ISOPERIMETRIC RESIDUES AND A MESOSCALE FLATNESS CRITERION FOR HYPERSURFACES WITH BOUNDED MEAN CURVATURE

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Abstract. We obtain a full resolution result for minimizers in the exterior isoperimetric problem with respect to a compact obstacle in the large volume regime $v \to \infty$. This is achieved by the study of a Plateau-type problem with free boundary (both on the compact obstacle and at infinity) which is used to identify the first obstacle-dependent term (called *isoperimetric residue*) in the energy expansion, as $v \to \infty$, of the exterior isoperimetric problem. A crucial tool in the analysis of isoperimetric residues is a new "mesoscale flatness criterion" for hypersurfaces with bounded mean curvature, which we obtain as a development of ideas originating in the theory of minimal surfaces with isolated singularities.

Contents

1. Introduction

1.1. **Overview.** Given a compact set $W \subset \mathbb{R}^{n+1}$ ($n \geq 1$), we consider the classical **exterior isoperimetric problem** associated to W, namely

$$\psi_W(v) = \inf \Big\{ P(E;\Omega) : E \subset \Omega \,, |E| = v \Big\}\,, \qquad v > 0\,, \quad \Omega = \mathbb{R}^{n+1} \setminus W\,, \tag{1.1}$$

in the large volume regime $v \to \infty$. Here $|E|$ denotes the volume (Lebesgue measure) of E, and $P(E;\Omega)$ the (distributional) perimeter of E relative to Ω, so that $P(E;\Omega) = \mathcal{H}^n(\Omega \cap \partial E)$ whenever ∂E is locally Lipschitz. Relative isoperimetric problems are well-known for their analytical [Maz11, Sections 6.4-6.6] and geometric [Cha01, Chapter V] relevance. They also provide important models in physical applications, which obviously include capillarity theory [Fin86], but are not limited to it. To make an example related to the large volume regime considered here, in general relativity, (unique) "foliations at infinity" by hypersurfaces with constant mean curvature can be constructed by solving exterior isoperimetric problems at large volumes; for this beautiful approach to the Huisken-Yau theorem, see [EM13].

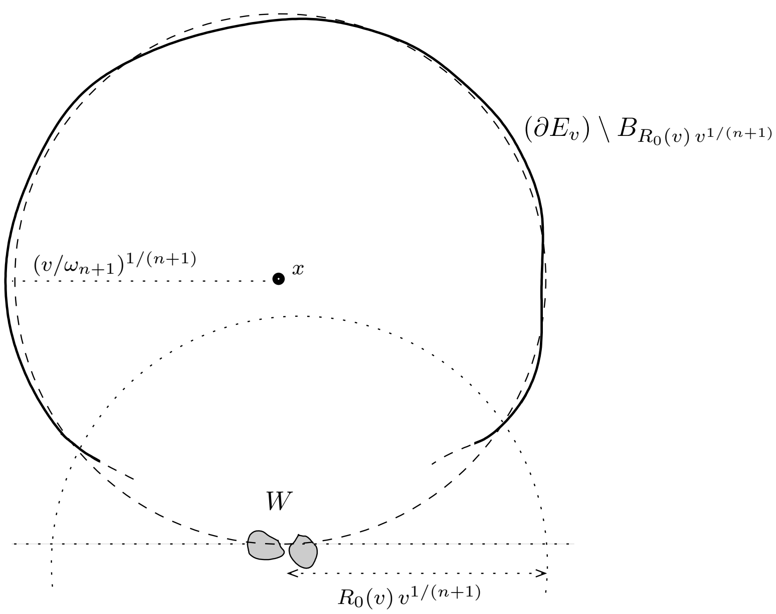

FIGURE 1.1. Quantitative isoperimetry gives no information on how W affects $\psi_W(v)$ for v large. This is true both at the level of energy asymptotics and of resolution formulas, see (1.2) and (1.4), since minimizers of $\psi_W(v)$ are close to balls of volume v only in the exterior of a ball with radius $R_0(v)\, v^{1/(n+1)} \to \infty$ (with $R_0(v) \to 0^+$) as $v \to \infty$.

When $v \to \infty$, we expect minimizers E_v in (1.1) to closely resemble balls of volume v. Indeed, by a direct comparison and by the Euclidean isoperimetric inequality, denoting by $B^{(v)}(x)$ the ball of center x and volume v, and with $B^{(v)} = B^{(v)}(0)$, we find that

$$\lim_{v\to\infty} \frac{\psi_W(v)}{P(B^{(v)})} = 1\,. \tag{1.2}$$

Additional information can be obtained by combining (1.2) with the quantitative Euclidean isoperimetric inequality [FMP08]: if $0 < |E| < \infty$, then

$$\frac{P(E)}{P(B^{(|E|)})} - 1 \geq c(n) \inf_{x\in\mathbb{R}^{n+1}} \Big(\frac{|E\Delta B^{(|E|)}(x)|}{|E|}\Big)^2\,. \tag{1.3}$$

The combination of (1.2) and (1.3) shows that minimizers E_v in $\psi_W(v)$ are close in L^1-distance to balls. In turn, this information can be considerably strengthened by a somehow classical argument (see, e.g. [FM11]) based on the local regularity theory of perimeter minimizers: in this way one shows the existence of a positive constant v_0 and of a function $R_0(v)$, both depending on n and W, with $R_0(v) \to 0^+$ and $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, and such that, if E_v is a minimizer of (1.1) with $v > v_0$, then (see Figure 1.1)

$$\begin{aligned}&(\partial E_v) \setminus B_{R_0\, v^{1/(n+1)}} \text{ is contained in a } C^1\text{-small normal graph over } \partial B^{(v)}(x),\\ &\text{for some } x \in \mathbb{R}^{n+1} \text{ with } |x| = (v/\omega_{n+1})^{1/(n+1)} + \mathrm{o}(v^{1/(n+1)}) \text{ as } v \to \infty\,;\end{aligned} \tag{1.4}$$

here ω_m stands for the volume of the unit ball in $\mathbb{R}^m$, $B_r(x)$ is the ball of center x and radius r in $\mathbb{R}^{n+1}$, and $B_r = B_r(0)$. The picture of the situation offered by (1.2) and (1.4) is thus incomplete under one important aspect: it offers no information related to the specific "obstacle" W under consideration – in other words, *two different obstacles are completely unrecognizable from* (1.2) *and* (1.4) *alone*. The first goal of this paper is characterizing the leading order obstacle-dependent terms in the exterior isoperimetric problem, thus improving on both the energy expansion (1.2) and the "resolution formula" (1.4).

The first step in order to obtain obstacle-dependent information on ψ_W is studying $L^1_{\rm loc}$-subsequential limits F of exterior isoperimetric sets E_v as $v\to\infty$. Since the mean curvature of E_v has order $v^{-1/(n+1)}$ as $v\to\infty$, such limit sets F are easily seen to have minimal (zero mean curvature) boundaries. A finer analysis is required to give a more detailed characterization of such limits F as minimizers in a "Plateau's problem with free boundary on the obstacle and at infinity", whose negative is precisely defined in (1.10) below and denoted by $\mathcal{R}(W)$. We call $\mathcal{R}(W)$ the **isoperimetric residue of** W because it captures the "residual effect" of W in (1.2), as expressed by the limit

$$\lim_{v\to\infty}\,\psi_W(v)-P(B^{(v)})=-\mathcal{R}(W)\,. \tag{1.5}$$

The study of the geometric information about W stored in $\mathcal{R}(W)$ is particularly interesting. We shall prove here its close relation to the maximal hyperplane sectional area and the maximal hyperplane projection area of W (see (1.11) below – which shows, in particular, that $\mathcal{R}(W)>0$ as soon as $|W|>0$), and we shall discuss its maximization under a diameter constraint (see (1.17) below). The proof of (1.5) requires proving a blow-down result for exterior minimal hypersurfaces, and then extracting sharp decay information towards hyperplane blowdown limits. In particular, in the process of proving (1.5), we shall prove the existence of a positive R_2 (depending on n and W only) such that for every maximizer F of $\mathcal{R}(W)$, $(\partial F)\setminus B_{R_2}$ is the graph of a smooth solution to the minimal surfaces equation. An application of Allard's regularity theorem [All72] leads then to complement (1.4) with the following "local" resolution formula: for every $S>R_2$, if v is large enough in terms of n, W and S, then

$$\begin{aligned}&(\partial E_v)\cap\big(B_S\setminus B_{R_2}\big)\text{ is contained in a }C^1\text{-small normal graph over }\partial F,\\&\text{where }F\text{ is optimal for the isoperimetric residue }\mathcal{R}(W)\text{ of }W\,.\end{aligned} \tag{1.6}$$

Unfortunately, (1.4) and (1.6) contain no information on ∂E_v in the "mesoscale" transition region between the resolution models ∂F and $\partial B^{(v)}(x)$, i.e., inside $B_{R_0(v)\,v^{1/(n+1)}}\setminus B_S$.

To address this last issue, we are compelled to develop what is another main result of our paper, and namely, a **mesoscale flatness criterion** for hypersurfaces with bounded mean curvature. This kind of statement is qualitatively novel with respect to the flatness criteria typically used in the study of blowups and blowdowns of minimal surfaces – although it is clearly related to those tools at the mere technical level – and holds promise for applications to other geometric variational problems. In the study of the exterior isoperimetric problem, it allows us to prove the existence of positive constants v_0 and R_1, depending on n and W only, such that if $v>v_0$ and E_v is a minimizer of $\psi_W(v)$, then

$$\begin{aligned}&(\partial E_v)\cap\big(B_{R_1\,v^{1/(n+1)}}\setminus B_{R_2}\big)\text{ is contained in a }C^1\text{-small normal graph over }\partial F,\\&\text{where }F\text{ is optimal for the isoperimetric residue }\mathcal{R}(W)\text{ of }W\,.\end{aligned} \tag{1.7}$$

The key difference between (1.6) and (1.7) is that the domain of resolution given in (1.7) *overlaps* with that of (1.4): indeed, $R_0(v)\to 0^+$ as $v\to\infty$ implies that $R_0\,v^{1/(n+1)}<R_1\,v^{1/(n+1)}$ for $v>v_0$. As a by-product of this overlapping and of the graphicality of ∂F outside of B_{R_2}, we deduce that *boundaries of exterior isoperimetric sets, outside of* B_{R_2}*, are diffeomorphic to* n*-dimensional disks.* Finally, when $n\le 6$, and maximizers F of $\mathcal{R}(W)$ have locally smooth boundaries in Ω, (1.7) can be propagated up to the obstacle itself by a standard application of Allard's regularity theorem; see Remark 1.8 below.

The rest of this introduction is organized as follows. In section 1.2 we rigorously define and collect in one statement all the properties of isoperimetric residues proved in this paper, see Theorem 1.1. In section 1.3 we gather all our results concerning exterior isoperimetric sets with large volumes, see Theorem 1.7. Finally, in section 1.4 we introduce the mesoscale flatness criterion, and in section 1.5 we present the organization of the paper.

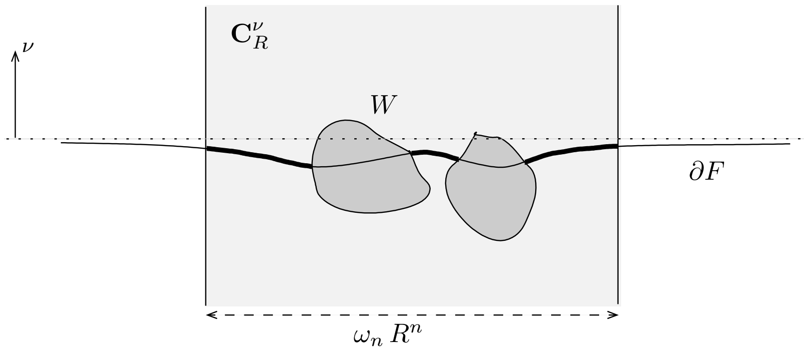

FIGURE 1.2. If $(F,\nu)\in\mathcal{F}$ then F is contained in a slab around $\nu^\perp$ and is such that ∂F has full projection over $\nu^\perp$. The behavior of ∂F inside of W is irrelevant, since only the perimeter of F in Ω matters in the computation of $\mathrm{res}_W(F,\nu)$. The perimeter of F in the open set $\mathbf{C}_R^\nu\setminus W$ (depicted as a bold line) is compared to $\omega_n\,R^n$ (that is, the perimeter of a half-space orthogonal to ν in $\mathbf{C}_R^\nu$); the corresponding "residual" perimeter as $R\to\infty$, that is $\mathrm{res}_W(F,\nu)$, is optimized to define $\mathcal{R}(W)$.

1.2. **Isoperimetric residues.** We now define the isoperimetric residue $\mathcal{R}(W)$ of a compact obstacle $W\subset\mathbb{R}^{n+1}$. We introduce the class

$$\mathcal{F}$$

of those pairs (F,ν) with $\nu\in\mathbb{S}^n$ (= the unit sphere of $\mathbb{R}^{n+1}$) and $F\subset\mathbb{R}^{n+1}$ a set of locally finite perimeter in Ω (i.e., $P(F;\Omega')<\infty$ for every $\Omega'\subset\subset\Omega$), contained in slab around $\nu^\perp=\{x:x\cdot\nu=0\}$, and whose boundary (see Remark 1.6 below) has full projection over $\nu^\perp$ itself, that is to say, for some $\alpha,\beta\in\mathbb{R}$,

$$\partial F\subset\big\{x:\alpha<x\cdot\nu<\beta\big\}\,,\tag{1.8}$$

$$\mathbf{p}_{\nu^\perp}(\partial F)=\nu^\perp:=\big\{x:x\cdot\nu=0\big\}\,,\tag{1.9}$$

where $\mathbf{p}_{\nu^\perp}(x)=x-(x\cdot\nu)\,\nu$, $x\in\mathbb{R}^{n+1}$. In correspondence to the obstacle W, we introduce the **residual perimeter functional**, $\mathrm{res}_W:\mathcal{F}\to\mathbb{R}\cup\{\pm\infty\}$, by setting

$$\mathrm{res}_W(F,\nu)=\limsup_{R\to\infty}\omega_n\,R^n-P(F;\mathbf{C}_R^\nu\setminus W)\,,\qquad (F,\nu)\in\mathcal{F}\,,$$

where $\mathbf{C}_R^\nu=\{x\in\mathbb{R}^{n+1}:|\mathbf{p}_{\nu^\perp}(x)|<R\}$ denotes the (unbounded) cylinder of radius R with axis along ν – and where the limsup is actually a monotone decreasing limit thanks to (1.8) and (1.9) (see (4.11) below for a proof). For a reasonably "well-behaved" F, e.g. if ∂F is the graph of a Lipschitz function over $\nu^\perp$, $\omega_n\,R^n$ is the (obstacle-independent) leading order term of the expansion of $P(F;\mathbf{C}_R^\nu\setminus W)$ as $R\to\infty$, while $\mathrm{res}_W(F,\nu)$ is expected to capture the first obstacle-dependent "residual perimeter" contribution of $P(F;\mathbf{C}_R^\nu\setminus W)$ as $R\to\infty$. The **isoperimetric residue** of W is then defined by maximizing res_W over $\mathcal{F}$, so that

$$\mathcal{R}(W)=\sup_{(F,\nu)\in\mathcal{F}}\mathrm{res}_W(F,\nu)\,;\tag{1.10}$$

see Figure 1.2. To get an impression of the geometric meaning of this quantity, we notice that $\mathcal{R}(\lambda\,W)=\lambda^n\,\mathcal{R}(W)$ for every $\lambda>0$, and $\mathcal{R}(W)$ can be trapped in between the largest areas of the hyperplane sections and of the directional projections of the obstacle (see (1.11) below); moreover, by exploiting (1.19) and (1.20) below, we can also show that $\mathcal{R}(W)=\mathrm{diam}\,(W)$ if $n=1$ and W is connected. In general, however, $\mathcal{R}(W)$ does not seem to admit a simple characterization, as it is finely tuned to the near-to-the-obstacle behavior of "plane-like" minimal surfaces with free boundary on W. Our first main result collects these (and other) properties of isoperimetric residues and of their maximizers.

Theorem 1.1 (Isoperimetric residues)**.** *Let $W \subset \mathbb{R}^{n+1}$ be compact, and set $\Omega = \mathbb{R}^{n+1} \setminus W$.*

(i): *We always have,*

$$\mathcal{S}(W) \le \mathcal{R}(W) \le \mathcal{P}(W)\,, \tag{1.11}$$

where

$$\mathcal{S}(W) = \sup\big\{\mathcal{H}^n(W \cap \Pi) : \Pi \text{ is a hyperplane in } \mathbb{R}^{n+1}\big\} \tag{1.12}$$

$$\mathcal{P}(W) = \sup\big\{\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) : \nu \in \mathbb{S}^n\big\}\,. \tag{1.13}$$

In particular, $\mathcal{R}(W) > 0$ as soon as $|W| > 0$.

(ii): *There exist maximizers (F, ν) of $\mathcal{R}(W)$, and if (F, ν) is a maximizer of $\mathcal{R}(W)$, then F is a* **perimeter minimizer with free boundary in** *Ω, i.e.*

$$P(F; \Omega \cap B) \le P(G; \Omega \cap B)\,, \qquad \forall F \Delta G \subset\subset B,\ B \text{ a ball}. \tag{1.14}$$

Moreover, if $\mathcal{R}(W) > 0$, then there exist $R_2 = R_2(W)$, $C_0 = C_0(W)$, and a smooth function $f : \nu^\perp \to \mathbb{R}$ such that

$$(\partial F) \setminus \mathbf{C}^\nu_{R_2} = \big\{x + f(x)\,\nu : x \in \nu^\perp\,, |x| > R_2\big\}\,, \tag{1.15}$$

and, for some $a, b \in \mathbb{R}$, and $c \in \nu^\perp$, and every $x \in \nu^\perp$, $|x| > R_2$,

$$\begin{aligned} &f(x) = a\,, && (n = 1)\\ &\Big|f(x) - \Big(a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n}\Big)\Big| \le \frac{C_0}{|x|^n}\,, && (n \ge 2)\\ &\max\Big\{|x|^{n-1}\,|\nabla f(x)|, |x|^n\,|\nabla^2 f(x)|\Big\} \le C_0\,, \end{aligned} \tag{1.16}$$

with $\max\{|a|, |b|, |c|\} \le C_0$*; finally, ∂F is contained in the smallest slab $\{x : \alpha \le x \cdot \nu \le \beta\}$ containing W.*

(iii): *At fixed diameter, isoperimetric residues are maximized by balls, i.e.*

$$\mathcal{R}(W) \le \omega_n \Big(\frac{\operatorname{diam} W}{2}\Big)^n = \mathcal{R}\big(\operatorname{cl}(B_{\operatorname{diam} W/2})\big)\,, \tag{1.17}$$

where $\operatorname{cl}(X)$ *denotes topological closure of $X \subset \mathbb{R}^{n+1}$. Moreover, if equality holds in* (1.17) *and (F, ν) is a maximizer of $\mathcal{R}(W)$, then* (1.16) *holds with $b = 0$ and $c = 0$, and setting $\Pi = \big\{y : y \cdot \nu = a\big\}$, we have*

$$(\partial F) \setminus W = \Pi \setminus \operatorname{cl}\big(B_{\operatorname{diam} W/2}(x)\big)\,, \tag{1.18}$$

for some $x \in \Pi$. Finally, equality holds in (1.17) *if and only if there exist a hyperplane Π and a point $x \in \Pi$ such that*

$$\partial B_{\operatorname{diam} W/2}(x) \cap \Pi \subset W\,, \tag{1.19}$$

i.e., W contains an $(n-1)$-dimensional sphere of diameter $\operatorname{diam}(W)$*, and such that*

$$\Omega \setminus \Big(\Pi \setminus \operatorname{cl}\big(B_{\operatorname{diam} W/2}(x)\big)\Big) \text{ has exactly two unbounded connected components.} \tag{1.20}$$

Remark 1.2. See Figure 1.3 for the role of the topological condition (1.20).

Remark 1.3. The assumption $\mathcal{R}(W) > 0$ in statement (ii) is *really* weak. In appendix E we prove that if $\mathcal{R}(W) = 0$, then W is purely $\mathcal{H}^n$-unrectifiable; see Proposition E.1.

Remark 1.4 (Regularity of isoperimetric residues)**.** In the physical dimension $n = 2$, and provided Ω has boundary of class $C^{1,1}$, maximizers of $\mathcal{R}(W)$ are $C^{1,1/2}$-regular up to the obstacle, and smooth away from it. More generally, condition (1.14) implies that $M = \operatorname{cl}(\Omega \cap \partial F)$ is a smooth hypersurface with boundary in $\Omega \setminus \Sigma$, where Σ is a closed set such that $\Sigma \cap \Omega$ is empty if $1 \le n \le 6$, is locally discrete in Ω if $n = 7$, and is locally $\mathcal{H}^{n-7}$-rectifiable in Ω if $n \ge 8$; see, e.g. [Mag12, Part III], [NV20]. Of course, by

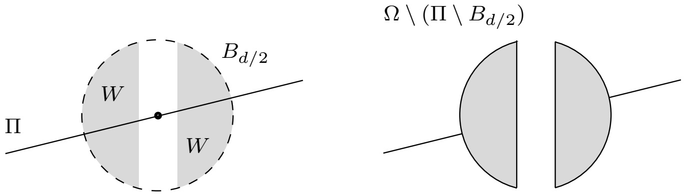

FIGURE 1.3. The obstacle W (depicted in grey) is obtained by removing a cylinder $\mathbf{C}_r^{e_{n+1}}$ from a ball $B_{d/2}$ with $d/2 > r$. In this way $d = \mathrm{diam}\,(W)$ and $B_{d/2}$ is the only ball such that (1.19) can hold. Hyperplanes Π satisfying (1.19) are exactly those passing through the center of $B_{d/2}$, and intersecting W on a $(n-1)$-dimensional sphere of radius $d/2$. For every such Π, $\Omega \setminus (\Pi \setminus B_{d/2})$ has exactly one unbounded connected component, and (1.20) does not hold.

(1.15), $\Sigma \setminus B_{R_2} = \emptyset$ in every dimension. Moreover, justifying the initial claim concerning the case $n = 2$, if we assume that Ω is an open set with $C^{1,1}$-boundary, then M is a $C^{1,1/2}$-hypersurface with boundary in $\mathbb{R}^{n+1} \setminus \Sigma$, with boundary contained in $\partial\Omega$, $\Sigma \cap \partial\Omega$ is $\mathcal{H}^{n-3+\varepsilon}$-negligible for every $\varepsilon > 0$, and Young's law $\nu_F \cdot \nu_\Omega = 0$ holds on $(M \cap \partial\Omega) \setminus \Sigma$; see, e.g. [Grü87, GJ86, DPM15, DPM17].

Remark 1.5. It would be interesting to find geometric information on $\mathcal{R}(W)$ in addition to the one provided by (1.11) and (1.17), for example in the class of convex obstacles.

Remark 1.6 (Normalization of competitors)**.** We work under the standard convention according to which sets of locally finite perimeter F in an open set Ω are automatically modified on and by a set of zero Lebesgue measure so to entail $\Omega \cap \partial F = \Omega \cap \mathrm{cl}\,(\partial^* F)$, where cl denotes topological closure and where $\partial^* F$ is the reduced boundary of F in Ω; see [Mag12, Proposition 12.19]. Under this normalization, local perimeter minimality conditions like (1.14) (or (3.1) below, satisfied by minimizers E_v of $\psi_W(v)$) additionally imply that $F \cap \Omega$ is open in $\mathbb{R}^{n+1}$; see, e.g. [DPM15, Lemma 2.16].

1.3. **Resolution of exterior isoperimetric sets.** Our second main result concerns the resolution of minimizers with large volumes in the exterior isoperimetric problem.

Theorem 1.7 (Resolution of exterior isoperimetric sets)**.** *If $W \subset \mathbb{R}^{n+1}$ is compact, then for every $v > 0$ there exist minimizers of $\psi_W(v)$. Moreover, if $\mathcal{R}(W) > 0$, then*

$$\lim_{v\to\infty} \psi_W(v) - P(B^{(v)}) = -\mathcal{R}(W)\,, \tag{1.21}$$

and there exist positive constants v_0, C_0, R_1, and R_2, depending on n and W only, and R_0, depending on n, W, and v only and with $R_0(v) \to 0^+$ and $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, such that, if E_v is a minimizer of $\psi_W(v)$ with $v > v_0$, then:

(i): *E_v determines $x \in \mathbb{R}^{n+1}$ and $u \in C^\infty(\partial B^{(1)})$ such that*

$$\frac{|E_v \Delta B^{(v)}(x)|}{v} \le \frac{C_0}{v^{1/[2(n+1)]}}\,, \tag{1.22}$$

$$\begin{aligned}&(\partial E_v) \setminus B_{R_0(v)\, v^{1/(n+1)}} &&(1.23)\\ &= \Big\{ y + v^{1/(n+1)}\, u\Big(\frac{y-x}{v^{1/(n+1)}}\Big)\, \nu_{B^{(v)}(x)}(y) : y \in \partial B^{(v)}(x) \Big\} \setminus B_{R_0(v)\, v^{1/(n+1)}}\,,\end{aligned}$$

where, for any $G \subset \mathbb{R}^{n+1}$ with locally finite perimeter, ν_G is the outer unit normal to G;

(ii): *E_v determines a maximizer (F, ν) of $\mathcal{R}(W)$ and $f \in C^\infty((\partial F) \setminus B_{R_2})$ such that*

$$(\partial E_v) \cap A_{R_2}^{R_1\, v^{1/(n+1)}} = \Big\{ y + f(y)\, \nu_F(y) : y \in \partial F \Big\} \cap A_{R_2}^{R_1\, v^{1/(n+1)}}\,, \tag{1.24}$$

where $A_r^s = B_s \setminus \mathrm{cl}\,(B_r)$;

(iii): $(\partial E_v)\setminus B_{R_2}$ *is diffeomorphic to an n-dimensional disk;*

(iv): *Finally,*

$$\lim_{v\to\infty}\sup_{E_v}\max\Big\{\Big|\frac{|x|}{v^{1/(n+1)}}-\frac{1}{\omega_{n+1}^{1/(n+1)}}\Big|,\Big|\nu-\frac{x}{|x|}\Big|,\|u\|_{C^1(\partial B^{(1)})}\Big\}=0\,,\qquad(1.25)$$

$$\lim_{v\to\infty}\sup_{E_v}\|f\|_{C^1(B_M\cap\partial F)}=0\,,\qquad\qquad\forall M>R_2\,,\qquad(1.26)$$

where $\sup_{E_v}$ *ranges over the set of all the minimizers of* $\psi_W(v)$ *with* (x,u) *as in* (1.23) *and* (F,ν,f) *as in* (1.24).

Remark 1.8 (Resolution up to the obstacle)**.** By combining Remark 1.4 with a standard covering argument, we see that, in dimension $n\le 6$, for every $\delta>0$, if $v>v_0(n,W,\delta)$, then the resolution formula (1.24) holds with $B_{R_1\,v^{1/(n+1)}}\setminus I_\delta(W)$ in place of $A_{R_2}^{R_1\,v^{1/(n+1)}}$, where $I_\delta(W)$ denotes the open δ-neighborhood of W. Similarly, when Ω has smooth boundary and $n=2$ (and thus $\Omega\cap\partial F$ is regular up to the obstacle), we can find v_0 (depending on n and W only) such that (1.24) holds with $B_{R_1\,v^{1/(n+1)}}\cap\Omega$ in place of $A_{R_2}^{R_1\,v^{1/(n+1)}}$, that is, graphicality over ∂F holds up to the obstacle itself.

Remark 1.9 (Comparison inequalities)**.** In [CGR07, FM21] it is proved that, if W is convex and J is an half-space in $\mathbb{R}^{n+1}$, then

$$\psi_W(v)\ge\psi_J(v)\qquad\forall v>0\,,\qquad(1.27)$$

with equality for some $v>0$ if and only if ∂W contains a flat facet, large enough to support an half-ball of volume v. Since $\psi_J(v)=P(B^{(2\,v)})/2=P(B^{(v)})/2^{1/(n+1)}$ and $\psi_W(v)-P(B^{(v)})\to-\mathcal{R}(W)$ as $v\to\infty$, (1.27) is far from being optimal if W is a compact convex set and v is large. It would thus be interesting to understand if sharper global bounds than (1.27) are valid on convex obstacles.

Remark 1.10 (Sharp rates of convergence)**.** An interesting problem, which we do not attempt to discuss here, is that of obtaining sharp rates of convergence for minimizers E_v towards the limit model provided by large balls $B^{(v)}(x)$ and by isoperimetric residue maximizers F. In this direction, we do not expect the explicit rate provided in (1.22) to be sharp. Similarly, it would be interesting to further explore the energy expansion in (1.21), and, in particular, to give explicit rates of convergence towards $\mathcal{R}(W)$.

1.4. **The mesoscale flatness criterion.** In our final main result we work with with hypersurfaces M whose mean curvature is bounded by $\Lambda\ge0$ in an annular region $B_{1/\Lambda}\setminus\overline{B}_R$, $R\in(0,1/\Lambda)$. Even without any information on the behavior of M inside B_R (where M could have a non-trivial boundary, or topology, etc.) the classical argument leading to the monotonicity formula still shows that the density-type quantity

$$\begin{aligned}\Theta_{M,R,\Lambda}(r)\;=\;&\frac{\mathcal{H}^n\big(M\cap(B_r\setminus B_R)\big)}{r^n}\\&+\frac{R}{n\,r^n}\int_{M\cap\partial B_R}\frac{|x^{TM}|}{|x|}\,d\mathcal{H}^{n-1}+\Lambda\int_R^r\frac{\mathcal{H}^n\big(M\cap(B_\rho\setminus B_R)\big)}{\rho^n}\,d\rho\,,\end{aligned}\qquad(1.28)$$

is increasing for $r\in(R,1/\Lambda)$ (where x^{TM} denotes the projection of x over T_xM), and that, if $\Theta_{M,R,\Lambda}$ is constant over a sub-interval $(a,b)\subset(R,1/\Lambda)$, then $M\cap(B_b\setminus\overline{B}_a)$ is a cone. Since the constant density value corresponding to $M=H\setminus B_R$ for H a hyperplane through the origin is ω_n (as a result of a double cancellation which also involves the "boundary term" in $\Theta_{H\setminus B_R,R,0}$), we consider the **area deficit**

$$\delta_{M,R,\Lambda}(r)=\omega_n-\Theta_{M,R,\Lambda}(r)\,,\qquad r\in(R,1/\Lambda)\,,\qquad(1.29)$$

which defines a decreasing quantity on $(R,1/\Lambda)$. Here we use the term "deficit", rather than the more usual term "excess", since $\delta_{M,R,\Lambda}$ does not necessarily have non-negative sign (which is one of the crucial property of "excess quantities" typically used in ε-regularity theorems, see, e.g., [Mag12, Lemma 22.11]). Recalling that $A_r^s = B_s \setminus \mathrm{cl}\,(B_r)$ if $s > r > 0$, we are now ready to state the following "smooth version" of our mesoscale flatness criterion. For the sake of clarity, we postpone the statement of the more general version for rectifiable varifolds (which is the one actually needed in our analysis of ψ_W, and whose statement is necessarily more technical) to Theorem 2.3 below.

Theorem 1.11 (Mesoscale flatness criterion (smooth version))**.** *If $n \ge 2$, $\Gamma \ge 0$, and $\sigma > 0$, then there are positive constants M_0 and ε_0, depending on n, Γ and σ only, with the following property.*

Let $\Lambda \ge 0$, $R \in (0, 1/\Lambda)$, and let M be a smooth hypersurface with mean curvature bounded by Λ in $B_{1/\Lambda} \setminus \overline{B}_R$, and with

$$\mathcal{H}^{n-1}\big(M \cap \partial B_R\big) \le \Gamma\, R^{n-1}\,, \qquad \sup_{\rho \in (R,1/\Lambda)} \frac{\mathcal{H}^n\big(M \cap (B_\rho \setminus B_R)\big)}{\rho^n} \le \Gamma\,. \tag{1.30}$$

If there exists $s > 0$ such that

$$\max\{M_0, 64\}\, R < s < \frac{\varepsilon_0}{4\,\Lambda}\,, \qquad \mathcal{H}^n\big(M \cap A_{s/4}^{s/6}\big) > 0\,, \tag{1.31}$$

if, for a hyperplane H with $0 \in H$ and unit normal ν_H, the flatness conditions

$$|\delta_{M,R,\Lambda}(s/8)| \le \varepsilon_0\,, \qquad \frac{1}{s^n} \int_{M \cap A_{s/2}^{s/8}} \Big(\frac{|y \cdot \nu_H|}{|\mathbf{p}_H y|}\Big)^2 \, d\,\mathcal{H}^n_y \le \varepsilon_0\,, \tag{1.32}$$

hold (with $\mathbf{p}_H y = y - (y \cdot \nu_H)\,\nu_H$), and if, setting,

$$R_* = \sup\Big\{\rho \ge \frac{s}{8} : \delta_{M,R,\Lambda}(\rho) \ge -\varepsilon_0\Big\}\,, \qquad S_* = \min\Big\{R_*, \frac{\varepsilon_0}{\Lambda}\Big\}\,, \tag{1.33}$$

we have

$$R_* > 4\,s\,,$$

(and thus $S_* > 4\,s$), then

$$\begin{aligned} M \cap A_{s/16}^{S_*/32} &= \Big\{x + f(x)\,\nu_K : x \in K\Big\} \cap A_{s/16}^{S_*/32}\,, \\ \sup\Big\{\frac{|f(x)|}{|x|} &+ |\nabla f(x)| : x \in K\Big\} \le C(n)\,\sigma\,, \end{aligned} \tag{1.34}$$

for a hyperplane K with $0 \in K$ and unit normal ν_K, and for a function $f \in C^1(K)$.

Remark 1.12 (Structure of the statement)**.** The first condition in (1.31) implicitly requires R to be sufficiently small in terms of $1/\Lambda$, as it introduces a mesoscale s which is both small with respect to $1/\Lambda$ and large with respect to R. The two conditions in (1.32) express the flatness of M at the mesoscale s, both in terms of its area deficit, and in terms of its "angular variance" with respect to a specific hyperplane through the origin H (notice that this is different from the L^2-excess often used in similar contexts, and which would correspond to take $(|y \cdot \nu_H|/s)^2$ in place of $(|y \cdot \nu_H|/|\mathbf{p}_H y|)^2$; see, e.g., [Sim83b, Definition 1.12(i)]). The final key assumption, $R_* > 4\,s$, express the requirement that the area deficit does not decrease too abruptly, and stays above $-\varepsilon_0$ at least up to the scale $4\,s$. Under these assumptions, graphicality with respect to (a possibly different) hyperplane K is inferred on an annulus whose lower radius $s/16$ has the order of the mesoscale s, and whose upper radius $S_*/32$ can be as large as the decay of the area deficit allows (potentially up to $\varepsilon_0/32\,\Lambda$ if $R_* = \infty$), but in any case not too large with respect to $1/\Lambda$.

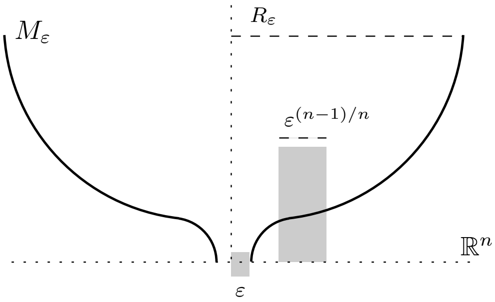

FIGURE 1.4. A half-period of an unduloid with mean curvature n and waist size ε in $\mathbb{R}^{n+1}$. By exploiting the graphicality with respect to the horizontal hyperplane $\mathbb{R}^n$, see (1.35), we see that the flatness of M_ε is no smaller than $\mathrm{O}(\varepsilon^{2(n-1)/n})$, and is exactly $\mathrm{O}(\varepsilon^{2(n-1)/n})$ on an annulus sitting in the mesoscale $\mathrm{O}(\varepsilon^{(n-1)/n})$. This mesoscale is both very large with respect to waist size ε, and very small with respect to the size of the inverse mean curvature, which is order one.

Remark 1.13 (Sharpness of the statement)**.** The statement is sharp in the sense that for a surface "with bounded mean curvature and non-trivial topology inside a hole", flatness can only be established on a mesoscale which is both large with respect to the size of the hole and small with respect to the size of the inverse mean curvature. An example is provided by unduloids M_ε with waist size ε and mean curvature n in $\mathbb{R}^{n+1}$; see Figure 1.4. A "half-period" of M_ε is the graph $\{x + f_\varepsilon(x)\, e_{n+1} : x \in \mathbb{R}^n\,, \varepsilon < |x| < R_\varepsilon\}$ of

$$f_\varepsilon(x) = \int_\varepsilon^{|x|} \Big\{\Big(\frac{r^{n-1}}{r^n - \varepsilon^n + \varepsilon^{n-1}}\Big)^2 - 1\Big\}^{-1/2} dr\,, \qquad \varepsilon < |x| < R_\varepsilon\,, \tag{1.35}$$

where ε and R_ε are the only solutions of $r^{n-1} = r^n - \varepsilon^n + \varepsilon^{n-1}$. One can directly check that f_ε solves $-\mathrm{div}\,(\nabla f_\varepsilon/\sqrt{1+|\nabla f_\varepsilon|^2}) = n$ with $f_\varepsilon = 0$ and $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = \varepsilon\}$, and that $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = R_\varepsilon\}$ with $R_\varepsilon = 1 - \mathrm{O}(\varepsilon^{n-1})$. The minimum of $|\nabla f_\varepsilon|$ is achieved on a radius $\mathrm{O}(\varepsilon^{(n-1)/n})$, and indeed if $r \in (a\,\varepsilon^{(n-1)/n}, b\,\varepsilon^{(n-1)/n})$ for some $b > a > 0$, then we have $|\nabla f_\varepsilon| = \mathrm{O}_{a,b}(\varepsilon^{2(n-1)/n})$. Thus, the flatness of M_ε with respect of $\mathbb{R}^n$ can be no smaller than $\mathrm{O}(\varepsilon^{2(n-1)/n})$, and has that exact order on a scale which is both very large with respect to the size of the hole ($\varepsilon^{(n-1)/n} \gg \varepsilon$) and very small with respect to the size of the inverse mean curvature ($\varepsilon^{(n-1)/n} \ll 1$).

Remark 1.14 (On the application to exterior isoperimetry)**.** Exterior isoperimetric sets E_v with large volume v have small constant mean curvature of order $\Lambda = \Lambda_0(n, W)/v^{1/(n+1)}$. We will work with "holes" of size $R = R_3(n, W)$, for some R_3 sufficiently large with respect to the radius R_2 appearing in Theorem 1.1-(ii), and determined through the sharp decay rates (1.16). The hyperplane H such that the second condition in (1.32) holds is $H = \nu^\perp$ for (F, ν) a maximizer of $\mathcal{R}(W)$ such that E_v is close to F. The decay properties of F towards $\{x : x \cdot \nu = a\}$, the C^1-proximity of ∂E to $\partial B^{(v)}(x)$ for $|x| \approx (\omega_{n+1}/v)^{1/(n+1)}$, and the C^1-proximity of ∂E to ∂F on bounded annuli of the form $A^{2\,R_3}_{R_2}$ are all crucial in checking that (1.30) holds with $\Gamma = \Gamma(n, W)$, that E_v is flat in the sense of (1.32), and, most importantly, that the area deficit $\delta_{M,R,\Lambda}$ of $M = (\partial E_v) \setminus B_{R_3}$ lies above $-\varepsilon_0$ up to scale $r = \mathrm{O}(v^{1/(n+1)})$ – which is the key information to deduce that R_* is comparable to $1/\Lambda$, and thus obtain overlapping domains of resolutions in terms of $\partial B^{(v)}(x)$ and ∂F.

Remark 1.15 (Further applications and criteria)**.** While Theorem 1.11 seems clearly applicable to other geometric problems, there are situations where one may need to develop considerably finer "mesoscale flatness criteria". For example, consider the problem of "resolving" boundaries with almost constant mean curvature undergoing bubbling [CM17, DMMN18, DM19]. When the oscillation of the mean curvature around a constant Λ is small, such boundaries are close to finite unions of mutually tangent spheres of radius n/Λ, and can be covered by C^1-small normal graphs over such spheres away from their tangency points. The application of Allard's regularity theorem gives that graphicality property only up to distance ε/Λ from the tangency points, with $\varepsilon = \varepsilon(n)$, and provided the mean curvature oscillation is small in terms of ε. In order to propagate this flatness information up to a distance directly related to the oscillation of the mean curvature (which, in turn, seems a key step in addressing the conjecture, based on [Sch83, BM21], according to which, near limit tangency points, boundaries with almost constant mean curvature converge to catenoids), one would need a version of Theorem 1.11 for "double" spherical graphs; in the setting of blow-up/blow-down theorems, this would be equivalent to extending known results to limit hyperplanes with multiplicity larger than one.

