_0)$, so we get that $|x_1 - x_0| = \text{dist}_{B_1\setminus\Omega}(x_0)$.
Using this and (c), we obtain that

$$|x_0 - x_1| = \text{dist}_{B_1\setminus\Omega}(x_0) \le \frac{1}{\kappa}\phi(x_0) \le \frac{\delta}{\kappa}.$$

If $x_1 \in \{\phi \le \delta\}$, we repeat the same procedure to obtain a point $x_2$. Iterating this argument, we obtain a sequence of points $x_n$ such that

$$x_n \in B_{r_n} \cap \Big\{\frac{\delta}{2}(1+\sigma)^n < \phi \le \delta\Big\} \qquad \text{with} \qquad r_n := \frac{1}{2} + n\frac{\delta}{\kappa}$$

and

$$\frac{1}{\mathcal{H}^n} w(x_n) \le w(x_0) \le \mathcal{H}^n w(x_n)$$

where $\mathcal{H} > 1$ is the Harnack constant from part (ii) of Lemma 2.1. Now, define $N$ to be the largest index for which $x_N \in B_1 \cap \{\phi \le \delta\}$ and to which we can apply Lemma 2.1 to obtain $x_{N+1}$. Thus, necessarily

$$\frac{1}{2}(1+\sigma)^N \le 1,$$

which means that

$$N \le \frac{1}{\log_2(1+\sigma)}.$$

Thus, we have also that

$$r_{N+1} \le \frac{1}{2} + (N+1)\frac{\delta}{\kappa} \le \frac{1}{2} + \Big(\frac{1}{\log_2(1+\sigma)} + 1\Big)\frac{\delta_0}{\kappa},$$

so by choosing $\delta_0$ small enough, we can suppose that $r_{N+1} \le 3/4$ and that we can still apply Lemma 2.1 to $x_{N+1}$ to obtain $x_{N+2}$. Thus, the procedure stops because

$$x_{N+1} \in \{\phi > \delta\}.$$

Hence

$$\frac{1}{\mathcal{H}^{N+1}} \min_{B_1\cap\{\phi>\delta\}} w \le \frac{1}{\mathcal{H}^{N+1}} w(x_{N+1}) \le w(x_0)$$

and

$$w(x_0) \le \mathcal{H}^{N+1} w(x_{N+1}) \le \mathcal{H}^{N+1} \max_{B_1\cap\{\phi>\delta\}} w.$$

The claim follows by taking $A := \mathcal{H}^{N+1}$ and $x_0$ as the point at which the maximum (resp. the minimum) $w$ is achieved in $B_1 \cap \{\phi \ge \delta/2\}$. □

2.2. **Harnack chains and Harnack inequality away from the boundary.** The main result of this subsection is the following interior Harnack inequality away from the boundary, which we will use in Step 2 (Section 3.2) of the proof of Theorem 1.3.

**Proposition 2.3** (Interior Harnack inequality away from the boundary). *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (d) *and* (e) *of Theorem 1.2. Then, for every $\delta > 0$ there is $R_0$ for which the following holds.*

*For every $R \in (0, R_0]$, there is a constant $c_{\mathcal{H}} = c_{\mathcal{H}}(\delta, R) > 0$ such that for every positive harmonic function*

$$w : \Omega \cap B_1 \to \mathbb{R}\,, \qquad w \geq 0 \quad in \quad \Omega \cap B_1\,, \qquad \Delta w = 0 \quad in \quad \Omega \cap B_1$$

*we have*

$$\inf_{\{\phi > \delta R\} \cap B_R} w \geq c_{\mathcal{H}} \sup_{\{\phi > \delta R\} \cap B_R} w.$$

In order to prove Proposition 2.3 it is sufficient to show that there are constants $N > 0$ and $r > 0$ (depending also on $\delta$ and $R$) such that, for every pair of points $x_0, y_0 \in \{\phi > \delta R\} \cap B_R$, there exists a curve $\gamma : [0, 1] \to B_1$ such that

$$\gamma(0) = x_0\,; \quad \gamma(1) = y_0.$$

and a family of balls $\big\{B_r(x_j) \,:\, j = 1, \dots, N\big\}$ such that:

- $x_j \in \gamma([0, 1])$ for every $j = 1, \dots, N$;
- $B_{2r}(x_j) \subset \Omega$ for every $j = 1, \dots, N$;
- the family $\big\{B_r(x_j) \,:\, j = 1, \dots, N\big\}$ is an open covering of $\gamma([0, 1])$.

The existence of such a family is an immediate consequence of the following lemma (and a covering theorem), in which we prove the existence of an Harnack chain by combining (d) with the monotonicity formula of Alt-Caffarelli-Friedman (see [3]).

**Lemma 2.4** (Harnack chains away from the boundary). *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (d) *and* (e) *of Theorem 1.2. For every $\delta \in (0, 2L)$ there is $\tau \in (0, 1)$ such that the following holds. For every $R \in (0, 1)$ and every couple of points $x_1, x_2 \in B_{\tau R} \cap \{\phi > \delta R\}$, there is a curve connecting $x_1$ to $x_2$ in $B_R \cap \{\phi > \frac{\delta}{2} R\}$.*

