

Discrete Variational Problems with Interfaces

Roberto Alicandro, Andrea Braides, Marco Cicalese, and Margherita Solci

Cambridge University Press, 2022

Contents

Notation	1
1 Introduction	3
Bibliographical notes to the Introduction	13
2 Preliminaries	15
2.1 A prototypical example: nearest-neighbour ferromagnetic energies	15
2.2 Γ -convergence	17
2.2.1 Definition and properties	18
2.2.2 Computation of Γ -limits	19
2.2.3 Upper and lower bounds	19
2.3 Surface energies	22
2.3.1 Sets of finite perimeter	22
2.3.2 Perimeter functionals	24
2.4 Discrete-to-continuous convergence of lattice functions	25
2.5 Γ -limit of nearest-neighbour ferromagnetic energies	27
Bibliographical notes to Chapter 2	29
3 Homogenization of pairwise ferromagnetic systems	31
3.1 Homogeneous systems	31
3.2 The blow-up method for discrete energies	39
3.3 Periodic homogenization	45
3.3.1 The asymptotic homogenization formula	47
3.3.2 Upper bound	49
3.3.3 Plane-like minimizers	50
3.3.4 Multi-Bravais lattices	51
3.3.5 Perforated domains	53
3.3.6 A generalization: almost-periodic homogenization	56
3.4 Aperiodic lattices	58
3.4.1 Quasicrystals	58
3.4.2 Penrose tilings	68

3.5	Thin objects	71
3.5.1	An example: brittle nanotubes	73
3.6	Optimal bounds for two-phase systems	75
3.6.1	The general case	75
3.6.2	Two-dimensional nearest-neighbour mixtures	77
3.7	Random systems and percolation	84
3.7.1	Random mixtures	84
3.7.2	Extreme cases	88
	Bibliographical notes to Chapter 3	99
4	A compactness and integral-representation theorem	101
4.1	Some examples of general interactions	101
4.2	Partitions into sets of finite perimeter	110
4.3	Admissible lattices	113
4.4	Convergence of discrete functions	114
4.5	The localization method	116
4.6	The Compactness Theorem	119
4.7	Convergence of minimum problems	127
4.7.1	Boundary-value problems	127
4.7.2	Problems with measure constraint. The Wulff problem	128
4.8	Homogenization	130
4.9	The pairwise ferromagnetic case	135
4.10	An application: design of networks	138
	Bibliographical notes to Chapter 4	139
5	Random lattices	141
5.1	Stochastic lattices	141
5.1.1	Properties of stochastic lattices and an ergodic theorem	144
5.1.2	A Γ -convergence result	145
5.2	Poisson random sets in the plane	153
	Bibliographical notes to Chapter 5	159
6	Extensions	161
6.1	Surfactant ternary systems	161
6.1.1	One-dimensional ternary systems	162
6.1.2	Ternary systems in dimension d	165
6.1.3	The Blume-Emery-Griffiths model	166
6.2	Double porosity	172
6.2.1	Weak interactions between strong media. Multi-phase limits	172
6.2.2	Weak inclusions. Lower-order effects	175
6.2.3	General double-porosity limits	177

6.3	Ferromagnetic systems with modulated phases: chiral molecules	178
6.4	Non-local limits	180
	Bibliographical notes to Chapter 6	184
7	Frustrated systems	185
7.1	Model cases	186
7.2	An integral representation result on patterns	190
7.2.1	Assumptions on the energy density	191
7.2.2	Discrete-to-continuum analysis	192
7.2.3	Simplified variables and application of the Compactness Theorem	194
7.2.4	Examples of homogenized energy densities	195
7.2.5	Total frustration	196
7.3	Analysis of inhomogeneous energies	196
7.4	Some remarks on boundary conditions	204
7.4.1	Non-local effects of boundary conditions	204
7.4.2	Frustrated thin films	206
	Bibliographical notes to Chapter 7	209
8	Perspectives towards dense graphs	211
8.1	Locally dense graphs: coarse graining	211
8.1.1	Coarse graining in dimension one	212
8.1.2	Coarse graining in dimension d	215
8.1.3	An example of a diffuse interface for a sparse graph sequence	215
8.2	Graphons	218
	Bibliographical notes to Chapter 8	227
	Appendix	228
	Multiscale analysis	229
	Spin systems as a limit case of nonconvex elastic interactions	235
	Bibliographical notes to the Appendix	237
	Bibliography	239
	Index	251