Remark 1.16 (Comparison with blow-up/blow-down results)**.** From the technical viewpoint, Theorem 1.11 fits into the framework set up by Allard and Almgren in [AA81] for the study of blow-ups and blow-downs of minimal surfaces with tangent integrable cones. At the same time, as clearly exemplified by Remark 1.13, Theorem 1.11 really points in a different direction with respect to [AA81], since it pertains to situations where neither blow-up or blow-down limits make sense. A less immediate but actually crucial point is that in [AA81], the area deficit $\delta_{M,R,\Lambda}$ is considered with a sign, non-positive for blow-ups, and non-negative for blow-downs, see [AA81, Theorem 5.9(4), Theorem 9.6(4)]. These signs restrictions are used there to deduce continuous decay towards limit tangent cones, and, thus, their uniqueness. A key insight here is that sign restrictions *are not needed in propagating graphicality*; and, the more, that dismissing them is actually *crucial* for obtaining overlapping domains of resolutions in statements like (1.4) and (1.7); see Remark 2.5 for additional comments on this point.

Remark 1.17 (Extension to general minimal cones)**.** Proving Theorem 1.11 in higher codimension and with arbitrary *integrable* minimal cones in place of hyperplanes should be possible with essentially the same proof presented here. We do not pursue this extension for two reasons: first, the case of hypersurfaces and hyperplanes is all that is needed in our study of exterior isoperimetry; and, second, in going for generality, one should definitely work in the more powerful framework set up by Simon in [Sim83b, Sim85, Sim96], since, at variance with the more elementary Allard–Almgren's framework used here, it allows one to dispense with the integrability assumption on the reference minimal cones. In this direction, we notice that Theorem 1.11 with $\Lambda = 0$ and $R_* = +\infty$ is a blow-down result for exterior minimal surfaces (see also Theorem 2.3-(ii, iii)). A blow-down result for exterior minimal surfaces is outside the scope of [AA81, Theorem 9.6] which pertains to *entire* minimal surfaces, but it is claimed, with a sketch of proof, on [Sim85, Page 269] as a modification of [Sim85, Theorem 5.5, $m < 0$]. It should be mentioned that, in order to cover the case of exterior minimal surfaces, an additional term of the form $C \int_\Sigma (\dot{u}(t))^2$ should be added on the right side of one of the assumptions of that theorem, namely, of [Sim85, 5.3, $m < 0$]. The addition of this term seems not to cause any serious difficulty with the remainder of the arguments leading to [Sim85, Theorem 5.5, $m < 0$]. For these reasons, we expect that Simon's approach, besides giving a viable approach to the blow-down analysis of exterior minimal surfaces, should also be suitable for proving our mesoscale flatness criterion in the generality described in Remark 1.17.

1.5. **Organization of the paper.** In section 2 we reformulate and prove Theorem 1.11 in a suitable class of varifolds, see Theorem 2.3 below. In section 3 we prove those parts of Theorem 1.7 which are directly descending from quantitative isoperimetry, and do not require the introduction of isoperimetric residues and of a mesoscale flatness analysis; see Theorem 3.1. Section 4 is devoted to the study of isoperimetric residues and of their maximizers, and contains the proof Theorem 1.1. We also present there a statement, repeatedly used in our analysis, which summarizes the fruits of some ideas contained in [Sch83] concerning the decay rates of exterior minimal surfaces towards hyperplanes; see Proposition 4.1. Finally, in section 5, we prove the main energy expansion (1.21) as well as those parts of Theorem 1.7 left out in section 3 (i.i, statements (ii, iii, iv)). This final section is, from a certain viewpoint, the most interesting part of the paper: indeed, it is only the detailed examination of those arguments that clearly illustrates the degree of fine tuning of the preliminary analysis of exterior isoperimetric sets and of maximizers of isoperimetric residues which is needed in order to allow for the application of the mesoscale flatness criterion, and for the consequent full resolution of the exterior isoperimetric problem at large volumes presented in this paper.

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2. A mesoscale flatness criterion for varifolds

In section 2.1 we introduce a class $\mathcal{V}_n(\Lambda,R,S)$ of "varifolds with bounded mean curvature and with boundary" and reformulate Theorem 1.11 in that class, see Theorem 2.3. In sections 2.2-2.3 we present two reparametrization lemmas (Lemma 2.6 and Lemma 2.8) and some "energy estimates" (Theorem 2.9) for spherical graphs, while in section 2.4 we state the monotonicity formula for varifolds in $\mathcal{V}_n(\Lambda,R,S)$ and some more energy estimates involving the monotonicity gap. Finally, in section 2.5, we prove Theorem 2.3.

2.1. **Statement of the criterion.** Following the general notation and terminology of [Sim83a], given an n-dimensional integer rectifiable varifold $V=\mathbf{var}\,(M,\theta)$ in $\mathbb{R}^{n+1}$, defined by a locally $\mathcal{H}^n$-rectifiable set M, and by a (Borel) multiplicity function $\theta:M\to\mathbb{N}$, we denote by $\|V\|=\theta\,\mathcal{H}^n \llcorner M$ the weight of V, and by δV the first variation of V, so that

$$\delta V(X)=\int \operatorname{div}^T X(x)\,dV(x,T)=\int_M \operatorname{div}^M X(x)\,\theta\,d\mathcal{H}^n_x\,,\qquad \forall X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})\,.$$

Given $S>R>0$ and $\Lambda\ge 0$ we consider the family

$$\mathcal{V}_n(\Lambda,R,S)\,, \tag{2.1}$$

of those n-dimensional integral varifolds V in $\mathbb{R}^{n+1}$ such that $\operatorname{spt}V\subset\mathbb{R}^{n+1}\setminus B_R$ and

$$\delta V(X)=\int X\cdot\vec H\,d\|V\|+\int X\cdot\nu_V^{\rm co}\,d\,\mathrm{bd}_V\,,\qquad \forall X\in C^1_c(B_S;\mathbb{R}^{n+1})\,,$$

holds for a Radon measure bd_V in $\mathbb{R}^{n+1}$ supported in ∂B_R, and Borel vector fields $\vec H:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ with $|\vec H|\le\Lambda$ and $\nu_V^{\rm co}:\partial B_R\to\mathbb{R}^{n+1}$ with $|\nu_V^{\rm co}|=1$.

Remark 2.1 (Smooth case)**.** When $\theta\equiv 1$ and M is a smooth hypersurface with boundary such that $M\subset\mathbb{R}^{n+1}\setminus B_R$, $\mathrm{bdry}\,(M)\subset\partial B_R$, and $|H_M|\le\Lambda$, then $V=\mathbf{var}\,(M,1)\in\mathcal{V}_n(\Lambda,R,S)$, with $\vec H$ given by the mean curvature vector of M, $\mathrm{bd}_V=\mathcal{H}^{n-1}\llcorner\mathrm{bdry}\,(M)$, and $\nu_V^{\rm co}$ equal to the outer unit conormal to M along ∂B_R.

Area deficit: Given $V\in\mathcal{V}_n(\Lambda,R,S)$, the quantity

$$\Theta_{V,R,\Lambda}(r)=\frac{\|V\|(B_r\setminus B_R)}{r^n}-\frac{1}{n\,r^n}\int x\cdot\nu_V^{\rm co}\,d\,\mathrm{bd}_V+\Lambda\int_R^r\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n}\,d\rho\,, \tag{2.2}$$

is increasing for $r\in(R,S)$ (see Theorem 2.10-(i) below), and agrees with (1.28) when V is smooth as in Remark 2.1. We define the area deficit of $V\in\mathcal{V}_n(\Lambda,R,S)$ as

$$\delta_{V,R,\Lambda}(r)=\omega_n-\Theta_{V,R,\Lambda}(r)\,,\qquad r\in(R,S)\,.$$

Remark 2.2. Since, by assumption, every $V \in \mathcal{V}_n(\Lambda, S, R)$ is such that $\operatorname{spt} V \subset \mathbb{R}^{n+1} \setminus B_R$, we actually have $\|V\|(B_r) = \|V\|(B_r \setminus B_R)$ for every $r > 0$, and we could have used $\|V\|(B_r)$ in place of $\|V\|(B_r \setminus B_R)$ in the definition of $\Theta_{V,R,\Lambda}$ and in subsequent formulas.

Angular flatness: Given a hyperplane H in $\mathbb{R}^{n+1}$ with $0 \in H$, we call the quantity

$$\int_{A_r^s} \omega_H(y)^2 \, d\|V\|_y \,,$$

the angular flatness of V on the annulus $A_r^s = B_s \setminus \operatorname{cl}(B_r)$ with respect to H, where

$$\omega_H(y) = \operatorname{dist}_{\mathbb{S}^n}\Big(\frac{y}{|y|}, H \cap \mathbb{S}^n\Big) = \arctan\Big(\frac{|y \cdot \nu_H|}{|\mathbf{p}_H y|}\Big) \,, \tag{2.3}$$

for $y \in \mathbb{R}^{n+1} \setminus \{0\}$, $\mathbf{p}_H(y) = y - (y \cdot \nu_H)\, \nu_H$, and ν_H a unit normal to H. (See Remark 1.12 for comparison with the notion of L^2-flatness typically used in similar contexts.)

Theorem 2.3 (Mesoscale flatness criterion)**.** *If $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$ then there are positive constants M_0 and ε_0, depending on n, Γ and σ only, with the following property. If $\Lambda \geq 0$, $R \in (0, 1/\Lambda)$, $V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda)$,*

$$\|\mathrm{bd}_V\|(\partial B_R) \leq \Gamma\, R^{n-1} \,, \qquad \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \leq \Gamma \,. \tag{2.4}$$

and for some hyperplane $H \subset \mathbb{R}^{n+1}$ with $0 \in H$ and for some $s > 0$ we have

$$\frac{\varepsilon_0}{4\,\Lambda} > s > \max\{M_0, 64\}\, R \,, \tag{2.5}$$

$$|\delta_{V,R,\Lambda}(s/8)| \leq \varepsilon_0 \,, \tag{2.6}$$

$$R_* := \sup\Big\{\rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq -\varepsilon_0\Big\} \geq 4\, s \,, \tag{2.7}$$

$$\frac{1}{s^n} \int_{A_{s/8}^{s/2}} \omega_H^2 \, d\|V\| \leq \varepsilon_0 \,, \tag{2.8}$$

$$\|V\|\big(A_{s/6}^{s/4}\big) > 0 \,, \tag{2.9}$$

then **(i):** *if*

$$S_* = \min\Big\{R_*, \frac{\varepsilon_0}{\Lambda}\Big\} < \infty \,, \tag{2.10}$$

then there exists a hyperplane $K \subset \mathbb{R}^{n+1}$ with $0 \in K$ and a function $u \in C^1((K \cap \mathbb{S}^n) \times (s/32, S_*/16))$ with

$$\begin{aligned} &(\operatorname{spt} V) \cap A_{s/32}^{S_*/16} = \Big\{r\,\frac{\omega + u(r,\omega)\,\nu_K}{\sqrt{1 + u(r,\omega)^2}} : \omega \in K \cap \mathbb{S}^n \,, r \in \big(s/32, S_*/16\big)\Big\} \\ &\sup\Big\{|u| + |\nabla^{K \cap \mathbb{S}^n} u| + |r\,\partial_r u| : (\omega, r) \in (K \cap \mathbb{S}^n) \times \big(s/32, S_*/16\big)\Big\} \leq C(n)\,\sigma \,; \end{aligned} \tag{2.11}$$

(ii): *if $\Lambda = 0$ and $\delta_{V,R,0} \geq -\varepsilon_0$ on $(s/8, \infty)$, then (2.11) holds with $S_* = \infty$;*

(iii): *if $\Lambda = 0$ and $\delta_{V,R,0} \geq 0$ on $(s/8, \infty)$, then (2.11) holds with $S_* = \infty$, and one has decay estimates, continuous in the radius, of the form*

$$\delta_{V,R,0}(r) \;\leq\; C(n)\Big(\frac{s}{r}\Big)^\alpha \delta_{V,R,0}\Big(\frac{s}{8}\Big) \,, \qquad \forall r > \frac{s}{4} \,, \tag{2.12}$$

$$\frac{1}{r^n} \int_{A_r^{2\,r}} \omega_K^2 \, d\|V\| \;\leq\; C(n)\,(1 + \Gamma)\Big(\frac{s}{r}\Big)^\alpha \delta_{V,R,0}\Big(\frac{s}{8}\Big) \,, \qquad \forall r > \frac{s}{4} \,, \tag{2.13}$$

for some $\alpha(n) \in (0, 1)$.

Remark 2.4. In Theorem 2.3, graphicality is formulated in terms of the notion of *spherical graph* (further discussed in the subsequent sections) which is more natural than the usual notion of "cylindrical graph" in setting up the iteration procedure behind Theorem 2.3. Spherical graphicality in terms a function u with C^1-small norm as in (2.11) translates into cylindrical graphicality in terms of a function f as in (1.34) with

$$\frac{f(x)}{|x|} \approx u(|x|, \hat{x}) \,, \qquad \nabla_{\hat{x}} f(x) - \frac{f(x)}{|x|} \approx |x|\, \partial_r\, u(|x|, \hat{x}) \,,$$

for $x \neq 0$ and $\hat{x} = x/|x|$; see, in particular, Lemma D.1 in appendix D.

Remark 2.5 (Decay rates and negativity of the area deficit)**.** Even in the case $\Lambda > 0$ one can obtain estimates analogous to (2.12) and (2.13), with the same proofs but with the limitation that they will hold only on the *bounded* range of radii r such that

$$\frac{s}{4} < r < \frac{1}{16} \min\Big\{R_{**}, \frac{\varepsilon_0}{\Lambda}\Big\} \,, \qquad R_{**} = \sup\Big\{\rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq 0\Big\} \,.$$

In particular, without the possibility of sending $r \to \infty$, the resulting estimates will hold for several possible choices of K. In the framework provided by [AA81] the non-negativity of $\delta_{V,R,\Lambda}$ is necessary to set up continuous-in-r decay estimates like (2.12) and (2.13) (see, e.g. (2.98) below), but it is actually dispensable if one is just interested in the iteration scheme needed for propagating "flat graphicality" (see (2.65)–(2.70) below). The gain is not just theoretical (because of the obvious fact that $R_* > R_{**}$). In our application to exterior isoperimetry one can see that R_{**} of $V = \mathbf{var}\,((\partial E_v) \setminus \partial B_R, 1)$ with v large can be at most of order $\mathrm{O}(v^\alpha)$ for some $\alpha < 1/(n+1)$: indeed, on a scale $\mathrm{O}(v^{1/(n+1)})$, the smooth proximity of ∂E_v to a sphere $\partial B^{(v)}(x)$ will force $\delta_{V,R,\Lambda}$ to be negative. As a consequence, the overlapping of the domain of resolution of (1.4) and (1.7) would be lost if working with R_{**} (and with it, the complete resolution of exterior isoperimetric sets).

2.2. **Spherical graphs.** We start setting up some notation. We denote by

$$\mathcal{H}$$

the family of the oriented hyperplanes $H \subset \mathbb{R}^{n+1}$ such that $0 \in \mathcal{H}$; in particular, the choice of $H \in \mathcal{H}$ implies the choice of a unit normal vector ν_H to H. Given $H \in \mathcal{H}$, we set

$$\Sigma_H = H \cap \mathbb{S}^n \,,$$

for the $(n-1)$-dimensional equatorial sphere defined by H on $\mathbb{S}^n$, and we denote by

$$\mathbf{p}_H : \mathbb{R}^{n+1} \to H \,, \qquad \mathbf{q}_H : \mathbb{R}^{n+1} \to H^\perp \,,$$

the orthogonal projections of $\mathbb{R}^{n+1}$ onto H and onto the orthogonal complement $H^\perp = \{t\,\nu_H : t \in \mathbb{R}\}$ to H in $\mathbb{R}^{n+1}$. Given $\sigma > 0$, we set

$$\mathcal{X}_\sigma(\Sigma_H) = \big\{u \in C^1(\Sigma_H) : \|u\|_{C^1(\Sigma_H)} < \sigma\big\} \,.$$

Clearly there exists $\sigma_0 = \sigma_0(n) > 0$ such that if $H \in \mathcal{H}$ and $u \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, then the map

$$f_u(\omega) = \frac{\omega + u(\omega)\,\nu_H}{\sqrt{1 + u(\omega)^2}} \,, \qquad \omega \in \Sigma_H \,,$$

defines a diffeomorphism of Σ_H into an hypersurface $\Sigma_H(u) \subset \mathbb{S}^n$, namely

$$\Sigma_H(u) = f_u(\Sigma_H) = \Big\{\frac{\omega + u(\omega)\,\nu_H}{\sqrt{1 + u(\omega)^2}} : \omega \in \Sigma_H\Big\} \,. \tag{2.14}$$

We call $\Sigma_H(u)$ a **spherical graph** over Σ_H. Exploiting the fact that Σ_H is a minimal hypersurface in $\mathbb{S}^n$ and that if $\{\tau_i\}_i$ is a local orthonormal frame on Σ_H then $\nu_H \cdot \nabla_{\tau_i} \tau_j = 0$, a second variation computation (see, e.g., [ESV19, Lemma 2.1]) gives, for $u \in \mathcal{X}_\sigma(\Sigma_H)$,

$$\Big|\mathcal{H}^{n-1}(\Sigma_H(u)) - n\,\omega_n - \frac{1}{2} \int_{\Sigma_H} |\nabla^{\Sigma_H} u|^2 - (n-1)\,u^2\Big| \leq C(n)\,\sigma \int_{\Sigma_H} u^2 + |\nabla^{\Sigma_H} u|^2 \,, \tag{2.15}$$

(where, of course, $n\,\omega_n = \mathcal{H}^{n-1}(\Sigma_H) = \mathcal{H}^{n-1}(\Sigma_H(0))$). We also recall that $u \in L^2(\Sigma_H)$ is a unit norm Jacobi field of Σ_H (i.e., a zero eigenvector of $\Delta^{\Sigma_H} + (n-1)\,\mathrm{Id}$ with unit $L^2(\Sigma_H)$-norm) if and only if there is $\tau \in \mathbb{S}^n$ with $\tau \cdot \nu_H = 0$ such that

$$u(\omega) = c_0(n)\,(\omega \cdot \tau)\,, \qquad \forall \omega \in \Sigma_H\,, \quad \text{where } c_0(n) = \sqrt{\frac{n}{\mathcal{H}^{n-1}(\Sigma_H)}}\,. \tag{2.16}$$

We denote by $E^0_{\Sigma_H}$ the orthogonal projection operator of $L^2(\Omega)$ onto the span of the Jacobi fields of Σ_H. The following lemma provides a way to reparameterize spherical graphs over equatorial spheres so that the projection over Jacobi fields is annihilated.

Lemma 2.6. *There exist constants C_0, ε_0 and σ_0, depending on the dimension n only, with the following properties:*

(i): *if $H, K \in \mathcal{H}$, $|\nu_H - \nu_K| \le \varepsilon < \varepsilon_0$, and $u \in \mathcal{X}_\sigma(\Sigma_H)$ for $\sigma < \sigma_0$, then the map $T^K_u : \Sigma_H \to \Sigma_K$ defined by*

$$T^K_u(\omega) = \frac{\mathbf{p}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H}{|\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H|}\,, \qquad \omega \in \Sigma_H\,,$$

is a diffeomorphism between Σ_H and Σ_K, and the function $v^K_u : \Sigma_K \to \mathbb{R}$ defined by

$$v^K_u(T^K_u(\omega)) = \frac{\mathbf{q}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\nu_K \cdot (\omega + u(\omega)\,\nu_H)}{|\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H|}\,, \qquad \omega \in \Sigma_H\,, \tag{2.17}$$

is such that

$$v^K_u \in \mathcal{X}_{C(n)\,(\sigma+\varepsilon)}(\Sigma_K)\,, \qquad \Sigma_H(u) = \Sigma_K(v^K_u)\,, \tag{2.18}$$

and

$$\Big|\int_{\Sigma_K} (v^K_u)^2 - \int_{\Sigma_H} u^2\Big| \le C(n)\,\Big\{|\nu_H - \nu_K|^2 + \int_{\Sigma_H} u^2\Big\}\,. \tag{2.19}$$

(ii): *if $H \in \mathcal{H}$ and $u \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, then there exist $K \in \mathcal{H}$ with $|\nu_H - \nu_K| < \varepsilon_0$ and $v \in \mathcal{X}_{C_0\,\sigma_0}(\Sigma_K)$ such that*

$$\begin{aligned}
&\Sigma_H(u) = \Sigma_K(v)\,, \qquad E^0_{\Sigma_K}[v] = 0\,,\\
&|\nu_K - \nu_H|^2 \le C_0(n) \int_{\Sigma_H} \big(E^0_{\Sigma_H}[u]\big)^2\,,\\
&\Big|\int_{\Sigma_K} v^2 - \int_{\Sigma_H} u^2\Big| \le C_0(n) \int_{\Sigma_H} u^2\,.
\end{aligned}$$

Proof. See appendix A. □

Remark 2.7. It may seem unnecessary to present a detailed proof of Lemma 2.6, as we do in appendix A, given that, when Σ_H is replaced by a generic integrable minimal surface Σ in $\mathbb{S}^n$, similar statements are found in the first four sections of [AA81, Chapter 5]. However, two of those statements, namely [AA81, 5.3(4), 5.3(5)], seem to be not correctly formulated; and the issue requires clarification, since those statements are used in the iteration arguments behind the blow-up theorem [AA81, Theorem 5.9] and its blow-down counterpart [AA81, Theorem 9.6]; see, for example, the second displayed chain of inequalities on [AA81, Page 254]. To explain this issue **we momentarily adopt the notation of** [AA81]. In [AA81, Chapter 5] they consider a family of minimal surfaces in $\mathbb{S}^n$, denoted by $\{M_t\}_{t\in U}$, and obtained as diffeomorphic images of a minimal surface $M = M_0$. The parameter t ranges in an open ball $U \subset \mathbb{R}^j$, where j is the dimension of the space of Jacobi fields of M. Given a vector field Z in $\mathbb{S}^n$, defined on and normal to M_t, they denote by $F_t(Z)$ the diffeomorphism of M_t into $\mathbb{S}^n$ obtained by combining Z with the exponential map of $\mathbb{S}^n$ (up to lower than second order corrections in Z, this is equivalent to taking the graph of Z over M_t, and then projecting it back on $\mathbb{S}^n$, which is what we do, following [Sim83b], in (2.14)). Then, in [AA81, 5.2(2)], they define Λ_t as the family of those Z such that $\mathrm{Image}(F_t(Z)) = \mathrm{Image}(F_0(W))$ for some vector field W normal to M, and, given $t, u \in U$ and $Z \in \Lambda_t$, they define $F_t^u : \Lambda_t \to \Lambda_u$ as the map between such classes of normal vector fields with the property that $\mathrm{Image}(F_t(Z)) = \mathrm{Image}(F_u(F_t^u(Z)))$: in particular, $F_t^u(Z)$ is the vector field that takes M_u to the same surface to which Z takes M_t. With this premise, in [AA81, 5.3(5)] they say that if $t, u \in U$, and $Z \in \Lambda_t$, then

$$\Big| \int_{M_u} |F_t^u(Z)|^2 - \int_{M_t} |Z|^2 \Big| \le C\,|t-u| \int_{M_t} |Z|^2 \,, \tag{2.20}$$

for a constant C depending on M only. Testing this with $Z = 0$ (notice that $0 \in \Lambda_t$ by [AA81, 5.3(1)]) one finds $F_t^u(0) = 0$, and thus $M_t = \mathrm{Image}(F_t(0)) = \mathrm{Image}(F_u(F_t^u(0))) = \mathrm{Image}(F_u(0)) = M_u$. In particular, $M_u = M_t$ for every $t, u \in U$, that is, $\{M_t\}_{t\in U}$ consists of a single surface, M itself. But this is never the case since $\{M_t\}_{t\in U}$ always contains, to the least, every sufficiently small rotation of M in $\mathbb{S}^n$. An analogous problem is contained in [AA81, 5.3(4)]. Coming back to our notation, with reference to Lemma 2.6-(i), the analogous estimate to (2.20) in our setting would be equivalent to claiming that, for every $H, K \in \mathcal{H}$ with $|\nu_K - \nu_H| < \varepsilon_0$ and $u \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, v_u^K defined in (2.17) satisfies

$$\Big| \int_{\Sigma_K} (v_u^K)^2 - \int_{\Sigma_H} u^2 \Big| \le C(n)\,|\nu_H - \nu_K| \int_{\Sigma_H} u^2 \,, \tag{2.21}$$

which again leads to a contradiction if tested with $u = 0$. A correct estimate, analogous in spirit to (2.21) and still sufficiently precise to be used in iterations, is (2.19) in Lemma 2.6. There should be no difficulty in adapting the arguments from appendix A to the more general context of integrable cones, and then in using the resulting generalization of (2.19) to implement the iterations needed in [AA81, Theorem 5.9, Theorem 9.6].

2.3. **Energy estimates for spherical graphs over annuli.** We now introduce the notion of spherical graph over an annulus, together with some basic energy estimates for spherical graphs with bounded mean curvature.

Given $H \in \mathcal{H}$ and $0 < r_1 < r_2$ we let $\mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ be the class of those $u \in C^1(\Sigma_H \times (r_1, r_2))$ such that, setting $u_r = u(\cdot, r)$, one has

$$\begin{aligned} u_r &\in \mathcal{X}_\sigma(\Sigma_H)\,, && \forall r \in (r_1, r_2)\,,\\ |r\,\partial_r u| &\le \sigma\,, && \text{on } \Sigma_H \times (r_1, r_2)\,. \end{aligned}$$

If $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, then the spherical graph of u over $\Sigma_H \times (r_1, r_2)$, given by

$$\Sigma_H(u, r_1, r_2) = \Big\{ r\,\frac{\omega + u_r(\omega)\,\nu_H}{\sqrt{1 + u_r(\omega)^2}} : \omega \in \Sigma_H\,, r \in (r_1, r_2) \Big\}\,,$$

is an hypersurface in $A_{r_1}^{r_2}$. It is useful to keep in mind that

$$\Sigma_H(0, r_1, r_2) = \big\{ r\,\omega : \omega \in \Sigma\,, r \in (r_1, r_2) \big\} = H \cap A_{r_1}^{r_2}\,,$$

is a flat annular region with $\mathcal{H}^n(\Sigma_H(0, r_1, r_2)) = \omega_n\,(r_2^n - r_1^n)$, and that

$$\frac{1}{C(n)} \int_{\Sigma_H(u, r_1, r_2)} \omega_H^2\, d\mathcal{H}^n \le \int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\, u^2 \le C(n) \int_{\Sigma_H(u, r_1, r_2)} \omega_H^2\, d\mathcal{H}^n \tag{2.22}$$

whenever $u \in \mathcal{X}_{\sigma_1}(\Sigma_H, r_1, r_2)$ for a sufficiently small $\sigma_1 = \sigma_1(n)$. The following reparametrization lemma is analogous to (and easily obtained from) Lemma 2.6.

Lemma 2.8. *There exist positive constants ε_0, σ_0 and C_0, depending on the dimension n only, with the following properties:*

(i): *if $H, K \in \mathcal{H}$, $\nu_H \cdot \nu_K > 0$, $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, for $\sigma < \sigma_0$, $v \in \mathcal{X}_{\sigma_0}(\Sigma_K, r_1, r_2)$, and $|\nu_H - \nu_K| = \varepsilon < \varepsilon_0$, then there exists $w \in \mathcal{X}_{C_0(\sigma+\varepsilon)}(\Sigma_H, r_1, r_2)$ such that*

$$\Sigma_K(v, r_1, r_2) = \Sigma_H(w, r_1, r_2)\,. \tag{2.23}$$

(ii): *if* $H \in \mathcal{H}$, $u \in \mathcal{X}_{\sigma_0}(\Sigma_H, r_1, r_2)$, *and* $(a,b) \subset\subset (r_1, r_2)$, *then there exist* $K \in \mathcal{H}$, $v \in \mathcal{X}_{C_0\,\sigma_0}(\Sigma_K, r_1, r_2)$, *and* $r_* \in [a,b]$ *such that*

$$\Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2)\,, \tag{2.24}$$

$$E^0_{\Sigma_K}(v_{r_*}) = 0\,, \tag{2.25}$$

$$|\nu_H - \nu_K|^2 \le C_0(n)\,\min_{\rho\in[a,b]}\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u_\rho]\big)^2\,. \tag{2.26}$$

Moreover, for every $r \in (r_1, r_2)$,

$$\Big|\int_{\Sigma_K}(v_r)^2 - \int_{\Sigma_H}(u_r)^2\Big| \le C_0(n)\Big\{\min_{\rho\in[a,b]}\int_{\Sigma_H}(u_\rho)^2 + \int_{\Sigma_H}(u_r)^2\Big\}\,. \tag{2.27}$$

Proof. We first prove statement (i). If $|\nu_H - \nu_K| = \varepsilon < \varepsilon_0$, since for every $r \in (r_1, r_2)$ we have $u_r \in \mathcal{X}_\sigma(\Sigma_H)$, we can apply Lemma 2.6-(i) to deduce that the map $T_r : \Sigma_H \to \Sigma_K$,

$$T_r(\omega) = \frac{\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H]}{|\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H]|} \qquad \omega \in \Sigma_H\,, \tag{2.28}$$

is a diffeomorphism between Σ_H and Σ_K, and the function $v_r : \Sigma_K \to \mathbb{R}$,

$$v_r(T_r(\omega)) = \frac{\nu_K\cdot(\omega + u_r(\omega)\,\nu_H)}{|\mathbf{p}_K[\omega + u_r(\omega)\,\nu_H]|}\,, \qquad \omega \in \Sigma_H\,, \tag{2.29}$$

satisfies $v_r \in \mathcal{X}_{C_0\,(\sigma+\varepsilon)}(\Sigma_K)$ as well as

$$\Sigma_H(u_r) = \Sigma_K(v_r)\,, \qquad \Big|\int_{\Sigma_K}(v_r)^2 - \int_{\Sigma_H}(u_r)^2\Big| \le C(n)\Big\{|\nu_H - \nu_K|^2 + \int_{\Sigma_H}(u_r)^2\Big\}\,. \tag{2.30}$$

Since $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$, and since the definitions of T_r and v_r depend smoothly on u_r, setting $v(\omega, r) := v_r(\omega)$ we define $v : \Sigma_K \times (r_1, r_2) \to \mathbb{R}$ such that

$$\Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2)\,,$$

by (2.30), and $v \in \mathcal{X}_{C_0\,(\sigma+\varepsilon)}(\Sigma_H, r_1, r_2)$ (where $|r\,\partial_r v_r| \le C_0(\sigma+\varepsilon)$ is deduced by differentiation in (2.28) and (2.29), and thanks to $|u_r|, |r\,\partial_r u_r| < \sigma$). This proves (2.23).

Step two: We prove statement (ii). Let us set

$$\gamma = \min_{\rho\in[a,b]}\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u_\rho]\big)^2\,,$$

and let $r_* \in [a,b]$ be such that the minimum γ is achieved at $r = r_*$. If $\gamma = 0$, then we prove the lemma with $K = H$ and $v = u$. If $\gamma > 0$, then we apply Lemma 2.6-(ii) to $u_{r_*} \in \mathcal{X}_{\sigma_0}(\Sigma_H)$, and correspondingly we find $K \in \mathcal{H}$ with $|\nu_K - \nu_H| < \varepsilon_0$ and $v_{r_*} \in \mathcal{X}_{C_0\,s_0}(\Sigma_K)$ such that $\Sigma_H(u_{r_*}) = \Sigma_K(v_{r_*})$ and

$$E^0_{\Sigma_K}[v_{r_*}] = 0\,, \tag{2.31}$$

$$|\nu_K - \nu_H|^2 \le C_0(n)\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u_{r_*}]\big)^2 = C_0(n)\,\gamma\,, \tag{2.32}$$

$$\Big|\int_{\Sigma_K}(v_{r_*})^2 - \int_{\Sigma_H}(u_{r_*})^2\Big| \le C_0(n)\int_{\Sigma_H}(u_{r_*})^2\,. \tag{2.33}$$

Since $v_{r_*} = v(\cdot, r_*)$ for the function v constructed as in step one starting from u, H and K, we see that combining (2.30) and (2.32) we find (2.27), while (2.31) is (2.25). $\square$

We will use two basic "energy estimates" for spherical graphs over annuli. In order to streamline the later application of these estimates to diadic families of annuli we introduce the following terminology: the intervals (r_1, r_2) and (r_3, r_4) are (η, η_0)**-related** if

$$r_2 = r_0(1+\eta_0)\,, \quad r_1 = r_0(1-\eta_0)\,, \quad r_4 = r_0(1+\eta)\,, \quad r_3 = r_0(1-\eta)\,, \tag{2.34}$$

for some $\eta_0 > \eta > 0$, and with $r_0 = (r_1 + r_2)/2 = (r_3 + r_4)/2$; in particular, (r_3, r_4) is contained in, and concentric to, (r_1, r_2). The case $\Lambda = 0$ of the following statement is the codimension one, equatorial spheres case of [AA81, Lemma 7.14, Theorem 7.15].

Theorem 2.9 (Energy estimates for spherical graphs)**.** *If $n \geq 2$ and $\eta_0 > \eta > 0$, then there are $\sigma_0 = \sigma_0(n, \eta_0, \eta)$ and $C_0 = C_0(n, \eta_0, \eta)$ positive, with the following property.*

If $H \in \mathcal{H}$, $\Lambda \geq 0$, and $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ is such that $\max\{1, \Lambda\, r_2\}\, \sigma \leq \sigma_0$ and

$$\Sigma_H(u, r_1, r_2) \text{ has mean curvature bounded by } \Lambda \text{ in } A_{r_1}^{r_2}\,, \tag{2.35}$$

then

$$\Big|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\Big| \leq C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \left(u^2 + \Lambda\, r\, |u|\right), \tag{2.36}$$

whenever (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.34)*. Moreover, if*

$$\exists\, r \in (r_1, r_2) \text{ s.t. } E^0_{\Sigma_H} u_r = 0 \text{ on } \Sigma_H\,, \tag{2.37}$$

then we also have

$$\int_{\Sigma_H \times (r_3, r_4)} r^{n-1}\, u^2 \leq C(n)\, \Lambda\, r_2\, (r_2^n - r_1^n) + C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1}\, (r\, \partial_r u)^2\,. \tag{2.38}$$

Proof. See appendix B. □

2.4. **Monotonicity for exterior varifolds with bounded mean curvature.** The following theorem states the monotonicity of $\Theta_{V,R,\Lambda}$ for $V \in \mathcal{V}_n(\Lambda, R, S)$, and provides, when V corresponds to a spherical graph, a quantitative lower bound for the gap in the associated monotonicity formula; in the case $\Lambda = 0$, $R = 0$, it reduces to the codimension one, equatorial spheres case of [AA81, Lemma 7.16] and [AA81, Theorem 7.17].

Theorem 2.10. (i): *If $V \in \mathcal{V}_n(\Lambda, R, S)$, then*

$$\Theta_{V,R,\Lambda} \text{ is increasing on } (R, S)\,, \tag{2.39}$$

where $\Theta_{V,R,\Lambda}$ is defined as in (2.2)*.*

(ii): *There exists $\sigma_0(n)$ such that, if the assumptions of part (i) hold and, for some $H \in \mathcal{H}$, $u \in \mathcal{X}_\sigma(\Sigma, r_1, r_2)$ with $\sigma \leq \sigma_0(n)$, and $(r_1, r_2) \subset (R, S)$, we have*

$$V \text{ corresponds to } \Sigma_H(u, r_1, r_2) \text{ in } A_{r_1}^{r_2}\,, \tag{2.40}$$

then

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r\, \partial u_r)^2 \leq C(n)\, r_2^n \Big\{\Theta_{V,R,\Lambda}(r_2) - \Theta_{V,R,\Lambda}(r_1)\Big\}\,. \tag{2.41}$$

(iii): *Finally, given $\eta_0 > \eta > 0$, there exist σ_0 and C_0 depending on n, η_0, and η only, such that if the assumptions of part (i) and part (ii) hold and, in addition to that, we also have $\max\{1, \Lambda\, r_2\}\, \sigma \leq \sigma_0$ and*

$$\exists\, r \in (r_1, r_2) \text{ s.t. } E^0_{\Sigma_H} u_r = 0 \text{ on } \Sigma_H\,, \tag{2.42}$$

then, whenever (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.34)*, we have*

$$\begin{aligned} &\Big|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\Big| \\ &\qquad \leq C_0\, r_2^n \Big\{\Theta_{V,R,\Lambda}(r_2) - \Theta_{V,R,\Lambda}(r_1) + (\Lambda\, r_2)^2\Big\}\,. \end{aligned} \tag{2.43}$$

Proof. Step one: We prove part (i) and part (ii). For the sake of both brevity and clarity, we give the proof in the case when V is the multiplicity one varifold associated to a smooth hypersurface M with boundary, whose boundary contained in ∂B_R, see Remark 2.1: the general case is then addressed by the classical argument with rescaled radial test vector fields, and is presented in appendix C. By the coarea formula, by the tangential divergence theorem and by the fact that $|\vec{H}| \le \Lambda$, we find that, for a.e. $\rho > R$,

$$\begin{aligned}\frac{d}{d\rho}\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n} &= \frac{1}{\rho^n}\int_{M\cap\partial B_\rho}\frac{|x|}{|x^{TM}|}\,d\mathcal{H}^{n-1} - \frac{n\,\mathcal{H}^n(M\cap(B_\rho\setminus B_R))}{\rho^{n+1}} \\ &= \frac{1}{\rho^n}\int_{M\cap\partial B_\rho}\frac{|x|}{|x^{TM}|}\,d\mathcal{H}^{n-1} - \frac{1}{\rho^n}\int_{M\cap(B_\rho\setminus B_R)}\frac{x}{\rho}\cdot\vec{H}\,d\mathcal{H}^n \\ &\quad -\frac{1}{\rho^{n+1}}\Big\{\int_{M\cap\partial B_\rho}\nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1} + \int_{M\cap\partial B_R}\nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1}\Big\} \\ &\ge \frac{1}{\rho^n}\int_{M\cap\partial B_\rho}\Big(\frac{|x|}{|x^{TM}|} - \frac{|x^{TM}|}{|x|}\Big)\,d\mathcal{H}^{n-1} \\ &\quad -\frac{1}{\rho^{n+1}}\int_{M\cap\partial B_R}\nu_M^{\rm co}\cdot x\,d\mathcal{H}^{n-1} - \Lambda\,\frac{\mathcal{H}^n(M\cap(B_\rho\setminus B_R))}{\rho^n} \\ &= \mathrm{Mon}(V,\rho) + \frac{d}{d\rho}\frac{1}{n\,\rho^n}\int x\cdot\nu_V^{\rm co}\,d\,\mathrm{bd}_V - \Lambda\,\frac{\|V\|(B_\rho\setminus B_R)}{\rho^n} \qquad (2.44)\end{aligned}$$

where we have set

$$\mathrm{Mon}(V,\rho) = \frac{d}{d\rho}\int_{B_\rho\setminus B_R}\frac{|x^\perp|^2}{|x|^{n+2}}\,d\|V\|\,. \qquad (2.45)$$

Since $\mathrm{Mon}(V,\rho)\ge 0$, this proves (2.39). Assuming now that (2.40) holds, by using [AA81, Lemma 3.5(6)] as done in the proof of [AA81, Lemma 7.16], we see that, under (2.40), we have

$$C(n)\,r_2^n\int_{r_1}^{r_2}\mathrm{Mon}(V,\rho)\,d\rho \ge \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\,,$$

thus completing the proof of (2.41).

Step two: We prove part (iii). Let us set

$$a = r_0\Big(1-\frac{\eta+\eta_0}{2}\Big)\,,\qquad b = r_0\Big(1+\frac{\eta+\eta_0}{2}\Big)\,,$$

so that (a,b) and (r_3,r_4) are $(\eta,(\eta+\eta_0)/2)$-related, and (r_1,r_2) and (a,b) are $((\eta+\eta_0)/2,\eta_0)$-related (in particular, $(r_3,r_4)\subset(a,b)\subset(r_1,r_2)$). By suitably choosing σ_0 in terms of n, η and η_0, we can apply (2.36) in Theorem 2.9 with (r_3,r_4) and (a,b), so to find

$$\begin{aligned}\Big|\mathcal{H}^n(\Sigma(u,r_3,r_4)) - \mathcal{H}^n(\Sigma(0,r_3,r_4))\Big| &\le C(n,\eta_0,\eta)\int_{\Sigma_H\times(a,b)} r^{n-1}\big(u^2+\Lambda\,r\,|u|\big) \\ &\le C(n,\eta_0,\eta)\Big\{(\Lambda\,b)^2\,(b^n-a^n) + \int_{\Sigma_H\times(a,b)} r^{n-1}\,u^2\Big\}\,.\end{aligned}$$

Thanks to (2.42) we can apply (2.38) in Theorem 2.9 with (a,b) and (r_1,r_2) to find

$$\int_{\Sigma_H\times(a,b)} r^{n-1}\,u^2 \le C(n,\eta_0,\eta)\Big\{(\Lambda\,r_2)^2\,(r_2^n-r_1^n) + \int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\Big\}\,.$$

We find (2.43) thanks to (2.41) and $(\Lambda\,b)^2\,(b^n-a^n)\le(\Lambda\,r_2)^2\,r_2^n$. ☐

2.5. **Proof of the mesoscale flatness criterion.** As a final preliminary result to the proof of Theorem 2.3, we prove the following lemma, where Allard's regularity theorem is combined with a compactness argument to provide the basic graphicality criterion used throughout the iteration. The statement should be compared to [AA81, Lemma 5.7].