*Proof.* Fix $\tau \in (0, 1)$ and suppose that $x_1$ and $x_2$ are two points in $B_{\tau R} \cap \{\phi > \delta R\}$ that lie in two different connected components, $\Omega_1$ and $\Omega_2$, of $B_R \cap \{\phi > \frac{\delta}{2} R\}$ . Let $\phi_1$ and $\phi_2$ be the restrictions of the function $(\phi - \frac{\delta}{2} R)_+$ respectively on $\Omega_1$ and $\Omega_2$. Then, $\phi_1$ and $\phi_2$ are both $L$-Lipschitz, $L$ being the constant from (b), and $\phi_j(x_j) \geq \frac{\delta}{2} R$ for $j = 1, 2$. Moreover, for every radius $r \in [\tau R, R]$, there is a point $x_r \in \partial B_r$ such that $\phi_1(x_r) = \phi_2(x_r) = 0$. Define now the functions $\psi_j = (\phi_j - \frac{3\delta}{4} R)_+$ for $j = 1, 2$. Again $\psi_j$ are $L$-Lipschitz and harmonic where they are positive; we have that, $\psi_j(x_j) \geq \frac{\delta}{4} R$ and moreover

$$\psi_j \equiv 0 \quad \text{on} \quad B_{\frac{\delta R}{4L}}(x_r) \quad \text{for every} \quad r \in [\tau R, R].$$

Now, if $\delta$ is small enough, this implies the density estimate

$$\alpha(r) := \frac{\mathcal{H}^{d-1}\Big(\{\psi_1 = \psi_2 = 0\} \cap \partial B_r\Big)}{\mathcal{H}^{d-1}(\partial B_r)} \geq F\left(\frac{\delta^{d-1}}{(4L)^{d-1}}\right) \quad \text{for every} \quad r \in [\tau R, R],$$

$F : [0, +\infty) \to \mathbb{R}$ being a continuously differentiable increasing function depending only on the dimension and such that $F(0) = 0$ and $F'(0) > 0$. Now, for every $r \in [\tau R, R]$, let

$$\Phi(r) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla \psi_1|^2}{|x|^{d-2}}\, dx \int_{B_r} \frac{|\nabla \psi_2|^2}{|x|^{d-2}}\, dx.$$

Now, by [3] (see also [4, Lemma 4.3]), we have that

$$\frac{d}{dr}\Big[\ln(\Phi(r))\Big] \geq \frac{1}{r} G(\alpha(r)),$$

where $G : [0, +\infty) \to \mathbb{R}$ is a positive increasing convex function with $G(0) = 0$ and $G'(0) > 0$. Combining the two estimates, we have that for $\delta$ small enough ($\delta \leq \delta_0$ for some dimensional $\delta_0 > 0$),

$$\frac{d}{dr}\Big[\ln(\Phi(r))\Big] \geq C_d \frac{1}{r} \frac{\delta^{d-1}}{(4L)^{d-1}}.$$

since $\phi_1(0) = 0 = \phi_2(0)$ we get that

$$\fint_{B_r} |\nabla\psi_1|^2\,dx \fint_{B_r} |\nabla\psi_2|^2\,dx \le \Phi(r) \le \left(\frac{r}{R}\right)^\alpha \Phi(R), \quad \text{for } r\in[\tau R, R],$$

with $\alpha = C_d(\delta/(4L))^{d-1}$. Moreover, using again the density estimate (e), by the Poincaré inequality we deduce that

$$\frac{1}{r^4}\fint_{B_r} \psi_1^2\,dx \fint_{B_r} \psi_2^2\,dx \le \left(\frac{r}{R}\right)^\alpha \Phi(R), \quad \text{for } r\in[\tau R, R].$$

We will next estimate the left-hand side from below. By the Lipschitz continuity (b) of $\psi_1$ and $\psi_2$, we have that

$$\psi_i \ge \frac{\delta}{4}R - |x-x_i|L \quad \text{in} \quad B_\rho(x_i).$$

Next, we choose $\rho = \frac{\delta R}{4L}$ and we notice that since $\frac{\delta}{L}\le 2$, we have that $\rho \le R/2$. Thus,

$$\int_{B_r}\psi_i^2\,dx \ge \int_{B_r\cap B_\rho(x_i)}\psi_i^2\,dx \ge \int_{B_r\cap B_\rho(x_i)}\left(\frac{\delta}{4}R - |x-x_i|L\right)^2 dx \ge c_d\int_{B_\rho(x_i)}\left(\frac{\delta}{4}R - |x-x_i|L\right)^2 dx\,,$$

where $c_d$ is a dimensional constant. Now, a straightforward computation gives

$$\begin{aligned}\int_{B_\rho(x_i)}\left(\frac{\delta}{4}R - |x-x_i|L\right)^2 dx &\ge \frac{1}{|B_\rho|}\left(\int_{B_\rho}\left(\frac{\delta}{4}R-|x|L\right)dx\right)^2\\ &= \frac{(d\omega_d)^2}{\omega_d\rho^d}\left(\int_0^\rho s^{d-1}\left(\frac{\delta}{4}R - sL\right)ds\right)^2\\ &= \frac{(d\omega_d)^2}{\omega_d\rho^d}\left(L\int_0^\rho s^{d-1}(\rho-s)\,ds\right)^2 = \frac{\omega_d d^2}{(d+1)^2}L^2\rho^{d+2}.\end{aligned}$$

Thus,

$$\frac{1}{r^2}\fint_{B_r}\psi_i^2\,dx \ge C_d\frac{\delta^{d+2}}{L^d}\left(\frac{R}{r}\right)^{d+2}$$

and so, by the Lipschitz continuity of $\psi_1, \psi_2$, we obtain the inequality

$$C_d\frac{\delta^{2d+4}}{L^{2d}} \le \left(\frac{r}{R}\right)^{d+2+\alpha}\Phi(R) \le \left(\frac{r}{R}\right)^{d+2+\alpha}\frac{\omega_d^2L^4}{4} \quad \text{for } r\in[\tau R, R].$$