Lemma 2.11 (Graphicality lemma)**.** *Let $n \ge 2$. For every $\sigma > 0$, $\Gamma \ge 0$, and $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2) \subset\subset (0,1)$ with $\lambda_1 \ge 1/32$, there are positive constants ε_1 and M_1, depending only on n, σ, Γ, λ_1, λ_2, λ_3, and λ_4 with the following property.*

If $\Lambda \ge 0$, $R \in (0, 1/\Lambda)$, $V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda)$,

$$\|\mathrm{bd}_V\|(\partial B_R) \le \Gamma\, R^{n-1}\,, \qquad \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \le \Gamma\,,$$

and there exists $r > 0$ such that,

$$\max\{M_1, 64\}\, R \le r \le \frac{\varepsilon_1}{\Lambda}\,, \tag{2.46}$$

$$|\delta_{V,R,\Lambda}(r)| \le \varepsilon_1\,, \tag{2.47}$$

$$\|V\|(A^{\lambda_4 r}_{\lambda_3 r}) > 0\,, \tag{2.48}$$

and if, for some $K \in \mathcal{H}$, we have

$$\frac{1}{r^n} \int_{A^{\lambda_2 r}_{\lambda_1 r}} \omega_K^2 \, d\|V\| \le \varepsilon_1\,, \tag{2.49}$$

then there exists $u \in \mathcal{X}_\sigma(\Sigma_K, r/32, r/2)$ such that

$$V \text{ corresponds to } \Sigma_K(u, r/32, r/2) \text{ on } A^{r/2}_{r/32}\,.$$

Proof. We argue by contradiction. Should the lemma be false, then we could find $\sigma > 0$, $\Gamma \ge 0$, $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2) \subset\subset (0,1)$ with $\lambda_1 \ge 1/32$, $K_j \in H$, positive numbers R_j, $\Lambda_j < 1/R_j$, r_j, and $W_j \in \mathcal{V}_n(\Lambda_j, R_j, 1/\Lambda_j)$ such that

$$\|W_j\|\big(A^{\lambda_4 r_j}_{\lambda_3 r_j}\big) > 0\,, \qquad \frac{\|\mathrm{bd}_{W_j}\|(\partial B_{R_j})}{R_j^{n-1}} \le \Gamma\,, \qquad \sup_{\rho \in (R_j, 1/\Lambda_j)} \frac{\|W_j\|(B_\rho \setminus B_{R_j})}{\rho^n} \le \Gamma\,,$$

$$\lim_{j\to\infty} \max\Big\{\rho_j = \frac{R_j}{r_j}\,,\ r_j\,\Lambda_j\,,\ \delta_{W_j,R_j,\Lambda_j}(r_j)\,,\ \frac{1}{r_j^n}\int_{A^{\lambda_2 r_j}_{\lambda_1 r_j}} \omega_{K_j}^2\, d\|W_j\|\Big\} = 0\,,$$

such that there exists no $u \in \mathcal{X}_\sigma(\Sigma_{K_j}, r_j/32, r_j/2)$ with the property that

$$W_j \text{ corresponds to } \Sigma_{K_j}(u, r_j/32, r_j/2) \text{ on } A^{r_j/2}_{r_j/32}\,.$$

Setting $V_j = W_j/r_j$, we would then find that no $u \in \mathcal{X}_\sigma(\Sigma_{K_j}, 1/32, 1/2)$ exists such that

$$V_j \text{ corresponds to } \Sigma_{K_j}(u, 1/32, 1/2) \text{ on } A^{1/2}_{1/32}\,,$$

despite the fact that each $V_j \in \mathcal{V}_n(r_j\,\Lambda_j, \rho_j, 1/(r_j\,\Lambda_j))$ satisfies

$$\|V_j\|(A^{\lambda_4}_{\lambda_3}) > 0\,, \quad \frac{\|\mathrm{bd}_{V_j}\|(\partial B_{\rho_j})}{\rho_j^{n-1}} \le \Gamma\,, \qquad \sup_{\rho \in (\rho_j, 1/(\Lambda_j\, r_j))} \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^n} \le \Gamma\,,$$

$$\lim_{j\to\infty} \max\Big\{\delta_{V_j,\rho_j,r_j\,\Lambda_j}(1)\,,\ \int_{A^{\lambda_2}_{\lambda_1}} \omega_{K_j}^2\, d\|V_j\|\Big\} = 0\,. \tag{2.50}$$

Clearly we can find $K \in \mathcal{H}$ such that, up to extracting subsequences and as $j \to \infty$, $K_j \cap B_1 \to K \cap B_1$ in $L^1(\mathbb{R}^{n+1})$. Similarly, by (2.50), we can find an n-dimensional integer rectifiable varifold V such that $V_j \rightharpoonup V$ as varifolds in $B_1 \setminus \{0\}$. Since the bound on the distributional mean curvature of V_j on $B_{1/(\Lambda_j\, r_j)} \setminus \overline{B}_{\rho_j}$ is $r_j\,\Lambda_j$, and since $\rho_j \to 0^+$ and $r_j\,\Lambda_j \to 0^+$, it also follows that V is stationary in $B_1 \setminus \{0\}$, and thus, by a standard argument and since $n \geq 2$, on B_1. By $\|V_j\|(A_{\lambda_3}^{\lambda_4}) > 0$, for every j there is $x_j \in A_{\lambda_3}^{\lambda_4} \cap \operatorname{spt} V_j$, so that, up to extracting subsequences, $x_j \to x_0$ as $j \to \infty$ for some

$$x_0 \in \overline{A}_{\lambda_3}^{\lambda_4} \cap \operatorname{spt} V\,.$$

By $(\lambda_3, \lambda_4) \subset\subset (\lambda_1, \lambda_2)$ we can find $\rho > 0$ such that $B_\rho(x_0) \subset A_{\lambda_1}^{\lambda_2}$, and conclude

$$\|V\|(A_{\lambda_1}^{\lambda_2}) \geq \|V\|(B_\rho(x_0)) \geq \omega_n\, \rho^n > 0\,,$$

thus proving that $V \llcorner A_{\lambda_1}^{\lambda_2} \neq \emptyset$. This last fact, combined with $\omega_K = 0$ on $(\operatorname{spt} V) \cap A_{\lambda_1}^{\lambda_2}$, allows to use the constancy theorem [Sim83a, Theorem 41.1] to deduce that

$$A_{\lambda_1}^{\lambda_2} \cap \operatorname{spt} V = A_{\lambda_1}^{\lambda_2} \cap K\,.$$

Since V is stationary in B_1, we conclude that $B_1 \cap K \subset B_1 \cap \operatorname{spt} V$, so that, in particular,

$$\|V\|(B_1) \geq \omega_n\,.$$

At the same time, since $\|\mathrm{bd}_{V_j}\|(\partial B_{\rho_j}) \leq \Gamma\, \rho_j^{n-1}$ and $\|V_j\|(B_\rho \setminus B_{\rho_j}) \leq \Gamma\, \rho^n$ for every $\rho \in (\rho_j, 1/(\Lambda_j\, r_j)) \supset (\rho_j\, 1)$, by (2.50) we have

$$\begin{aligned}
\omega_n &= \lim_{j\to\infty} \|V_j\|(B_1 \setminus B_{\rho_j}) - \frac{\rho_j}{n}\, \|\delta V_j\|(\partial B_{\rho_j}) + \Lambda_j\, r_j \int_{\rho_j}^1 \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^n}\, d\rho \\
&\geq \|V\|(B_1) - \Gamma \limsup_{j\to\infty} \Big(\rho_j^n + \Lambda_j\, r_j\Big) = \|V\|(B_1)\,,
\end{aligned}$$

so that $\|V\|(B_1) = \omega_n$ and thus $B_1 \cap K = B_1 \cap (\mathrm{spt} V)$. By Allard's regularity theorem and by $V_j \rightharpoonup V$ as $j \to \infty$ we deduce the existence of a sequence $\{u_j\}_j$, with $u_j \in \mathcal{X}_{\sigma_j}(\Sigma_K, 1/32, 1/2)$ for some $\sigma_j \to 0$ as $j \to \infty$, such that V_j corresponds to $\Sigma_K(u_j, 1/32, 1/2)$ in $A_{1/32}^{1/2}$ for j large enough. As soon as j is large enough to give $\sigma_j < \sigma$, we have reached a contradiction. $\square$

Proof of Theorem 2.3. We start by imposing some constraints on the constants ε_0 and M_0 appearing in the statement. For the finite set

$$J = \Big\{\Big(\frac{1}{3}, \frac{1}{6}\Big), \Big(\frac{2}{3}, \frac{1}{3}\Big)\Big\} \subset \big\{(\eta_0, \eta) : \eta_0 > \eta > 0\big\}\,, \tag{2.51}$$

we let $\sigma_0 = \sigma_0(n)$ such that Lemma 2.8-(ii), Theorem 2.9, and Theorem 2.10-(ii,iii) hold for every $(\eta_0, \eta) \in J$, and such that

$$\sigma_0 \leq \frac{\sigma_1}{C_0} \qquad \text{for } \sigma_1(n) \text{ as in (2.22), and } C_0(n) \text{ as in Lemma 2.8-(ii)}\,; \tag{2.52}$$

we shall henceforth assume, without loss of generality, that

$$\sigma < \sigma_0\,.$$

Moreover, for ε_1 and M_1 as in Lemma 2.11, we let

$$M_0 \geq \max\Big\{M_1\Big(n, \sigma, \Gamma, \Big(\frac{1}{8}, \frac{1}{2}\Big), \Big(\frac{1}{6}, \frac{1}{4}\Big)\Big), M_1\Big(n, \sigma, \Gamma, \Big(\frac{1}{16}, \frac{1}{8}\Big), \Big(\frac{3}{32}, \frac{7}{64}\Big)\Big)\Big\}\,, \tag{2.53}$$

$$\varepsilon_0 \leq \min\Big\{\varepsilon_1\Big(n, \sigma, \Gamma, \Big(\frac{1}{8}, \frac{1}{2}\Big), \Big(\frac{1}{6}, \frac{1}{4}\Big)\Big), \varepsilon_1\Big(n, \sigma, \Gamma, \Big(\frac{1}{16}, \frac{1}{8}\Big), \Big(\frac{3}{32}, \frac{7}{64}\Big)\Big)\Big\}\,. \tag{2.54}$$

We also assume that ε_0 is smaller than the n-dependent ε_0's appearing in Lemma 2.6 and Lemma 2.8.

Let us now recall that, by assumption, $V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda)$ is such that

$$\|\mathrm{bd}_V\|(\partial B_R) \leq \Gamma\, R^{n-1}\,, \qquad \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \leq \Gamma\,; \tag{2.55}$$

in particular, by Theorem 2.10-(i),

$$\delta_{V,R,\Lambda} \text{ is decreasing on } (R, 1/\Lambda)\,. \tag{2.56}$$

Moreover, we are assuming the existence of s with $\max\{64, M_0\}\,R < s < \varepsilon_0/4\,\Lambda$ such that

$$|\delta_{V,R,\Lambda}(s/8)| \le \varepsilon_0\,, \tag{2.57}$$

$$R_* = \sup\Big\{\rho \ge \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \ge -\varepsilon_0\Big\} \ge 4\,s\,, \tag{2.58}$$

$$\frac{1}{s^n}\int_{A_{s/8}^{s/2}} \omega_H(y)^2\,d\|V\|_y \le \varepsilon_0\,, \tag{2.59}$$

$$\|V\|(A_{s/6}^{s/4}) > 0\,. \tag{2.60}$$

By (2.56), (2.57) and (2.58) we have

$$|\delta_{V,R,\Lambda}(r)| \le \varepsilon_0\,, \qquad \forall r \in (s/8, R_*)\,. \tag{2.61}$$

By (2.53), (2.54), (2.55), (2.59), (2.60) and (2.61) we can apply Lemma 2.11 with $(\lambda_1,\lambda_2) = (1/8, 1/2)$, $(\lambda_3,\lambda_4) = (1/6, 1/4)$, and $r = s$. Setting $H_0 = H$, we thus find $u_0 \in \mathcal{X}_\sigma(\Sigma_{H_0}, s/32, s/2)$ such that

$$V \text{ corresponds to } \Sigma_{H_0}(u_0, s/32, s/2) \text{ on } A_{s/32}^{s/2}\,, \tag{2.62}$$

and letting

$$T_0 = \frac{1}{(s/4)^n}\int_{s/8}^{s/4} r^{n-1}\,dr \int_{\Sigma_{H_0}} [u_0]_r^2\,,$$

we have, thanks to (2.22), (2.52), (2.62), and (2.59),

$$T_0 = \frac{1}{(s/4)^n}\int_{s/8}^{s/4} r^{n-1}\,dr \int_{\Sigma_{H_0}} [u_0]_r^2 \le \frac{C(n)}{s^n}\int_{A_{s/8}^{s/4}} \omega_H^2\,d\|V\| \le C(n)\,\varepsilon_0\,. \tag{2.63}$$

Let

$$s_j = 2^{j-3}\,s\,, \qquad j \in \mathbb{N}\,.$$

By (2.58) and by $s < \varepsilon_0/4\,\Lambda$ there exists $N \in \{j \in \mathbb{N} : j \ge 2\} \cup \{+\infty\}$ such that

$$\{0, 1, .., N\} = \Big\{j \in \mathbb{N} : 8\,s_j \le S_* = \min\big\{R_*, \frac{\varepsilon_0}{\Lambda}\big\}\Big\}\,. \tag{2.64}$$

Notice that if $\Lambda > 0$ then it must be $N < \infty$. We are now in the position to make the following:

Claim: There exist $\tau = \tau(n) \in (0,1)$ and $\{(H_j, u_j)\}_{j=0}^{N-2}$ with $H_j \in \mathcal{H}$ and $u_j \in \mathcal{X}_\sigma(\Sigma_{H_j}, s/32, 4\,s_j)$, such that, setting,

$$T_j = \frac{1}{s_{j+1}^n}\int_{s_j}^{s_{j+1}} r^{n-1}\,dr \int_{\Sigma_{H_j}} [u_j]_r^2\,,$$

we have, for every $j = 0, ..., N-2$,

$$V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4\,s_j) \text{ on } A_{s/32}^{4\,s_j}\,, \tag{2.65}$$

$$|\delta_{V,R,\Lambda}(s_j)| \le \varepsilon_0\,, \tag{2.66}$$

$$T_j \le C(n)\,\varepsilon_0\,, \tag{2.67}$$

and, for every $j = 1, ..., N-2$,

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \le C(n)\,T_{j-1}\,, \tag{2.68}$$

$$\delta_{V,R,\Lambda}(s_j) \le \tau\left\{\delta_{V,R,\Lambda}(s_{j-1}) + (1+\Gamma)\,\Lambda\,s_{j-1}\right\}, \tag{2.69}$$

$$T_j \le C(n)\Big\{\delta_{V,R,\Lambda}(s_{j-1}) - \delta_{V,R,\Lambda}(s_{j+2}) + \Lambda\,s_{j-1}\Big\}\,. \tag{2.70}$$

Proof of the claim: We argue by induction. Clearly $(2.65)_{j=0}$, $(2.66)_{j=0}$ and $(2.67)_{j=0}$ are, respectively, (2.62), (2.57) and (2.63). This concludes the proof of the claim if $N=2$, therefore we shall assume $N\ge 3$ for the rest of the argument.

To set up the inductive argument, we consider $\ell\in\mathbb{N}$ such that: either $\ell=0$; or $1\le\ell\le N-3$ and (2.65), (2.66), and (2.67) hold for $j=0,...,\ell$, and (2.68), (2.69) and (2.70) hold for $j=1,...,\ell$; and prove that all the conclusions of the claim holds with $j=\ell+1$.

The validity of $(2.66)_{j=\ell+1}$ is of course immediate from (2.61) and (2.64). Also, after proving $(2.70)_{j=\ell+1}$, we will be able to combine with $(2.66)_{j=\ell+1}$ and (2.64) to deduce $(2.67)_{j=\ell+1}$. We now prove, in the order, (2.68), (2.65), (2.69), and (2.70) with $j=\ell+1$.

To prove $(2.68)_{j=\ell+1}$: Let $[a,b]\subset\subset(s_\ell,s_{\ell+1})$ with $(b-a)=(s_{\ell+1}-s_\ell)/2$, so that

$$\frac{1}{C(n)}\min_{r\in[a,b]}\int_{\Sigma_{H_\ell}}[u_\ell]_r^2\le\frac{1}{s_{\ell+1}^n}\int_{s_\ell}^{s_{\ell+1}}r^{n-1}\,dr\int_{\Sigma_{H_\ell}}[u_\ell]_r^2=T_\ell\,. \tag{2.71}$$

Keeping in mind $(2.65)_{j=\ell}$, we can apply Lemma 2.8-(ii) with $(r_1,r_2)=(s/32,4\,s_\ell)$ and $[a,b]$ to find $H_{\ell+1}\in\mathcal{H}$, $u_{\ell+1}\in\mathcal{X}_{C_0\,\sigma_0}(\Sigma_{H_{\ell+1}},s/32,4\,s_\ell)$ (with C_0 as in Lemma 2.8-(ii)) and

$$s_\ell^*\in[a,b]\subset(s_\ell,s_{\ell+1})\,, \tag{2.72}$$

such that, thanks also to (2.71),

$$\Sigma_{H_\ell}(u_\ell,s/32,4\,s_\ell)=\Sigma_{H_{\ell+1}}(u_{\ell+1},s/32,4\,s_\ell)\,, \tag{2.73}$$
$$E^0_{\Sigma_{H_{\ell+1}}}([u_{\ell+1}]_{s_\ell^*})=0\,, \tag{2.74}$$
$$|\nu_{H_\ell}-\nu_{H_{\ell+1}}|^2\le C(n)\,T_\ell\,, \tag{2.75}$$
$$\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\le C(n)\left(T_\ell+\int_{\Sigma_{H_\ell}}[u_\ell]_r^2\right)\,,\qquad\forall r\in(s/32,4\,s_\ell)\,. \tag{2.76}$$

In particular, (2.75) is $(2.68)_{j=\ell+1}$.

To prove $(2.65)_{j=\ell+1}$: Notice that (2.73) and $(2.65)_{j=\ell}$ do not imply $(2.65)_{j=\ell+1}$, since, in $(2.65)_{j=\ell+1}$, we are claiming the graphicality of V inside $A^{4\,s_{\ell+1}}_{s/32}$ (which is strictly larger than $A^{4\,s_\ell}_{s/32}$), and we are claiming that $u_{\ell+1}$ has C^1-norm bounded by σ, and not just by $C_0\,\sigma_0$ (with C_0 as in Lemma 2.8-(ii)).

We want to apply Lemma 2.11 with

$$r=8\,s_{\ell+1}\,,\qquad(\lambda_1,\lambda_2)=\left(\frac{1}{16},\frac{1}{8}\right),\qquad(\lambda_3,\lambda_4)=\left(\frac{3}{32},\frac{7}{64}\right),\qquad K=H_{\ell+1}\,. \tag{2.77}$$

We check the validity of (2.46), (2.47), (2.49) and (2.48) for these choices of r, λ_1, λ_2, λ_3, λ_4 and K.

Since $r=8\,s_{\ell+1}\ge s\ge\max\{M_0,64\,R\}$, and since (2.64) and $\ell+1\le N$ give $r=8\,s_{\ell+1}\le\varepsilon_0/\Lambda$, we deduce the validity of (2.46) with $r=8\,s_{\ell+1}$. The validity of (2.47) with $r=8\,s_{\ell+1}$ is immediate from (2.61) by our choice (2.54) of ε_0. Next we notice that

$$\|V\|(A^{\lambda_4\,r}_{\lambda_3\,r})=\|V\|(A^{7\,[8\,s_{\ell+1}]/64}_{3\,[8\,s_{\ell+1}]/32})=\|V\|(A^{7\,s_\ell/4}_{3\,s_\ell/2})>0$$

thanks to $(2.65)_{j=\ell}$, so that (2.48) holds for r, λ_3 and λ_4 as in (2.77). Finally, by (2.22) (which can be applied to $u_{\ell+1}$ thanks to (2.52)), (2.73) and $(2.65)_{j=\ell}$, and, then by (2.76), we have

$$
\begin{aligned}
\frac{1}{r^n}\int_{A_{\lambda_1 r}^{\lambda_2 r}} \omega_{H_{\ell+1}}^2\, d\|V\| &\le \frac{C(n)}{s_{\ell+1}^n}\int_{s_\ell}^{s_{\ell+1}} r^{n-1}\, dr \int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]_r^2 \\
&\le C(n)\, T_\ell + \frac{C(n)}{s_{\ell+1}^n}\int_{s_\ell}^{s_{\ell+1}} r^{n-1}\, dr \int_{\Sigma_{H_\ell}} [u_\ell]_r^2 \\
&\le C(n)\, T_\ell \le C(n)\, \varepsilon_0 \,,
\end{aligned}
$$

where in the last inequality we have used $(2.67)_{j=\ell}$. Again by our choice (2.54) of ε_0, we deduce that (2.49) holds with r, λ_1 and λ_2 as in (2.77).

We can thus apply Lemma 2.11, and find $v \in \mathcal{X}_\sigma(\Sigma_{H_{\ell+1}}, s_{\ell+1}/4, 4\, s_{\ell+1})$ such that

$$
V \text{ corresponds to } \Sigma_{H_{\ell+1}}(v, s_{\ell+1}/4, 4\, s_{\ell+1}) \text{ on } A_{s_{\ell+1}/4}^{4\, s_{\ell+1}} \,. \tag{2.78}
$$

By (2.73), $(2.65)_{j=\ell}$, and (2.78), $v = u_{\ell+1}$ on $\Sigma_{H_{\ell+1}} \times (s_{\ell+1}/4, 4\, s_\ell)$. We can thus use v to extend $u_{\ell+1}$ from $\Sigma_{H_{\ell+1}} \times (s/32, 4\, s_\ell)$ to $\Sigma_{H_{\ell+1}} \times (s/32, 4\, s_{\ell+1})$, and, thanks to (2.73), $(2.65)_{j=\ell}$ and (2.78), the resulting extension is such that $(2.65)_{j=\ell+1}$ holds.

To prove $(2.69)_{j=\ell+1}$: We set

$$
r_0 = \frac{s_\ell + s_{\ell+1}}{2} \,,
$$

and notice that for $\eta_0 = 1/3$ we have

$$
r_1 = r_0\,(1-\eta_0) = s_\ell \,, \qquad r_2 = r_0\,(1+\eta_0) = s_{\ell+1} \,. \tag{2.79}
$$

For $\eta = 1/6$ we correspondingly set

$$
r_3 = r_0\,(1-\eta) =: s_\ell^- \,, \qquad r_4 = r_0\,(1+\eta) =: s_\ell^+ \,, \tag{2.80}
$$

and notice that $(\eta_0, \eta) \in J$, see (2.51). With the aim of applying Theorem 2.10-(iii) to these radii, we notice that $(2.65)_{j=\ell+1}$ implies that assumption (2.40) holds with $H = H_{\ell+1}$ and $u = u_{\ell+1}$, while, by (2.74), $r = s_\ell^* \in (s_\ell, s_{\ell+1})$ is such that (2.42) holds. Taking into account that $\Lambda\, s_{\ell+1} \le \varepsilon_0 \le 1$ by (2.64), we thus find by (2.43) that

$$
\begin{aligned}
&\frac{1}{s_{\ell+1}^n}\Big|\|V\|\big(B_{s_\ell^+} \setminus B_{s_\ell^-}\big) - \omega_n\left((s_\ell^+)^n - (s_\ell^-)^n\right)\Big| \\
&= \frac{1}{s_{\ell+1}^n}\Big|\mathcal{H}^n(\Sigma_{H_{\ell+1}}(u_{\ell+1}, s_\ell^-, s_\ell^+)) - \mathcal{H}^n(\Sigma_{H_{\ell+1}}(0, s_\ell^-, s_\ell^+))\Big| \\
&\le C(n)\Big\{(\Lambda\, s_{\ell+1})^2 + \Theta_{V,R,\Lambda}(s_{\ell+1}) - \Theta_{V,R,\Lambda}(s_\ell)\Big\}
\end{aligned} \tag{2.81}
$$

where $C(n) = C_0(n, 1/6, 1/3)$ for C_0 as in Theorem 2.10-(iii). Setting for brevity $\delta = \delta_{V,R,\Lambda}$ and $\Theta = \Theta_{V,R,\Lambda}$, and recalling that

$$
\begin{aligned}
r^n\,\delta(r) &= \omega_n\, r^n - \Theta(r)\, r^n \\
&= \omega_n\, r^n - \|V\|(B_r \setminus B_R) - \Lambda\, r^n \int_R^r \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n}\, d\rho + \frac{R\,\|\delta V\|(\partial B_R)}{n}
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{s_\ell^n}\Big|(s_\ell^-)^n\,\delta(s_\ell^-) - (s_\ell^+)^n\,\delta(s_\ell^+)\Big| \le C(n)\Big\{(\Lambda\, s_\ell)^2 + \Theta(s_{\ell+1}) - \Theta(s_\ell)\Big\} \\
&\qquad + C(n)\,\frac{\Lambda}{s_\ell^n}\Big\{(s_\ell^+)^n \int_R^{s_\ell^+} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n}\, d\rho - (s_\ell^-)^n \int_R^{s_\ell^-} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n}\, d\rho\Big\} \\
&\le C(n)\Big\{(\Lambda\, s_\ell)^2 + \Theta(s_{\ell+1}) - \Theta(s_\ell)\Big\} + C(n)\,\Lambda \int_R^{s_\ell^+} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n}\, d\rho \,.
\end{aligned}
$$

By $\Lambda\,s_\ell\le 1$ and since $s_\ell^+\le s_\ell\le\varepsilon_0/8\,\Lambda$ thanks to $\ell<N$, we can use the upper bound $\|V\|(B_\rho\setminus B_R)\le\Gamma\,\rho^n$ with $\rho\in(R,s_\ell^+)\subset(R,1/\Lambda)$, to find that

$$\Big|\frac{(s_\ell^-)^n}{s_\ell^n}\,\delta(s_\ell^-)-\frac{(s_\ell^+)^n}{s_\ell^n}\,\delta(s_\ell^+)\Big|\le C_*(n)\,\big\{\delta(s_\ell)-\delta(s_{\ell+1})\big\}+C_*(n)\,(\Gamma+1)\,\Lambda\,s_\ell\,,$$

for a constant $C_*(n)$ depending on the dimension n only. By rearranging terms we thus find

$$C_*(n)\,\delta(s_{\ell+1})+\frac{(s_\ell^+)^n}{s_\ell^n}\,\delta(s_\ell^+)\le C_*(n)\,\delta(s_\ell)+\frac{(s_\ell^-)^n}{s_\ell^n}\,\delta(s_\ell^-)+C_*(n)\,(1+\Gamma)\,\Lambda\,s_\ell\,.$$

Using the monotonicity of δ on (R,∞) and $(s_\ell^-,s_\ell^+)\subset(s_\ell,s_{\ell+1})$, we conclude that

$$\Big(C_*(n)\,+\frac{(s_\ell^+)^n}{s_\ell^n}\Big)\,\delta(s_{\ell+1})\le\Big(C_*(n)\,+\frac{(s_\ell^-)^n}{s_\ell^n}\Big)\,\delta(s_\ell)+C_*(n)\,(1+\Gamma)\,\Lambda\,s_\ell,.\tag{2.82}$$

We finally notice that by (2.79), (2.80), $\eta_0=1/3$, and $\eta=1/6$, we have

$$\frac{s_\ell^-}{s_\ell}=\frac{r_0\,(1-\eta)}{r_0\,(1-\eta_0)}=\frac{5}{4}\,,\qquad\frac{s_\ell^+}{s_\ell}=2\,\frac{s_\ell^+}{s_{\ell+1}}=2\,\frac{1+\eta}{1+\eta_0}=\frac{7}{4}\,,$$

so that, setting

$$\tau=\tau(n)=\frac{C_*(n)+(5/4)^n}{C_*(n)+(7/4)^n}\,,\qquad\tau_*=\tau_*(n)=\frac{C_*(n)}{C_*(n)+(7/4)^n}<\tau\,,$$

we find

$$\delta(s_{\ell+1})\le\tau\,\big\{\delta(s_\ell)+(1+\Gamma)\,\Lambda\,s_\ell\big\}\,,\tag{2.83}$$

which is $(2.69)_{j=\ell+1}$.

To prove $(2.70)_{j=\ell+1}$: We want to prove

$$\frac{1}{s_{\ell+1}^n}\int_{s_{\ell+1}}^{2\,s_{\ell+1}}r^{n-1}\,dr\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\le C(n)\,\Big\{\delta_{V,R,\Lambda}(s_\ell)-\delta_{V,R,\Lambda}(s_{\ell+3})+\Lambda\,s_\ell\Big\}\,.\tag{2.84}$$

By $(2.65)_{j=\ell+1}$ we know that

$$V\text{ corresponds to }\Sigma_{H_{\ell+1}}(u_{\ell+1},s/32,4\,s_{\ell+1})\text{ on }A_{s/32}^{4\,s_{\ell+1}}\,.\tag{2.85}$$

Let us set

$$\begin{aligned}r_1&=s_\ell=3\,s_\ell-2\,s_\ell\,,\qquad &r_2&=5\,s_\ell=3\,s_\ell+2\,s_\ell\,,\\ r_3&=s_{\ell+1}=3\,s_\ell-s_\ell\,,\qquad &r_4&=2\,s_{\ell+1}=3\,s_\ell+s_\ell\,,\end{aligned}$$

so that (2.34) holds with $r_0=3\,s_\ell$ and $(\eta_0,\eta)=(2/3,1/3)\in J$, see (2.51). Since

$$s_\ell^*\in(s_\ell,s_{\ell+1})\subset(r_1,r_2)$$

by (2.85), (2.74) and $(r_1,r_2)\subset(s/32,4\,s_{\ell+1})$ we can apply (2.38) in Theorem 2.9 to deduce that

$$\int_{s_{\ell+1}}^{2\,s_{\ell+1}}r^{n-1}\int_{\Sigma_{H_{\ell+1}}}[u_{\ell+1}]_r^2\le C(n)\int_{s_\ell}^{5\,s_\ell}r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r u_{\ell+1})_r^2+C(n)\,\Lambda\,(s_\ell)^{n+1}\,.$$

Again by (2.85) we can apply Theorem 2.10-(ii) with $(r_1,r_2)=(s_\ell,8\,s_\ell)$, and find that

$$\begin{aligned}\int_{s_\ell}^{5\,s_\ell}r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r[u_{\ell+1}])_r^2&\le\int_{s_\ell}^{8\,s_\ell}r^{n+1}\int_{\Sigma_{H_{\ell+1}}}(\partial_r[u_{\ell+1}])_r^2\\&\le C(n)\,s_\ell^n\,\Big\{\Theta_{V,R,\Lambda}(8\,s_\ell)-\Theta_{V,R,\Lambda}(s_\ell)\Big\}\\&\le C(n)\,s_\ell^n\,\big\{\delta_{V,R,\Lambda}(s_\ell)-\delta_{V,R,\Lambda}(s_{\ell+3})\big\}\,.\end{aligned}$$

The last two estimates combined give (2.84). This completes the proof of the **claim**.

Proof of statement (i): We assume $S_* < \infty$ (that is either $\Lambda > 0$ or $R_* < \infty$). In this case N (as defined in (2.64)) is finite, with

$$2^N \le \frac{S_*}{s} < 2^{N+1}\,.$$

By $(2.65)_{j=N-2}$, V corresponds to $\Sigma_{H_{N-2}}(u_{N-2}, s/32, 4\,s_{N-2})$ on $A^{4\,s_{N-2}}_{s/32}$, and since

$$4\,s_{N-2} = 4\,2^{N-2-3}\,s = \frac{2^{N+1}\,s}{16} > \frac{S_*}{16}$$

we deduce (2.11) with $K = H_{N-2}$ and $u = u_{N-2}$.

Proof of statement (ii): We assume $\Lambda = 0$ and $R_* = +\infty$ (that is, $\delta_{V,R,0} \ge -\varepsilon_0$ on $(s/8, \infty)$). In this case, we have $N = +\infty$, and the iteration procedure set up in the claim actually defines a sequence $\{(H_j, u_j)\}_{j=0}^\infty$ with $u_j \in \mathcal{X}_\sigma(\Sigma_{H_j}, s/32, 4\,s_j)$ and

$$V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4\,s_j) \text{ on } A^{4\,s_j}_{s/32} \tag{2.86}$$

for every $j \ge 0$. By compactness of $\mathbb{S}^n$, we can find $K \in \mathcal{H}$ such that, along a subsequence $\{H_{j(k)}\}_k$, we have $\varepsilon_k = |\nu_K - \nu_{H_{j(k)}}| \to 0$ as $k \to \infty$. In particular, for k large enough, we have $\varepsilon_k < \varepsilon_0$, and thus, by Lemma 2.8-(i) and by (2.86) we can find $v_k \in \mathcal{X}_{C(n)\,(\sigma+\varepsilon_k)}(\Sigma_K; s/32, 4\,s_{j(k)})$ such that

$$V \text{ corresponds to } \Sigma_K(v_{j(k)}, s/32, 4\,s_{j(k)}) \text{ on } A^{4\,s_{j(k)}}_{s/32}\,. \tag{2.87}$$

By (2.87), $v_{j(k)+1} = v_{j(k)}$ on $\Sigma_K \times (s/32, 4\,s_{j(k)})$. Since $s_{j(k)} \to \infty$ we have thus found $u \in \mathcal{X}_{C(n)\,\sigma}(\Sigma_K; s/32, \infty)$ such that V corresponds to $\Sigma_K(u, s/32, \infty)$ on $A^\infty_{s/32}$. This proves statement (ii).

Proof of statement (iii): We finally assume that $\Lambda = 0$ and that

$$\delta(r) \ge 0\,, \qquad \forall r \ge \frac{s}{8}\,, \tag{2.88}$$

where we have set for brevity $\delta = \delta_{V,R,0}$. As in the case of statement (ii) we have $N = +\infty$, and there is a sequence $\{(H_j, u_j)\}_{j=0}^\infty$ satisfying

$$V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4\,s_j) \text{ on } A^{4\,s_j}_{s/32}\,, \tag{2.89}$$

for every $j \ge 0$, as well as

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \;\le\; C(n)\,T_{j-1}\,, \qquad \text{if } j \ge 1\,, \tag{2.90}$$

$$\delta(s_j) \;\le\; \begin{cases} \varepsilon_0\,, & \text{if } j = 0\,,\\ \tau\,\delta(s_{j-1})\,, & \text{if } j \ge 1\,,\end{cases} \tag{2.91}$$

$$T_j \;\le\; \begin{cases} C(n)\,\varepsilon_0\,, & \text{if } j = 0\,,\\ C(n)\,\delta(s_{j-1})\,, & \text{if } j \ge 1\,.\end{cases} \tag{2.92}$$

Notice that, in asserting the validity of (2.92) with $j \ge 1$, we have used (2.88) to estimate $-\delta(s_{j+2}) \le 0$ in $(2.70)_j$. By iterating (2.91) we find

$$\delta(s_j) \le \tau^j\,\delta(s/8) \le \tau^j\,\varepsilon_0\,, \qquad \forall j \ge 1\,, \tag{2.93}$$

which, combined with (2.92) and (2.90), gives, for every $j \ge 1$,

$$T_j \le C(n)\,\min\{1, \tau^{j-1}\}\,\delta(s/8) \le C(n)\,\tau^j\,\delta(s/8)\,, \tag{2.94}$$

$$|\nu_{H_j} - \nu_{H_{j-1}}|^2 \le C(n)\,\min\{1, \tau^{j-2}\}\,\delta(s/8) \le C(n)\,\tau^j\,\delta(s/8)\,, \tag{2.95}$$

thanks also to $\tau=\tau(n)$ and, again, to (2.88). By (2.95), for every $j\ge 0$, $k\ge 1$, we have

$$|\nu_{H_{j+k}}-\nu_{H_j}|\le C(n)\,\sqrt{\delta(s/8)}\sum_{h=1}^{k+1}|\nu_{H_{j-1+h}}-\nu_{H_{j-2+h}}|\le C(n)\,\sqrt{\delta(s/8)}\sum_{h=1}^{k+1}\big(\sqrt{\tau}\big)^{j-1+h}\,,$$

so that there exists $K\in\mathcal{H}$ such that

$$|\nu_K-\nu_{H_j}|^2\le C(n)\,\tau^j\,\delta(s/8)\,,\qquad\forall j\ge 1\,,\tag{2.96}$$

and, in particular, $|\nu_{H_j}-\nu_K|\to 0$ as $j\to\infty$. By arguing as in the proof of statement (ii) we see find $u\in\mathcal{X}_{\sigma'}(\Sigma_K,s/32,\infty)$ for every $\sigma'>\sigma$ such that, for every j large enough,

$$\Sigma_K(u,s/32,4\,s_j)=\Sigma_{H_j}(u_j,s/32,4\,s_j)\,,$$

and hence, by (2.65),

$$V\text{ corresponds to }\Sigma_K(u,s/32,\infty)\text{ on }\mathbb{R}^{n+1}\setminus B_{s/32}\,,\tag{2.97}$$

which is (2.11) with $S_*=+\infty$.

To prove (2.12), we notice that if $r\in(s_j,s_{j+1})$ for some $j\ge 1$, then, setting $\tau=(1/2)^\alpha$ (i.e., $\alpha=\log_{1/2}(\tau)\in(0,1)$) and noticing that $r/s\le 2^{j+1-3}$, by (2.56) and (2.93) we have

$$\delta(r)\le\delta(s_j)\le\tau^j\,\delta\Big(\frac{s}{8}\Big)=\Big(\frac{1}{2^j}\Big)^\alpha\,\delta\Big(\frac{s}{8}\Big)=4^{-\alpha}\,\Big(\frac{1}{2^{j-2}}\Big)^\alpha\,\delta\Big(\frac{s}{8}\Big)\le C(n)\,\Big(\frac{s}{r}\Big)^\alpha\,\delta\Big(\frac{s}{8}\Big)\,,\tag{2.98}$$

where in the last inequality (2.88) was used again; this proves (2.12). To prove (2.13), we recall that $\omega_K(y)=\arctan(|\nu_K\cdot\hat y|/|\mathbf{p}_K\,\hat y|)$, provided arctan is defined on $\mathbb{R}\cup\{\pm\infty\}$, and where $\hat y=y/|y|$, $y\ne 0$. Now, by (2.97),

$$y=|y|\,\frac{\mathbf{p}_K\,\hat y+u(\mathbf{p}_K\,\hat y,|y|)\,\nu_K}{\sqrt{1+u(\mathbf{p}_K\,\hat y,|y|)^2}}\,,\qquad\forall y\in(\mathrm{spt}\,V)\setminus B_{s/32}\,,$$

so that $|\mathbf{p}_K\,\hat y|\ge 1/2$ for $y\in(\mathrm{spt}V)\setminus B_{s/32}$; therefore, by (2.96), up to further decrease the value of ε_0, and recalling $\delta(s/8)\le\varepsilon_0$, we conclude that

$$|\mathbf{p}_{H_j}\,\hat y|\ge\frac{1}{3}\,,\qquad\forall y\in(\mathrm{spt}V)\setminus B_{s/32}\,,\tag{2.99}$$

for every $j\in\mathbb{N}\cup\{+\infty\}$ (if we set $H_\infty=K$). By (2.99) we easily find

$$|\omega_K(y)-\omega_{H_j}(y)|\le C\,|\nu_{H_j}-\nu_K|\,,\qquad\forall y\in(\mathrm{spt}V)\setminus B_{s/32}\,,\forall j\ge 1\,,$$

from which we deduce that, if $j\ge 1$ and $r\in(s_j,s_{j+1})$, then

$$\begin{aligned}\frac{1}{r^n}\int_{A_r^{2\,r}}\omega_K^2\,d\|V\|&\le C(n)\Big\{\frac{1}{s_j^n}\int_{A_{s_j}^{s_{j+1}}}\omega_K^2\,d\|V\|+\frac{1}{s_{j+1}^n}\int_{A_{s_{j+1}}^{s_{j+2}}}\omega_K^2\,d\|V\|\Big\}\\&\le C(n)\Big\{\frac{1}{s_j^n}\int_{A_{s_j}^{s_{j+1}}}\omega_{H_j}^2\,d\|V\|+\frac{1}{s_{j+1}^n}\int_{A_{s_{j+1}}^{s_{j+2}}}\omega_{H_{j+1}}^2\,d\|V\|\Big\}\\&\quad+C(n)\,\Gamma\,\Big\{|\nu_K-\nu_{H_j}|^2+|\nu_K-\nu_{H_{j+1}}|^2\Big\}\,,\end{aligned}$$

where we have also used (2.55) to bound $\|V\|(A_\rho^{2\,\rho})\le\Gamma\,(2\,\rho)^n$ with $\rho=s_j,s_{j+1}\in(R,1/\Lambda)$. By (2.89) we can exploit (2.22) on the first two integrals, so that taking (2.96) into account we find that, if $j\ge 1$ and $r\in(s_j,s_{j+1})$, then

$$\frac{1}{r^n}\int_{A_r^{2\,r}}\omega_K^2\,d\|V\|\le C(n)\{T_j+T_{j+1}\}+C(n)\,\Gamma\,\tau^j\,\delta(s/8)\le C(n)\,(1+\Gamma)\,\tau^j\,\delta(s/8)\,,$$

where in the last inequality we have used (2.94). Since $\tau^j\le C(n)\,(s/r)^\alpha$, we conclude the proof of (2.13), and thus, of the theorem. □

3. Application of quantitative isoperimetry

In this section we prove the existence of minimizers in $\psi_W(v)$ and then apply quantitative isoperimetry to prove Theorem 1.7-(i) and some of the estimates in Theorem 1.7-(iv).

Theorem 3.1. *If $W \subset \mathbb{R}^{n+1}$ is compact, and $v > 0$, then $\psi_W(v)$ admits minimizers. Moreover, there exist positive constants v_0, C_0, Λ_0, $s_0 \in (0,1)$, depending on n and W only, and a function $R_0(v)$ depending on n and W only, with $R_0(v) \to 0^+$ and $R_0(v)\, v^{1/(n+1)} \to \infty$ as $v \to \infty$, such that, if $v > v_0$ and E_v is a minimizer of $\psi_W(v)$, then:*

(i): *E_v is a* $(\Lambda_0/v^{1/(n+1)}, s_0\, v^{1/(n+1)})$**-perimeter minimizer with free boundary in** *Ω, that is*

$$P(E_v; \Omega \cap B_r(x)) \le P(F; \Omega \cap B_r(x)) + \frac{\Lambda_0}{v^{1/(n+1)}} \left| E_v \Delta F \right| , \tag{3.1}$$

holds whenever $F \subset \Omega = \mathbb{R}^{n+1} \setminus W$ with $E_v \Delta F \subset\subset B_r(x)$ and $r < s_0\, v^{1/(n+1)}$;

(ii)*: E_v determines $x \in \mathbb{R}^{n+1}$ such that*

$$\frac{|E_v \Delta B^{(v)}(x)|}{v} \le \frac{C_0}{v^{1/[2(n+1)]}} , \tag{3.2}$$

and, assuming in addition that $\mathcal{R}(W) > 0$, E_v also determines $u \in C^\infty(\partial B^{(1)})$ such that

$$\begin{aligned}&(\partial E_v) \setminus B_{R_0\, v^{1/(n+1)}} \\ &= \Big\{ y + v^{1/(n+1)} u\Big(\frac{y-x}{v^{1/(n+1)}}\Big)\, \nu_{B^{(v)}(x)}(y) : y \in \partial B^{(v)}(x) \Big\} \setminus B_{R_0\, v^{1/(n+1)}} \, ;\end{aligned} \tag{3.3}$$

(iii): *finally, if $\mathcal{R}(W) > 0$, then*

$$\lim_{v\to\infty} \sup_{E_v} \max \Big\{ \Big| \frac{|x|}{v^{1/(n+1)}} - \frac{1}{\omega_{n+1}^{1/(n+1)}} \Big| , \|u\|_{C^1(\partial B^{(1)})} \Big\} = 0 , \tag{3.4}$$

where x and u depend on E_v as in (3.2) *and* (3.3)*, and where E_v ranges among all minimizers of $\psi_W(v)$.*

Remark 3.2 (Improved convergence)**.** In the proof of Theorem 3.1 we will make repeated use of the following well–known (see, e.g. [FM11, CL12, FFM+15]) fact: *If Ω is an open set, $\Lambda \ge 0$, $s > 0$, if $\{F_j\}_j$ are* (Λ, s)**-perimeter minimizers in** Ω*, i.e. if it holds that*

$$P(F_j; B_r(x)) \le P(G_j; B_r(x)) + \Lambda\, |F_j \Delta G_j| , \tag{3.5}$$

whenever $G_j \Delta F_j \subset\subset B_r(x) \subset\subset \Omega$ and $r < s$, and if F is an open set with smooth boundary in Ω such that $F_j \to F$ in $L^1_{\rm loc}(\Omega)$ as $j \to \infty$, then for every $\Omega' \subset\subset \Omega$ we can find $j(\Omega')$ such that, if $j \ge j(\Omega')$, then

$$(\partial F_j) \cap \Omega' = \Big\{ y + u_j(y)\, \nu_F(y) : y \in \Omega \cap \partial F \Big\} \cap \Omega'$$

for a sequence $\{u_j\}_j \subset C^1(\Omega \cap \partial F)$ with $\|u_j\|_{C^1(\Omega\cap\partial F)} \to 0$ as $j \to \infty$. (A proof is quickly obtained by combining [CLM16, Lemma 4.4] with a covering argument.) Notice also the terminology used in (3.1) and (3.5): when we add "with free boundary", the "localizing balls" $B_r(x)$ are not required to be compactly contained in Ω, and the perimeters are computed in $B_r(x) \cap \Omega$. This allows competitors in (3.1) to differ from the minimizer near $\partial\Omega$, while being constrained to be subsets of Ω. In particular, (Λ, s)-perimeter minimality as defined in (3.5) is weaker than its "with free boundary" variant defined in (3.1).

Proof of Theorem 3.1. Step one: We prove the existence of minimizers. Since W is compact, suitable translations of $B^{(v)}$ are competitors in $\psi_W(v)$, and therefore we can find a sequence $\{E_j\}_j$ with $E_j \subset \Omega$, $|E_j| = v$, and

$$P(E_j; \Omega) \le \min \Big\{ P(B^{(v)}), P(F; \Omega) \Big\} + \frac{1}{j} , \tag{3.6}$$

for every $F \subset \Omega$ with $|F| = v$. By standard compactness theorems for sets of finite perimeter we find that, up to extracting subsequences, $E_j \to E$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$ with $P(E;\Omega) \le \liminf_{j\to\infty} P(E_j;\Omega)$, where $E \subset \Omega$ and $|E| \le v$. We now make three remarks concerning E:

(a): We notice that, if $\{\Omega_i\}_{i\in I}$ are the connected components of Ω, then

$$\text{either } \Omega \cap \partial^* E \ne \emptyset\,, \qquad \text{or } \exists\ I_0 \subset I \text{ s.t. } E = \textstyle\bigcup_{i\in I_0}\Omega_i\,. \tag{3.7}$$

Indeed, if $\Omega \cap \partial^* E = \emptyset$, then, by $\mathrm{cl}\,(\partial^* E) \cap \Omega = \partial E \cap \Omega$, we find $\partial E \subset \partial\Omega$, and thus the second possibility in (3.7) occurs; viceversa, if the second possibility in (3.7) occurs, then, trivially, $\Omega \cap \partial^* E = \emptyset$ holds.

(b): We notice that, if $\Omega \cap \partial^* E \ne \emptyset$, then we can construct a system of "volume–fixing variations" for $\{E_j\}_j$. Indeed, if $\Omega \cap \partial^* E \ne \emptyset$, then there exist $B_{S_0}(x_0) \subset\subset \Omega$ with $P(E;\partial B_{S_0}(x_0)) = 0$ and a vector field $X \in C^\infty_c(B_{S_0}(x_0);\mathbb{R}^{n+1})$ such that $\int_E \mathrm{div}\,X = 1$. By the volume-fixing variation construction (see [Mag12, Theorem 29.14]), there exist constants $C_0, c_0 > 0$, depending on E itself, with the following property: whenever $|(F\Delta E) \cap B_{S_0}(x_0)| < c_0$, then there exists a smooth function $\Phi^F : \mathbb{R}^n \times (-c_0, c_0) \to \mathbb{R}^n$ such that, for each $|t| < c_0$, the map $\Phi^F_t = \Phi^F(\cdot, t)$ is a smooth diffeomorphism with $\{\Phi^F_t \ne \mathrm{id}\} \subset\subset B_{S_0}(x_0)$ and

$$|\Phi^F_t(F)| = |F| + t\,, \qquad P(\Phi^F_t(F); B_{S_0}(x_0)) \le \big(1 + C_0\,|t|\big)\,P(F; B_{S_0}(x_0))\,.$$

For j large enough, we evidently have $|(E_j\Delta E) \cap B_{S_0}(x_0)| < c_0$, and thus we can construct smooth functions $\Phi^j : \mathbb{R}^n \times (-c_0, c_0) \to \mathbb{R}^n$ such that, for each $|t| < c_0$, the map $\Phi^j_t = \Phi^j(\cdot, t)$ is a smooth diffeomorphism with $\{\Phi^j_t \ne \mathrm{id}\} \subset\subset B_{S_0}(x_0)$ and

$$|\Phi^j_t(E_j)| = |E_j| + t\,, \qquad P(\Phi^j_t(E_j); B_{S_0}(x_0)) \le \big(1 + C_0\,|t|\big)\,P(E_j; B_{S_0}(x_0))\,.$$

(c): We notice that, if $\Omega \cap \partial^* E \ne \emptyset$, then E is bounded. Since $|E| \le v < \infty$, it is enough to prove that $\Omega \cap \partial^* E$ is bounded. In turn, taking $x_0 \in \Omega \cap \partial^* E$, and since W is bounded and $|E| < \infty$, the boundedness of $\Omega \cap \partial^* E$ descends immediately by the following lower volume-density estimate: there exists $r_1 > 0$ such that

$$\begin{aligned}&|E \cap B_r(x)| \ge c(n)\,r^{n+1}\\ &\forall\ x \in \Omega \cap \partial^* E\,,\ r < r_1\,,\ B_r(x) \subset\subset \mathbb{R}^{n+1} \setminus \big(I_{r_1}(W) \cup B_{S_0}(x_0)\big)\,.\end{aligned} \tag{3.8}$$

To prove (3.8), let $r_1 > 0$ be such that $|B_{r_1}| < c_0$, let x and r be as in (3.8), and set

$$F_j = \big(\Phi^j_t(E_j) \cap B_{S_0}(x_0)\big) \cup \Big(E_j \setminus \big(B_r(x) \cup B_{S_0}(x_0)\big)\Big)\,,$$

for $t = |E_j \cap B_r(x)|$ (which is an admissible value of t by $|B_{r_1}| < c_0$). In this way, $|F_j| = |E_j| = v$, and thus we can exploit (3.6) to find that, for a.e. $r < r_1$,

$$\begin{aligned}P(E_j;\Omega) \le P(F_j;\Omega) + \frac{1}{j} &\le P\Big(E_j; \Omega \setminus \big(B_r(x) \cup B_{S_0}(x_0)\big)\Big) + P(E_j; B_{S_0}(x_0))\\ &\quad + C_0\left(\psi_W(v) + \frac{1}{j}\right)|E_j \cap B_r(x)| + \mathcal{H}^n(E_j \cap \partial B_r(x)) + \frac{1}{j}\,,\end{aligned}$$

or, setting for the sake of brevity $u_j(r) = |E_j \cap B_r(x)|$,

$$P(E_j; B_r(x)) \le u_j'(r) + C_0\,(\psi_W(v) + 1)\,u_j(r) + \frac{1}{j}\,, \qquad \text{for a.e. } r < r_1\,,$$

where we have used that $u_j'(r) = \mathcal{H}^n(E_j \cap \partial B_r(x))$ for a.e. $r > 0$. Adding $u_j'(r)$ on both sides and applying the Euclidean isoperimetric inequality to $E_j \cap B_r(x)$, we find

$$c(n)\,u_j(r)^{n/(n+1)} \le 2\,u_j'(r) + C_0\,(\psi_W(v) + 1)\,u_j(r) + \frac{1}{j}\,, \qquad \text{for a.e. } r < r_1\,.$$

Since $C_0\,(\psi_W(v)+1)\,u_j(r)\le[c(n)/2]u_j(r)^{n/(n+1)}$ is equivalent to

$$u_j(r)\le\Big(\frac{c(n)}{2\,C_0\,(\psi_W(v)+1)}\Big)^{n+1}\,,$$

and since $u_j(r)\le\omega_{n+1}\,r_1^{n+1}$, up to further decreasing the value of r_1 we can assume that $C_0\,(\psi_W(v)+1)\,u_j\le[c(n)/2]u_j^{n/(n+1)}$ on $(0,r_1)$, and thus deduce

$$\frac{c(n)}{2}\,u_j(r)^{n/(n+1)}\le u_j'(r)+\frac{1}{j}\,,\qquad\text{for a.e. }r\in(0,r_1)\,.$$

Letting $j\to\infty$ and setting $u(r)=|E\cap B_r(x)|$, we find $u^{n/(n+1)}\le C(n)\,u'$ a.e. on $(0,r_1)$, and thus (by $x\in\partial^*E\cap\Omega$) that $u(r)\ge c_0\,r^{n+1}$ for every $r\in(0,r_1)$, which is (3.8).

We are ready to prove the existence of a minimizer of $\psi_W(v)$. Since $\partial\Omega\subset W$ is bounded, every connected component of Ω with finite volume is bounded. Thus, by combining (a), (b) and (c) above, we conclude that there exists $R>0$ such that

$$W\cup E\subset\subset B_R\,. \tag{3.9}$$

Since $|E\cap[B_{R+1}\setminus B_R]|=0$, we can pick $T\in(R,R+1)$ such that

$$\lim_{j\to\infty}\mathcal{H}^n(E_j\cap\partial B_T)=0\,,\qquad P(E_j\setminus B_T)=\mathcal{H}^n(E_j\cap\partial B_T)+P(E_j;\Omega\setminus B_T)\,, \tag{3.10}$$

and consider the sets

$$F_j=\big(E_j\cap B_T\big)\cup B_{\rho_j}(y)\,,\qquad\rho_j=\Big(\frac{|E_j\setminus B_T|}{\omega_{n+1}}\Big)^{1/(n+1)}\,,$$

corresponding to a $y\in\mathbb{R}^{n+1}$ which is independent from j and such that $|y|>\rho_j+T$ (notice that $\sup_j\rho_j\le C(n)\,v^{1/(n+1)}$). Since $|F_j|=|E_j|=v$, (3.6) gives

$$\begin{aligned}P(E_j;\Omega)&\le P(F_j;\Omega)+\frac{1}{j}\le P(E_j;\Omega\cap B_T)+\mathcal{H}^n(E_j\cap\partial B_T)+P(B_{\rho_j})+\frac{1}{j}\\&\le P(E_j;\Omega)+2\,\mathcal{H}^n(E_j\cap\partial B_T)+\frac{1}{j}=P(E_j;\Omega)+\mathrm{o}(1)\,,\qquad\text{as }j\to\infty\,,\end{aligned}$$

thanks to (3.10) and $P(B_{\rho_j})\le P(E_j\setminus B_T)$. Thus $\{F_j\}_j$ is a minimizing sequence for $\psi_W(v)$, with $F_j\subset B_{T^*}$, with T^* independent of j. In particular, up to extracting subsequences, F_j converges in $L^1(\mathbb{R}^{n+1})$ to a set E^* with $|E^*|=v$, so that E^* is a minimizer of $\psi_W(v)$.

Step two: We prove (3.2). If E_v a minimizer of $\psi_W(v)$ and $R>0$ is such that $W\subset\subset B_R$, then by $P(E_v;\Omega)\le P(B^{(v)})$ we have, for $v>v_0$, and v_0 and C_0 depending on n and W,

$$P(E_v\setminus B_R)\le P(E_v;\Omega)+n\,\omega_n\,R^n\le P(B^{(v)})+C_0\le\Big(1+\frac{C_0}{v}\Big)\,P(B^{(|E_v\setminus B_R|)})+C_0\,, \tag{3.11}$$

where we have used that, if $v>2\,b>0$ and $\alpha=n/(n+1)$, then

$$\frac{P(B^{(v)})}{P(B^{(v-b)})}-1=\Big(\frac{v}{v-b}\Big)^\alpha-1\le\frac{\alpha\,b}{v-b}\le\frac{2\,\alpha\,b}{v}\,.$$

By combining (1.3) and (3.11) we conclude that

$$c(n)\,\inf_{x\in\mathbb{R}^{n+1}}\Big(\frac{|(E_v\setminus B_R)\Delta B^{(|E_v\setminus B_R|)}(x)|}{|E_v\setminus B_R|}\Big)^2\le\frac{P(E_v\setminus B_R)}{P(B^{(|E_v\setminus B_R|)})}-1\le\frac{C_0}{v^{n/(n+1)}}\,, \tag{3.12}$$

provided $v>v_0$. If $x\in\mathbb{R}^{n+1}$ achieves the above infimum, then we find that

$$\begin{aligned}|E_v\Delta B^{(v)}(x)|&=2\,|E_v\setminus B^{(v)}(x)|\le C_0+2\,\big|(E_v\setminus B_R)\setminus B^{(v)}(x)\big|\\&\le C_0+2\,\big|(E_v\setminus B_R)\setminus B^{(|E_v\setminus B_R|)}(x)\big|\le C_0+|E_v\setminus B_R|\,\frac{C_0}{v^{n/2\,(n+1)}}\,,\end{aligned}$$

which immediately implies (3.2).

Step three: We prove the existence of v_0, Λ_0, and s_0 such that every minimizer E_v of $\psi_W(v)$ with $v>v_0$ satisfies (3.1). To this end, we argue by contradiction, and assume the existence of $v_j\to\infty$, minimizers E_j in $\psi_W(v_j)$, sets $F_j\subset\Omega$ with $F_j\Delta E_j\subset\subset B_{r_j}(x_j)$ for some $x_j\in\mathbb{R}^{n+1}$ and $r_j=v_j^{1/(n+1)}/j$, $|F_j\Delta E_j|>0$, and

$$P(E_j;\Omega\cap B_{r_j}(x_j))\geq P(F_j;\Omega\cap B_{r_j}(x_j))+\frac{j}{v_j^{1/(n+1)}}\left|E_j\Delta F_j\right|.$$

Denoting by E_j^*, F_j^* and Ω_j the sets obtained by scaling E_j, F_j and Ω by a factor $v_j^{-1/(n+1)}$, we find that $F_j^*\Delta E_j^*\subset\subset B_{1/j}(y_j)$ for some $y_j\in\mathbb{R}^{n+1}$, and

$$P(E_j^*;\Omega_j\cap B_{1/j}(y_j))\geq P(F_j^*;\Omega_j\cap B_{1/j}(y_j))+j\left|E_j^*\Delta F_j^*\right|. \tag{3.13}$$

By (3.2) there exist $z_j\in\mathbb{R}^{n+1}$ such that $|E_j^*\Delta B^{(1)}(z_j)|\to 0$ as $j\to\infty$. We can therefore use the volume-fixing variations of $B^{(1)}$ to find constants $c(n)$ and $C(n)$ and diffeomorphisms $\Phi_t^j:\mathbb{R}^n\to\mathbb{R}^n$ such that, for every $|t|<c(n)$, one has $\{\Phi^j_t\neq\mathrm{id}\}\subset\subset U_j$ for some open ball U_j with $U_j\subset\subset\Omega_j\setminus B_{1/j}(y_j)$, and

$$|\Phi_t^j(E_j^*)\cap U_j|=|E_j^*\cap U_j|+t\,,\qquad P(\Phi_t^j(E_j^*);U_j)\leq(1+C(n)\,|t|)\,P(E_j^*;U_j)\,.$$

Since $F_j^*\Delta E_j^*\subset\subset B_{1/j}(y_j)$ implies that for j large enough $||F_j^*|-|E_j^*||<c(n)$, if we take $t=|E_j^*|-|F_j^*|$, then the resulting set $G_j^*=\Phi_t^j(F_j^*)$ is such that $|G_j^*|=|E_j^*|$, and therefore the minimality property of E_j in $\psi_W(v_j)$ can be used to infer

$$\begin{aligned}P(E_j^*;\Omega_j)&\leq P(G_j^*;\Omega_j)\\&\leq P\big(E_j^*;\Omega_j\setminus(U_j\cup B_{1/j}(y_j))\big)+P(F_j^*;\Omega_j\cap B_{1/j}(y_j))\\&\quad+P(E_j^*;U_j)+C(n)\,P(E_j^*;U_j)\left|E_j^*\Delta F_j^*\right|.\end{aligned}$$

Taking into account $P(E_j^*;U_j)\leq\psi_W(v_j)/v_j^{n/(n+1)}\leq C(n)$, we thus find

$$P(E_j^*;\Omega_j\cap B_{1/j}(y_j))\leq P(F_j^*;\Omega_j\cap B_{1/j}(y_j))+C(n)\left|E_j^*\Delta F_j^*\right|,$$

which, combined with (3.13), gives $j\left|E_j^*\Delta F_j^*\right|\leq C(n)\left|E_j^*\Delta F_j^*\right|$. Since $|E_j^*\Delta F_j^*|>0$, this is a contradiction for j large enough.

Step four: We now prove that, if $\mathcal{R}(W)>0$, then

$$\lim_{v\to\infty}\sup_{E_v}\left|\frac{|x|}{v^{1/(n+1)}}-\frac{1}{\omega_{n+1}^{1/(n+1)}}\right|=0\,, \tag{3.14}$$

where x is related to the minimizer E_v of $\psi_W(v)$ under consideration by (3.2).

In proving (3.14) we will use the assumption $\mathcal{R}(W)>0$ and the energy upper bound

$$\limsup_{v\to\infty}\psi_W(v)-P(B^{(v)})\leq-\mathcal{R}(W)\,. \tag{3.15}$$

A proof of (3.15) is given in step one of the proof of Theorem 1.7, see section 5; in turn, that proof is solely based on the results from section 4, where no part of Theorem 3.1 (not even the existence of minimizers in $\psi_W(v)$) is ever used. This said, we notice that when $|W|>0$, and thus $\mathcal{S}(W)>0$, one can replace the use of (3.15) in the proof of (3.14) by the use of the simpler upper bound

$$\limsup_{v\to\infty}\psi_W(v)-P(B^{(v)})\leq-\mathcal{S}(W)\,, \tag{3.16}$$

where, we recall, $\mathcal{S}(W) = \sup\{\mathcal{H}^n(W \cap \Pi) : \Pi \text{ is a hyperplane in } \mathbb{R}^{n+1}\}$. To prove (3.16), we notice that, for every given Π, we can construct competitors for $\psi_W(v)$ by intersecting Ω with balls $B^{(v')}(x_v)$ with $v' > v$ and x_v such that

$$|B^{(v')}(x_v) \setminus W| = v\,, \qquad \lim_{v\to\infty} \mathcal{H}^n(W \cap \partial B^{(v')}(x_v)) = \mathcal{H}^n(W \cap \Pi)\,,$$

and

$$\limsup_{v\to\infty} \psi_W(v) - P(B^{(v)}) \le -\mathcal{H}^n(W \cap \Pi)\,,$$

thus giving (3.16). The proof of (3.15) is identical in spirit to that of (3.16), with the difference that to glue a large ball to a maximizer (F, ν) in $\mathcal{R}(W)$ we will need to establish the decay of ∂F towards a hyperplane parallel to $\nu^\perp$ to the high degree of precision expressed in (1.16).

Coming to the proof of (3.14), we argue by contradiction and consider $v_j \to \infty$, minimizers E_j of $\psi_W(v_j)$, and $x_j \in \mathbb{R}^{n+1}$ with

$$\inf_{x\in\mathbb{R}^{n+1}} \frac{|E_j \Delta B^{(v_j)}(x)|}{v_j} = \frac{|E_j \Delta B^{(v_j)}(x_j)|}{v_j}\,, \tag{3.17}$$

such that

$$\liminf_{j\to\infty} \left| \frac{|x_j|}{v_j^{1/(n+1)}} - \frac{1}{\omega_{n+1}^{1/(n+1)}} \right| > 0\,, \tag{3.18}$$

and then set

$$E_j^* = \frac{E_j - x_j}{v_j^{1/(n+1)}}\,, \qquad W_j^* = \frac{W - x_j}{v_j^{1/(n+1)}}\,, \qquad \Omega_j^* = \frac{\Omega - x_j}{v_j^{1/(n+1)}}\,.$$

By (3.1), each E_j^* is a (Λ_0, s_0)-perimeter minimizer with free boundary in the open set Ω_j^*. By (3.2) and (3.17), $E_j^* \to B^{(1)}$ in $L^1(\mathbb{R}^{n+1})$ as $j \to \infty$. Moreover, $\operatorname{diam}(W_j^*) \to 0$ and, by (3.18),

$$\liminf_{j\to\infty} \operatorname{dist}\big(W_j^*, \partial B^{(1)}\big) > 0\,. \tag{3.19}$$

Correspondingly, we can find $z_0 \notin \partial B^{(1)}$ such that, for every $\rho < \operatorname{dist}(z_0, \partial B^{(1)})$, there is $j(\rho)$ such that $\{E_j^*\}_{j\ge j(\rho)}$ is a sequence of (Λ_0, s_0)-perimeter minimizers in $\mathbb{R}^{n+1} \setminus B_{\rho/2}(z_0)$. By Remark 3.2, up to increasing the value of $j(\rho)$, we ensure that $(\partial E_j^*) \setminus B_\rho(z_0)$ is contained in the normal graph over $\partial B^{(1)}$ of a function u_j with $\|u_j\|_{C^1(\partial B^{(1)})} \to 0$; in particular, by (3.19), $(\partial E_j^*) \setminus B_\rho(z_0)$ is disjoint from W_j^*. By the constant mean curvature condition satisfied by $\Omega \cap \partial E_j^*$, and by Alexandrov's theorem [Ale62], we conclude that $(\partial E_j^*) \setminus B_\rho(z_0)$ is a sphere M_j^* for $j \ge j(\rho)$. Let B_j^* be the ball bounded by M_j^*. Since $M_j^* \cap W_j^* = \emptyset$, we have either one of the following two cases:

Case one: $W_j^* \subset B_j^*$. In this case we have

$$\partial[B_j^* \cup E_j^*] \subset M_j^* \cup [(\partial E_j^*) \setminus \operatorname{cl}(B_j^*)] \subset (\partial E_j^*) \setminus W_j^*\,,$$

so that, thanks to $|B_j^* \cup E_j^*| \ge |E_j^*| + |W_j^*| \ge 1$, we conclude that

$$P(E_j^*; \mathbb{R}^{n+1} \setminus W_j^*) \ge P(B_j^* \cup E_j^*) \ge P(B^{(1)})\,,$$

that gives $\psi_W(v_j) \ge P(B^{(1)})$, in contradiction with (3.15).

Case two: $W_j^* \cap B_j^* = \emptyset$. In this case we first see that

$$E_j^* = B_j^* \cup G_j^*\,,$$

where G_j^* is the union of the connected components of E_j^* whose boundaries have non-empty intersection with W_j^*: in other words, we are claiming that B_j^* is the only connected component of E_j^* whose closure is disjoint from W_j^*. Indeed, if this were not the case, we could recombine all the connected components of E_j^* with closure disjoint from W_j^* into a single ball of same total volume, centered far away from W_j^*, in such a way to strictly decrease $P(E_j^*;\Omega_j^*)$ – thus violating the minimality of E_j in $\psi_W(v_j)$. Let us now set

$$G_j = x_j + v_j^{1/(n+1)}\,G_j^*\,,\qquad U_j = x_j + v_j^{1/(n+1)}\,B_j^*\,,$$

so that $E_j = G_j\cup U_j$ and G_j and U_j are at positive distance.

If we start sliding U_j from infinity towards $G_j\cup W$ along arbitrary directions in $\mathbb{S}^n$, then we must find at least one direction such that the resulting "contact point" z_j belongs to $\Omega\cap\partial G_j$: if this were not the case, then we would find that G_j is contained in the convex envelope of W, so that $|B_j| = |E_j| - |G_j| \ge v_j - C(W)$, and thus

$$\psi_W(v_j) = P(E_j;\Omega) \ge P(B_j;W) = P(B_j) \ge P(B^{(v_j - C(W))}) \ge P(B^{(v_j)}) - \frac{C(W)}{v_j^{1/(n+1)}}\,,$$

which contradicts (3.15) for j large enough.

By construction, there is a half-space H_j such that $G_j\subset H_j$, $z_j\in(\partial G_j)\cap(\partial H_j)$, and G_j is a perimeter minimizer in $B_r(z_j)$ for some $r>0$ sufficiently small. By the strong maximum principle, see, e.g. [DPM15, Lemma 2.13], G_j has a unique blow-up at z_j, which is given by $H_j - z_j$. By De Giorgi's regularity theorem, see e.g. [Mag12, Part III], G_j is an open set with smooth boundary in a neighborhood of z_j. Therefore, if we denote by U_j' the translation of U_j constructed in the sliding argument, then, $E_j' = G_j\cup U_j'$ is a minimizer of $\psi_W(v)$, which, in a neighborhood of z_j, is given by the union of two disjoint sets with smooth boundary which touch tangentially at z_j. In particular,

$$\lim_{r\to 0^+}\frac{|E_j'\cap B_r(z_j)|}{|B_r|} = 1\,,$$

thus contradicting the upper volume density estimate which descends from (3.1), see, e.g. [Mag12, Theorem 21.11, Equation (21.9)].

Step five: We complete the proof of the theorem by showing the existence of $v_0>0$ and $R_0(v)$ with $R_0(v)\to 0^+$ and $R_0(v)\,v^{1/(n+1)}\to\infty$, such that every minimizer E_v of $\psi_W(v)$ with $v>v_0$ determines x and $u\in C^\infty(\partial B^{(1)})$ with

$$(\partial E_v) = \Big\{y + v^{1/(n+1)}u\Big(\frac{y-x}{v^{1/(n+1)}}\Big)\,\nu_{B^{(v)}(x)}(y) : y\in\partial B^{(v)}(x)\Big\}\setminus B_{R_0\,v^{1/(n+1)}}\,,\tag{3.20}$$

and

$$\lim_{v\to\infty}\sup_{E_v}\|u\|_{C^1(\partial B^{(1)})} = 0\,.\tag{3.21}$$

To this end, let us consider $v_j\to\infty$, minimizers E_j in $\psi_W(v_j)$, and define x_j, E_j^* and W_j^* as in step four. Thanks to (3.14), we have that

$$\exists\ z_0\in\partial B^{(1)}\ \text{s.t.}\ \operatorname{dist}(z_0,W_j^*)\to 0\,,$$

as $j\to\infty$. In particular, for every $\rho>0$ (no matter how small), we can find $j(\rho)\in\mathbb{N}$ such that if $j\ge j(\rho)$, then E_j^* is a (Λ_0,s_0)-perimeter minimizer in $\mathbb{R}^{n+1}\setminus B_\rho(z_0)$, with $E_j^*\to B^{(1)}$ as $j\to\infty$. By Remark 3.2, we can then find functions $u_j\in C^1(\partial B^{(1)})$ such that

$$(\partial E_j^*) = \Big\{y + u_j(y)\,\nu_{B^{(1)}}(y) : y\in\partial B^{(1)}\Big\}\setminus B_{2\,\rho}(z_0)\,,\qquad \forall j\ge j(\rho)\,,$$

and with $\|u_j\|_{C^1(\partial B^{(1)})}\to 0$ as $j\to\infty$. By the arbitrariness of ρ and by a contradiction argument, we conclude that (3.20) and (3.21) hold (with some $R_0(v)$ such that $R_0(v)\to 0^+$ and $R_0(v)\,v^{1/(n+1)}\to\infty$ as $v\to\infty$). $\square$

4. Properties of isoperimetric residues

This section is devoted to the proof of Theorem 1.1. It will be convenient to introduce some notation for cylinders and slabs in $\mathbb{R}^{n+1}$: precisely, given $r>0$, $\nu\in\mathbb{S}^n$ and $I\subset\mathbb{R}$, and setting $\mathbf{p}_{\nu^\perp}(x)=x-(x\cdot\nu)\,\nu$ ($x\in\mathbb{R}^{n+1}$), we let

$$\begin{aligned}\mathbf{D}_r^\nu &= \big\{x\in\mathbb{R}^{n+1}:|\mathbf{p}_{\nu^\perp}x|<r\,,x\cdot\nu=0\big\}\,,\\ \mathbf{C}_r^\nu &= \big\{x\in\mathbb{R}^{n+1}:|\mathbf{p}_{\nu^\perp}x|<r\big\}\,,\\ \mathbf{C}_{r,I}^\nu &= \big\{x\in\mathbb{R}^{n+1}:|\mathbf{p}_{\nu^\perp}x|<r\,,x\cdot\nu\in I\big\}\,,\\ \partial_\ell\mathbf{C}_{r,I}^\nu &= \big\{x\in\mathbb{R}^{n+1}:|\mathbf{p}_{\nu^\perp}x|=r\,,x\cdot\nu\in I\big\}\,,\\ \mathbf{S}_I^\nu &= \big\{x\in\mathbb{R}^{n+1}:x\cdot\nu\in I\big\}\,.\end{aligned}\tag{4.1}$$

In each case, given $x\in\mathbb{R}^{n+1}$, we also set $\mathbf{D}_r^\nu(x)=x+\mathbf{D}_r^\nu$, $\mathbf{C}_r^\nu(x)=x+\mathbf{C}_r^\nu$, etc. It is also convenient to premise the following proposition, which will be used in the proof of both Theorem 1.1 and Theorem 1.7, and which is based on [Sch83, Proposition 1 and Proposition 3].

Proposition 4.1. *Let $n\ge 2$, $\nu\in\mathbb{S}^n$, and let f be a Lipschitz solution to the minimal surface equation on $\nu^\perp\setminus\mathrm{cl}\,(\mathbf{D}_R^\nu)$. If $n=2$, assume in addition that $M=\{x+f(x)\,\nu:|x|>R\}$ is stable and has quadratic area growth in $\mathbb{R}^3\setminus B_R$, i.e.*

$$\int_M|\nabla^M\varphi|^2-|A|^2\,\varphi^2\ge 0\,,\qquad\qquad \forall\varphi\in C_c^1(\mathbb{R}^3\setminus B_R)\,,\tag{4.2}$$

$$\mathcal{H}^2(M\cap B_r)\le C\,r^2\,,\qquad\qquad \forall r>R\,.\tag{4.3}$$

Then there exist $a,b\in\mathbb{R}$ and $c\in\nu^\perp$ such that, for every $|x|>R$,

$$\Big|f(x)-\Big(a+\frac{b}{|x|^{n-2}}+\frac{c\cdot x}{|x|^n}\Big)\Big|\le\frac{C}{|x|^n}\,,\qquad\qquad (n\ge 3)\tag{4.4}$$

$$\Big|f(x)-\Big(a+b\,\log|x|+\frac{c\cdot x}{|x|^2}\Big)\Big|\le\frac{C}{|x|^2}\,,\qquad\qquad (n=2)\tag{4.5}$$

$$\max\Big\{|x|^{n-1}\,|\nabla f(x)|,|x|^n\,|\nabla^2 f(x)|:|x|>R\Big\}\le C\,,\qquad\qquad (\textit{every } n)\,.\tag{4.6}$$

Proof. If $n\ge 3$, the fact that ∇f is bounded allows one to represent f as the convolution with a singular kernel which, by a classical result of Littman, Stampacchia, and Weinberger [LSW63], is comparable to the Green's function of $\mathbb{R}^n$; (4.4) is then deduced starting from that representation formula. For more details, see [Sch83, Proposition 3].

In the case $n=2$, by (4.2) and (4.3), we can exploit a classical "logarithmic cut-off argument" to see that M has finite total curvature, i.e.

$$\int_M|K|\,d\mathcal{H}^2<\infty\,,\tag{4.7}$$

where K is the Gaussian curvature of M. Thanks to (4.7) (see, e.g. [PR02, Section 1.2]) the compactification $\overline{M}$ of M is a Riemann surface with boundary, and M is conformally equivalent to $\overline{M}\setminus\{p_1,...,p_m\}$, where p_i are interior points of $\overline{M}$. One can thus repeat the argument in [Sch83, Proposition 1] to see that, for each $i=1,...,m$, there exist a plane Π_i and a solution of the minimal surfaces equation f_i over an exterior domain in Π_i such that M contains the graph of f_i, and such that f_i satisfies

$$\Big|f_i(x)-\Big(a_i+b_i\,\log|x|+\frac{c_i\cdot x}{|x|^2}\Big)\Big|\le\frac{C}{|x|^2}\,,\qquad \forall x\in\Pi_i\,,|x|\ge R_i\,,\tag{4.8}$$

for suitable $a_i,b_i\in\mathbb{R}$ and $c_i\in\Pi$. Evidently, the fact that $M=\{x+f(x)\,\nu:|x|>R\}$ implies $m=1$, and (4.8) implies (4.5). $\square$

Proof of Theorem 1.1. Step one: Given a hyperplane Π in $\mathbb{R}^{n+1}$, if F is a half-space with $\partial F = \Pi$ and ν is a unit normal to Π, then $\mathrm{res}_W(F,\nu) = \mathcal{H}^n(W \cap \Pi)$. Therefore,

$$\mathcal{R}(W) \geq \mathcal{S}(W) = \sup\left\{\mathcal{H}^n(\Pi \cap W) : \Pi \text{ an hyperplane in } \mathbb{R}^{n+1}\right\}, \tag{4.9}$$

and thus obtain the lower bound in (1.11).

Step two: We notice that, if $(F,\nu) \in \mathcal{F}$, then by (1.8), (1.9), and the divergence theorem (see, e.g., [Mag12, Lemma 22.11]), we can define a Radon measure on the open set $\nu^\perp \setminus \mathbf{p}_{\nu^\perp}(W)$ by setting

$$\mu(U) = P\big(F; (\mathbf{p}_{\nu^\perp})^{-1}(U)\big) - \mathcal{H}^n(U), \qquad U \subset \nu^\perp \setminus \mathbf{p}_{\nu^\perp}(W). \tag{4.10}$$

In particular, setting $R' = \inf\{\rho : W \subset \mathbf{C}_\rho^\nu\}$, $\mu(\mathbf{D}_R^\nu \setminus \mathbf{p}_{\nu^\perp}(W)) \geq 0$ gives

$$P(F; \mathbf{C}_R^\nu \setminus W) \geq \omega_n R^n - \mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)), \qquad \forall R > R',$$

while the identity

$$\omega_n R^n - P(F; \mathbf{C}_R^\nu \setminus W) = -\mu(\mathbf{D}_R^\nu \setminus \mathbf{D}_{R'}^\nu) + \omega_n (R')^n - P(F; \mathbf{C}_{R'}^\nu \setminus W)$$

(which possibly holds as $-\infty = -\infty$ if $P(F; \mathbf{C}_{R'}^\nu \setminus W) = +\infty$) gives that

$$R \in (R',\infty) \mapsto \omega_n R^n - P(F; \mathbf{C}_R^\nu \setminus W) \text{ is a } \textit{decreasing} \text{ function on } (R',\infty). \tag{4.11}$$

In particular, the limsup defining the residual perimeter functional always exists as a limit.

Step three: We prove the existence of a maximizer (F,ν) in $\mathcal{R}(W)$ and (1.14). We first claim that if $\{(F_j,\nu_j)\}_j$ is a maximizing sequence for $\mathcal{R}(W)$, then, in addition to $\mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$, one can modify (F_j,ν_j), preserving the optimality in the limit $j \to \infty$, so that[1]

$$\partial F_j \subset \mathbf{S}_{[A_j,B_j]}^{\nu_j}, \qquad \mathbf{S}_{(-\infty,A_j)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} F_j, \qquad \mathbf{S}_{(B_j,\infty)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F_j, \tag{4.12}$$

where $[A_j,B_j]$ is such that

$$[A_j,B_j] = \bigcap\left\{(\alpha,\beta) : W \subset \mathbf{S}_{(\alpha,\beta)}^{\nu_j}\right\}. \tag{4.13}$$

Indeed, since $(F_j,\nu_j) \in \mathcal{F}$, for some $\alpha_j < \beta_j \in \mathbb{R}$ we have

$$\partial F_j \subset \mathbf{S}_{[\alpha_j,\beta_j]}^{\nu_j}, \qquad \mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp. \tag{4.14}$$

Would it be that

$$\text{either} \quad \mathbf{S}_{(-\infty,\alpha_j)\cup(\beta_j,\infty)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} F_j, \qquad \text{or} \qquad \mathbf{S}_{(-\infty,\alpha_j)\cup(\beta_j,\infty)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F_j,$$

then, by the divergence theorem and by $\mathbf{p}_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$, we would find

$$P(F_j; \mathbf{C}_R^{\nu_j} \cap \Omega) \geq 2\left(\omega_n R^n - \mathcal{H}^n(\mathbf{p}_{\nu_j^\perp}(W))\right), \qquad \forall R > 0,$$

and thus $\mathrm{res}_W(F_j,\nu_j) = -\infty$; in particular, $(F_j,\nu_j) \in \mathcal{F}$ being a maximizing sequence, we would have $\mathcal{R}(W) = -\infty$, against (4.9). This proves the validity (up to switching F_j with $\mathbb{R}^{n+1} \setminus F_j$), of the inclusions

$$\mathbf{S}_{(-\infty,\alpha_j)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} F_j, \qquad \mathbf{S}_{(\beta_j,\infty)}^{\nu_j} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F_j. \tag{4.15}$$

Thanks to (4.15) (and by exploiting basic set operations on sets of finite perimeter, see, e.g., [Mag12, Theorem 16.3]), we see that the sets

$$F_j^* = \big(F_j \cup \mathbf{S}_{(-\infty,A_j-1/j)}^{\nu_j}\big) \cap \mathbf{S}_{(-\infty,B_j+1/j)}^{\nu_j}$$

satisfy

$$(F_j^*,\nu_j) \in \mathcal{F}, \qquad P\big(F_j^*; \mathbf{C}_R^{\nu_j} \setminus W\big) \leq P\big(F_j; \mathbf{C}_R^{\nu_j} \setminus W\big), \qquad \forall R > 0; \tag{4.16}$$

in particular, $\{(F_j^*,\nu_j)\}_j$ is also a maximizing sequence for $\mathcal{R}(W)$.