In particular, by taking $r=\tau R$ we deduce

$$\tau^{d+2+\alpha} \ge C_d\left(\frac{\delta}{L}\right)^{2d+4},$$

which is a contradiction if $\tau$ is small enough. □

## 3. Boundary Harnack Inequality: proof of Theorem 1.3

In this section we prove Theorem 1.3. We follow step-by-step the recent proof of De Silva and Savin of the Boundary Harnack inequality in Lipschitz and NTA domains [8]. The proof is divided in three main steps. In Step 1 (Section 3.1), the main result is Lemma 3.1 from which Theorem 1.3 follows by an iteration procedure; in our case Lemma 3.1 is an immediate consequence of Lemma 2.2 from Section 2. In Step 2 (Section 3.2), we prove Proposition 3.3, which allows to start the iteration procedure from Step 1. The proof of Proposition 3.3 is a consequence of Lemma 3.4 from Section 2 and on the Harnack-type estimate Lemma 3.5; for general operators Lemma 3.5 is contained the proof of the Krylov-Safonov's Theorem [16] (see also [5, Theorem 4.8] and [8, Theorem 1.3]), while in our case it is a consequence of the mean-value formula. Finally, in Step 3 (Section 3.3), we simply combine the results from Step 1 and Step 2.

3.1. **Step 1.** The main result of this step is Lemma 3.1; the proof is based on Lemma 2.1 and the oscillation lemma from the De Giorgi's theorem.

**Lemma 3.1.** *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (c), (d), *and* (e) *of Theorem 1.2. Then, there are constants $\delta_1 > 0$ and $a > 0$, depending on the dimension and the constants from* (b), (c) *and* (e), *for which that the following holds. Suppose that $w : B_1 \to \mathbb{R}$ is a continuous function satisfying*

$$\begin{cases} \Delta w = 0 & \text{in} \quad B_1 \cap \{\phi > 0\} \\ w = 0 & \text{on} \quad B_1 \cap \{\phi = 0\} \\ w \geq M & \text{on} \quad B_1 \cap \{\phi > \delta\} \\ w \geq -1 & \text{on} \quad B_1 \cap \{0 < \phi \leq \delta\}, \end{cases} \tag{2}$$

*for some $\delta \in (0, \delta_1]$ and some $M > 0$. Then, in $B_{1/2}$,*

$$\begin{cases} \Delta w = 0 & \text{in} \quad B_{1/2} \cap \{\phi > 0\} \\ w = 0 & \text{on} \quad B_{1/2} \cap \{\phi = 0\} \\ w \geq aM & \text{on} \quad B_{1/2} \cap \{\phi > \frac{\delta}{2}\} \\ w \geq -a & \text{on} \quad B_{1/2} \cap \{0 < \phi \leq \frac{\delta}{2}\}. \end{cases} \tag{3}$$

*Therefore, we get that*

$$\sup_{B_{1/2}\cap\Omega} w^- \leq a \quad \text{and} \quad \inf_{B_{1/2}\cap\{\phi>\delta/2\}} w^+ \geq aM$$

*where $w^+, w^-$ are respectively the positive and negative part of $w$.*

*Proof.* We consider the function $w + 1$, which is positive and harmonic on $B_1 \cap \Omega$.
Taking $\delta_1$ to be smaller than the constant $\delta_0$ from Lemma 2.2, we have

$$\min_{B_{1/2}\cap\{\phi>\delta/2\}} (w+1) \geq \frac{1}{A} \min_{B_1\cap\{\phi>\delta\}} (w+1) \geq \frac{1}{A}(M+1),$$

and so, if we choose

$$a \leq \frac{1}{2A} \qquad \text{and} \qquad M = 2A,$$

we get

$$\min_{B_{1/2}\cap\{\phi>\delta/2\}} w \geq \frac{1}{A}(M+1) - 1 \geq 1 \geq aM.$$

In order to prove the bound from below on $B_{1/2} \cap \{0 < \phi \leq \frac{\delta}{2}\}$, we use the density bound from (e) and the classical De Giorgi's oscillation lemma (see [8, Theorem 1.2] and Remark 3.2). In fact, if we fix a point

$$x_0 \in B_{1/2} \cap \partial\{\phi > 0\}$$

and if we apply Remark 3.2 to the negative part of $w$, then we get that

$$\sup_{B_{2^{-n}}(x_0)} w_- \leq (1-c)^{n-1} \sup_{B_{1/2}(x_0)} w_- \leq (1-c)^{n-1} \sup_{B_1} w_- \leq (1-c)^{n-1},$$

where $c \in (0, 1)$ is the dimensional constant from Remark 3.2 below. Now, choosing $n$ to be such that

$$(1-c)^{n-1} \leq \frac{1}{2A},$$

we get

$$\sup_{B_{2^{-n}}(x_0)} w_- \leq \frac{1}{2A}. \tag{4}$$

We now choose the second bound on $\delta_1$ to be

$$\delta_1 \leq 2\kappa\, 2^{-n}.$$

Thus, by the bound from below (c), we have that the set $B_{1/2} \cap \partial\{0 < \phi < \delta_1/2\}$ is contained in the union of all balls $B_{2^{-n}}(x_0)$ with centers $x_0 \in \partial\Omega \cap B_{1/2}$. Thus, choosing $a$ to be precisely $\frac{1}{2A}$ and using (4), we get that

$$w_- \leq a \qquad \text{on} \qquad B_{1/2} \cap \partial\{0 < \phi < \delta/2\},$$

for any $\delta \leq \delta_1$, which concludes the proof. ∎

**Remark 3.2** (De Giorgi's oscillation lemma for the Laplacian)**.** Suppose that $w : B_1 \to \mathbb{R}$ is a subharmonic function bounded between 0 and 1, and such that $|\{w = 0\} \cap B_{1/4}| \geq \mu |B_{1/4}|$ for some constant $\mu > 0$. Then,

$$w \leq 1 - c \quad \text{on} \quad B_{1/2}, \tag{5}$$

where $c > 0$ depends only on $\mu$ and the dimension $d$. Indeed, by the mean value formula, for every $x_0 \in B_{1/4}$

$$w(x_0) \leq \frac{1}{|B_{1/2}|}\int_{B_{1/2}(x_0)} w(x)\,dx \leq \frac{1}{|B_{1/2}|}\Big(|B_{1/2}| - \mu|B_{1/4}|\Big) = 1 - \frac{\mu}{2^d}.$$