[1]Here $X \subset_{\mathcal{L}^{n+1}} Y$ means $|X \setminus Y| = 0$.

By standard compactness theorems for sets of finite perimeter we can find a set of locally finite perimeter $F \subset \mathbb{R}^{n+1}$ and $\nu \in \mathbb{S}^n$ such that $F_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$ and $\nu_j \to \nu$ as $j \to \infty$. If A is an open set compactly contained in $\mathbf{C}_R^\nu \setminus W$, then, for j large enough, $A \subset\subset \mathbf{C}_R^{\nu_j} \setminus W$, and thus

$$P(F; \mathbf{C}_R^\nu \setminus W) = \sup_{A \subset\subset \mathbf{C}_R^\nu \setminus W} P(F; A) \le \liminf_{j \to \infty} P(F_j; \mathbf{C}_R^{\nu_j} \setminus W) . \tag{4.17}$$

By (4.11), $R \mapsto \omega_n R^n - P(F_j; \mathbf{C}_R^{\nu_j} \setminus W)$ is decreasing on $R > R_j = \inf\{\rho : W \subset \mathbf{C}_\rho^{\nu_j}\}$. Since, evidently, $\sup_j R_j \le C(W) < \infty$, we deduce from (4.17) that

$$\omega_n R^n - P(F; \mathbf{C}_R^\nu \setminus W) \ge \limsup_{j \to \infty} \omega_n R^n - P(F_j; \mathbf{C}_R^{\nu_j} \setminus W) \ge \limsup_{j \to \infty} \mathrm{res}_W(F_j, \nu_j) ,$$

for every $R > C(W)$; in particular, letting $R \to \infty$,

$$\mathrm{res}_W(F, \nu) \ge \limsup_{j \to \infty} \mathrm{res}_W(F_j, \nu_j) = \mathcal{R}(W) . \tag{4.18}$$

By $F_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$, $\partial F = \mathrm{cl}\,(\partial^* F)$ is contained in the set of accumulation points of sequences $\{x_j\}_j$ with $x_j \in \partial F_j$, so that (4.12) gives

$$\partial F \subset \mathbf{S}^\nu_{[A,B]} , \qquad \mathbf{S}^\nu_{(-\infty,A)} \subset_{\mathcal{L}^{n+1}} F , \qquad \mathbf{S}^\nu_{(B,\infty)} \subset_{\mathcal{L}^{n+1}} \mathbb{R}^{n+1} \setminus F , \tag{4.19}$$

provided $[A,B] = \bigcap\{(\alpha,\beta) : W \subset \mathbf{S}^\nu_{(\alpha,\beta)}\}$. Therefore $(F,\nu) \in \mathcal{F}$, and thus, by (4.18), (F,ν) is a maximizer of $\mathcal{R}(W)$.

We now show that (4.18) implies (1.14), that is

$$P(F; \Omega \cap B) \le P(G; \Omega \cap B) , \qquad \forall F \Delta G \subset\subset B, \ B \text{ a ball} . \tag{4.20}$$

Indeed, should (4.20) fail, we could find $\delta > 0$ and $G \subset \mathbb{R}^{n+1}$ with $F \Delta G \subset\subset B$ for some ball B, such that $P(G; B \setminus W) + \delta \le P(F; B \setminus W)$. For R large enough to entail $B \subset\subset \mathbf{C}_R^\nu$ we would then find

$$\mathrm{res}_W(F,\nu) + \delta \le \omega_n R^n - P(F; \mathbf{C}_R^\nu \setminus W) + \delta \le \omega_n R^n - P(G; \mathbf{C}_R^\nu \setminus W) ,$$

which, letting $R \to \infty$, would violate the maximality of (F,ν) in $\mathcal{R}(W)$.

Step four: We now assume $\mathcal{R}(W) > 0$, and begin the proof of the rest of statement (ii) by showing that if (F,ν) is a maximizer in $\mathcal{R}(W)$, then $\partial F \subset \mathbf{S}^\nu_{[A,B]}$ for A, B as in (4.19). Indeed, if this were not the case, then we could repeate the same truncation procedure leading to (4.16), and deduce this time by the maximality of F that

$$\omega_n R^n - P\big(F^*; \mathbf{C}_R^{\nu_j} \setminus W\big) \ge \omega_n R^n - P\big(F; \mathbf{C}_R^{\nu_j} \setminus W\big) \ge \mathcal{R}(W) \qquad \forall R > 0 ,$$

so that (F^*, ν) is also a maximizer. Now $P\big(F; \mathbf{C}_R^{\nu_j} \setminus W\big) - P\big(F^*; \mathbf{C}_R^{\nu_j} \setminus W\big)$ is increasing in R, and since $\mathrm{res}_W(F,\nu) = \mathrm{res}_W(F^*,\nu)$, it follows that $P\big(F; \mathbf{C}_R^{\nu_j} \setminus W\big) = P\big(F^*; \mathbf{C}_R^{\nu_j} \setminus W\big)$ for all large R. But this equality can hold only if $\partial F \cap \Omega$ is a plane that does not intersect W, in which case $\mathcal{R}(W) = \mathrm{res}_W(F,\nu) = 0$, a contradiction.

Step five: Still assuming $\mathcal{R}(W) > 0$, we complete the proof of statement (ii) by proving (1.16). By (4.19), if (F,ν) is a maximizer of $\mathcal{R}(W)$, then $F/R \to H^- = \{x \in \mathbb{R}^{n+1} : x \cdot \nu < 0\}$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$ as $R \to \infty$. By (4.20) and by improved convergence (i.e., Remark 3.2 – notice carefully that ∂F is bounded in the direction ν thanks to step four), we find $R_F > 0$ and functions $\{f_R\}_{R > R_F} \subset C^1(\mathbf{D}_2^\nu \setminus \mathbf{D}_1^\nu)$ such that

$$\big(\mathbf{C}_2^\nu \setminus \mathbf{C}_1^\nu\big) \cap \partial(F/R) = \big\{x + f_R(x)\,\nu : x \in \mathbf{D}_2^\nu \setminus \mathbf{D}_1^\nu\big\} , \qquad \forall R > R_F .$$

with $\|f_R\|_{C^1(\mathbf{D}_2^\nu \setminus \mathbf{D}_1^\nu)} \to 0$ as $R \to \infty$. Scaling back to F we deduce that

$$(\partial F) \setminus \mathbf{C}_{R_F}^\nu = \big\{x + f(x)\,\nu : x \in \nu^\perp \setminus \mathbf{D}_{R_F}^\nu\big\} , \tag{4.21}$$

for a (necessarily smooth) solution f to the minimal surfaces equation with

$$\|f\|_{C^0(\nu^\perp\setminus\mathbf{D}^\nu_{R_F})}\le B-A\,,\qquad \lim_{R\to\infty}\|\nabla f\|_{C^0(\mathbf{D}^\nu_{2\,R}\setminus\mathbf{D}^\nu_R)}=0\,,\tag{4.22}$$

thanks to the fact that $f(x)=R\,f_R(x/R)$ if $x\in\mathbf{D}^\nu_{2\,R}\setminus\mathbf{D}^\nu_R$.

If $n\ge3$, then we deduce (1.16) thanks to (4.21) and Proposition 4.1.

If $n=2$, we notice that (4.2) holds by (4.20), while (4.3) holds thanks to $\mathrm{res}_W(F,\nu)\ge0$. Indeed the latter condition implies that we can find $R'>R_F$ such that $\omega_n\,R^n\ge P(F;\mathbf{C}^\nu_R\setminus W)-1$ if $R>R'$. In particular, setting $M=(\partial F)\setminus B_{R_F}$ for $R>R'$ we have

$$\mathcal{H}^2(M\cap B_R)\le\mathcal{H}^2(M\cap W)+P(F;\mathbf{C}^\nu_R\setminus W)\le\omega_n\,R^n+1+\mathcal{H}^2(M\cap W)\le C\,R^n\,,$$

provided $C=\omega_n+[(1+\mathcal{H}^2(M\cap W))/(R')^n]$; while if $R\in(R_F,R')$, then $\mathcal{H}^2(M\cap B_R)\le C\,R^n$ with $C=\mathcal{H}^2(M\cap B_{R'})/R_F^n$. This said, we can apply Proposition 4.1 to deduce the validity of (4.5). Since ∂F is bounded in a slab, the logarithmic term in (4.5) must vanish (i.e. (4.5) holds with $b=0$), and thus (1.16) is proved.

If $n=1$, then (4.21) and (4.22) imply the existence of $a_1,a_2\in\mathbb{R}$, and $x_1<x_2$, $x_1,x_2\in\nu^\perp\equiv\mathbb{R}$, such that $f(x)=a_1$ for $x\in\nu^\perp$, $x<x_1$, and $f(x)=a_2$ for $x\in\nu^\perp$, $x>x_2$. We want to prove that $a_1=a_2$. Indeed, setting $M_1=\{x+a_1\,\nu:x\in\nu^\perp,x<x_1\}$ and $M_2=\{x+a_2\,\nu:x\in\nu^\perp,x>x_2\}$, we have that

$$P(F;\mathbf{C}^\nu_R\setminus W)=\mathcal{H}^n\big(\mathbf{C}^\nu_R\cap(\partial F)\setminus(W\cup M_1\cup M_2)\big)+2\,R-|x_2-x_1|\,;$$

while, if L denotes the line through $x_1+a_1\,\nu$ and $x_2+a_2\,\nu$, then we can find $\nu_L\in\mathbb{S}^1$ and a set F_L such that $(F_L,\nu_L)\in\mathcal{F}$ with

$$\partial F_L=\Big(\big((\partial F)\setminus(M_1\cup M_2)\big)\cup(L_1\cup L_2)\Big)$$

where L_1 and L_2 are the two half-lines obtained by removing from L the segment joining $x_1+a_1\,\nu$ and $x_2+a_2\,\nu$. In this way

$$P(F_L;\mathbf{C}^{\nu_L}_R\setminus W)=\mathcal{H}^n\big(\mathbf{C}^\nu_R\cap(\partial F)\setminus(W\cup M_1\cup M_2)\big)+2\,R-\big|(x_1+a_1\,\nu)-(x_2+a_2\,\nu)\big|\,.$$

We thus conclude that $\mathrm{res}_W(F_L,\nu_L)-\mathrm{res}_W(F,\nu)=\big|(x_1+a_1\,\nu)-(x_2+a_2\,\nu)\big|-|x_2-x_1|>0$, against the maximality of (F,ν) in $\mathcal{R}(W)$.

We are left to prove that (4.21) holds with $R_2=R_2(W)$ in place of R_F, and the constants a, b, c and C_0 appearing in (1.16) can be bounded in terms of W only. To this end, we notice that the argument presented in step one shows that the set of maximizers (F,ν) of $\mathcal{R}(W)$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Using this fact and a contradiction argument based on improved convergence (Remark 3.2), we conclude the proof of statement (ii).

Step six: We complete the proof of statement (i) and begin the proof of statement (iii) by showing that, setting for brevity $d=\mathrm{diam}\,(W)$, it holds

$$\mathcal{H}^n(W\cap\Pi)\le\mathcal{R}(W)\le\sup_{\nu\in\mathbb{S}^n}\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W))\le\omega_n\Big(\frac{d}{2}\Big)^n\,,\tag{4.23}$$

whenever Π is a hyperplane in $\mathbb{R}^{n+1}$. We have already proved the first inequality in step one. To prove the others, we notice that, if $(F,\nu)\in\mathcal{F}$, then $\mathbf{p}_{\nu^\perp}(\partial F)=\nu^\perp$ and (4.11) (that is, the monotone increasing character of $R\mapsto P(F;\mathbf{C}^\nu_R\setminus W)-\omega_n\,R^n$ over $R>R'=\inf\{\rho:W\subset\mathbf{C}^\nu_\rho\}$), give, for every $R>R'$,

$$\begin{aligned}-\mathrm{res}_W(F,\nu)&\ge P(F;\mathbf{C}^\nu_R\setminus W)-\omega_n\,R^n\\&\ge\mathcal{H}^n\big(\mathbf{p}_{\nu^\perp}(\partial F\setminus W)\cap\mathbf{D}^\nu_R\big)-\omega_n\,R^n\\&=-\mathcal{H}^n\big(\mathbf{D}^\nu_R\setminus\mathbf{p}_{\nu^\perp}(\partial F\setminus W)\big)\\&\ge-\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W))\ge-\omega_n\Big(\frac{d}{2}\Big)^n\,,\end{aligned}\tag{4.24}$$

where in the last step we have used the isodiametric inequality. Maximizing over (F,ν) in (4.24) we complete the proof of (4.23). Moreover, if $W = \mathrm{cl}\,(B_{d/2})$, then, since $\mathcal{S}(\mathrm{cl}\,(B_{d/2})) = \mathcal{H}^n(\mathrm{cl}\,(B_{d/2})\cap\Pi) = \omega_n\,(d/2)^n$ for any hyperplane Π through the origin, we find that $\mathcal{R}(\mathrm{cl}\,(B_{d/2})) = \omega_n\,(d/2)^n$; in particular, (4.23) implies (1.17).

Step seven: We continue the proof of statement (iii) by showing (1.18). Let $\mathcal{R}(W) = \omega_n\,(d/2)^n$ and let (F,ν) be a maximizer in $\mathcal{R}(W)$. Since every inequality in (4.24) holds as an equality, we find in particular that

$$\sup_{R>R'} P(F;\mathbf{C}_R^\nu\setminus W) - \mathcal{H}^n\big(\mathbf{p}_{\nu^\perp}(\partial F\setminus W)\cap\mathbf{D}_R^\nu\big) = 0\,, \tag{4.25}$$

$$\mathcal{H}^n(\mathbf{p}_{\nu^\perp}(W)) = \omega_n\,\Big(\frac{d}{2}\Big)^n\,. \tag{4.26}$$

By (4.26) and the discussion of the equality cases for the isodiametric inequality (see, e.g. [MPP14]), we see that, for some $x_0\in\nu^\perp$,

$$\mathbf{p}_{\nu^\perp}(W) = \mathrm{cl}\,(\mathbf{D}_{d/2}^\nu(x_0))\,,\qquad \text{so that } W\subset\mathbf{C}_{d/2}^\nu(x_0)\,. \tag{4.27}$$

Condition (4.25) implies that (1.16) holds with $u\equiv a$ for some $a\in[A,B]=\bigcap\{(\alpha,\beta) : W\subset\mathbf{S}_{(\alpha,\beta)}^\nu\}$; in particular, since $(\partial F)\setminus W$ is a minimal surface and $W\subset\mathbf{C}_{d/2}^\nu(x_0)$, by analytic continuation we find that

$$(\partial F)\setminus\mathbf{C}_{d/2}^\nu(x_0) = \Pi\setminus\mathbf{C}_{d/2}^\nu(x_0)\,,\qquad \Pi=\big\{x : x\cdot\nu = a\big\}\,. \tag{4.28}$$

By (4.28), we have that for $R>R'$,

$$P(F;\mathbf{C}_R^\nu\setminus W) - \omega_n\,R^n = P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W) - \omega_n\,\Big(\frac{d}{2}\Big)^n\,.$$

Going back to (4.24), this implies $P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W)=0$. However, since $(\partial F)\setminus W$ is (distributionally) a minimal surface, $P(F;B_\rho(x)\setminus W)\ge\omega_n\,\rho^n$ whenever $x\in(\partial F)\setminus W$ and $\rho<\mathrm{dist}(x,W)$, so that $P(F;\mathbf{C}_{d/2}^\nu(x_0)\setminus W)=0$ gives

$$((\partial F)\setminus W)\cap\mathbf{C}_{d/2}^\nu(x_0) = \emptyset\,. \tag{4.29}$$

By (4.28) and (4.29) we find $(\partial F)\setminus W = \Pi\setminus\mathrm{cl}\,\big(B_{d/2}(x)\big)$ for some $x\in\Pi$, that is (1.18).

Step eight: We finally prove that $\mathcal{R}(W)=\omega_n\,(d/2)^n$ if and only if there exist a hyperplane Π and a point $x\in\Pi$ such that

$$\Pi\cap\partial B_{d/2}(x)\subset W\,, \tag{4.30}$$

$$\Omega\setminus(\Pi\setminus B_{d/2}(x)) \text{ has exactly two unbounded connected components}\,. \tag{4.31}$$

We first prove that the two conditions are sufficient. Let ν be a unit normal to Π and let Π^+ and Π^- be the two open half-spaces bounded by Π. The condition $\Pi\cup\partial B_{d/2}(x)\subset W$ implies $W\subset\mathbf{C}_{d/2}^\nu(x)$, and thus

$$\Omega\setminus\mathrm{cl}\,\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big] = (\Pi^+\cup\Pi^-)\setminus\mathrm{cl}\,\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big]\,.$$

In particular, $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ has a connected component F which contains

$$\Pi^+\setminus\mathrm{cl}\,\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big]\,;$$

and since $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ contains exactly two unbounded connected components, it cannot be that F contains also $\Pi^-\setminus\mathrm{cl}\,[\mathbf{C}_{d/2,(-d,d)}^\nu(x)]$, therefore

$$\Pi^+\setminus\mathrm{cl}\,\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big]\subset F\,,\qquad \Pi^-\setminus\mathrm{cl}\,\big[\mathbf{C}_{d/2,(-d,d)}^\nu(x)\big]\subset\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(F)\,. \tag{4.32}$$

As a consequence ∂F is contained in the slab $\{y : |(y-x)\cdot\nu|<d\}$, and is such that $\mathbf{p}_{\nu^\perp}(\partial F)=\nu^\perp$, that is, $(F,\nu)\in\mathcal{F}$. Moreover, (4.32) implies

$$\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))\subset\Omega\cap\partial F\,,$$

while, at the same time, the fact that F is a connected component of $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ implies $\Omega\cap\partial F\subset\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. In conclusion, $\Omega\cap\partial F=\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, which gives

$$\omega_n\left(\frac{d}{2}\right)^n=\lim_{r\to\infty}\omega_n r^n-P(F;\mathbf{C}_r^\nu\setminus W)\le\mathcal{R}(W)\le\omega_n\left(\frac{d}{2}\right)^n, \tag{4.33}$$

and $\mathcal{R}(W)=\omega_n\,(d/2)^n$, as claimed.

We prove that the two conditions are necessary. Let (F,ν) be a maximizer in $\mathcal{R}(W)$. As proved in step seven, there is a hyperplane Π and $x\in\Pi$ such that $\Omega\cap\partial F=\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. If $z\in\Pi\cap\partial B_{d/2}(x)$ but $z\in\Omega$, then there is $\rho>0$ such that $B_\rho(z)\subset\Omega$, and since ∂F is a minimal surface in Ω, we would obtain that $\Pi\cap B_\rho(z)\subset\Omega\cap\partial F$, against $\Omega\cap\partial F=\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$. So it must be $\Pi\cap\partial B_{d/2}(x)\subset W$, and the necessity of (4.30) is proved. To prove the necessity of (4.31), we notice that since $\Pi^+\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ and $\Pi^-\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ are both open, connected, and unbounded subsets of $\Omega\setminus(\Pi\setminus B_{d/2}(x))$, and since the complement in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ of their union is bounded, it must be that $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ has *at most* two unbounded connected components: therefore we just need to exclude that *it has only one*. Assuming by contradiction that this is the case, we could then connect any point $x^+\in\Pi^+\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ to any point $x^-\in\Pi^-\setminus\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]$ with a continuous path γ entirely contained in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. Now, recalling that $\Omega\cap\partial F=\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, we can pick $x_0\in\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$ and $r>0$ so that

$$B_r(x_0)\cap\Pi^+\subset F\,,\qquad B_r(x_0)\cap\Pi^-\subset\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(F)\,, \tag{4.34}$$

and $B_r(x_0)\cap\mathrm{cl}\,[\mathbf{C}^\nu_{d/2,(-d,d)}(x)]=\emptyset$. We can then pick $x^+\in B_r(x_0)\cap\Pi^+$, $x^-\in B_r(x_0)\cap\Pi^-$, and then connect them by a path γ entirely contained in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. By (4.34), γ must intersect ∂F, and since γ is contained in Ω, we see that γ must intersect $\Omega\cap\partial F=\Pi\setminus\mathrm{cl}\,(B_{d/2}(x))$, which of course contradicts the containment of γ in $\Omega\setminus(\Pi\setminus B_{d/2}(x))$. We have thus proved that $\Omega\setminus(\Pi\setminus B_{d/2}(x))$ has exactly two unbounded connected components. □

5. Resolution theorem for exterior isoperimetric sets

In this section we complete the proof of Theorem 1.7. We recall that the parts of Theorem 1.7 related to quantitative isoperimetry, namely Theorem 1.7-(i) and the estimate for $|v^{-1/(n+1)}\,|x|-\omega_{n+1}^{-1/(n+1)}|$ in (1.25), have already been proved in Theorem 3.1-(ii, iii). The notation set in (4.1) is also used in this section.

Proof of Theorem 1.7. We recall that, throughout the proof, $\mathcal{R}(W)>0$.

Step one: We prove that

$$\limsup_{v\to\infty}\psi_W(v)-P(B^{(v)})\le-\mathcal{R}(W)\,. \tag{5.1}$$

To this end, let (F,ν) be a maximizer of $\mathcal{R}(W)$, so that by (1.15) and (1.16), we have

$$F\setminus\mathbf{C}^\nu_{R_2}=\big\{x+t\,\nu:x\in\nu^\perp\,,|x|>R_2\,,t<f(x)\big\}\,, \tag{5.2}$$

for a function $f\in C^1(\nu^\perp)$ satisfying

$$\begin{aligned}&\Big|f(x)-\Big(a+\frac{b}{|x|^{n-2}}+\frac{c\cdot x}{|x|^n}\Big)\Big|\le\frac{C_0}{|x|^n}\,,\\ &\max\Big\{|x|^{n-1}\,|\nabla f(x)|,|x|^n\,|\nabla^2 f(x)|\Big\}\le C_0\,,\qquad\forall x\in\nu^\perp\,,|x|>R_2\,,\end{aligned} \tag{5.3}$$

and for some $a,b\in\mathbb{R}$ and $c\in\nu^\perp$ such that $\max\{|a|,|b|,|c|\}\le C(W)<\infty$ (moreover, we can take $b=0$, $c=0$ and $C_0=0$ if $n=1$). We are going to construct competitors for $\psi_W(v)$ with v large by gluing a large sphere S to ∂F along $\partial\mathbf{C}^\nu_r$ for $r>R_2$. This operation comes at the price of an area error located on the cylinder $\partial\mathbf{C}^\nu_r$. We can make this error negligible thanks to the fact that (5.3) determines the distance (inside of $\partial\mathbf{C}^\nu_r$)

of ∂F from a hyperplane (namely, ∂G_r for the half-space G_r defined below) up to $\mathrm{o}(r^{1-n})$ as $r \to \infty$. Thus, the asymptotic expansion (1.16) is just as precise as needed in order to perform this construction, i.e. our construction would not be possible with a less precise information.

We now discuss the construction in detail. Given $r > R_2$, we consider the half-space $G_r \subset \mathbb{R}^{n+1}$ defined by the condition that

$$G_r \cap \partial \mathbf{C}_r^\nu = \Big\{ x + t\,\nu : x \in \nu^\perp \,, |x| = r \,, t < a + \frac{b}{r^{n-2}} + \frac{c \cdot x}{r^n} \Big\} \,, \tag{5.4}$$

so that G_r is the "best half-space approximation" of F on $\partial \mathbf{C}_r^\nu$ according to (5.3). Denoting by $\mathrm{hd}\,(X,Y)$ the Hausdorff distance between $X, Y \subset \mathbb{R}^{n+1}$, for every $r > R_2$ and $v > 0$ we can define $x_{r,v} \in \mathbb{R}^{n+1}$ in such a way that $v \mapsto x_{r,v}$ is continuous and

$$\lim_{v \to \infty} \mathrm{hd}\,(B^{(v)}(x_{r,v}) \cap K, G_r \cap K) = 0 \qquad \forall\, K \subset\subset \mathbb{R}^{n+1} \,. \tag{5.5}$$

Thus, the balls $B^{(v)}(x_{r,v})$ have volume v and are locally converging in Hausdorff distance, as $v \to \infty$, to the optimal half-space G_r. Finally, we notice that by (5.3) we can find $\alpha < \beta$ such that

$$\Big((\partial F) \cup (\partial G_r) \cup (G_r \Delta F) \Big) \cap \mathbf{C}_r^\nu \;\subset\; \mathbf{C}_{r,(\alpha+1,\beta-1)}^\nu \,, \tag{5.6}$$

and then define $F_{r,v}$ by setting

$$F_{r,v} = \big(F \cap \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) \cup \big(B^{(v)}(x_{r,v}) \setminus \mathrm{cl}\,\big[\mathbf{C}_{r,(\alpha,\beta)}^\nu \big] \big) \,, \tag{5.7}$$

see Figure 5.1. We claim that, by using $F_{r,v}$ as comparisons for $\psi_W(|F_{r,v}|)$, and then sending first $v \to \infty$ and then $r \to \infty$, one obtains (5.1).

We begin by noticing that, thanks to (5.5) and (5.6) (see, e.g. [Mag12, Theorem 16.16]), we have

$$\begin{aligned} P(F_{r,v}; \Omega) &= P(F; \mathbf{C}_{r,(\alpha,\beta)}^\nu \setminus W) + P\big(B^{(v)}(x_{r,v}); \mathbb{R}^{n+1} \setminus \mathrm{cl}\,\big[\mathbf{C}_{r,(\alpha,\beta)}^\nu \big] \big) \\ &\quad + \mathcal{H}^n \big((F \Delta B^{(v)}(x_{r,v})) \cap \partial_\ell \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) \,, \end{aligned} \tag{5.8}$$

where the last term is the "gluing error" generated by the mismatch between the boundaries of ∂F and $\partial B^{(v)}(x_{r,v})$ along $\partial_\ell \mathbf{C}_{r,(\alpha,\beta)}^\nu$. Now, thanks to (5.3) we have

$$\mathrm{hd}\,(G_r \cap \partial \mathbf{C}_r^\nu, F \cap \partial \mathbf{C}_r^\nu) \le \frac{C_0}{r^n} \,,$$

so that

$$\mathcal{H}^n \big((F \Delta G_r) \cap \partial \mathbf{C}_r^\nu \big) \le n\, \omega_n\, r^{n-1}\, \mathrm{hd}\,(G_r \cap \partial \mathbf{C}_r^\nu, F \cap \partial \mathbf{C}_r^\nu) \le \frac{C(n,W)}{r} \,. \tag{5.9}$$

At the same time, by (5.5),

$$\lim_{v \to \infty} \mathcal{H}^n \big((G_r \Delta B^{(v)}(x_{r,v})) \cap \partial_\ell \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) = 0 \,, \tag{5.10}$$

and thus we have the following estimate for the gluing error,

$$\limsup_{v \to \infty} \mathcal{H}^n \big((F \Delta B^{(v)}(x_{r,v})) \cap \partial_\ell \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) \le \frac{C(n,W)}{r} \,, \qquad \forall r > R_2 \,. \tag{5.11}$$

Again by (5.5), we find

$$\lim_{v \to \infty} P\big(B^{(v)}(x_{r,v}); \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) = P\big(G_r; \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) \tag{5.12}$$

where

$$\omega_n\, r^n \le P\big(G_r; \mathbf{C}_{r,(\alpha,\beta)}^\nu \big) = \int_{\mathbf{D}_r^\nu} \sqrt{1 + \Big| \frac{c}{r^n} \Big|^2} \le \omega_n\, r^n \Big(1 + \frac{C_0}{r^{2\,n}} \Big) \tag{5.13}$$

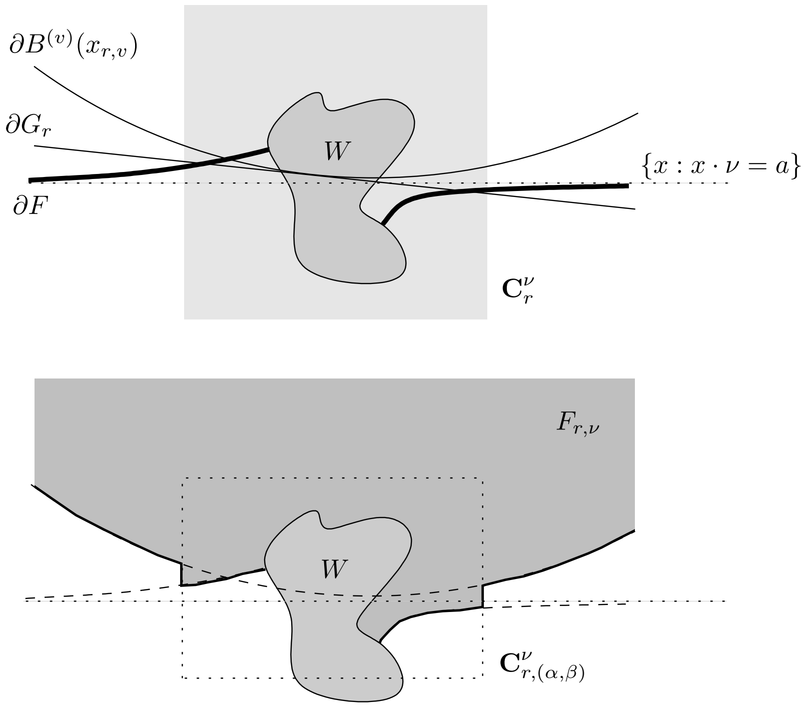

FIGURE 5.1. The competitors $F_{r,v}$ constructed in (5.7). A maximizer F in the isoperimetric residue $\mathcal{R}(W)$ is joined to a ball of volume v, whose center $x_{r,v}$ is determined by looking at best hyperplane approximating ∂F on the "lateral" cylinder $\partial \mathbf{C}_r^\nu$. In order for the area error made in joining this large sphere to ∂F to be negligible, the distance between ∂F and the sphere inside $\partial \mathbf{C}_r^\nu$ must be $\mathrm{o}(r^{1-n})$ as $r \to \infty$. The asymptotic expansion (5.3) gives a hyperplane ∂G_r which is close to ∂F up to $\mathrm{O}(r^{-n})$, and is thus just as precise as needed to perform the construction.

so that, by (5.12) and by the lower bound in (5.13),

$$\limsup_{v\to\infty} P\big(B^{(v)}(x_{r,v}); \mathbb{R}^{n+1} \setminus \mathrm{cl}\,\big[\mathbf{C}_{r,(\alpha,\beta)}^\nu\big]\big) - P(B^{(v)}) \le -\omega_n\, r^n\,, \qquad \forall r > R_2\,. \tag{5.14}$$

Combining (5.11) and (5.14) with (5.8) and the fact that $\mathbf{C}_{r,(\alpha,\beta)}^\nu \cap \partial F = \mathbf{C}_r^\nu \cap \partial F$ (see (5.6)), we find that for every $r > R_2$,

$$\begin{aligned}\limsup_{v\to\infty} P(F_{r,v};\Omega) - P(B^{(v)}) \ &\le\ P(F;\mathbf{C}_r^\nu \setminus W) - \omega_n\, r^n + \frac{C(n,W)}{r} \\ &\le\ -\mathrm{res}_W(F,\nu) + \frac{C(n,W)}{r} = -\mathcal{R}(W) + \frac{C(n,W)}{r}\,.\end{aligned} \tag{5.15}$$

where in the last inequality we have used (4.11). Now, combining the elementary estimates

$$\big||F_{r,v}| - v\big| \le C(n)\, r^{n+1}\,, \qquad \big|P(B^{(v)}) - P(B^{(|F_{r,v}|)})\big| \le C(n)\,\frac{r^{n+1}}{v^{1/(n+1)}}\,, \tag{5.16}$$

with (5.15), we see that

$$\limsup_{v\to\infty} \psi_W(|F_{r,v}|) - P(B^{(|F_{r,v}|)}) \le -\mathcal{R}(W) + \frac{C(n,W)}{r}\,, \qquad \forall r > R_2\,. \tag{5.17}$$

Again by (5.16) and since $v \mapsto |F_{r,v}|$ is a continuous function, we see that

$$\limsup_{v\to\infty} \psi_W(|F_{r,v}|) - P(B^{(|F_{r,v}|)}) = \limsup_{v\to\infty} \psi_W(v) - P(B^{(v)})\,.$$

This last identity combined with (5.17) implies (5.1) in the limit $r \to \infty$.

Step two: Let us now consider minimizers E_j of $\psi_W(v_j)$ corresponding to a sequence $v_j \to \infty$ as $j \to \infty$. By (3.1) and a standard argument (see, e.g. [Mag12, Theorem 21.14]), there exists a local perimeter minimizer with free boundary F in Ω such that, up to extracting subseqences,

$$\begin{aligned}
&E_j \to F \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^{n+1})\,,\\
&\mathcal{H}^n \llcorner \partial E_j \rightharpoonup \mathcal{H}^n \llcorner \partial F \qquad \text{as Radon measures in } \Omega\,,\\
&{\rm hd}\,(K \cap \partial E_j; K \cap \partial F) \to 0 \qquad \text{for every } K \subset\subset \Omega\,,
\end{aligned} \tag{5.18}$$

as $j \to \infty$. Notice that it is not immediate to conclude from the minimality of E_j in $\psi_W(v_j)$ that, for some $\nu \in \mathbb{S}^n$, (F, ν) is a maximizer of $\mathcal{R}(W)$ (or even that $(\nu, F) \in \mathcal{F}$), nor that $P(E_j; \Omega) - P(B^{(v_j)})$ is asymptotically bounded from below by $-{\rm res}_W(F, \nu)$. In this step we prove some preliminary properties of F (see statements (a), (b), and (c) below), and we exploit the blow-down result for exterior minimal surfaces contained in Theorem 2.3-(ii) to prove that F satisfies (5.2) and (5.3) (see statement (c) below). Then, in step three, we shall use the asymptotic expansion (5.3) to show that E_j can be "glued" to F in a similar construction to the one used in step one, and then derive from the corresponding energy estimates the lower bound matching (5.1) and the optimality of F in $\mathcal{R}(W)$.

(a) $\Omega \cap \partial F \cap \partial B_\rho \neq \emptyset$ *for every* ρ *such that* $W \subset\subset B_\rho$: If not there would be $\varepsilon > 0$ such that $W \subset\subset B_{\rho-\varepsilon}$ and $\Omega \cap \partial F \cap A^{\rho+\varepsilon}_{\rho-\varepsilon} = \emptyset$ (recall that $A^s_r = \{x : s > |x| > r\}$). By (5.18) and the constant mean curvature condition satisfied by $\Omega \cap \partial E_j$, we would then find that each E_j (with j large enough) has a connected component of the form $B^{(w_j)}(x_j)$, with $B^{(w_j)}(x_j) \subset\subset \mathbb{R}^{n+1} \setminus B_{\rho+\varepsilon}$ and $w_j \geq v_j - C(n)\,(\rho+\varepsilon)^{n+1}$. In particular,

$$\psi_W(v_j) = P(E_j; \Omega) \geq P(B^{(v_j - C(n)\,(\rho+\varepsilon)^{n+1})}) \geq P(B^{(v_j)}) - C(n)\,\frac{(\rho+\varepsilon)^{n+1}}{v_j^{1/(n+1)}}\,,$$

against $\mathcal{R}(W) > 0$.

(b) *Sharp area bound*: We combine the upper energy bound (5.1) with the perimeter inequality for spherical symmetrization, to prove the area growth estimate

$$P(F; \Omega \cap B_r) \leq \omega_n r^n - \mathcal{R}(W)\,, \qquad \text{for every } r \text{ s.t. } W \subset\subset B_r\,. \tag{5.19}$$

Notice carefully that (5.19) does not immediately imply an analogous estimate for $P(F; \Omega \cap \mathbf{C}^\nu_r)$ (for some $\nu \in \mathbb{S}^n$), which would be helpful to connect $\mathcal{R}(W)$ and the residual perimeter ${\rm res}_W(F, \nu)$ of (F, ν).

To prove (5.19) we argue by contradiction, and consider the existence of $\delta > 0$ and r with $W \subset\subset B_r$ such that $P(F; \Omega \cap B_r) \geq \omega_n\, r^n - \mathcal{R}(W) + \delta$. In particular, for j large enough, we would then have

$$P(E_j; \Omega \cap B_r) \geq \omega_n r^n - \mathcal{R}(W) + \delta\,. \tag{5.20}$$

Again for j large enough, it must be $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$: indeed, by (3.1), $\Omega \cap \partial E_j$ has mean curvature of order ${\rm O}(v_j^{-1/(n+1)})$ as $j \to \infty$, while of course ∂B_r has constant mean curvature equal to n/r. Thanks to $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$, we find

$$P(E_j; \Omega) = P(E_j; \Omega \cap B_r) + P\big(E_j; \mathbb{R}^{n+1} \setminus {\rm cl}\,(B_r)\big)\,. \tag{5.21}$$

Let now E^s_j denote the spherical symmetral of E_j (with respect to the origin, i.e. the center of B_r, and with respect to some fixed direction, say e_{n+1}); in particular, $E^s_j \cap \partial B_\rho$ is a spherical cap in ∂B_ρ, centered at $\rho\, e_{n+1}$, with area equal to $\mathcal{H}^n(E_j \cap \partial B_\rho)$, and we have the perimeter inequality

$$P\big(E_j; \mathbb{R}^{n+1} \setminus {\rm cl}\,(B_r)\big) \geq P\big(E^s_j; \mathbb{R}^{n+1} \setminus {\rm cl}\,(B_r)\big)\,; \tag{5.22}$$

see [CPS20]. Now, we can find a half-space J orthogonal to e_{n+1} and such that $\mathcal{H}^n(J\cap\partial B_r)=\mathcal{H}^n(E_j\cap\partial B_r)$. In this way, using that $|E_j^s\setminus B_r|=|E_j\setminus B_r|$ (by Fubini's theorem in spherical coordinates), and that $\mathcal{H}^n(B_r\cap\partial J)\le\omega_n\,r^n$ (by the fact that ∂J is a hyperplane), we find

$$\begin{aligned}P\big(E_j^s;\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(B_r)\big) &= P\big((E_j^s\setminus\mathrm{cl}\,(B_r))\cup(J\cap B_r)\big)-\mathcal{H}^n(B_r\cap\partial J)\\ &\ge P\big(B^{(|E_j|-|E_j\cap B_r|+|J\cap B_r|)}\big)-\omega_n\,r^n\\ &\ge P(B^{(v_j)})-C(n)\frac{r^{n+1}}{v_j^{1/(n+1)}}-\omega_n\,r^n\end{aligned}$$

which, combined with (5.20), (5.21) and (5.22) finally gives

$$P(E_j;\Omega)-P(B^{(v_j)})>-\mathcal{R}(W)+\delta-C(n)\frac{r^{n+1}}{v_j^{1/(n+1)}}\,,$$

for j large enough. Letting $j\to\infty$ we obtain a contradiction with (5.1).