Now let $y_0 \in B_{1/2}$. Since $B_{1/2}(y_0) \cap B_{1/4}$ contains at least a ball of radius $1/8$, by the previous estimate in $B_{1/4}$ we get that

$$w(y_0) \leq \frac{1}{|B_{1/2}|}\int_{B_{1/2}(y_0)} w(x)\,dx \leq \frac{1}{|B_{1/2}|}\Big(|B_{1/2}| - \frac{\mu}{2^d}|B_{1/8}|\Big) = 1 - \frac{\mu}{8^d},$$

which is precisely (5) with $c = 8^{-d}\mu$.

3.2. **Step 2.** In this section we prove a bound which allows to start the iterative procedure based on Lemma 3.1 from Step 1. This is the only point of the proof in which we use the hypothesis (f) of Theorem 1.3.

**Proposition 3.3.** *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and that $\phi : B_1 \to \mathbb{R}$ satisfies the conditions* (a), (b), (c), (d) *and* (f) *of Theorem 1.2. Then, there are constants $C > 0$ and $\delta_2 > 0$ depending on $d$ and the constants from* (b), (c), (f), *for which the following holds. If $w : B_1 \to \mathbb{R}$ is a nonnegative continuous function satisfying*

$$\begin{cases} \Delta w = 0 & \text{in} \quad B_1 \cap \{\phi > 0\} \\ w = 0 & \text{on} \quad B_1 \cap \{\phi = 0\} \\ w \leq 1 & \text{on} \quad B_1 \cap \{\phi \geq \delta_2\}. \end{cases} \tag{6}$$

*Then,*

$$w \leq C \quad \text{in} \quad B_{1/4}.$$

We first prove the following lemma which is a consequence of the Harnack inequality close to the boundary (Lemma 2.2). We notice that the constants $\delta_2$ from Lemma 3.4 and Proposition 3.3 are the same.

**Lemma 3.4** (A pointwise estimate up to the boundary)**.** *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (c) *and* (d) *of Theorem 1.2. There are constants $\delta_2 > 0$, $C$ and $p$ depending on $d$, $L$ and $\kappa$ from* (b) *and* (c), *for which the following holds. For every $\delta \in (0, \delta_2]$ and every positive harmonic function $w : \Omega \to \mathbb{R}$, satisfying*

$$w \leq 1 \quad \text{on} \quad B_1 \cap \{\phi > \delta\},$$

*we have*

$$w \leq C\phi^{-p} \quad \text{on} \quad B_{1/2} \cap \Omega.$$

*Proof.* It is sufficient to prove the claim for $\delta = \delta_2$. Let $x_0 \in B_{1/2} \cap \Omega$ and $\ell \geq 1$ be a fixed constant that we will choose later. If $\phi(x_0) \geq \ell\delta_2$, then it is enough to choose $C \geq L^p$. In fact, by the Lipschitz bound (b) and the fact that $x_0 \in \{\phi > \delta_2\}$, we have

$$w(x_0) \leq 1 \leq L^p\Big(\max_{B_{1/2}} \phi\Big)^{-p} \leq L^p \phi(x_0)^{-p}.$$

Therefore, suppose that $\phi(x_0) \leq \ell\delta_2$. Let $z_0$ be the projection of $x_0$ on $\partial\Omega \cap B_1$. By (c), we have that

$$r := |x_0 - z_0| \leq \frac{1}{\kappa}\phi(x_0) \leq \frac{\ell\delta_2}{\kappa}.$$

Thus, if $\ell\delta_2$ is small enough, such that $\ell\delta_2 \leq \kappa/8$, we have that $r \leq 1/8$ and, in particular, $B_{2r}(z_0) \subset B_1$. Moreover, by the bound from below (c) we have that

$$\frac{\kappa}{2} 2r = \kappa|x_0 - z_0| \leq \phi(x_0)$$

and so, since $\ell\delta_2 \leq \frac{\kappa}{2}$, we get that

$$x_0 \in B_{2r}(z_0) \cap \big\{\phi > \kappa r\big\} \subset B_{2r}(z_0) \cap \big\{\phi > 2\ell\delta_2 r\big\}.$$

Let now $\ell\delta_2 \leq \delta_0$, $\delta_0$ being the threshold from Lemma 2.2, and let $n \geq 1$ be such that

$$2^n r \leq \frac{1}{4} < 2^{n+1} r. \tag{7}$$

Then, $B_{2^n r}(z_0) \subset B_1$ and we can iterate the estimate from Lemma 2.2 obtaining

$$w(x_0) \leq \max_{B_{2r}(z_0)\cap\{\phi>2r\ell\delta_2\}} w \leq A^{n-1} \max_{B_{2^n r}(z_0)\cap\{\phi>2^n\ell\delta_2 r\}} w \leq A^{n-1} \max_{B_1\cap\{\phi>\frac{\ell\delta_2}{8}\}} w.$$

Thus, let us choose $\ell = 8$.
Next, since

$$2 \leq \kappa\phi(x_0)^{-1},$$

by choosing $p > 0$ such that $A^{n-1} = 2^p > 1$, we get that

$$w(x_0) \leq A^{n-1} \max_{B_1\cap\{\phi>\delta_2\}} w \leq 2^p \max_{B_1\cap\{\phi>\delta_2\}} w \leq \kappa^p\phi(x_0)^{-p},$$

which gives the claim. We notice that it is enough to choose $\delta_2$ and $C$ as

$$\delta_2 \leq \min\left\{\frac{\kappa}{64}, \frac{\delta_0}{8}\right\} \qquad \text{and} \qquad C = \max\{\kappa^p, L^p\}. \qquad \square$$

In the proof of Proposition 3.3 we will need the following Krylov-Safonov-type estimate, which was also used in [8] (see [8, Theorem 1.3]). In our specific case, there is a simple proof based only on the mean-value fromula for harmonic functions, which still uses the idea from the conclusion of the Krylov-Safonov's Theorem.

**Lemma 3.5** (A Krylov-Safonov-type estimate)**.** *Suppose that $\Omega$ is an open set in $B_1$ and that the continuous*[1] *function $w : B_1 \to \mathbb{R}$ is such that:*

- *$w$ is nonnegative on $B_1$ and vanishes identically on $B_1 \setminus \Omega$;*
- *$w$ is harmonic in $\Omega$ and subharmonic on $B_1$;*
- *$\Omega$ satisfies the exterior denisity bound* (e) *in $B_1$;*
- *there is $\varepsilon > 0$ such that $\displaystyle\int_{B_1} w^\varepsilon \, dx \leq 1$.*

*Then, there is a constant $M > 0$ depending on the dimension, the density bound $\mu$ from* (e) *and on $\varepsilon$, such that*

$$w \leq M \quad \text{in} \quad B_{1/2}.$$

*Proof.* Let $x_0 \in B_{1/2} \cap \Omega$, $R := \operatorname{dist}(x_0, \partial\Omega)$ and $M := w(x_0) > 0$. We also fix $\delta := \varepsilon/2d$. We consider two cases.
**Case 1.** Assume that $2R \geq M^{-\delta}$. Notice that, in $B_R(x_0)$ the function $w$ is harmonic (and positive). Thus, by the Harnack inequality in $B_R(x_0)$, there is a dimensional constant $c_{\mathcal{H}} > 0$ such that

$$w \geq c_{\mathcal{H}} M \quad \text{in} \quad B_{R/2}(x_0).$$

But then,

$$1 \geq \int_{B_1} w^\varepsilon \, dx \geq \int_{B_{R/2}(x_0)} w^\varepsilon \, dx \geq |B_{R/2}|(c_{\mathcal{H}} M)^\varepsilon \geq \frac{\omega_d \, c_{\mathcal{H}}^\varepsilon}{4^d} M^{-d\delta+\varepsilon} = \frac{\omega_d \, c_{\mathcal{H}}^\varepsilon}{4^d} M^{\varepsilon/2},$$

which means that in this case there is a constant $C_{d,\varepsilon}$ depending only on $d$ and $\varepsilon$ such that $M \leq C_{d,\varepsilon}$.
**Case 2.** Suppose now that $2R \leq M^{-\delta}$ and $M > C_{d,\varepsilon}$, $C_{d,\varepsilon}$ being the constant from the previous case.
Let $z_0$ be the projection of $x_0$ on $\partial\Omega \cap B_1$. Then, the ball $B_{M^{-\delta}}(x_0)$ contains $B_{M^{-\delta}/2}(z_0)$ and is contained in $B_1$. In particular, since $\Delta w \geq 0$ in $B_1$,

$$M = w(x_0) \leq \frac{1}{|B_{M^{-\delta}}|} \int_{B_{M^{-\delta}}(x_0)} w(x) \, dx \leq \frac{|B_{M^{-\delta}}(x_0) \cap \Omega|}{|B_{M^{-\delta}}|} \|w\|_{L^\infty(B_{M^{-\delta}}(x_0))},$$

which by the density estimate in the ball $B_{M^{-\delta}/2}(z_0)$ gives

$$M \leq (1 - 2^{-d}\mu)\|w\|_{L^\infty(B_{M^{-\delta}}(x_0))} \leq \frac{1}{1 + 2^{-d}\mu} \|w\|_{L^\infty(B_{M^{-\delta}}(x_0))},$$

which means that there exists a point $x_1 \in B_{M^{-\delta}}(x_0)$ such that

$$w(x_1) \geq (1 + 2^{-d}\mu)M.$$

Iterating the same procedure, we obtain a sequence of points $x_n \in \Omega \cap B_1$ such that

$$w(x_{n+1}) \geq (1 + 2^{-d}\mu)w(x_n) \geq M(1 + 2^{-d}\mu)^n \qquad \text{and} \qquad |x_{n+1} - x_n| \leq \frac{1}{M^\delta(1 + 2^{-d}\mu)^{n\delta}}.$$

[1] We notice that this assumption is not restricitive as below we will also assume $w$ is harmonic in $\Omega$ and that $\Omega$ saisfies an exterior density bound.

Now, if we choose $M$ large enough, then

$$\sum_{n=0}^{+\infty} \frac{1}{M^\delta (1+2^{-d}\mu)^{n\delta}} \leq \frac{1}{4},$$

so $x_n$ is defined for every $n \geq 1$ (it never leaves $\Omega \cap B_{3/4}$). But this is impossible since $w(x_n) \to \infty$. □