(c) *Asymptotic behavior of ∂F*: We prove that there are $\nu\in\mathbb{S}^n$, a smooth function $f:\nu^\perp\to\mathbb{R}$, and positive constants $R'>\sup\{\rho:W\subset\mathbf{C}^\nu_\rho\}$ and C such that

$$\partial F\setminus\mathbf{C}^\nu_{R'}=\big\{x+f(x)\,\nu:x\in\nu^\perp\,,|x|>R'\big\}\,,\tag{5.23}$$

and, for some $a,b\in\mathbb{R}$, and $c\in\nu^\perp$, and every $x\in\nu^\perp$, $|x|>R'$,

$$\begin{aligned}&f(x)=a\,, && (n=1)\\ &\Big|f(x)-\Big(a+\frac{b}{|x|^{n-2}}+\frac{c\cdot x}{|x|^n}\Big)\Big|\le\frac{C}{|x|^n}\,, && (n\ge2)\,,\\ &\max\Big\{|x|^{n-1}\,|\nabla f(x)|,|x|^n\,|\nabla^2 f(x)|\Big\}\le C_0\,.\end{aligned}\tag{5.24}$$

To this end, we start noticing that, by a standard argument exploiting the local perimeter minimality of F in Ω, given $r_j\to\infty$ there exists $J\subset\mathbb{R}^{n+1}$ such that, up to extracting subsequences,

$$F/r_j\overset{\mathrm{loc}}{\to}J\qquad\text{in }L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})\text{ as }j\to\infty\,,\tag{5.25}$$

where J is a perimeter minimizer in $\mathbb{R}^{n+1}\setminus\{0\}$, $0\in\partial J$ (thanks to property (a)), J is a cone with vertex at 0, i.e., $\lambda\,J=J$ for every $\lambda>0$ (thanks to Theorem 2.10 and, in particular to (2.44)), and $P(J;B_1)\le\omega_n$ (by (5.19)). If $n\ge2$, then ∂J has vanishing distributional mean curvature in $\mathbb{R}^{n+1}$ (as points are removable singularities for the mean curvature operator in dimension $n\ge2$), thus $P(J;B_1)\ge\omega_n$ by upper semicontinuity of area densities of stationary varifolds, and finally $P(J;B_1)=\omega_n$ can be used with Allard's regularity theorem to conclude that J is indeed a half-space. If $n=1$, then the properties listed above imply that ∂J is the union of two half-lines ℓ_1 and ℓ_2 meeting at the origin. If ℓ_1 and ℓ_2 are not opposite (i.e., if J is not a half-space), then we can find a half-space J^* such that $(J\cap J^*)\Delta J\subset\subset B\subset\subset\mathbb{R}^2\setminus\{0\}$ for some ball B, and $P(J\cap J^*;B)<P(J;B)$, thus violating the fact that J is a perimeter minimizer in $\mathbb{R}^{n+1}\setminus\{0\}$.

In the case $n=1$ it is immediate to conclude from the above information that, for some $R'>0$, $F\setminus B_{R'}=J\setminus B_{R'}$; this proves (5.23) and (5.24) in the case $n=1$.

To prove (5.23) and (5.24) when $n\ge2$, we let M_0 and ε_0 be as in Theorem 2.3-(ii) corresponding to n, $\Gamma=2\,n\,\omega_n$ and $\sigma=1$. Since J is a half-space we can apply improved convergence (i.e., Remark 3.2) to (5.25) on the annulus $A^{2\,L}_{1/2}$, for some $L>\max\{M_0,64\}$ to be chosen later on depending also on ε_0, and find that

$$(\partial F)\cap A^{4\,L\,r_j}_{r_j/2}=\Big\{x+r_j\,f_j\Big(\frac{x}{r_j}\Big)\,\nu:x\in\nu^\perp\Big\}\cap A^{4\,L\,r_j}_{r_j/2}\,,\qquad\nu^\perp=\partial J\,,\tag{5.26}$$

for functions $f_j \in C^1(\nu^\perp)$ with $\|f_j\|_{C^1(\nu^\perp)} \to 0$ as $j \to \infty$. By (5.26), denoting by

$$V_j = \mathbf{var}\big((\partial F) \setminus B_{r_j}, 1\big)$$

the multiplicity one varifold associated to $(\partial F) \setminus B_{r_j}$, we see that $V_j \in \mathcal{V}_n(0, r_j, \infty)$ with

$$\begin{aligned} \frac{1}{r_j^n} \int x \cdot \nu_{V_j}^{\rm co} \, d\mathrm{bd}_{V_j} &= -n\,\omega_n + \mathrm{o}(1) && (5.27) \\ \frac{\|\mathrm{bd}_{V_j}\|(\partial B_{r_j})}{r_j^{n-1}} &= n\,\omega_n + \mathrm{o}(1)\,, && (5.28) \\ \sup_{r \in (r_j, 3\,L\,r_j)} \Big| \frac{\|V_j\|(B_r \setminus B_{r_j})}{r^n - r_j^n} - \omega_n \Big| &= \mathrm{o}(1)\,, && (5.29) \end{aligned}$$

where $\mathrm{o}(1) \to 0$ as $j \to \infty$. By our choice of Γ, by (5.19) and (5.27) we see that, for j large enough, we have

$$\|\mathrm{bd}_{V_j}\|(\partial B_{r_j}) \le \Gamma\, r_j^{n-1}\,, \qquad \sup_{\rho > r_j} \frac{\|V_j\|(B_\rho \setminus B_{r_j})}{\rho^n} \le \Gamma\,. \tag{5.30}$$

Moreover, we claim that setting

$$s_j = 2\,L\,r_j$$

(so that, in particular, $s_j > \max\{M_0, 64\}\, r_j$), then

$$|\delta_{V_j, r_j, 0}(s_j/8)| \le \varepsilon_0\,, \qquad \inf_{r > s_j/8} \delta_{V_j, r_j, 0}(r) \ge -\varepsilon_0\,, \tag{5.31}$$

provided j and L are taken large enough depending on ε_0. To check the first inequality in (5.31) we notice that, by (5.27) and (5.29),

$$\begin{aligned} \delta_{V_j, r_j, 0}(s_j/8) &= \omega_n - \frac{\|V_j\|(B_{s_j/8} \setminus B_{r_j})}{(s_j/8)^n} + \frac{1}{n\,(s_j/8)^n} \int x \cdot \nu_{V_j}^{\rm co}\, d\,\mathrm{bdry}_{\,V_j} \\ &= \omega_n - \big(\omega_n + \mathrm{o}(1)\big) \frac{(s_j/8)^n - r_j^n}{(s_j/8)^n} - \frac{\omega_n\, r_j^n}{(s_j/8)^n} \big(1 + \mathrm{o}(1)\big) \\ &= \mathrm{o}(1) \Big(1 + \Big(\frac{r_j}{s_j}\Big)^n\Big) = \mathrm{o}(1)\,, \end{aligned}$$

so that $|\delta_{V_j, r_j, 0}(s_j/8)| \le \varepsilon_0$ as soon as j is large enough with respect to ε_0. Similarly, if $r > s_j/8 = (L\,r_j)/4$, then by (5.27), (5.29), (5.19), and $r_j/r \le 4/L$,

$$\begin{aligned} \delta_{V_j, r_j, 0}(r) &= \omega_n - \frac{\|V_j\|(B_r \setminus B_{2\,r_j})}{r^n} - \frac{\|V_j\|(B_{2\,r_j} \setminus B_{r_j})}{r^n} - \frac{\omega_n\, r_j^n}{r^n} \big(1 + \mathrm{o}(1)\big) \\ &\ge \omega_n - \frac{\omega_n\, r^n - \mathcal{R}(W)}{r^n} - \big(\omega_n + \mathrm{o}(1)\big) \frac{(2\,r_j)^n - r_j^n}{r^n} - \frac{\omega_n\, r_j^n}{r^n} \big(1 + \mathrm{o}(1)\big) \\ &\ge \frac{\mathcal{R}(W)}{r^n} - 2 \Big(\frac{4}{L}\Big)^n \big(\omega_n + \mathrm{o}(1)\big) - \Big(\frac{4}{L}\Big)^n \mathrm{o}(1) \ge -3 \Big(\frac{4}{L}\Big)^n \omega_n\,, \end{aligned}$$

provided j is large enough; hence the second inequality in (5.31) holds if L is large enough in terms of ε_0. Having proved (5.31), we now claim that

$$\frac{1}{s_j^n} \int_{A_{s_j/8}^{s_j/2}} \omega_H^2\, d\|V_j\| \le \varepsilon_0\,, \qquad \|V_j\|\big(A_{s_j/6}^{s_j/4}\big) \ge c(n)\,, \tag{5.32}$$

with $H = \partial J = \nu^\perp$. The second condition in (5.32) is immediate from (5.26), which also implies that if

$$y \in (\mathrm{spt}\, V_j) \cap A_{s_j/8}^{s_j/2} = (\partial F) \cap A_{L\,r_j/4}^{L\,r_j}\,,$$

then, taking (2.3) and $y = x + r_j\, f_j(x/r_j)\,\nu$ for some $x \in \nu^\perp \cap A^{L\,r_j}_{L\,r_j/4}$ into account, we find

$$\omega_H(y) = \arctan\Big(\frac{|y\cdot\nu|}{|\mathbf{p}_{\nu^\perp}(y)|}\Big) \le \arctan\Big(\frac{r_j\,\|f_j\|_{C^0(\nu^\perp)}}{|x|}\Big) \le \arctan\Big(4\,\frac{\|f_j\|_{C^0(\nu^\perp)}}{L}\Big)$$

so that, by the second inequality in (5.30),

$$\frac{1}{s_j^n}\int_{A^{s_j/2}_{s_j/8}} \omega_H^2\, d\|V_j\| \le \arctan\Big(\frac{\|f_j\|_{C^0(\nu^\perp)}}{4\,L}\Big)^2\, 8^n\,\Gamma\,;$$

in particular, the first inequality in (5.32) holds provided j is large enough. Combining (5.30), (5.31) and (5.32) with see that Theorem 2.3-(ii) can be applied to $(V, R, \Lambda, s) = (V_j, r_j, 0, s_j)$ (provided j is large enough). As a consequence, passing from spherical graphs to cylindrical graphs with the aid of Lemma D.1, we find that, for some sufficiently large value of j,

$$(\partial F)\setminus B_{s_j/16} = \Big\{x + f(x)\,\nu : x\in\nu^\perp\Big\}\setminus B_{s_j/16}\,, \tag{5.33}$$

where $f:\nu^\perp\to\mathbb{R}$ is a smooth function which solves the minimal surfaces equation on $\nu^\perp\setminus B_{s_j/16}$. Since ∂F admits at least one sequential blow-down limit hyperplane (namely, $\nu^\perp = \partial J$), by a theorem of Simon [Sim87, Theorem 2] we find that ∇f has a limit as $|x|\to\infty$; in particular, $|\nabla f|$ is bounded. Moreover, by (5.33) (or by the fact that F is a local perimeter minimizer in Ω), ∂F is a stable minimal surface in $\mathbb{R}^{n+1}\setminus B_{s_j/16}$, which, thanks to (5.19), satisfies an area growth bound like (4.3). We can thus apply Proposition 4.1 to deduce the validity of (5.24) when $n\ge 3$, and of

$$\Big|f(x) - \Big(a + b\,\log|x| + \frac{c\cdot x}{|x|^2}\Big)\Big| \le \frac{C}{|x|^2}\,,\qquad \forall |x| > R'\,, \tag{5.34}$$

when $n=2$ (and for some $R' > s_j$). Recalling that F is a local perimeter minimizer with free boundary in Ω (that is, $P(F;\Omega\cap B)\le P(F';\Omega\cap B)$ whenever $F\Delta F'\subset\subset B\subset\subset\mathbb{R}^3$) it must be that $b=0$, as it can be seen by comparing F with the set F' obtained by changing F inside $\mathbf{C}^\nu_r$ ($r >> R'$) with the half-space G_r bounded by the plane $\{x + t\,\nu : x\in\nu^\perp, t = a + b\,\log(r) + c\cdot x/r^2\}$ and such that $\mathcal{H}^2((F\Delta G_r)\cap\partial\mathbf{C}^\nu_r)\le C/r^2$ (we omit the details of this standard comparison argument). Since (5.34) holds with $b=0$, the proof is complete.

(d) *$F\cup W$ defines an element of $\mathcal{F}$*: With $R > R'$ as in (5.23) and (5.24), we see that $V_R = \mathbf{var}\,((\partial F)\cap(B_R\setminus W))$ is a stationary varifold in $\mathbb{R}^{n+1}\setminus K_R$ for

$$K_R = W\cup\big\{x + f(x)\,\nu : x\in\nu^\perp\,, |x| = R\big\}\,,$$

and has bounded support. By the convex hull property [Sim83a, Theorem 19.2], we deduce that, for every $R > R'$, $\mathrm{spt}V_R$ is contained in the convex hull of K_R, for every $R > R'$. Taking into account that $f(x)\to a$ as $|x|\to\infty$ we conclude that $\Omega\cap\partial F$ is contained in the smallest slab $\mathbf{S}^\nu_{[\alpha,\beta]}$ containing both W and $\{x : x\cdot\nu = a\}$. Now set

$$F' = F\cup W\,.$$

Clearly F' is a set of locally finite perimeter in Ω (since $P(F';\Omega') = P(F;\Omega')$ for every $\Omega'\subset\subset\Omega$). Second, $\partial F'$ is contained in $\mathbf{S}^\nu_{[\alpha,\beta]}$ (since $\partial F'\subset[(\partial F)\cap\Omega]\cup W$). Third, by (5.23) and (5.24),

$$\Big\{x + t\,\nu : x\in\nu^\perp\,, |x| > R'\,, t < \alpha\Big\}\subset F'\,, \tag{5.35}$$

$$\Big\{x + t\,\nu : x\in\nu^\perp\,, |x| > R'\,, t > \beta\Big\}\subset \mathbb{R}^{n+1}\setminus F'\,, \tag{5.36}$$

$$\Big\{x + t\,\nu : x\in\nu^\perp\,, |x| < R'\,, t\in\mathbb{R}\setminus[\alpha,\beta]\Big\}\cap(\partial F') = \emptyset\,. \tag{5.37}$$

By combining (5.35) and (5.37) we see that $\{x+t\,\nu:x\in\nu^\perp\,,t<\alpha\}\subset F'$, and by combining (5.36) and (5.37) we see that $\{x+t\,\nu:x\in\nu^\perp\,,t>\beta\}\subset\mathbb{R}^{n+1}\setminus F'$: in particular, $\mathbf{p}_{\nu^\perp}(\partial F')=\nu^\perp$, and thus $(F',\nu)\in\mathcal{F}$.

Step three: We prove that

$$\liminf_{v\to\infty}\psi_W(v)-P(B^{(v)})\geq-\mathcal{R}(W)\,. \tag{5.38}$$

For a sequence $\{v_j\}_j$ with $v_j\to\infty$ as $j\to\infty$ and achieving the liminf in (5.38), let E_j be a minimizer of $\psi_W(v_j)$, and let F be a (sub-sequential) limit of E_j, so that properties (a), (b), (c) and (d) proved in step two hold for F. In particular, properties (5.23) and (5.24) from (c) are entirely analogous to properties (5.2) and (5.3) exploited in step one: therefore, the family of half-spaces $\{G_r\}_{r>R'}$ defined by (5.4) is such that

$$\Big((\partial F)\cup(\partial G_r)\cup(G_r\Delta F)\Big)\cap\mathbf{C}_r^\nu\ \subset\ \mathbf{C}^\nu_{r,(\alpha+1,\beta-1)}\,, \tag{5.39}$$

$$\mathcal{H}^n\big((F\Delta G_r)\cap\partial\mathbf{C}_r^\nu\big)\leq\frac{C(n,W)}{r}\,, \tag{5.40}$$

$$\big|P\big(G_r;\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)-\omega_n\,r^n\big|\leq\frac{C(n,W)}{r^n}\,, \tag{5.41}$$

(compare with (5.6), (5.9), and (5.13) in step one). In particular, by (5.41) we find

$$-\mathrm{res}_W(F',\nu)=\lim_{r\to\infty}P(F;\mathbf{C}_r^\nu\setminus W)-\omega_n\,r^n=\lim_{r\to\infty}P(F;\mathbf{C}_r^\nu\setminus W)-P(G_r;\mathbf{C}^\nu_{r,(\alpha,\beta)})\,. \tag{5.42}$$

In order to relate the residue of (F',ν) to $\psi_W(v_j)-P(B^{(v_j)})$ we consider the sets

$$Z_j=\big(G_r\cap\mathbf{C}^\nu_{r,(\alpha,b)}\big)\cup\big(E_j\setminus\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)\,,$$

which, by the Euclidean isoperimetric inequality, satisfy

$$P(Z_j)\geq P(B^{(|E_j\setminus\mathbf{C}^\nu_{r,(\alpha,\beta)}|)})\geq P(B^{(v_j)})-C(n)\,\frac{r^n\,(\beta-\alpha)}{v_j^{1/(n+1)}}\,. \tag{5.43}$$

Since for a.e. $r>R'$ we have

$$P(Z_j)=P(E_j;\mathbb{R}^{n+1}\setminus\mathbf{C}^\nu_{r,(\alpha,\beta)})+P(G_r;\mathbf{C}^\nu_{r,(\alpha,b)})+\mathcal{H}^n\big((E_j\Delta G_r)\cap\partial\mathbf{C}^\nu_{r,(\alpha,b)}\big)\,,$$

we conclude that

$$\begin{aligned}\psi_W(v_j)-P(B^{(v_j)})&=P(E_j;\mathbf{C}^\nu_{r,(\alpha,\beta)}\setminus W)+P(E_j;\mathbb{R}^{n+1}\setminus\mathbf{C}^\nu_{r,(\alpha,\beta)})-P(B^{(v_j)})\\&=P(E_j;\mathbf{C}^\nu_{r,(\alpha,\beta)}\setminus W)+P(Z_j)-P(B^{(v_j)})\\&\quad-P(G_r;\mathbf{C}^\nu_{r,(\alpha,b)})-\mathcal{H}^n\big((E_j\Delta G_r)\cap\partial\mathbf{C}^\nu_{r,(\alpha,b)}\big)\end{aligned}$$

so that $E_j\to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and (5.43) give, for a.e. $r>R'$,

$$\begin{aligned}&\liminf_{j\to\infty}\psi_W(v_j)-P(B^{(v_j)})\\&\geq P(F;\mathbf{C}^\nu_{r,(\alpha,\beta)}\setminus W)-P(G_r;\mathbf{C}^\nu_{r,(\alpha,b)})-\mathcal{H}^n\big((F\Delta G_r)\cap\partial\mathbf{C}^\nu_{r,(\alpha,\beta)}\big)\,,\\&\geq P(F;\mathbf{C}^\nu_r\setminus W)-P(G_r;\mathbf{C}^\nu_r)-\frac{C(n,W)}{r}\,,\end{aligned}$$

where in the last inequality we have used (5.40) and the fact that $(F\Delta G_r)\cap\partial\mathbf{C}^\nu_r=(F\Delta G_r)\cap\partial\mathbf{C}^\nu_{r,(\alpha,\beta)}$. Letting $r\to\infty$ and recalling (5.42) we find that

$$\liminf_{j\to\infty}\psi_W(v_j)-P(B^{(v_j)})\geq-\mathrm{res}_W(F',\nu)\geq-\mathcal{R}(W)\,,$$

where in the last inequality we have used $(F',\nu)\in\mathcal{F}$. This completes the proof of (5.38), which in turn, combined with (5.1), gives (1.21), and also shows that L^1_{loc}-subsequential limits F of minimizers E_j of $\psi_W(v_j)$ for $v_j\to\infty$ as $j\to\infty$ are such that, for some $\nu\in\mathbb{S}^n$, $(F\cup W,\nu)\in\mathcal{F}$ and $F\cup W$ is a maximizer of $\mathcal{R}(W)$.

Step four: Moving towards the proof of (1.24), we prove the validity, uniformly among varifolds associated to maximizers of $\mathcal{R}(W)$, of estimates analogous to (5.30), (5.31) and (5.32). For a constant $\Gamma > 2\,n\,\omega_n$ to be determined later on (see (5.61), (5.62), and (5.63) below) in dependence of n and W, and for $\sigma > 0$, we let $M_0 = M_0(n, 2\,\Gamma, \sigma)$ and $\varepsilon_0 = \varepsilon_0(n, 2\,\Gamma, \sigma)$ be determined by Theorem 2.3. If (F, ν) is a maximizer of $\mathcal{R}(W)$, then by Theorem 1.1-(ii) we can find $R_2 > 0$ and a smooth $f : \nu^\perp \to \mathbb{R}$ such that

$$(\partial F) \setminus \mathbf{C}^\nu_{R_2} = \Big\{ x + f(x)\,\nu : x \in \nu^\perp\,, |x| > R_2 \Big\}\,, \tag{5.44}$$

and such that (1.16) holds with coefficients a, b, and c satisfying $\max\{|a|, |b|, |c|\} \le C(W)$, and with the bound $|\nabla f(x)| \le C_0/|x|^{n-1}$ for $|x| > R_2$ in force. This gives,

$$\lim_{r\to\infty} \sup_{(F,\nu)} \|\nabla f\|_{C^0(\nu^\perp \setminus \mathbf{D}^\nu_r)} = 0\,,$$

and thus we can find $R_3 > \max\{2\,R_2, 1\}$ depending on W only such that, setting

$$V_F = \mathbf{var}\,((\partial F) \setminus B_{R_3}, 1)$$

we have $V_F \in \mathcal{V}_n(0, R_3, \infty)$, and

$$\|\mathrm{bd}_{V_F}\|(\partial B_{R_3}) \le \Gamma\, R_3^{n-1}\,, \qquad \sup_{\rho > R_3} \frac{\|V_F\|(B_\rho \setminus B_{R_3})}{\rho^n} \le \Gamma\,, \tag{5.45}$$

(compare with (5.30)). Then, arguing as in step three-(c), or more simply by exploiting (5.44) and the decay estimates (1.16), we can show the existence of a suitably large constant $L > \max\{M_0, 64\}$, depending on n, W and σ only, such that, setting

$$s_W(\sigma) = 2\,L\,R_3 \tag{5.46}$$

we have

$$|\delta_{V_F,R_3,0}(s_W(\sigma)/8)| \le \frac{\varepsilon_0}{2}\,, \qquad \inf_{r > s_W(\sigma)/8} \delta_{V_F,R_3,0}(r) \ge -\frac{\varepsilon_0}{2}\,, \tag{5.47}$$

(compare with (5.31)), as well as

$$\frac{1}{s_W(\sigma)^n} \int_{A^{s_W(\sigma)/2}_{s_W(\sigma)/8}} \omega^2_{\nu^\perp}\, d\|V_F\| \le \frac{\varepsilon_0}{2}\,, \qquad \|V_F\|\big(A^{s_W(\sigma)/4}_{s_W(\sigma)/6}\big) \ge c(n)\,, \tag{5.48}$$

(compare with (5.32)) for some $c(n) > 0$.

Step five: We now consider a sequence $\{(v_j, E_j)\}_j$ with $v_j \to \infty$ and E_j a minimizer of $\psi_W(v_j)$, and prove the existence of a maximizer (F, ν) of $\mathcal{R}(W)$ and of functions $h_j \in C^\infty((\partial F) \setminus B_{R_2})$ such that

$$(\partial E_j) \cap A^{R_1\,v_j^{1/(n+1)}}_{4\,R_2} = \Big\{ y + h_j(y)\,\nu_F(y) : y \in \partial F \Big\} \cap A^{R_1\,v_j^{1/(n+1)}}_{4\,R_2}\,, \tag{5.49}$$

with

$$\lim_{j\to\infty} \|h_j\|_{C^1((\partial F) \cap A^M_{4\,R_2})} = 0\,, \qquad \forall M < \infty\,; \tag{5.50}$$

moreover, we show that if x_j is defined by $|E_j \Delta B^{(v_j)}(x_j)| = \inf_x |E_j \Delta B^{(v_j)}(x)|$, then

$$\lim_{j\to\infty} \Big| \frac{x_j}{|x_j|} - \nu \Big| = 0\,; \tag{5.51}$$

finally, we prove that

$$(\partial E_j) \setminus B_{R_2} \text{ is diffeomorphic to an } n\text{-dimensional disk}\,. \tag{5.52}$$

We start by noticing that, by step three, there is (F, ν) a maximizer of $\mathcal{R}(W)$ such that, up to extracting subsequences, (5.18) holds. By (5.18) and (5.44), and with $s_W(\sigma)$ defined as in step four (see (5.46)) starting from F, we can apply improved convergence (Remark 3.2) to find smooth functions $f_j:\nu^\perp\to\mathbb{R}$ such that

$$(\partial E_j)\cap A^{s_W(\sigma)}_{2\,R_2}=\Big\{x+f_j(x)\,\nu:x\in\nu^\perp\Big\}\cap A^{s_W(\sigma)}_{2\,R_2}\,,\tag{5.53}$$

for j large enough (in terms of σ, n, W, and F), and such that

$$\lim_{j\to\infty}\|f_j-f\|_{C^1(\mathbf{D}^\nu_{s_W(\sigma)}\setminus\mathbf{D}^\nu_{2\,R_2})}=0\,.\tag{5.54}$$

With R_3 as in step four and with the goal of applying Theorem 2.3 to the varifolds

$$V_j=\mathbf{var}\big((\partial E_j)\setminus B_{R_3},1\big)\,,$$

we start noticing that $V_j\in\mathcal{V}_n(\Lambda_j,R_3,\infty)$, for some Λ_j satisfying (thanks to (3.1))

$$\Lambda_j\le\frac{\Lambda_0}{v_j^{1/(n+1)}}\,.\tag{5.55}$$

By (5.46) and (5.55), $s_W(\sigma)$ satisfies the "mesoscale bounds" (compare with (2.5))

$$\frac{\varepsilon_0}{4\,\Lambda_j}>s_W(\sigma)>\max\{M_0,64\}\,R_3\tag{5.56}$$

provided j is large enough. Moreover, thanks to $R_3>2\,R_2$ and to $s_W(\sigma)/8>2\,R_2$, by (5.44), (5.53) and (5.54) we can exploit (5.45), (5.47), and (5.48) to deduce that

$$\begin{aligned}\|\mathrm{bd}_{V_j}\|(\partial B_{R_3})&\le(2\,\Gamma)\,R_3^{n-1}\,,&&(5.57)\\ |\delta_{V_j,R_3,0}(s_W(\sigma)/8)|&\le\frac{2}{3}\,\varepsilon_0\,,&&(5.58)\\ \frac{1}{s_W(\sigma)^n}\int_{A^{s_W(\sigma)/2}_{s_W(\sigma)/8}}\omega^2_{\nu^\perp}\,d\|V_j\|&\le\varepsilon_0\,,&&(5.59)\\ \|V_j\|\big(A^{s_W(\sigma)/4}_{s_W(\sigma)/6}\big)&\ge\frac{c(n)}{2}\,,\qquad\text{for } j \text{ large enough}\,.&&(5.60)\end{aligned}$$

We claim that, up to increasing the value of Γ (depending on n and W), we can entail

$$\sup_{\rho>R_3}\frac{\|V_j\|(B_\rho\setminus B_{R_3})}{\rho^n}\le\Gamma\,.\tag{5.61}$$

To prove this, we resort to Theorem 3.1-(i), which asserts that, for some positive constants Λ_0 and s_0 depending on W only, E_j is a $(\Lambda_0\,v_j^{-1/(n+1)},s_0\,v_j^{1/(n+1)})$-perimeter minimizer with free boundary in Ω. In particular, we can use (3.1) to compare E_j to $E_j\setminus B_r$, and find that for every $r<s_0\,v_j^{1/(n+1)}$ one has

$$P(E_j;\Omega\cap B_r)\le C(n)\Big(r^n+\frac{\Lambda_0}{v_j^{1/(n+1)}}\,r^{n+1}\Big)\le C(n,W)\,r^n\,;\tag{5.62}$$

since, at the same time, if $r>s_0\,v_j^{1/(n+1)}$, then

$$P(E_j;\Omega\cap B_r)\le P(E_j;\Omega)=\psi_W(v_j)\le P(B^{(v_j)})\le\frac{C(n)}{s_0^n}\,r^n\,,\tag{5.63}$$

by combining (5.62) and (5.63) we find (5.61). With (5.57) and (5.61) at hand, we can also show that

$$|\delta_{V_j,R_3,\Lambda_j}(s_W(\sigma)/8)|\le\varepsilon_0\,.\tag{5.64}$$

Indeed, by $s_W(\sigma)=2\,L\,R_3$ and by (5.55),

$$\begin{aligned}\Big|\delta_{V_j,R_3,\Lambda_j}\Big(\frac{s_W(\sigma)}{8}\Big)-\delta_{V_j,R_3,0}\Big(\frac{s_W(\sigma)}{8}\Big)\Big| &\le \frac{\Lambda_0}{v_j^{1/(n+1)}}\int_{R_3}^{s_W(\sigma)/8}\frac{\|V_j\|(B_\rho\setminus B_{R_3})}{\rho^n}\,d\rho\\ &\le \frac{\Lambda_0}{v_j^{1/(n+1)}}\Big(\frac{L}{4}-1\Big)\,R_3\,\Gamma\le\frac{\varepsilon_0}{3}\,,\end{aligned}$$

provided j is large enough. To complete checking that Theorem 2.3 can be applied to every V_j with j large enough, we now consider the quantities

$$R_{*j}=\sup\Big\{\rho>\frac{s_W(\sigma)}{8}:\delta_{V_j,R_3,\Lambda_j}(\rho)\ge-\varepsilon_0\Big\}\,,$$

and prove that, for a constant τ_0 depending on n and W only, we have

$$R_{*j}\ge\tau_0\,v_j^{1/(n+1)}\,; \tag{5.65}$$

in particular, provided j is large enough, (5.65) implies immediately

$$R_{*j}\ge 4\,s_W(\sigma)\,, \tag{5.66}$$

which was the last assumption in Theorem 2.3 that needed to be checked. To prove (5.65), we pick τ_0 such that

$$\Big|\frac{\mathcal{H}^n\big(B_{\tau_0}(z)\cap\partial B^{(1)}\big)}{\tau_0^n}-\omega_n\Big|\le\frac{\varepsilon_0}{2}\,,\qquad\forall z\in\partial B^{(1)}\,. \tag{5.67}$$

(Of course this condition only requires τ_0 to depend on n.) By definition of x_j and by (3.4), and up to extracting a subsequence, we have $x_j\to z_0$ as $j\to\infty$ for some $z_0\in\partial B^{(1)}$. In particular, setting $\rho_j=\tau_0\,v_j^{1/(n+1)}$, we find

$$\begin{aligned}\frac{\|V_j\|(B_{\rho_j}\setminus B_{R_3})}{\rho_j^n} &= \frac{1}{\tau_0^n}\,P\Big(\frac{E_j-x_j}{v_j^{1/(n+1)}}\,;\,B_{\tau_0}(-x_j)\setminus B_{R_3/\rho_j}(-x_j)\Big)\\ &\to\frac{\mathcal{H}^n\big(B_{\tau_0}(-z_0)\cap\partial B^{(1)}\big)}{\tau_0^n}\le\omega_n+\frac{\varepsilon_0}{2}\,,\end{aligned}$$

as $j\to\infty$, thus proving that, for j large enough,

$$\begin{aligned}\delta_{V_j,R_3,\Lambda_j}(\rho_j) &\ge -\frac{\varepsilon_0}{2}+\frac{1}{n\,\rho_j^n}\int x\cdot\nu_{V_j}^{\rm co}\,d\,{\rm bd}_{V_j}-\Lambda_j\int_{R_3}^{\rho_j}\frac{\|V_j\|(B_\rho\setminus B_{R_3})}{\rho^n}\,d\rho\\ &\ge -\frac{\varepsilon_0}{2}-\frac{2\,\Gamma\,R_3^n}{n\,\tau_0^n}\,\frac{1}{v_j^{1/(n+1)}}-\Lambda_0\,\Gamma\,\frac{(\rho_j-R_3)}{v_j^{1/(n+1)}}\\ &\ge -\frac{\varepsilon_0}{2}-\frac{C_*(n,W)}{\tau_0^n\,v_j^{1/(n+1)}}-C_{**}(n,W)\,\tau_0\,,\end{aligned}$$

where we have used (5.57), ${\rm spt}\,{\rm bd}_{V_j}\subset\partial B_{R_3}$, and (5.61). Therefore, provided we pick τ_0 depending on n and W so that $C_{**}\,\tau_0\le\varepsilon_0/4$, and then we pick j large enough to entail $(C_*(n,W)/\tau_0^n)v_j^{-1/(n+1)}\le\varepsilon_0/4$, we conclude that if $r\in(R_3,\rho_j]$, then

$$\delta_{V_j,R_3,\Lambda_j}(r)\ge\delta_{V_j,R_3,\Lambda_j}(\rho_j)\ge-\varepsilon_0\,,$$

where in the first inequality we have used Theorem 2.10-(i) and the fact that $V_j\in\mathcal{V}_n(\Lambda_j,R_3,\infty)$.

In summary, by (5.57) and (5.61) (which give (2.4)), by (5.56) (which gives (2.5) with $s=s_W(\sigma)/8$), and by (5.64), (5.66), (5.59) and (5.60) (which imply, respectively, (2.6), (2.7), (2.8), and (2.9)), we see that Theorem 2.3-(i) can be applied with $V=V_j$ and

$s = s_W(\sigma)/8$ provided j is large enough with respect to σ, n, W and the limit F of the E_j's. As a consequence, setting

$$S_{*j} = \min\Big\{R_{*j}, \frac{\varepsilon_0}{\Lambda_j}\Big\}\,, \tag{5.68}$$

we conclude that, for j large enough, there exist hyperplanes $K_j \subset \mathbb{R}^{n+1}$ with $0 \in K_j$, and functions $u_j \in C^1(\Sigma_{K_j} \times (s_W(\sigma)/32, S_{*j}/16))$, such that

$$\begin{aligned}
&(\operatorname{spt} V_j) \cap A^{S_{*j}/16}_{s_W(\sigma)/32} = \Big\{r\,\frac{\omega + u_j(r,\omega)\,\nu_{K_j}}{\sqrt{1+u_j(r,\omega)^2}} : \omega \in \Sigma_{K_j}\,, r \in \Big(\frac{s_W(\sigma)}{32}, \frac{S_{*j}}{16}\Big)\Big\}\\
&\sup\Big\{|u_j| + |\nabla^{K_j\cap\mathbb{S}^n} u_j| + |r\,\partial_r u_j| : (\omega, r) \in \Sigma_{K_j} \times \Big(\frac{s_W(\sigma)}{32}, \frac{S_{*j}}{16}\Big)\Big\} \le \sigma\,,
\end{aligned} \tag{5.69}$$

where, thanks to (5.65) and (5.55) we can assume to have

$$S_{*j} \ge 16\,R_1\,v_j^{1/(n+1)}\,, \tag{5.70}$$

for a constant R_1 depending on n and W only. Using the more compact notation for spherical graphs introduced in section 2, we can rewrite (5.69) as

$$(\partial E_j) \cap A^{R_1\,v_j^{1/(n+1)}}_{s_W(\sigma)/32} = \Sigma_{K_j}\Big(u_j, \frac{s_W(\sigma)}{32}, R_1\,v_j^{1/(n+1)}\Big)\,, \tag{5.71}$$

for $u_j \in \mathcal{X}_\sigma(\Sigma_{K_j}, \sigma_W(\sigma)/32, R_1\,v_j^{1/(n+1)})$. Similarly, by (5.45), (5.47), and (5.48), thanks to Theorem 2.3-(ii) we have that

$$(\partial F) \cap \big(\mathbb{R}^{n+1} \setminus B_{s_W(\sigma)/32}\big) = \Sigma_{\nu^\perp}\Big(u, \frac{s_W(\sigma)}{32}, \infty\Big)\,, \tag{5.72}$$

for $u \in \mathcal{X}_{\sigma'}(\Sigma_{\nu^\perp}, s_W(\sigma)/32, \infty)$ for every $\sigma' > \sigma$. Now, by $E_j \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, (5.71) and (5.72) can hold only if

$$|\nu_{K_j} - \nu| \le \zeta(\sigma)$$

for a function ζ, depending on n and W only, such that $\zeta(\sigma) \to 0$ as $\sigma \to 0^+$. In particular, if we denote for the sake of clarity by σ_0^*, ε_0^* and C_0^* the dimension dependent constants that we originally introduced in Lemma 2.8 as σ_0, ε_0 and C_0, then we can find $\sigma_1 = \sigma_1(n, W) \le \sigma_0^*$ such that if $\sigma < \sigma_1$, then $\varepsilon_0^* \ge \zeta(\sigma) \ge |\nu_{K_j} - \nu|$, and correspondingly, Lemma 2.8-(i) can be used to infer the existence of

$$u_j^* \in \mathcal{X}_{C_0\,(\sigma+\zeta(\sigma))}(\Sigma_{\nu^\perp}, s_W(\sigma)/32, 2\,R_1\,v_j^{1/(n+1)}) \ \text{ such that}$$

$$\begin{aligned}
\Sigma_{\nu^\perp}\Big(u_j^*, \frac{s_W(\sigma)}{32}, 2\,R_1\,v_j^{1/(n+1)}\Big) &= \Sigma_{K_j}\Big(u_j, \frac{s_W(\sigma)}{32}, 2\,R_1\,v_j^{1/(n+1)}\Big)\\
&= (\partial E_j) \cap A^{2\,R_1\,v_j^{1/(n+1)}}_{s_W(\sigma)/32}\,,
\end{aligned} \tag{5.73}$$

for every j large enough. We can now use Lemma D.1 in the appendix to translate (5.73) in terms of cylindrical graphs: more precisely, if σ_1 is sufficiently small, then, by using Lemma D.1 and keeping in mind (5.53), we can find functions $g_j \in C^1(\nu^\perp)$ such that

$$\sup_{x\in\nu^\perp}\Big\{\frac{|g_j(x)|}{|x|}, |\nabla g_j(x)|\Big\} \le C\,\big(\sigma + \zeta(\sigma)\big)\,, \tag{5.74}$$

$$(\partial E_j) \cap A^{R_1\,v_j^{1/(n+1)}}_{2\,R_2} = \Big\{x + g_j(x)\,\nu : x \in \nu^\perp\Big\} \cap A^{R_1\,v_j^{1/(n+1)}}_{2\,R_2}\,. \tag{5.75}$$

At the same time, by (5.44) and the decay properties in (1.16), we see that, up to further increase the value of R_2, and up to further decrease the value of σ_1, we can exploit Lemma

D.2 in the appendix to find functions $h_j \in C^1(G(f))$, $G(f) = \{x + f(x)\,\nu : x \in \nu^\perp\}$, such that

$$\Big\{x + g_j(x)\,\nu : x \in \nu^\perp\Big\} \setminus B_{4\,R_2} = \Big\{z + h_j(z)\,\nu_F(z) : z \in G(f)\Big\} \setminus B_{4\,R_2}$$

which, combined with (5.44) and (5.75) shows that

$$(\partial E_j) \cap A_{4\,R_2}^{R_1\,v_j^{1/(n+1)}} = \Big\{z + h_j(z)\,\nu_F(z) : z \in \partial F\Big\} \cap A_{4\,R_2}^{R_1\,v_j^{1/(n+1)}}$$

that is (5.49). By $E_j \to F$ in $L^1_{\rm loc}(\mathbb{R}^{n+1})$, we find $h_j \to 0$ in $L^1((\partial F) \cap A^M_{4\,R_2})$ for every $M < \infty$, so that, by elliptic regularity, (5.50) follows. We now recall that by Theorem 3.1-(ii) we have

$$\begin{aligned}&(\partial E_j) \setminus B_{R_0(v_j)\,v_j^{1/(n+1)}} \\ &= \Big\{y + v_j^{1/(n+1)} w_j\Big(\frac{y - x_j}{v_j^{1/(n+1)}}\Big)\,\nu_{B^{(v_j)}(x_j)}(y) : y \in \partial B^{(v_j)}(x_j)\Big\} \setminus B_{R_0(v_j)\,v_j^{1/(n+1)}}\,,\end{aligned} \tag{5.76}$$

for functions $w_j : \partial B^{(1)} \to \mathbb{R}$ such that $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$ and for $R_0(v_j) \to 0$ as $j \to \infty$. The overlapping of (5.75) and (5.76) (i.e., the fact that $R_0(v_j) < R_1$ if j is large enough) imply (5.52). Finally, combining (5.74) and (5.75) with (5.76) and $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$ we deduce the validity of (5.51). More precisely, rescaling by $\lambda_j = v_j^{1/(n+1)}$ in (5.74) and (5.75) and setting $E_j^* = E_j/\lambda_j$, we find that there are functions $g_j^* \in C^1(\nu^\perp)$ with

$$\sup_{x \in \nu^\perp} \Big\{\frac{|g_j^*(x)|}{|x|}, |\nabla g_j^*(x)|\Big\} \le C\,\big(\sigma + \zeta(\sigma)\big)\,, \tag{5.77}$$

$$(\partial E_j^*) \cap A_{2\,R_2/\lambda_j}^{R_1} = \Big\{x + g_j^*(x)\,\nu : x \in \nu^\perp\Big\} \cap A_{2\,R_2/\lambda_j}^{R_1}\,, \tag{5.78}$$

for every $j \ge j_0(\sigma)$ and for every $\sigma < \sigma_1$, while rescaling by λ_j in (5.76) and setting $z_j = x_j/\lambda_j$ we find

$$\begin{aligned}&(\partial E_j^*) \setminus B_{R_0(v_j)} \\ &= \Big\{z_j + z + w_j(z)\,\nu_{B^{(1)}}(z) :: y \in \partial B^{(1)}(z_j)\Big\} \setminus B_{R_0(v_j)}\,,\end{aligned} \tag{5.79}$$

where $||z_j| - \omega_{n+1}^{1/(n+1)}| \to 0$ thanks to (3.4). Up to extracting a further subsequence, $z_j \to z_0$ as $j \to \infty$, where $|z_0| = \omega_{n+1}^{1/(n+1)}$. Should $z_0 \ne |z_0|\,\nu$, then picking σ small enough in terms of $|\nu - (z_0/|z_0|)| > 0$ and picking j large enough, we would then be able to exploit (5.77) to get a contradiction with $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$.