**Proof of Proposition 3.3.** We first show that there are $\alpha > 0$ and $C > 0$ such that

$$\int_{B_{1/2}} w^\alpha \, dx \leq C.$$

Indeed, by Lemma 3.4 and (f) of Theorem 1.2, we have that

$$\begin{aligned}
\int_{B_{1/2}} (C\phi^{-p})^\alpha \, dx &= C^\alpha p\alpha \int_0^{+\infty} t^{\alpha p-1} |\{\phi^{-1} > t\} \cap B_{1/2}| \, dt \\
&\leq C^\alpha p\alpha \left( |B_{1/2}| \int_0^1 t^{\alpha p-1} \, dt + \int_1^{+\infty} t^{\alpha p-1} |\{\phi^{-1} > t\} \cap B_{1/2}| \, dt \right) \\
&\leq C^\alpha p\alpha \left( \frac{1}{\alpha p} |B_{1/2}| + \int_1^{+\infty} t^{\alpha p-1} |\{\phi < 1/t\} \cap B_{1/2}| \, dt \right) \\
&\leq C^\alpha p\alpha \left( \frac{1}{\alpha p} |B_{1/2}| + |B_{1/2}| \Lambda \int_1^{+\infty} t^{\alpha p-2} \, dt \right) = C^\alpha |B_{1/2}| \left( 1 + \frac{\Lambda \alpha p}{1-\alpha p} \right),
\end{aligned}$$

so it is sufficient to choose $\alpha = \frac{1}{2p}$. Now, the conclusion follows from Lemma 3.5. □

3.3. **Step 3.** We first show that we can choose the constants $M$ from Theorem 1.3 and a level $\delta$ in such a way that we can start the iterative procedure from Lemma 3.1

**Lemma 3.6.** *Suppose that $\Omega \subset B_1$, $0 \in \partial\Omega$, and $\phi : B_1 \to \mathbb{R}$ satisfy the conditions* (a), (b), (c), (d), (e) *and* (f) *of Theorem 1.2. Let $R \in (0, R_0]$ where $R_0$ is the radius from Proposition 2.3. Then, there are constants $C_* > 0$ and $\delta \leq \min\{\eta, \delta_1, \delta_2\}$*[2]*, depending on the dimension $d$, the radius $R$, and the constants from* (b), (c), (e) *and* (f)*, such that for every couple*

$$u, v : B_1 \to \mathbb{R}$$

*of nonnegative continuous functions satisfying*

$$\begin{cases}
\Delta u = \Delta v = 0 \quad \text{in} \quad \Omega \cap B_1; \\
u = v = 0 \quad \text{on} \quad B_1 \setminus \Omega; \\
u(P) = v(P) \quad \text{for some point} \quad P \in B_R \cap \{\phi > \delta R\},
\end{cases}$$

*we have that*

$$C_* u - v \quad \text{and} \quad C_* v - u$$

*fulfill the assumptions of Lemma 3.1.*

**Proof of Lemma 3.6.** Indeed, by Proposition 2.3, there is a constant $C$ (depending also on $R$) such that

$$\frac{1}{C} \leq u, v \leq C \quad \text{on} \quad B_R \cap \{\phi > \delta R\}.$$

Thus, by Proposition 3.3, there is a constant $\Lambda > 0$ such that

$$v \leq \Lambda \quad \text{in} \quad B_{R/4},$$

and a constant $\lambda > 0$ such that

$$u \leq \lambda \quad \text{in} \quad B_{R/4} \cap \{\phi > \frac{\delta}{4} R\}.$$

Thus, for some $C_1 > 0$ large enough, the function $C_1 u - v$ satisfies the conditions of Lemma 3.1.
Relabeling the previous inequalities, we easily deduce the existence of $C_2 > 0$ large enough, such that $C_2 v - u$ satisfies the assumption of Lemma 3.1 too. Finally, the result follows by taking $C_* = \max\{C_1, C_2\}$. □

[2] $\delta_1$ and $\delta_2$ are the constants from Lemma 3.1 and Proposition 3.3, while $\eta$ is the constant from (g) of Theorem 1.2.

**Proof of Theorem 1.3.** We first notice that, by choosing $R$ and $\delta$ small enough, we can apply Lemma 3.6 in a neighborhood of the origin. Precisely, there are $R > \rho > 0$ and $\delta$, such that

$$\sup_{B_{R/4}(x_0)\cap\Omega} (C_* u - v)^- \leq a \quad \text{and} \quad \inf_{B_{R/4}(x_0)\cap\{\phi > \frac{R}{4}\delta\}} (C_* u - v)^+ > 0,$$

for every $x_0 \in B_\rho$. Iterating Lemma 3.1 (up to a dilatation and rescaling), we get that for every $n \geq 0$

$$C_* u - v \geq 0 \quad \text{in} \quad B_{r_n}(x_0) \cap \{\phi > r_n \delta\},$$

where $r_n := R2^{-2-n}$. Now, it is sufficient to notice that for $\rho$ small enough the family of sets

$$\Big\{B_{r_n}(x_0) \cap \{\phi > r_n\delta\} \ : \ x_0 \in \partial\Omega \cap B_\rho,\ n \geq 0\Big\},$$

is a covering of $B_\rho$. By repeating the same argument with $C_* v - u$, we get the claimed result. □

## 4. Hölder continuity up to the boundary. Proof of Theorem 1.2

In this section, we show how the Boundary harnack inequality (Theorem 1.3) implies that the ratio of two harmonic functions vanishing simultaneously on $\partial\Omega$ is Hölder continuous up to the boundary (Theorem 1.2). Our main theorem is a consequence of the following proposition, which is well-known (see for instance [11, Corollary 1.3.8]); we give here the detailed proof for the sake of completeness.