Conclusion: Theorem 3.1 implies Theorem 1.7-(i), and (1.21) was proved in step three. Should Theorem 1.7-(ii), (iii), or (iv) fail, then we could find a sequence $\{(E_j, v_j)\}_j$ contradicting the conclusions of either step five or Theorem 3.1. We have thus completed the proof of Theorem 1.7. □

Appendix A. Proof of Lemma 2.6

The two dimension dependent constants ε_0 and σ_0 involved in the statement will be taken in the relation

$$\sigma_0 = \frac{\varepsilon_0}{C_*}$$

for a sufficiently large dimension dependent constant C_*, to be determined in the course of the proof.

Step one: We prove statement (i): we show that if $H, K \in \mathcal{H}$, $|\nu_H - \nu_K| \le \varepsilon < \varepsilon_0$ and $u \in \mathcal{X}_\sigma(\Sigma_H)$ with $\sigma < \sigma_0$, then the map $T_u^K : \Sigma_H \to \Sigma_K$ defined by

$$T_u^K(\omega) = \frac{\mathbf{p}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H}{|\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H|}\,, \qquad \omega \in \Sigma_H\,,$$

is a diffeomorphism between Σ_H and Σ_K; and that the function $v_u^K : \Sigma_K \to \mathbb{R}$ defined by

$$v_u^K(T_u^K(\omega)) = \frac{\mathbf{q}_K(f_u(\omega))}{|\mathbf{p}_K(f_u(\omega))|} = \frac{\nu_K \cdot (\omega + u(\omega)\,\nu_H)}{|\mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H|}\,, \qquad \omega \in \Sigma_H\,, \tag{A.1}$$

is such that

$$v_u^K \in \mathcal{X}_{C(n)\,(\sigma+\varepsilon)}(\Sigma_K)\,, \qquad \Sigma_H(u) = \Sigma_K(v_u^K)\,, \tag{A.2}$$

and

$$\Big|\int_{\Sigma_K} (v_u^K)^2 - \int_{\Sigma_H} u^2\Big| \le C(n)\Big\{|\nu_H - \nu_K|^2 + \int_{\Sigma_H} u^2\Big\}\,. \tag{A.3}$$

Indeed, let us set, for $\omega \in \Sigma_H$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$g_u^K(\omega) = \mathbf{p}_K\omega + u(\omega)\,\mathbf{p}_K\nu_H\,, \qquad \Phi(x) = \frac{x}{|x|}\,,$$

so that $T_u^K = \Phi \circ g_u^K$, and let us notice that, when u is identically 0, we have

$$g_0^K(\omega) = \mathbf{p}_K\omega\,, \qquad T_0^K(\omega) = \frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|}\,, \qquad \forall \omega \in \Sigma_H\,.$$

By $|\mathbf{p}_K\nu_H|^2 = 1 - (\nu_H\cdot\nu_K)^2 \le 2\,(1 - (\nu_H\cdot\nu_K)) = |\nu_H - \nu_K|^2$, we easily see that

$$\begin{aligned}
|g_u^K - g_0^K| &= |u|\,|\mathbf{p}_K\nu_H| \le |u|\,|\nu_H - \nu_K|\,,\\
|\nabla^{\Sigma_H} g_u^K - \nabla^{\Sigma_H} g_0^K| &\le |\nabla^{\Sigma_H} u|\,|\nu_H - \nu_K|\,.
\end{aligned}$$

In particular, $|g_u^K| \ge 1 - \sigma_0\,\varepsilon_0 \ge 1/2$, and since Φ and $\nabla\Phi$ are Lipschitz continuous on $\{|x| \ge 1/2\}$, we find

$$\max\big\{\|g_u^K - g_0^K\|_{C^1(\Sigma_H)}, \|T_u^K - T_0^K\|_{C^1(\Sigma_H)}\big\} \le C(n)\,\|u\|_{C^1(\Sigma_H)}\,|\nu_H - \nu_K|\,. \tag{A.4}$$

Similarly, since $\omega\cdot\nu_K = \omega\cdot(\nu_K - \nu_H)$ for $\omega \in \Sigma_H$, we find that

$$\|g_0^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\,|\nu_H - \nu_K|\,, \qquad \|T_0^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\,|\nu_H - \nu_K|\,, \tag{A.5}$$

and we thus conclude that T_u^K is a diffeomorphism between Σ_H and Σ_K. As a consequence, the definition (A.1) of v_u^K makes sense, and (A.2) immediately follows (in particular, $\Sigma_H(u) = \Sigma_K(v_u^K)$ is deduced easily from (A.1) and the definition of spherical graph (2.14)). Finally, if we set

$$F_u^K(\omega) = v_u^K(T_u^K(\omega))^2\,J^{\Sigma_H}\,T_u^K(\omega)\,, \qquad \omega \in \Sigma_H\,,$$

then

$$\int_{\Sigma_K} (v_u^K)^2 - \int_{\Sigma_H} u^2 = \int_{\Sigma_H} \Big(\frac{\nu_K\cdot(\omega + u\,\nu_H)}{|g_u^K(\omega)|}\Big)^2 J^{\Sigma_H}\,T_u^K(\omega) - u^2\,,$$

where, using again $|\omega\cdot\nu_K| \le |\nu_H - \nu_K|$ for every $\omega \in \Sigma_H$, we find

$$\begin{aligned}
|J^{\Sigma_H} T_u^K(\omega) - 1| &\le C(n)\,\|T_u^K - \mathrm{id}\|_{C^1(\Sigma_H)} \le C(n)\,|\nu_H - \nu_K|\,,\\
\big|1 - |g_u^K(\omega)|^2\big| &\le \big|1 - |\mathbf{p}_K\omega|^2\big| + |\mathbf{p}_K\nu_H|\,u^2 + 2\,|u|\,|\mathbf{p}_K\nu_H|\,|\mathbf{p}_K\omega|\\
&\le C\,\big(|\nu_H - \nu_K|^2 + u^2\big)\,,\\
\big|(\nu_K\cdot(\omega + u\,\nu_H))^2 - u^2\big| &\le |\nu_K\cdot\omega|^2 + u^2\,(1 - (\nu.\nu_K)^2) + 2\,|u|\,|\nu_H\cdot\nu_K|\,|\omega\cdot\nu_K|\\
&\le |\nu_K - \nu_K|^2 + 2\,u^2\,|\nu_H - \nu_K| + 2\,|u|\,|\nu_H - \nu_K|\\
&\le C\,\big(|\nu_H - \nu_K|^2 + u^2\big)
\end{aligned}$$

therefore

$$
\begin{aligned}
\Big|\int_{\Sigma_K}(v_u^K)^2-\int_{\Sigma_H}u^2\Big| \;\; &\le \;\; \int_{\Sigma_H}|J^{\Sigma_H}\,T_u^K-1|\,u^2+2\int_{\Sigma_H}\frac{|(\nu_K\cdot(\omega+u\,\nu_H))^2-u^2|}{|g_u^K|^2}\\
&\quad +2\int_{\Sigma_H}\Big|1-\frac{1}{|g_u^K|^2}\Big|\,u^2\le C(n)\Big(|\nu_H-\nu_K|^2+\int_{\Sigma_H}u^2\Big)\,,
\end{aligned}
$$

that is (A.3).

Step two: We prove statement (ii): if $H\in\mathcal{H}$ and $u\in\mathcal{X}_{\sigma_0}(\Sigma_H)$, then there exist $K\in\mathcal{H}$ and $v\in\mathcal{X}_{C_0\,\sigma_0}(\Sigma_K)$ such that

$$
\Sigma_H(u)=\Sigma_K(v)\,, \tag{A.6}
$$

$$
E^0_{\Sigma_K}[v]=0\,, \tag{A.7}
$$

$$
|\nu_K-\nu_H|^2\le C(n)\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u]\big)^2\,, \tag{A.8}
$$

$$
\Big|\int_{\Sigma_K}v^2-\int_{\Sigma_H}u^2\Big|\le C(n)\int_{\Sigma_H}u^2\,, \tag{A.9}
$$

We first notice that, if $E^0_{\Sigma_H}[u]=0$, then we can just set $K=H$, $v=u$. Therefore, we can assume that

$$
\gamma^2=\int_{\Sigma_H}\big(E^0_{\Sigma_H}[u]\big)^2>0\,,
$$

and pick an $L^2(\Sigma_H)$-orthonormal basis $\{\phi^i_H\}_{i=1}^n$ of $L^2(\Sigma_H)\cap\{E^0_{\Sigma_H}=0\}$ so that

$$
E^0_{\Sigma_H}[u]=\gamma\,\phi^1_H\,,\qquad \gamma=\int_{\Sigma_H}u\,\phi^1_H\neq 0\,.
$$

Notice that the choice of $\{\phi^i_H\}_{i=1}^n$ actually corresponds to the choice of an orthonormal basis $\{\tau^i_H\}_{i=1}^n$ of H with the property that

$$
\phi^i_H(\omega)=c_0(n)\,\omega\cdot\tau^i_H\,,\qquad \omega\in\Sigma_H\,, \tag{A.10}
$$

for $c_0(n)$ as in (2.16). For each $K\in\mathcal{H}$ with $\mathrm{dist}_{\mathbb{S}^n}(\nu_H,\nu_K)<\varepsilon_0$ we define an orthonormal basis $\{\tau^i_K\}_{i=1}^n$ of K by parallel transport of $\{\tau^i_H\}_{i=1}^n\subset H\equiv T_{\nu_H}\mathbb{S}^n$ to $K\equiv T_{\nu_K}\mathbb{S}^n$. The maps $\nu\mapsto\tau^i(\nu):=\tau^i_{K(\nu)}$ define an orthonormal frame $\{\tau^i\}_{i=1}^n$ of $\mathbb{S}^n$ on the open set A given by

$$
A=B^{\mathbb{S}^n}_{\varepsilon_0}(\nu_H)=\big\{\nu\in\mathbb{S}^n:\mathrm{dist}_{\mathbb{S}^n}(\nu,\nu_H)<\varepsilon_0\big\}\,.
$$

We denote by ρ^K_H the rotation of $\mathbb{R}^{n+1}$ which takes H into K by setting

$$
\rho^K_H(\tau^i_H)=\tau^i_K\,,\qquad \rho^K_H(\nu_H)=\nu_K\,.
$$

By the properties of parallel transport we have that

$$
\|\rho^K_H-\mathrm{Id}\|_{C^0(\Sigma_K)}\le C(n)\,\mathrm{dist}_{\mathbb{S}^n}(\nu_H,\nu_K)\le C(n)\,\varepsilon_0\,. \tag{A.11}
$$

Finally, we define an $L^2(\Sigma_K)$-orthonormal basis $\{\phi^i_K\}_{i=1}^n$ of $L^2(\Sigma_K)\cap\{E^0_{\Sigma_K}=0\}$ by setting

$$
\phi^i_K(\omega)=c_0(n)\,\omega\cdot\tau^i_K\,,\qquad \omega\in\Sigma_K\,,
$$

and correspondingly we consider the map $\Psi_u:A\to\mathbb{R}^n$ defined by setting

$$
\Psi_u(\nu)=\Big(\int_{\Sigma_{K(\nu)}}v_u^{K(\nu)}\,\phi^1_{K(\nu)},\dots,\int_{\Sigma_{K(\nu)}}v_u^{K(\nu)}\,\phi^n_{K(\nu)}\Big)\,,\qquad \nu\in A\,,
$$

where $v_u^{K(\nu)}$ is well-defined for every $\nu\in A$ thanks to step one.

We now claim the existence of $\nu_*\in A$ such that

$$
\Psi_u(\nu_*)=0\,. \tag{A.12}
$$

By the area formula, (A.1), and $\mathbf{q}_{K(\nu)}[e]=\nu\cdot e$, we find

$$
\begin{aligned}
(e_j\cdot\Psi_u)(\nu) &:= \int_{\Sigma_{K(\nu)}} v_u^{K(\nu)}\,\phi^j_{K(\nu)}\\
&= \int_{\Sigma_H} v_u^{K(\nu)}(T_u^{K(\nu)})\,\phi^j_{K(\nu)}(T_u^{K(\nu)})\,J^{\Sigma_H}T_u^{K(\nu)}\\
&= c_0(n)\int_{\Sigma_H}\nu\cdot(\omega+u\,\nu_H)\,\Big(\rho_H^{K(\nu)}[\tau^j_H]\cdot\frac{\mathbf{p}_K(\omega+u\,\nu_H)}{|\mathbf{p}_K(\omega+u\,\nu_H)|^2}\Big)\,J^{\Sigma_H}T_u^{K(\nu)}\,d\mathcal{H}^{n-1}_\omega\,,
\end{aligned}
$$

so that (A.4) gives

$$\|\Psi_u-\Psi_0\|_{C^1(A)}\le C(n)\,\sigma_0\,, \tag{A.13}$$

where

$$e_j\cdot\Psi_0(\nu)=c_0(n)\int_{\Sigma_H}(\nu\cdot\omega)\,\Big(\rho_H^{K(\nu)}[\tau^j_H]\cdot\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|^2}\Big)\,J^{\Sigma_H}\Big[\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|}\Big]\,d\mathcal{H}^{n-1}_\omega\,.$$

By definition of A and by (A.5) and (A.11),

$$\sup_{\nu\in A}\sup_{\omega\in\Sigma_H}\Big|\tau^j_H\cdot\omega-\Big(\rho_H^{K(\nu)}[\tau^j_H]\cdot\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|^2}\Big)\,J^{\Sigma_H}\Big[\frac{\mathbf{p}_K\omega}{|\mathbf{p}_K\omega|}\Big]\Big|\le C(n)\,\varepsilon_0\,,$$

so that

$$\|\Psi_0-\Psi_*\|_{C^1(A)}\le C(n)\,(\sigma_0+\varepsilon_0)\,, \tag{A.14}$$

where $\Psi_*:A\to\mathbb{R}^n$ is defined by

$$e_j\cdot\Psi_*(\nu)=c_0(n)\int_{\Sigma_H}(\nu\cdot\omega)\,(\tau^j_H\cdot\omega)\,d\mathcal{H}^{n-1}_\omega\,,\qquad \nu\in A\,.$$

Recalling that $\{\tau^i\}_{i=1}^n$ is an orthonormal frame of $\mathbb{S}^n$ on A, with $\nabla_{\tau^i}\nu=\tau^i(\nu)=\tau^i_{K(\nu)}=\rho_H^{K(\nu)}[\tau^i_H]$, we find

$$e_j\cdot\nabla_{\tau^i}\Psi_*(\nu)=c_0(n)\int_{\Sigma_H}(\rho_H^{K(\nu)}[\tau^i_H]\cdot\omega)\,(\tau^j_H\cdot\omega)\,d\mathcal{H}^{n-1}_\omega\,,$$

so that, at $\nu=\nu_H$,

$$e_j\cdot\nabla_{\tau^i}\Psi_*(\nu_H)=c_0(n)\int_{\Sigma_H}(\tau^i_H\cdot\omega)\,(\tau^j_H\cdot\omega)\,d\mathcal{H}^{n-1}_\omega=\frac{\delta_{ij}}{c_0(n)}\,.$$

By (A.11), (A.13) and (A.14) we conclude that

$$\|\Psi_u-\Psi_*\|_{C^1(A)}\le C(n)\,(\sigma_0+\varepsilon_0)\,, \tag{A.15}$$

$$\Big\|\nabla^{\mathbb{S}^n}\Psi_u-\frac{1}{c_0(n)}\sum_{j=1}^n e_j\otimes\tau^j\Big\|_{C^0(A)}\le C(n)\,(\sigma_0+\varepsilon_0)\,. \tag{A.16}$$

Let us finally consider the map $h:A\times[0,1]\to\mathbb{R}^n$,

$$h(\nu,t)=h_t(\nu)=t\,\Psi_*(\nu)+(1-t)\,\Psi_u(\nu)\,,\qquad(\nu,t)\in A\times[0,1]\,,$$

which defines an homotopy between Ψ_* and Ψ_u. By (A.15) and (A.16) we see that if $\nu\in\partial A$, that is, if $\mathrm{dist}_{\mathbb{S}^n}(\nu,\nu_H)=\varepsilon_0$, then, denoting by $[\nu_H,\nu]_s$ the unit-speed length minimizing geodesic from ν_H to ν, considering that $[\nu_H,\nu]_s\in A$ for every $s\in(0,\varepsilon_0)$, and that $\mathbb{S}^n$ is close to be flat in A, we find

$$
\begin{aligned}
|h_t(\nu)| &\ge \Big|\int_0^{\varepsilon_0}\frac{d}{ds}\,h_t([\nu_H,\nu]_s)\,ds\Big|-|h_t(\nu_H)|\\
&\ge \Big(\frac{1}{c_0(n)}-C(n)\,(\varepsilon_0+\sigma_0)\Big)\,\varepsilon_0-C(n)\,\sigma_0\ge\frac{\varepsilon_0}{2\,c_0(n)}\,,
\end{aligned}
$$

provided $\sigma_0 = \varepsilon_0/C_*$ is small enough with respect to ε_0 (which can always be arranged by the choice of C_*), ε_0 is small in terms of c_0, and where we have used $\Psi_*(\nu_H) = 0$ and

$$|\Psi_u(\nu_H)| = |\gamma| = \Big|\int_{\Sigma_H} u\,\phi_H^1\Big| \le C(n)\,\sigma_0\,, \tag{A.17}$$

to deduce $|h_t(\nu_H)| \le C(n)\,\sigma_0$. This proves that

$$0 \notin \partial\, h_t(\partial A) \qquad \forall t \in [0,1]\,,$$

so that $\deg(h_t, A, 0)$ is independent of $t \in [0,1]$. In particular, since $h_0 = \Psi_u$ and $h_1 = \Psi_*$, we find

$$\deg(\Psi_u, A, 0) = \deg(\Psi_*, A, 0) = 1\,,$$

where in the last identity we have used $\Psi_*(\nu_H) = 0$ as well as the fact that, up to decreasing the value of ε_0, Ψ_* is injective on A. The fact that $\deg(\Psi_u, A, 0) = 1$ implies the existence of $\nu_* \in A$ such that $\Psi_u(\nu_*) = 0$, as claimed in (A.12).

Setting $K = K(\nu_*)$ and $v = v_u^K$ we deduce (A.6) from (A.2) and (A.7) from $\Psi_u(\nu_*) = 0$. Again by (A.16) and (A.17) we find that

$$\begin{aligned}
\Big(\int_{\Sigma_H} \big(E^0_{\Sigma_H}[u]\big)^2\Big)^{1/2} = |\gamma| &= |\Psi_u(\nu_H)| = |\Psi_u(\nu_H) - \Psi_u(\nu_*)| \\
&= \Big|\int_0^{\mathrm{dist}_{\mathbb{S}^n}(\nu_*,\nu_H)} \frac{d}{ds}\,\Psi_u([\nu_H,\nu_*]_s)\,ds\Big| \\
&\ge \Big(\frac{1}{c_0(n)} - C(n)\,(\varepsilon_0 + \sigma_0)\Big)\,\mathrm{dist}_{\mathbb{S}^n}(\nu_*,\nu_H) \\
&\ge \frac{1}{2\,c_0(n)}\,|\nu_* - \nu_H|\,,
\end{aligned}$$

that is (A.8). Finally, (A.9) follows from (A.8) and (A.3). This completes the proof of Lemma 2.6-(ii).

Appendix B. Proof of Theorem 2.9

Throughout this appendix, $H \in \mathcal{H}$, $\Lambda \ge 0$, $\eta_0 > \eta > 0$, (r_1, r_2) and (r_3, r_4) are (η, η_0)-related as in (2.34), and $u \in \mathcal{X}_\sigma(\Sigma_H, r_1, r_2)$ is such that

$$\Sigma_H(u, r_1, r_2) \text{ has mean curvature bounded by } \Lambda \text{ in } A_{r_1}^{r_2}\,. \tag{B.1}$$

We want to prove the existence of σ_0 and C_0, depending on n, η_0, and η only, such that if $\max\{1, \Lambda\, r_2\}\,\sigma \le \sigma_0$, then

$$\Big|\mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4))\Big| \le C_0 \int_{\Sigma_H \times (r_1,r_2)} r^{n-1}\,\big(u^2 + \Lambda\, r\,|u|\big)\,; \tag{B.2}$$

and such that, if, in addition to (B.1), we also assume

$$\exists\, r \in (r_1, r_2) \text{ s.t. } E^0_{\Sigma_H}[u_r] = 0 \text{ on } \Sigma_H\,, \tag{B.3}$$

then

$$\int_{\Sigma_H \times (r_3,r_4)} r^{n-1}\,u^2 \le C(n)\,\Lambda\, r_2\,(r_2^n - r_1^n) + C_0 \int_{\Sigma_H \times (r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\,. \tag{B.4}$$

We make three preliminary considerations:

(i): For the sake of brevity, it will be convenient to set

$$Q_\zeta(u,v) = \int_{\Sigma_H \times (r_1,r_2)} r^{n-1}\,\zeta(r)^2\,u\,v\,, \qquad Q_\zeta(u) = Q_\zeta(u,u)\,.$$

$$Q_\zeta(X,Y) = \int_{\Sigma_H \times (r_1,r_2)} r^{n-1}\,\zeta(r)^2\,X \cdot Y\,, \qquad Q_\zeta(X) = Q_\zeta(X,X)\,,$$

whenever $\zeta:(r_1,r_2)\to\mathbb{R}$ is a radial function, $u,v:\Sigma_H\times(r_1,r_2)\to\mathbb{R}$, and $X,Y:\Sigma_H\times(r_1,r_2)\to\mathbb{R}^m$.

(ii): The second order expansion of the area functional at $\Sigma_H(u,r_1,r_2)$ takes the form

$$
\begin{aligned}
&\Big|\mathcal{H}^n(\Sigma_H(u,r_1,r_2))-\mathcal{H}^n(\Sigma_H(0,r_1,r_2)) \\
&\qquad -\frac{1}{2}\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\left(|\nabla^{\Sigma_H}u|^2+(r\,\partial_r u)^2-(n-1)\,u^2\right)\Big| \\
\le\;& C(n)\,\sigma\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\big(u^2+|\nabla^{\Sigma_H}u|^2+(r\,\partial_r u)^2\big)\,,
\end{aligned}
\tag{B.5}
$$

see [AA81, 4.5(8)]. Similarly, by combining the last displayed formula on [AA81, Page 236] with [AA81, Lemma 4.9(1)], we find that if $\varphi=\psi^2\,w$ for $w\in C^1(\Sigma_H\times(r_1,r_2))$ and $\psi\in C^1(r_1,r_2)$, then

$$
\begin{aligned}
&\Big|\frac{d}{dt}\Big|_{t=0}\mathcal{H}^n(\Sigma_H(u+t\,\varphi,r_1,r_2)) \\
&\qquad -\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\Big\{\nabla^{\Sigma_H}u\cdot\nabla^{\Sigma_H}\varphi+(r\,\partial_r u)\,(r\,\partial_r\varphi)-(n-1)\,u\,\varphi\Big\}\Big| \\
\le\;& C(n)\,\sigma\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,\psi^2\left(|\nabla^{\Sigma_H}u|^2+|\nabla^{\Sigma_H}w|^2+(r\,\partial_r u)^2+(r\,\partial_r w)^2+u^2+w^2\right) \\
&+C(n)\,\sigma\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\psi')^2\,w^2\,,
\end{aligned}
\tag{B.6}
$$

which is the second order expansion of the first variation of the area at $\Sigma_H(u,r_1,r_2)$ along outer variations in spherical coordinates of the form $\varphi=\psi^2\,w$, $\psi=\psi(r)$.

(iii): The following two estimates (whose elementary proof is contained in [AA81, Lemma 7.13]) hold: whenever $v\in C^1(\Sigma_H\times(r_1,r_2))$, we have

$$
\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,v^2\le C(n,\eta,\eta_0)\Big\{\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r v)^2+\int_{\Sigma_H\times(r_3,r_4)} r^{n-1}\,v^2\Big\}\,, \tag{B.7}
$$

and, provided there exists $r\in[r_1,r_2]$ such that $v_r=0$ on Σ_H, we have

$$
\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,v^2\le C(n,\eta_0)\int_{\Sigma_H\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r v)^2\,. \tag{B.8}
$$

We are now ready for the proof. Compared to the series of lemmas in [AA81, Chapter 4], the main difference is that we need to replace [AA81, Lemma 4.10] with (B.9) (which indeed boils down to [AA81, Lemma 4.10] when $\Lambda=0$).

Step one: We prove that for every $w\in C^1(\Sigma_H\times(r_1,r_2))$ and $\psi\in C^1(r_1,r_2)$ we have

$$
\begin{aligned}
&\Big|T_\psi(u,w)-\int_{\Sigma_H\times(r_1,r_2)} r^n\,\psi^2\,w\,h\Big| \\
&\le C(n)\,\sigma_0\,\Big(Q_\psi(u)+Q_\psi(w)+Q_\psi(\nabla^{\Sigma_H}u)+Q_\psi(\nabla^{\Sigma_H}w) \\
&\qquad\qquad +Q_{r\,\psi}(\partial_r u)+Q_{r\,\psi}(\partial_r w)+Q_{r\,\psi'}(w)\Big)\,.
\end{aligned}
\tag{B.9}
$$

where $h:\Sigma_H\times(r_1,r_2)\to[-\Lambda,\Lambda]$ and where

$$
\begin{aligned}
T_\psi(u,w)&=Q_\psi(\nabla^{\Sigma_H}u,\nabla^{\Sigma_H}w)+Q_r\big(\partial_r u,\partial_r[\psi^2 w]\big)-(n-1)\,Q_\psi(u,w) \\
&=Q_\psi(\nabla^{\Sigma_H}u,\nabla^{\Sigma_H}w)+Q_{r\,\psi}(\partial_r u,\partial_r w)-(n-1)\,Q_\psi(u,w)+2\,Q_r(\psi\,\partial_r u,\psi' w)\,.
\end{aligned}
\tag{B.10}
$$

We start rewriting (B.6) as

$$\begin{aligned}
&\Big|T_\psi(u,w)-\frac{d}{dt}\Big|_{t=0}\mathcal{H}^n(\Sigma(u+t\,\psi^2\,w,r_1,r_2))\Big| \qquad\qquad \text{(B.11)}\\
&\le C(n)\,\sigma\,\Big(Q_\psi(u)+Q_\psi(w)+Q_\psi(\nabla^{\Sigma_H}u)+Q_\psi(\nabla^{\Sigma_H}w)\\
&\qquad\qquad +Q_{r\,\psi}(\partial_r u)+Q_{r\,\psi}(\partial_r w)+Q_{r\,\psi'}(w)\Big)\,.
\end{aligned}$$

Next we define diffeomorphisms $F_{u+t\,\varphi}:\Sigma_H\times(r_1,r_2)\to\Sigma_H(u+t\,\varphi,r_1,r_2)$, $\varphi=\psi^2\,w$, by setting

$$F_{u+t\,\varphi}(\omega,r)=r\,\frac{\omega+(u(\omega,r)+t\varphi(\omega,r))\,\nu_H}{\sqrt{1+(u(\omega,r)+t\varphi(\omega,r))^2}}\,,$$

In particular, $\{\Phi_t=F_{u+t\,\varphi}\circ(F_u)^{-1}\}_{t\in[0,1]}$ defines a one-parameter family of diffeomorphisms on $\Sigma_H(u,r_1,r_2)$, with $\Phi_t(\Sigma_H(u,r_1,r_2))=\Sigma_H(u+t\,\varphi,r_1,r_2)$, and initial velocity

$$\dot\Phi_0=\frac{d}{dt}\Big|_{t=0}\Phi_t\,.$$

Then, by (B.1), for some bounded function $h:\Sigma_H\times(r_1,r_2)\to[-\Lambda,\Lambda]$ we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\mathcal{H}^n(\Sigma(u+t\varphi,r_1,r_2)) &= \Lambda\int_{\Sigma_H(u,r_1,r_2)} h(F_u^{-1})\,\dot\Phi_0\cdot\nu_{\Sigma_H(u,r_1,r_2)}\\
&= \Lambda\int_{\Sigma_H\times(r_1,r_2)} h\,\dot\Phi_0\big(F_u\big)\cdot\star\Big(\partial_rF_u\wedge\bigwedge_{i=1}^{n-1}\partial_iF_u\Big)\,,
\end{aligned}$$

where $\partial_i=\nabla_{\tau_i}$ for a local orthonormal frame $\{\tau_i\}_{i=1}^{n-1}$ in Σ_H, and where $\star$ is the Hodge star-operator (so that $\star(v_1\wedge v_2...\wedge v_n)$ is a normal vector to the hyperplane spanned by the v_is, with length equal to the n-dimensional volume of the parallelogram defined by the v_is, and whose orientation depends on the ordering of the v_is themselves). We can compute the initial velocity of $\{\Phi_t\}_{t\in[0,1]}$ by noticing that

$$\Phi_t\big(F_u(\omega,r)\big)=r\,\frac{\omega+(u+t\,\varphi)\,\nu_H}{\sqrt{1+(u+t\,\varphi)^2}}\,,$$

so that,

$$\begin{aligned}
\dot\Phi_0(F_u) &= \frac{d}{dt}\Big|_{t=0}r\,\frac{\omega+(u+t\,\varphi)\,\nu_H}{\sqrt{1+(u+t\,\varphi)^2}}=r\,\Big(-\frac{u\,\varphi}{(1+u^2)^{3/2}}\,\omega+\frac{\varphi}{(1+u^2)^{3/2}}\,\nu_H\Big)\\
&= r\,\big(-u\,\varphi\,\omega+\varphi\,\nu_H\big)+r\,\sigma\,\mathrm{O}\big(\psi^2\,(u^2+w^2)\big)\,.
\end{aligned}$$

At the same time

$$\begin{aligned}
\partial_rF_u &= \frac{\omega+u\,\nu_H}{\sqrt{1+u^2}}+r\,\partial_r\Big(\frac{\omega+u\,\nu_H}{\sqrt{1+u^2}}\Big)=\frac{\omega+u\,\nu_H}{\sqrt{1+u^2}}-\frac{r\,u\,\partial_ru}{(1+u^2)^{3/2}}\,\omega+\frac{r\,\partial_ru}{(1+u^2)^{3/2}}\,\nu_H\\
&= \Big(1-\frac{u^2}{2}-u\,r\,\partial_ru\Big)\,\omega+\Big(u+r\,\partial_ru\Big)\,\nu_H+\sigma\,\mathrm{O}(u^2+(r\partial_ru)^2)\\
&= A\,\omega+B\,\nu_H+\sigma\,\mathrm{O}(u^2+(r\partial_ru)^2)\,,
\end{aligned}$$

while

$$\begin{aligned}
\frac{\partial_iF_u}{r} &= \partial_i\Big(\frac{\omega+u\,\nu_H}{\sqrt{1+u^2}}\Big)=\frac{\tau_i}{\sqrt{1+u^2}}-\frac{u\,\partial_iu}{(1+u^2)^{3/2}}\,\omega+\frac{\partial_iu}{(1+u^2)^{3/2}}\,\nu_H\\
&= \Big(1-\frac{u^2}{2}\Big)\,\tau_i-u\,\partial_iu\,\omega+\partial_iu\,\nu_H+\sigma\,\mathrm{O}(u^2+(\partial_iu)^2)\\
&= C\,\tau_i+E_i\,\omega+F_i\,\nu_H+\sigma\,\mathrm{O}(u^2+(\partial_iu)^2)
\end{aligned}$$

so that, setting $\hat\tau_i = \bigwedge_{j\neq i}\tau_j$, we find

$$
\begin{aligned}
\frac{\partial_r F_u \wedge \bigwedge_{i=1}^{n-1}\partial_i F_u}{r^{n-1}} &= \big(A\,\omega + B\,\nu_H\big)\wedge\bigwedge_{i=1}^{n-1}\big(C\,\tau_i + E_i\,\omega + F_i\,\nu_H\big)\\
&\quad +\sigma\,\mathrm{O}\big(u^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2\big)\\
&= A\,C^{n-1}\Big(\omega\wedge\bigwedge_{i=1}^{n-1}\tau_i\Big)+B\,C^{n-1}\Big(\nu_H\wedge\bigwedge_{i=1}^{n-1}\tau_i\Big)\\
&\quad +G_i\,\big(\omega\wedge\nu_H\wedge\hat\tau_i\big)+\sigma\,\mathrm{O}\big(u^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2\big)\,,
\end{aligned}
$$

for some coefficient G_i which we shall not need to compute. Indeed, since $\star(\omega\wedge\nu_H\wedge\hat\tau_i)$ is parallel to τ_i, and thus perpendicular to both ω and ν_H, we conclude that

$$
\begin{aligned}
& r^{-n}\,\dot\Phi_0\big(F_u(r,\omega)\big)\cdot\star\Big(\partial_r F_u\wedge\bigwedge_{i=1}^{n-1}\partial_i F_u\Big)\\
=\;& \Big[\big(-u\,\varphi\,\omega+\varphi\,\nu_H\big)+\sigma\,\mathrm{O}\big(\psi^2(u^2+v^2)\big)\Big]\cdot\\
&\qquad\cdot\Big[A\,C^{n-1}\,\nu_H - B\,C^{n-1}\,\omega+\sigma\,\mathrm{O}\big(u^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2\big)\Big]\\
=\;& C^{n-1}\Big[\Big(1-\frac{u^2}{2}-u\,r\,\partial_r u\Big)\,\varphi+\Big(u+r\,\partial_r u\Big)\,u\,\varphi\Big]\\
&\qquad +\sigma\,\mathrm{O}\Big(\psi^2\,\big(u^2+w^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2\big)\Big)\\
=\;& \varphi+\sigma\,\mathrm{O}\Big(\psi^2\,\big(u^2+w^2+|\nabla^{\Sigma_H}u|^2+(r\partial_r u)^2\big)\Big)
\end{aligned}
$$

In particular, since $|h|\le\Lambda$,

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\mathcal{H}^n(\Sigma(u+t\varphi,r_1,r_2)) = \int_{\Sigma\times(r_1,r_2)} h\;\dot\Phi_0\big(F_u\big)\cdot\star\Big(\partial_r F_u\wedge\bigwedge_{i=1}^{n-1}\partial_i F_u\Big)\\
=\;& \int_{\Sigma_H\times(r_1,r_2)} r^n\,\psi^2\,w\,h+\sigma\,\Lambda\,r_2\,\mathrm{O}\Big(Q_\psi(u)+Q_\psi(w)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\Big)\,.
\end{aligned}
$$

Plugging this estimate into (B.11), and taking into account $\max\{1,\Lambda\,r_2\}\,\sigma\le\sigma_0$, we find (B.9).

Step two: We prove that

$$
Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\le Q_\psi(|u|,\Lambda\,r)+C(n)\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)\big)\,. \tag{B.12}
$$

Indeed, by

$$
Q_r(\psi\,\partial_r u,\psi' u)\le\frac{Q_{r\,\psi}(\partial_r u)}{4}+C\,Q_{r\,\psi'}(u)
$$

and by (B.9) with $w=u$ we find

$$
\begin{aligned}
& Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\le Q_\psi(|u|,\Lambda\,r)+C(n)\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)\big)\\
& +C(n)\,\sigma_0\,\Big(Q_\psi(u)+Q_{r\,\psi'}(u)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\Big)\,.
\end{aligned}
$$

which implies (B.12) provided σ_0 is small enough.

Step three: We prove that, if $w:\Sigma_H\times(r_1,r_2)\to\mathbb{R}$ is such that, for every $r\in(r_1,r_2)$,

$$
\int_{\Sigma_H} w_r\,(u_r-w_r)=\int_{\Sigma_H}\partial_r w_r\,(\partial_r u_r-\partial_r w_r)=\int_{\Sigma_H}\nabla^{\Sigma_H}w_r\cdot(\nabla^{\Sigma_H}u_r-\nabla^{\Sigma_H}w_r)=0\,, \tag{B.13}
$$

then

$$|T_\psi(u,w)|\le Q_\psi(|w|,\Lambda\,r)+C(n)\,\sigma_0\,\Big(Q_\psi(u)+Q_{r\,\psi'}(u)+Q_\psi(|u|,\Lambda\,r)\Big)\,. \tag{B.14}$$

Indeed, by (B.13), we find that

$$Q_\zeta(w)\le Q_\zeta(u)\,,\qquad Q_\zeta(\partial_r w)\le Q_\zeta(\partial_r u)\,,\qquad Q_\zeta(\nabla^{\Sigma_H}w)\le Q_\zeta(\nabla^{\Sigma_H}u)\,,$$

whenever $\zeta:(r_1,r_2)\to\mathbb{R}$ is radial. Therefore (B.9) gives

$$|T_\psi(u,w)|\le Q_\psi(|w|,\Lambda\,r)+C(n)\,\sigma_0\,\Big(Q_\psi(u)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)+Q_{r\,\psi'}(u)\Big)\,,$$

which we combine with (B.12) to get (B.14).

Step four: We prove (B.2). Let now ψ be a cut-off function between (r_3,r_4) and (r_1,r_2), so that

$$\Big|\int_{\Sigma_H\times(r_3,r_4)}r^{n-1}\big\{|\nabla^{\Sigma_H}u|^2-(n-1)\,u^2+(r\,\partial_r u)^2\big\}\Big|\le Q_\psi(\nabla^{\Sigma_H}u)+Q_\psi(u)+Q_{r\,\psi}(\partial_r u)\,.$$

Then, by (B.5) with (r_3,r_4) in place of (r_1,r_2), we find

$$\begin{aligned}
&\Big|\mathcal{H}^n(\Sigma_H(u,r_3,r_4))-\mathcal{H}^n(\Sigma_H(0,r_3,r_4))\Big|\\
&\le Q_\psi(\nabla^{\Sigma_H}u)+Q_\psi(u)+Q_{r\,\psi}(\partial_r u)+C(n)\,\sigma\,\Big\{Q_\psi(u)+Q_\psi(\nabla^{\Sigma_H}u)+Q_{r\,\psi}(\partial_r u)\Big\}\\
&\le Q_\psi(|u|,\Lambda\,r)+C(n)\,\big(Q_\psi(u)+Q_{r\,\psi'}(u)\big)\\
&+C(n)\,\sigma\,\Big\{Q_\psi(u)+Q_\psi(|u|,\Lambda\,r)+C(n)\,\Big(Q_\psi(u)+Q_{r\,\psi'}(u)\Big)\Big\}\,,
\end{aligned}$$

where in the last inequality we have used (B.12). We deduce

$$\Big|\mathcal{H}^n(\Sigma_H(u,r_3,r_4))-\mathcal{H}^n(\Sigma_H(0,r_3,r_4))\Big|\le C(n)\,\big(Q_\psi(|u|,\Lambda\,r)+Q_\psi(u)+Q_{r\,\psi'}(u)\big)\,, \tag{B.15}$$

and then (B.2) follows (with C_0 dependent on n, η_0 and η by the properties of ψ).

Step five: We prove that, if $E^0_{\Sigma_H}[u_{r_*}]=0$ for some $r_*\in(r_1,r_2)$, see (B.3), then (B.4) holds, that is

$$\int_{\Sigma_H\times(r_3,r_4)}r^{n-1}\,u^2\le C(n)\,\Lambda\,r_2\,(r_2^n-r_1^n)+C(n,\eta_0,\eta)\int_{\Sigma\times(r_1,r_2)}r^{n-1}\,(r\,\partial_r u)^2\,. \tag{B.16}$$

To this end, we define $u^+,u^-,u^0:\Sigma_H\times(r_1,r_2)\to\mathbb{R}$ by setting, for each $r\in(r_1,r_2)$,

$$(u^+)_r=E^+_{\Sigma_H}[u_r]\,,\qquad (u^-)_r=E^-_{\Sigma_H}[u_r]\,,\qquad (u^0)_r=E^0_{\Sigma_H}[u_r]\,,$$

where $E^\pm_{\Sigma_H}$ denote the $L^2(\Sigma_H)$-orthogonal projections on the spaces of positive/negative eigenvectors of the Jacobi operator of Σ_H, and where $E^0_{\Sigma_H}$ is the $L^2(\Sigma_H)$-orthogonal projection onto the space of the Jacobi fields of Σ_H. Since $(u^0)_{r_*}=0$, we can directly apply (B.8) with $v=u^0$ and deduce that

$$\int_{\Sigma_H\times(r_1,r_2)}r^{n-1}\,(u^0)^2\le C(n,\eta_0)\int_{\Sigma_H\times(r_1,r_2)}r^{n-1}\,(r\,\partial_r u^0)^2\,. \tag{B.17}$$

By the orthogonality relations between u^0_r, u^+_r and u^-_r we have that

$$\int_{\Sigma_H\times(r_3,r_4)}r^{n-1}\,u^2=\int_{\Sigma_H\times(a,b)}r^{n-1}\,\Big((u^0)^2+(u^+)^2+(u^-)^2\Big) \tag{B.18}$$

$$\int_{\Sigma_H\times(r_1,r_2)}r^{n+1}\,(\partial_r u)^2=\int_{\Sigma_H\times(r_1,r_2)}r^{n+1}\,\Big((\partial_r u^0)^2+(\partial_r u^+)^2+(\partial_r u^-)^2\Big)\,. \tag{B.19}$$

By the spectral decomposition theorem we have that

$$\frac{1}{C_1(n)}\int_{\Sigma_H}(u^-)_r^2\le\int_{\Sigma_H}(n-1)\,(u^-)_r^2-|\nabla^{\Sigma_H}(u^-)_r|^2\qquad\forall r\in(r_1,r_2)\,,$$

which, multiplied by $r^{n-1}\,\psi^2$, gives

$$\begin{aligned}\frac{Q_\psi(u^-)}{C_1(n)} &\le (n-1)\,Q_\psi(u^-) - Q_\psi(\nabla^{\Sigma_H} u^-)\\ &= (n-1)\,Q_\psi(u^-,u) - Q_\psi(\nabla^{\Sigma_H} u^-,\nabla^{\Sigma_H} u)\\ &= -T_\psi(u^-,u) + Q_r(\partial_r u, \partial_r(\psi^2\,u^-))\,,\end{aligned}$$

where in the second to last identity we have used the validity of the orthogonality relations (B.13) for $w = u^-$; in particular, by (B.14) with $w = u^-$, we find

$$\begin{aligned}\frac{Q_\psi(u^-)}{C_1(n)} &\le Q_\psi(|u^-|,\Lambda\,r) + C(n)\,\sigma_0\,\Big(Q_\psi(u) + Q_{r\,\psi'}(u) + Q_\psi(|u|,\Lambda\,r)\Big)\\ &\quad + Q_r(\partial_r u, \partial_r(\psi^2\,u^-))\,.\end{aligned} \tag{B.20}$$

Again by (B.13) with $w = u^-$ we have

$$\begin{aligned}Q_r(\partial_r u, \partial_r(\psi^2\,u^-)) &= Q_r(\partial_r u^-, \partial_r(\psi^2\,u^-)) = Q_{r\,\psi}(\partial_r u^-) + 2\,Q_r(\psi'\,\partial_r u^-, \psi\,u^-)\\ &\le Q_{r\,\psi}(\partial_r u^-) + \frac{Q_\psi(u^-)}{2\,C_1(n)} + C(n)\,Q_{r\,\psi'}(\partial_r u^-)\,,\end{aligned}$$

which combined into (B.20) gives

$$\begin{aligned}\frac{Q_\psi(u^-)}{2\,C_1(n)} &\le Q_\psi(|u^-|,\Lambda\,r) + C(n)\,\sigma_0\,\Big(Q_\psi(u) + Q_{r\,\psi'}(u) + Q_\psi(|u|,\Lambda, r)\Big)\\ &\quad + Q_{r\,\psi}(\partial_r u^-) + C(n)\,Q_{r\,\psi'}(\partial_r u^-)\,.\end{aligned}$$

Using Hölder inequality again we have

$$\begin{aligned}&Q_\psi(|u^-|,\Lambda\,r) \le \frac{Q_\psi(u^-)}{4\,C_1(n)} + C(n)\,\Lambda\,r_2\,(r_2^n - r_1^n)\,,\\ &Q_\psi(|u|,\Lambda\,r) \le 2\,Q_\psi(u) + C(n)\,\Lambda\,r_2\,(r_2^n - r_1^n)\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{Q_\psi(u^-)}{4\,C_1(n)} &\le C(n)\,\sigma_0\,\Big(Q_\psi(u) + Q_{r\,\psi'}(u) + \Lambda\,r_2\,(r_2^n - r_1^n)\Big)\\ &\quad + Q_{r\,\psi}(\partial_r u^-) + C(n)\,Q_{r\,\psi'}(\partial_r u^-) + C(n)\,\Lambda\,r_2\,(r_2^n - r_1^n)\,.\end{aligned}$$

By taking ψ to be a cut-off function between (r_3, r_4) and (r_1, r_2), we thus find

$$\begin{aligned}\int_{\Sigma\times(r_3,r_4)} r^{n-1}\,(u^-)^2 &\le C(n)\,\Lambda\,r_2\,(r_2^n - r_1^n) + C(n,\eta_0,\eta)\int_{\Sigma\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u^-)^2\\ &\quad + C(n,\eta_0,\eta)\,\sigma_0\int_{\Sigma\times(r_1,r_2)} r^{n-1}\,u^2\,.\end{aligned} \tag{B.21}$$

By combining (B.17), (B.21), and the analogous estimate to (B.21) for u^+ with (B.18) and (B.19) we find

$$\begin{aligned}\int_{\Sigma\times(r_3,r_4)} r^{n-1}\,u^2 &\le C(n)\,\Lambda\,r_2\,(r_2^n - r_1^n) + C(n,\eta_0,\eta)\int_{\Sigma\times(r_1,r_2)} r^{n-1}\,(r\,\partial_r u)^2\\ &\quad + C(n,\eta_0,\eta)\,\sigma_0\int_{\Sigma\times(r_1,r_2)} r^{n-1}\,u^2\,,\end{aligned}$$

which, thanks to (B.7), finally gives (B.16). This completes the proof of Theorem 2.9.

Appendix C. Proof of the monotonicity formula

We want to prove (see the proof of Theorem 2.10-(i), that, if $V \in \mathcal{V}_n(\Lambda, R, S)$, that is, if V is an n-dimensional integral varifolds in $\mathbb{R}^{n+1}$ such that

$$\begin{aligned}
&\operatorname{spt} V \subset \mathbb{R}^{n+1} \setminus B_R\,,\\
&\int \operatorname{div}^T X\, dV = \int X \cdot \vec{H}\, d\|V\| + \int X \cdot \nu_V^{\rm co}\, d\,\mathrm{bd}_V\,, \qquad \forall X \in C^1_c(B_S; \mathbb{R}^{n+1}) \quad \text{(C.1)}
\end{aligned}$$

hold with $\vec{H} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ a Borel vector field such that $|\vec{H}| \le \Lambda$, bd_V a Radon measure in $\mathbb{R}^{n+1}$, and $\nu_V^{\rm co} : \partial B_R \to \mathbb{R}^{n+1}$ a Borel vector field with

$$\operatorname{spt} \mathrm{bd}_V \subset \partial B_R\,, \qquad |\nu_V^{\rm co}| = 1 \ \mathrm{bd}_V\text{-a.e.}\,;$$

then

$$\Theta_{V,R,\Lambda}(r) = \frac{\|V\|(B_r \setminus B_R)}{r^n} - \frac{1}{n\, r^n} \int x \cdot \nu_V^{\rm co}\, d\,\mathrm{bd}_V + \Lambda \int_R^r \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n}\, d\rho\,,$$

is increasing on (R, S), with

$$\frac{d\Theta_{V,R,\Lambda}}{d\rho} \ge \frac{d}{d\rho} \int_{B_\rho \setminus B_R} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\|\,,$$

for a.e. $\rho \in (R, S)$ (compare to (2.45)). Denoting by M a locally $\mathcal{H}^n$-rectifiable set such that $V = \mathbf{var}\,(M, \theta)$ for some multiplicity function θ, we take $\zeta \in C^1_c([0,1))$ and $s \in (R, S)$, set $r(x) = |x|$ and $X(x) = \zeta(r(x)/s)\, x$, so that $X \in C^1_c(B_S; \mathbb{R}^{n+1})$, and test (C.1) with X. If $\nu_M : M \to \mathbb{S}^n$ is such that $T_x M = \nu_M(x)^\perp$ at $\mathcal{H}^n$-a.e. $x \in M$, then

$$\operatorname{div}^M X = \frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big)\big(1 - (\hat{x} \cdot \nu_M)^2\big) + n\, \zeta\Big(\frac{r}{s}\Big)\,, \qquad \hat{x} = \frac{x}{|x|}\,,$$

and (C.1) gives

$$\begin{aligned}
&\int_{B_S \setminus B_R} \Big\{\frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big) + n\, \zeta\Big(\frac{r}{s}\Big)\Big\}\, d\|V\| - \int_{B_S \setminus B_R} \frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big)\, (\hat{x} \cdot \nu_M)^2\, d\|V\|\\
=\;& \int_{B_S \setminus B_R} (x \cdot \vec{H})\, \zeta\Big(\frac{r}{s}\Big)\, d\|V\| + \int_{\partial B_R} (x \cdot \nu_V^{\rm co})\, \zeta\Big(\frac{R}{s}\Big)\, d\,\mathrm{bd}_V\,.
\end{aligned}$$

We multiply by s^{-n-1} and integrate in s over $(\sigma, \rho) \subset\subset (R, S)$ to obtain

$$\begin{aligned}
&\int_{B_S \setminus B_R} d\|V\| \int_\sigma^\rho \Big\{\frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big) + n\, \zeta\Big(\frac{r}{s}\Big)\Big\} \frac{ds}{s^{n+1}} - \int_{B_S \setminus B_R} (\hat{x} \cdot \nu_M)^2\, d\|V\| \int_\sigma^\rho \frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big) \frac{ds}{s^{n+1}}\\
=\;& \int_{B_S \setminus B_R} (x \cdot \vec{H})\, d\|V\| \int_\sigma^\rho \zeta\Big(\frac{r}{s}\Big) \frac{ds}{s^{n+1}} + \int_{\partial B_R} (x \cdot \nu_V^{\rm co})\, d\,\mathrm{bd}_V \int_\sigma^\rho \zeta\Big(\frac{R}{s}\Big) \frac{ds}{s^{n+1}}\,.
\end{aligned}$$

We notice that

$$\frac{d}{ds}\, \frac{1}{s^n}\, \zeta\Big(\frac{r}{s}\Big) = -\frac{n}{s^{n+1}}\, \zeta\Big(\frac{r}{s}\Big) - \frac{r}{s^{n+2}}\, \zeta'\Big(\frac{r}{s}\Big)$$

implies

$$\int_{B_S \setminus B_R} d\|V\| \int_\sigma^\rho \Big\{\frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big) + n\, \zeta\Big(\frac{r}{s}\Big)\Big\} \frac{ds}{s^{n+1}} = \int_{B_S \setminus B_R} \Big\{\frac{1}{\sigma^n}\, \zeta\Big(\frac{r}{\sigma}\Big) - \frac{1}{\rho^n}\, \zeta\Big(\frac{r}{\rho}\Big)\Big\}\, d\|V\|$$

and

$$\begin{aligned}
&- \int_{B_S \setminus B_R} (\hat{x} \cdot \nu_M)^2\, d\|V\| \int_\sigma^\rho \frac{r}{s}\, \zeta'\Big(\frac{r}{s}\Big) \frac{ds}{s^{n+1}}\\
&= \int_{B_S \setminus B_R} \Big\{\frac{1}{\rho^n}\, \zeta\Big(\frac{r}{\rho}\Big) - \frac{1}{\sigma^n}\, \zeta\Big(\frac{r}{\sigma}\Big) + n \int_\sigma^\rho \zeta\Big(\frac{r}{s}\Big) \frac{ds}{s^{n+1}}\Big\} (\hat{x} \cdot \nu_M)^2\, d\|V\|
\end{aligned}$$

Therefore, letting $\zeta \to 1_{[0,1)}$ in

$$\begin{aligned}
&\int_{B_S\setminus B_R} \Big\{\frac{1}{\sigma^n}\,\zeta\Big(\frac{r}{\sigma}\Big) - \frac{1}{\rho^n}\,\zeta\Big(\frac{r}{\rho}\Big)\Big\}\, d\|V\| \\
&\quad + \int_{B_S\setminus B_R} \Big\{\frac{1}{\rho^n}\,\zeta\Big(\frac{r}{\rho}\Big) - \frac{1}{\sigma^n}\,\zeta\Big(\frac{r}{\sigma}\Big) + n\int_\sigma^\rho \zeta\Big(\frac{r}{s}\Big)\,\frac{ds}{s^{n+1}}\Big\}(\hat{x}\cdot\nu_M)^2\, d\|V\| \\
= &\int_{B_S\setminus B_R} (x\cdot\vec{H})\, d\|V\| \int_\sigma^\rho \zeta\Big(\frac{r}{s}\Big)\frac{ds}{s^{n+1}} + \int_{\partial B_R} (x\cdot\nu_V^{\mathrm{co}})\, d\,\mathrm{bd}_V \int_\sigma^\rho \zeta\Big(\frac{R}{s}\Big)\frac{ds}{s^{n+1}}\,,
\end{aligned}$$

one obtains

$$\begin{aligned}
&\int_{B_S\setminus B_R} \Big\{\frac{1_{B_\sigma}}{\sigma^n} - \frac{1_{B_\rho}}{\rho^n}\Big\}\, d\|V\| \\
&\quad + \int_{B_S\setminus B_R} \Big\{\frac{1_{B_\rho}}{\rho^n} - \frac{1_{B_\sigma}}{\sigma^n} + n\int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\,\frac{ds}{s^{n+1}}\Big\}(\hat{x}\cdot\nu_M)^2\, d\|V\| \\
= &\int_{B_S\setminus B_R} (x\cdot\vec{H})\, d\|V\| \int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\frac{ds}{s^{n+1}} + \int_{\partial B_R} (x\cdot\nu_V^{\mathrm{co}})\, d\,\mathrm{bd}_V \int_\sigma^\rho 1_{[0,1)}\Big(\frac{R}{s}\Big)\frac{ds}{s^{n+1}}\,.
\end{aligned}$$

We compute that

$$\int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\frac{ds}{s^{n+1}} = 1_{B_\rho}\int_{\max\{r,\sigma\}}^\rho \frac{ds}{s^{n+1}} = \Big(\frac{1}{\max\{r,\sigma\}^n} - \frac{1}{\rho^n}\Big)\frac{1_{B_\rho}}{n}\,,$$

so that

$$\begin{aligned}
&\int_{B_S\setminus B_R} \Big\{\frac{1_{B_\rho}}{\rho^n} - \frac{1_{B_\sigma}}{\sigma^n} + n\int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\,\frac{ds}{s^{n+1}}\Big\}(\hat{x}\cdot\nu_M)^2\, d\|V\| \\
&= \int_{B_S\setminus B_R} \Big\{\frac{1_{B_\rho}}{\max\{r,\sigma\}^n} - \frac{1_{B_\sigma}}{\sigma^n}\Big\}\,(\hat{x}\cdot\nu_M)^2\, d\|V\| = \int_{B_\rho\setminus B_\sigma} \frac{(\hat{x}\cdot\nu_M)^2}{r^n}\, d\|V\|\,,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_\rho\setminus B_\sigma} \frac{(\hat{x}\cdot\nu_M)^2}{r^n}\, d\|V\| = \frac{\|V\|(B_\rho\setminus B_R)}{\rho^n} - \frac{\|V\|(B_\sigma\setminus B_R)}{\sigma^n} \\
&\quad + \int_{B_S\setminus B_R} (x\cdot\vec{H})\, d\|V\| \int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\frac{ds}{s^{n+1}} + \frac{1}{n}\Big(\frac{1}{\sigma^n} - \frac{1}{\rho^n}\Big)\int_{\partial B_R} (x\cdot\nu_V^{\mathrm{co}})\, d\,\mathrm{bd}_V\,,
\end{aligned}$$

where we have used $1_{[0,1)}(R/s) = 1$ for $s \in (\sigma,\rho)$. Finally, by

$$\begin{aligned}
\Big|\int_{B_S\setminus B_R} (x\cdot\vec{H})\, d\|V\| \int_\sigma^\rho 1_{[0,1)}\Big(\frac{r}{s}\Big)\frac{ds}{s^{n+1}}\Big| \;&\le\; \Lambda\int_\sigma^\rho \frac{1}{s^{n+1}}\int_{B_s\setminus B_R} r\, d\|V\| \\
&\le\; \Lambda\int_\sigma^\rho \frac{\|V\|(B_s\setminus B_R)}{s^n}\, ds\,,
\end{aligned}$$

we conclude as claimed that

$$\Theta_{V,\Lambda,R}(\rho) - \Theta_{V,\Lambda,R}(\sigma) \ge \int_{B_\rho\setminus B_\sigma} \frac{(\hat{x}\cdot\nu_M)^2}{r^n}\, d\|V\| = \int_{B_\rho\setminus B_\sigma} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\|V\|\,,$$

whenever $(\sigma,\rho) \subset (R,S)$.

Appendix D. Auxiliary facts on spherical and cylindrical graphs

In this appendix we prove for the sake of completeness two technical lemmas concerning spherical and cylindrical graphs. They are both used in the last step of the proof of Theorem 1.7.

Lemma D.1 (Spherical graphs as cylindrical graphs)**.** *There exists dimension independent positive constants C and η_0 with the following property. If $n \ge 1$, $H \in \mathcal{H}$ and $u \in \mathcal{X}_\eta(\Sigma_H, r_1, r_2)$ with $\eta < \eta_0$, then we have*

$$\mathbf{D}^{\nu_H}_{(1-C\,\eta^2)\,r_2} \setminus \mathbf{D}^{\nu_H}_{r_1} \subset \mathbf{p}_H\big(\Sigma_H(u, r_1, r_2)\big) \subset \mathbf{D}^{\nu_H}_{r_2} \setminus \mathbf{D}^{\nu_H}_{(1-C\,\eta^2)\,r_1}\,, \tag{D.1}$$

and there exists $g \in C^1(H)$ such that

$$\Sigma_H(u, r_1, r_2) = \Big\{x + g(x)\,\nu_H : x \in \mathbf{p}_H\big(\Sigma_H(u, r_1, r_2)\big)\Big\}\,, \tag{D.2}$$

$$\sup\Big\{\frac{|g(x)|}{|x|} + |\nabla g(x)| : x \in H\Big\} \le C\,\eta\,. \tag{D.3}$$

Moreover, if $(\rho_1, \rho_2) \subset ((1 + C\,\eta)\,r_1, (1 - C\,\eta^2)\,r_2)$, then we have

$$\Sigma_H(u, \rho_1, \rho_2) = \Big\{x + g(x)\,\nu_H : x \in H\Big\} \cap A^{\rho_2}_{\rho_1}\,. \tag{D.4}$$

Proof. For brevity, let us set $S = \Sigma_H(u, r_1, r_2)$. The maps $\ell_u, f_u : \Sigma_H \times (r_1, r_2) \to \mathbb{R}$ and $P_u : \Sigma_H \times (r_1, r_2) \to H$ defined by setting

$$\ell_u(\omega, r) = \frac{r}{\sqrt{1 + u(\omega, r)^2}}\,, \qquad P_u = \frac{r\,\omega}{\sqrt{1 + u(\omega, r)^2}}\,, \qquad f_u = \frac{r\,u(\omega, r)}{\sqrt{1 + u(\omega, r)^2}}\,,$$

are such that $S = \{P_u(\omega, r) + f_u(\omega, r)\,\nu_H : (\omega, r) \in \Sigma_H \times (r_1, r_2)\}$. By elementary computations,

$$|\ell_u - r| \le C\,\eta^2\,r\,, \qquad |\partial_r \ell_u - 1| \le C\,\eta^2\,, \qquad \forall(\omega, r) \in \Sigma_H \times (r_1, r_2)\,, \tag{D.5}$$

from which (D.1) follows, as well as the fact that $P_u = \ell_u\,\omega$ is a C^1-diffeomorphism between $\Sigma_H \times (r_1, r_2)$ and $\mathbf{p}_H(S)$. In particular, $\mathbf{p}_H(S)$ is an open subset of H, and $g_u = ((\mathbf{p}_H)|_S)^{-1} : \mathbf{p}_H(S) \to \mathbb{R}$ defines C^1-function with $S = \{x + g_u(x)\,\nu_H : x \in \mathbf{p}_H(S)\}$ and

$$g_u\big(P_u(\omega, r)\big) = f_u(\omega, r)\,, \qquad \forall(\omega, r) \in \Sigma_H \times (r_1, r_2)\,. \tag{D.6}$$

From (D.6) we find

$$\frac{g_u(x)}{|x|} = \frac{f_u(\omega, r)}{|P_u(\omega, r)|} = u(\omega, r)\,, \qquad \forall x \in \mathbf{p}_H(S) = P_u\big(\Sigma_H \times (r_1, r_2)\big)\,,$$

which gives $|g_u(x)| \le \eta |x|$ for $x \in \mathbf{p}_H(S)$. Similarly, by

$$\frac{\partial_r P_u}{|\partial_r P_u|} = \omega\,, \qquad \big||\partial_r P_u| - 1\big| \le C\,\eta^2\,, \qquad |\partial_r f_u| \le C\,\eta\,, \tag{D.7}$$

$$\big|\nabla_\tau P_u - r\,\tau\big| \le C\,r\,\eta^2\,, \qquad |\nabla_\tau f_u| \le C\,r\,\eta\,, \tag{D.8}$$

differentiating in (D.6) along r we find $\partial_r f_u = (\nabla g_u(P_u))\cdot\partial_r P_u$, and thus $|\nabla g_u(x)\cdot\omega| \le C\,\eta$, while differentiating in (D.6) along $\tau \in \Sigma_H \cap \omega^\perp$, we find $\nabla_\tau f_u = (\nabla g_u(P_u)) \cdot \nabla_\tau P_u$, and thus $|\nabla g_u(x) \cdot \tau| \le C\,\eta$, so that $|\nabla g_u(x)| \le C\,\eta$ for every $x \in \mathbf{p}_H(S)$. Hence, if we define g by suitably extending g_u from $\mathbf{p}_H(S)$ to H in a way that preserves the C^1-bound on g_u, we prove (D.2) and (D.3). Intersecting in (D.2) with $A^{\rho_2}_{\rho_1}$ for $(\rho_1, \rho_2) \subset ((1 + C\,\eta)\,r_1, (1 - C\,\eta^2)\,r_2)$, we see that (D.4) follows by the inclusion

$$\Big\{x + g(x)\,\nu_H : x \in H\Big\} \cap A^{\rho_2}_{\rho_1} \subset \Big\{x + g(x)\,\nu_H : x \in \mathbf{p}_H\big(\Sigma_H(u, r_1, r_2)\big)\Big\}\,.$$

To prove this inclusion, we first notice that $|x + g(x)\,\nu_H| < \rho_2$ implies $(1 - C\,\eta^2)\,r_2 > |x|$, while $\rho_1 \le |x + g(x)\,\nu_H| \le (1 + C\,\eta)\,|x|$ gives $|x| \ge r_1$; therefore $\rho_1 < |x + g(x)\,\nu_H| < \rho_2$ implies $r_1 < |x| < (1 - C\,\eta^2)\,r_2$, which, by (D.1) implies $x \in \mathbf{p}_H(\Sigma_H(u, r_1, r_2))$, as desired. ☐

The mesoscale flatness criterion produces graphicality with respect to hyperplanes with C^1-bounds of the form (D.3). Given two such graphs, corresponding to functions f and g, if one of the two functions, say f, satisfies a mild polynomial C^2-decay at infinity, like (D.9) below, then it is possible to parameterize the graph of g as a normal graph over the graph of f. This idea is precisely formulated in the following lemma, where for $H \in \mathcal{H}$ and a C^1-function $f : H \to \mathbb{R}$ we set

$$
\begin{aligned}
G_H(f) &= \Big\{ x + f(x)\,\nu_H : x \in H \Big\}\,,\\
\nu_f(z) &= \frac{-\nabla f(x) + \nu_H}{\sqrt{1+|\nabla f(x)|^2}}\,, \qquad \text{if } z = x + f(x)\,\nu_H\,,
\end{aligned}
$$

stand for the cylindrical graph of f over H and the unit normal to $G_H(f)$ (pointing in the positive ν_H direction).

Lemma D.2 (Cylindrical graphs as normal graphs over cylindrical graphs)**.** *There exist $\eta \in (0,1)$ with the following property. If $H \in \mathcal{H}$, $R > 1$, $f \in C^2(H)$, and $g \in C^1(H)$ are such that*

$$
\max\Big\{ |f(x)|\,,\ |x|\,|\nabla f(x)|\,,\ |x|\,|\nabla^2 f(x)| : x \in H\,, |x| > R \Big\} < \eta\,, \tag{D.9}
$$

$$
\max\Big\{ \frac{|g(x)|}{|x|}\,,\, |\nabla g(x)| : x \in H \Big\} < \eta\,, \tag{D.10}
$$

then there exists $h \in C^1(G_H(f))$ such that

$$
G_H(g) \setminus B_{4\,R} = \big\{ z + h(z)\,\nu_f(z) : z \in G_H(f) \big\} \setminus B_{4\,R}\,. \tag{D.11}
$$

Proof. Step one: We show that if $z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}$, then there exists a unique $y \in G_H(g)$ such that $y = z + t\,\nu_f(z)$ for some $t \in \mathbb{R}$. To prove this, let

$$
Y_S = \Big\{ y \in \mathbb{R}^{n+1} : |\mathbf{p}_H(y)| > S\,,\ |y \cdot \nu_H| < \eta\,|\mathbf{p}_H(y)| \Big\}\,,
$$

so that, by (D.9) and (D.10), we have

$$
\big(G_H(f) \cup G_H(g)\big) \setminus \mathbf{C}_R^{\nu_H} \subset Y_R\,. \tag{D.12}
$$

Given $z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}$, let $L(z) = \mathrm{cl}\,(Y_R) \cap \{z + t\,\nu_f(z) : t \in \mathbb{R}\}$. Since $\mathrm{cl}\,(Y_R)$ is a closed convex set and $|\nu_f(z) - \nu_H| \le C\,\eta/|\mathbf{p}_H(z)|$ by (D.9), if η is small enough, then $L(z)$ is a compact segment in $\mathbb{R}^{n+1}$, with end-points on $W = (\partial Y_R) \setminus \partial \mathbf{C}_R^{\nu_H}$. Correspondingly, $\ell(z) = \mathbf{p}_H(L(z))$ is a compact segment in $H \cap \{|x| > 2\,R\}$, and there is $[a,b] \subset \mathbb{R}$ such that $\mathbf{p}_H(z + t\,\nu_f(z))$ is a parametrization of $\ell(z)$. If we set $\varphi(t) = \nu_H \cdot (z + t\,\nu_f(z))$ and $\psi(t) = g(\mathbf{p}_H(z + t\,\nu_f(z)))$, then by the above considerations we have $\varphi(a) < \varphi(b)$ and $\{\psi(t) : t \in [a,b]\} \subset\subset (\varphi(a), \varphi(b))$. By the mean value theorem, there exists $t \in \mathbb{R}$ such that $\psi(t) = \varphi(t)$, which means $y = z + t\,\nu_f(z) \in G_H(g)$.

We now want to prove that if $z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}$ and $z + t_i\,\nu_f(z) \in G_H(g)$ for $i = 1, 2$, then $t_1 = t_2$. Indeed, let $x, w_i \in H$ be such that $|x| > R$, $z = x + f(x)\,\nu_H$, and

$$
w_i + g(w_i)\,\nu_H = z + t_i\,\nu_f(z) = x - t_i\,\frac{\nabla f(x)}{\sqrt{1+|\nabla f(x)|^2}} + \Big(f(x) + \frac{t_i}{\sqrt{1+|\nabla f(x)|^2}} \Big)\,\nu_H\,;
$$

then, by (D.9) we have

$$
|w_1 - w_2| \le C\,\eta\,|t_1 - t_2|\,, \qquad |g(w_1) - g(w_2)| \ge (1 - C\,\eta^2)\,|t_1 - t_2|\,.
$$

so that $\mathrm{Lip}(g; H) < \eta$ implies $(1 - C\,\eta^2)\,|t_1 - t_2| \le C\,\eta^2\,|t_1 - t_2|$, and thus $t_1 = t_2$ as soon as η is small enough. This proves the claim.

Step two: By step one there is a function $h : G_H(f) \setminus \mathbf{C}_R^{\nu_H} \to \mathbb{R}$ such that

$$
\big\{ z + h(z)\,\nu_f(z) : z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H} \big\} \subset G(g)\,. \tag{D.13}
$$

Let us now set, for $x \in H$ with $|x| > R$ and $t \in \mathbb{R}$,

$$\Phi(x,t) = \Big(x - t\,\frac{\nabla f(x)}{\sqrt{1+|\nabla f(x)|^2}}\Big) + \Big(f(x) + \frac{t}{\sqrt{1+|\nabla f(x)|^2}}\Big)\,\nu_H\,,$$

and notice that by (D.9) we have

$$\big|\Phi - (x + t\,\nu_H)\big| \le C\,\big(|f| + |t|\,|\nabla f|\big)\,, \tag{D.14}$$
$$\big|\partial_t\Phi - \nu_H\big| \le C\,|\nabla f|\,, \qquad |\nabla^H\Phi - \mathrm{Id}\,_H| \le C\,\big(|\nabla f| + |t|\,|\nabla^2 f|\big)\,. \tag{D.15}$$

We claim that, if we pick C_0 large enough (independent of R), then $Y_{2\,R} \subset \Phi(U)$ where

$$U = \Big\{(x,t) : |x| > R\,,\ |t|\,|\nabla f(x)| < C_0\,\eta^2\,,\ \ |t|\,|\nabla^2 f(x)| < C_0\,\eta^2\Big\}\,.$$

Thanks to (D.14) and (D.15) we can then choose η in terms of C_0 so to entail that Φ is a C^1-diffeomorphism between U and $\Phi(U)$. To prove $Y_{2\,R} \subset \Phi(U)$, we notice that if $y \in Y_{2\,R}$, then by $|\mathbf{p}_H(y)| > 2\,R$ and (D.9) we have

$$\begin{aligned} \mathrm{dist}\big(y, G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big) &\le \big|y - \big(\mathbf{p}_H(y) + f(\mathbf{p}_H(y))\,\nu_H\big)\big| \\ &\le |y - \mathbf{p}_H(y)| + |f(\mathbf{p}_H(y))| \le 2\,\eta\,|\mathbf{p}_H(y)|\,. \end{aligned} \tag{D.16}$$

In particular, if $x \in H$, $|x| \ge R$ is such that $\mathrm{dist}(y, G_H(f) \setminus \mathbf{C}_R^{\nu_H}) = |y - (x + f(x)\,\nu_H)|$, then we have $|x| > R$ thanks to $|\mathbf{p}_H(y)| > 2\,R$ and to

$$|x - \mathbf{p}_H(y)| \le |y - (x + f(x)\,\nu_H)| \le 2\,\eta\,|\mathbf{p}_H(y)|\,, \tag{D.17}$$

provided η is small enough. Since $|x| > R$, we can differentiate $x' \mapsto |y - (x' + f(x')\,\nu_H)|$ at $x' = x$ and deduce from the minimality of x that

$$y = z + t\,\nu_f(z) = \Phi(x,t)\,, \qquad \begin{array}{l} \text{where } z = x + f(x)\,\nu_H \\ \text{and } |t| = \mathrm{dist}\big(y, G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big)\,. \end{array} \tag{D.18}$$

Since, by (D.17), $|\mathbf{p}_H(y)|/|x| \le C$, we can use (D.16) and (D.9) to infer

$$|t|\,|\nabla f(x)| \le 2\,\eta\,|\mathbf{p}_H(y)|\,\frac{\eta}{|x|} \le C_0\,\eta^2\,,$$

and, analogously, that $|t|\,|\nabla^2 f(x)| \le C_0\,\eta^2$, for a suitable constant C_0.

Having proved that $Y_{2\,R} \subset \Phi(U)$ and that Φ is a C^1-diffeomorphism between U and $\Phi(U)$, we then notice that by (D.10) we have $G_H(g) \setminus \mathbf{C}_{2R}^{\nu_H} \subset Y_{2\,R}$, and thus conclude that for every $y \in G_H(g) \setminus \mathbf{C}_{2R}^{\nu_H}$ there exists a unique $(x,t) \in U$ such that $y = \Phi(x,t) = z + t\,\nu_f(z)$ for $z = x + f(x)\,\nu_H$. Since $(x,t) \in U$ implies $|x| > R$, we have $z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}$, and thus it must be, by step one, that $t = h(z)$. We have thus proved that

$$G_H(g) \setminus \mathbf{C}_{2\,R}^{\nu_H} \subset \big\{z + h(z)\,\nu_f(z) : z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big\}\,, \tag{D.19}$$

and, taking into account (D.16) and (D.18), that $|h(z)| \le 2\,\eta\,|\mathbf{p}_H(z + h(z)\,\nu_f(z))|$, and thus that

$$|h(z)| \le C\,\eta\,|\mathbf{p}_H(z)|, \qquad \forall z \in G(f) \setminus \mathbf{C}_R^{\nu_H}\,. \tag{D.20}$$

Combining (D.13) and (D.19) we thus find that

$$G_H(g) \setminus \mathbf{C}_{2\,R}^{\nu_H} = \big\{z + h(z)\,\nu_f(z) : z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big\} \setminus \mathbf{C}_{2\,R}^{\nu_H}\,. \tag{D.21}$$

By (D.9), (D.10), and (D.20) we see that, if η is small enough, then

$$\begin{aligned} &G_H(g) \cap \mathbf{C}_{2\,R}^{\nu_H} \cap \big\{y : |y \cdot \nu_H| > 2\,R\big\} = \emptyset\,, \\ &\big\{z + h(z)\,\nu_f(z) : z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big\} \cap \mathbf{C}_{2\,R}^{\nu_H} \cap \big\{y : |y \cdot \nu_H| > 2\,R\big\} = \emptyset\,, \end{aligned}$$

which, combined with (D.21) gives

$$G_H(g) \setminus \mathbf{C}_{2\,R,2\,R}^{\nu_H} = \big\{z + h(z)\,\nu_f(z) : z \in G_H(f) \setminus \mathbf{C}_R^{\nu_H}\big\} \setminus \mathbf{C}_{2\,R,2\,R}^{\nu_H}\,. \tag{D.22}$$

Since $\mathbf{C}^{\nu_H}_{2R,2R} \subset B_{4R}$, upon extending h from $G_H(f) \setminus \mathbf{C}^{\nu_H}_R$ to the whole $G_H(f)$ so that $|h(z)| \le 2R$ on $G(f) \cap \mathbf{C}^{\nu_H}_R$, we deduce (D.11) from (D.22). □

Appendix E. Obstacles with zero isoperimetric residue

We conclude with some remarks on the case $\mathcal{R}(W) = 0$, which can be largely addressed through a synoptical reading of the main arguments of the paper. To exemplify this point, let us explain why (1.21) holds even when $\mathcal{R}(W) = 0$. On the one hand, by comparison with balls of volume v we trivially have

$$\limsup_{v\to\infty} \psi_W(v) - P(B^{(v)}) \le 0 = \mathcal{R}(W)\,.$$

To prove the matching lower bound one argues by contradiction, and assumes that, for some sequence $\{(E_j, v_j)\}_j$ with $v_j \to \infty$ and E_j minimizer of $\psi_W(v_j)$, it holds that

$$\liminf_{v\to\infty} \psi_W(v) - P(B^{(v)}) = \lim_{j\to\infty} P(E_j;\Omega) - P(B^{(v_j)}) < 0\,. \tag{E.1}$$

With (E.1) replacing $\mathcal{R}(W) > 0$, one can repeat *verbatim* the proof of property (a) in step two of the proof of Theorem 1.1; said property can then be used to derive the asymptotic expansion for F as in the proof of property (c), which is then the key fact used in step three to derive that

$$\lim_{j\to\infty} P(E_j;\Omega) - P(B^{(v_j)}) \ge -\mathrm{res}_W(F \cup W, \nu) \ge -\mathcal{R}(W)\,;$$

the latter inequality is of course in contradiction with (E.1) if $\mathcal{R}(W) = 0$.

This said, the class of obstacles such that $\mathcal{R}(W) = 0$ seems of rather limited interest from a physical/geometric viewpoint, because of the following proposition.

Proposition E.1. *If W is compact and $\mathcal{R}(W) = 0$, then W is purely $\mathcal{H}^n$-unrectifiable, in the sense that W cannot contain an $\mathcal{H}^n$-rectifiable set of $\mathcal{H}^n$-positive measure. In a partial converse, if W is purely $\mathcal{H}^n$-unrectifiable and $\mathcal{H}^n(W) < \infty$, then $\mathcal{R}(W) = 0$.*

Proof. Step one: We prove the first statement. We argue by contradiction, and assume the existence of an $\mathcal{H}^n$-rectifiable set S with $\mathcal{H}^n(W \cap S) > 0$. By [Sim83a, Lemma 11.1], we can assume without loss of generality that S is a C^1-embedded hypersurface in $\mathbb{R}^{n+1}$. Let x be a point of tangential differentiability for $W \cap S$ so that, in particular, $\mathcal{H}^n(W \cap S \cap B_\rho(x)) = \omega_n\,\rho^n + \mathrm{o}_x(\rho^n)$ as $\rho \to 0^+$. Since S is a C^1-embedded hypersurface, there is $\nu \in \mathbb{S}^n$ such that for every $\varepsilon > 0$ there is $\rho_* = \rho_*(x,\varepsilon) > 0$ with

$$S \cap \mathbf{C}^\nu_{\rho_*,\rho_*}(x) = \{y + g(y)\,\nu : y \in \mathbf{D}^\nu_{\rho_*}(x)\}\,,$$

where $g : (x + \nu^\perp) \to \mathbb{R}$ a C^1-function with $g(x) = 0$ and $\mathrm{Lip}(g) \le \varepsilon$. Denoting $G(g) = \{y + g(y)\,\nu : y \in (x + \nu^\perp)\}$, and up to decrease the value of ρ_*, we can entail

$$\mathcal{H}^n\big(G(g) \cap W \cap \mathbf{C}^\nu_{\rho_*}(x)\big) \ge \mathcal{H}^n(W \cap S \cap B_{\rho_*}(x)) \ge (1-\varepsilon)\,\omega_n\,\rho_*^n\,. \tag{E.2}$$

Since $|g| \le \varepsilon\,\rho_*$ on $\partial\mathbf{D}^\nu_{\rho_*}(x)$, we can define $f : (x+\nu^\perp) \to \mathbb{R}$ so that $f = g$ on $\mathbf{D}^\nu_{\rho_*}(x)$, $f = 0$ on $(x + \nu^\perp) \setminus \mathbf{D}^\nu_{2\rho_*}(x)$, and $\mathrm{Lip}(f) \le \varepsilon$. Denoting by F the epigraph of f, we have that $(F,\nu) \in \mathcal{F}$ and we compute, for R large enough to entail $\mathbf{C}^\nu_{2\rho_*}(x) \cup W \subset\subset \mathbf{C}^\nu_R$,

$$\begin{aligned}
\omega_n\,R^n - P(F;\mathbf{C}^\nu_R \setminus W) &\ge \omega_n\,(2\rho_*)^n - P(F;\mathbf{C}^\nu_{2\rho_*}(x) \setminus W)\\
&= \int_{\mathbf{D}^\nu_{2\rho_*}(x)} 1 - \sqrt{1+|\nabla f|^2} + P(F;\mathbf{C}^\nu_{2\rho_*}(x) \cap W)\\
&\ge -\omega_n\,(2\rho_*)^n\,\varepsilon^2 + (1-\varepsilon)\,\omega_n\,{\rho_*}^n
\end{aligned}$$

where in the first identity we have used $f = 0$ on $\nu^\perp \setminus \mathbf{D}^\nu_{2\rho_*}(x)$ and in the last inequality we have used (E.2) and $\sqrt{1+\varepsilon^2} \le 1+\varepsilon^2$. Up to taking $\varepsilon < \varepsilon(n)$, we thus find $\mathrm{res}_W(F,\nu) > 0$, and thus deduce $\mathcal{R}(W) > 0$.

Step two: We now assume that W is purely $\mathcal{H}^n$-unrectifiable and such that $\mathcal{H}^n(W) < \infty$. Let (F, ν) be a maximizer for $\mathcal{R}(W)$. Since F is a local perimeter minimizer in Ω, we can assume that F is open in Ω with $\Omega \cap \partial F = \mathrm{cl}\,(\partial^* F)$ ($\partial^* F =$ the reduced boundary of F as a set of locally finite perimeter in Ω). Now, $\omega_n\, R^n - P(F; \mathbf{C}_R^\nu \setminus W)$ is decreasing towards $\mathcal{R}(W) \ge \mathcal{S}(W) \ge 0$, therefore $P(F; \mathbf{C}_R^\nu \setminus W) < \infty$ for every R. In particular, $\mathcal{H}^n \llcorner (\Omega \cap \partial F)$ is a Radon measure on $\mathbb{R}^{n+1}$. Now, $\partial F \subset (\Omega \cap \partial F) \cup W$, so that $\mathcal{H}^n(W) < \infty$ implies that $\mathcal{H}^n \llcorner \partial F$ is a Radon measure on $\mathbb{R}^{n+1}$ and, since F is open, that F is a set of finite perimeter in $\mathbb{R}^{n+1}$ thanks to [Fed69, Theorem 4.5.11]. Therefore we can use the purely $\mathcal{H}^n$-unrectifiability of W to conclude that $P(F; \mathbf{C}_R^\nu \setminus W) = P(F; \mathbf{C}_R^\nu)$, where $P(F; \mathbf{C}_R^\nu) \ge \omega_n\, R^n$ by (1.8) and (1.9), and thus $\mathcal{R}(W) = \mathrm{res}_W(F, \nu) \le 0$. This proves $\mathcal{R}(W) = 0$. □

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