**Proposition 4.1.** *Let $\Omega \subset B_1$ be an open set with the following property.*

$$\text{(8)} \quad \begin{cases} \text{There is a constant } M > 0 \text{ such that for every } x_0 \in \partial\Omega \cap B_{1/2}, \\ \text{every } r \in (0, 1 - |x_0|), \text{ there is a point } P_r(x_0) \in B_r(x_0) \cap \Omega \text{ for which the following holds.} \\ \text{For every pair of continuous non-negative functions} \\ \qquad u, v : B_r(x_0) \to \mathbb{R} \\ \text{satisfying} \\ \qquad \Delta u = \Delta v = 0 \quad \text{in} \quad B_r(x_0) \cap \Omega\,, \\ \qquad u = v = 0 \quad \text{on} \quad B_r(x_0) \setminus \Omega\,, \\ \qquad u(P_r(x_0)) = v(P_r(x_0))\,, \\ \text{we have that} \\ \qquad \dfrac{1}{M} \leq \dfrac{u(x)}{v(x)} \leq M \quad \text{for every} \quad x \in B_{r/2}(x_0) \cap \Omega. \end{cases}$$

*Then, there are constants $\alpha > 0$ and $C > 0$, depending on $c$, $M$ and the dimension, such that for every pair of continuous non-negative functions*

$$u, v : B_1 \to \mathbb{R}$$

*satisfying*

$$\text{(9)} \quad \begin{cases} \Delta u = \Delta v = 0 \quad \text{in} \quad B_1 \cap \Omega \\ u = v = 0 \quad \text{on} \quad B_1 \setminus \Omega \\ u(P_1(0)) = v(P_1(0)) > 0\,, \end{cases}$$

*the following Hölder estimate holds*

$$\text{(10)} \quad \left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| \leq C|x-y|^\alpha \qquad \text{for every} \qquad x, y \in B_{1/4} \cap \Omega\,.$$

In order to prove the proposition it is sufficient to estimate the oscillation of $\frac{u}{v}$ from one scale to another. The main lemma is the following.

**Lemma 4.2.** *Let $\Omega \subset B_1$ be an open set with the property (8). Then, for every $x_0 \in \partial\Omega \cap B_{1/2}$, every $r \leq 1/2$, and every pair of continuous and non-negative functions $u, v : B_r(x_0) \to \mathbb{R}$ satisfying*

$$\Delta u = \Delta v = 0 \quad \text{in} \quad B_r(x_0) \cap \Omega\,, \qquad u = v = 0 \quad \text{on} \quad B_r(x_0) \setminus \Omega,$$

*we have that*

$$\operatorname*{osc}_{\Omega \cap B_{r/2}(x_0)} \frac{u}{v} \leq \Big(1 - \frac{1}{2M}\Big) \operatorname*{osc}_{\Omega \cap B_r(x_0)} \frac{u}{v},$$

*where $M$ is the constant from (8).*

*Proof of Lemma 4.2.* For simplicity, we set

$$P_r := P_r(x_0)\ , \qquad M_r := \sup_{\Omega\cap B_r(x_0)} \frac{u}{v} \qquad \text{and} \qquad m_r := \inf_{\Omega\cap B_r(x_0)} \frac{u}{v}\,.$$

Suppose that $\dfrac{u(P_r)}{v(P_r)} \geq \dfrac{M_r+m_r}{2}$. Then, the functions $u - m_r v$ and $v$ are harmonic and non-negative in $B_r(x_0)\cap\Omega$ and satisfy

$$u(P_r) - m_r v(P_r) \geq \frac{M_r - m_r}{2} v(P_r).$$

Thus, by the hypothesis (8), we have

$$u - m_r v \geq \frac{1}{M}\frac{M_r - m_r}{2} v \qquad \text{in} \qquad B_{r/2}(x_0),$$

where $M$ is the constant from (8). Thus,

$$\inf_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \geq m_r + \frac{1}{M}\frac{M_r - m_r}{2},$$

and so

$$\operatorname*{osc}_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \leq M_r - \left(m_r + \frac{1}{M}\frac{M_r - m_r}{2}\right) = (M_r - m_r)\left(1 - \frac{1}{2M}\right).$$

Analogously, if $\dfrac{u(P_r)}{v(P_r)} \leq \dfrac{M_r+m_r}{2}$, then

$$M_r v - u \geq \frac{1}{M}\frac{M_r - m_r}{2} v \qquad \text{in} \qquad B_{r/2}(x_0),$$

which implies that

$$\sup_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \leq M_r - \frac{1}{M}\frac{M_r - m_r}{2},$$

and

$$\operatorname*{osc}_{\Omega\cap B_{r/2}(x_0)} \frac{u}{v} \leq \left(M_r - \frac{1}{M}\frac{M_r - m_r}{2}\right) - m_r = (M_r - m_r)\left(1 - \frac{1}{2M}\right).$$

which concludes the proof of Lemma 4.2. □

*Proof of Proposition 4.1.* We will prove the following claim.

(11)
$$\begin{cases} \text{There is a constant } c\in(0,1) \text{ such that for every } x_0\in\overline{\Omega}\cap B_{1/2}, \text{ every } r\leq 1/2, \\ \text{and every pair of continuous and non-negative functions} \\ \qquad u, v : B_r(x_0) \to \mathbb{R} \\ \text{satisfying} \\ \qquad \Delta u = \Delta v = 0 \quad \text{in} \quad B_r\cap\Omega\ , \qquad u = v = 0 \quad \text{on} \quad B_r\setminus\Omega, \\ \text{we have that} \\ \qquad \operatorname*{osc}_{\Omega\cap B_{r/16}(x_0)} \frac{u}{v} \leq (1-c) \operatorname*{osc}_{\Omega\cap B_r(x_0)} \frac{u}{v}. \end{cases}$$

In order to prove (11), we consider two cases.

Suppose that there is a point $y_0 \in \partial\Omega\cap B_{r/8}(x_0)$. Then, we have that $B_{r/2}(y_0)\subset B_r(x_0)$, and by Lemma 4.2, we have

$$\operatorname*{osc}_{B_{r/4}(y_0)\cap\Omega} \frac{u}{v} \leq \left(1 - \frac{1}{2M}\right) \operatorname*{osc}_{B_{r/2}(y_0)\cap\Omega} \frac{u}{v} \leq \left(1 - \frac{1}{2M}\right) \operatorname*{osc}_{B_r(x_0)\cap\Omega} \frac{u}{v}.$$

Now, since $B_{r/8}(x_0)\subset B_{r/4}(y_0)$, we get that

$$\operatorname*{osc}_{B_{r/8}(x_0)\cap\Omega} \frac{u}{v} \leq (1-c) \operatorname*{osc}_{B_r(x_0)\cap\Omega} \frac{u}{v} \qquad \text{with} \qquad c = \frac{1}{2M}.$$

Conversely, suppose that $B_{r/8}(x_0)\subset\Omega$. Then, by the classical (interior) Harnack inequality, we have

$$\operatorname*{osc}_{B_{r/16}(x_0)\cap\Omega} \frac{u}{v} \leq (1-c_{\mathcal{H}}) \operatorname*{osc}_{B_{r/8}(x_0)\cap\Omega} \frac{u}{v} \leq (1-c_{\mathcal{H}}) \operatorname*{osc}_{B_r(x_0)\cap\Omega} \frac{u}{v},$$

where $c_{\mathcal{H}}\in(0,1)$ is a dimensional constant. This concludes the proof of (11). The Hölder estimate (10) now follows by a standard argument. □

## 5. Applications

In this section we briefly discuss two examples of domains satisfying the conditions of Theorem 1.2.

**5.1. The vectorial free boundary problem.** Let $B_1 \subset \mathbb{R}^d$. For every vector-valued function $U : B_1 \to \mathbb{R}^k$ we define the functional

$$\mathcal{F}(U) := \int_{B_1} |\nabla U|^2 \, dx + \big|\{|U| > 0\}\big|.$$

We say that a function $U : B_1 \to \mathbb{R}^k$ is a (variational) solution of the vectorial problem if it minimizes $\mathcal{F}$ among all $\mathbb{R}^k$-valued functions with prescribed values on $\partial B_1$. We say that $U = (u_1, \dots, u_k)$ is *non-degenerate*, if there is a component, say $u_1$, which is strictly positive in $\{|U > 0|\} \cap B_1$. If this is not the case, we say that $U$ is *degenerate*. The non-degenerate case was first studied in [6], [14] and [18], while the regularity of the flat free boundaries in the degenerate case was first obtained in [15]; see also [7] for a different approach and [20] for an analysis of the singular part of the free boundaries in dimension two.

We notice that the proofs in [6], [18] and [19], of the $C^{1,\alpha}$ regularity of the flat free boundaries, are all based on the Boundary Harnack principle, which allows to transform the free boundary condition

$$\sum_{j=1}^{k} |\nabla u_j|^2 = 1 \quad \text{on} \quad \partial\{|U > 0|\} \cap B_1,$$

into a condition of the form

$$|\nabla u_j| = g(x) \quad \text{on} \quad \partial\{|U > 0|\} \cap B_1,$$

involving just one of the components of $U$ and an auxiliary Hölder continuous function $g : \partial\Omega \to \mathbb{R}$. In order to prove that the Boundary Harnack principle holds on $\Omega_U := \{|U| > 0\}$, in [6] it was shown that $\Omega_U$ is an NTA domain, while in [18] it was proved that $\Omega_U$ is Reifenberg-flat; in both cases the conclusion followed from [10]. In this case Theorem 1.2 offers an alternative approach. In fact, the modulus $|U|$ of a variational solution $U$ satisfies the conditions of Theorem 1.2. In fact, (a) and (d) are clearly satisfied. For the Lipschitz continuity (b) and the non-degeneracy (g) of $|U|$ we refer to [18], while (f) was proved in [19, Section 2.2]. Moreover, in the non-degenerate case, in [14] it was shown that up to a constant one can bound $|U|$ from above with $u_1$. Thus, (c) is an immediate consequence from the classical interior Harnack inequality and the non-degeneracy of $|U|$. Finally, the exterior density estimate (e) was proved in [18].

**5.2. Subsolutions and supersolutions.** For every $\Lambda > 0$ and every non-negative function $u : B_1 \to \mathbb{R}$ we define the functional

$$\mathcal{F}_\Lambda(u) := \int_{B_1} |\nabla u|^2 \, dx + \Lambda\big|\{u > 0\}\big|.$$

We say that $u$ is a supersolution (subsolution) of $\mathcal{F}_\Lambda$, if

$$\mathcal{F}_\Lambda(u) \le \mathcal{F}_\Lambda(v),$$

for every non-negative $v : B_1 \to \mathbb{R}$ with the same boundary data as $u$ and such that $u \le v$ $(u \ge v)$ in $B_1$. It is easy to show that if $u$ is at the same time a sub- and a supersolution of $\mathcal{F}_\Lambda$, then $u$ is actually a minimizer of $\mathcal{F}_\Lambda$ and so, by the classical result of Alt and Caffarelli [2], the free boundary $\partial\{u > 0\}$ is smooth in $B_1$ up to a set of small Hausdorff dimension. On the other hand, if $u$ is a subsolution for some $\mathcal{F}_\lambda$ and a supersolution for some $\mathcal{F}_\Lambda$, then nothing is known about the local structure of the free boundary. Still, from the analysis in [2, Sections 3 and 4] (see also [21]), one can easily check that we have the following result.

**Proposition 5.1.** *Suppose that* $0 < \lambda < \Lambda$ *are two constants and that* $u \in H^1(B_1)$ *is a non-negative function, which is a subsolution for* $\mathcal{F}_\lambda$ *and a supersolution for* $\mathcal{F}_\Lambda$*. Then,* $u$ *satisfies the conditions* (a), (b), (c), (d), (e), (f) *and* (g) *of Theorem 1.2 and the Boundary Harnack principle holds on the set* $\Omega_u = \{u > 0\} \cap B_1$.

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